MATH 6931 MATHEMATICAL MODELLING — Prof. Madras

Some Examples of Markov Chains

1. Weather Models

In an effort to model weather patterns, some meteorologists have tried to describe the sequence of "wet" and "dry" days as a Markov chain. The numerical values of the transition probabilities will depend upon the time of year and the location.

Formally, we consider a two-state Markov chain, with state space

$$\mathcal{S} = \{ \text{ dry, wet } \}.$$

The transition probabilities are then given by a 2×2 matrix, such as the following:

$$\begin{array}{c} \\ \text{dry} \\ \text{wet} \end{array} \begin{array}{c} \begin{array}{cc} \text{dry} & \text{wet} \end{array} \\ \begin{pmatrix} 0.817 & 0.183 \\ 0.414 & 0.586 \end{pmatrix} \end{array}$$

(With this matrix, would you expect dry periods to last longer than wet periods, or vice versa?) The values of the transition probabilities must be estimated using statistical methods based on historical data. The above matrix was estimated for January weather in Berkeley, California by R.W. Katz and M.B. Parlange (in the journal *Water Resources Research*, vol. 29, pp. 2335–2344, July 1993).

Let P be the above 2×2 matrix. Then one can evaluate the m-day transition probabilities by computing powers of P, for example:

$$P^2 = \begin{pmatrix} 0.7433 & 0.2567 \\ 0.5808 & 0.4192 \end{pmatrix} \qquad P^6 = \begin{pmatrix} 0.6948 & 0.3052 \\ 0.6905 & 0.3095 \end{pmatrix}$$

Thus we see that after 6 days, the knowledge of the first day has a very minor effect on the probability. That is, the second column of P^6 shows that

$$\Pr\{ \text{ Day 6 is wet} \mid \text{Day 0 is wet } \} \approx \Pr\{ \text{ Day 6 is wet} \mid \text{Day 0 is dry } \}$$

In fact,

$$\lim_{m \to \infty} P^m = \begin{pmatrix} 0.6935 & 0.3065 \\ 0.6935 & 0.3065 \end{pmatrix}.$$

2. Credit Risk

In finance, it is important to estimate the risk associated with bonds issued by (and loans to) companies (and governments). One way to do this is to assign *credit ratings*. One major company, Standard & Poor's (S&P), has seven main categories: from best to worst, these are AAA, AA, A, BBB, BB, B, and CCC. A firm rated AAA is judged to be very safe; at the other extreme, a firm rated CCC has a significant risk of *defaulting* (i.e., being unable to repay the loan). Since circumstances change over time, the firms are frequently reassessed, and their ratings can change over time. The *CreditMetrics* system assumes that the rating of a particular firm changes from year to year according to a Markov chain, until that firm defaults.

S&P publishes frequent tables of these probabilities. For example, the following is from Standard & Poor's CreditWeek April 15, 1996:

Initial	Rating at year-end							
Rating	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	.9081	.0833	.0068	.0006	.0012	0	0	0
AA	.0070	.9065	.0779	.0064	.0006	.0014	.0002	0
A	.0009	.0227	.9105	.0552	.0074	.0026	.0001	.0006
BBB	.0002	.0033	.0595	.8693	.0530	.0117	.0012	.0018
BB	.0003	.0014	.0067	.0773	.8053	.0884	.0010	.0106
B	0	.0011	.0024	.0043	.0648	.8346	.0407	.0520
CCC	.0022	0	.0022	.0130	.0238	.1124	.6486	.1979

To formalize this as a Markov chain, we use a state space with eight states:

$$\mathcal{S} = \{\text{AAA, AA, A, BBB, BB, B, CCC, Default}\}.$$

The transition probability matrix is an 8×8 matrix, whose first seven rows are given by the table above. The eighth row are the probabilities of moving from "default" to other states. This is not relevant to typical applications: if a firm defaults on a loan you have made, then your money is gone, and that is the end of it. In such a situation, the natural thing to do is to make $P_{Default,Default} = 1$, so that the last row of the transition matrix is

	AAA	AA	A	BBB	BB	B	CCC	Default
Default	0	0	0	0	0	0	0	1

We say that "Default" is an "absorbing state": once the chain enters an absorbing state, it can never leave.

With this transition matrix, one can compute the probability that a firm with rating BBB, say, will default in the next 15 years. This is very relevant to anyone holding a 15-year bond that this firm has issued! Or, one can ask the expected number of years until a firm defaults, given that it is presently rated AA.

(For anyone who wishes to learn more about CreditMetrics, see *CreditMetrics*™ — *Technical Document* by G.M. Gupton, C.C. Finger, and M. Bhatia (J.P. Morgan and Co., 1997), at `www.riskmetrics.com/cmtdovv.html`. This is not required for the present course, however!)

So, taking our 8×8 one-year transition probability matrix P to be

	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	.9081	.0833	.0068	.0006	.0012	0	0	0
AA	.0070	.9065	.0779	.0064	.0006	.0014	.0002	0
A	.0009	.0227	.9105	.0552	.0074	.0026	.0001	.0006
BBB	.0002	.0033	.0595	.8693	.0530	.0117	.0012	.0018
BB	.0003	.0014	.0067	.0773	.8053	.0884	.0010	.0106
B	0	.0011	.0024	.0043	.0648	.8346	.0407	.0520
CCC	.0022	0	.0022	.0130	.0238	.1124	.6486	.1979
Default	0	0	0	0	0	0	0	1

we then find that the 15-year transition matrix by computing P^{15}, which turns out to be

	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	.2549	.3563	.2526	.0823	.0258	.0160	.0018	.0089
AA	.0321	.3104	.3852	.1587	.0515	.0330	.0038	.0225
A	.0103	.1175	.4137	.2423	.0919	.0609	.0071	.0494
BBB	.0050	.0574	.2600	.2849	.1408	.1083	.0135	.1130
BB	.0030	.0275	.1278	.1903	.1567	.1605	.0223	.2577
B	.0020	.0134	.0554	.0968	.1170	.1667	.0264	.5044
CCC	.0030	.0098	.0356	.0564	.0619	.0873	.0148	.7230
Default	0	0	0	0	0	0	0	1

Hence, for example, suppose that a firm has rating AA now. The probability that it will default within the next 15 years is $P_{AA,Default}$, which we see is 0.0225. For a firm that has rating BBB now, the corresponding probability is 0.1130.