15) Determine whether the following systems of linear equations is consistent. In case of consistent, find the dimension of their solution set.

1) $x_1 - 2x_2 + x_3 + 2x_4 = 1$
$x_1 + x_2 - x_3 + x_4 = 2$
$x_1 + 7x_2 - 5x_3 - x_4 = 3$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{array}\right]$$

$\therefore x_1 - 2x_2 + x_3 + 2x_4 = 1$
& $3x_2 - 2x_3 - x_4 = 1$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$= \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -3 & 2 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1) $x_1 - 2x_2 + x_3 + 2x_4 = 1$
$x_1 + x_2 - x_3 + x_4 = 2$
$x_1 + 7x_2 - 5x_3 - x_4 = 3$

In matrix form -

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{array}\right]$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 9 & -6 & -3 & 2 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right]$$

at row 3

$0x_1 + 0x_2 + 0x_3 + 0x_4 = -1$

$\therefore 0 = -1 \leftarrow$ not possible

$\therefore$ System is inconsistent and has no sol^n. Ans

2) $x_1 - x_2 + 2x_3 = 1$
$2x_1 + 2x_3 = 1$
$x_1 - 3x_2 + 4x_3 = 2$

In matrix form -

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{array}\right]$$

$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & -3 & 2 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 1 & 0 & 0 \\ -2 & -2 & 0 & 0 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

at row 3

$0x_1 + 0x_2 + 0x_3 = 0$

$\boxed{0 = 0}$

$\therefore$ System will have ~~unique~~ more than one sol^n.

$\therefore$ System is consistent.

Now, $\dim(A) = col(A) - nonzero\ rows(A)$

$= 3 - 2$

$[\dim(A) = 1.]$

3) $2x_1-3x_2-7x_3+5x_4+2x_5=-2$
$x_1-2x_2-4x_3+3x_4+x_5=-2$
$2x_1-4x_3+2x_4+x_5=3$
$x_1-5x_2-7x_3+6x_4+2x_5=-7$

In matrix form-

$$\left[\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ 1 & -2 & -4 & 3 & 1 & -2 \\ 2 & 0 & -4 & 2 & 1 & 3 \\ 1 & -5 & -7 & 6 & 2 & -7 \end{array}\right]$$

$R_2 \to R_2-R_1$, $R_3 \to R_3-R_1$, $R_4 \to R_4-R_1$

$$\to \left[\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ -1 & 1 & 3 & -2 & -1 & 0 \\ 0 & 3 & 3 & -3 & -1 & 5 \\ -1 & -2 & 0 & 1 & 0 & -5 \end{array}\right]$$

$R_3 \to R_3-R_2$, $R_4 \to R_4-R_2$

$$\to \left[\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ -1 & 1 & 3 & -2 & -1 & 0 \\ 1 & 2 & 0 & -1 & 0 & 5 \\ 0 & -3 & -3 & 3 & 1 & -5 \end{array}\right]$$

$R_4 \to R_4+R_2$

$$\to \left[\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ -1 & 1 & 3 & -2 & -1 & 0 \\ 1 & 2 & 0 & -1 & 0 & 5 \\ -1 & -2 & 0 & 1 & 0 & -5 \end{array}\right]$$

$R_4 \to R_4+R_3$

$$\to \left[\begin{array}{ccccc|c} 2 & -3 & -7 & 5 & 2 & -2 \\ -1 & 1 & 3 & -2 & -1 & 0 \\ 1 & 2 & 0 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

• in row 4 -
$0x_1+0x_2+0x_3+0x_4+0x_5=0$
$\therefore \boxed{0=0}$

$\therefore$ system will have more than one sol^n.
$\therefore$ System is consistent.

$\dim(A)=5-3$
$=2.$

Ans

16) Determine whether the following matrices are positive definite, positive semi-definite, negative definite, negative semi-definite or indefinite.

1) $A=\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ -1 & -1 & -3 \end{bmatrix}$

Leading Principal minor of matrix A ->

$d_1=[-2]=-2$. —(1)

$d_2=\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}=4$ —(2)

$d_3=\begin{vmatrix} -2 & 0 & -1 \\ 0 & -2 & -1 \\ -1 & -1 & -3 \end{vmatrix} = -2(6-1)-0(\;)+(-1)(0-2)$
$=-10+2=-8$. —(3)

$\therefore$ Some are -ve & some are +ve leading principal minors.
$\therefore$ Given matrix has ~~Indefinite~~ $d_1, d_3 \to$ -ve & $d_2 \to$ +ve
$\therefore$ It is negative definite.

2) $A=\begin{bmatrix} -2 & 4 & -1 \\ 4 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$

Leading Principal minors of matrix A ->

$d_1=[-2]=-2$. —(1)

$d_2=\begin{bmatrix} -2 & 4 \\ 4 & -2 \end{bmatrix}=4-16=-12$ —(2)

$d_3=\begin{vmatrix} -2 & 4 & -1 \\ 4 & -2 & -1 \\ -1 & -1 & -2 \end{vmatrix} = -2(4-1)-4(-8+1)-1(-4+2)$
$=-6+28+2$
$=24$ —(3)

Here, $d_1, d_2 \to$ -ve & $d_3 \to$ +ve
$\therefore$ Matrix is Indefinite.

3) $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & -2 \\ -1 & -2 & -4 \end{bmatrix}$

Leading Principal minors of A ⇒

$d_1 = |2| = 2.$ —(1)

$d_2 = \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 1 = 7$ —(2)

$d_3 = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 4 & -2 \\ -1 & -2 & -4 \end{vmatrix} = 2(-16-4) - 1(-4-2) - 1(-2+4)$

$= -40 + 6 - 2$

$= -36.$ —(3)

∵ $d_1, d_2 \to +ve$ & $d_3 \to -ve$

∴ Given matrix is Indefinite.

4) $A \Rightarrow \begin{bmatrix} 2 & -1 & 3 \\ -1 & 5 & 3 \\ 3 & 3 & 9 \end{bmatrix}$

Leading Principal minors of A ⇒

$d_1 = |2| = 2.$ —(1)

$d_2 = \begin{vmatrix} 2 & -1 \\ -1 & 5 \end{vmatrix} = 10 - 1 = 9$ —(2)

$d_3 = \begin{vmatrix} 2 & -1 & 3 \\ -1 & 5 & 3 \\ 3 & 3 & 9 \end{vmatrix} = 2(45-9) + 1(-9-9) + 3(-3-15)$

$= 72 - 18 - 54$

$= 72 - 72$

$= 0.$

∵ $d_1, d_2 \to +ve$, $d_3 \to 0$

∴ Given matrix is Positive semidefinite.

5) $A \cdot \begin{bmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -5 \end{bmatrix}$

Leading Principal Minor of A ⇒

$d_1 = |-2| = -2$ —(1)

$d_2 = \begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix} = 6 - 1 = 5$ —(2)

$d_3 = \begin{vmatrix} -2 & 1 & -1 \\ 1 & -3 & -2 \\ -1 & -2 & -5 \end{vmatrix} = -2(15-4) - 1(-5-2) - 1(-2-3)$

$= -22 + 7 + 5$

$= -10.$

∵ $d_1, d_3 \to -ve$ & $d_2 \to +ve$

∴ Given matrix is negative definite.

17) For what value(s) of b, the following matrix is +ve definite.

$\begin{bmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{bmatrix}$

A matrix to be +ve definite, its leading principal minor must be strictly greater than 0.

∴ $d_1 = 2 > 0$ → true

$d_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$ → true

$d_3 = \begin{vmatrix} 2 & -1 & b \\ -1 & 2 & -1 \\ b & -1 & 2 \end{vmatrix} > 0$

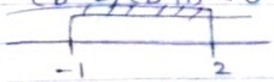
$= 2(4-1) + 1(-2+b) + b(1-2b) > 0$

$= 6 - 2 + b + b - 2b^2 > 0$

$= 2b^2 - 2b - 4 < 0$

$= b^2 - b - 2 < 0$

$= (b-2)(b+1) < 0$



[∵ $-1 \leq b < 2$] Ans

18) Let S_+^2 denote the set of all 2×2 positive semidefinite matrices. Then show that -

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S_+^2 \iff x \geq 0,\ z \geq 0,\ xz \geq y^2$$

<u>Sol</u>ⁿ - Here, the given matrix is +ve semi-definite

∴ Its Principal Minors are greater than or equal to 0.

$\therefore\ d_1 = x \geq 0$

$d_2 = z \geq 0$

$d_3 = \begin{vmatrix} x & y \\ y & z \end{vmatrix} = xz - y^2 \geq 0 \Rightarrow xz \geq y^2$

$$\therefore \begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S_+^2 \iff x \geq 0,\ z \geq 0,\ xz \geq y^2.$$

[illegible]

<u>Hence, Proved</u>