Examples of discrete probability distributions:

The binomial and Poisson distributions

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Binomial Probability Distribution

- A fixed number of observations (trials), n
  - e.g., 15 tosses of a coin; 20 patients; 1000 people surveyed
- A binary random variable
  - e.g., head or tail in each toss of a coin; defective or not defective light bulb
  - Generally called "success" and "failure"
  - Probability of success is p, probability of failure is 1 – p
- Constant probability for each observation
  - e.g., Probability of getting a tail is the same each time we toss the coin

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Binomial example

Take the example of 5 coin tosses. What's the probability that you flip exactly 3 heads in 5 coin tosses?

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Binomial distribution

Solution:
One way to get exactly 3 heads: HHHTT

What's the probability of this exact arrangement?
P(heads)xP(heads) xP(heads)xP(tails)xP(tails)
$=(1/2)^3 \, x \, (1/2)^2$

Another way to get exactly 3 heads: THHHT
Probability of this exact outcome = $(1/2)^1 \, x \, (1/2)^3 \, x \, (1/2)^1 = (1/2)^3 \, x \, (1/2)^2$

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Binomial distribution

In fact, $(1/2)^3 \, x \, (1/2)^2$ is the probability of each unique outcome that has exactly 3 heads and 2 tails.

So, the overall probability of 3 heads and 2 tails is:
$(1/2)^3 \, x \, (1/2)^2 + (1/2)^3 \, x \, (1/2)^2 + (1/2)^3 \, x \, (1/2)^2 + \ldots$.. for as many unique arrangements as there are—but how many are there??

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Outcome	Probability
THHHT	$(1/2)^3 \, x \, (1/2)^2$
HHHTT	$(1/2)^3 \, x \, (1/2)^2$
TTHHH	$(1/2)^3 \, x \, (1/2)^2$
HTTHH	$(1/2)^3 \, x \, (1/2)^2$
HHTTH	$(1/2)^3 \, x \, (1/2)^2$
THTHH	$(1/2)^3 \, x \, (1/2)^2$
HTHTH	$(1/2)^3 \, x \, (1/2)^2$
HHTHT	$(1/2)^3 \, x \, (1/2)^2$
THHTH	$(1/2)^3 \, x \, (1/2)^2$
HTHHT	$(1/2)^3 \, x \, (1/2)^2$

10 arrangements $x \, (1/2)^3 \, x \, (1/2)^2$

$\binom{5}{3}$ ways to arrange 3 heads in 5 trials

The probability of each unique outcome (note: they are all equal)

${}_5C_3 = 5!/3!2! = 10$

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$\therefore$ **P(3 heads and 2 tails) =** $\binom{5}{3}$ ***x P(heads)***^{3} ***x P(tails)***^{2} **=**

10 x (½)^5***=31.25%***

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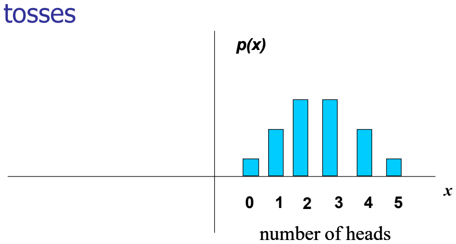
Binomial distribution function:

X= the number of heads tossed in 5 coin tosses



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Binomial distribution, generally

Note the general pattern emerging → if you have only two possible outcomes (call them 1/0 or yes/no or success/failure) in *n* independent trials, then the probability of exactly *X* "successes"=

n = number of trials

$$\boxed{\binom{n}{X} p^X (1-p)^{n-X}}$$

1-p = probability of failure

X = # successes out of *n* trials

p = probability of success

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Definitions: Binomial

- **Binomial:** Suppose that n independent experiments, or trials, are performed, where n is a fixed number, and that each experiment results in a "success" with probability p and a "failure" with probability $1-p$. The total number of successes, X, is a binomial random variable with parameters n and p.
- We write: ***$X \sim Bin(n, p)$*** *{reads: "X is distributed binomially with parameters n and p}*
- And the probability that $X=r$ (i.e., that there are exactly r successes) is:

$$P(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$$

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Definitions: Bernouilli

Bernouilli trial: If there is only 1 trial with probability of success p and probability of failure $1-p$, this is called a Bernouilli distribution. (special case of the binomial with n=1)

Probability of success: $P(X=1) = \binom{1}{1} p^1 (1-p)^{1-1} = p$

Probability of failure: $P(X=0) = \binom{1}{0} p^0 (1-p)^{1-0} = 1-p$

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Binomial distribution: example

- If I toss a coin 20 times, what's the probability of getting exactly 10 heads?

$$\binom{20}{10} (.5)^{10} (.5)^{10} = .176$$

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Binomial distribution: example

- If I toss a coin 20 times, what's the probability of getting of getting 2 or fewer heads?

$$\binom{20}{0}(.5)^0(.5)^{20} = \frac{20!}{20!0!}(.5)^{20} = 9.5x10^{-7} +$$
$$\binom{20}{1}(.5)^1(.5)^{19} = \frac{20!}{19!1!}(.5)^{20} = 20x9.5x10^{-7} = 1.9x10^{-5} +$$
$$\binom{20}{2}(.5)^2(.5)^{18} = \frac{20!}{18!2!}(.5)^{20} = 190x9.5x10^{-7} = 1.8x10^{-4}$$
$$= 1.8x10^{-4}$$

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****All probability distributions are characterized by an expected value and a variance:**

<u>If</u> *X* follows a binomial distribution with parameters *n* and *p*: ***X ~ Bin (n, p)***

<u>Then:</u>

$\mu_x = E(X) = np$

$\sigma_x^2 = Var(X) = np(1-p)$

$\sigma_x = SD(X) = \sqrt{np(1-p)}$

Note: the variance will always lie between
0*N-.25 *N
p(1-p) reaches maximum at p=.5
P(1-p)=.25

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Characteristics of Bernouilli distribution

<u>For Bernouilli ($n=1$)</u>

$E(X) = p$

$Var(X) = p(1-p)$

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Recall coin toss example

- X= number of heads in 100 tosses of a coin
- *X ~ Bin (100, .5)*
- *E(x) = 100*.5=50*
- *Var(X) = 100*.5*.5 = 25*
- *SD(X) = 5*

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Multinomial distribution

The multinomial is a generalization of the binomial. It is used when there are more than 2 possible outcomes (for ordinal or nominal, rather than binary, random variables).

- Instead of partitioning *n* trials into 2 outcomes (yes with probability *p* / no with probability *1-p*), you are partitioning *n* trials into 3 or more outcomes (with probabilities: p_1, p_2, p_3,..)
  - General formula for 3 outcomes:

$$P(D=x, R=y, G=z) = \frac{n!}{x!\,y!\,z!} p_D^x p_R^y (1-p_D-p_R)^z$$

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Multinomial example

Specific Example: if you are randomly choosing 8 people from an audience that contains 50% democrats, 30% republicans, and 20% green party, what's the probability of choosing exactly 4 democrats, 3 republicans, and 1 green party member?

$$P(D=4, R=3, G=1) = \frac{8!}{4!\,3!\,1!}(.5)^4(.3)^3(.2)^1$$

You can see that it gets hard to calculate very fast! The multinomial has many uses in genetics where a person may have 1 of many possible alleles (that occur with certain probabilities in a given population) at a gene locus.

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Introduction to the **Poisson Distribution**

- Poisson distribution is for counts—if events happen at a constant rate over time, the Poisson distribution gives the probability of X number of events occurring in time T.

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Poisson Mean and Variance

- Mean $\mu = \lambda$

For a Poisson random variable, the variance and mean are the same!

- Variance and Standard Deviation

$$\sigma^2 = \lambda$$

$$\sigma = \sqrt{\lambda}$$

where λ = expected number of hits in a given time period

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Poisson Distribution, example

The Poisson distribution models counts, such as the number of new cases of SARS that occur in women in New England next month.

The distribution tells you the probability of all possible numbers of new cases, from 0 to infinity.

If X= # of new cases next month and $X \sim$ Poisson (λ), then the probability that $X=k$ (a particular count) is:

$$p(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

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Example

- For example, if new cases of West Nile Virus in New England are occurring at a rate of about 2 per month, then these are the probabilities that: 0,1, 2, 3, 4, 5, 6, to 1000 to 1 million to… cases will occur in New England in the next month:

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Poisson Probability table

X	P(X)
0	$\frac{2^0 e^{-2}}{0!}$ =.135
1	$\frac{2^1 e^{-2}}{1!}$ =.27
2	$\frac{2^2 e^{-2}}{2!}$ =.27
3	$\frac{2^3 e^{-2}}{3!}$ =.18
4	=.09
5	
…	…

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more on Poisson…

"Poisson Process" (rates)

Note that the Poisson parameter λ can be given as the mean number of events that occur in a defined time period OR, equivalently, λ can be given as a rate, such as λ=2/month (2 events per 1 month) that must be multiplied by *t*=time (called a "Poisson Process") →

X ~ Poisson (λ)

$$P(X=k)=\frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

E(X) = λt

Var(X) = λt

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Example

For example, if new cases of West Nile in New England are occurring at a rate of about 2 per month, then what's the probability that exactly 4 cases will occur in the next 3 months?

X ~ Poisson (λ=2/month)

$$P(X = 4 \text{ in } 3 \text{ months}) = \frac{(2*3)^4 e^{-(2*3)}}{4!} = \frac{6^4 e^{-(6)}}{4!} = 13.4\%$$

Exactly 6 cases?

$$P(X = 6 \text{ in } 3 \text{ months}) = \frac{(2*3)^6 e^{-(2*3)}}{6!} = \frac{6^6 e^{-(6)}}{6!} = 16\%$$

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Practice problems

1a. If calls to your cell phone are a Poisson process with a constant rate λ=2 calls per hour, what's the probability that, if you forget to turn your phone off in a 1.5 hour movie, your phone rings during that time?

1b. How many phone calls do you expect to get during the movie?

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Answer

1a. If calls to your cell phone are a Poisson process with a constant rate λ=2 calls per hour, what's the probability that, if you forget to turn your phone off in a 1.5 hour movie, your phone rings during that time?

X ~ Poisson (λ=2 calls/hour)

P(X≥1)=1 – P(X=0)

$$P(X=0) = \frac{(2*1.5)^0 e^{-2(1.5)}}{0!} \frac{(3)^0 e^{-3}}{0!} = e^{-3} = .05$$

$\therefore$P(X≥1)=1 – .05 = 95% chance

1b. How many phone calls do you expect to get during the movie?

E(X) = λt = 2(1.5) = 3

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