4.3. Cramer's rule; Formula for A^{-1}; Applications of Determinants

Theorem

If the entries in an row (column) of a square matrix are multiplied by the cofactors of the corresponding entries in a different row (column), then the sum of the product is zero.

Adjoint of a matrix

Definition

If A is an $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the matrix of cofactors from A.
The transpose of this matrix is called the adjoint of A and is denoted by $\mathrm{adj}(A)$.

Adjoint of a matrix

Example

Find the adjoint of $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

Inverse of a matrix

Theorem

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)} adj(A).$$

Inverse of a matrix

Example

Find the inverse of $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$

Cramer's rule

Theorem

If $A\mathbf{x} = \mathbf{b}$ *is a linear system of* n *equations in* n *unknowns, then the system has a unique solution if and only if* $\det(A) \neq 0$*, in which case the solution is*

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j *is the matrix that results when the* j*th column of* A *is replaced by* $\mathbf{b}$*.*

Cramer's rule

Example

Use Cramer's rule to solve the system

$$\begin{aligned} x_1 \quad\quad\quad + \quad 2x_2 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Geometric interpretation of determinants

Cross Products