7.5 The Rank Theorem and Its Implications

Theorem (The Rank Theorem)

The row space and column space of a matrix have the same dimension.

The Rank Theorem

Theorem

If A is an $m \times n$ matrix, then

(a) $rank(A) = rank(A^T)$.

(b) $rank(A) + nullity(A^T) = m$.

Corollary

If A is an $m \times n$ matrix with rank k, then

$$\dim(row(A)) = k, \quad \dim(null(A)) = n - k$$

$$\dim(col(A)) = k, \quad \dim(null(A^T)) = m - k$$

Relationship between consistency and rank

Theorem (The consistency theorem)

If $A\mathbf{x} = \mathbf{b}$ is a linear system of m equations in n unknowns, then the followings are equivalent:

(a) *$A\mathbf{x} = \mathbf{b}$ is consistent.*

(b) *$\mathbf{b}$ is in the column space of A.*

(c) *The coefficient matrix A and the augmented matrix $[A|\mathbf{b}]$ have the same rank.*

Relationship between consistency and rank

Example

Consider the linear system

$$\begin{array}{rcrcrcr}
x_1 & - & 2x_2 & - & 3x_3 & = & -4 \\
-3x_1 & + & 7x_2 & - & x_3 & = & -3 \\
2x_1 & - & 5x_2 & + & 4x_3 & = & 7 \\
-3x_1 & + & 6x_2 & + & 9x_3 & = & -1
\end{array}$$

Matices with full row (column) rank

Definition

An $m \times n$ matrix A is said to have full column rank if its column vectors are linearly independent, and it is said to have full row rank if its row vectors are linearly independent.

Theorem

Let A be an $m \times n$ matrix.

(a) *A has full column rank if and only if its column vectors form a basis for the column space, that is, if and only if $\mathrm{rank}(A) = n$.*

(b) *A has full row rank if and only if its row vectors form a basis for the row space, that is, if and only if $\mathrm{rank}(A) = m$.*

Matices with full row (column) rank

Example

Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -3 & 1 \end{bmatrix}$

Matices with full row (column) rank

Theorem (The consistency theorem)

If A is an $m \times n$ matrix, then the followings are equivalent:

(a) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(b) $A\mathbf{x} = \mathbf{b}$ *has at most one solution for every* $\mathbf{b}$ *in* $\mathbb{R}^m$*.*

(c) *A has full column rank.*

Matices with full row (column) rank

Example

Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -3 & 1 \end{bmatrix}$

Overdetermined and underdetermined linear systems

Theorem

Let A be an $m \times n$ matrix.

(a) (**Overdetermined case**) If $m > n$, then the system $A\mathbf{x} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$ in $\mathbb{R}^m$.

(b) (**Underdetermined case**) If $m < n$, then for every vector $\mathbf{b}$ in $\mathbb{R}^m$ the system $A\mathbf{x} = \mathbf{b}$ is either inconsistent or has infinitely many solutions.

Matrices of the form $A^T A$ and AA^T

Theorem

If A is an $m \times n$ matrix, then

(a) *A and $A^T A$ have the same null space.*

Matrices of the form A^TA and AA^T

(b) A and A^TA have the same row space.

(c) A^T and A^TA have the same column space.

(d) A and A^TA have the same rank.

Theorem

If A is an $m \times n$ matrix, then

(a) *A^T and AA^T have the same null space.*

(b) *A^T and AA^T have the same row space.*

(c) *A and AA^T have the same column space.*

(d) *A and AA^T have the same rank.*

Some unifying theorems

Theorem

If A is an $m \times n$ matrix, then the followings are equivalent:

(a) $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(b) $A\mathbf{x} = \mathbf{b}$ *has at most one solution for every* $\mathbf{b}$ *in* $\mathbb{R}^m$*.*

(c) A *has full column rank.*

(d) $A^T A$ *is invertible.*

Some unifying theorems

Theorem

If A is an $m \times n$ matrix, then the followings are equivalent:

(a) $A^T\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*

(b) $A^T\mathbf{x} = \mathbf{b}$ *has at most one solution for every* $\mathbf{b}$ *in* $\mathbb{R}^n$*.*

(c) A *has full row rank.*

(d) AA^T *is invertible.*

Example

Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -3 & 1 \end{bmatrix}$