7.6 The Pivot Theorem and Its Implications

Let W be the subspace spanned by a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$. There are two problems of finding a basis for W:

1. Find any basis for W.

2. Find a basis for W consisting of vectors from S.

Finding a basis for W consisting of vectors from S

Theorem

Let A and B be row equivalent matrices.

(a) *If some subset of column vectors from A is linearly independent, then the corresponding column vectors from B are linearly independent, and conversely.*

(b) *If some subset of column vectors from A is linearly dependent, then the corresponding column vectors from B are linearly dependent, and conversely. Moreover, the column vectors in the two matrices have the same dependency relationships.*

Finding a basis for W consisting of vectors from S

Example

Find a subset of the column vectors of

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

that forms a basis for $\mathrm{col}(A)$.

The pivot theorem

Definition

The column vectors of a matrix A that lie in the column positions where the leading 1's occur in the row echelon forms of A are called the pivot columns of A.

Theorem

The pivot columns of a nonzero matrix A form a basis for the column space of A.

Algorithm 1

If W is the subspace of $\mathbb{R}^n$ spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$, then the following procedure extracts a basis for W from S.

Step 1. Form a matrix A that has $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ as successive column vectors.

Step 2. Reduce A to a row echelon form U, and identify the columns with the leading 1's to determine the pivot columns of A.

Step 3. Extract the pivot columns of A to obtain a basis for W.

Algorithm 1

If W is the subspace of $\mathbb{R}^n$ spanned by $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s\}$, then the following procedure expresses the vectors of S that are not in the basis as linear combinations of the basis vectors.

Step 4. If it is desired to express the vectors of S that are not in the basis as linear combinations of the basis vectors, then continue reducing U to obtain the reduced row echelon form R of A.

Step 5. Express each column vector of R that does not contain a leading 1 as a linear combination of preceding column vectors that contain leading 1's. Replace the column vectors in these linear combinations by corresponding column vectors of A to obtain equations that express the column vector of A that are not in the basis as linear combinations of basis vectors.

Algorithm 1

Example

Let W be the subspace of $\mathbb{R}^4$ that is spanned by

$$\begin{aligned} &\mathbf{v}_1 = (1, -2, 0, 3), \quad \mathbf{v}_2 = (2, -5, -3, 6), \quad \mathbf{v}_3 = (0, 1, 3, 0), \\ &\mathbf{v}_4 = (2, -1, 4, -7), \quad \mathbf{v}_5 = (5, -8, 1, 2) \end{aligned}$$

Find a subset of these vectors that forms a basis for W.

Algorithm 1

Example

Let W be the subspace of $\mathbb{R}^4$ that is spanned by

$$\begin{aligned}&\mathbf{v}_1 = (1, -2, 0, 3), \quad \mathbf{v}_2 = (2, -5, -3, 6), \quad \mathbf{v}_3 = (0, 1, 3, 0),\\ &\mathbf{v}_4 = (2, -1, 4, -7), \quad \mathbf{v}_5 = (5, -8, 1, 2)\end{aligned}$$

Express those vectors that are not in the basis as linear combinations of those vectors in the basis.

Bases for the fundamental spaces of a matrix

Finding bases for three of four fundamental spaces of a matrix A

Reduce A to a row echelon form U or its reduced row echelon form R:

1. The nonzero rows of U form a basis for row(A).

2. The columns of U with leading 1's identify the pivot columns of A, and these form a basis for col(A).

3. The canonical solution of $A\mathbf{x} = \mathbf{0}$ form a basis for null(A), and these are readily obtained from the system $R\mathbf{x} = \mathbf{0}$.

Algorithm for finding a basis for null(A^T)

If A is an $m \times n$ matrix with rank k, and if $k < m$, then the following procedure produces a basis for null(A^T) by elementary row operations on A.

Step 1. Adjoin the $m \times m$ identity matrix I_m to the right of A to create a partitioned matrix $\begin{bmatrix} A & I_m \end{bmatrix}$.

Step 2. Apply elementary row operations $\begin{bmatrix} A & I_m \end{bmatrix}$ until A is reduced to a row echelon form U, and let the resulting partitioned matrix be $\begin{bmatrix} U & E \end{bmatrix}$.

Step 3. Repartition $\begin{bmatrix} U & E \end{bmatrix}$ by adding a horizontal rule to split off the zero rows of U. This yields a matrix of the form $\begin{bmatrix} V & E_1 \\ 0 & E_2 \end{bmatrix}$.

Step 4. The row vectors of E_2 form a basis for null(A^T).

Algorithm for finding a basis for null(A^T)

Find a basis for null(A^T) if

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Column-row factorization

Theorem

If A is an $m \times n$ matrix of rank k, then A can be factored as

$$A = CR$$

where C is the $m \times k$ matrix whose column vectors are the pivot columns of A and R is the $k \times n$ matrix whose row vectors are the nonzero rows in the reduced row echelon form of A.

Column-row expansion

Theorem

If A is a nonzero matrix of rank k, then A can be expressed as

$$A = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \cdot + \mathbf{c}_k\mathbf{r}_k$$

where $\mathbf{c}_1, \mathbf{c}_2, \ldots, c_k$ are the successive pivot columns of A and $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_k$ are the successive row vectors of in the reduced row echelon form of A.