4.4 A First Look at Eigenvalues and Eigenvectors

Theorem (Fixed points)

If A is an $n \times n$ matrix, then the followings are equivalent:

(a) *A has nontrivial fixed points.*

(b) *$I - A$ is singular.*

(c) $\det(I - A) = 0$.

Problem

If A is an $n \times n$ matrix, for what values of the scalar λ, if any, are there nonzero vectors in $\mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$?

Definition

If A is an $n \times n$ matrix, then the scalar λ is called an eigenvalue of A if there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. If λ is an eigenvalue of A, then every nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ is called an eigenvector of A corresponding to λ.

Eigenvalues and Eigenvectors

Theorem

If A is an $n \times n$ matrix and λ is a scalar, the followings are equivalent:

(a) *λ is an eigenvalue of A.*

(b) *The linear equation $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solution.*

(c) *λ is a solution of $\det(\lambda I - A) = 0$.*

Definition

The equation $\det(\lambda I - A) = 0$ is called the characteristic equation of A.

Eigenvalues and Eigenvectors

Example

Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$$

Eigenvalues of triangular matrices

Theorem

If A is a triangular matrix, then the eigenvalues of A are the entries on the main diagonal of A.

Eigenvalues of powers of a matrices

Theorem

If λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector, and if k is any positive integer, then λ^k is an eigenvalue of A^k and $\mathbf{x}$ is a corresponding eigenvector.

The unifying theorem

Theorem

If A is an $n \times n$ matrix, then the followings are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A is expressible as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
7. *The column vectors of A are linearly independent.*
8. *The row vectors of A are linearly independent.*
9. $\det(A) =\neq 0$.
10. *$\lambda = 0$ is not an eigenvalues of A.*

Expressions for determinant and trace in terms of eigenvalues

Theorem

If A is an $n \times n$ matrix with $\lambda_1, \lambda_2, ..., \lambda_n$, then

(a) $det(A) = \lambda_1 \lambda_2 ... \lambda_n$

(b) $tr(A) = \lambda_1 + \lambda_2 + ... + \lambda_n$