3.1 Operations on Matrices

- We will use capital letters to denote matrices and lowercase letters to denote entries. A general $m \times n$ matrix might be denoted as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

  or more compactly as

$$A = \left[a_{ij}\right]_{m\times n} \qquad \text{or} \qquad A = \left[a_{ij}\right]$$

- A matrix with n rows and n columns is called a square matrix of order n.

Operations on matrices

- Two matrices are equal if they have the same size and their corresponding entries are equal.
- If $A = [a_{ij}]$ and $B = [b_{ij}]$ have the same size, their sum and difference are defined by
  - $(A + B)_{ij} = [A]_{ij} + [B]_{ij} = a_{ij} + b_{ij}$
  - $(A - B)_{ij} = [A]_{ij} - [B]_{ij} = a_{ij} - b_{ij}$

Example

Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & -5 \\ -1 & 2 & 3 \end{bmatrix}$

a. $A + B =$

b. $A - B =$

Operations on matrices

- If A is any matrix and c is any scalar, then the product cA is defined by

$$(cA)_{ij} = c(A)_{ij} = ca_{ij}$$

Example

Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 1 & 2 \end{bmatrix}$

a. $3A =$

b. $-A =$

Row and column vectors

- A row vector is a matrix with one row.
- A column vector is a matrix with one column.
- A matrix can be subdivided into row vectors or column vectors.

Example

A matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$

can be subdivided into row vectors as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix}$$

or into column vectors as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{bmatrix}$$

The product $A\mathbf{x}$

Definition

If A is an $m \times n$ matrix whose column vectors are $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and $\mathbf{x}$ is an $n \times 1$ column vector, then the **product $A\mathbf{x}$** is defined as

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Example

$$\begin{bmatrix} 1 & -3 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} =$$

The product AB

Theorem (Linearity properties)

If A is an $m \times n$ matrix, then the following relationships hold for all column vector $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ and for every scalar c:

1. $A(c\mathbf{u}) = c(A\mathbf{u})$
2. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

Definition

If A is an $m \times s$ matrix and B is an $s \times n$ matrix and if the column vectors of B are $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$, then the product AB is the $m \times n$ matrix defined as

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_n \end{bmatrix}$$

The product AB

Theorem (The Row-Column rule)

The entry in row i and column j of AB is the product of the ith row vector of A and jth column vector of B.

Example

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -1 \\ 3 & 2 \\ -1 & 3 \end{bmatrix}$. Compute AB.

Transpose of a matrix

Definition

If A is an $m \times n$ matrix, then the transpose of A, denoted by A^T, is the $n \times m$ matrix obtained by making the rows of A into columns.

Example

Find A^T if $A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 2 \end{bmatrix}$

Trace of a square matrix

Definition

If A is a square matrix, then the trace of A, denoted by $\operatorname{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A.

Example

Find $\operatorname{tr}(A)$ if $A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 3 & 1 \\ -2 & 0 & -2 \end{bmatrix}$