7.9 Orthonormal Bases and the Gram-Schmidt Process

Definition

- A set of vectors in $\mathbb{R}^n$ is said to be orthogonal if each pair of distinct vectors in the set is orthogonal.
- A set of vectors in $\mathbb{R}^n$ is said to be orthonormal if it is orthogonal and each vector has norm 1.

Orthogonal and orthonormal bases

Example

Show that the vectors

$$\mathbf{v}_1 = (0, 2, 0), \mathbf{v}_2 = (3, 0, 3), \mathbf{v}_3 = (-4, 0, 4)$$

form an orthogonal basis for $\mathbb{R}^3$ and convert it into an orthonormal basis.

Orthogonal and orthonormal bases

Theorem

An orthogonal set of nonzero vectors in $\mathbb{R}^n$ is linearly independent.

Orthogonal projections using orthonormal bases

Recall that if W is a nonzero subspace of $\mathbb{R}^n$, and $\mathbf{v}$ is a column vector in $\mathbb{R}^n$, then

$$\text{proj}_W\mathbf{x} = M(M^TM)^{-1}M^T\mathbf{x}$$

for any matrix M whose column vectors form a basis for W.
In particular, if the column vectors of M are orthonormal, then $M^TM = I$, so

$$\text{proj}_W\mathbf{x} = MM^T\mathbf{x}$$

Orthogonal projections using orthonormal bases

Example

Find the standard matrix P for the orthogonal projection of $\mathbb{R}^3$ onto the plane through the origin that is spanned by thr orthonormal vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$.

Orthogonal projections using orthonormal bases

Theorem

(a) *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is an orthonormal basis for a subspace* W *of* $\mathbb{R}^n$*, then the orthogonal projection of a vector* $\mathbf{x}$ *in* $\mathbb{R}^n$ *onto* W *can be expressed as*

$$proj_W \mathbf{x} = (\mathbf{x} \cdot \mathbf{v}_1) v_1 + (\mathbf{x} \cdot \mathbf{v}_2) v_2 + \cdots (\mathbf{x} \cdot \mathbf{v}_k) v_k$$

(b) *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is an orthogonal basis for a subspace* W *of* $\mathbb{R}^n$*, then the orthogonal projection of a vector* $\mathbf{x}$ *in* $\mathbb{R}^n$ *onto* W *can be expressed as*

$$proj_W \mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2} \mathbf{v}_1 + \frac{\mathbf{x} \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2} \mathbf{v}_2 + \cdot + \frac{\mathbf{x} \cdot \mathbf{v}_k}{\|\mathbf{v}_k\|^2} \mathbf{v}_k$$

Orthogonal projections using orthonormal bases

Example

Find the orthogonal projection of $\mathbf{x} = (1, 1, 1)$ onto the plane W in $\mathbb{R}^3$ that is spanned by thr orthonormal vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$.

Linear combinations of orthonormal basis vectors

Theorem

(a) *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is an orthonormal basis for a subspace* W *of* $\mathbb{R}^n$*, and if* $\mathbf{w}$ *is a vector in* W*, then*

$$\mathbf{w} = (\mathbf{w} \cdot \mathbf{v}_1)v_1 + (\mathbf{w} \cdot \mathbf{v}_2)v_2 + \cdots (\mathbf{w} \cdot \mathbf{v}_k)v_k$$

(b) *If* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is an orthogonal basis for a subspace* W *of* $\mathbb{R}^n$*, and if* $\mathbf{w}$ *is a vector in* W*, then*

$$\mathbf{w} = \frac{\mathbf{w} \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 + \frac{\mathbf{w} \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 + \cdot + \frac{\mathbf{w} \cdot \mathbf{v}_k}{\|\mathbf{v}_k\|^2}\mathbf{v}_k$$

Linear combinations of orthonormal basis vectors

Example

Express the vector $\mathbf{w} = (1, 1, 1)$ as a linear combination of the orthonormal basis vectors

$$\mathbf{v}_1 = \left(\frac{3}{7}, -\frac{6}{7}, \frac{2}{7}\right), \mathbf{v}_2 = \left(\frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right), \mathbf{v}_3 = \left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)$$

Finding orthogonal and orthonormal bases

(Gram-Schmidt Process)

Step 1. Choose any basis $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ of W and let $\mathbf{v}_1 = \mathbf{w}_1$.

Step 2. Let $W_1 = \text{span}\{\mathbf{v}_1\}$ and choose $\mathbf{v}_2$ as the vector component of $\mathbf{w}_2$ orthogonal to W_1, i.e.,

$$\mathbf{v}_2 = \mathbf{w}_2 - \text{proj}_{W_1}\mathbf{w}_2 = \mathbf{w}_2 - \frac{\mathbf{w}_2 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1$$

Step 3. Let $W_2 = \text{span}\{w_1, \mathbf{w}_2\}$ and choose $\mathbf{v}_3$ as the vector component of $\mathbf{w}_3$ orthogonal to W_2, i.e.,

$$\mathbf{v}_3 = \mathbf{w}_3 - \text{proj}_{W_2}\mathbf{w}_3 = \mathbf{w}_3 - \frac{\mathbf{w}_3 \cdot \mathbf{v}_1}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\mathbf{w}_3 \cdot \mathbf{v}_2}{\|\mathbf{v}_2\|^2}\mathbf{v}_2$$

Step 4. Continue in this way to produce an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

Step 5. Normalize each vector to get an orthonormal basis

$$\left\{\frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}, \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}, \ldots, \frac{\mathbf{v}_k}{\|\mathbf{v}_k\|}\right\}$$

Finding orthogonal and orthonormal bases

Example

Use the Gram-Schmidt process to construct an orthonormal basis for the plane $x + y + z = 0$ in $\mathbb{R}^3$.