7.7 The Projection Theorem and Its Implications

Orthogonal projection onto lines in $\mathbb{R}^2$

Orthogonal projection onto lines in $\mathbb{R}^n$

Theorem

If $\mathbf{a}$ is a nonzero vector in $\mathbb{R}^n$, then every vector $\mathbf{x}$ in $\mathbb{R}^n$ can be expressed in exactly one way as

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$$

where $\mathbf{x}_1$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{x}_2$ is orthogonal to $\mathbf{a}$. The vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are given by

$$\mathbf{x}_1 = \frac{\mathbf{x} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}, \qquad \mathbf{x}_2 = \mathbf{x} - \mathbf{x}_1 = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$

Orthogonal projection onto lines in $\mathbb{R}^n$

Definition

If $\mathbf{a}$ is a nonzero vector in $\mathbb{R}^n$, and if $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then the orthogonal projection of $\mathbf{x}$ onto span$\{\mathbf{a}\}$ is denoted by $\text{proj}_{\mathbf{a}}\mathbf{x}$ and is defined as

$$\text{proj}_{\mathbf{a}}\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$

The vector $\text{proj}_{\mathbf{a}}\mathbf{x}$ is called the vector component of $\mathbf{x}$ along $\mathbf{a}$, and $\mathbf{x} - \text{proj}_{\mathbf{a}}\mathbf{x}$ is called the vector component of $\mathbf{x}$ orthogonal to $\mathbf{a}$.

Orthogonal projection onto lines in $\mathbb{R}^n$

Example

Let $\mathbf{x} = (2, -1, 3)$ and $\mathbf{a} = (4, -1, 2)$. Find the vector components of $\mathbf{x}$ along $\mathbf{a}$ and orthogonal to $\mathbf{a}$.

Projection operators on $\mathbb{R}^n$

Definition

Define an operator $T : \mathbb{R}^n \to \mathbb{R}^n$ by

$$T(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a}$$

This is called the orthogonal projection of $\mathbb{R}^n$ onto span$\{\mathbf{a}\}$.

Projection operators on $\mathbb{R}^n$

Theorem

If $\mathbf{a}$ *is a nonzero vector in* $\mathbb{R}^n$*, and if* $\mathbf{a}$ *is expressed in column form, then the standard matrix for the linear operator* $T(\mathbf{x}) = \mathrm{proj}_{\mathbf{a}}\mathbf{x}$ *is*

$$P = \frac{1}{\mathbf{a}^T\mathbf{a}}\mathbf{a}\mathbf{a}^T$$

This matrix is symmetric and has rank 1*.*

Orthogonal projections onto general subspaces

Theorem (Projection theorem for subspaces)

If W is a subspace of $\mathbb{R}^n$, then every vector $\mathbf{x}$ in $\mathbb{R}^n$ can be expressed exactly one way as

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$$

where $\mathbf{x}_1$ is in W and $\mathbf{x}_2$ is in $W^\perp$.

Orthogonal projections onto general subspaces

Definition

If $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ where $\mathbf{x}_1$ is in W and $\mathbf{x}_2$ is in $W^\perp$, $\mathbf{x}_1$ is called the orthogonal projection of $\mathbf{x}$ on W, and $\mathbf{x}_2$ is called the orthogonal projection of $\mathbf{x}$ on $W^\perp$. They are denoted by $\text{proj}_W\mathbf{x}$ and $\text{proj}_{W^\perp}\mathbf{x}$, respectively.

Theorem

If W is a nonzero subspace of $\mathbb{R}^n$, and if M is any matrix whose column vectors form a basis for W, then

$$proj_W\mathbf{x} = M(M^TM)^{-1}M^T\mathbf{x}$$

for every column vector $\mathbf{x}$ in $\mathbb{R}^n$

Orthogonal projections onto general subspaces

Example

(a) Find the standard matrix P for the orthogonal projection of $\mathbb{R}^3$ onto the plane $x - 4y + 2z = 0$.

(b) Use the matrix P to find the orthogonal projection of the vector $x = (1, 0, 4)$ onto the plane.