3.6 Matrices with Special Forms

Definition

A square matrix in which all entries off the main diagonal are zero is called a diagonal matrix.

Properties of diagonal matrices

If D is a diagonal matrix $D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$ and k is a positive integer, then

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix} \qquad D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

Triangular matrices

Definition

A square matrix in which all entries above the main diagonal are zero is called lower triangular, and a square matrix in which all entries below the main diagonal are zero is called upper triangular. A matrix that is either upper triangular or lower triangular is called triangular.

Triangular matrices

Properties of triangular matrices

(a) The transpose of a lower (upper) triangular matrix is upper (lower) triangular.

(b) A product of lower (upper) triangular matrices is lower (upper) triangular.

(c) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.

(d) The inverse of an invertible lower (upper) triangular matrix is lower (upper) triangular.

Symmetric and Skew-symmetric matrices

Definition

A square matrix is called symmetric if $A^T = A$ and skew-symmetric if $A^T = -A$.

Symmetric and Skew-symmetric matrices

Theorem

If A and B are symmetric matrices with the same size and k is any scalar, then

(a) *A^T is symmetric.*

(b) *$A + B$ and $A - B$ are symmetric.*

(c) *kA is symmetric.*

Symmetric and Skew-symmetric matrices

Theorem

The product of two symmetric matrices is symmetric if and only if they commute.

Theorem

If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Matrices of the form AA^T and A^TA

Theorem

If A is a square matrix, then the matrices A, A^TA, and AA^T are either all invertible or all singular.

Fixed point of a matrix

Definition

If A is a square matrix of order n, the solution of the linear system $A\mathbf{x} = \mathbf{x}$ is called the fixed points of A.

Example

Find the fixed points of the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Inverting $I - A$ when A is nilpotent

Definition

A square matrix such that $A^k = 0$ for some positive integer k is called nilpotent.

Theorem

If A is a nilpotent square matrix such that $A^k = 0$, then the matrix $I - A$ is invertible and

$$(I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}.$$