3.5. The Geometry of Linear Systems

Definition

If $\mathbf{x}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ are vectors in $\mathbb{R}^n$ and $W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$, then the set of vectors of the form

$$\mathbf{x} = \mathbf{x}_0 + t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_s\mathbf{v}_s$$

is called the translation of W by $\mathbf{x}_0$ and denoted by

$$\mathbf{x}_0 + W \qquad \text{or} \qquad \mathbf{x}_0 + \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}.$$

Relationship between $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$

Theorem

If $A\mathbf{x} = \mathbf{b}$ is a consistent nonhomogeneous linear system, and if W is the solution space of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, then the solution set of $A\mathbf{x} = \mathbf{b}$ is the translated subspace $\mathbf{x}_0 + W$, where $\mathbf{x}_0$ is any solution of the nonhomogeneous system $A\mathbf{x} = \mathbf{b}$.

Relationship between $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$

Theorem

A general solution of a consistent linear system $A\mathbf{x} = \mathbf{b}$ can be obtained by adding a particular solution of $A\mathbf{x} = \mathbf{b}$ to a general solution of $A\mathbf{x} = \mathbf{0}$.

Theorem

If A is an $m \times n$ matrix, then the followings are equivalent:

(a) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

(b) *$A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b}$ in $\mathbb{R}^m$.*

Theorem

A nonhomogeneous linear system with more unknowns than equations is either inconsistent or has infinitely many solutions.

Consistency of linear system

Definition

Let A be an $m \times n$ matrix. The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called the column space of A and is denoted by $\mathrm{col}(A)$.

Theorem

A linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Consistency of linear system

Example

Determine whether the vector $\mathbf{w} = (9, 1, 0)$ can be expressed as a linear combination of the vectors

$$\mathbf{v}_1 = (1, 2, 3), \mathbf{v}_2 = (1, 4, 6), \mathbf{v}_3 = (2, -3, -5)$$

and, if so, find such a linear combination.

Hyperplanes

Dot product

If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ are vectors in $\mathbb{R}^n$, then the dot product of $\mathbf{v}$ and $\mathbf{w}$ is denoted by $\mathbf{v} \cdot \mathbf{w}$ and is defined by

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n$$

Orthogonality

Two vectors $\mathbf{v}$ and $\mathbf{w}$ are said to be orthogonal if $\mathbf{v} \cdot \mathbf{w} = 0$.

Hyperplanes

Definition

The set of points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ satisfying

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

is called a hyperplane in $\mathbb{R}^n$.

Geometric property of hyperplane

When $b = 0$, the hyperplane passes through the origin, and the hyperplane $\mathbf{a} \cdot \mathbf{x} = 0$ is called the orthogonal complement of $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and denoted by $\mathbf{a}^\perp$.

Geometric interpretation of solution spaces

Theorem

If A is an $m \times n$ matrix, then the solution space of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in $\mathbb{R}^n$ that are orthogonal to every row vector of A.