# 8.2 Similarity and Diagonalizability

## Definition

If $A$ and $C$ are square matrices with the same size, then we say that $C$ is similar to $A$ if there is an invertible matrix $P$ such that $C = P^{-1}AP$.

## Theorem

*Two square matrices are similar if and only if there exist bases with respect to which the matrices represent the same linear operator.*

# Similarity invariants

## Theorem

(a) *Similar matrices have the same determinant.*

(b) *Similar matrices have the same rank.*

(c) *Similar matrices have the same nullity.*

(d) *Similar matrices have the same trace.*

(e) *Similar matrices have the same characteristic equation and hence have the same eigenvalues with the same algebraic multiplicities.*

# Eigenvalues and eigenvectors of similar matrices

## Definition

Let $A$ be an $n \times n$ matrix and $\lambda_0$ is an eigenvalue of $A$.

(a) The solution space of $(\lambda_0 I - A)\mathbf{x} = \mathbf{0}$ is called the eigenspace of $A$ corresponding to $\lambda_0$.

(b) The dimension of this eigenspace is called the geometric multiplicity of $\lambda_0$.

## Algebraic multiplicities

The algebraic multiplicity of $\lambda_0$ is the number of repetitions of the factor $\lambda - \lambda_0$ in the complete factorization of the characteristic polynomial of $A$.

# Eigenvalues and eigenvectors of similar matrices

## Example

Find the algebraic and geometric multiplicities of the eigenvalues of

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 5 & 3 \end{bmatrix}$$

# Eigenvalues and eigenvectors of similar matrices

## Example

Find the algebraic and geometric multiplicities of the eigenvalues of

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

# Eigenvalues and eigenvectors of similar matrices

## Theorem

*Similar matrices have the same eigenvalues and those eigenvalues have the same algebraic and geometric multiplicities for both matrices.*

## Theorem

*Suppose $C = P^{-1}AP$ and that $\lambda$ is an eigenvalue of $A$ and $C$.*

(a) *If $\mathbf{x}$ is an eigenvector of $C$ corresponding to $\lambda$, then $P\mathbf{x}$ is an eigenvector of $A$ corresponding to $\lambda$.*

(b) *If $\mathbf{x}$ is an eigenvector of $A$ corresponding to $\lambda$, then $P^{-1}\mathbf{x}$ is an eigenvector of $C$ corresponding to $\lambda$.*

# Diagonalization

## Definition (The diagonalization problem)

Given a square matrix $A$, does there exist an invertible matrix $P$ for which $P^{-1}AP$ is a diagonal matrix, and if so, how does one find such a $P$? If such a matrix $P$ exists, then $A$ is said to be diagonalizable, and $P$ is said to diagonalize $A$.

## Theorem

*An $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors.*

# Diagonalization

## Diagonalizing an $n \times n$ matrix with $n$ linearly independent eigenvectors

Step 1. Find $n$ linearly independent eigenvectors of $A$, say $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$.

Step 2. Form the matrix $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}$.

Step 3. The matrix $P^{-1}AP$ will be diagonal and will have eigenvalues corresponding to $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$, respectively, as its successive diagonal entries.

# Diagonalization

Determine whether $A$ is diagonalizable. If so, find a matrix $P$ that diagonalizes $A$ and determine $P^{-1}AP$.

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

# Diagonalization

Determine whether $A$ is diagonalizable. If so, find a matrix $P$ that diagonalizes $A$ and determine $P^{-1}AP$.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 5 & 3 \end{bmatrix}$$

# Diagonalization

### Theorem

*If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are eigenvectors of a matrix $A$ that correspond to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is linearly independent.*

### Theorem

*An $n \times n$ matrix with $n$ distinct real eigenvalues is diagonalizable.*

### Theorem

*An $n \times n$ matrix $A$ is diagonalizable if and only if the sum of the geometric multiplicities of its eigenvalues is $n$.*

### Theorem

*If $A$ is a square matrix, then:*

(a) *The geometric multiplicity of an eigenvalue of $A$ is less than or equal to its algebraic multiplicity.*

(b) *$A$ is diagonalizable if and only if the geometric multiplicity of each eigenvalue of $A$ is the same as its algebraic multiplicity.*

# Diagonalization

## Theorem

*If $A$ is an $n \times n$ matrix, then the followings are equivalent:*

(a) *$A$ is diagonalizable.*

(b) *$A$ has $n$ linearly independent eigenvectors.*

(c) *$\mathbb{R}^n$ has a basis consisting of eigenvectors of $A$.*

(d) *The sum of geometric multiplicities of the eigenvalues of $A$ is $n$.*

(e) *The geometric multiplicity of each eigenvalue of $A$ is the same as the algebraic multiplicity.*