6.1 Matrices as Transformations

Definition

- A function whose inputs and outputs are vectors is called a transformation.
- A transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is called an operator on $\mathbb{R}^n$.

Example

Let T be the transformation that maps a vector $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$ into the vector $2\mathbf{x} = (2x_1, 2x_2)$ in $\mathbb{R}^2$.

Transformation

Example

Consider the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$ and let T_A be the transformation which maps a 2×1 column vector $\mathbf{x}$ in $\mathbb{R}^2$ into the 3×1 column vector $A\mathbf{x}$ in $\mathbb{R}^3$.

Matrix transformations

Definition

If A is an $m \times n$ matrix, and if $\mathbf{x}$ is a column vector in $\mathbb{R}^n$, then the product $A\mathbf{x}$ is a vector in $\mathbb{R}^m$. So, multiplying $\mathbf{x}$ by A creates a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ and this transformation is called the multiplication by A or the transformation A, and is denoted by T_A to emphasize the matrix A.

Matrix transformations

Example

- Zero transformation
- Identity operator

Matrix transformations

Example

Let T_A be the matrix transformation where $A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$. Find a vector in $\mathbb{R}^2$, if any, whose image under T_A is $\mathbf{b} = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix}$.

Linear transformation

Definition

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ if the following two properties hold for all vectors **u** and **v** in $\mathbb{R}^n$ and for all scalars c:

(i) $T(c\mathbf{u}) = cT(\mathbf{u})$ [Homogeniety Property]

(ii) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ [Additivity Property]

In the special case where $m = n$, the linear transformation T is called a linear operator on $\mathbb{R}^n$.

Superposition principle

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are vectors in $\mathbb{R}^n$ and $c_1, c_2, \ldots, c_k$ are any scalars, then

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \cdots + c_k T(\mathbf{v}_k).$$

Linear transformation

Example

Show that a matrix transformation T_A is linear.

Linear transformation

Example

Show that the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (x_1^2, x_2^2, x_3^2)$$

is not linear.

Linear transformation

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then

(a) $T(\mathbf{0}) = \mathbf{0}$.

(b) $T(-\mathbf{u}) = -T(\mathbf{u})$.

(c) $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$.

Linear transformation

Example

Show that the operator $T : \mathbb{R}^n \to \mathbb{R}^n$, defined by $T(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$ for some vector $\mathbf{x}_0$ in $\mathbb{R}^n$, is not linear

Linear transformation

Theorem

All linear transformations are matrix transformations.

Linear transformation

Theorem (Precise version)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. If $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_n$ are standard unit vectors in $\mathbb{R}^n$, and if $\mathbf{x}$ is any vector in $\mathbb{R}^n$, then $T(\mathbf{x})$ can be expressed as

$$T(\mathbf{x}) = A\mathbf{x}$$

where

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

The matrix A is called the standard matrix for T and denoted by $A = [T]$.

Linear transformation

Example

Show that the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_2 - x_3)$ is linear and find its standard matrix.

Linear operators on $\mathbb{R}^2$

Rotations about the origin

The standard matrix for the rotation about the origin through an angle θ is

$$R_\theta = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Linear operators on $\mathbb{R}^2$

Example

Find the image of $\mathbf{x} = (1, 1)$ under a rotation of $\pi/6$ about the origin.

Linear operators on $\mathbb{R}^2$

Reflection about the line through the origin

The standard matrix for the reflection about the line through the origin that makes an angle θ with the positive x-axis is

$$H_\theta = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \cos(\frac{\pi}{2} - 2\theta) \\ \sin 2\theta & -\sin(\frac{\pi}{2} - 2\theta) \end{bmatrix} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Linear operators on $\mathbb{R}^2$

Example

Find the image of $\mathbf{x} = (1, 1)$ under a reflection about the line through the origin that makes an angle of $\pi/6$ with the positive x-axis.

Linear operators on $\mathbb{R}^2$

Orthogonal projection

The standard matrix for the orthogonal projection onto the line through the origin that makes an angle θ with the positive x-axis is

$$P_\theta = \begin{bmatrix} \frac{1}{2}(1+\cos 2\theta) & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \frac{1}{2}(1-\cos 2\theta) \end{bmatrix} = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

Linear operators on $\mathbb{R}^2$

Example

Find the orthogonal projection of $\mathbf{x} = (1, 1)$ on the line through the origin that makes an angle of $\pi/12$ with the x-axis.