7.4 The Dimension Theorem and Its Implications

Theorem (The Dimension Theorem for Matrices)

If A is an $m \times n$ matrix, then

$$\mathit{rank}(A) + \mathit{nullity}(A) = n.$$

The Dimension Theorem

Example

Find rank and nullity of

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ -2 & 1 & -3 & -2 & -4 \\ 0 & 5 & -14 & -9 & 0 \\ 2 & 10 & -28 & -18 & 4 \end{bmatrix}$$

Extending a linearly independent set to a basis

Every linearly independent set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ can be enlarged to a basis for $\mathbb{R}^n$.

Step 1. Form a matrix A that has $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ as row vectors.

Step 2. Find a basis $\mathbf{w}_{k+1}, \ldots, \mathbf{w}_n$ for the null space of A by solving $A\mathbf{x} = \mathbf{0}$.

Step 3. Then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{w}_{k+1}, \ldots, \mathbf{w}_k\}$ is a basis for $\mathbb{R}^n$.

Extending a linearly independent set to a basis

Example

Find a basis for $\mathbb{R}^4$ containing $\mathbf{v}_1 = (1, 3, -1, 1)$ and $\mathbf{v}_2 = (0, 1, 1, 6)$.

Consequences of Dimension Theorem

Theorem

If an $m \times n$ matrix A has rank k, then:

(a) A has nullity $n - k$.

(b) Every row echelon form of A has k nonzero rows.

(c) Every row echelon form of A has $m - k$ zero rows.

(d) The homogeneous system $A\mathbf{x} = \mathbf{0}$ has k leading (pivot) variables and $n - k$ free variables.

Example

Can a 5×7 matrix A have a one-dimensional null space?

Dimension Theorem

Theorem (The Dimension Theorem for Subspaces)

If W is a subspace of $\mathbb{R}^n$, then

$$\dim(W) + \dim(W^{\perp}) = n.$$

The unifying theorem

Theorem

If A is an $n \times n$ matrix, then the followings are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A is expressible as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
7. *The column vectors of A are linearly independent.*
8. *The row vectors of A are linearly independent.*
9. $\det(A) \neq 0$.
10. *T_A is one-to-one.*
11. *T_A is onto.*

The unifying theorem

14. The column vectors of A span $\mathbb{R}^n$.
15. The row vectors of A span $\mathbb{R}^n$.
16. The column vectors of A form a basis for $\mathbb{R}^n$.
17. The row vectors of A form a basis for $\mathbb{R}^n$.
18. $\text{rank}(A) = n$.
19. $\text{nullity}(A) = 0$.

Hyperplanes

Theorem

If W is a subspace of $\mathbb{R}^n$ with dimension $n-1$, then there is a nonzero vector $\mathbf{a}$ for which $W = \mathbf{a}^\perp$; that is, W is a hyperplane through the origin of $\mathbb{R}^n$.

Rank 1 matrices

Let A be an $m \times n$ matrix. Then the followings are equivalent:

- $\text{rank}(A) = 1$
- $\text{nullity}(A) = n - 1$.
- $\text{row}(A)$ is a line through the origin of $\mathbb{R}^n$.
- $\text{null}(A)$ is a hyperplane through the origin of $\mathbb{R}^n$.
- The row vectors of A are all scalar multiples of some nonzero vector $\mathbf{a}$.

Rank 1 matrices

Theorem

If $\mathbf{u}$ *is a nonzero* $m \times 1$ *matrix and* $\mathbf{v}$ *is a nonzero* $n \times 1$ *matrix, then the product*

$$A = \mathbf{u}\mathbf{v}^T$$

has rank 1*. Conversely, if* A *is an* $m \times n$ *matrix with rank* 1*, then* A *can be factored into a product of the above form.*

Rank 1 matrices

Example

Let $\mathbf{u} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \\ -1 \end{bmatrix}$. Show that $\mathbf{u}\mathbf{v}^T$ has rank 1.

Rank 1 matrices

Example

Factor the rank 1 matrix $A = \begin{bmatrix} 2 & -4 & -6 & 0 \\ -3 & 6 & 9 & 0 \end{bmatrix}$ into a product of the form $\mathbf{u}\mathbf{v}^T$.

Symmetric rank 1 matrices

Theorem

If $\mathbf{u}$ *is a nonzero* $n \times 1$ *column vector, then the product* $\mathbf{u}\mathbf{u}^T$ *is a symmetric matrix of rank* 1*. Conversely, if* A *is a symmetric* $n \times n$ *matrix of rank* 1*, then it can be factored as* $\mathbf{u}\mathbf{u}^T$ *or else as* $-\mathbf{u}\mathbf{u}^T$ *for some nonzero* $n \times 1$ *column vector* $\mathbf{u}$*.*