4.2 Properties of Determinants

Theorem

If A is a square matrix, then $\det(A) = \det(A^T)$.

Effect of elementray row operations

If B is the matrix that results when a single row or single column of A is multiplied by a scalar k, then $\det(B) = k \det(A)$.

Effect of elementray row operations

If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.

Effect of elementray row operations

If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then $\det(B) = \det(A)$.

Effect of elementray row operations

Theorem

Let A be an $n \times n$ matrix.

(a) *If A has two identical rows or columns, then* $\det(A) = 0$.

(b) *If A has two proportional rows or columns, then* $\det(A) = 0$.

(c) $\det(kA) = k^n \det(A)$.

Simplifying cofactor expansions

Example

Use a cofactor expansion to find the determinant of

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Determinants by Gaussian elimination

Example

Use a cofactor expansion to find the determinant of $A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$

A determinant test for invertibility

Theorem

A square matrix A is invertible if and only if $\det(A) \neq 0$*.*

Determinant of a product

Theorem

If A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B)$$

Determinant of the inverse

Theorem

If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

The unifying theorem

Theorem

If A is an $n \times n$ matrix, then the followings are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A is expressible as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
7. *The column vectors of A are linearly independent.*
8. *The row vectors of A are linearly independent.*
9. $\det(A) =\neq 0$.