6.2 Geometry of Linear Operators

Norm of a vector

If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in $\mathbb{R}^n$, the norm of $\mathbf{v}$ is denoted by $\|\mathbf{v}\|$ and is defined by

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

Norm as a dot product

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}.$$

Orthogonal operators

Definition

A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ with the norm-preserving property $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ is called an orthogonal operator or a linear isometry.

Orthogonal operators

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator on $\mathbb{R}^n$, then the following statements are equivalent:

(a) $\|T(\mathbf{x})\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* $\mathbb{R}^n$.

(b) $T(\mathbf{x}) \cdot T(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^n$.

Orthogonal operators

Theorem

An orthogonal operator preserves angles.

Orthogonal matrices

Definition

A square matrix A is said to be orthogonal if $A^{-1} = A^T$.

Example

Show that $A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$ is orthogonal.

Orthogonal matrices

Theorem

(a) The transpose of an orthogonal matrix is orthogonal.

(b) The inverse of an orthogonal matrix is orthogonal.

(c) A product of orthogonal matrices is orthogonal.

(d) If A is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.

Orthognal matrices

Theorem

If A is an $m \times n$ matrix, then the followings are equivalent:

(a) $A^T A = I$.

(b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* $\mathbb{R}^n$.

(c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^n$.

(d) *The column vectors of A are orthonormal.*

Orthognal matrices

Theorem

If A is an $n \times n$ matrix, then the followings are equivalent:

(a) *A is orthogonal.*

(b) $\|A\mathbf{x}\| = \|\mathbf{x}\|$ *for all* $\mathbf{x}$ *in* $\mathbb{R}^n$.

(c) $A\mathbf{x} \cdot A\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbb{R}^n$.

(d) *The column vectors of A are orthonormal.*

(e) *The row vectors of A are orthonormal.*

Orthognal operators

Theorem

A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if its standard matrix is orthogonal.

Example

Show that rotations about the origin and reflections about a line through the origin on $\mathbb{R}^2$ are orthogonal.

Orthogonal operators

Theorem

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an orthogonal linear operator, then the standard matrix for T is expressible in the form

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \text{or} \qquad H_{\theta/2} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

That is, T is either a rotation about the origin or a reflection about a line through the origin. Moreover, T_A is a rotation if $\det(A) = 1$ and T_A is a reflection if $\det(A) = -1$.

Non-orthogonal linear operators

Definition

If k is a nonnegative scalar, then $T(x, y) = (kx, ky)$ is called the scaling operator with factor k.

- T is called a contraction if $0 \leq k < 1$.
- T is called a dilation if $k > 1$.

Non-orthogonal linear operators

Definition

If k is a nonnegative scalar, then $T(x, y) = (kx, y)$ is called

- a compression in the x-direction with factor k if $0 \leq k < 1$.
- a expansion in the x-direction with factor k if $k > 1$.

$T(x, y) = (x, ky)$ is the compression (or expansion) in the y-direction with factor k.

Non-orthogonal linear operators

Definition

If k is a nonnegative scalar, then $T(x, y) = (x + ky, y)$ is called the shear in the x-direction with factor k and $T(x, y) = (x, y + kx)$ is the shear in the y-direction with factor k.

Operators in $\mathbb{R}^3$

Theorem

All 3×3 orthogonal matrices correspond to linear operators in $\mathbb{R}^3$ of the following types:

Type 1: *Rotations about lines through the origin.*

Type 2: *Reflections about planes through the origin.*

Type 3: *A rotation about a line through the origin followed by a reflection about the plane through the origin that is perpendicular to the line.*

Theorem

If A is a 3×3 orthogonal matrix, then A represents

- *a rotation (i.e., of type 1) if $\det(A) = 1$.*
- *a type 2 or type 3 operator if $\det(A) = -1$.*