3.2. Inverses; Algebraic Properties of Matrices

Addition and Scalar multiplication

If a and b are scalars and matrices A, B, and C have the same size, then

(a) $A + B = B + A$ [Commutative law for addition]

(b) $A + (B + C) = (A + B) + C$ [Associative law for addition]

(c) $(ab)A = a(bA)$

(d) $(a + b)A = aA + bA$

(e) $(a - b)A = aA - bA$

(f) $a(A + B) = aA + aB$

(g) $a(A - B) = aA - aB$

Matrix multiplication

$AB \neq BA$ in general

- AB may defined and BA may not.
- AB and BA may both be defined but they may have different sizes.
- AB and BA may both be defined and have the same size, but they may be different.

Properties of matrix multiplication

(a) $A(BC) = (AB)C$ [Associative law for multiplication]

(b) $A(B + C) = AB + AC$ [Left distributive law]

(c) $(B + C)A = BA + CA$ [Right distributive law]

(d) $A(B - C) = AB - AC$

(e) $(B - C)A = BA - CA$

(f) $a(BC) = (aB)C = B(aC)$

Zero matrices

Definition

A matrix whose entries are all zero is called a zero matrix and is denoted by $\mathit{0}$.

Properties of zero matrices

(a) $A + \mathit{0} = \mathit{0} + A = A$

(b) $A - \mathit{0} = A$

(c) $A - A = A + (-A) = \mathit{0}$

(d) $0A = \mathit{0}$

(e) If $cA = \mathit{0}$, then $c = 0$ or $A = \mathit{0}$.

Properties of matrix multiplication

Cancellation law is not true

$AB = AC$ does not imply $B = C$.

Example

$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$.

Properties of matrix multiplication

Nonzero matrices can have a zero product

$AB = 0$ does not imply $A = 0$ or $B = 0$.

Example

$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$.

Identity matrices

Definition

A square matrix with 1's on the main diagonal and zeros elsewhere is called an identity matrix and is denoted by I_n or simply I.

Properties of identity matrices

If A is an $m \times n$ matrix, then

$$AI_n = A \qquad \text{and} \qquad I_mA = A$$

Identity matrices

Theorem
If R is the reduced row echelon form of a square matrix A of order n, then either R has a row of zeros or R is the identity matrix I_n.

Inverse of a matrix

Definition

If A is a square matrix and there is a matrix B with the same size as A such that $AB = BA = I$, then A is said to be invertible (or nonsingular), and B is called an inverse of A. If there is no matrix B with this property, then A is said to be singular.

Example

$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

Inverse of a matrix

Theorem

If A is an invertible matrix, and B and C are both inverses of A, then $B = C$.

The inverse of an invertible matrix A is denoted by A^{-1}.

Inverse of a matrix

Theorem

The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$, in which case A^{-1} is given by

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example

Determine whether $A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$ is invertible. If so, find its inverse.

Inverse of a matrix

Theorem

If A and B are invertible matrices with the same size, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

Corollary

A product of any number of invertible matrices is invertible, and its inverse is the product of the inverses in the reverse order.

Powers of a matrix

Definition

If A is a square matrix and n is a positive integer, then

- $A^0 = I$
- $A^n = \underbrace{AA \cdots A}_{n \text{ factors}}$

If A is invertible, then

- $A^{-n} = (A^{-1})^n = \underbrace{A^{-1}A^{-1} \cdots A^{-1}}_{n \text{ factors}}$

Theorem

$A^r A^s = A^{r+s}$ and $(A^r)^s = A^{rs}$.

Powers of a matrix

Theorem

If A is invertible and n is a nonnegative integer, then

(a) A^{-1} *is invertible and* $(A^{-1})^{-1} = A$.

(b) A^n *is invertible and* $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.

(c) kA *is invertible for any nonzero scalar k and* $(kA)^{-1} = k^{-1}A^{-1}$.

Square of a matrix sum

$(A + B)^2 = A^2 + AB + BA + B^2$.

Matrix polynomials

Definition

If A is a square matrix and if $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$, then $p(A)$ is a matrix defined by

$$p(x) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m.$$

An expression of this form is called a matrix polynomial in A.

Example

Find $p(A)$ for $p(x) = x^2 - 2x - 3$ and $A = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}$

Properties of transpose

Theorem

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(A - B)^T = A^T - B^T$

(d) $(kA)^T = kA^T$

(e) $(AB)^T = B^T A^T$

Corollary

The transpose of a product of matrices is the product of the transposes in the reverse order.

Theorem

If A is invertible, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$.

Properties of the trace

Theorem

If A and B are square matrices with the same size, then

(a) $\mathrm{tr}(A^T) = \mathrm{tr}(A)$

(b) $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$

(c) $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$

(d) $\mathrm{tr}(A - B) = \mathrm{tr}(A) - \mathrm{tr}(B)$

(e) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

Example

Compute AB and BA and find their traces if $A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$

Properties of the trace

Theorem

If $\mathbf{r}$ *is a* $1 \times n$ *row vector and* $\mathbf{c}$ *is an* $n \times 1$ *column vector, then*

$$\mathbf{rc} = \operatorname{tr}(\mathbf{cr}).$$

Example

$\mathbf{r} = \begin{bmatrix} 1 & 3 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$