7.11 Coordinates with respect to a Basis

Nonrectangular coordinate system in $\mathbb{R}^n$

Nonrectangular coordinate system in $\mathbb{R}^n$

Definition

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an ordered basis for a subspace W of $\mathbb{R}^n$, and if

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$$

is the expression for a vector $\mathbf{w}$ in W as a linear combination of the vectors in B, then we call

$$a_1, a_2, \ldots, a_k$$

the coordinates of $\mathbf{w}$ with respect to B; and more specifically, we call a_j the $\mathbf{v}_j$-coordinate of $\mathbf{w}$. We denote the ordered k-tuple of coordinates by

$$(\mathbf{w})_B = (a_1, a_2, \ldots, a_k)$$

and call it the coordiate vector for $\mathbf{w}$ with respect to B; and we denote the column vector of coordinates by

$$[\mathbf{w}]_B = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix}^T$$

and call it the coordinate matrix for $\mathbf{w}$ with respect to B.

Coordinates with respect to an orthonormal basis

Theorem

If $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is an orthonormal basis for W and if $\mathbf{w}$ is a vector in W, then the coordinate vector for $\mathbf{w}$ with respect to B is

$$(\mathbf{w})_B = (\mathbf{w} \cdot \mathbf{v}_1, \mathbf{w} \cdot \mathbf{v}_2, \ldots, \mathbf{w} \cdot \mathbf{v}_k)$$

Theorem

If B is an orthonormal basis for a k-dimensional subspace W of $\mathbb{R}^n$, and if $\mathbf{v}$ and $\mathbf{w}$ are vectors in W with coordinate vectors

$$(\mathbf{v})_B = (v_1, v_2, \ldots, v_k) \qquad (\mathbf{w})_B = (w_1, w_2, \ldots, w_k)$$

then

(a) $\|\mathbf{w}\| = \sqrt{w_1^2 + w_2^2 + \cdots + w_k^2} = \|(\mathbf{w})_B\|$

(b) $\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \cdots + v_k w_k = (\mathbf{v})_B \cdot (\mathbf{w})_B$

Change of basis for $\mathbb{R}^n$

The Change of Basis Problem

If $\mathbf{w}$ is a vector in $\mathbb{R}^n$, and if we change the basis for $\mathbb{R}^n$ from a basis B to a basis B', how are the coordinate matrices $[\mathbf{w}]_B$ and $[\mathbf{w}]_{B'}$ related?

Change of basis for $\mathbb{R}^n$

Theorem

If $\mathbf{w}$ is a vector in $\mathbb{R}^n$, and if $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ are bases for $\mathbb{R}^n$, then the coordinate matrices of $\mathbf{w}$ with respect to two bases are related by the equation

$$[\mathbf{w}]_B = P_{B \to B'}[\mathbf{w}]_B$$

where

$$P_{B \to B'} = \begin{bmatrix} [\mathbf{v}_1]_{B'} & [\mathbf{v}_2]_{B'} & \cdots & [\mathbf{v}_n]_{B'} \end{bmatrix}$$

This matrix is called the transition matrix (or the change of coordinate matrix) from B to B'.

Change of basis for $\mathbb{R}^n$

Example

Consider the bases $B_1 = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $B_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$ where

$$\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1), \mathbf{v}_1 = (1, 1), \mathbf{v}_2 = (2, 1)$$

(a) Find the transition matrix from B_1 to B_2.

Change of basis for $\mathbb{R}^n$

(b) Find $[\mathbf{w}]_{B_2}$ given that $[\mathbf{w}]_{B_1} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$.

(c) Find the transition matrix from B_2 to B_1.

(d) Recover the vector $[\mathbf{w}]_{B_1}$ from the vector $[\mathbf{w}]_{B_2}$.

Invertibility of transition matrices

Theorem

If B and B' are bases for $\mathbb{R}^n$, then the transition matrices $P_{B \to B'}$ and $P_{B' \to B}$ are invertible and are inverses of one another.

Procedure for computing $P_{B \to B'}$

Step 1. Form the matrix $[B' \mid B]$

Step 2. Use elementary row operations to reduce the matrix in Step 1 to reduced row echelon form.

Step 3. The resulting matrix will be $[I \mid P_{B \to B'}]$.

Step 4. Extract the matrix $P_{B \to B'}$ from the right side of the matrix in Step 3.

Procedure for computing $P_{B \to B'}$

Example

Consider the bases $B_1 = \{\mathbf{e}_1, \mathbf{e}_2\}$ and $B_2 = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$ where

$$\mathbf{e}_1 = (1, 0), \mathbf{e}_2 = (0, 1), \mathbf{v}_1 = (1, 1), \mathbf{v}_2 = (2, 1)$$

Find the transition matrices $P_{B_1 \to B_2}$ and $P_{B_2 \to B_1}$.

Coordinate maps

Definition

If B is a basis for $\mathbb{R}^n$, then the transformation

$$\mathbf{x} \to (\mathbf{x})_B \quad \text{or in column notation,} \mathbf{x} \to [\mathbf{x}]_B$$

is called the coordinate map for B.

Theorem

If B is a basis for $\mathbb{R}^n$, then the coordinate map is a one-to-one linear operator on $\mathbb{R}^n$. Moreover, if B is an orthonormal basis for $\mathbb{R}^n$, then it is an orthogonal operator.

Transition between orthonormal bases

Theorem

If B and B' are orthonormal bases for $\mathbb{R}^n$, then the transition matrices $P_{B \to B'}$ and $P_{B' \to B}$ are orthogonal.