3.4. Subspaces and Linear Independence

Definition

If W is a nonempty set of vectors in $\mathbb{R}^n$, then

1. W is closed under scalar multiplication if $c\mathbf{v} \in W$ for any scalar c and any vector $\mathbf{v} \in W$.
2. W is closed under addition if $\mathbf{v}_1 + \mathbf{v}_2 \in W$ for any vectors $\mathbf{v}_1, \mathbf{v}_2 \in W$.
3. A nonempty set W of vectors in $\mathbb{R}^n$ is called a subspace of $\mathbb{R}^n$ if it is closed under scalar multiplication and addition.

Examples

- lines through the origin in $\mathbb{R}^2$
- planes through the origin in $\mathbb{R}^3$

Properties of subspaces

- $\{\mathbf{0}\}$ is a subspace of $\mathbb{R}^n$, called zero subspace or trivial subspace.
- $\mathbb{R}^n$ itself is a subspace of $\mathbb{R}^n$.
- Every subspace of $\mathbb{R}^n$ must contain $\mathbf{0}$.

Properties of subspaces

Theorem

If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ *are vectors in* $\mathbb{R}^n$*, then the set of all linear combinations*

$$\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_s\mathbf{v}_s$$

is a subspace of $\mathbb{R}^n$ *called the* *span* *of* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ *and denoted by* *span* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$*.*

Properties of subspaces

Example

$\{\mathbf{0}\} = \text{span}\{\mathbf{0}\}$.

Example

$\mathbb{R}^n = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$.

Properties of subspaces

Example

List all subspaces of $\mathbb{R}^2$.

Example

List all subspaces of $\mathbb{R}^3$.

Solution space of a linear system

Theorem

If $A\mathbf{x} = \mathbf{0}$ *is a homogeneous linear system with* n *unknowns, then its solution set is a subspace of* $\mathbb{R}^n$*.*

Solution space of a linear system

Theorem

1. *If A is a matrix with n columns, then the solution space of the homogeneous system $A\mathbf{x} = \mathbf{b}$ is all of $\mathbb{R}^n$ if and only if $A = 0$.*
2. *If A and B are matrices with n columns, then $A = B$ if and only if $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x}$ in $\mathbb{R}^n$.*

Linear independence

Definition

A nonempty set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s\}$ in $\mathbb{R}^n$ is said to be linearly independent if the only scalar $c_1, c_2, \dots, c_n$ satisfyng the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_s\mathbf{v}_s = \mathbf{0}$$

are $c_1 = 0, c_2 = 0, \dots, c_s = 0$.
If there are scalars, not all zero, that satisfy this equation, then the set is said to be linearly dependent.

Linear independence

Example

A vector $\mathbf{v}$ is linearly independent if and only if it is not the zero vector.

Example

A nonempty set of vectors in $\mathbb{R}^n$ containing the zero vector is linearly dependent.

Linear independence

Theorem

A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ *in* $\mathbb{R}^n$ *with two or more vectors is linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of other vectors in S.*

Linear independence

Example

Two vectors in $\mathbb{R}^n$ are linearly dependent if they are colinear and linearly independent if they are not.

Example

Three vectors in $\mathbb{R}^n$ are linearly dependent if they lie in a plane through the origin and are linearly independent if they are not.

Linear independence

Theorem

A homogeneous linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if the column vectors of A are linearly independent.

Linear independence

Example

Determine whether the following vectors are linearly independent or not.

$$\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 5, 0), \mathbf{v}_3 = (3, 3, 8)$$

Linear independence

Example

Determine whether the following vectors are linearly independent or not.

$$\mathbf{v}_1 = (2, -4, 6), \mathbf{v}_2 = (0, 7, -5), \mathbf{v}_3 = (6, 9, 8), \mathbf{v}_4 = (5, 0, 1)$$

Theorem

A set with more than n vectors in $\mathbb{R}^n$ is linearly dependent.

The unifying theorem

Theorem

If A is an $n \times n$ matrix, then the followings are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A is expressible as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
7. *The column vectors of A are linearly independent.*
8. *The row vectors of A are linearly independent.*