8.3 Orthogonal Diagonalizablity; Functions of a Matrix

Definition

If A and C are square matrices with the same size, then we say that C is orthogonally similar to A if there exists an orthogonal matrix P such that $C = P^T AP$.

Theorem

Two matrices are orthogonally similar if and only if there exist orthonormal bases with respect to which the matrices represent the same linear operator.

Orthogonal diagonalization

The Orthogonal Diagonalization Problem

Given a square matrix A, does there exist an orthogonal matrix P for which $P^T AP$ is a diagonal matrix, and if so, how does one find such a P? If such a matrix P exists, then A is said to be orthogonally diagonalizable, and P is said to orthogonally diagonalize A.

Theorem

An $n \times n$ matrix A is orthogonally diagonalizable if and only if there exists an orthonormal set of n eigenvectors of A.

Theorem

(a) *A matrix is orthogonally diagonalizable if and only if it is symmetric.*

(b) *If A is a symmetric matrix, then eigenvectors from different eigenspaces are orthogonal.*

Orthogonal diagonalization

Orthogonally diagonalizing an $n \times n$ symmetric matrix

Step 1. Find a basis for each eigenspace of A.

Step 2. Apply the Gram-Schmidt process to each of these bases to produce orthonormal bases for the eigenspaces.

Step 3. From the matrix $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \cdots & \mathbf{p}_n \end{bmatrix}$ whose columns are the vectors constructed in Step 2. The matrix P will orthogonally diagonalize A, and the eigenvalues on the diagonal of $D = P^T AP$ will be in the same order as their corresponding eigenvectors in P.

Orthogonal diagonalization

Example

Find a matrix P that orthogonally diagonalizes the symmetric matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Spectral decomposition

Theorem

If A is symmetric matrix that is orthogonally diagonalized by

$$P = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_n \end{bmatrix}$$

and if $\lambda_1, \lambda_2, \ldots, \lambda_n$ *are the eigenvalues of A corresponding to* $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$*, then A can be written as*

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdot + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$

which is called a spectral decomposition of A.

Spectral decomposition

Example

Find the spectral decomposition of

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Powers of a diagonalizable matrix

Theorem

If A is diagonalizable and $P^{-1}AP = D$ is a diagonal matrix, then

$$A^k = PD^kP^{-1}$$

for any positive integer k.

Powers of a diagonalizable matrix

Example

Find A^{13} for the diagonalizable matrix

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Cayley-Hamilton Theorem

Theorem (Cayley-Hamilton Theorem)

Every square matrix satisfies its characteristic equation; that is, if A is an $n \times n$ matrix whose characteristic equation is

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

then

$$A^n + c_1 A^{n-1} + \cdots + c_n I_n = 0$$

Cayley-Hamilton Theorem

Example

Find A^4 using the Cayley-Hamilton theorem when

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 5 & 3 \end{bmatrix}$$

Cayley-Hamilton Theorem

Example

Find A^{-1} using the Cayley-Hamilton theorem when

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 5 & 3 \end{bmatrix}$$

Exponential of a matrix

Theorem

Suppose that A is an $n \times n$ diagonalizable matrix that is diagonalized by P and that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A corresponding to the successive column vector of P. If f is a real-valued function whose Maclaurin series converges on some interval containing the eigenvalues of A, then

$$f(A) = P \begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & f(\lambda_n) \end{bmatrix} P^{-1}$$

Exponential of a matrix

Example

Find e^{tA} for the diagonalizable matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ -3 & 5 & 3 \end{bmatrix}$$