7.2 Properties of Bases

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is a basis for a subspace* V *of* $\mathbb{R}^n$*, then every vector* $\mathbf{v}$ *in* V *can be expressed in exactly one way as a linear combination of the vectors in* S*.*

Properties of bases

Theorem

Let S be a finite set of vectors in a nonzero subspace V of $\mathbb{R}^n$. If S spans V, but is not a basis for V, then a basis for V can be obtained by removing appropriate vectors from S.

Properties of bases

Theorem

Let S be a finite set of vectors in a nonzero subspace V of $\mathbb{R}^n$. If S is linearly independent, but is not a basis for V, then a basis for V can be obtained by addinging appropriate vectors from V to S.

Properties of bases

Theorem

If V is a nonzero subspace of $\mathbb{R}^n$, then $\dim(V)$ is the maximum number of linearly independent vectors in V.

Subspaces of subspaces

Theorem

If V and W are subspaces of $\mathbb{R}^n$, and if V is a subspace of W, then:

(a) $0 \leq \dim(V) \leq \dim(W) \leq n$.

(b) $V = W$ if and only if $\dim(V) = \dim(W)$.

Subspaces of subspaces

Theorem

Let S be a nonempty set of vectors in $\mathbb{R}^n$, and let S' be a set that results by adding additional vectors in $\mathbb{R}^n$ to S.

(a) If the additional vectors are in $\text{span}(S)$, then $\text{span}(S') = \text{span}(S)$.

(b) If $\text{span}(S') = \text{span}(S)$, then the additional vectors are in $\text{span}(S)$.

(c) If $\text{span}(S')$ and $\text{span}(S)$ have the same dimension, then the additional vectors are in $\text{span}(S)$ and $\text{span}(S') = \text{span}(S)$.

Sometimes spanning implies linear independence, and conversely

Theorem

(a) A set of k linearly independent vectors in a nonzero k-dimensional subspace of $\mathbb{R}^n$ is a basis for that space.

(b) A set of k vectors that span a nonzero k-dimensional subspace of $\mathbb{R}^n$ is a basis for that space.

(c) A set of fewer than k vectors in a nonzero k-dimensional subspace of $\mathbb{R}^n$ cannot span that subspace.

(d) A set with more than k vectors in a nonzero k-dimensional subspace of $\mathbb{R}^n$ is linearly dependent.

Properties of bases

Example

Show that the vectors $\mathbf{v}_1 = (1, 2, 1)$, $\mathbf{v}_2 = (1, -1, 3)$, and $\mathbf{v}_3 = (1, 1, 4)$ form a basis for $\mathbb{R}^3$.

The unifying theorem

Theorem

If A is an $n \times n$ matrix, then the followings are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A is expressible as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
7. *The column vectors of A are linearly independent.*
8. *The row vectors of A are linearly independent.*
9. $\det(A) \neq 0$.
10. *T_A is one-to-one.*
11. *T_A is onto.*

The unifying theorem

12. The column vectors of A are linearly independent.
13. The row vectors of A are linearly independent.
14. The column vectors of A span $\mathbb{R}^n$.
15. The row vectors of A span $\mathbb{R}^n$.
16. The column vectors of A form a basis for $\mathbb{R}^n$.
17. The row vectors of A form a basis for $\mathbb{R}^n$.