8.6 Singular Value Decomposition

Theorem

If A is an $n \times n$ matrix of rank k, then A can be factored as

$$A = U\Sigma V^T$$

where U and V are $n \times n$ orthogonal matrices and Σ is an $n \times n$ diagonal matrix whose main diagonal has k positive entires and $n - k$ zeros.

Singular Value Decomposition of Square Matrices

Theorem

(Singular Value Decomposition of a Square Matrix) If A is an $n \times n$ matrix of rank k, then A has a singular value decomposition $A = U\Sigma V^T$ in which:

(a) $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$ *orthogonally diagonalizes* A^TA.

(b) *The nonzero diagonal entries of* Σ *are*

$$\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \ldots, \sigma_k = \sqrt{\lambda_k}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ *are the nonzero eigenvalues of* A^TA *corresponding to the column vectors of* V*.*

Singular Value Decomposition of Square Matrices

(c) The column vectors of V are ordered so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$.

(d) $\mathbf{u}_i = \frac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|} = \frac{1}{\sigma_i} A\mathbf{v}_i$ $(i = 1, 2, \ldots, k)$

(e) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is an orthonormal basis for $col(A)$.

(f) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_k + 1, \ldots, \mathbf{u}_n\}$ is extension of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ to an orthonormal basis for$\mathbb{R}^n$.

Singular Value Decomposition of Square Matrices

Example

Find the singular value decomposition of the matrix

$$A = \begin{bmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{bmatrix}$$

Singular Value Decomposition of Nonsquare Matrices

Theorem

(Singular Value Decomposition of a General Matrix) If A is an $m \times n$ matrix of rank k, then A can be factored as

$$A = U\Sigma V^T (12)$$

in which U, Σ and V have sizes $m \times m$, $m \times n$, and $n \times n$, respectively, and in which:

Singular Value Decomposition of Nonsquare Matrices

(a) $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$ orthogonally diagonalizes $A^T A$.

(b) The nonzero diagonal entries of Σ are $\sigma_1 = \sqrt{\lambda_1}, \sigma_2 = \sqrt{\lambda_2}, \ldots, \sigma_k = \sqrt{\lambda_k}$, where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the nonzero eigenvalues of $A^T A$ corresponding to the column vectors of V.

(c) The column vectors of V are ordered so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k > 0$.

(d) $\mathbf{u}_i = \frac{A\mathbf{v}_i}{\|A\mathbf{v}_i\|} = \frac{1}{\sigma_i} A\mathbf{v}_i \ (i = 1, 2, \ldots, k)$

(e) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is an orthonormal basis for $col(A)$.

(f) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_k + 1, \ldots, \mathbf{u}_m\}$ is extension of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ to an orthonormal basis for$\mathbb{R}^m$.

Singular Value Decomposition of Nonsquare Matrices

Example

Find the singular value decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Singular Value Decomposition And The Fundamental Spaces of A Matrix

Theorem

If A is an $m \times n$ matrix of rank k, and if $A = U\Sigma V^T$ is the singular value decomposition given in Formula (12) *, then:*

(a) $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ *is an orthonormal basis for col*(A).

(b) $\{\mathbf{u}_{k+1}, \ldots, \mathbf{u}_m\}$ *is an orthonormal basis for col*$(A)^\perp =$ *null*(A^T).

(c) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is an orthonormal basis for row*(A).

(d) $\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ *is an orthonormal basis for row*$(A)^\perp =$ *null*(A).

Reduced Singular Value Decomposition

$$A = U_1 \Sigma_1 V_1^T$$

which is called a reduced singular value decomposition of A.

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T + \cdot + \sigma_k \mathbf{u}_k \mathbf{v}_k^T$$

which is called a reduced singular value expansion of A.