6.4 Composition and Invertibility of Linear Transformations

If $T_1 : \mathbb{R}^n \to \mathbb{R}^k$ and $T_2 : \mathbb{R}^k \to \mathbb{R}^m$ are linear transformations in which the codomain of T_1 is the same as the domain of T_2, then for each x in $\mathbb{R}^n$ we can first compute $T_1(x)$ to produce a vector in $\mathbb{R}^k$, and then we can compute $T_2(T_1(x))$ to produce a vector in $\mathbb{R}^m$. Thus first applying T_1 and then applying T_2 to the output of T_1 produces a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. This transformation, called the composition of T_2 with T_1, is denoted by $T_2 \circ T_1$ (read, " T_2 circle T_1")

$$(T_2 \circ T_1)(\mathbf{x}) = T_2(T_1(\mathbf{x}))$$

Compositions of linear transformations

Theorem

If $T_1 : \mathbb{R}^n \to \mathbb{R}^k$ *and* $T_2 : \mathbb{R}^k \to \mathbb{R}^m$ *are both linear transformations, then* $(T_2 \circ T_1) : \mathbb{R}^n \to \mathbb{R}^m$ *is also a linear transformation.*

Compositions of linear transformations

Theorem

If A is a $k \times n$ matrix and B is an $m \times k$ matrix, then $m \times n$ matrix BA is the standard matrix for the composition of the linear transformation corresponding to B with the linear transformation corresponding to A.

$$T_B \circ T_A = T_{BA}$$

Compositions of three or more linear transformations

Specifically, if $T_1 : \mathbb{R}^n \to \mathbb{R}^k$, $T_2 : \mathbb{R}^k \to \mathbb{R}^l$, $T_3 : \mathbb{R}^l \to \mathbb{R}^m$ then we define the composition $(T_3 \circ T_2 \circ T_1)(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}^m$ by

$$(T_3 \circ T_2 \circ T_1)(\mathbf{x}) = T_3(T_2(T_1(\mathbf{x})))$$

$$[T_3 \circ T_2 \circ T_1] = [T_3][T_2][T_1]$$

$$T_C \circ T_B \circ T_A = T_{CBA}$$

Inverse of linear transformation

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a one to one linear transformation, then each vector $\mathbf{w}$ in the range of T is the image of a unique vector $\mathbf{x}$ in the domain of T; we call $\mathbf{x}$ the preimage of $\mathbf{w}$. The uniqueness of the preimage allows us to create a new function that maps $\mathbf{w}$ into $\mathbf{x}$; we call this function the inverse of T and denote it by T^{-1}. Thus,

$$T^{-1}(\mathbf{w}) = \mathbf{x} \text{ if and only if } T(\mathbf{x}) = \mathbf{w}$$

Theorem

If T is a one to one linear transformation, then so is T^{-1}.

Invertible of linear operator

Theorem

If T is a one to one linear operator on $\mathbb{R}^n$, then the standard matrix for T is invertible and its inverse is the standard matrix for T^{-1}.