6.3 Kernel and Range

Definition

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the set of vectors in $\mathbb{R}^n$ that T maps into $\mathbf{0}$ is called the kernel of T and is denoted by $\ker(T)$.

Kernel of a linear transformation

Example

Find the kernels of the following linear operators on $\mathbb{R}^3$.

(a) The zero operator

(b) The identity operator

(c) The orthogonal projection onto the xy-plane

(d) A rotation about a line through the origin

Kernel of a linear transformation

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the kernel of T is a subspace of $\mathbb{R}^n$.

Kernel of a linear transformation

Theorem

If A is an $m \times n$ matrix, then the kernel of the corresponding linear transformation is the solution space of $A\mathbf{x} = \mathbf{0}$.

Definition

If A is an $m \times n$ matrix, then the solution space of the linear system $A\mathbf{x} = \mathbf{0}$, or, equivalently, the kernel of the transformation T_A, is called the null space of A and is denoted by $\text{null}(A)$.

Kernel of a linear transformation

Example

Find the null space of $A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$

Kernel of a linear transformation

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T maps subspaces of $\mathbb{R}^n$ into subspaces of $\mathbb{R}^m$.

Range of a linear transformation

Definition

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the range of T, denoted by $\operatorname{ran}(T)$, is the set of all vectors in $\mathbb{R}^m$ that are images of at least one vector in $\mathbb{R}^n$.

Range of a linear transformation

Example

Find the ranges of the following linear operators on $\mathbb{R}^3$.

(a) The zero operator

(b) The identity operator

(c) The orthogonal projection onto the xy-plane

(d) A rotation about a line through the origin

Range of a linear transformation

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then ran(T) is a subspace of $\mathbb{R}^m$.

Theorem

If A is an $m \times n$ matrix, then the range of the corresponding linear transformation is the column space of A.

Existence and Uniqueness

Existence question

Is every vector in $\mathbb{R}^m$ the image of at least one vector in $\mathbb{R}^n$?

Uniqueness question

Can two different vectors in $\mathbb{R}^n$ have the same image in $\mathbb{R}^m$?

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto if its range is the entire codomain $\mathbb{R}^m$.

Definition

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one if T maps distinct vectors in $\mathbb{R}^n$ into distinct vectors in $\mathbb{R}^m$.

Existence and Uniqueness

Example

Chech whether the following transformations are onto or one-to-one.

(a) Rotation about the origin in $\mathbb{R}^2$.

(b) Orthgonal projection on the xy-plane in $\mathbb{R}^3$.

(c) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x, y, 0)$.

(d) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (x, y)$.

Existence and Uniqueness

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then the followings are equivalent:

(a) *T is one-to-one.*

(b) $\ker(T) = \{\mathbf{0}\}$.

Existence and Uniqueness

Theorem

If A is an $m \times n$ matrix, then the corresponding linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if and only if the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Theorem

If A is an $m \times n$ matrix, then the corresponding linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ is onto if and only if the linear system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathbb{R}^n$.

Linear operator

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator on $\mathbb{R}^n$, then T is one-to-one if and only if it is onto.

The unifying theorem

Theorem

If A is an $n \times n$ matrix, then the followings are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A is expressible as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
7. *The column vectors of A are linearly independent.*
8. *The row vectors of A are linearly independent.*
9. $\det(A) \neq 0$.
10. *T_A is one-to-one.*
11. *T_A is onto.*