8.1 Matrix Representations of Linear Transformations

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator, let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a basis for $\mathbb{R}^n$, and let

$$A = \Big[[T(\mathbf{v}_1)]_B \mid [T(\mathbf{v}_2)]_B \mid \cdots \mid [T(\mathbf{v}_n)]_B \Big]$$

Then

$$[T(\mathbf{x})]_B = A[\mathbf{x}]_B$$

for every vector $\mathbf{x}$ in $\mathbb{R}^n$.

Definition

The matrix A in the above theorem is called the matrix for T with respect to the basis B and is denoted by

$$[T]_B = \Big[[T(\mathbf{v}_1)]_B \mid [T(\mathbf{v}_2)]_B \mid \cdots \mid [T(\mathbf{v}_n)]_B \Big]$$

Matrix of a linear operator w.r.t. a basis

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator whose standard matrix is

$$[T] = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

Find the matrix for T with respect to the basis $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$

Changing bases

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator, and if $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ are bases for $\mathbb{R}^n$, then $[T]_B$ and $[T]_{B'}$ are related by the equation

$$[T]_{B'} = P[T]_B P^{-1}$$

in which

$$P = P_{B \to B'} = \Big[[\mathbf{v}_1]_{B'} \mid [\mathbf{v}_2]_{B'} \mid \cdots \mid [\mathbf{v}_n]_{B'} \Big]$$

is the transition matrix from B to B'.
In the special case where B and B' are orthonormal bases the matrix P is orthonormal, so $[T]_B$ and $[T]_{B'}$ are related by the equation

$$[T]_{B'} = P[T]_B P^T$$

Changing bases

Theorem

If $T : \mathbb{R}^n \to \mathbb{R}^n$ *is a linear operator, and if* $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *is bases for* $\mathbb{R}^n$*, then* $[T]$ *and* $[T]_B$ *are related by the equation*

$$[T] = P[T]_B P^{-1}$$

in which

$$P = [\mathbf{v}_1 \mid \mathbf{v}_2 \mid \cdots \mid \mathbf{v}_n]$$

is the transition matrix from B to the standard basis.
In the special case where B is orthonormal bases the matrix P is orthonormal, so $[T]$ *and* $[T]_B$ *are related by the equation*

$$[T] = P[T]_B P^T$$

Changing bases

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator whose standard matrix is

$$[T] = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

and let B be the basis $B = \left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right\}$ for $\mathbb{R}^2$.

Verify $[T] = P[T]_B P^{-1}$ where P is an appropriate transition matrix.

Matrix of a linear transformation w.r.t. a pair of bases

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ *be a linear transformation, let* $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ *and* $B' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ *be bases for* $\mathbb{R}^n$ *and* $\mathbb{R}^m$*, respectively, and let*

$$A = \Big[[T(\mathbf{v}_1)]_{B'} \mid [T(\mathbf{v}_2)]_{B'} \mid \cdots \mid [T(\mathbf{v}_n)]_{B'} \Big]$$

Then

$$[T(\mathbf{x})]_{B'} = A[\mathbf{x}]_B$$

for every vector $\mathbf{x}$ *in* $\mathbb{R}^n$*.*

Definition

The matrix A in the above theorem is called the matrix for T with respect to the bases B and B' and is denoted by

$$[T]_{B',B} = \Big[[T(\mathbf{v}_1)]_B \mid [T(\mathbf{v}_2)]_B \mid \cdots \mid [T(\mathbf{v}_n)]_B \Big]$$

Matrix of a linear transformation w.r.t. a pair of bases

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{bmatrix}$$

Find the matrix for T with respect to the bases $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ for $\mathbb{R}^2$ and $B' = \{\mathbf{v}_1', \mathbf{v}_2', \mathbf{v}_3'\}$ for $\mathbb{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \mathbf{v}_1' = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2' = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \mathbf{v}_3' = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$