7.3 The Fundamental Spaces of a Matrix

If A is an $m \times n$ matrix, then there are three important spaces associated with A:

1. The row space of A, denoted by row(A), is a subspace of $\mathbb{R}^n$ spanned by the row vectors of A.
2. The column space of A, denoted by col(A), is a subspace of $\mathbb{R}^m$ spanned by the column vectors of A.
3. The null space of A, denoted by null(A), is the solution space of $A\mathbf{x} = \mathbf{0}$. This is a subspace of $\mathbb{R}^n$.

Fundamental spaces of A

Definition

The four subspaces $\text{row}(A), \text{col}(A), \text{null}(A), \text{null}(A^T)$ are called the fundamental spaces of A.

Definition

- The dimension of $\text{row}(A)$ is called the rank of A and is denoted by $\text{rank}(A)$.
- The dimension of $\text{null}(A)$ is called the nullity of A and is denoted by $\text{nullity}(A)$.

Orthogonal complements

Definition

If S is a nonempty set in $\mathbb{R}^n$, then the orthogonal complement of S, denoted by $S^\perp$, is defined to be the set of al vectors in $\mathbb{R}^n$ that are orthogonal to every vector in S.

Example

Find the orthogonal complement of the following sets:

(a) A line L through the origin of $\mathbb{R}^3$.

(b) A set S of row vectors of an $m \times n$ matrix A.

Orthogonal complement

Theorem

If S is a nonempty set in $\mathbb{R}^n$, then $S^\perp$ is a subspace of $\mathbb{R}^n$.

Orthogonal complement

Example

Find the orthogonal complement in an *xyz*-coordinate system of the set $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ where

$$\mathbf{v}_1 = (1, -2, 1), \mathbf{v}_2 = (3, -7, 5).$$

Properties of orthogonal complements

Theorem

(a) If W is a subspace of $\mathbb{R}^n$, then $W^\perp \cap W = \{\mathbf{0}\}$.

(b) If S is a nonempty subset of $\mathbb{R}^n$, then $S^\perp = \text{span}(S)^\perp$.

(c) If W is a subspace of $\mathbb{R}^n$, then $(W^\perp)^\perp = W$.

Orthogonal complement

Theorem

If A is an $m \times n$ matrix, then the row space of A and the null space of A are orthogonal complements.

Orthogonal complement

Theorem

If A is an $m \times n$ matrix, then the column space of A and the null space of A^T are orthogonal complements.

Theorem

Let A be an $m \times n$ matrix.

(a) Elementary row operations do not change the row space of a matrix.

(b) Elementary row operations do not change the null space of a matrix.

(c) The nonzero row vectors in any row echelon form of a matrix form a basis for the row space of the matrix.

Relationship between the fundamental spaces of two matrices

Theorem

If A and B are matrices with the same number of columns, then the followings are equivalent:

(a) *A and B have the same row space.*

(b) *A and B have the same null space.*

(c) *The row vectors of A are linear combinations of the row vectors of B, and conversely.*

Finding bases by row reduction

Example

Let $\mathbf{v}_1 = (1, 0, 0, 0, 2)$, $\mathbf{v}_2 = (-2, 1, -3, -2, -4)$,
$\mathbf{v}_3 = (0, 5, -14, -9, 0)$, and $\mathbf{v}_4 = (2, 10, -28, -18, 4)$.

(a) Find a basis for the subspace W spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$.

Finding bases by row reduction

Example

(b) Find a basis for $W^{\perp}$.

Finding bases by row reduction

Example

(c) Find a homogeneous linear system $B\mathbf{x} = \mathbf{0}$ whose solution space is W.

Determining whether a vector is in a given subspace

Problem 1. Given a set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in $\mathbb{R}^m$, find conditions on the numbers $b_1, b_2, \ldots, b_m$ under which $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ will lie in span(S).

Problem 2. Given an $m \times n$ matrix A, find conditions on the numbers $b_1, b_2, \ldots, b_m$ under which $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ will lie in col(A).

Problem 3. Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, find conditions on the numbers $b_1, b_2, \ldots, b_m$ under which $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ will lie in ran(T).

Determining whether a vector is in a given subspace

Example

Let $\mathbf{v}_1 = (1, 0, 0, 0, 2)$, $\mathbf{v}_2 = (-2, 1, -3, -2, -4)$,
$\mathbf{v}_3 = (0, 5, -14, -9, 0)$, and $\mathbf{v}_4 = (2, 10, -28, -18, 4)$.

(a) Find conditions when a vector $\mathbf{b} = (b_1, b_2, b_3, b_4, b_5)$ lies in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Determining whether a vector is in a given subspace

Example

(b) Determine which of the vectors $\mathbf{b}_1 = (7, -2, 5, 3, 14), \mathbf{b}_2 = (7, -2, 5, 3, 6)$, and $\mathbf{b}_3 = (0, -1, 3, -2, 0)$, if any, lie in span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.