4.1 Determinants; Cofactor Expansion

Determinant of 2×2 matrix

The determinant of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is defined by

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}.$$

Example

Find $\det(A)$ if $A = \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix}$

Determinant

Determinant of 3×3 matrix

The determinant of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is defined by

$$\begin{aligned} \det(A) = & a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ & - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}. \end{aligned}$$

Determinant

Example

Find $\det(A)$ if $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$

Determinant

Determinant of $n \times n$ matrix

The determinant of $A = \left[a_{ij}\right]$ is defined by

$$\det(A) = \pm a_{1j_1} a_{2j_2} \cdots a_{nj_n}$$

where the sign is $+$ if the permutation $\{j_1, j_2, \ldots, j_n\}$ is even and $-$ if it is odd.

Computation of determinants

Theorem

If A is a square matrix with a row or a column of zeros, then $\det(A) = 0$*.*

Theorem

If A is a triangular matrix, then $\det(A)$ *is the product of the entries on the main diagonal.*

Minors and cofactors

Definition

If A is a square matrix, then the minor of entry a_{ij} is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains when the ith row and jth column of A are deleted. The number $C_{ij} = (-1)^{i+j} M_{ij}$ is called the cofactor of entry a_{ij}.

Example

Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$

1. Find the minor of a_{12}.

2. Find the cofactor of a_{22}.

Cofactor expansion

Theorem

The determinant of an $n \times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products.

Cofactor expansion

Example

Use a cofactor expansion to find the determinant of

$$A = \begin{bmatrix} 2 & 0 & 0 & 5 \\ -1 & 2 & 4 & 1 \\ 3 & 0 & 0 & 3 \\ 8 & 6 & 0 & 0 \end{bmatrix}$$