2.1. Introduction to Systems of Linear Equations

Definition

- A linear equation in the n variables $x_1, x_2, \ldots, x_n$ is one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

  where $a_1, a_2, \ldots, a_n$, and b are constants and the a's are not all zero.
- If $b = 0$, the linear equation is called homogeneous.

Examples

- $x + 2y = 5$.
- $2x - y + 3z = 10$.
- $x_1 - x_2 + 3x_3 - 4x_4 = 0$.
- $x_1 + x_2 + \cdots + x_n = 1$.

System of linear equations

Definition

- A finite set of linear equations is called a system of linear equations or a linear system.
- The variables in a linear system is called the unknowns.
- A general linear system of m equations in the n unknowns $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

- A solution of a linear system in the unknowns $x_1, x_2, \ldots, x_n$ is a sequence of n numbers $s_1, s_2, \ldots, s_n$ that makes every equation true.

System of linear equations

Examples

a. $\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$

b. $\begin{aligned} x + y &= 4 \\ 3x + 3y &= 6 \end{aligned}$

c. $\begin{aligned} 4x - 2y &= 4 \\ 2x - y &= 2 \end{aligned}$

Theorem

Every system of linear equations has zero, one, or infinitely many solutions; there are no other possibilities.

System of linear equations

Example

Solve the linear system

$$\begin{array}{rcrcrcr} x & + & y & + & 3z & = & 0 \\ 2x & + & 3y & - & z & = & 9 \\ 3x & - & 2y & + & 3z & = & -4 \end{array}$$

Augmented matrices

A general linear system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

can be abbreviated by writing only the rectangular array of numbers

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

which is called the augmented matrix for the system.

Elementary row operations

Operations on equations

1. Multiply an equation through by a nonzero number.
2. Interchange two equations.
3. Add a multiple of one equation to another.

Operations on the rows of the augmented matrix

1. Multiply a row through by a nonzero number.
2. Interchange two rows.
3. Add a multiple of one row to another.

These are called elementary row operations on a matrix.

System of linear equations

Example

Solve the linear system

$$\begin{array}{rcrcrcr} x & + & y & + & 3z & = & 0 \\ 2x & + & 3y & - & z & = & 9 \\ 3x & - & 2y & + & 3z & = & -4 \end{array}$$

by operating on the rows of the augmented matrix.