2.2. Solving Linear Systems by Row Reduction

Definition

A matrix is said to be in reduced row echelon form if it satisfies

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. This is called a leading 1.
2. If there are any rows consisting entirely of zeros, then they are at the bottom of the matrix.
3. The leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
4. Each column containing a leading 1 has zeros everywhere else.

Example

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & 0 & 0 & * & * \\ 0 & 0 & 1 & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row echelon form

Definition

A matrix satisfying the first three properties is said to be in row echelon form:

1. If a row does not consist entirely of zeros, then the first nonzero number in the row is a 1. This is called a leading 1.
2. If there are any rows consisting entirely of zeros, then they are at the bottom of the matrix.
3. The leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

Example

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 3 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving a linear system

Example

Suppose that the augmented matrix for a linear system in x, y, and z has been reduced by elementary row operation to the given reduced row echelon form. Solve the system.

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Solving a linear system

Example

Suppose that the augmented matrix for a linear system in x, y, and z has been reduced by elementary row operation to the given reduced row echelon form. Solve the system.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solving a linear system

Example

Suppose that the augmented matrix for a linear system in x, y, and z has been reduced by elementary row operation to the given reduced row echelon form. Solve the system.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving a linear system

Example

Suppose that the augmented matrix for a linear system in x, y, and z has been reduced by elementary row operation to the given reduced row echelon form. Solve the system.

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Gauss-Jordan and Gaussian elimination

Step 1. Locate the leftmost column that does not consist entirely of zero.

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

Step 3. If the top entry of the column in Step 2 is a, multiply the first row by $1/a$ to introduce a leading 1.

Step 4. Add suitable multiple of the top row to the rows below so that all entries below the leading 1 become zeros.

Step 5. Now cover the top row and begin again with Step 1 to the submatrix that remains. Continue in this way until the entire matrix is in row echelon form.

Step 6. Beginning with the last nonzero row and working upward, add suitable multiple of each row to the rows above to introduce zeros above the leading 1's.

Gauss-Jordan and Gaussian elimination

Example

Reduce the following matrix to the reduced echelon form by Gauss-Jordan elimination.

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 0 & 3 \\ 2 & 4 & 4 & 2 & 4 & 6 \\ 1 & 2 & 0 & -3 & 2 & -3 \\ 2 & 4 & 0 & -4 & -2 & 0 \end{bmatrix}$$

(Reduced) row echelon forms

Theorem

Every matrix has a unique reduced row echelon form.

Theorem

Row echelon forms are not unique. However, all of the row echelon forms have their leading 1's in the same positions and all have the same number of zero rows at the bottom.

Gauss-Jordan elimination

Example

Solve the following linear system by Gauss-Jordan elimination.

$$
\begin{array}{rcrcrcrcrcr}
 & & & & 2x_3 & + & 4x_4 & + & 4x_5 & = & 0 \\
2x_1 & - & 4x_2 & - & 2x_3 & + & 2x_4 & + & 2x_5 & = & -8 \\
3x_1 & - & 6x_2 & & & + & 9x_4 & + & 9x_5 & = & -12 \\
2x_1 & - & 4x_2 & & & + & 9x_4 & + & 6x_5 & = & -5
\end{array}
$$

Gaussian elimination

Example

Solve the following linear system by Gaussian elimination and back substitution.

$$
\begin{array}{rcrcrcrcrcr}
 & & & & 2x_3 & + & 4x_4 & + & 4x_5 & = & 0 \\
2x_1 & - & 4x_2 & - & 2x_3 & + & 2x_4 & + & 2x_5 & = & -8 \\
3x_1 & - & 6x_2 & & & + & 9x_4 & + & 9x_5 & = & -12 \\
2x_1 & - & 4x_2 & & & + & 9x_4 & + & 6x_5 & = & -5
\end{array}
$$

Homogeneous linear system

Definition

- A linear system is called homogeneous if each of its equation is homogeneous.
- A general homogeous system is

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0 \end{array}$$

- $x_1 = x_2 = \cdots = x_n = 0$ is a solution, called the trivial solution.
- All other solutions, if any, are called nontrivial solutions.

Theorem

A homogeneous linear system has only the trivial solution or it has infinitely many solutions.

Homogeneous linear system

Example

Solve the following homogeneous linear system by Gauss-Jordan elimination.

$$
\begin{array}{rcrcrcrcrcr}
 & & & & 2x_3 & + & 4x_4 & + & 4x_5 & = & 0 \\
2x_1 & - & 4x_2 & - & 2x_3 & + & 2x_4 & + & 2x_5 & = & 0 \\
3x_1 & - & 6x_2 & & & + & 9x_4 & + & 9x_5 & = & 0 \\
2x_1 & - & 4x_2 & & & + & 9x_4 & + & 6x_5 & = & 0
\end{array}
$$

Homogeneous linear system

Theorem (Dimension theorem for homogeneous linear systems)

If a homogeneous linear system has n unknowns and if the reduced row echelon form of its augmented matrix has r nonzero rows, then the system has $n - r$ free variables.

Theorem

A homogeneous linear system with more unknowns than equations has infinitely many solutions.