7.1 Basis and Dimension

Definition

A set of vectors in a subspace V of $\mathbb{R}^n$ is said to be a basis for V if it is linearly independent and spans V.

Basis of a subspace

Example

$\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is the standard basis for $\mathbb{R}^n$

Basis of a subspace

Theorem

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ *is a set of two or more vectors in* $\mathbb{R}^n$*, then* S *is linearly dependent if and only if some vector in* S *is a linear combination of its predecessor.*

Basis of a subspace

Example

Show that the vectors

$$\mathbf{v}_1 = (0, 2, 0), \mathbf{v}_2 = (3, 0, 3), \mathbf{v}_3 = (-4, 0, 4)$$

are linearly independent by showing that no vectors in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linear combination of predecessors.

Basis of a subspace

Example

The nonzero row vectors of a matrix in row echelon form are linearly independent.

Basis of a subspace

Theorem (Existence of a basis)

If V is a nonzero subspace of $\mathbb{R}^n$, then there exists a basis for V that has at most n vectors.

Basis of a subspace

Theorem

All bases for a nonzero subspace of $\mathbb{R}^n$ have the same number of vectors.

Dimension of a subspace

Definition

If V is a nonzero subspace of $\mathbb{R}^n$, then the dimension of V, written $\dim(V)$, is defined to be the number of vectors in a basis for V. In addition, we define the zero subspace to have dimension 0.

Dimension of a solution space

Definition

- The general solution of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ is of the form
$$\mathbf{x} = r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots r_s\mathbf{v}_s$$
where the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ are linearly independent.
- The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s$ are called the canonical solution of $A\mathbf{x} = \mathbf{0}$.
- Since the canonical solution vectors span the solution space and are linearly independent, they form a basis for the solution space. This basis is called the canonical basis for the solution space.

Dimension of a solution space

Example

Find a canonical basis for the solution space of the linear system

$$\begin{array}{rcrcrcrcrcrcl}
x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\
2x_1 & + & 6x_2 & - & 5x_3 & - & 2x_4 & + & 4x_5 & - & 3x_6 & = & 0 \\
 & & & & 5x_3 & + & 10x_4 & & & + & 15x_6 & = & 0 \\
2x_1 & + & 6x_2 & & & + & 8x_4 & + & 4x_5 & + & 18x_6 & = & 0
\end{array}$$

and state the dimension of the solution space.

Dimension of a hyperplane

Theorem

If $\mathbf{a}$ *is a nonzero vector in* $\mathbb{R}^n$*, then* $\dim(\mathbf{a}^\perp) = n - 1$.