3.3 Elementary matrices; A method for finding A^{-1}

Elementary row operations

1. Multiply a row by nonzero constant.
2. Interchange two rows.
3. Add a multiple of one row to another.

Definition

A matrix that results from applying a single elementary row operation to an identity matrix is called an elementary matrix.

Examples

a. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Elementary matrix

Theorem

If A is an $m \times n$ matrix and if the elementary matrix E results by performing a certain row operation on the $m \times m$ identity matrix, then the product EA is the matrix that results when the same row operation is performed on A.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$. Find an elementary matrix E such that EA is the matrix that results by adding 3 times the first row of A to the second row.

Inverse operations of the elementary row operations

Row operations on I that gives E	Row operations on E that gives I
1. Multiply row i by $c \neq 0$	1. Multiply row I by $\frac{1}{c}$

Inverse operations of the elementary row operations

Row operations on I that gives E	Row operations on E that gives I
2. Interchange rows i and j	2. Interchange rows i and j

Inverse operations of the elementary row operations

Row operations on I that gives E	Row operations on E that gives I
3. Add c times row i to row j	3. Add $-c$ times row i to row j

Invertibility

Theorem

An elementary matrix is invertible, and its inverse is also an elementary matrix.

Theorem

If A is an $n \times n$ matrix, then the followings are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A is expressible as a product of elementary matrices.*
3. *A is invertible.*

Row equivalence

Definition

Two matrices that can be obtained from one another by finite sequence of elementary row operations are said to be row equivalent.

Theorem

If A and B are square matrices of the same size, then the followings are equivalent:

1. *A and B are row equivalent.*
2. *There is an invertible matrix E such that $B = EA$.*
3. *There is an invertible matrix F such that $A = FB$.*

The inversion algorithm

Theorem

To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to I, and then perform the same sequence of operations on I to obtain A^{-1}.

The inversion algorithm

Example

Find the inverse of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$

The inversion algorithm

Example

Find the inverse of $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$

Solving linear systems by matrix inversion

Theorem

If $A\mathbf{x} = \mathbf{b}$ is a linear system of n equations in n unknowns, and if the coefficient matrix A is invertible, then the system has a unique solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

Solving linear systems by matrix inversion

Example

Solve the linear system:

$$\begin{aligned} x_1 + \quad 2x_2 + \quad 3x_3 &= \quad 5 \\ 2x_1 + \quad 5x_2 + \quad 3x_3 &= \quad 3 \\ x_1 \qquad\quad + \quad 8x_3 &= \quad 17 \end{aligned}$$

Solving linear systems by matrix inversion

Theorem

If $A\mathbf{x} = \mathbf{0}$ is a homogeneous system of n equations in n unknowns, then the system has only the trivial solution if and only if the coefficient matrix A is invertible.

Invertibility

Theorem

If A is an $n \times n$ matrix, then the followings are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A is expressible as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*

Invertibility

Theorem

1. *If A and B are square matrices such that $AB = I$ or $BA = I$, then A and B are both invertible, and each is the inverse of the other.*
2. *If A and B are square matrices whose product AB is invertible, then A and B are invertible.*

Invertibility

Theorem

If A is an $n \times n$ matrix, then the followings are equivalent.

1. *The reduced row echelon form of A is I_n.*
2. *A is expressible as a product of elementary matrices.*
3. *A is invertible.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *$A\mathbf{x} = \mathbf{b}$ is consistent for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*
6. *$A\mathbf{x} = \mathbf{b}$ has exactly one solution for every vector $\mathbf{b}$ in $\mathbb{R}^n$.*

Consistency of linear systems

Theorem (The consistency problem)

For a given matrix A find all vectors $\mathbf{b}$ for which the linear system $A\mathbf{x} = \mathbf{b}$ is consistent.

Consistency of linear systems

Example

What conditions must b_1, b_2, and b_3 satisfy for the following linear system to be consistent?

$$\begin{aligned} x_1 + \quad x_2 + \quad 2x_3 &= \quad b_1 \\ x_1 \qquad\quad + \quad x_3 &= \quad b_2 \\ 2x_1 + \quad x_2 + \quad 3x_3 &= \quad b_3 \end{aligned}$$