Supplementary Information: Cyclically symmetric Thomas oscillators as swarmalators: a model for active fluids and pattern formation

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1. The Zindler Curve

Consider the following parametric equation with a single real parameter ϵ for a simple mathematical illustration.

$$\begin{aligned} Z(w) &= X(w) + iY(w) \\ &= e^{i2w} + 2e^{-iw} + \epsilon e^{iw/2} \qquad w \in [0, 4\pi] \end{aligned} \tag{1}$$

The Zindler curve is defined as a closed curve in a plane such that all chords which cut the curve length into equal halves have the same length. For $\epsilon > 4$, the curve is a Zindler Curve. In order to prove this the derivative of Equation(1) and its absolute value are considered as given below.

$$\begin{aligned} Z'(w) &= i(2e^{i2w} - 2e^{-iw} + \frac{\epsilon}{2} e^{iw/2}) \qquad (2) \\ |Z'(w)|^2 &= Z'(w)Z'(w)^* = 8 + \frac{\epsilon^2}{2} - 8cos3 \end{aligned}$$

From Equation(3) it is clear that $|Z'(w)|$ is 2π-periodic. Then for any fixed value of the variable, say w_0, the following equation holds, which is half the length of the entire curve.

$$\int_{w_0}^{w_0+2\pi} |Z'(w)| dw = \int_0^{2\pi} |Z'(w)| dw \tag{3}$$

The length of the straight line, which divides the curve into two halves, can be shown to be independent of w_0, the reference point. These straight lines are bounded by the points $Z(w_0)$ and $Z(w_0 + 2\pi)$ for any choice of the reference point $w_0 \in [0, 4\pi]$. The length of such a straight line is

$$|Z(w_0 + 2\pi) - Z(w_0)| = |2\epsilon e^{iw_0/2}| = 2\epsilon \tag{4}$$

The following plots show the curves for a different choice of the parameter ϵ. For $\epsilon = 4$, the chord meets the curve at one additional point and is not a Zindler curve. The simplest of all Zindler curves is a circleBracho et al. (2004); Rochera (2022). Figure (1) shows the behaviour of Equation (1) for four different values of the parameter ϵ.

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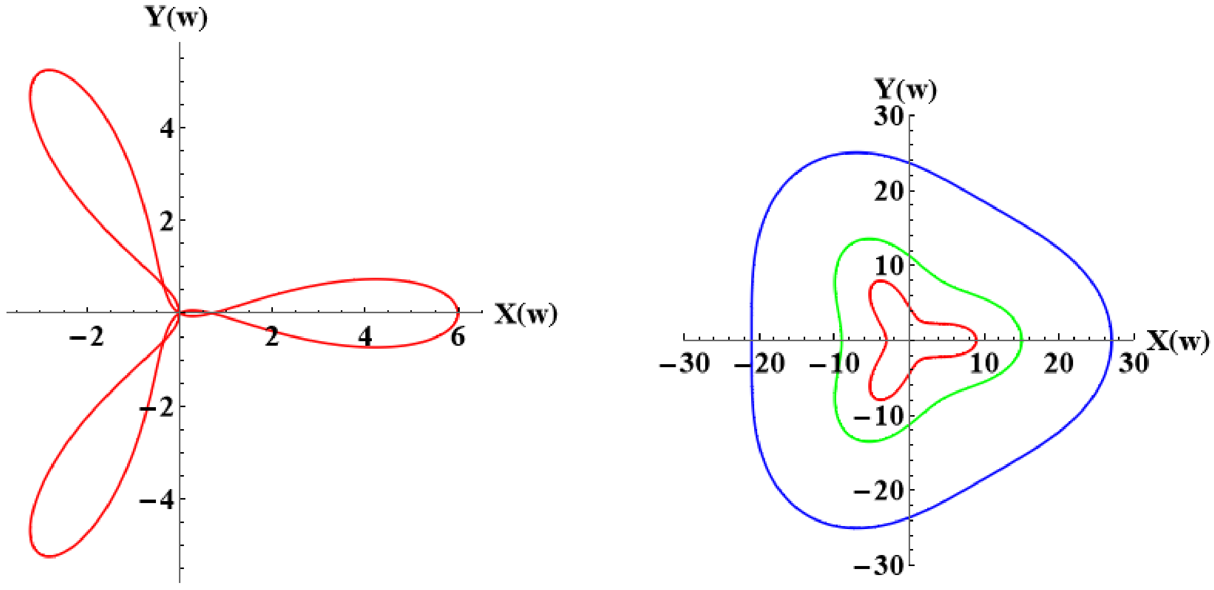

(a) (b)

Figure 1: (a) Shows the curve for $\epsilon = 3$ and is not a Zindler curve and (b) depicts the Zindler curves for $\epsilon = 8, 16, 24$(red,green,blue) respectively.

2. Principal Component Analysis (PCA) and Angular Momentum

To investigate the intrinsic dynamics of particles within swarming clusters, we employed Principal Component Analysis (PCA). For each cluster, the dominant two-dimensional (2D) plane of motion was first identified using PCA. The center of mass (COM) of each cluster was then determined, followed by an analysis of the rotational dynamics about the COM. This approach effectively reduces the three-dimensional (3D) particle motion to a 2D representation that is aware of the underlying geometry, enabling accurate and robust computation of angular momentum within each cluster.

Given the position of the i^{th} particle, denoted as $\mathbf{r}_i = [x_i, y_i, z_i]^T$, in a given cluster, we first center the data by subtracting the mean position (center of mass) of the cluster:

$$\bar{\mathbf{r}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{r}_i, \tag{5}$$

$$\mathbf{r}_c = \{\mathbf{r}_i - \bar{\mathbf{r}}\}_{i=1}^{N}, \tag{6}$$

and this translation ensures that the subsequent motions are invariant to the global position. An empirical covariance matrix is constructed from the centered particle positions, and the corresponding eigenvalues and eigenvectors are subsequently computed to identify the principal directions of motion.

$$\mathbf{\Omega} = \frac{1}{N-1}\sum_{i=1}^{N}(\mathbf{r}_i - \bar{\mathbf{r}})(\mathbf{r}_i - \bar{\mathbf{r}})^T \in \mathcal{R}^{3\times3} \tag{7}$$

The eigenvalues are $\lambda_1 \geq \lambda_2 \geq \lambda_3$ and the corresponding eigenvectors are V_1, V_2, V_3 satisfying:

$$\mathbf{\Omega} V_j = \lambda_j V_j \tag{8}$$

The two leading principal components—corresponding to the largest eigenvalues—are retained to construct an orthonormal basis for the intrinsic two-dimensional plane associated with the cluster.

$$\mathbf{V}_{2D} = (V_1, V_2) \in \mathcal{R}^{3\times 2} \tag{9}$$

Each 3D position vector $\mathbf{r}_i$ is then projected onto the identified 2D plane using the principal components as basis vectors. The projection is given by:

$$\tilde{\mathbf{r}}_i = \mathbf{V}_{2D}^T(\mathbf{r}_i - \bar{\mathbf{r}}) \in \mathcal{R}^2 \tag{10}$$

where $\bar{\mathbf{r}}$ is the center of mass of the cluster, and $\mathbf{P} \in \mathbb{R}^{3\times 2}$ is the matrix whose columns are the two leading eigenvectors of the covariance matrix.

The corresponding velocity vector v_i of each particle is projected onto the same 2D plane using the same principal basis. The projection is expressed as:

$$\tilde{v}_i = \mathbf{V}_{2D}^T v_i \tag{11}$$

where $\mathbf{P} \in \mathbb{R}^{3\times 2}$ is the matrix of principal components used to define the cluster's plane, and $\tilde{\mathbf{v}}_i$ is the projected velocity in the reduced 2D space. This yields the position and velocity in the 2D subspace that best captures the shape and motion within the clusters. In the projected plane the angular momentum of each particle relative to the clusters COM is computed using the 2D cross product. In the projected plane, the angular momentum of each particle relative to the cluster's center of mass is computed using the 2D cross product between the position and velocity vectors. Specifically,

$$\ell_i = \tilde{\mathbf{r}}_i \times \tilde{\mathbf{v}}_i = \tilde{r}_{i,x}\tilde{v}_{i,y} - \tilde{r}_{i,y}\tilde{v}_{i,x}, \tag{12}$$

where $\tilde{\mathbf{r}}_i$ and $\tilde{\mathbf{v}}_i$ are the projected position and velocity vectors of the i-th particle in the PCA-defined plane. The scalar quantity ℓ_i represents the out-of-plane component of angular momentum.

The total and average angular momentum of the cluster are computed as

$$L_{\text{total}} = \sum_{i=1}^{N} \ell_i, \quad \tilde{L} = \frac{1}{N}\sum_{i=1}^{N} \ell_i, \tag{13}$$

where N is the number of particles in the cluster and ℓ_i denotes the angular momentum of the i-th particle relative to the cluster's center of mass in the projected plane. A positive value of $\tilde{L}$ indicates a net counter clockwise rotation within the cluster's intrinsic plane, whereas a negative value corresponds to a net clockwise rotation.

References

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