Propositions and operations

Info

The course is split into 4 blocks, each ending with a *workshop* in which you and your group spend 6 hours working on a massive sheet about the contents of the block. During the exam you will present one of the four workshops, so be sure to write things down!

Propositions are declarative sentences that can be true or false. These are typically denoted with small letters, usually p, q, and r.

Operations

Negation is the "not" operator, using the symbol $\neg$. $\neg p$ is "not p", and this flips the truth value of the proposition p.
Conjunction is the "and" operator, which connects two propositions with the symbol $\wedge$. The conjunction of p and q is subsequently $p \wedge q$. $p \wedge q$ is true if, and only if, both p and q have a truth value of T, ie $v(p) = T \wedge v(q) = T$.
Disjunction is the "or" operator, connecting two propositions with the symbol $\vee$. The conjunction of p and q is $p \vee q$, and is true if p or q have a truth value of T, ie $v(p) = T \vee v(q) = T$.
Implication is the "if ... then" operator, connecting two propositions with the symbol $\rightarrow$. The implication $p \rightarrow q$ is true if p "implicates" q. This is best understood by the truth tables.
Bi-implication is the "if and only if" operator, connecting two propositions with the symbol $\leftrightarrow$. The bi-implication $p \leftrightarrow q$ is true if the truth values of p and q are identical, ie $v(p) = v(q)$.

Truth tables and bit operations

Truth tables

A truth table can be used to show the relations between different propositions.

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F

p	q	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
F	T	F	T	T	F
F	F	F	F	T	T

Bit operations

It's common to associate the bit value 1 with T and 0 with F. These operations are extended to bit strings of the same length by applying them to each component separately:

$$01100 \oplus 00111 = 01011$$

Logical equivalence, tautologies and contradictions

Tautology and contradiction

A compound proposition that is always true is called a **tautology**.
A compound proposition that is always false is called a **contradiction**.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical equivalence

If two compound propositions p and q have the same truth values for all possible cases, we call these propositions equivalent. This is denoted by $p \equiv q$. For example:

$$(p \to q) \wedge (q \to p) \equiv (p \leftrightarrow q)$$

There exist many useful, established logical equivalences which can be proven.

De Morgan's laws

$$\neg(q \vee p) \equiv \neg p \wedge \neg q$$

$$\neg(q \wedge p) \equiv \neg p \vee \neg q$$

Conditional disjunction equivalence

$$(p \to q) \equiv (\neg p \vee q)$$

Distributive laws

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$
$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Quantors and propositional functions

Propositional functions

A propositional function is an expression, containing one of more variables, which becomes a proposition when each of the variables is replaced by some one of its values from a discourse domain (universe) of individuals.

Quantors

Universal quantor

The universal quantification of a propositional function $P(x)$ is the proposition

> $P(x)$ is true for all values of x in the domain.

We denote this by $\forall x P(x)$. This proposition is true if and only if $P(x)$ is true for all possible values of x. It is false as soon as in the domain one counterexample x exists for which $P(x)$ is false.

Existential quantor

The existential quantification of a propositional function $P(x)$ is the proposition

> There exists an element in x in the domain such that $P(x)$ is true.

We denote this by $\exists x P(x)$. This proposition is true if and only if $P(x)$ is true for at least one possible value of x. It is false if $P(x)$ is false for all x in the domain. $\exists! x P(x)$ means there exists *exactly* one x where $P(x)$ is true.

Sets

Sets

A set is an unordered collection of distinct objects, called elements or members. We write $e \in S$ if the element e is in the set S. A set is usually denoted by listing its elements:

$$B = \{0, 1\}$$

The empty set is the set that contains no elements:

$$\emptyset = \{\}$$

A set can contain other sets:

$$S = \{\{\}, \{0\}, \{1\}, \{0, 1\}\}$$

A set containing a single element is often referred to as a singleton. For example, $\{a\}$ is a singleton of a.

Important sets

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Set builder notation

Not all sets can be described by listing the elements. The set builder notation can be used to describe which elements belong to the set in this case. Let D be a domain of discourse and P be a propositional function, then

$$S = \{e \in D | Q(e)\}$$

is the set of all elements in D for which $Q(e)$ is true.

Subsets, cardinality of sets, and set operations

Subsets

The set A is a subset of the set B if and only if every element of A is also an element of B.

$$\forall x \in A, x \in B$$

$$\forall x (x \in A \to x \in B)$$

For any set S it holds that $\emptyset \subseteq S$ and $S \subseteq S$.

Proper subsets

A is a proper subset of B if $A \subseteq B$, but $A \neq B$. Here we use the notation $A \subset B$.

Cardinality of sets

A set S is finite if it contains a finite number n of elements. The size $|S|$ (cardinality) of a finite set S is then defined to be the number n of elements it contains. For example:

$$S = \{1, 2, 3\}$$

$$|S| = 3$$

Set operations

Union

The union of two sets A and B is the set containing all elements that are in A or B, denoted by $A \cup B$. In set builder notation (*SBN*):

$$A \cup B = \{x | x \in A \vee x \in B\}$$

Intersection

The intersection of two sets A and B is the set containing all elements that are in A and B, denoted by $A \cap B$. In SBN:

$$A \cap B = \{x | x \in A \wedge x \in B\}$$

Cardinality of intersection and union

If A and B are finite sets, then the following equation holds:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Difference

The difference of two sets A and B is the set containing all elements that are in A but not in B, denoted by $A - B$ or $A \setminus B$. In SBN:

$$A \setminus B = \{x | x \in A \wedge x \notin B\}$$

Complement

Let U be the universal set. The complement of a set A is the set containing all elements in U that are not in A ($U \setminus A$), denoted by $\overline{A}$. In SBN:

$$\overline{A} = \{x \in U | x \notin A\}$$

It always holds true that $\overline{\overline{S}} = S$, ie the complement of the complement of a set is merely the set.

Set identities

Set identities are analogous to predicate logic, so sets follow De Morgan's laws, the conditional disjunction equivalence, the distributive laws and all other laws of logic.

Cartesian products and ordered pairs

Ordered pairs

An ordered pair (a, b) is a set

$$\{\{a\}, \{a, b\}\}.$$

This structure ensures that $(a, b) \neq (b, a)$, since

$$\{\{a\}, \{a, b\}\} \neq \{\{b\}, \{b, a\}\}.$$

This holds true despite sets being unordered, since only one singleton, $\{a\}$ or $\{b\}$, is in the set. So while the subsets are the same ($\{a, b\} = \{b, a\}$), the standalone elements are unique. Effectively, this means the pair is *ordered*, since the order of appearance in (a, b) changes the meaning.

Cartesian products

The cartesian product $A \times B$ is the set

$$\{(a, b) | a \in A \land b \in B\}.$$

In other words, it's the set containing all possible ordered pairs constructed from the elements of two sets, where the element of the first set must come first in the ordered pair. Example:

$$(A = \{0, 1, 2\}) \land (B = \{0, 1\}) \iff A \times B = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}$$

Note that $A \neq B \implies A \times B \neq B \times A$:

$$\begin{aligned} A \times B \neq B \times A \iff & \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\} \neq \\ & \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\} \end{aligned}$$

This is once again due to the order of appearance being relevant in an ordered pair.

Cardinality of cartesian products

The following statement about cardinalities of cartesian products holds true:

$$|A| = n \land |B| = m \implies |A \times B| = nm$$

In other words:

> If the cardinality of a set A is n and the cardinality of a set B is m, then the cardinality of the cartesian product between A and B (in either order) must be nm.

Cartesian products of >2 sets

An ordered pair is also called a tuple of two elements. A tuple can contain more elements, however. A tuple is simply an ordered collection of things. Cartesian products between more than 2 sets produce tuples with more than 2 elements:

$$A \times B \times C = \{(a, b, c) | a \in A \land b \in B \land c \in C\}$$

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_1 \in A_1 \land a_2 \in A_2 \land \ldots \land a_n \in A_n\}$$

Properties of tuples

> ⚠ **Not from course: This section contains information gathered from sources other than what the DTG course provides**

The general rule for the identity of two n-tuples is

$$(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n) \iff a_1 = b_1 \land a_2 = b_2 \land \ldots \land a_n = b_n.$$

This means that

1. A tuple can contain duplicates, ie $(1, 2, 2, 3) \neq (1, 2, 3)$; conversely, a set $\{1, 2, 2, 3\} = \{1, 2, 3\}$.
2. Tuple elements are ordered, ie $(1, 2, 3) \neq (3, 2, 1)$; conversely, a set $\{1, 2, 3\} = \{3, 2, 1\}$.
3. A tuple has a finite number of elements; conversely, a set may have an infinite number of elements, for example the set of all natural numbers.

Construction of n-tuples as ordered pairs

A tuple of 3 elements can be constructed as ordered pairs and therefore notated as a set:

$$(a, b, c) := ((a, b), c) = (\{\{a\}, \{a, b\}\}, c) = \{\{\{\{a\}, \{a, b\}\}\}, \{\{\{a\}, \{a, b\}\}, c\}\}$$

This process can be applied to a tuple of any n elements, for example a tuple of 6 elements has the recursive definition

$$(a, b, c, d, e, f) := (((((a, b), c), d), e), f).$$

Note that the "reverse" can also be used to represent an n-tuple as an ordered pair:

$$(a, b, c, d, e, f) := (a, (b, (c, (d, (e, f)))))$$

Cartesian square

$A \times A$ can be written as A^2. For example, let $A = \{a, b\}$, then

$$A^2 = \{(a, a), (a, b), (b, a), (b, b)\}$$

Cartesian products involving the empty set

When the empty set is involved in a cartesian product, we get the empty set:

$$\begin{gathered} \text{Example: } A = \{a, b, c\} \\ A \times \emptyset = \emptyset \\ \emptyset \times A = \emptyset \end{gathered}$$