Workshop 1

Workshop 1 - Social Networks

A group of people want to create a new social media app and are considering how it should be designed. For now the set of users is given by

$$U = \{\text{Abe, Bob, Cam, Dee, Eve, Fey, Gil}\}.$$

As a starting point it is set up such that you can follow other users (similar to e.g. Instagram or Twitter). This introduces a relation S on the set of users given by $(a, b) \in S$ if person a follows person b. Figure 1 shows which users are following whom at this point (as a starting point, everyone is following themselves, since you can see your own posts).

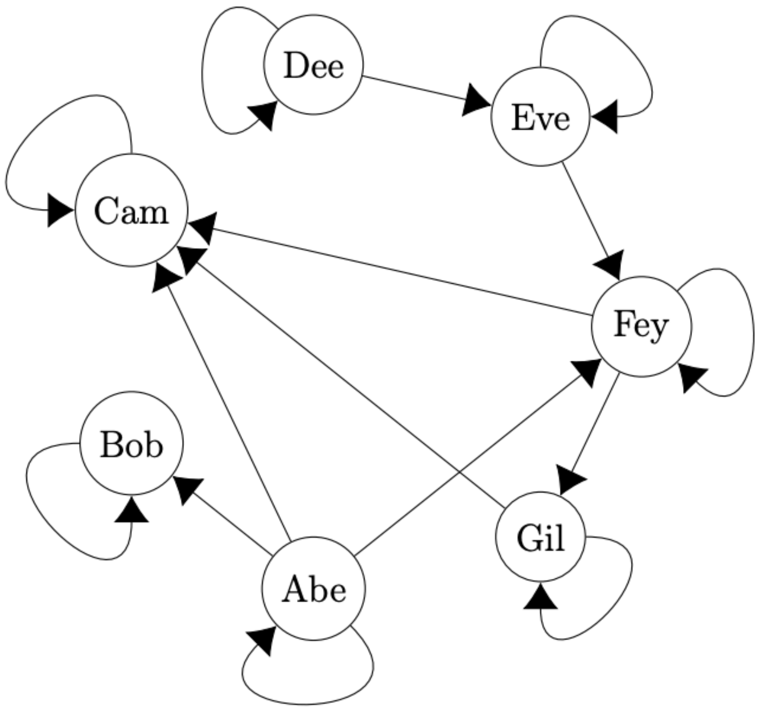


Figure 1: An arrow from one person to another means the first is following the second. E.g. Dee follows Eve.

Exercise 1

1. Write down the relation S given in figure 1 as a set.

$$\begin{aligned} S = \{&(Dee, Eve), (Eve, Fey), (Fey, Gil), (Fey, Cam), \\ &(Gil, Cam), (Abe, Fey), (Abe, Bob), (Abe, Cam), \\ &(Dee, Dee), (Eve, Eve), (Fey, Fey), (Gil, Gil), \\ &(Abe, Abe), (Bob, Bob), (Cam, Cam)\}. \end{aligned}$$

2. What is the cardinality of the relation?

$$|S| = 15$$

Since there are 15 elements in S, each element being an ordered pair.

3. The relation can be described as a subset of a cartesian product. Which cartesian product does this refer to here and what is its cardinality?

The relation is between the universe U and itself, so it is a subset of the cartesian product $U \times U$ or U^2. The cardinality of a cartesian product is defined by

$$|A| = n \wedge |B| = m \implies |A \times B| = nm.$$

So here we can apply

$$|U| = 7 \Rightarrow |U^2| = 7^2 = 49$$

4. Decide whether the relation above is reflexive, symmetric, transitive, or antisymmetric. Explain why it has or does not have these properties.

   Consider what it means for a social network to have these properties.

Is it reflexive?

A homogenous relation R on a set A is **reflexive** if

$$\forall a \in A : a\,R\,a.$$

For all elements a in the set A, a is R-related to a. So a reflexive relation is one wherein there is an ordered pair (a, a) for every element a in A. Every element of A is R-related to itself.

Our relation S is reflexive, as every element $\forall a \in U$ is S-related to itself:

$$\forall a \in U : a\,S\,a.$$

Is it symmetric?

A relation R is **symmetric** if

$$\forall a, b \in A : a\,R\,b \iff b\,R\,a.$$

A relation is therefore symmetric whenever the existence of an ordered pair (a, b) in the relation implies the presence of the reverse, (b, a).

S is not symmetric, as we can see in Figure 1 there are actually no bidirectional arrows.

Is it antisymmetric?

A homogenous relation R on a set A is **antisymmetric** if

$$\forall a, b \in A : a\,R\,b \land b\,R\,a \implies a = b$$

In other words, a relation on a set is considered antisymmetric when there are no symmetric, distinct pairs in the relation.

S is antisymmetric since there are no symmetric, distinct pairs. There are only symmetric, nondistinct pairs, since everyone follows themselves.

Is it transitive?

A relation R on a set A is **transitive** if

$$\forall a, b, c, \in A : a\,R\,b \land b\,R\,c \implies a\,R\,c.$$

This is the same logic as a syllogism. If a is R-related to b, and b is R-related to c, then a is R-related to c.

S is not transitive; we can prove this by counterexample. Dee is S-related to Eve, and Eve is S-related to Fey, but Dee is not S-related to Fey. This violates the definition of transitivity and therefore S is not transitive.

Exercise 2

One topic for discussion for the social media app is the idea that if someone publishes a post, it is re-published by everyone who follows them (a form of automatic retweet). The makers of the app can not decide whether this should

a) only happen with immediate followers or

b) generate a chain reaction.

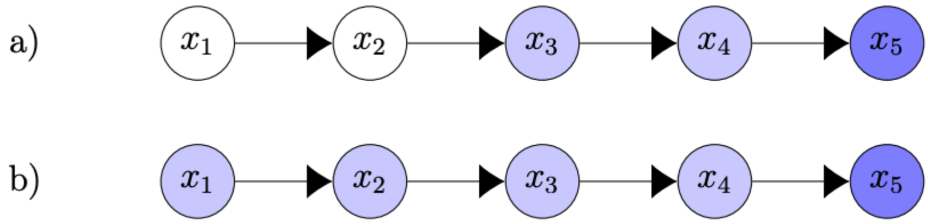


Figure 2: Representation of who sees posts by user x_5 in the different models

For option b) this means that if $x_1 S x_2, x_2 S x_3, \ldots, x_{n-1} S x_n$ then $x_1, x_2, \ldots, x_{n-1}$ will also see all posts by x_n and in effect also follow them. While in option a) posts by x_n are only visible to x_{n-1} and x_{n-2}.

1. Explain the two options in terms of relations. The second option is related to a closure operation, which one?

For option a), we can use the above text to infer that the ability to see ones post is analog to following them -- therefore, anytime x_n posts something, then x_{n-1} (who follows x_n) re-publishes it; then, x_{n-2} can see anytime x_n posts through the re-publishing of x_{n-1}.

$$x_{n-1}\ S\ ; x_n\ \wedge x_{n-2}\ S\ x_{n-1}\ \Rightarrow x_{n-2}\ S\ x_n.$$

We operate here under the assumption that "x follows y" is entirely equivalent to "y's posts are visible to x."

For option b), where we have a chain reaction, it becomes inductive in nature

$$x_{n-k}\ ; S\ ; x_{n-k+1}\ \wedge x_{n-k+1}\ S\ x_{n-k+2}\ \wedge \ldots \wedge x_{n-1}\ S\ x_n\ \Rightarrow x_{n-k}\ S\ x_n.$$

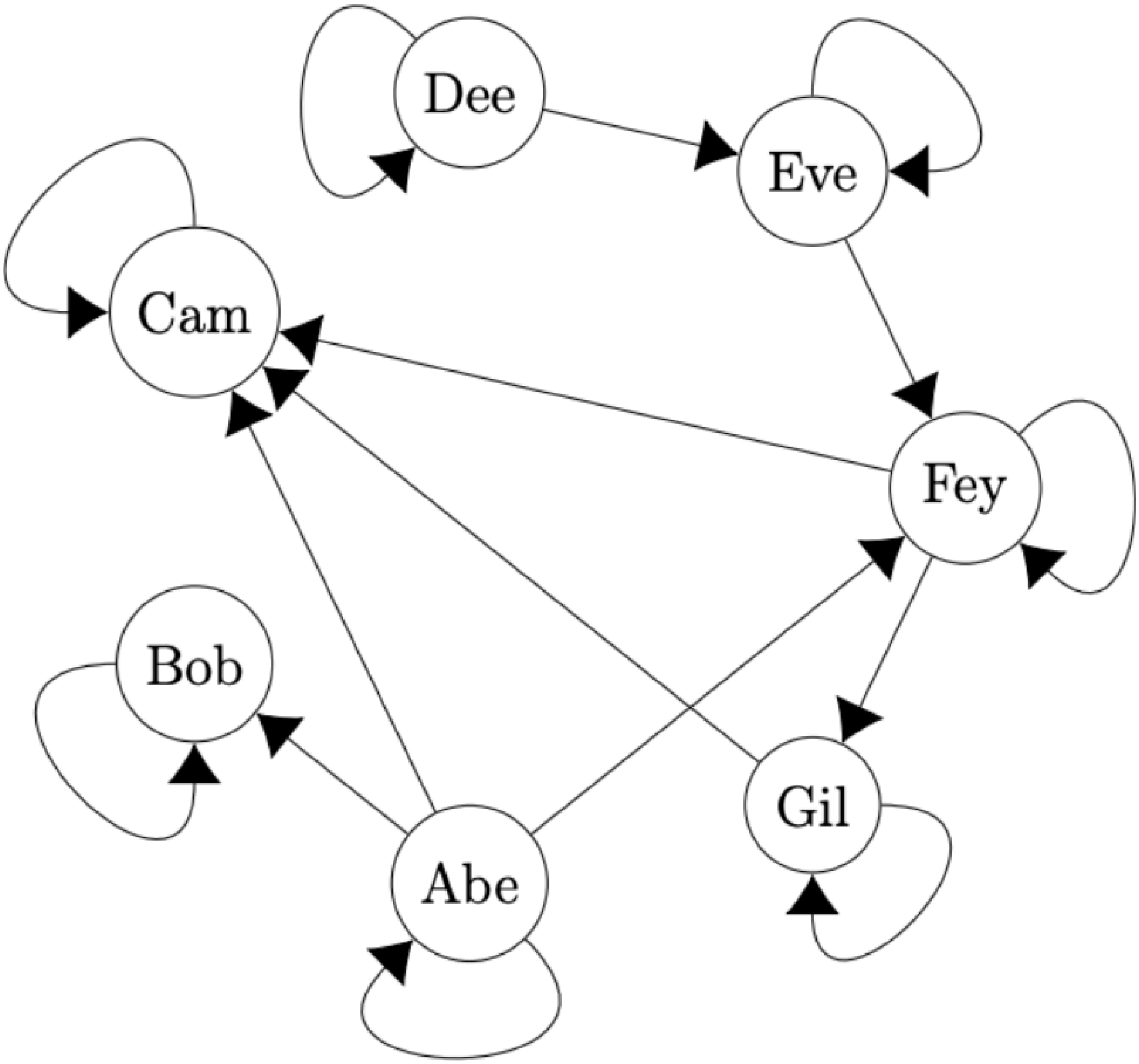
This option is related to the transitive closure S^* of our relation S, which is the smallest transitive relation containing S. The definition of the transitive closure is as follows:

$$S^* = \bigcup_{n=1}^{\infty} S^n$$

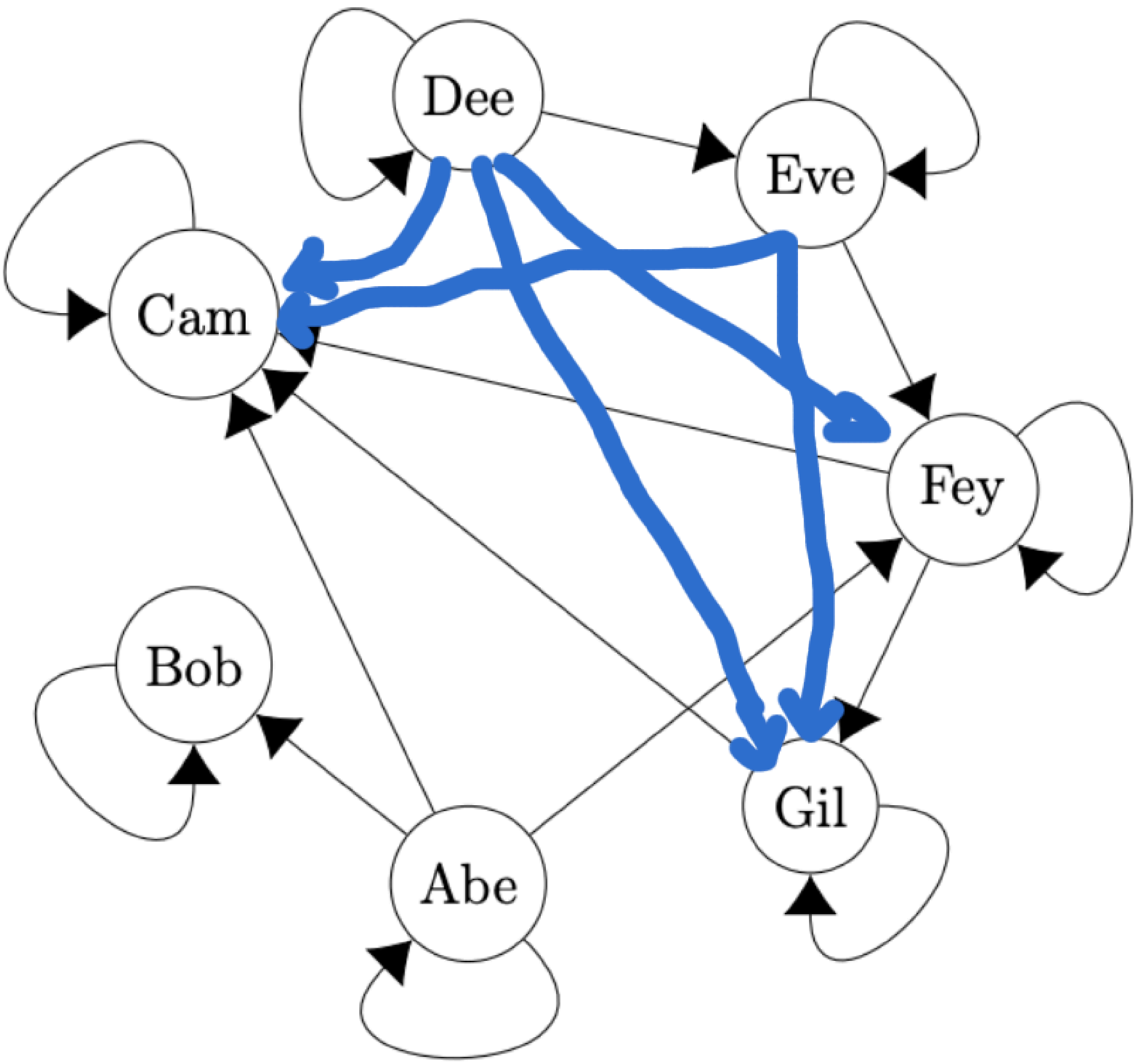
So it is the union of every power of the relation.

2. Find this closure for the relation in figure 1. We will call this new relation S^*.

So this is figure 1



In order to draw S^* we must draw arrows to from every node to every node that has a path to it



Beautiful.

Exercise 3

1. Consider the following sets

$$F_{\text{Cam}} = \{u \in U \mid (u, \text{Cam}) \in S^*\}$$
$$F_{\text{Fey}} = \{u \in U \mid (u, \text{Fey}) \in S^*\}.$$

What do these sets describe? Which elements are in the sets F_{Cam}, F_{Fey} and $F_{\text{Cam}} \cap F_{\text{Fey}}$.

The set F_{name} contains every person who can see name's posts (name's effective "followers"). Since we are using the relation S^*, it is of course where we go with option b) in exercise 2, where there is this chain reaction such that the whole follower chain can see your posts.

$$F_{Cam} = \{Cam, Abe, Gil, Fey, Eve, Dee\}$$
$$F_{Fey} = \{Fey, Abe, Eve, Dee\}$$
$$F_{Cam} \cap F_{Fey} = \{Fey, Abe, Eve, Dee\}$$

2. Let $T \subseteq U$ and consider $G_{\text{Cam}} = \{t \in T \mid (t, \text{Cam}) \in S^*\}$. Show that $G_{\text{Cam}} \subseteq F_{\text{Cam}}$.

$$T \subseteq U \Rightarrow \{t \in T | (t, Cam) \in S^*\} \subseteq \{u \in U | (u, Cam) \in S^*\}$$

So to prove this, let $x \in G_{Cam}$ be arbitrary. It then follows that $x \in T$ and $(x, Cam) \in S^*$. Since $T \subseteq U$ and $x \in T$ then it must follow that $x \in U$. By the definition of F_{Cam}

$$F_{Cam} = \{x \in U | (x, Cam) \in S^*\},$$

and the fact that x is arbitrary, we can conclude that $G_{Cam} \subseteq F_{Cam}$.

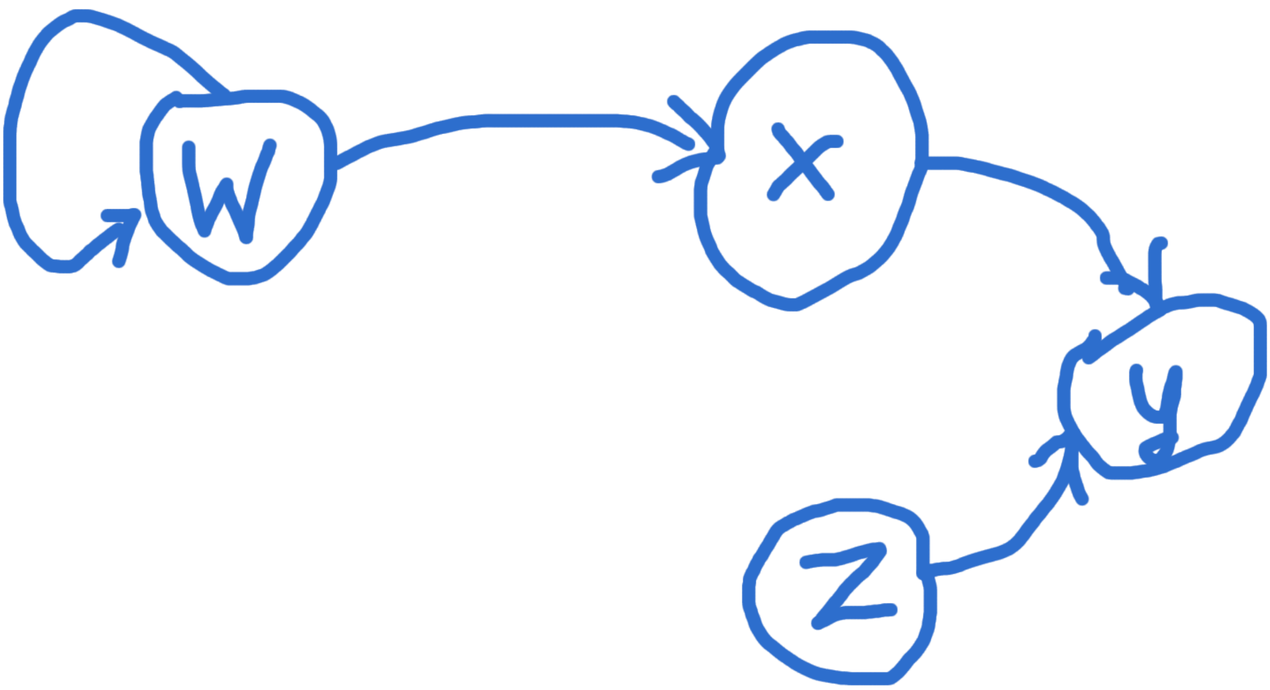
Exercise 4

1. Consider the relation R on $A = \{w, x, y, z\}$ given by

$$R = \{(w, w), (w, x), (x, y), (z, y)\}.$$

Draw the directed graph that represents this relation and write down its matrix representation.

This is the directed graph for R:



Its matrix representation is

$$M_R = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

2. Find the reflexive closure $\tilde{R}$ of R.

The reflexive closure of a relation R on a set A is defined by $R \cup \Delta$ where

$$\begin{aligned} \Delta &= \{(a, a) \mid a \in A\} \\ \tilde{R} &= R \cup \Delta \\ &= \{(w, w), (w, x), (x, y), (z, y)\} \\ &\cup \{(w, w), (x, x), (y, y), (z, z)\} \\ &= \{(w, w), (x, x), (y, y), (z, z), (w, x), (x, y), (z, y)\} \end{aligned}$$

We will now continue with constructing succesive closures of this relation, but consider two different orders.

3. a) Construct the symmetric closure S of the reflexive closure $\tilde{R}$ of R. You may use whichever representation of the relation you prefer.

So we have

$$\tilde{R} = \{(w, w), (x, x), (y, y), (z, z), (w, x), (x, y), (z, y)\}$$

To create the symmetric closure S of this, we must find $\tilde{R} \cup \tilde{R}^{-1}$. We define $\tilde{R}^{-1}$ by

$$\tilde{R}^{-1} = \{(b, a) \mid (a, b) \in \tilde{R}\},$$

so in our case it is

$$\tilde{R}^{-1} = \{(w, w), (x, x), (y, y), (z, z), (x, w), (y, x), (y, z)\}.$$

Then the union becomes

$$\begin{aligned} \tilde{R} \cup \tilde{R}^{-1} &= \{(w, w), (x, x), (y, y), (z, z), (w, x), (x, y), (z, y)\} \\ &\cup \{(w, w), (x, x), (y, y), (z, z), (x, w), (y, x)(y, z)\} \\ &= \{(w, w), (x, x), (y, y), (z, z), (w, x), (x, y), (z, y), (x, w), (y, x), (y, z)\} \end{aligned}$$

This is the symmetric closure S

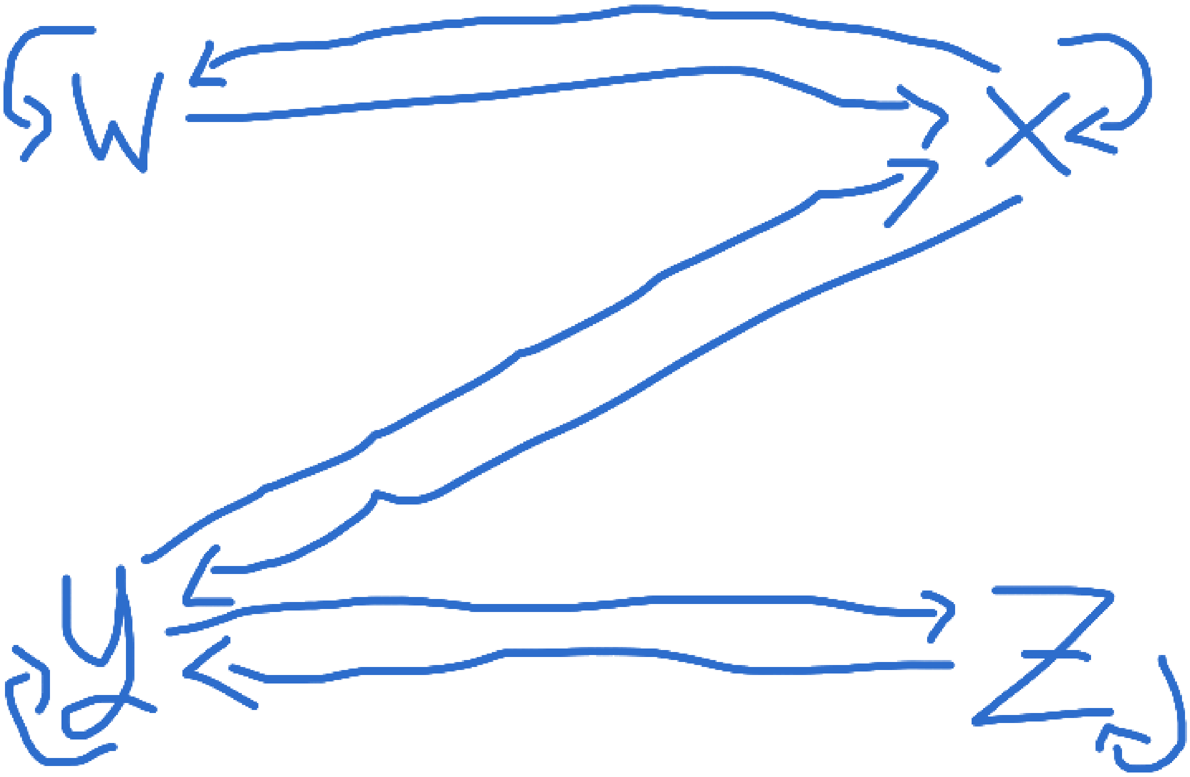
$$S = \tilde{R} \cup \tilde{R}^{-1} = \{(w, w), (x, x), (y, y), (z, z), (w, x), (x, y), (z, y), (x, w), (y, x), (y, z)\}.$$

b) Construct the transitive closure S^* of S.

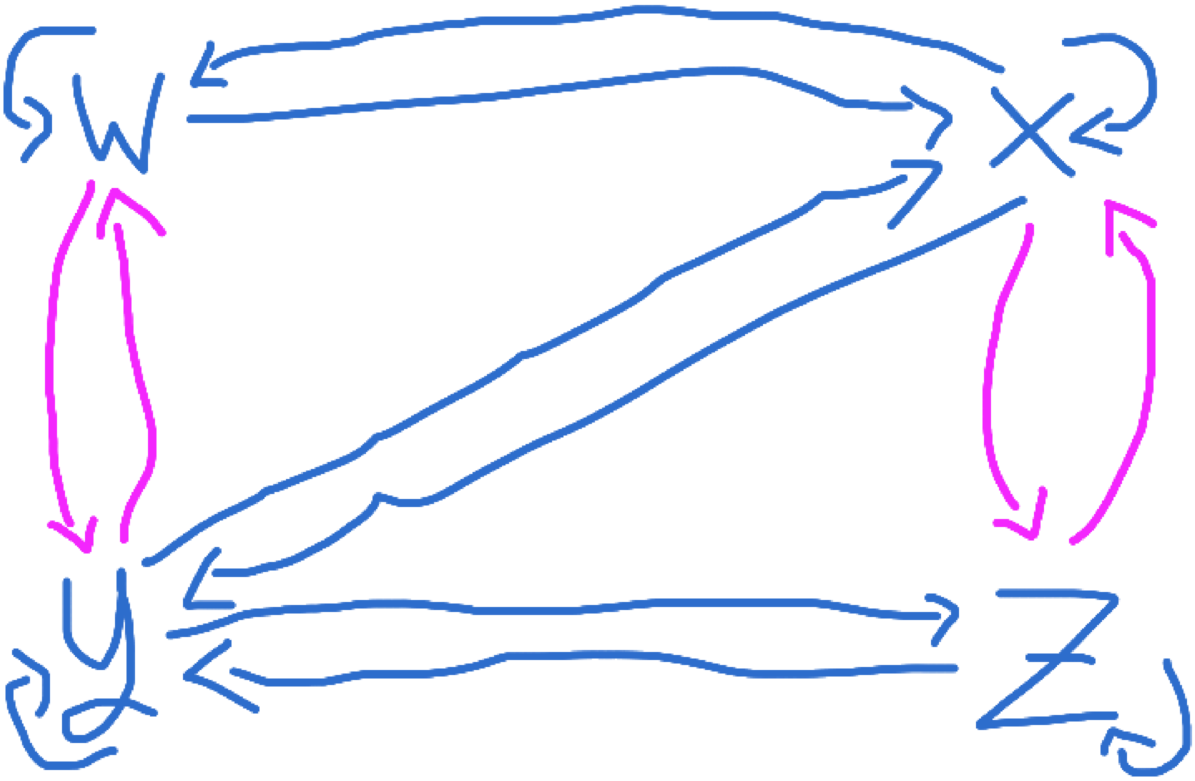
To construct the transitive closure we need to find

$$S^* = \bigcup_{n=1}^{\infty} S^n$$

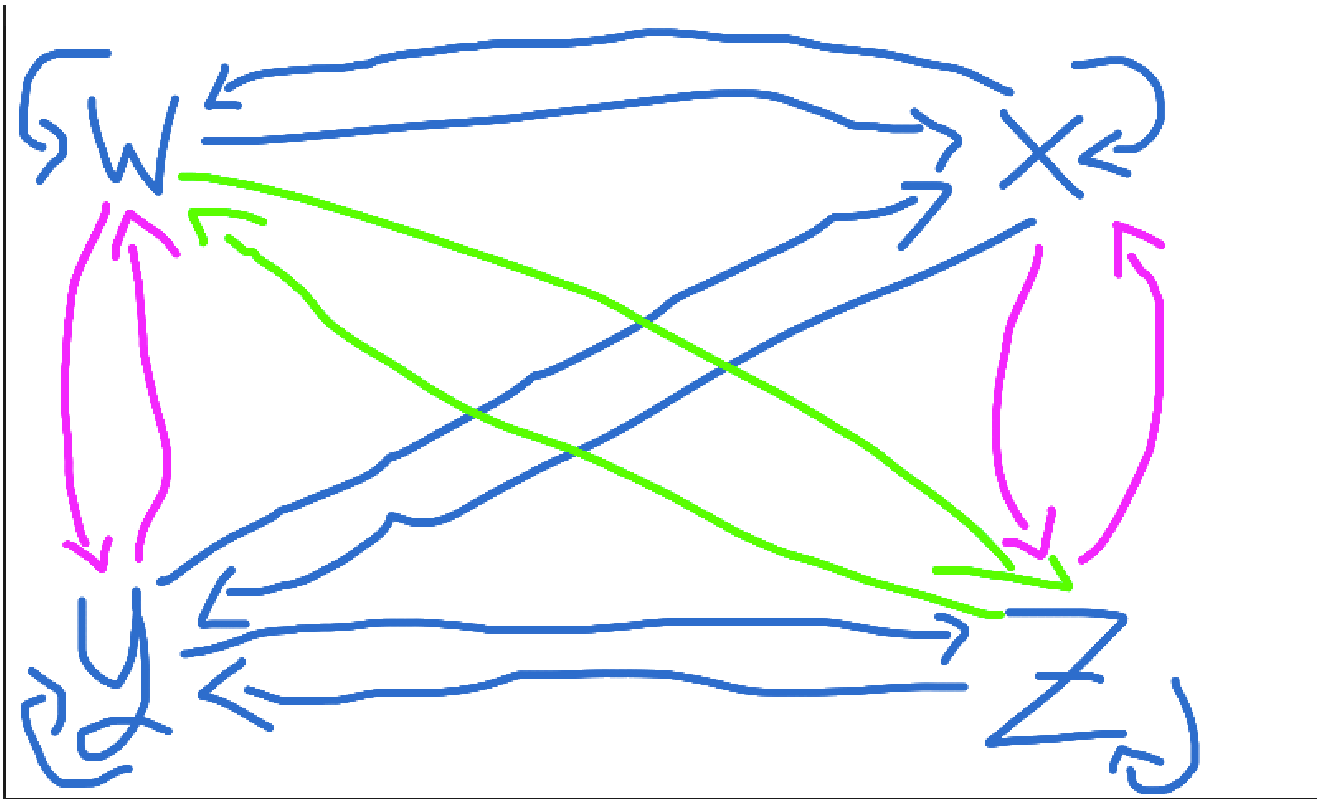
We'll stop at the point where $S^n = S^{n-1}$. It is wayyyy more simple to do this with a graph than to do this with just sets in text, so first we'll draw S^1:



Then we must draw S^2, which essentially means for every instance of an arrow from a to b and another arrow from b to c, we must ensure there is an arrow from a to c:



Then we must draw S^3, where we can transitivity-ify a pink arrow with a blue arrow:



Now, let's attempt to draw S^4, where we can transitivity-ify a pink/blue arrow with a green arrow... we cannot add more arrows that aren't already there. So in conclusion,

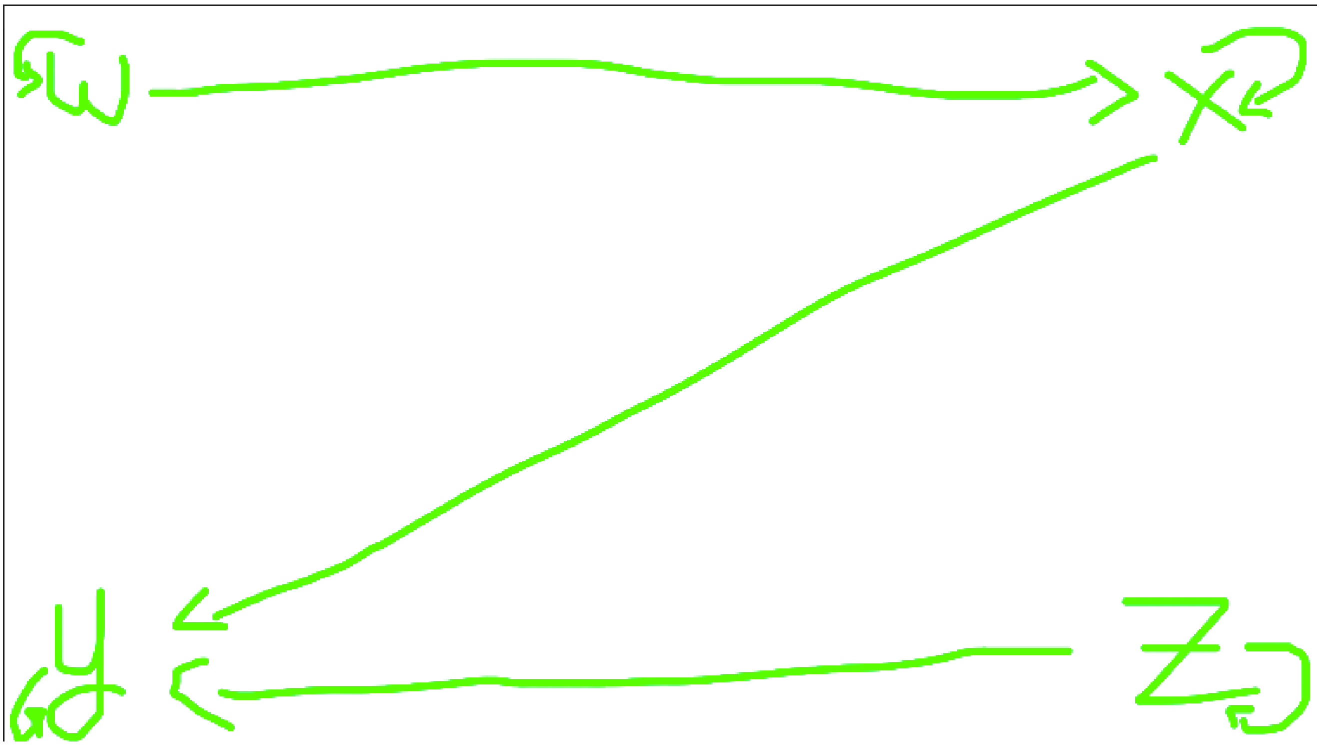
$$\begin{aligned} S^* = \{&(w,w),(y,y),(x,x),(z,z),(w,x),\\ &(x,w),(w,y),(y,w),(w,z),(z,w),\\ &(x,y),(y,x),(x,z),(z,x),(y,z),(z,y)\}. \end{aligned}$$

4. a) Now begin by constructing the transitive closure T of the reflexive closure $\tilde{R}$. You may use whichever representation of the relation you prefer.

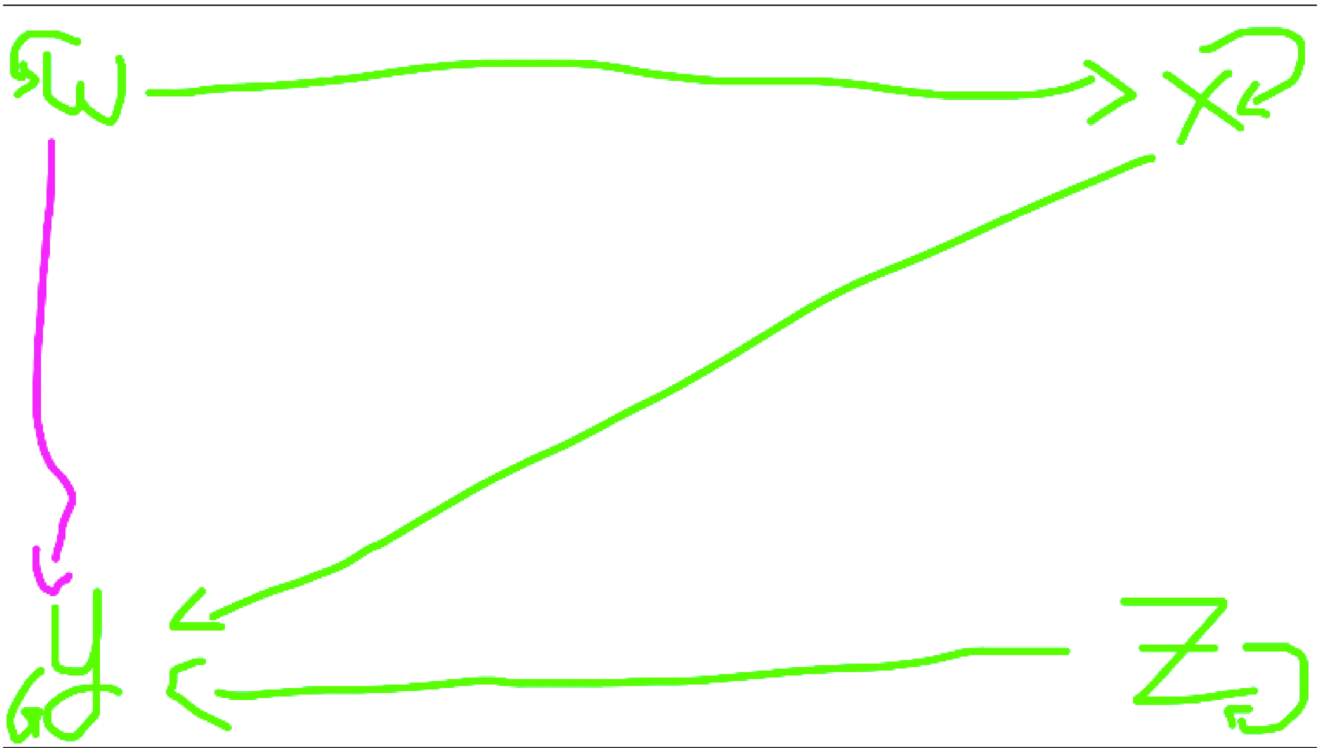
Again,

$$\tilde{R} = \{(w,w),(x,x),(y,y),(z,z),(w,x),(x,y),(z,y)\}.$$

It is once again much easier to construct T with a graph. We start by drawing $\tilde{R}$:



Then we draw $\tilde{R}^2$:



Now, observe that if we were to attempt a $\tilde{R}^3$ or $\tilde{R}^2 \circ \tilde{R}$, there will be no arrows added. The pink arrow that was added previously points to y, which does not itself point to anything else. Therefore we can conclude that the transitive closure T of the reflexive closure $\tilde{R}$ is

$$T = \{(w,w),(x,x),(y,y),(z,z),(w,x),(x,y),(z,y),(w,y)\}.$$

b) Next construct the symmetric closure T' of T.

To construct the symmetric closure T' of T, we must find $T \cup T^{-1}$. We define T^{-1} by

$$T^{-1} = \{(b,a) \mid (a,b) \in T\},$$

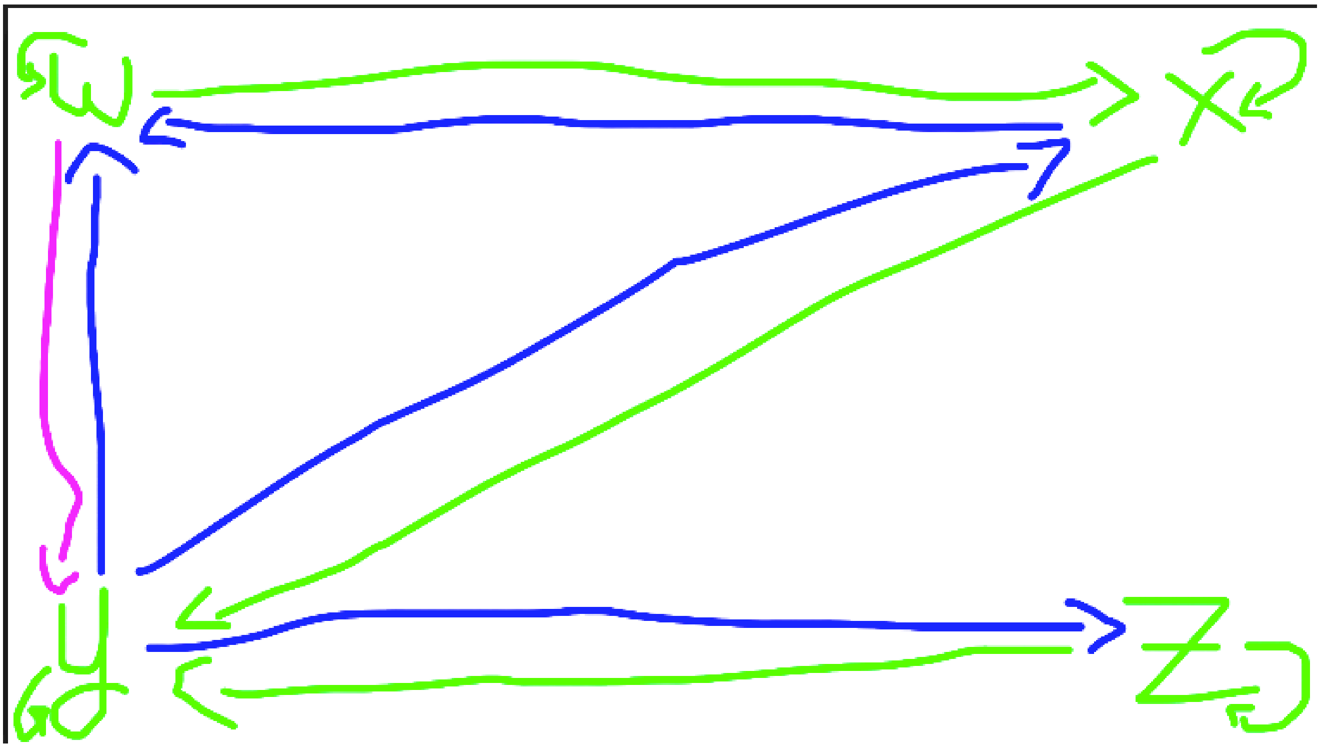
which then must be

$$T^{-1} = \{(w,w),(x,x),(y,y),(z,z),(x,w),(y,x),(y,z),(y,w)\}.$$

The symmetric closure must then be

$$\begin{aligned} T' =& T \cup T^{-1} \\ =& \{(w,w),(x,x),(y,y),(z,z),(w,x),(x,y),(z,y),(w,y)\} \\ \cup& \{(w,w),(x,x),(y,y),(z,z),(x,w),(y,x),(y,z),(y,w)\} \\ =& \{(w,w),(x,x),(y,y),(z,z),(w,x),(x,y),(z,y),(w,y),(x,w),(y,x),(y,z),(y,w)\}. \end{aligned}$$

To help with the next question, let's draw T':



5. Compare the results from the two previous tasks. Which of the two final relations S^* and T' is an equivalence relation?

An equivalence relation is a relation which is reflexive, symmetrical and transitive. Both S^* and T' are reflexive. They are also both symmetrical since any arrow has an arrow going the opposite way (ie $(a,b) \iff (b,a)$). However only S^* is transitive since it satisfies the condition $\forall a,b,c \in A : (a,b) \in R \wedge (b,c) \in R \Rightarrow (a,c) \in R$. Meanwhile, we can simply prove by counterexample that T' is *not* transitive since we have (x,y) and (y,z) but no (x,z).