

Workshop 1

Workshop 1 - Social Networks

A group of people want to create a new social media app and are considering how it should be designed. For now the set of users is given by

$$U = \{\text{Abe, Bob, Cam, Dee, Eve, Fey, Gil}\}.$$

As a starting point it is set up such that you can follow other users (similar to e.g. Instagram or Twitter). This introduces a relation S on the set of users given by $(a, b) \in S$ if person a follows person b . Figure 1 shows which users are following whom at this point (as a starting point, everyone is following themselves, since you can see your own posts).

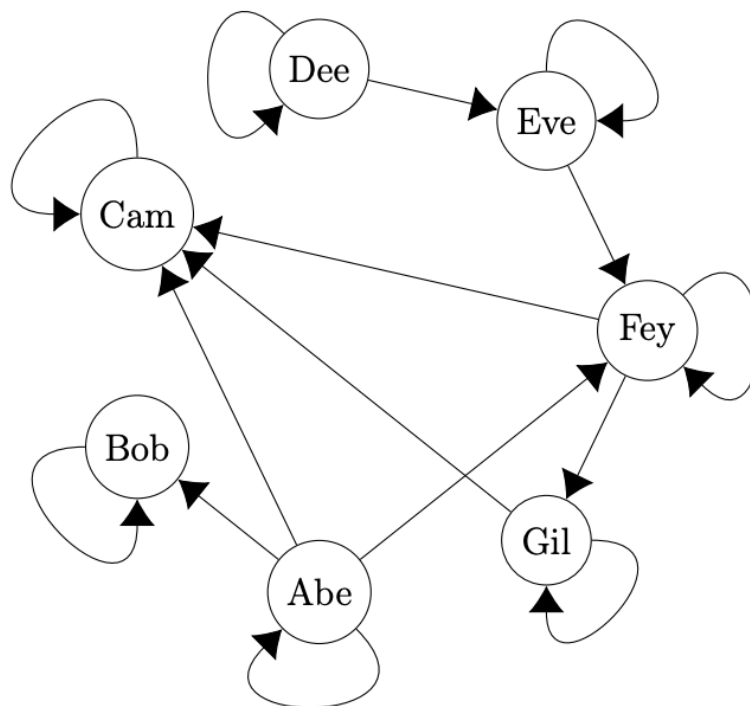


Figure 1: An arrow from one person to another means the first is following the second. E.g. Dee follows Eve.

Exercise 1

1. Write down the relation S given in figure 1 as a set.

$$S = \{(Dee, Eve), (Eve, Fey), (Fey, Gil), (Fey, Cam), (Gil, Cam), (Abe, Fey), (Abe, Bob), (Abe, Cam), (Dee, Dee), (Eve, Eve), (Fey, Fey), (Gil, Gil), (Abe, Abe), (Bob, Bob), (Cam, Cam)\}.$$

2. What is the cardinality of the relation?

$$|S| = 15$$

Since there are 15 elements in S , each element being an ordered pair.

3. The relation can be described as a subset of a cartesian product. Which cartesian product does this refer to here and what is its cardinality?

The relation is between the universe U and itself, so it is a subset of the cartesian product $U \times U$ or U^2 . The cardinality of a cartesian product is defined by

$$|A| = n \wedge |B| = m \implies |A \times B| = nm.$$

So here we can apply

$$|U| = 7 \Rightarrow |U^2| = 7^2 = 49$$

4. Decide whether the relation above is reflexive, symmetric, transitive, or antisymmetric. Explain why it has or does not have these properties.

Consider what it means for a social network to have these properties.

Is it reflexive?

A homogenous relation R on a set A is **reflexive** if

$$\forall a \in A : a R a.$$

For all elements a in the set A , a is R -related to a . So a reflexive relation is one wherein there is an ordered pair (a, a) for every element a in A . Every element of A is R -related to itself.

Our relation S is reflexive, as every element $\forall a \in U$ is S -related to itself:

$$\forall a \in U : a S a.$$

Is it symmetric?

A relation R is **symmetric** if

$$\forall a, b \in A : a R b \iff b R a.$$

A relation is therefore symmetric whenever the existence of an ordered pair (a, b) in the relation implies the presence of the reverse, (b, a) .

S is not symmetric, as we can see in Figure 1 there are actually no bidirectional arrows.

Is it antisymmetric?

A homogenous relation R on a set A is **antisymmetric** if

$$\forall a, b \in A : a R b \wedge b R a \implies a = b$$

In other words, a relation on a set is considered antisymmetric when there are no symmetric, distinct pairs in the relation.

S is antisymmetric since there are no symmetric, distinct pairs. There are only symmetric, nondistinct pairs, since everyone follows themselves.

Is it transitive?

A relation R on a set A is **transitive** if

$$\forall a, b, c \in A : a R b \wedge b R c \implies a R c.$$

This is the same logic as a syllogism. If a is R -related to b , and b is R -related to c , then a is R -related to c .

S is not transitive; we can prove this by counterexample. Dee is S -related to Eve, and Eve is S -related to Fey, but Dee is not S -related to Fey. This violates the definition of transitivity and therefore S is not transitive.

Exercise 2

One topic for discussion for the social media app is the idea that if someone publishes a post, it is re-published by everyone who follows them (a form of automatic retweet). The makers of the app can not decide whether this should

- a) only happen with immediate followers or
- b) generate a chain reaction.

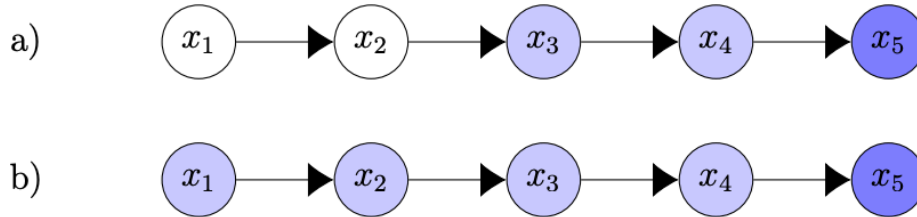


Figure 2: Representation of who sees posts by user x_5 in the different models

For option b) this means that if $x_1 S x_2, x_2 S x_3, \dots, x_{n-1} S x_n$ then x_1, x_2, \dots, x_{n-1} will also see all posts by x_n and in effect also follow them. While in option a) posts by x_n are only visible to x_{n-1} and x_{n-2} .

1. Explain the two options in terms of relations. The second option is related to a closure operation, which one?

For option a), we can use the above text to infer that the ability to see ones post is analog to following them -- therefore, anytime x_n posts something, then x_{n-1} (who follows x_n) re-publishes it; then, x_{n-2} can see anytime x_n posts through the re-publishing of x_{n-1} .

$$x_{n-1} S x_n \wedge x_{n-2} S x_{n-1} \Rightarrow x_{n-2} S x_n.$$

We operate here under the assumption that "x follows y" is entirely equivalent to "y's posts are visible to x."

For option b), where we have a chain reaction, it becomes inductive in nature

$$x_{n-k} S x_{n-k+1} \wedge x_{n-k+1} S x_{n-k+2} \wedge \dots \wedge x_{n-1} S x_n \Rightarrow x_{n-k} S x_n.$$

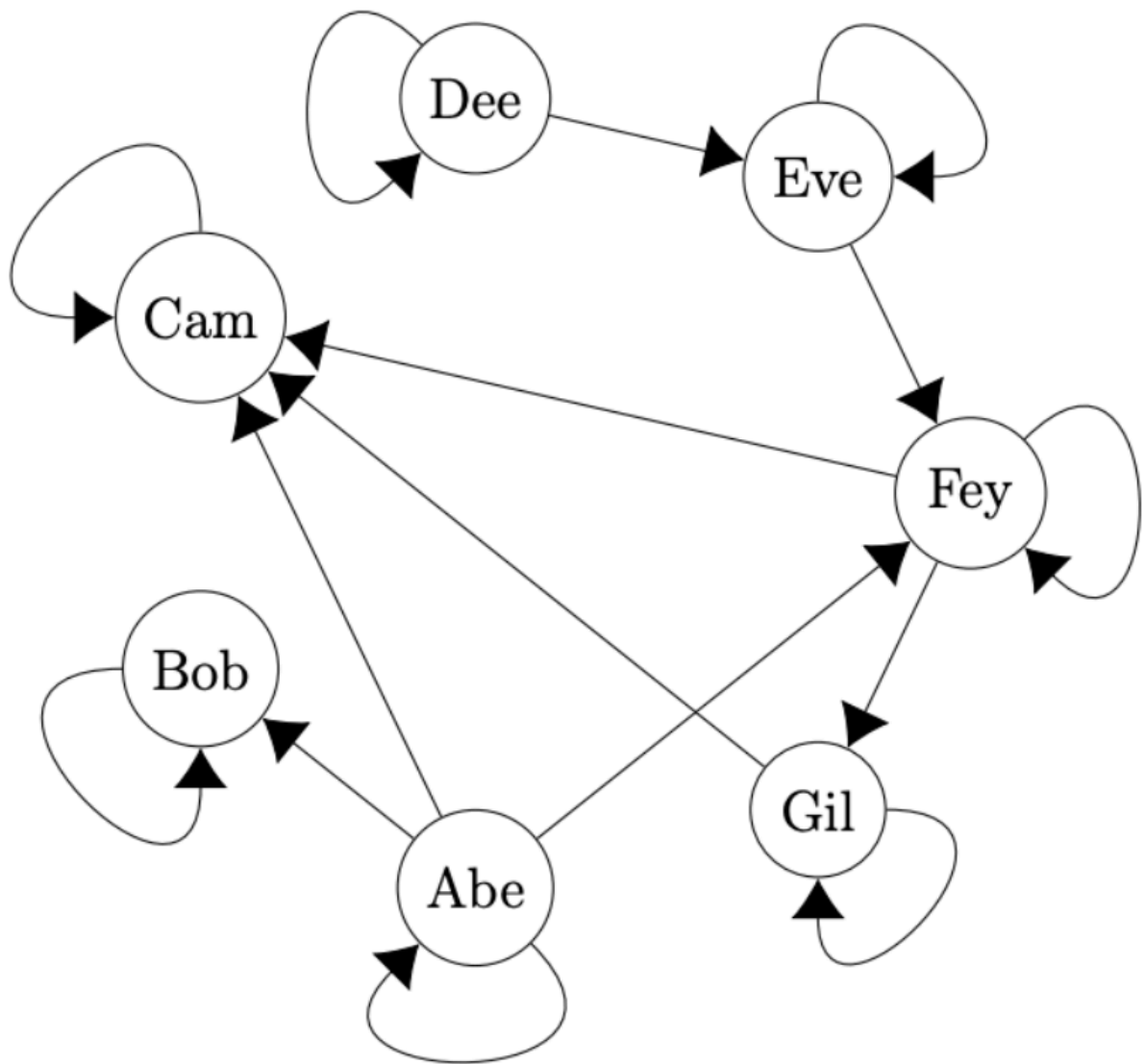
This option is related to the transitive closure S^* of our relation S , which is the smallest transitive relation containing S . The definition of the transitive closure is as follows:

$$S^* = \bigcup_{n=1}^{\infty} S^n$$

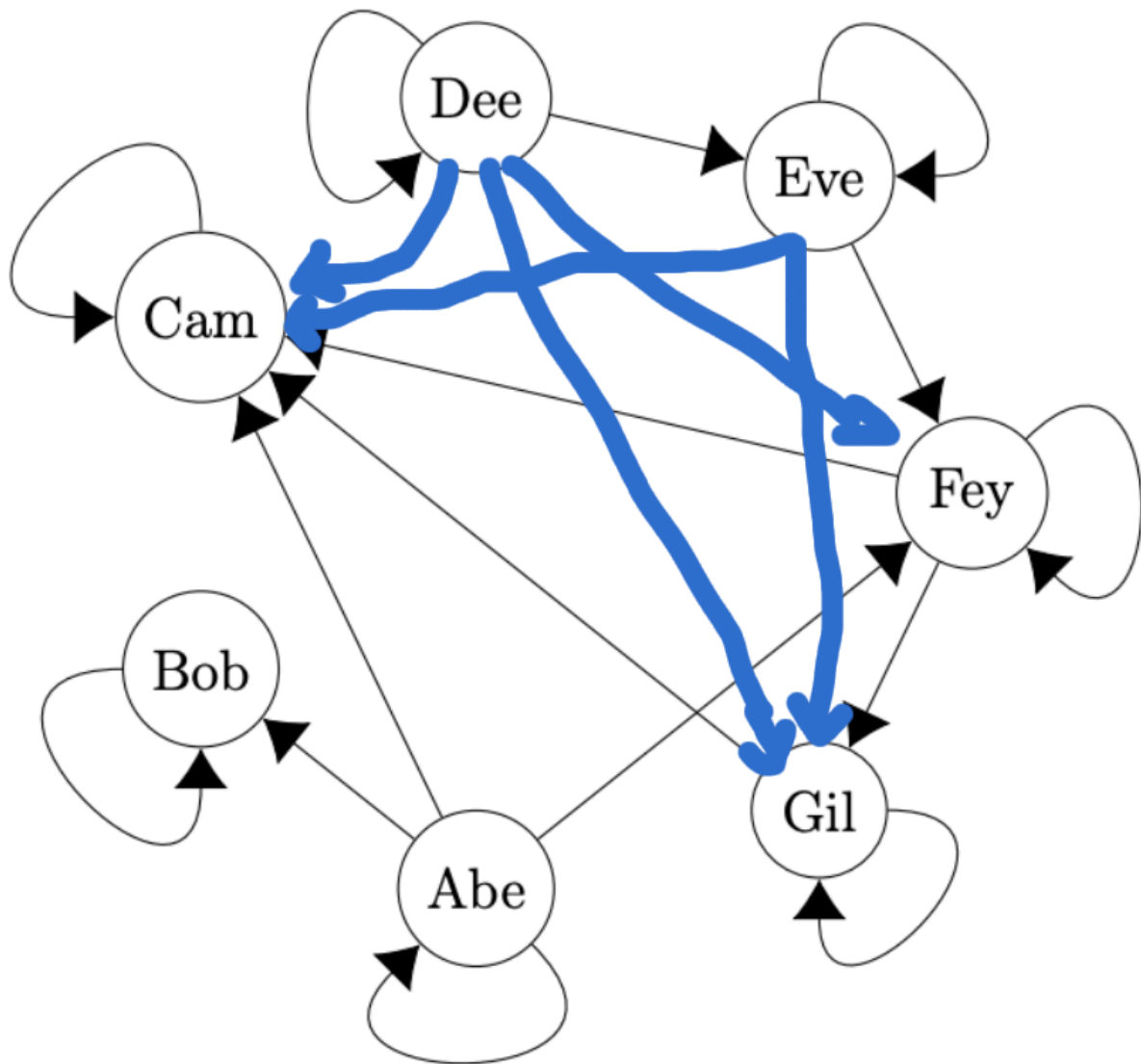
So it is the union of every power of the relation.

2. Find this closure for the relation in figure 1. We will call this new relation S^* .

So this is figure 1



In order to draw S^* we must draw arrows to from every node to every node that has a path to it



Beautiful.

Exercise 3

1. Consider the following sets

$$F_{\text{Cam}} = \{u \in U \mid (u, \text{Cam}) \in S^*\}$$

$$F_{\text{Fey}} = \{u \in U \mid (u, \text{Fey}) \in S^*\}.$$

What do these sets describe? Which elements are in the sets F_{Cam} , F_{Fey} and $F_{\text{Cam}} \cap F_{\text{Fey}}$.

The set F_{name} contains every person who can see name's posts (name's effective "followers"). Since we are using the relation S^* , it is of course where we go with option b) in exercise 2, where there is this chain reaction such that the whole follower chain can see your posts.

$$\begin{aligned}
 F_{Cam} &= \{Cam, Abe, Gil, Fey, Eve, Dee\} \\
 F_{Fey} &= \{Fey, Abe, Eve, Dee\} \\
 F_{Cam} \cap F_{Fey} &= \{Fey, Abe, Eve, Dee\}
 \end{aligned}$$

2. Let $T \subseteq U$ and consider $G_{Cam} = \{t \in T \mid (t, Cam) \in S^*\}$. Show that $G_{Cam} \subseteq F_{Cam}$.

$$T \subseteq U \Rightarrow \{t \in T \mid (t, Cam) \in S^*\} \subseteq \{u \in U \mid (u, Cam) \in S^*\}$$

So to prove this, let $x \in G_{Cam}$ be arbitrary. It then follows that $x \in T$ and $(x, Cam) \in S^*$. Since $T \subseteq U$ and $x \in T$ then it must follow that $x \in U$. By the definition of F_{Cam}

$$F_{Cam} = \{x \in U \mid (x, Cam) \in S^*\},$$

and the fact that x is arbitrary, we can conclude that $G_{Cam} \subseteq F_{Cam}$.

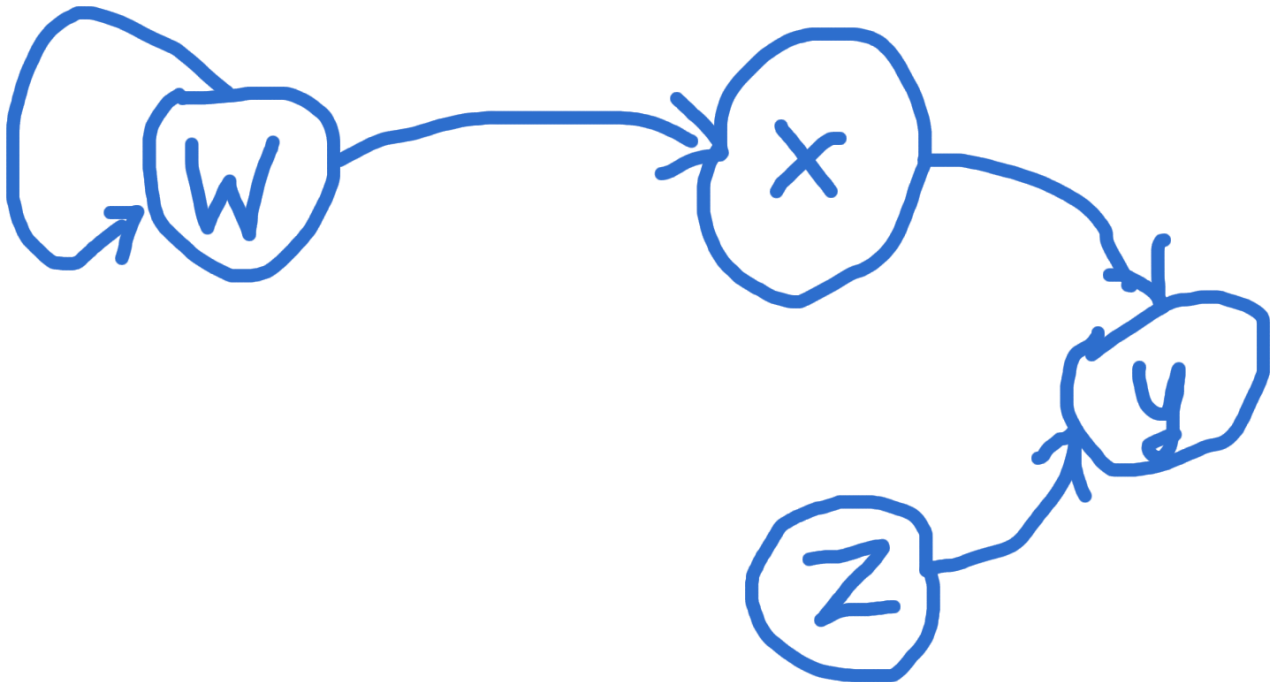
Exercise 4

1. Consider the relation R on $A = \{w, x, y, z\}$ given by

$$R = \{(w, w), (w, x), (x, y), (z, y)\}.$$

Draw the directed graph that represents this relation and write down its matrix representation.

This is the directed graph for R :



Its matrix representation is

$$M_R = \begin{pmatrix} & w & x & y & z \\ w & 1 & 1 & 0 & 0 \\ x & 0 & 0 & 1 & 0 \\ y & 0 & 0 & 0 & 0 \\ z & 0 & 0 & 1 & 0 \end{pmatrix}$$

2. Find the reflexive closure \tilde{R} of R .

The reflexive closure of a relation R is defined by $R \cup \Delta$ where

$$\begin{aligned} \Delta &= \{(a, a) \mid a \in R\} \\ \tilde{R} &= R \cup \Delta \\ &= \{(w, w), (w, x), (x, y), (z, y)\} \\ &\cup \{(w, w), (x, x), (y, y), (z, z)\} \\ &= \{(w, w), (x, x), (y, y), (z, z), (w, x), (x, y), (z, y)\} \end{aligned}$$

We will now continue with constructing successive closures of this relation, but consider two different orders.

3. a) Construct the symmetric closure S of the reflexive closure \tilde{R} of R .
You may use whichever representation of the relation you prefer.

So we have

$$\tilde{R} = \{(w, w), (x, x), (y, y), (z, z), (w, x), (x, y), (z, y)\}$$

To create the symmetric closure S of this, we must find $\tilde{R} \cup \tilde{R}^{-1}$. We define \tilde{R}^{-1} by

$$\tilde{R}^{-1} = \{(b, a) \mid (a, b) \in \tilde{R}\},$$

so in our case it is

$$\tilde{R}^{-1} = \{(w, w), (x, x), (y, y), (z, z), (x, w), (y, x), (y, z)\}.$$

Then the union becomes

$$\begin{aligned} \tilde{R} \cup \tilde{R}^{-1} &= \{(w, w), (x, x), (y, y), (z, z), (w, x), (x, y), (z, y)\} \\ &\cup \{(w, w), (x, x), (y, y), (z, z), (x, w), (y, x), (y, z)\} \\ &= \{(w, w), (x, x), (y, y), (z, z), (w, x), (x, y), (z, y), (x, w), (y, x), (y, z)\} \end{aligned}$$

This is the symmetric closure S

$$S = \tilde{R} \cup \tilde{R}^{-1} = \{(w, w), (x, x), (y, y), (z, z), (w, x), (x, y), (z, y), (x, w), (y, x), (y, z)\}.$$

- b) Construct the transitive closure S^* of S .

To construct the transitive closure we need to find

$$S^* = \bigcup_{n=1}^{\infty} S^n$$

We'll stop at the point where $S^n = S^{n-1}$. So first we'll find $S \circ S$:

$$S \circ S = \{\}$$

4. a) Now begin by constructing the transitive closure T of the reflexive closure \tilde{R} . You may use whichever representation of the relation you prefer.

b) Next construct the symmetric closure T' of T .

5. Compare the results from the two previous tasks. Which of the two final relations S^* and T' is an equivalence relation?