SC2001 Project 3

Dynamic Programming

Group 3

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01
Recursive definition

Recursive Definition of P(C)

C: capacity

i = 0, 1, … , n-1

Base Case:

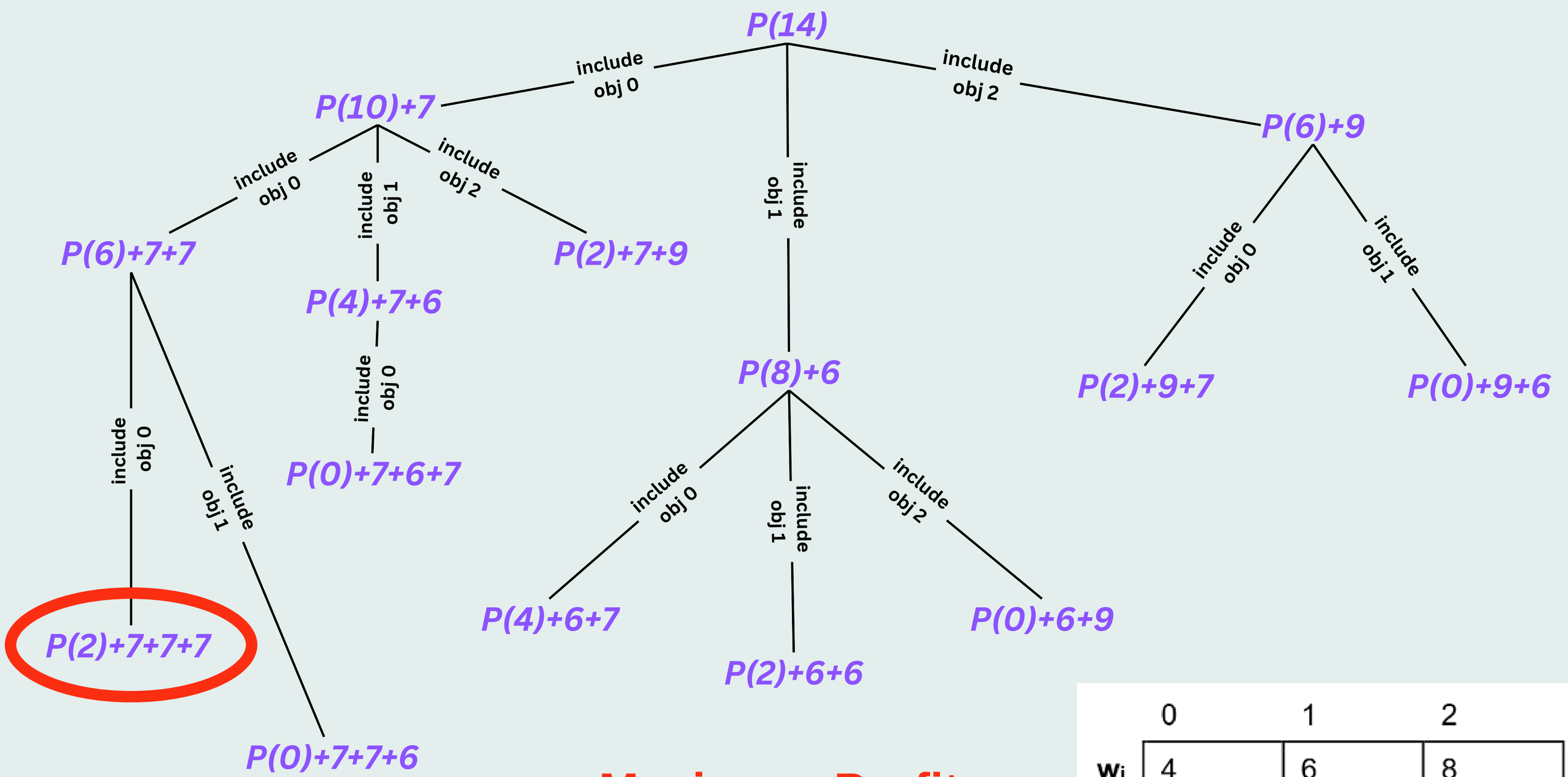
P(C)= 0 when C=0 or n=0

$$P(C) = \max(P(C), P(C-w_i) + p_i)$$

***ith* item is excluded**

***ith* item is included**

02
Subproblem graph of P(14)



Maximum Profit:
7+7+7 = 21

	0	1	2
w_i	4	6	8
p_i	7	6	9

03
Dynamic Programming

Dynamic Programming with Bottom-Up Approach:

1. **2D DP Array**: dp[i][j] stores the maximum profit for capacity j using items up to index i.
2. **Initialize DP Array**: Each row represents an item and each column represents a capacity from 0 to C.
3. **Outer Loop**:

- **iterates over each capacity** j from 1 to C.

4. **Inner Loop:**

- iterates over each item i.
- Excluding the item: If we exclude the item, dp[i][capacity] is set to dp[i - 1] [capacity] if i > 0, otherwise it's 0 (base case).
- Including the item: If weights[i] <= capacity, the code calculates the profit by including the item, **allowing reuse by taking dp[i][capacity - weights[i]] + profits[i]**

5. **Final Solution**:

- dp[n-1][C] holds the maximum profit for capacity C using all items.

Dynamic Programming with Bottom-Up Approach:

```python
def max_profit(C, weights, profits):
    n = len(weights)
    # Initialize a 2D DP array where rows are items and columns are capacities
    dp = [[0] * (C + 1) for _ in range(n)]

    # Fill the DP table
    for capacity in range(1, C + 1):
        for i in range(n):
            # Option 1: Exclude the item
            dp[i][capacity] = dp[i - 1][capacity] if i > 0 else 0

            # Option 2: Include the item (if capacity allows)
            if weights[i] <= capacity:
                dp[i][capacity] = max(dp[i][capacity], dp[i][capacity - weights[i]] + profits[i])

    # The maximum profit with the given capacity and all items is in dp[n-1][C]
    return dp[n - 1][C]
```

The code **allows item reuse** by **repeatedly updating dp[i][capacity] with dp[i][capacity-weights[i]] + profits[i] whenever an item fits.**

Time Complexity

Outer Loop (Capacities):

The outer loop iterates over each capacity from 1 to C, so it runs C times.

Inner Loop (Items):

For each capacity, the inner loop iterates over all n items.

In Each Iteration:

Within the inner loop, the program checks two options (excluding and including the item) and performs a constant-time update of the dp array.
Overall Time Complexity:

Since the outer loop runs C times and the inner loop runs n times for each capacity, the total time complexity is: **O(n×C)**

Space Complexity

DP Array:

The 2D dp array has n rows (one for each item) and C + 1 columns (one for each capacity from 0 to C), resulting in a space complexity of: O(n×C)

vs Recursive Solution (Without Memoization)

- try each item at every capacity level, making recursive calls for each item that can fit within the remaining capacity.
- The recursion continues until the remaining capacity is exhausted (or until there are no items left to consider).

Time Complexity

For each capacity C, we have n choices (one for each item) if it fits. For each choice, the function calls itself recursively with a smaller capacity (C - weight[i]).

Branching Factor: **At each step, the function can branch out n times (one for each item), and it can potentially reach down to a depth of C** (if each call reduces capacity by 1 in the worst case).

Worst-Case Time Complexity:**O(n^C), which is exponential**.

Space Complexity

Recursive Call Stack: The depth of the recursive calls depends on the capacity C.

worst case: each call reduces the capacity by the smallest weight, leading to a maximum depth of C.

-> O(C)

Code for running the DP algo and printing part 4 results

```python
def max_profit(C, weights, profits):
    n = len(weights)
    # Initialize a 2D DP array where rows are items and columns are capacities
    dp = [[0] * (C + 1) for _ in range(n)]

    # Fill the DP table
    for capacity in range(1, C + 1):
        for i in range(n):
            # Option 1: Exclude the item
            dp[i][capacity] = dp[i - 1][capacity] if i > 0 else 0

            # Option 2: Include the item (if capacity allows)
            if weights[i] <= capacity:
                dp[i][capacity] = max(dp[i][capacity], dp[i][capacity - weights[i]] + profits[i])

    # Print the DP table with labels
    print("DP Table (Items x Capacity):")
    print("      ", " ".join([f"C={j:2}" for j in range(C + 1)]))
    print("     ", "-" * (5 * (C + 1)))

    for i in range(n):
        print(f"Item {i} |", " ".join(f"{dp[i][capacity]:4}" for capacity in range(C + 1)))

    # The maximum profit with the given capacity and all items is in dp[n-1][C]
    return dp[n - 1][C]

# Test with the data
C = 14
weights = [5, 6, 8]
profits = [7, 6, 9]
max_profit = max_profit(C, weights, profits)
print("\nMaximum profit for P(14) with weights [5, 6, 8] and profits [7, 6, 9] is:", max_profit)
```

04
Results using Dynamic Programming

4(a) Results of P(14) with

	0	1	2
w_i	4	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ---------------------------------------------------------------------------
Item 0 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 1 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 2 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21

Maximum profit for P(14) with weights [4, 6, 8] and profits [7, 6, 9] is: 21
```

Maximum Profit = 21

4(a) Results of P(14) with

	0	1	2
w_i	4	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ------------------------------------------------------------------------------
Item 0 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 1 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 2 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21

Maximum profit for P(14) with weights [4, 6, 8] and profits [7, 6, 9] is: 21
```

Maximum Profit = 21

at C=4, item 0
will be placed in the bag
bag = [0]

4(a) Results of P(14) with

	0	1	2
w_i	4	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ------------------------------------------------------------------------------
Item 0 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 1 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 2 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21

Maximum profit for P(14) with weights [4, 6, 8] and profits [7, 6, 9] is: 21
```

Maximum Profit = 21

at C=8, another item 0
will be placed in the bag
bag = [0,0]

4(a) Results of P(14) with

	0	1	2
w_i	4	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ------------------------------------------------------------------------------
Item 0 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 1 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 2 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21

Maximum profit for P(14) with weights [4, 6, 8] and profits [7, 6, 9] is: 21
```

Maximum Profit = 21

at C=10,
if we consider including item 1

bag = [0,0]: profit = 7+7=14
bag = [0,1]: profit = 7+6=13<14

4(a) Results of P(14) with

	0	1	2
w_i	4	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ---------------------------------------------------------------------------
Item 0 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 1 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 2 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21

Maximum profit for P(14) with weights [4, 6, 8] and profits [7, 6, 9] is: 21
```

Maximum Profit = 21

at C=12, another item 0
will be placed in the bag
bag = [0,0, 0]

4(a) Results of P(14) with

	0	1	2
w_i	4	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ------------------------------------------------------------------------------
Item 0 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 1 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 2 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21

Maximum profit for P(14) with weights [4, 6, 8] and profits [7, 6, 9] is: 21
```

Maximum Profit = 21

at C=12,
if we consider including item 2

bag = [0,0, 0]: profit = 7+7+7=21
bag = [0,2]: profit = 7+9=15<21

4(a) Results of P(14) with

	0	1	2
w_i	4	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ---------------------------------------------------------------------------
Item 0 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 1 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 2 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21

Maximum profit for P(14) with weights [4, 6, 8] and profits [7, 6, 9] is: 21
```

Maximum Profit = 21

4(b) Results of P(14) with

	0	1	2
w_i	5	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ------------------------------------------------------------------------------
Item 0 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 1 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 2 |    0    0    0    0    0    7    7    7    9    9   14   14   14   16   16

Maximum profit for P(14) with weights [5, 6, 8] and profits [7, 6, 9] is: 16
```

Maximum Profit = 16

4(b)Results of P(14) with

	0	1	2
w_i	5	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ------------------------------------------------------------------------------
Item 0 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 1 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 2 |    0    0    0    0    0    7    7    7    9    9   14   14   14   16   16

Maximum profit for P(14) with weights [5, 6, 8] and profits [7, 6, 9] is: 16
```

Maximum Profit = 16

at C=5, item 0
will be placed in the bag
bag = [0]

4(b) Results of P(14) with

	0	1	2
w_i	5	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ------------------------------------------------------------------------------
Item 0 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 1 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 2 |    0    0    0    0    0    7    7    7    9    9   14   14   14   16   16

Maximum profit for P(14) with weights [5, 6, 8] and profits [7, 6, 9] is: 16
```

at C=8, only item 0
is in the bag
bag = [0]

Maximum Profit = 16

at C=8,
consider up till item 2

bag = [0] --> 7
bag = [2] --> 9>7

4(b) Results of P(14) with

	0	1	2
w_i	5	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ------------------------------------------------------------------------------
Item 0 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 1 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 2 |    0    0    0    0    0    7    7    7    9    9   14   14   14   16   16

Maximum profit for P(14) with weights [5, 6, 8] and profits [7, 6, 9] is: 16
```

Maximum Profit = 16

at C=10, another item 0
is added to the bag
bag = [0, 0] --> 7+7=14

at C=10,
consider **up till item 2**

bag = [0, 0] --> 7+7=14
bag = [2] --> 9<14

4(b) Results of P(14) with

	0	1	2
w_i	5	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ------------------------------------------------------------------------------
Item 0 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 1 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 2 |    0    0    0    0    0    7    7    7    9    9   14   14   14   16   16

Maximum profit for P(14) with weights [5, 6, 8] and profits [7, 6, 9] is: 16
```

Maximum Profit = 16

at C=11,
consider **item 1**

bag = [0, 0] --> 7+7=14

bag = [0,1] --> 7 + 6 = 13<14

4(b) Results of P(14) with

	0	1	2
w_i	5	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ---------------------------------------------------------------------------
Item 0 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 1 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 2 |    0    0    0    0    0    7    7    7    9    9   14   14   14   16   16

Maximum profit for P(14) with weights [5, 6, 8] and profits [7, 6, 9] is: 16
```

Maximum Profit = 16

at C=13, another item 0
is added to the bag
bag = [0, 0] --> 7+7=14

at C=13,
consider **up till item 2**

bag = [0, 0] --> 7+7=14
bag = [2, 0] --> 9 + 7 = 16 > 14

4(b)Results of P(14) with

	0	1	2
w_i	5	6	8
p_i	7	6	9

```
DP Table (Items x Capacity):
        C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ------------------------------------------------------------------------------
Item 0 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 1 |    0    0    0    0    0    7    7    7    7    7   14   14   14   14   14
Item 2 |    0    0    0    0    0    7    7    7    9    9   14   14   14   16   16

Maximum profit for P(14) with weights [5, 6, 8] and profits [7, 6, 9] is: 16
```

Maximum Profit = 16

Conclusion

Dynamic Programming improved the overall efficiency

time complexity (DP): O(Cn) vs time complexity(recursive function): O(n^C)

space complexity (DP): O(Cn) using 2D array

to further improve the programming,

we can implement the 1D array, we can improve the **space complexity to O(C)** but it decreases the readability.

using 2D array

```
DP Table (Items x Capacity):
       C= 0 C= 1 C= 2 C= 3 C= 4 C= 5 C= 6 C= 7 C= 8 C= 9 C=10 C=11 C=12 C=13 C=14
      ------------------------------------------------------------------------
Item 0 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 1 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21
Item 2 |    0    0    0    0    7    7    7    7   14   14   14   14   21   21   21

Maximum profit for P(14) with weights [4, 6, 8] and profits [7, 6, 9] is: 21
```

using 1D array

```
(base) gaoxinyue@Gaos-MacBook-Air Desktop % python unbounded_knapsack__Capacities.py
Capacity |   0   1   2   3   4   5   6   7   8   9  10  11  12  13  14
-------------------------------------------------------------------
C= 1     |   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
C= 2     |   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
C= 3     |   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
C= 4     |   0   0   0   0   7   0   0   0   0   0   0   0   0   0   0
C= 5     |   0   0   0   0   7   7   0   0   0   0   0   0   0   0   0
C= 6     |   0   0   0   0   7   7   7   0   0   0   0   0   0   0   0
C= 7     |   0   0   0   0   7   7   7   7   0   0   0   0   0   0   0
C= 8     |   0   0   0   0   7   7   7   7  14   0   0   0   0   0   0
C= 9     |   0   0   0   0   7   7   7   7  14  14   0   0   0   0   0
C=10     |   0   0   0   0   7   7   7   7  14  14  14   0   0   0   0
C=11     |   0   0   0   0   7   7   7   7  14  14  14  14   0   0   0
C=12     |   0   0   0   0   7   7   7   7  14  14  14  14  21   0   0
C=13     |   0   0   0   0   7   7   7   7  14  14  14  14  21  21   0
C=14     |   0   0   0   0   7   7   7   7  14  14  14  14  21  21  21

Final DP Array: [0, 0, 0, 0, 7, 7, 7, 7, 14, 14, 14, 14, 21, 21, 21]

Maximum profit for P(14) with weights [4, 6, 8] and profits [7, 6, 9] is: 21
```

THANK YOU