Appendix B

Complex Numbers

The ability to manipulate complex numbers is very handy in circuit analysis and in electrical engineering in general. Complex numbers are particularly useful in the analysis of ac circuits. Again, although calculators and computer software packages are now available to manipulate complex numbers, it is still advisable for a student to be familiar with how to handle them by hand.

B.1 Representations of Complex Numbers

A complex number z may be written in *rectangular form* as

$$z = x + jy \tag{B.1}$$

where $j = \sqrt{-1}$; x is the *real part* of z while y is the *imaginary part* of z; that is,

$$x = \text{Re}(z), \qquad y = \text{Im}(z) \tag{B.2}$$

The complex number z is shown plotted in the complex plane in Fig. B.1. Since $j = \sqrt{-1}$,

$$\begin{aligned} \frac{1}{j} &= -j \\ j^2 &= -1 \\ j^3 &= j \cdot j^2 = -j \\ j^4 &= j^2 \cdot j^2 = 1 \\ j^5 &= j \cdot j^4 = j \\ &\vdots \\ j^{n+4} &= j^n \end{aligned} \tag{B.3}$$

The complex plane looks like the two-dimensional curvilinear coordinate space, but it is not.

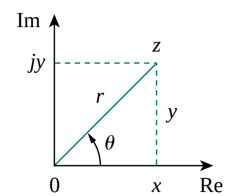

Figure B.1 Graphical representation of a complex number.

A second way of representing the complex number z is by specifying its magnitude r and the angle θ it makes with the real axis, as Fig. B.1 shows. This is known as the *polar form*. It is given by

$$z = |z|\underline{/\theta} = r\underline{/\theta} \tag{B.4}$$

where

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\frac{y}{x} \tag{B.5a}$$

or

$$x = r\cos\theta, \qquad y = r\sin\theta \tag{B.5b}$$

that is,

$$z = x + jy = r\underline{/\theta} = r\cos\theta + jr\sin\theta \tag{B.6}$$

In converting from rectangular to polar form using Eq. (B.5), we must exercise care in determining the correct value of θ. These are the four possibilities:

$$z = x + jy, \qquad \theta = \tan^{-1}\frac{y}{x} \qquad \text{(1st Quadrant)}$$

$$z = -x + jy, \qquad \theta = 180^\circ - \tan^{-1}\frac{y}{x} \qquad \text{(2nd Quadrant)}$$

$$z = -x - jy, \qquad \theta = 180^\circ + \tan^{-1}\frac{y}{x} \qquad \text{(3rd Quadrant)}$$
$$z = x - jy, \qquad \theta = 360^\circ - \tan^{-1}\frac{y}{x} \qquad \text{(4th Quadrant)} \tag{B.7}$$

assuming that x and y are positive.

In the exponential form, $z = re^{j\theta}$ so that $dz/d\theta = jre^{j\theta} = jz$.

The third way of representing the complex z is the *exponential form*:

$$z = re^{j\theta} \tag{B.8}$$

This is almost the same as the polar form, because we use the same magnitude r and the angle θ.

The three forms of representing a complex number are summarized as follows.

$$\boxed{\begin{array}{lll} z = x + jy, & (x = r\cos\theta,\ y = r\sin\theta) & \text{Rectangular form} \\ z = r\underline{/\theta}, & \left(r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\dfrac{y}{x}\right) & \text{Polar form} \\ z = re^{j\theta}, & \left(r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\dfrac{y}{x}\right) & \text{Exponential form} \end{array}} \tag{B.9}$$

The first two forms are related by Eqs. (B.5) and (B.6). In Section B.3 we will derive Euler's formula, which proves that the third form is also equivalent to the first two.

EXAMPLE B.1

Express the following complex numbers in polar and exponential form: (a) $z_1 = 6 + j8$, (b) $z_2 = 6 - j8$, (c) $z_3 = -6 + j8$, (d) $z_4 = -6 - j8$.

Solution:

Notice that we have deliberately chosen these complex numbers to fall in the four quadrants, as shown in Fig. B.2.

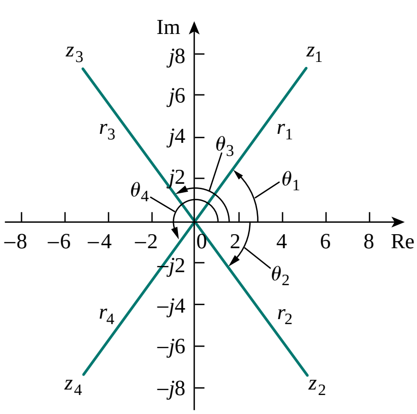

Figure B.2 For Example B.1.

(a) For $z_1 = 6 + j8$ (1st quadrant),

$$r_1 = \sqrt{6^2 + 8^2} = 10, \qquad \theta_1 = \tan^{-1}\frac{8}{6} = 53.13^\circ$$

Hence, the polar form is $10\underline{/53.13^\circ}$ and the exponential form is $10e^{j53.13^\circ}$.

(b) For $z_2 = 6 - j8$ (4th quadrant),

$$r_2 = \sqrt{6^2 + (-8)^2} = 10, \qquad \theta_2 = 360^\circ - \tan^{-1}\frac{8}{6} = 306.87^\circ$$

so that the polar form is $10\underline{/306.87^\circ}$ and the exponential form is $10e^{j306.87^\circ}$. The angle θ_2 may also be taken as -53.13°, as shown in Fig. B.2, so that the polar form becomes $10\underline{/-53.13^\circ}$ and the exponential form becomes $10e^{-j53.13^\circ}$.

(c) For $z_3 = -6 + j8$ (2nd quadrant),

$$r_3 = \sqrt{(-6)^2 + 8^2} = 10, \qquad \theta_3 = 180^\circ - \tan^{-1}\frac{8}{6} = 126.87^\circ$$

Hence, the polar form is $10\angle 126.87^\circ$ and the exponential form is $10e^{j126.87^\circ}$.

(d) For $z_4 = -6 - j8$ (3rd quadrant),

$$r_4 = \sqrt{(-6)^2 + (-8)^2} = 10, \qquad \theta_4 = 180^\circ + \tan^{-1}\frac{8}{6} = 233.13^\circ$$

so that the polar form is $10\angle 233.13^\circ$ and the exponential form is $10e^{j233.13^\circ}$.

PRACTICE PROBLEM B.1

Convert the following complex numbers to polar and exponential forms: (a) $z_1 = 3 - j4$, (b) $z_2 = 5 + j12$, (c) $z_3 = -3 - j9$, (d) $z_4 = -7 + j$.

Answer: (a) $5\angle 306.9^\circ$, $5e^{j306.9^\circ}$, (b) $13\angle 67.38^\circ$, $13e^{j67.38^\circ}$,

(c) $9.487\angle 251.6^\circ$, $9.487e^{j251.6^\circ}$, (d) $7.071\angle 171.9^\circ$, $7.071e^{j171.9^\circ}$.

EXAMPLE B.2

Convert the following complex numbers into rectangular form: (a) $12\angle -60^\circ$, (b) $-50\angle 285^\circ$, (c) $8e^{j10^\circ}$, (d) $20e^{-j\pi/3}$.

Solution:

(a) Using Eq. (B.6),

$$12\angle -60^\circ = 12\cos(-60^\circ) + j12\sin(-60^\circ) = 6 - j10.39$$

Note that $\theta = -60^\circ$ is the same as $\theta = 360^\circ - 60^\circ = 300^\circ$.

(b) We can write

$$-50\angle 285^\circ = -50\cos 285^\circ - j50\sin 285^\circ = -12.94 + j48.3$$

(c) Similarly,

$$8e^{j10^\circ} = 8\cos 10^\circ + j8\sin 10^\circ = 7.878 + j1.389$$

(d) Finally,

$$20e^{-j\pi/3} = 20\cos(-\pi/3) + j20\sin(-\pi/3) = 10 - j17.32$$

PRACTICE PROBLEM B.2

Find the rectangular form of the following complex numbers: (a) $-8\angle 210^\circ$, (b) $40\angle 305^\circ$, (c) $10e^{-j30^\circ}$, (d) $50e^{j\pi/2}$.

Answer: (a) $6.928 + j4$, (b) $22.94 - j32.77$, (c) $8.66 - j5$, (d) $j50$.

B.2 Mathematical Operations

Two complex numbers $z_1 = x_1 + jy_1$ and $z_2 = x_2 + jy_2$ are equal if and only if their real parts are equal and their imaginary parts are equal,

$$x_1 = x_2, \qquad y_1 = y_2 \tag{B.10}$$

We have used lightface notation for complex numbers—since they are not time- or frequency-dependent—whereas we use boldface notation for phasors.

The *complex conjugate* of the complex number $z = x + jy$ is

$$z^* = x - jy = r\underline{/-\theta} = re^{-j\theta} \tag{B.11}$$

Thus the complex conjugate of a complex number is found by replacing every j by $-j$.

Given two complex numbers $z_1 = x_1 + jy_1 = r_1\underline{/\theta_1}$ and $z_2 = x_2 + jy_2 = r_2\underline{/\theta_2}$, their sum is

$$z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \tag{B.12}$$

and their difference is

$$z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2) \tag{B.13}$$

While it is more convenient to perform addition and subtraction of complex numbers in rectangular form, the product and quotient of the two complex numbers are best done in polar or exponential form. For their product,

$$z_1 z_2 = r_1 r_2\underline{/\theta_1 + \theta_2} \tag{B.14}$$

Alternatively, using the rectangular form,

$$\begin{aligned} z_1 z_2 &= (x_1 + jy_1)(x_2 + jy_2) \\ &= (x_1 x_2 - y_1 y_2) + j(x_1 y_2 + x_2 y_1) \end{aligned} \tag{B.15}$$

For their quotient,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\underline{/\theta_1 - \theta_2} \tag{B.16}$$

Alternatively, using the rectangular form,

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} \tag{B.17}$$

We rationalize the denominator by multiplying both the numerator and denominator by z_2^*.

$$\frac{z_1}{z_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + j\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \tag{B.18}$$

EXAMPLE B.3

If $A = 2 + j5$, $B = 4 - j6$, find: (a) $A^*(A + B)$, (b) $(A + B)/(A - B)$.

Solution:

(a) If $A = 2 + j5$, then $A^* = 2 - j5$ and

$$A + B = (2 + 4) + j(5 - 6) = 6 - j$$

so that

$$A^*(A + B) = (2 - j5)(6 - j) = 12 - j2 - j30 - 5 = 7 - j32$$

(b) Similarly,

$$A - B = (2-4) + j(5 - -6) = -2 + j11$$

Hence,

$$\frac{A+B}{A-B} = \frac{6-j}{-2+j11} = \frac{(6-j)(-2-j11)}{(-2+j11)(-2-j11)}$$
$$= \frac{-12 - j66 + j2 - 11}{(-2)^2 + 11^2} = \frac{-23 - j64}{125} = -0.184 - j0.512$$

PRACTICE PROBLEM B.3

Given that $C = -3 + j7$ and $D = 8 + j$, calculate:
(a) $(C - D^*)(C + D^*)$, (b) D^2/C^*, (c) $2CD/(C+D)$.

Answer: (a) $-103 - j26$, (b) $-5.19 + j6.776$, (c) $6.054 + j11.53$.

EXAMPLE B.4

Evaluate:

(a) $\dfrac{(2+j5)(8e^{j10^\circ})}{2 + j4 + 2\underline{/-40^\circ}}$ (b) $\dfrac{j(3-j4)^*}{(-1+j6)(2+j)^2}$

Solution:

(a) Since there are terms in polar and exponential forms, it may be best to express all terms in polar form:

$$2 + j5 = \sqrt{2^2 + 5^2}\,\underline{/\tan^{-1} 5/2} = 5.385\underline{/68.2^\circ}$$
$$(2+j5)(8e^{j10^\circ}) = (5.385\underline{/68.2^\circ})(8\underline{/10^\circ}) = 43.08\underline{/78.2^\circ}$$
$$2 + j4 + 2\underline{/-40^\circ} = 2 + j4 + 2\cos(-40^\circ) + j2\sin(-40^\circ)$$
$$= 3.532 + j2.714 = 4.454\underline{/37.54^\circ}$$

Thus,

$$\frac{(2+j5)(8e^{j10^\circ})}{2 + j4 + 2\underline{/-40^\circ}} = \frac{43.08\underline{/78.2^\circ}}{4.454\underline{/37.54^\circ}} = 9.672\underline{/40.66^\circ}$$

(b) We can evaluate this in rectangular form, since all terms are in that form. But

$$j(3 - j4)^* = j(3 + j4) = -4 + j3$$
$$(2+j)^2 = 4 + j4 - 1 = 3 + j4$$
$$(-1+j6)(2+j)^2 = (-1+j6)(3+j4) = -3 - 4j + j18 - 24$$
$$= -27 + j14$$

Hence,

$$\frac{j(3-j4)^*}{(-1+j6)(2+j)^2} = \frac{-4+j3}{-27+j14} = \frac{(-4+j3)(-27-j14)}{27^2+14^2}$$
$$= \frac{108+j56-j81+42}{925} = 0.1622 - j0.027$$

PRACTICE PROBLEM B.4

Evaluate these complex fractions:

(a) $\dfrac{6\angle 30^\circ + j5 - 3}{-1+j+2e^{j45^\circ}}$ (b) $\left[\dfrac{(15-j7)(3+j2)^*}{(4+j6)^*(3\angle 70^\circ)}\right]^*$

Answer: (a) $1.213\angle 237.4^\circ$, (b) $2.759\angle -287.6^\circ$.

B.3 Euler's Formula

Euler's formula is an important result in complex variables. We derive it from the series expansion of e^x, $\cos\theta$, and $\sin\theta$. We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{B.19}$$

Replacing x by $j\theta$ gives

$$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \tag{B.20}$$

Also,

$$\begin{aligned} \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \\ \sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots \end{aligned} \tag{B.21}$$

so that

$$\cos\theta + j\sin\theta = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \cdots \tag{B.22}$$

Comparing Eqs. (B.20) and (B.22), we conclude that

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta} \tag{B.23}$$

This is known as *Euler's formula*. The exponential form of representing a complex number as in Eq. (B.8) is based on Euler's formula. From Eq. (B.23), notice that

$$\boxed{\cos\theta = \mathrm{Re}(e^{j\theta}), \qquad \sin\theta = \mathrm{Im}(e^{j\theta})} \tag{B.24}$$

and that

$$|e^{j\theta}| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Replacing θ by $-\theta$ in Eq. (B.23) gives

$$e^{-j\theta} = \cos\theta - j\sin\theta \tag{B.25}$$

Adding Eqs. (B.23) and (B.25) yields

$$\boxed{\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})} \tag{B.26}$$

Substracting Eq. (B.24) from Eq. (B.23) yields

$$\boxed{\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})} \tag{B.27}$$

B.4 Useful Identities

The following identities are useful in dealing with complex numbers. If $z = x + jy = r\underline{/\theta}$, then

$$zz^* = x^2 + y^2 = r^2 \tag{B.28}$$

$$\sqrt{z} = \sqrt{x + jy} = \sqrt{r}e^{j\theta/2} = \sqrt{r}\underline{/\theta/2} \tag{B.29}$$

$$z^n = (x + jy)^n = r^n\underline{/n\theta} = r^n e^{jn\theta} = r^n(\cos n\theta + j\sin n\theta) \tag{B.30}$$

$$\begin{gathered} z^{1/n} = (x + jy)^{1/n} = r^{1/n}\underline{/\theta/n + 2\pi k/n} \\ k = 0, 1, 2, \ldots, n-1 \end{gathered} \tag{B.31}$$

$$\begin{gathered} \ln(re^{j\theta}) = \ln r + \ln e^{j\theta} = \ln r + j\theta + j2k\pi \\ (k = \text{integer}) \end{gathered} \tag{B.32}$$

$$\begin{aligned} \frac{1}{j} &= -j \\ e^{\pm j\pi} &= -1 \\ e^{\pm j2\pi} &= 1 \\ e^{j\pi/2} &= j \\ e^{-j\pi/2} &= -j \end{aligned} \tag{B.33}$$

$$\begin{aligned} \text{Re}(e^{(\alpha+j\omega)t}) &= \text{Re}(e^{\alpha t}e^{j\omega t}) = e^{\alpha t}\cos\omega t \\ \text{Im}(e^{(\alpha+j\omega)t}) &= \text{Im}(e^{\alpha t}e^{j\omega t}) = e^{\alpha t}\sin\omega t \end{aligned} \tag{B.34}$$

EXAMPLE B.5

If $A = 6 + j8$, find: (a) $\sqrt{A}$, (b) A^4.

Solution:

(a) First, convert A to polar form:

$$r = \sqrt{6^2 + 8^2} = 10, \qquad \theta = \tan^{-1}\frac{8}{6} = 53.13^\circ, \qquad A = 10\underline{/53.13^\circ}$$

Then

$$\sqrt{A} = \sqrt{10}\underline{/53.13^\circ/2} = 3.162\underline{/26.56^\circ}$$

(b) Since $A = 10\angle 53.13^\circ$,

$$A^4 = r^4\angle 4\theta = 10^4\angle 4 \times 53.13^\circ = 10{,}000\angle 212.52^\circ$$

PRACTICE PROBLEM B.5

If $A = 3 - j4$, find: (a) $A^{1/3}$ (3 roots), and (b) $\ln A$.

Answer: (a) $1.71\angle 102.3^\circ$, $1.71\angle 222.3^\circ$, $1.71\angle 342.3^\circ$, (b) $1.609 + j5.356$.