CHAPTER 16 - FOURIER SERIES

List of topics for this chapter :
- Trigonometric Fourier Series
- Symmetry Considerations
- Circuit Applications
- Average Power and RMS Values
- Exponential Fourier Series
- Fourier Analysis with PSpice

TRIGONOMETRIC FOURIER SERIES

Problem 16.1 *[16.5]* A voltage source has a periodic waveform defined over its period as $v(t) = t(2\pi - t)$ V for $0 < t < 2\pi$. Find the Fourier series for this voltage.

$$v(t) = 2\pi t - t^2, \qquad 0 < t < 2\pi$$
$$T = 2\pi \qquad \omega_0 = 2\pi/T = 1$$

$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi}(2\pi t - t^2)\,dt = \left(\frac{1}{2\pi}\right)\left(\pi t^2 - \frac{t^3}{3}\right)\Bigg|_0^{2\pi}$$
$$a_0 = \left(\frac{4\pi^3}{2\pi}\right)\left(1 - \frac{2}{3}\right) = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{T}\int_0^T (2\pi t - t^2)\cos(nt)\,dt = \left(\frac{1}{\pi}\right)\left[\frac{2\pi}{n^2}\cos(nt) + \frac{2\pi t}{n}\sin(nt)\right]\Bigg|_0^{2\pi}$$
$$a_n = \frac{-1}{\pi n^3}[2nt\cos(nt) - 2\sin(nt) + n^2 t^2 \sin(nt)]\Big|_0^{2\pi}$$
$$a_n = \frac{2}{n^2}(1-1) - \frac{1}{\pi n^3}[4n\pi\cos(2\pi n)] = \frac{-4}{n^2}$$

$$b_n = \frac{2}{T}\int_0^T (2nt - t^2)\sin(nt)\,dt = \frac{1}{\pi}\int(2nt - t^2)\sin(nt)\,dt$$
$$b_n = \frac{2n}{\pi}\frac{1}{n^2}[\sin(nt) - nt\cos(nt)]\Big|_0^{\pi} - \frac{1}{\pi n^3}[2nt\sin(nt) + 2\cos(nt) - n^2 t^2\cos(nt)]\Big|_0^{2\pi}$$
$$b_n = \frac{-4\pi}{n} + \frac{4\pi}{n} = 0$$

Hence,

$$\underline{f(t) = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2}\cos(nt)}$$

Problem 16.2 Evaluate each of the following functions and determine if it is periodic. If it is periodic, find its period.

(a) f(t) = $\cos(\pi t/2) + \sin(\pi t) + \sqrt{3}\cos(2\pi t)$
(b) y(t) = $\sin(\sqrt{3}\,\pi t) + \cos(\pi t)$
(c) g(t) = $4 + \sin(\omega t)$
(d) h(t) = $2\sin(5t)\cos(3t)$
(e) z(t) = $e^{-t}\sin(\pi t)$

(a) This is a periodic function with a period of 4 seconds.
(b) This is a nonperiodic function since the first term has an irrational multiplier of πt while the second has a rational multiplier.
(c) The integral of this function goes to infinity because of the dc function. Thus this is a nonperiodic function.
(d) This is a periodic function with a period of π seconds.
(e) This is a nonperiodic function since it continuously changes as t goes to infinity.

SYMMETRY CONSIDERATIONS

Problem 16.3 Determine the type of function represented by the signal in Figure 16.1. Also, determine the Fourier series expansion.

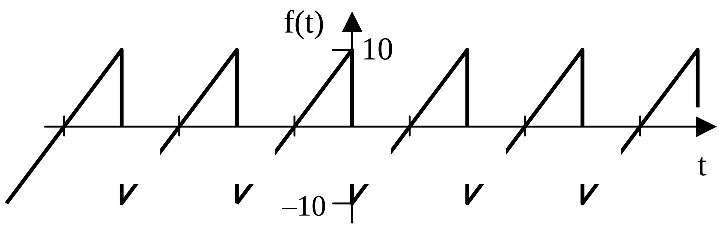


Figure 16.1

This is an odd function since f(t) = –f(–t). Therefore, $a_o = 0 = a_n$.

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_o t)dt\,, \text{ where } T = 1 \text{ sec and } \omega_o = 2\pi \text{ rad/sec.}$$

For $0 < t < 1$, $f(t) = 20t - 10$

Solving for $b_n = \frac{2}{1}\int_0^1 (20t - 10)\sin(2n\pi t)dt = 2\left[\int_0^1 20t\sin(2n\pi t)dt - \int_0^1 10\sin(2n\pi t)dt\right]$

$$= 2\left[\left(\frac{20}{4n^2\pi^2}\sin(2n\pi t) - \frac{20t}{2n\pi}\cos(2n\pi t)\right)\Bigg|_0^1 - \left(\frac{-10}{2n\pi}\cos(2n\pi t)\right)\Bigg|_0^1\right]$$

$$= 2\left[\frac{20}{4n^2\pi^2}(0-0) - \frac{20}{2n\pi}(1-0) - \frac{-10}{2n\pi}(1-1)\right] = \frac{-20}{n\pi}$$

Therefore, $\underline{\boldsymbol{f(t) = \frac{-20}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(2n\pi t)}}$

Problem 16.4 *[16.15]* Calculate the Fourier coefficients for the function in Figure 16.1.

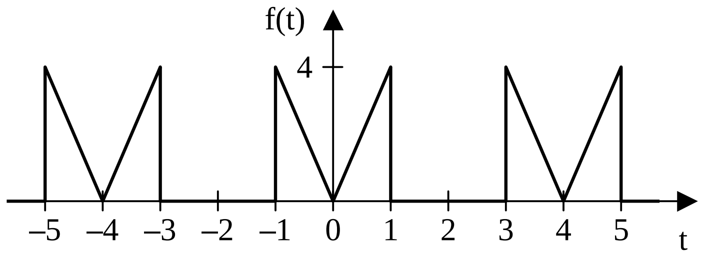


Figure 16.1

This is an even function, therefore $b_n = 0$. In addition, $T = 4$ and $\omega_0 = \omega/2$.

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt = \frac{2}{4}\int_0^1 4t\,dt = t^2\Big|_0^1 = \underline{1}$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(\omega_0 nt)\,dt = \frac{4}{4}\int_0^1 4t\cos(n\pi t/2)\,dt$$

$$a_n = 4\left[\frac{4}{n^2\pi^2}\cos(n\pi t/2) + \frac{2t}{n\pi}\sin(n\pi t/2)\right]\Bigg|_0^1$$

$$a_n = \frac{16}{n^2\pi^2}[\cos(n\pi/2)-1] + \frac{8}{n\pi}\sin(n\pi/2)$$

CIRCUIT APPLICATIONS

Problem 16.5 Figure 16.1 and $v_s(t)$ is periodic with a period equal to 2π msec and has the following values during that period,

$$V_s(t) = 10 \text{ volts} \qquad 0 < t < \pi \text{ msec}$$
$$= 0 \qquad \pi \text{ msec} < t < 2\pi \text{ msec}$$

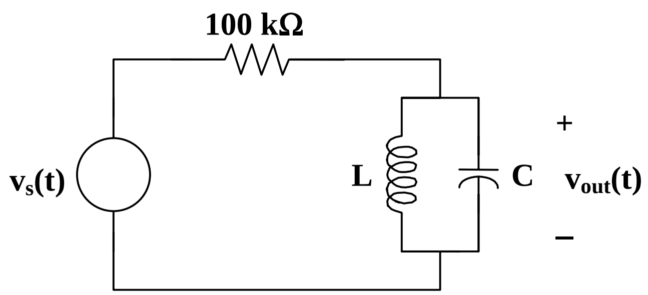


Figure 16.1

In addition, L = 1 H and C = 1 μF. Determine the value of $v_o(t)$.

The first step is to find the Fourier series for $v_s(t)$. $a_n = 0$ since this is an odd function.

$$f(t) = a_o + \sum_{n=1}^{\infty} b_n \sin(n\omega_o t)$$

$$T = 2\pi x10^{-3} \text{ and } \omega_o = 1000.$$

$$a_o = \frac{1}{2\pi 10^{-3}}\left[\int_0^{\pi 10^{-3}} 10dt + \int_{\pi 10^{-3}}^{2\pi 10^{-3}} 0dt\right] = \frac{1}{2\pi 10^{-3}}(10t-0)\Big|_0^{\pi 10^{-3}} = 5 \text{ volts}$$

$$b_n = \frac{2}{2\pi 10^{-3}}\int_0^T f(t)\sin(1000nt)dt = \frac{1}{\pi 10^{-3}}\left[\int_0^{\pi 10^{-3}} 10\sin(1000t)dt + 0\right]$$

$$= -\frac{1}{\pi 10^{-3} x 10^{3} n} 10\cos(1000t)\Big|_{0}^{\pi 10^{-3}}$$

$$= \frac{-10}{n\pi}\left(\cos(n\pi) - 1\right)$$

Thus, $$b_n = \begin{cases} \dfrac{20}{n\pi} \text{ for } n = \text{odd} \\ 0 \text{ for } n = \text{even} \end{cases}$$

Therefore, $$v_s(t) = \left[5 + \sum_{k=1}^{\infty} \frac{20}{(1+2k)\pi} \sin(1000(1+2k)t)\right] \text{ volts}$$

Now let us look at the first three terms.

Clearly, for the dc term, V_o = 0 since the inductor looks like a short for dc. For all the other values of n,

$$v_o = \frac{\dfrac{20}{n\pi}}{10^5 + \dfrac{L/C}{j(\omega L - 1/(\omega C))}}\left(\frac{L/C}{j(\omega L - 1/(\omega C))}\right), \ \omega = 1000n, \text{ for } n = \text{odd}$$

$$= \frac{\dfrac{20L}{n\pi C}}{j10^5(\omega L - 1/(\omega C)) + L/C} = \frac{\dfrac{20x10^6}{n\pi}}{j10^5(1000n - 1000/n) + 1000}$$

(1)

For n = 1, ω = 1000. Therefore, V_o = 20/π.

For n = 3, ω = 3000. Therefore,

$$\text{the value of } L\|C = \frac{L/C}{j(\omega L - 1/(\omega C))} = \frac{1000}{j2667} = -j0.375\Omega$$

This value of impedance is so much smaller than the value of the resistor that we can neglect this term and all of the others. Thus,

$$v_o(t) = \underline{\mathbf{\frac{20}{\pi}\sin(1000t) \text{ volts}}}$$

Does this answer make any sense? If we look at this term and the values of L and C, we find that L and C are in parallel resonance when $\omega = 1000$. Thus, this circuit is actually a filter that filters out a single sine wave from the input signal.

Problem 16.6 Refer to Figure 16.1. Change the value of L to (1/9) H. with everything else remaining the same. Now solve for $v_o(t)$. Everything remains the same as Problem 16.5 up till equation (a). The new value of L changes equation (a) as shown below.

Thus, our new equation for
$$V_o = \frac{\dfrac{20\text{x}10^6}{n\pi 9}}{j10^5\left(\dfrac{1000n}{9} - \dfrac{1000}{n}\right) + \dfrac{10^6}{9}}$$

For n = 1,

$$V_o = \frac{0.7074\text{x}10^6}{j10^5(111.11 - 1000) + 0.1111\text{x}10^6}$$
$$\cong \frac{0.7074\text{x}10^6}{-j888.9\text{x}10^5} = j0.007958$$

Clearly, this can be considered to be equal to zero.

For n = 3,

$$V_o = \frac{0.2358\text{x}10^6}{j10^5(333.3 - 333.3) + 0.11111\text{x}10^6} = 2.122 \text{ volts}$$

For all other values of n, V_o is essentially equal to zero. Therefore,

$$v_o(t) = \underline{\mathbf{\frac{20}{3\pi}\sin(3000t) = 2.122\sin(3000t)\ V}}$$

Problem 16.7 *[16.25]* If v_s in the circuit of Figure 16.1 is the same as function $f_2(t)$ in Figure 16.2, determine the dc component and the first three nonzero harmonics of $v_o(t)$.

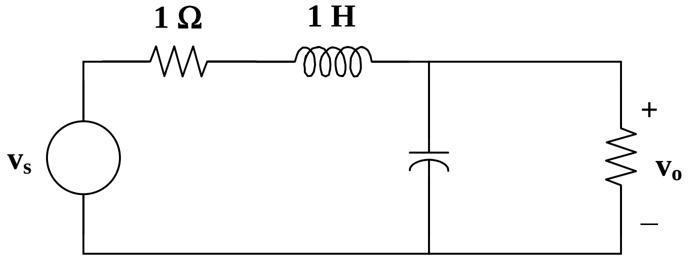


Figure 16.1

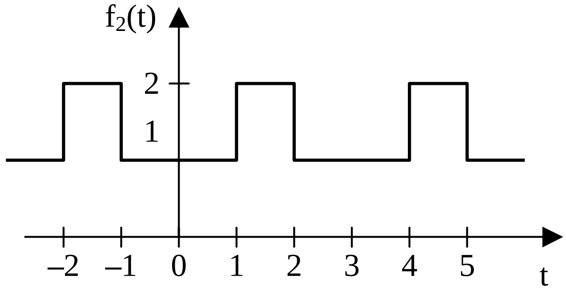


Figure 16.2

The signal is even, hence, $b_n = 0$. In addition, $T = 3$, $\omega_0 = 2\pi/3$.

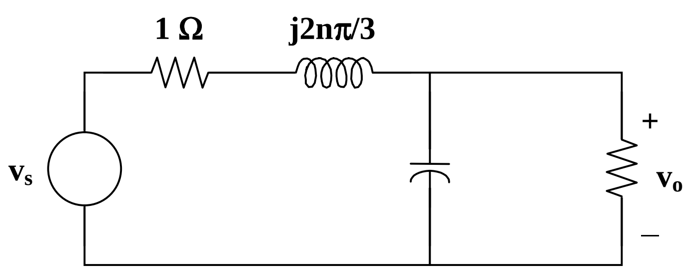
$$v_s(t) = 1 \qquad \text{for all } 0 < t < 1$$
$$v_s(t) = 2 \qquad \text{for all } 1 < t < 1.5$$

$$a_0 = \frac{2}{3}\left[\int_0^1 1\,dt + \int_1^{1.5} 2\,dt\right] = \frac{4}{3}$$

$$a_n = \frac{4}{3}\left[\int_0^1 \cos(2n\pi t/3)\,dt + \int_1^{1.5} 2\cos(2n\pi t/3)\,dt\right]$$
$$a_n = \frac{4}{3}\left[\frac{3}{2n\pi}\sin(2n\pi t/3)\Big|_0^1 + \frac{6}{2n\pi}\sin(2n\pi t/3)\Big|_1^{1.5}\right] = \frac{-2}{n\pi}\sin(2n\pi/3)$$

$$v_s(t) = \frac{4}{3} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(2n\pi/3)\cos(2n\pi t/3)$$

Now consider this circuit,



Let $\quad Z = \left(\frac{-j3}{2n\pi}\right)\left(\frac{1}{1 - j3/2n\pi}\right) = \frac{-j3}{2n\pi - j3}$

Thus, $\quad v_o = \dfrac{Z}{Z + 1 + j2n\pi/3} v_s$

Simplifying, we get

$$v_o = \frac{-j9}{12n\pi + j(4n^2\pi^2 - 18)} v_s$$

For the dc case, $n = 0$ and $v_s = 3/4$ V and $v_o = v_s/2 = 3/8$ V.

We can now solve for $v_o(t)$

$$\underline{v_o(t) = \left[\frac{3}{8} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2n\pi t}{3} + \Theta_n \right) \right] \text{V}}$$

where $\quad A_n = \underline{\dfrac{(6/n\pi)\sin(2n\pi/3)}{\sqrt{16n^2\pi^2 + [(4n^2\pi^2/3) - 6]^2}}}$

and $\quad \Theta_n = 90° - \tan^{-1}\left(\dfrac{n\pi}{3} - \dfrac{3}{2n\pi} \right)$

where we can further simplify A_n to $\quad A_n = \underline{\dfrac{9\sin(2n\pi/3)}{n\pi\sqrt{4n^4\pi^4 + 81}}}$

AVERAGE POWER AND RMS VALUES

Problem 16.8 Given the signal shown in Figure 16.6, determine the exact value of the rms

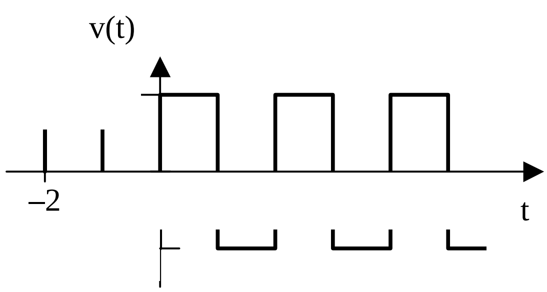


Figure 16.1

value of this wave shape. Using the Fourier series of the wave shape, calculate the estimated rms value using all the terms up to and including n = 5.

We can use the definition of V_{rms} to calculate the rms value of the wave shape.

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt} \quad \text{where } T = 2 \text{ sec.}$$

$$\frac{1}{2}\int_0^2 v^2(t)\,dt = \frac{1}{2}\left[\int_0^1 (10)^2\,dt + \int_1^2 (-10)^2\,dt\right] = \frac{1}{2}\left[100t\Big|_0^1 + 100t\Big|_1^2\right]$$

$$= 0.5[100 - 0 + 200 - 100] = 100$$

Thus, $V_{rms} = \sqrt{100} =$ **10 volts**.

We now proceed to the Fourier series. Please note, this is just the Fourier series of a standard square wave.

$$v(t) = \frac{40}{\pi}\sum_{k=1}^{\infty}\frac{1}{n}\sin(n\pi t), \quad n = 2k - 1.$$

For this problem, we want all the terms through and including n = 5 (k = 3).

For a Fourier series, we can solve for the rms value using,

$$F_{rms} = \sqrt{a_o^2 + \frac{1}{2}\sum_1^{\infty}(a_n^2 + b_n^2)}$$

Thus, $$V_{rms} \cong \sqrt{\frac{1}{2}\left[\left(\frac{40}{\pi}\right)^2 + \left(\frac{40}{3\pi}\right)^2 + \left(\frac{40}{5\pi}\right)^2\right]} = \frac{40}{\pi}\sqrt{\frac{1}{2}\left[1 + \frac{1}{9} + \frac{1}{25}\right]}$$

$= (40/\pi)(0.7587) =$ **9.66 volts**.

Although this answer is only within 5%, it is still significant enough for some cases. The reason that this is not closer to the actual value of 10 volts is that the coefficients for the Fourier series of a square wave do not decrease in value as fast as they do for other signals.

Problem 16.9 Given the triangular voltage wave shape shown in Figure 16.7, determine the exact value of the rms voltage. Then, calculate the approximate value of the rms value using the Fourier terms up to and including n = 5.

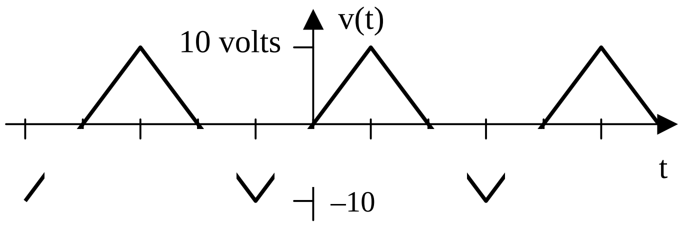


Figure 16.0

First we will calculate the exact value using,

$$V_{rms} = \sqrt{\frac{1}{2}\int_0^2 v^2(t)\,dt}\,,\ \text{where } v(t) = 20t \text{ for } 0 < t < 1/2.$$

Note that due to symmetry, we only need to use the range, 0 < t < 1/2.

$$\frac{1}{2}\int_0^2 v^2(t)\,dt = \frac{4}{2}\int_0^{1/2} 400t^2\,dt = \left.\frac{800t^3}{3}\right|_0^{1/2}$$

$$= (800/3)[(1/8) - 0] = 100/3$$

Therefore, $V_{rms} = 10/\sqrt{3}$ = **<u>5.774 volts</u>**.

Now we can solve the Fourier series. The student can verify that the Fourier series for this wave shape is given by,

$$v(t) = \frac{80}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{n^2}\sin(n\pi t),\ \text{where } n = 2k - 1.$$

Through n = 5 we get,

$$v(t) \cong \frac{80}{\pi^2}\left(\sin(\pi t) + \frac{1}{9}\sin(3\pi t) + \frac{1}{25}\sin(5\pi t)\right) \text{ volts.}$$

Therefore,

$$V_{rms} \cong \frac{80}{\pi^2}\sqrt{\frac{1}{2}\left(1 + \frac{1}{81} + \frac{1}{625}\right)}$$ = **<u>5.771 volts</u>**.

Clearly, this compares very favorably to the exact value of 5.774. The reason for this is because the Fourier series for a triangular wave shape converges very quickly.

Problem 16.10 *[16.31]* The voltage across the terminals of a circuit is

$$v(t) = 30 + 20\cos(120\pi t + 45°) + 10\cos(120\pi t - 45°) \text{ V}$$

The current entering the terminal at higher potential is

$$i(t) = 6 + 4\cos(120\pi t + 10°) - 2\cos(120\pi t - 60°) \text{ A}$$

Find:

(a) the rms value of the voltage,
(b) the rms value of the current,
(c) the average value of the power absorbed by the circuit.

(a) $$V_{rms} = \sqrt{a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)} = \sqrt{(30)^2 + \left(\frac{1}{2}\right)(20^2 + 10^2)} = \underline{\mathbf{33.91\ V}}$$

(b) $$I_{rms} = \sqrt{6^2 + \left(\frac{1}{2}\right)(4^2 + 2^2)} = \underline{\mathbf{6.782\ A}}$$

(c) $$P = V_{dc}\, I_{dc} + \frac{1}{2}\sum V_n\, I_n \cos(\Theta_n - \Phi_n)$$
$$P = (30)(60) + (0.5)[(20)(4)\cos(45° - 10°) - (10)(2)\cos(\text{-}45° + 60°)]$$
$$P = 180 + 32.76 - 9.659 = \underline{\mathbf{203.1\ W}}$$

Problem 16.11 Determine the rms value of a triangular wave shape with a peak-to-peak value of 40 volts. If this wave shape is placed across a 10-ohm resistor, determine the average power dissipated by that resistor.

As we saw in problem 16.9, the rms value of a triangular wave shape is given by,

$V_{rms} = V_{peak}/\sqrt{3} = 20/\sqrt{3} =$ **<u>11.547 volts</u>**.

Average power $= V_{rms}{}^2/R = (11.547)^2/10 =$ **<u>13.333 watts</u>**.

EXPONENTIAL FOURIER SERIES

Problem 16.12 Given the sawtooth voltage wave shape shown in Figure 16.8, find its exponential (complex) Fourier series.

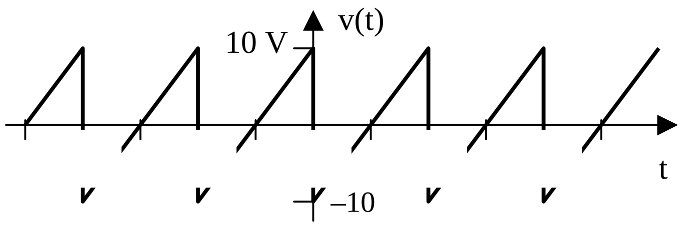


Figure 16.1

$$c_n = \frac{1}{T}\int_0^T v(t)e^{-jn\omega_o t}dt, \text{ where } T = 1 \text{ and } v(t) = (20t - 10) \text{ for } 0 < t < 1.$$

Since $T = 1$, $\omega_o = 2\pi$.

Therefore, $$c_n = \frac{1}{1}\int_0^1 (20t - 10)e^{-j2\pi nt}dt = 20\int_0^1 te^{-j2\pi nt}dt - 10\int_0^1 e^{-j2\pi nt}dt$$

$$= 20\left[\frac{te^{-j2\pi nt}}{-j2\pi n} - \frac{e^{-j2\pi nt}}{(-j2\pi n)^2}\right]_0^1 - 10\left.\frac{e^{-j2\pi nt}}{-j2\pi n}\right|_0^1$$

$$= 20\left[\frac{e^{-j2\pi n}}{-j2\pi n} + \frac{e^{j2\pi n}}{4\pi^2 n^2} - \frac{1}{4\pi^2 n^2}\right] - 10\left[\left(e^{-j2\pi n}\frac{j}{2\pi n}\right) - \frac{j}{2\pi n}\right]$$

$$= 20\left[\frac{j}{2\pi n} + \frac{1}{4\pi^2 n^2} - \frac{1}{4\pi^2 n^2}\right] - 10\left[\frac{j}{2\pi n} - \frac{j}{2\pi n}\right] = j\frac{10}{2\pi n}$$

In addition, $c_o = 0$.

Thus, $$\underline{v(t) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{j10}{n\pi}e^{j2n\pi t}}$$

Problem 16.13 *[16.37]* Determine the exponential Fourier series for $f(t) = t^2$, $-\pi < t < \pi$, with $f(t + 2\pi n) = f(t)$.

$$\omega_0 = 2\pi/T = 1$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\, e^{-jnt}\, dt$$

Integrating by parts twice gives,

$$c_n = 2\cos(n\pi/n^2) = (2)(-1^n/n^2)\,, \qquad n \neq 0$$

For $n = 0$,

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2\, dt = \frac{\pi^2}{3}$$

Hence,

$$f(t) = \frac{\pi^2}{3} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(2)(-1)^n}{n^2}\, e^{jnt}$$

FOURIER ANALYSIS WITH PSPICE

Problem 16.14 *[16.51]* Calculate the Fourier coefficients of the signal in Figure 16.1 using PSpice.

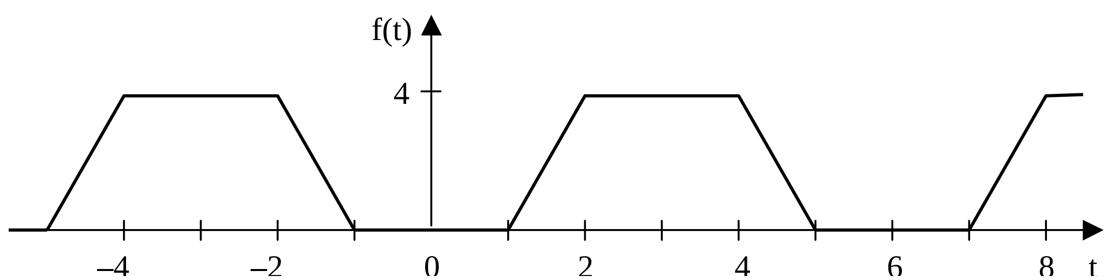
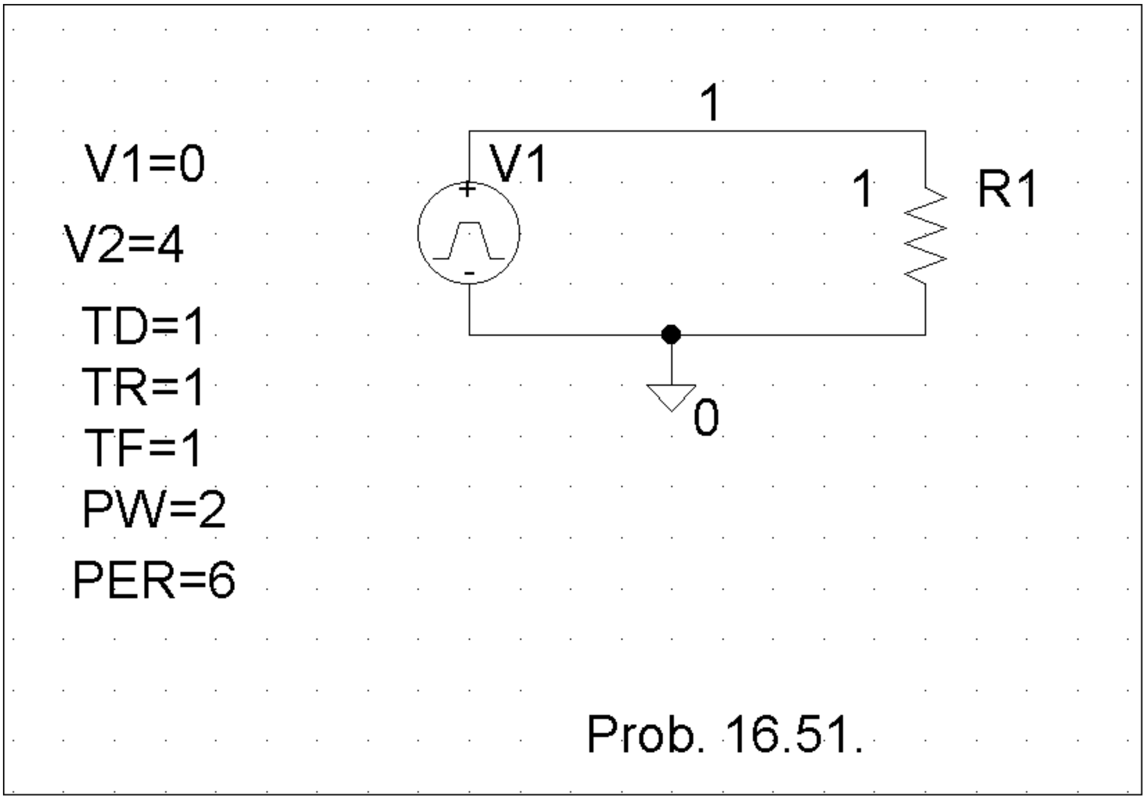


Figure 16.1

The Schematic is shown below. In the Transient dialog box, we type "Print step = 0.01s, Final time = 36s, Center frequency = 0.1667, Output vars = v(1)," and click Enable Fourier.



After simulation, the output file includes the following Fourier components,

FOURIER COMPONENTS OF TRANSIENT RESPONSE V(1)

DC COMPONENT = 2.000396E+00

HARMONIC NO	FREQUENCY (HZ)	FOURIER COMPONENT	NORMALIZED COMPONENT	PHASE (DEG)	NORMALIZED PHASE (DEG)
1	1.667E–01	2.432E+00	1.000E+00	–8.996E+01	0.000E+00
2	3.334E–01	6.576E–04	2.705E–04	–8.932E+01	6.467E–01
3	5.001E–01	5.403E–01	2.222E–01	9.011E+01	1.801E+02
4	6.668E–01	3.343E–04	1.375E–04	9.134E+01	1.813E+02
5	8.335E–01	9.716E–02	3.996E–02	–8.982E+01	1.433E–01
6	1.000E+00	7.481E–06	3.076E–06	–9.000E+01	–3.581E–02
7	1.167E+00	4.968E–02	2.043E–02	–8.975E+01	2.173E–01
8	1.334E+00	1.613E–04	6.634E–05	–8.722E+01	2.748E+00
9	1.500E+00	6.002E–02	2.468E–02	9.032E+01	1.803E+02