

CHAPTER 3 - METHODS OF ANALYSIS

List of topics for this chapter :

Solving Systems of Equations

Nodal Analysis

Nodal Analysis with Voltage Sources

Mesh Analysis

Mesh Analysis with Current Sources

Nodal and Mesh Analysis by Inspection

Circuit Analysis with PSpice

SOLVING SYSTEMS OF EQUATIONS

Problem 3.1 Invert a general $n \times n$ matrix.

The inverse of a nonsingular $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is given by

$$A^{-1} = \frac{C^T}{\Delta} = \frac{1}{\Delta} \begin{bmatrix} c_{11} & c_{21} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{n2} \\ \vdots & \vdots & \cdots & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{nn} \end{bmatrix}$$

where Δ is the determinant of the matrix A and c_{ij} is the cofactor of a_{ij} in Δ .

The value of the determinant, Δ , can be obtained by expanding along the i th row

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (-1)^{i+1} a_{i1} m_{i1} + (-1)^{i+2} a_{i2} m_{i2} + \cdots + (-1)^{i+n} a_{in} m_{in}$$

or the j th column

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = (-1)^{1+j} a_{1j} m_{1j} + (-1)^{2+j} a_{2j} m_{2j} + \cdots + (-1)^{n+j} a_{nj} m_{nj}$$

where m_{ij} , the minor of a_{ij} in Δ , is a $(n-1) \times (n-1)$ determinant of the submatrix of A obtained by removing the i th row and the j th column.

The cofactor of a_{ij} in Δ is $c_{ij} = (-1)^{i+j} m_{ij}$.

The transpose of the cofactor matrix is also known as the adjoint of the matrix; i.e., $\text{adj } A = C^T$.

Therefore,

$$A^{-1} = \frac{\text{adj } A}{\det A} = \frac{C^T}{\Delta}$$

Problem 3.2 Solve a general system of simultaneous equations using Cramer's rule.

Given a system of simultaneous equations having the form

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\ \vdots & \quad \quad \quad \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n &= b_n \end{aligned}$$

where there are n unknowns, x_1, x_2, \dots, x_n , to be determined.

The matrix representation of the system of simultaneous equations is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

or

$$AX = B$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Note that A is a square $(n \times n)$ matrix, while X and B are $(n \times 1)$ column matrices.

Cramer's rule states that the solution to the system of simultaneous equations, $AX = B$, is

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad \dots, \quad x_n = \frac{\Delta_n}{\Delta}$$

where the Δ 's are the determinants given by

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\Delta_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} \quad \Delta_2 = \begin{vmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{vmatrix} \quad \dots \quad \Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}$$

Notice that Δ is the determinant of matrix A and Δ_k is Δ with its k th column replaced with matrix B .

Obviously, Cramer's rule only applies when $\Delta \neq 0$. In the case that $\Delta = 0$, the set of equations has no unique solution because the equations are linearly dependent.

See Problem 3.1 to find out how to calculate the value of a determinant of a matrix.



NODAL ANALYSIS

Problem 3.3 [3.3] Find the currents i_1 through i_4 and the voltage v_o in Figure 3.1.

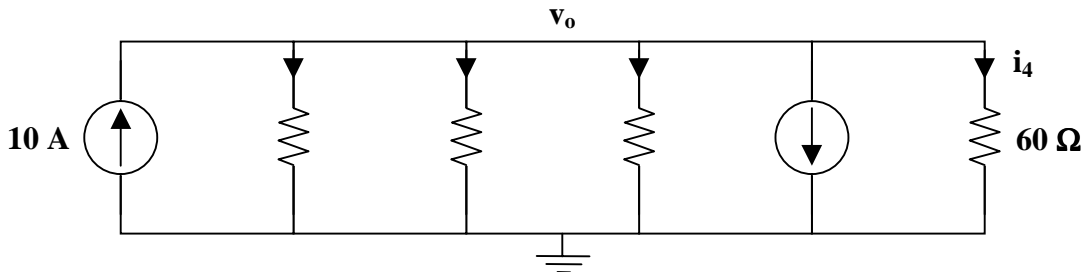


Figure 3.1

Applying KCL to the non-reference node,

$$10 = \frac{v_o}{10} + \frac{v_o}{20} + \frac{v_o}{30} + 2 + \frac{v_o}{60}$$

$$480 = 12 v_o \longrightarrow v_o = \underline{40 \text{ V}}$$

Thus,

$$i_1 = \frac{v_o}{10}$$

$$i_1 = \underline{4 \text{ A}}$$

$$i_2 = \frac{v_o}{20}$$

$$i_2 = \underline{2 \text{ A}}$$

$$i_3 = \frac{v_o}{30}$$

$$i_3 = \underline{1.3333 \text{ A}}$$

$$i_4 = \frac{v_o}{60}$$

$$i_4 = \underline{666.7 \text{ mA}}$$

Problem 3.4 Given the circuit in Figure 3.1, find I_x .

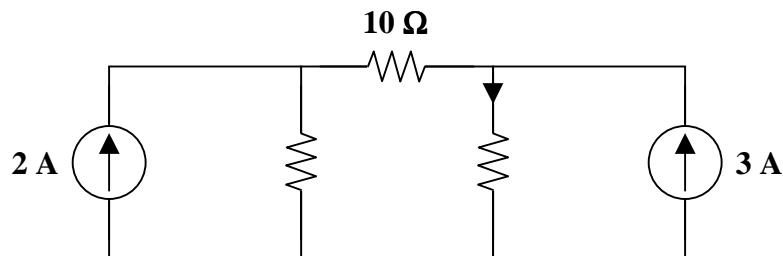


Figure 3.1

$$I_x = \underline{2 \text{ A}}$$

NODAL ANALYSIS WITH VOLTAGE SOURCES

Problem 3.5 Given the circuit in Figure 3.1, solve for V_x .

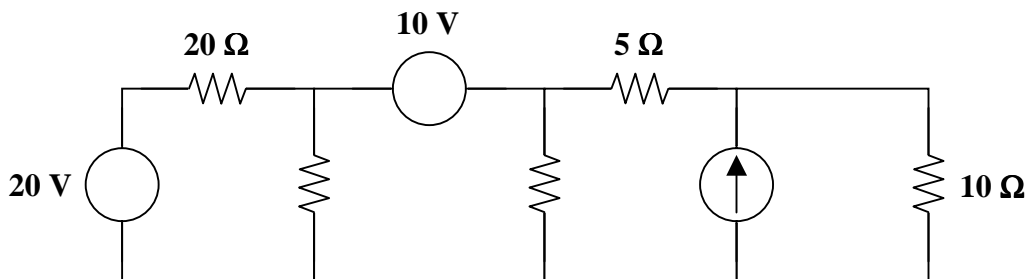


Figure 3.1

➤ **Carefully DEFINE the problem.**

Each component is labeled completely. The problem is clear.

➤ **PRESENT everything you know about the problem.**

The goal of the problem is to find V_x . Letting the lower node be the reference node, we need to find the voltage to the right of the 10-V voltage source.

A supernode is formed by enclosing a (dependent or independent) voltage source connected between two non-reference nodes and any elements connected in parallel with it. Hence, it is clear that the two nodes on either side of the 10-V voltage source form a supernode.

A supermesh results when two meshes have a (dependent or independent) current source in common. Hence, the two meshes on the right half of the circuit create a supermesh, with the 9-A current source in common.

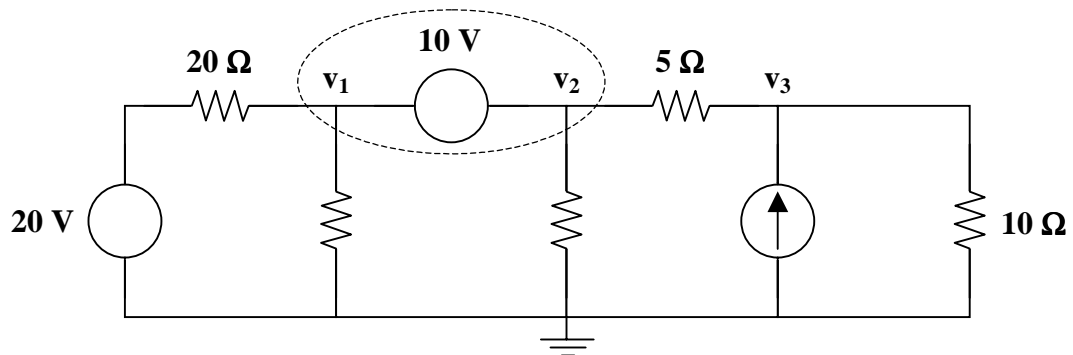
➤ **Establish a set of ALTERNATIVE solutions and determine the one that promises the greatest likelihood of success.**

The two methods of analysis of simple circuits, such as the one above, are nodal analysis and mesh analysis. Because the circuit contains three nodes in addition to the reference node, nodal analysis produces a set of three equations and three unknowns. Yet, the supernode changes the set to two equations and a constraint equation. On the other hand, the circuit has four loops. Thus, mesh analysis produces four equations and four unknowns. The supermesh changes the set to three equations and a constraint equation.

Because the goal of the problem is to find a voltage, the obvious choice is to use nodal analysis to find the voltages at each node of the circuit. If mesh analysis were used to find the mesh currents, Ohm's law would also be needed to find the voltage across the resistor.

➤ **ATTEMPT a problem solution.**

Begin the problem solution by identifying the nodes, including the supernode.



Clearly, $V_x = v_2$

Use nodal analysis to find v_1 , v_2 , and v_3 .

At the supernode (nodes 1 & 2):

$$\frac{v_1 - 20}{20} + \frac{v_1 - 0}{20} + \frac{v_2 - 0}{10} + \frac{v_2 - v_3}{5} = 0$$

At node 3:

$$\frac{v_3 - v_2}{5} - 9 + \frac{v_3 - 0}{10} = 0$$

Simplifying,

$$(v_1 - 20) + v_1 + 2v_2 + (4)(v_2 - v_3) = 0$$

$$(2)(v_3 - v_2) + v_3 = 90$$

$$2v_1 + 6v_2 - 4v_3 = 20$$

$$-2v_2 + 3v_3 = 90$$

$$v_1 + 3v_2 - 2v_3 = 10$$

This is a set of two equations and three unknowns. Thus, we must find a constraint equation. The supernode will provide the constraint equation.

$$v_2 = v_1 + 10$$

or

$$v_1 = v_2 - 10$$

Substitute the constraint equation into the simplified equation from the supernode. Then, this equation plus two times the simplified equation from node 3 will isolate v_3 .

$$(v_2 - 10) + 3v_2 - 2v_3 = 10$$

$$(2)[-2v_2 + 3v_3 = 90]$$

$$[4v_2 - 2v_3 = 20] + [-4v_2 + 6v_3 = 180] = [4v_3 = 200]$$

$$v_3 = 50 \text{ volts}$$

The equation at node 3 can be written,

$$2v_2 = 3v_3 - 90$$

$$v_2 = (1/2)[(3)(50) - 90] = (1/2)(150 - 90) = (1/2)(60) = 30 \text{ volts}$$

The constraint equation gives

$$v_1 = v_2 - 10 = 30 - 10 = 20 \text{ volts}$$

Therefore,

$$V_x = v_2 = 30 \text{ volts}$$

➤ **EVALUATE the solution and check for accuracy.**

This circuit can be analyzed using mesh analysis to verify the solution. This would provide practice analyzing a circuit with a supermesh. Mesh analysis will be discussed later in this chapter. So, we will check our solution using KCL at each node.

For the supernode,

$$\frac{v_1 - 20}{20} + \frac{v_1}{20} + \frac{v_2}{10} + \frac{v_2 - v_3}{5} = \frac{20 - 20}{20} + \frac{20}{20} + \frac{30}{10} + \frac{30 - 50}{5} = 0 + 1 + 3 - 4 = 0$$

For node 3,

$$\frac{v_3 - v_2}{5} - 9 + \frac{v_3}{10} = \frac{50 - 30}{5} - 9 + \frac{50}{10} = 4 - 9 + 5 = 0$$

KCL is not violated. Thus, our check for accuracy was successful.

- **Has the problem been solved SATISFACTORILY? If so, present the solution; if not, then return to “ALTERNATIVE solutions” and continue through the process again.**
This problem has been solved satisfactorily.

$$\underline{V_x = 30 \text{ V}}$$

Problem 3.6 [3.7] Using nodal analysis, find V_o in Figure 3.1.

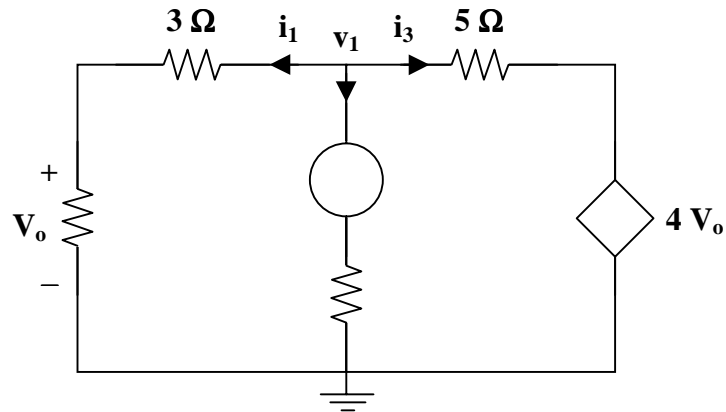


Figure 3.1

$$i_1 + i_2 + i_3 = 0 \longrightarrow \frac{v_1}{5} + \frac{v_1 - 3}{1} + \frac{v_1 - 4V_o}{5} = 0$$

$$7v_1 - 4V_o = 15$$

But $V_o = \frac{2}{5}v_1$ or $v_1 = \frac{5}{2}V_o$

So that $(7)\left(\frac{5}{2}\right)V_o - 4V_o = 15 \longrightarrow V_o = \frac{(2)(15)}{27}$

$$\underline{V_o = 1.1111 \text{ V}}$$

Problem 3.7 Given the circuit in Figure 3.1, solve for V_x using matrix inversion.

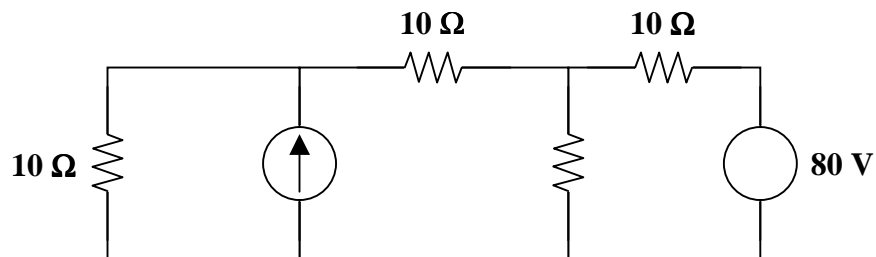


Figure 3.1

This answer is the same as that found in Problem 3.5.

MESH ANALYSIS

Problem 3.9 Given the circuit in Figure 3.1, solve for the loop currents, i_1 and i_2 , using mesh analysis.

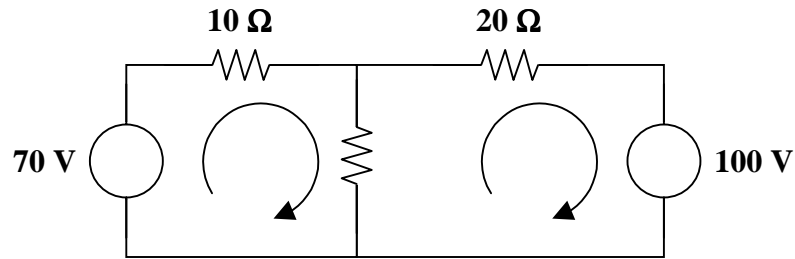


Figure 3.1

For loop 1 :

$$-70 + 10i_1 + (20)(i_1 - i_2) = 0$$

$$30i_1 - 20i_2 = 70$$

$$3i_1 - 2i_2 = 7$$

For loop 2 :

$$(20)(i_2 - i_1) + 20i_2 + 100 = 0$$

$$-20i_1 + 40i_2 = -100$$

$$-2i_1 + 4i_2 = -10$$

Thus, the system of simultaneous equations is

$$\begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -10 \end{bmatrix} \text{ or } \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \frac{1}{12-4} \begin{bmatrix} 4 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ -10 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 8 \\ -16 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Therefore, $i_1 = \underline{1 \text{ A}}$ and $i_2 = \underline{-2 \text{ A}}$

Problem 3.10 [3.33] Apply mesh analysis to find i in Figure 3.1.

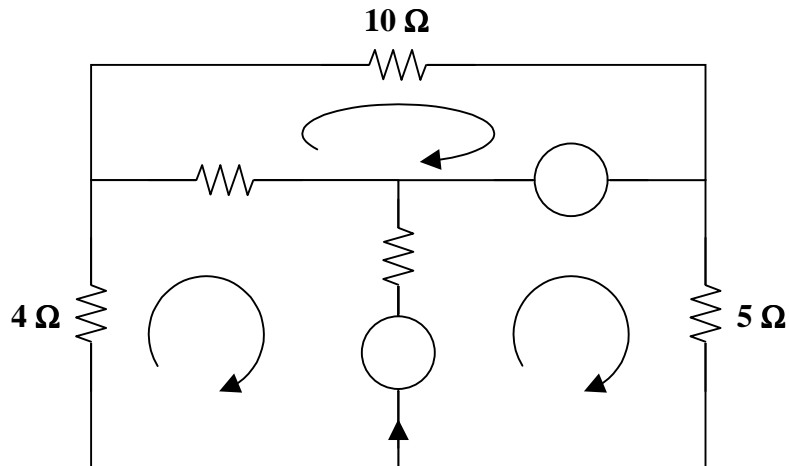


Figure 3.1

$$\text{For loop 1 : } 6 = 12i_1 - 2i_2 \longrightarrow 3 = 6i_1 - i_2 \quad (1)$$

$$\text{For loop 2 : } -8 = 7i_2 - 2i_1 - i_3 \longrightarrow 8 = 2i_1 - 7i_2 + i_3 \quad (2)$$

$$\text{For loop 3 : } -8 + 6 + 6i_3 - i_2 = 0 \longrightarrow 2 = 6i_3 - i_2 \quad (3)$$

Putting (1), (2), and (3) in matrix form,

$$\begin{bmatrix} 6 & -1 & 0 \\ 2 & -7 & 1 \\ 0 & -1 & 6 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 2 \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 6 & -1 & 0 \\ 2 & -7 & 1 \\ 0 & -1 & 6 \end{vmatrix} = -234 \quad \Delta_2 = \begin{vmatrix} 6 & 3 & 0 \\ 2 & 8 & 1 \\ 0 & 2 & 6 \end{vmatrix} = 240 \quad \Delta_3 = \begin{vmatrix} 6 & -1 & 3 \\ 2 & -7 & 8 \\ 0 & -1 & 2 \end{vmatrix} = -38$$

Clearly,

$$i = i_3 - i_2 = \frac{\Delta_3 - \Delta_2}{\Delta} = \frac{-38 - 240}{-234}$$

$$i = \underline{\underline{1.188 \text{ A}}}$$

MESH ANALYSIS WITH CURRENT SOURCES

Problem 3.11 Given the circuit shown in Figure 3.1, find I_x using mesh analysis.

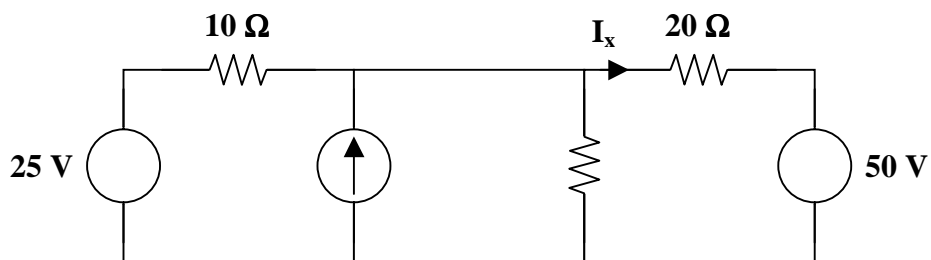


Figure 3.1

➤ **Carefully DEFINE the problem.**

Each component is labeled completely. The problem is clear.

➤ **PRESENT everything you know about the problem.**

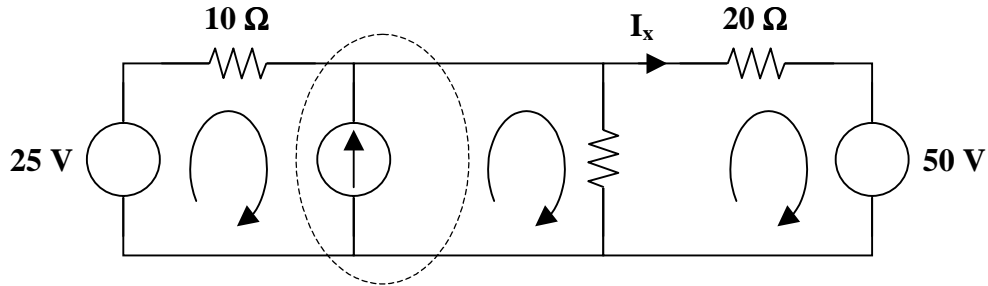
A supermesh results when two meshes have a (dependent or independent) current source in common. Hence, the leftmost mesh and the middle mesh create a supermesh, with the 1-A current source in common.

- **Establish a set of ALTERNATIVE solutions and determine the one that promises the greatest likelihood of success.**

The problem clearly states to use mesh analysis. This makes sense because the goal of the problem is to find a current, I_x , and mesh analysis produces the currents in each mesh, or loop, of a circuit.

- **ATTEMPT a problem solution.**

Begin the problem solution by identifying the meshes, including the supermesh.



Clearly, $I_x = i_3$

Use mesh analysis to find i_1 , i_2 , and i_3 .

For the supermesh (loops 1 & 2) : $-25 + 10i_1 + (20)(i_2 - i_3) = 0$

where $1 = i_2 - i_1$ or $i_1 = i_2 - 1$ (constraint equation)

For loop 3 : $50 + (20)(i_3 - i_2) + 20i_3 = 0$

Substitute the constraint equation into the equation for the supermesh and simplify,

$$-25 + 10i_2 - 10 + 20i_2 - 20i_3 = 0$$

$$50 + 20i_3 - 20i_2 + 20i_3 = 0$$

Simplifying further,

$$30i_2 - 20i_3 = 35 \quad \text{or} \quad 6i_2 - 4i_3 = 7$$

$$-20i_2 + 40i_3 = -50 \quad -2i_2 + 4i_3 = -5$$

The system of simultaneous equations is

$$\begin{bmatrix} 6 & -4 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$$

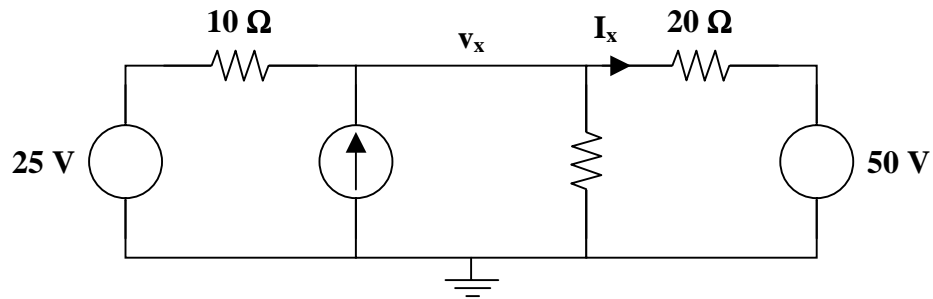
$$\begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = \frac{1}{24-8} \begin{bmatrix} 4 & 4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 8 \\ -16 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$$

and $i_1 = i_2 - 1 = 0.5 - 1 = -0.5$

Therefore, $I_x = i_3 = -1$ amp

➤ **EVALUATE the solution and check for accuracy.**

This circuit has only one unknown node after identifying the lower node as the reference node. Hence, it can easily be analyzed using nodal analysis.



$$\frac{v_x - 25}{10} - 1 + \frac{v_x}{20} + \frac{v_x - 50}{20} = 0$$

$$4v_x = 120$$

$$v_x = 30 \text{ volts}$$

Using Ohm's law,

$$I_x = \frac{v_x - 50}{20} = \frac{30 - 50}{20} = -1 \text{ amp}$$

This matches the answer that was obtained using mesh analysis. Our check for accuracy was successful.

➤ **Has the problem been solved SATISFACTORILY? If so, present the solution; if not, then return to “ALTERNATIVE solutions” and continue through the process again.**
This problem has been solved satisfactorily.

$$I_x = \underline{\underline{-1 \text{ A}}}$$

Problem 3.12 [3.35] Use mesh analysis to obtain i_o in the circuit of Figure 3.1.

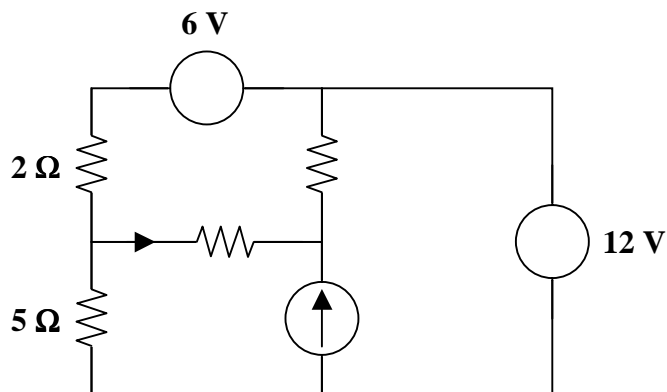
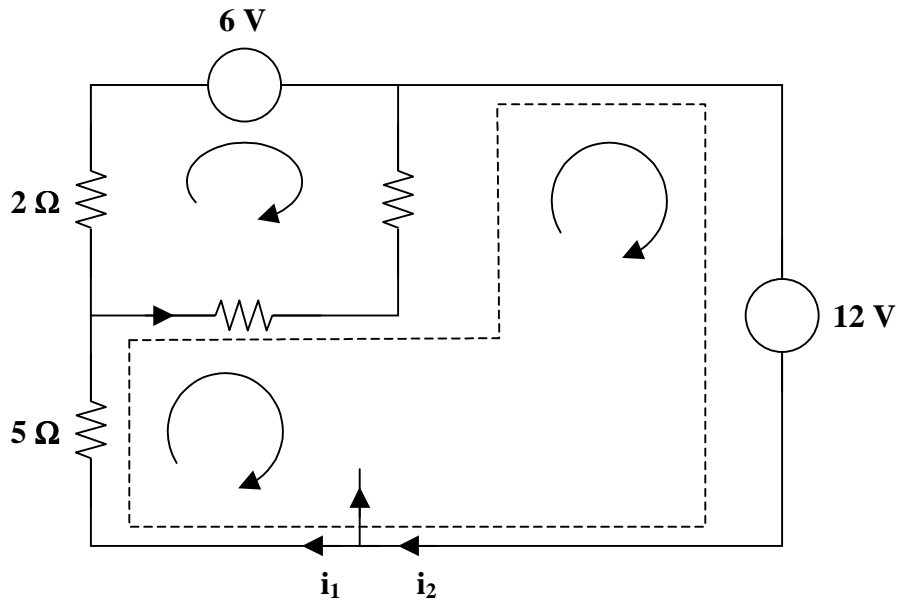


Figure 3.1



Loops 1 and 2 form a supermesh. For the supermesh,

$$6i_1 + 4i_2 - 5i_3 + 12 = 0 \quad (1)$$

For loop 3,

$$6 + 7i_3 - i_1 - 4i_2 = 0 \quad (2)$$

Also,

$$i_2 = 3 + i_1 \quad (3)$$

Putting (1), (2), and (3) into matrix form,

$$\begin{bmatrix} 6 & 4 & -5 \\ 1 & 4 & -7 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} -12 \\ 6 \\ 3 \end{bmatrix}$$

Solving this system of equations yields

$$i_1 = -3.067 \text{ amps}$$

$$i_2 = -0.06667 \text{ amps}$$

$$i_3 = -1.3333 \text{ amps}$$

Therefore,

$$i_o = i_1 - i_3 = -3.067 - (-1.3333)$$

$$i_o = \underline{\underline{-1.7333 \text{ A}}}$$

Problem 3.13 Given the circuit as shown in Figure 3.1, solve for I_x using mesh analysis.

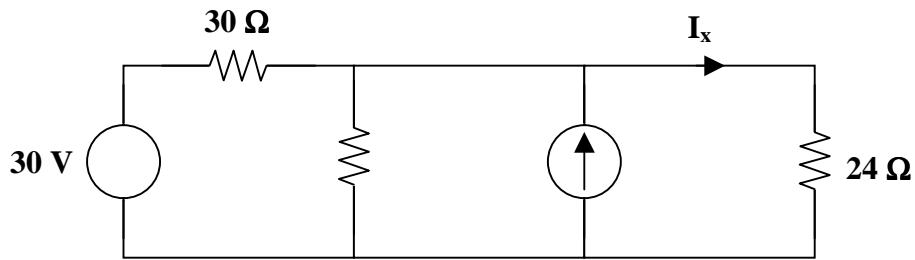


Figure 3.1

$$I_x = \underline{1 \text{ A}}$$

NODAL AND MESH ANALYSIS BY INSPECTION

Problem 3.14 [3.51] Obtain the node-voltage equations for the circuit shown in Figure 3.1 by inspection. Determine the node voltages v_1 and v_2 .

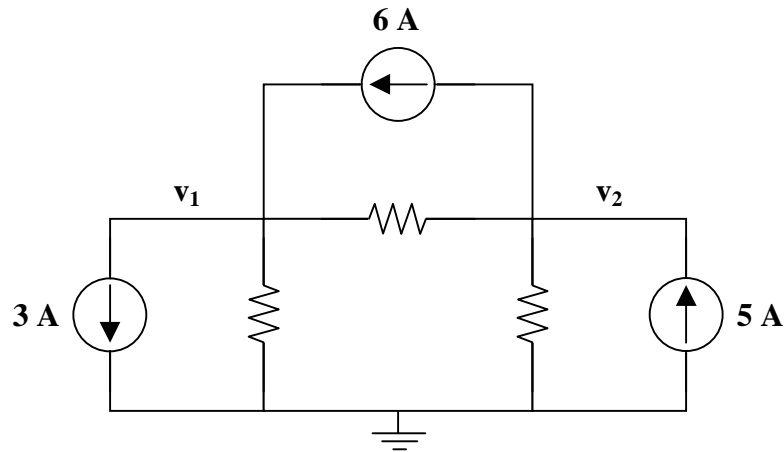


Figure 3.1

$$G_{11} = \frac{1}{1} + \frac{1}{4} = 1.25$$

$$i_1 = 6 - 3 = 3$$

$$G_{12} = -1 = G_{21}$$

$$i_2 = 5 - 6 = -1$$

$$G_{22} = \frac{1}{1} + \frac{1}{2} = 1.5$$

Hence, we have

$$\begin{bmatrix} 1.25 & -1 \\ -1 & 1.5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

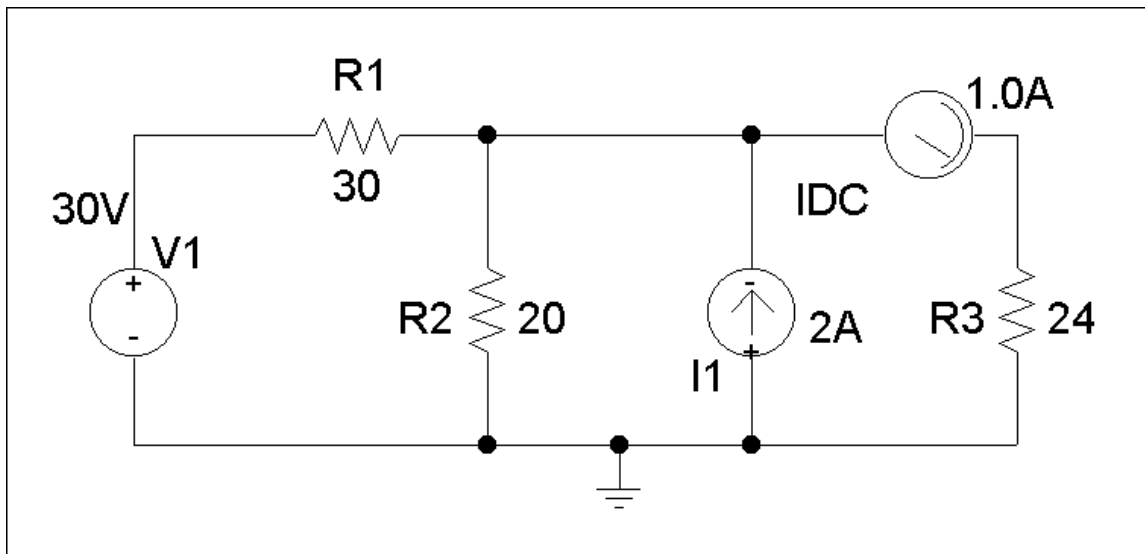
$$\begin{bmatrix} 1.25 & -1 \\ -1 & 1.5 \end{bmatrix}^{-1} = \frac{1}{\Delta} \begin{bmatrix} 1.5 & 1 \\ 1 & 1.25 \end{bmatrix} \quad \text{where } \Delta = (1.25)(1.5) - (-1)(-1) = 0.875$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1.7143 & 1.1429 \\ 1.1429 & 1.4286 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} (3)(1.7143) + (-1)(1.1429) \\ (3)(1.1429) + (-1)(1.4286) \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Clearly, $v_1 = \underline{4 \text{ V}}$ and $v_2 = \underline{2 \text{ V}}$

CIRCUIT ANALYSIS WITH PSpice

Problem 3.15 Solve Problem 3.13 using PSpice.



Clearly, I_x is the current flowing through R3 and the current probe reads $I_x = \underline{1.0 \text{ A}}$.

This answer is the same as the answer obtained in Problem 3.13.