Similarities Between Fourier and Power Series

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In Memory of Ralph P. Boas, Jr.

1. INTRODUCTION. In a paper titled "An Unorthodox Test" in the January 1992 issue of the MONTHLY, Abe Shenitzer of York University poses 16 questions that he feels are intellectually vital in the teaching of mathematics. The second of these asks:

> What are some basic differences between Taylor series and Fourier series?

In his response, in which he considers functions restricted to the reals, he points out that the infinite differentiability of a function does not itself assure that its power series will converge to that function, whereas mere periodicity and a little smoothness are enough to have the Fourier series converge uniformly to the function. Further, the terms of the Fourier series describe simple harmonic motion so that the function may be considered as a linear combination of harmonic motions, whereas the terms of a power series have no such physical interpretation.

While calling attention to such differences is instructive, an interesting and deeper question might also be posed.

> What are some basic similarities between power series and Fourier series beyond the fact that they are both infinite series?

One obvious answer to this question is that if the power series is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad -r < x < r,$$

and if it is continued to the complex plane, then

$$f(se^{i\theta}) = \sum_{n=0}^{\infty} a_n s^n e^{in\theta}, \qquad 0 \leq s < r,$$

is a Fourier series in θ. Like Shenitzer, however, we restrict ourselves to functions of a real variable.

In this context, we find that there is another connection that demonstrates more dramatic similarities, and at the same time, helps to explain some of the differences Shenitzer mentions. This is the existence of an expansion of a function f in a series of ultraspherical polynomials, $C_n^{\lambda}(x)$, $\lambda > 0$, that contains Fourier series of even functions and power series as special limiting cases. We establish this result by demonstrating that since the $C_n^{\lambda}(x)$ are orthogonal, the coefficients in the expansion can be represented by integrals, and if $\cos\theta$ is substituted for x, and we let $\lambda \to 0$, the resulting expansion becomes a Fourier series. On the other hand, if f is infinitely differentiable, n integrations by parts can be applied to obtain new integral representations of the coefficients, so that for analytic functions, by letting $\lambda \to \infty$, we obtain the classical formula for the coefficients as nth derivatives in the power series expansion of a function.

2. FOURIER SERIES AND POWER SERIES. Power series of a real variable have the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \tag{2.1}$$

where the coefficients are given by

$$c_n = f^{(n)}(0)/n!. \tag{2.2}$$

Since we will consider only even Fourier series, we have

$$g(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta \tag{2.3}$$

with

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos n\theta \, d\theta, \qquad n = 0, 1, \dots . \tag{2.4}$$

We note that the coefficients in a power series are given by the local formula (2.2) involving derivatives, while those in a Fourier series are represented by the global integral formula (2.4). The two are related to each other as we have noted and will demonstrate.

3. ULTRASPHERICAL POLYNOMIALS. In work on planetary motion, Legendre and Laplace introduced a set of orthogonal polynomials, $P_n(x)$, now called *Legendre polynomials* and generated by

$$(1 - 2xr + r^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) r^n. \tag{3.1}$$

These were later extended to the so-called *Gegenbauer* or *ultraspherical polynomials*, $C_n^\lambda(x)$, given by

$$(1 - 2xr + r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) r^n. \tag{3.2}$$

An explicit formula that clearly indicates that n is the degree of $C_n^\lambda(x)$ is

$$C_n^\lambda(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (\lambda)_{n-k}}{k!(n-2k)!}(2x)^{n-2k}, \tag{3.3}$$

where

$$(\alpha)_n = \Gamma(n+\alpha)/\Gamma(\alpha) \tag{3.4}$$

is the shifted factorial $\alpha(\alpha+1)\cdots(\alpha+n-1)$. This and other properties of the ultraspherical polynomials can be found in Chapter IV of **[6]** and Chapter 10 of **[1]**.

We may readily obtain a second representation of $C_n^\lambda(x)$ by setting $x = \cos\theta$ in (3.2), factoring

$$1 - 2r\cos\theta + r^2 = (1 - re^{i\theta})(1 - re^{-i\theta}),$$

expanding each factor on the right by the binomial theorem, and using the Cauchy product of two series. We find that

$$\begin{aligned} C_n^\lambda(\cos\theta) &= \sum_{k=0}^{n} \frac{(\lambda)_{n-k}(\lambda)_k}{(n-k)!k!} e^{i(n-2k)\theta} \\ &= \sum_{k=0}^{n} \frac{(\lambda)_{n-k}(\lambda)_k}{(n-k)!k!} \cos(n-2k)\theta, \end{aligned} \tag{3.5}$$

the last holding because $C_n^\lambda(\cos\theta)$ is real when λ is. A connection with Fourier series results from the fact that

$$\lim_{\lambda\to 0} \frac{C_n^\lambda(\cos\theta)}{\lambda} = \frac{2}{n}\cos n\theta, \qquad n = 1,2,3,\ldots, \tag{3.6}$$

which follows from (3.5) since all except the end terms have two factors of λ. From this limit relation, we immediately arrive at two other forms that are more useful for us since they have the advantage of holding for $n = 0$ as well as for positive n, while (3.6) holds only for positive n. One is

$$\lim_{\lambda\to 0} \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} = \cos n\theta, \qquad n = 0,1,\ldots, \tag{3.7}$$

and the other,

$$\lim_{\lambda\to 0} \frac{n+\lambda}{\lambda} C_n^\lambda(\cos\theta) = \begin{cases} 1 & n = 0 \\ 2\cos n\theta & n = 1,2,\ldots \end{cases}. \tag{3.8}$$

A different limiting case comes from (3.5). Since from (3.2) we have

$$\begin{aligned} \sum_{n=0}^{\infty} C_n^\lambda(1) r^n &= (1-r)^{-2\lambda} \\ &= \sum_{n=0}^{\infty} \frac{(2\lambda)_n}{n!} r^n, \end{aligned}$$

it follows that

$$C_n^\lambda(1) = \frac{(2\lambda)_n}{n!}. \tag{3.9}$$

Now from (3.5) and (3.9), we have

$$\begin{aligned} \lim_{\lambda\to\infty} \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} &= \lim_{\lambda\to\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{(\lambda)_k(\lambda)_{n-k}}{(2\lambda)_n} e^{i(n-2k)\theta} \\ &= \frac{e^{in\theta}}{2^n} \sum_{k=0}^{n} \binom{n}{k} e^{-2ik\theta} \\ &= \frac{e^{in\theta}\left(1+e^{-2i\theta}\right)^n}{2^n} \\ &= \cos^n\theta. \end{aligned} \tag{3.10}$$

Hence

$$\lim_{\lambda\to\infty} \frac{C_n^\lambda(x)}{C_n^\lambda(1)} = x^n. \tag{3.11}$$

We have now established that the building blocks of Fourier and of power series are limiting cases of ultraspherical polynomials. We determine the desired expansions in terms of these polynomials, well known to be orthogonal. Their orthogonality relation is given by

$$\int_{-1}^{1} C_n^\lambda(x) C_m^\lambda(x)(1-x^2)^{\lambda-1/2}\,dx = \frac{\lambda}{(n+\lambda)}\frac{(2\lambda)_n}{n!}A_\lambda\,\delta(m,n) \tag{3.12}$$

where

$$\begin{aligned} A_\lambda &= \int_{-1}^{1}(1-x^2)^{\lambda-1/2}\,dx \\ &= 2^{2\lambda}\frac{\Gamma(\lambda+1/2)^2}{\Gamma(2\lambda+1)} \end{aligned} \tag{3.13}$$

and $\delta(m,n)$ vanishes when $m \neq n$ and is 1 when $m = n$.

We will need also the familiar *Rodrigues formula*, which for the ultraspherical polynomials is given by

$$(1-x^2)^{\lambda-1/2} C_n^\lambda(x) = \frac{(-1)^n(2\lambda)_n}{2^n(\lambda+1/2)_n n!}\frac{d^n}{dx^n}(1-x^2)^{n+\lambda-1/2}. \tag{3.14}$$

Further, an extension of the familiar trigonometric addition formula, established in [3] by Gegenbauer and given in integral form as

$$\begin{aligned} &C_n^\lambda(\cos\theta) C_n^\lambda(\cos\phi) \\ &= \frac{C_n^\lambda(1)}{A_{\lambda-1/2}}\int_0^\pi C_n^\lambda(\cos\theta\cos\phi + \sin\theta\sin\phi\cos\psi)(\sin\psi)^{2\lambda-1}\,d\psi, \end{aligned} \tag{3.15}$$

reduces to

$$\cos n\theta \cos n\phi = [\cos n(\theta+\phi) + \cos n(\theta-\phi)]/2 \tag{3.16}$$

when $\lambda \to 0$, and becomes

$$(\cos\theta)^n(\cos\phi)^n = (\cos\theta\cos\phi)^n \tag{3.17}$$

when $\lambda \to \infty$.

4. A CONNECTION BETWEEN FOURIER AND POWER SERIES. We now turn to the orthogonal expansion in terms of ultraspherical polynomials that will provide the connection we seek between Fourier and power series. It is given by

$$f(x) = \sum_{n=0}^{\infty} a_n^{(\lambda)}\frac{(n+\lambda)}{\lambda}C_n^\lambda(x), \tag{4.1}$$

with $a_n^{(\lambda)}$ determined by

$$a_n^{(\lambda)} = \frac{1}{A_\lambda}\int_{-1}^{1} f(t)\frac{C_n^\lambda(t)}{C_n^\lambda(1)}(1-t^2)^{\lambda-1/2}\,dt \tag{4.2}$$

and A_λ defined in (3.13). Now, if we substitute $\cos\theta$ for x in (4.1) and $\cos\phi$ for t in (4.2), let $\lambda \to 0$, and set $a_n = \lim_{\lambda\to 0} a_n^{(\lambda)}$, we have, using (3.8) and (3.7), respectively,

$$f(\cos\theta) = a_0 + 2\sum_{n=1}^{\infty} a_n \cos n\theta, \tag{4.3}$$

where

$$a_n = \frac{1}{\pi}\int_0^{\pi} f(\cos\phi)\cos n\phi \, d\phi. \tag{4.4}$$

The series (4.3) is a result of interchanging the limit as $\lambda \to 0$ with the summation in (4.1), and the integral (4.4) is a consequence of permuting that limit with the integral in (4.2), both justifiable.

We now have shown that the Fourier series of even functions are contained in the ultraspherical expansion (4.1) when $\lambda \to 0$.

To see how power series arise, we first change the integral representation (4.2) into one that involves the nth derivative of $f(x)$. Assuming that f is infinitely differentiable and applying the Rodrigues formula (3.14) to (4.2), we have, after n integrations by parts,

$$a_n = \frac{1}{A_\lambda 2^n(\lambda+1)_n}\int_{-1}^{1} f^{(n)}(x)(1-x^2)^{n+\lambda-1/2}\, dx. \tag{4.5}$$

It follows that (4.1) becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{n+\lambda}{\lambda}\frac{(2\lambda)_n}{n!}\frac{1}{2^n(\lambda+1)_n}\frac{\int_{-1}^{1} f^{(n)}(t)(1-t^2)^{n+\lambda-1/2}\, dt}{\int_{-1}^{1}(1-t^2)^{n+\lambda-1/2}\, dt}\frac{C_n^\lambda(x)}{C_n^\lambda(1)}. \tag{4.6}$$

When $\lambda \to \infty$, and again the limit can be moved within the summation, (4.6) becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \tag{4.7}$$

since

$$\frac{C_n^\lambda(x)}{C_n^\lambda(1)} \to x^n \tag{4.8}$$

by (3.11),

$$\lim_{\lambda\to\infty} \frac{n+\lambda}{\lambda}\frac{(2\lambda)_n}{2^n(\lambda+1)_n} = 1 \tag{4.9}$$

and the measure

$$\frac{(1-x^2)^{n+\lambda-1/2}}{\int_{-1}^{1}(1-t^2)^{n+\lambda-1/2}\, dt}dx \tag{4.10}$$

is an approximation to the delta function. The latter results from the fact that it has mass one on $[-1, 1]$, and its distribution becomes singular when $\lambda \to \infty$, with all of its mass concentrated at $x = 0$.

By showing that the power series also comes from ultraspherical expansions, in this case when $\lambda \to \infty$, we have established our result using the set of ultraspherical polynomials to bridge the gap between Fourier and power series.

5. SMOOTHNESS CONDITIONS AND CONVERGENCE. Shenitzer remarked that Fourier series converge when the function represented has a little smoothness, but infinite differentiability is not sufficient to ensure the convergence of a power series to the function expanded. Let us examine the convergence of ultraspherical series to get a clearer understanding of the reason for this. Without including full details, we outline an idea introduced by Lebesgue for finding sufficient conditions for the uniform convergence of the ultraspherical series.

We begin with the partial sums

$$\begin{aligned} S_n(f;x) &= \sum_{k=0}^{n} a_k \frac{(k+\lambda)}{\lambda} C_k^\lambda(x) \\ &= \frac{1}{A_\lambda}\int_{-1}^{1} f(y) \sum_{k=0}^{n} \frac{(k+\lambda)}{\lambda} \frac{k!}{(2\lambda)_k} C_k^\lambda(x) C_k^\lambda(y)(1-y^2)^{\lambda-1/2}\,dy. \end{aligned}$$

Since, for an arbitrary polynomial $p_k(x)$ of degree k,

$$S_n(p_k;x) = p_k(x), \qquad n \ge k,$$

we note that

$$\begin{aligned} |f(x) - S_n(f;x)| &= |f(x) - p_k(x) - S_n(f-p_k;x)| \\ &\le |f(x) - p_k(x)| + |S_n(f-p_k;x)|. \end{aligned}$$

We have, however, that

$$|S_n(f-p_k;x)| \le \max_{-1\le x\le 1} |f(x) - p_k(x)| \max_{-1\le x\le 1} \rho_n^\lambda(x), \tag{5.1}$$

where the $\rho_n^\lambda(x)$ are defined by

$$\rho_n^\lambda(x) = \frac{1}{A_\lambda}\int_{-1}^{1}\left|\sum_{k=0}^{n} \frac{(k+\lambda)}{\lambda}\frac{k!}{(2\lambda)_k} C_k^\lambda(x)C_k^\lambda(y)\right|(1-y^2)^{\lambda-1/2}\,dy. \tag{5.2}$$

To establish our result, we first show that

$$\rho_n^\lambda(x) \le \rho_n^\lambda(1). \tag{5.3}$$

To this end, if we let $x = \cos\theta$, $y = \cos\phi$, and $z = \cos\theta\cos\phi + \sin\theta\sin\phi\cos\psi$, we can rewrite (3.15) in the form

$$C_n^\lambda(x)C_n^\lambda(y) = C_n^\lambda(1)\int_{-1}^{1} K_\lambda(x,y,z)C_n^\lambda(z)(1-z^2)^{\lambda-1/2}\,dz, \tag{5.4}$$

where

$$\begin{aligned} K_\lambda(x,y,z) &= \frac{1}{A_{\lambda-1/2}}(1-x^2-y^2-z^2+2xyz)^{\lambda-1} \\ &\quad \times\left[(1-x^2)(1-y^2)(1-z^2)\right]^{1/2-\lambda}, \end{aligned} \tag{5.5}$$

with $K_\lambda(x, y, z) = 0$ when $z < xy - (1-x^2)^{1/2}(1-y^2)^{1/2}$ or $z > xy + (1-x^2)^{1/2}(1-y^2)^{1/2}$.

We note from (5.4) that the kernel, $K_\lambda(x,y,z)$, may be represented by the series

$$K_\lambda(x,y,z) = \sum_{k=0}^{\infty} \frac{(k+\lambda)}{\lambda} C_k^\lambda(x)C_k^\lambda(y)C_k^\lambda(z)/C_k^\lambda(1). \tag{5.6}$$

It is non-negative, as is clear from (5.5), and

$$\int_{-1}^{1} K_\lambda(x,y,z)(1-z^2)^{\lambda-1/2}\,dz = 1, \tag{5.7}$$

as follows readily from (5.4) when $n = 0$.

Now, using (5.2), we establish (5.3). This follows from the fact that we have

$$\begin{aligned}
\rho_n^\lambda(x) &= \frac{1}{A_\lambda}\int_{-1}^{1}\left|\sum_{k=0}^{n}\frac{(k+\lambda)}{\lambda}\frac{k!}{(2\lambda)_k}C_k^\lambda(x)C_k^\lambda(y)\right|(1-y^2)^{\lambda-1/2}\,dy \\
&= \frac{1}{A_\lambda}\int_{-1}^{1}\left|\sum_{k=0}^{n}\frac{(k+\lambda)}{\lambda}\int_{-1}^{1}K_\lambda(x,y,z)C_k^\lambda(z)(1-z^2)^{\lambda-1/2}\,dz\right|(1-y^2)^{\lambda-1/2}\,dy \\
&\le \frac{1}{A_\lambda}\int_{-1}^{1}\left|\sum_{k=0}^{n}\frac{(k+\lambda)}{\lambda}C_k^\lambda(z)(1-z^2)^{\lambda-1/2}\right|\int_{-1}^{1}K_\lambda(x,y,z)(1-y^2)^{\lambda-1/2}\,dy\,dz \\
&= \frac{1}{A_\lambda}\int_{-1}^{1}\left|\sum_{k=0}^{n}\frac{(k+\lambda)}{\lambda}C_k^\lambda(z)\right|(1-z^2)^{\lambda-1/2}\,dz \\
&= \rho_n^\lambda(1).
\end{aligned}$$

Let us now use the estimate of the size of $\rho_n^\lambda(1)$ determined by H. Rau in [5]. He showed that, for each $\lambda > 0$, there is a positive constant B_λ such that

$$\rho_n^\lambda(1) = B_\lambda n^\lambda + o(n^\lambda). \tag{5.8}$$

It is a classical result that when $\lambda = 0$, the right-hand side of (5.8) becomes a constant times $\log n$ for the *Fourier series*.

Now, appealing to the inequality (5.1) as well as that immediately preceding it, and taking note of the bound for $\rho_n^\lambda(x)$ and Rau's result (5.8), we have, for $\lambda > 0$,

$$\begin{aligned}
|f(x) - S_n(f;x)| &\le |f(x) - p_k(x)| + \max_{-1\le t\le 1}|f(t) - p_k(t)| \max_{-1\le x\le 1}\rho_n^\lambda(t) \\
&\le An^\lambda \max_{-1\le t\le 1}|f(t) - p_k(t)|,
\end{aligned} \tag{5.9}$$

for all $k \ge n$. The right-hand side of (5.9) can be estimated from classical results on polynomials and trigonometric polynomials. This can most readily be accomplished by taking $x = \cos\theta$ and asking what smoothness conditions on a function on the unit circle will result in

$$|f(\cos\theta) - p_k(\cos\theta)| = o(n^{-\lambda}). \tag{5.10}$$

If $g(\theta) = f(\cos\theta)$ has $[\lambda] = j$ continuous derivatives, and if the jth derivative of $g(\theta)$ satisfies

$$\sup_{0\le\theta\le\pi}|g^{(j)}(\theta+\phi) - g^{(j)}(\theta)| = o(|\phi|^\alpha), \qquad 0 \le \alpha \le 1, \tag{5.11}$$

then

$$|f(\cos\theta) - p_k(\cos\theta)| = o(n^{-j-\alpha}), \qquad 0 \le \theta \le \pi. \tag{5.12}$$

Now take $\alpha = \lambda - [\lambda]$. When $0 < \alpha < 1$, if (5.11) is satisfied, then $f(\cos\theta) = g(\theta)$ has j continuous derivatives and (5.10) holds. For $\alpha = 0$, a slightly weaker condition is satisfied. See Zygmund [7, Chapter 3, section 13].

The results for uniform convergence are best possible. Thus, the degree of smoothness, which implies uniform convergence, increases as λ does.

6. THE JACOBI POLYNOMIALS.

All the above results may be extended by considering a more general set of orthogonal polynomials, the Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$, defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left[(1-x)^{n+\alpha}(1+x)^{n+\beta}\right], \quad (6.1)$$

which include the $C_n^{\lambda}(x)$ as the special case where $\alpha = \beta = \lambda - 1/2 > -1$. These polynomials satisfy slightly more complicated versions of all the formulas developed for the ultraspherical polynomials. For these polynomials, we have, for example, that

$$\frac{P_n^{(-1/2,-1/2)}(\cos\theta)}{P_n^{(-1/2,-1/2)}(1)} = \cos n\theta, \quad (6.2)$$

so that the even Fourier series occur directly and not as limiting cases. Further, we can also find sufficient conditions for the uniform convergence of the Jacobi series, establishing results analogous to those we have for the ultraspherical case. For this, we need an extension of the Gegenbauer formula for Jacobi polynomials, found by G. Gasper in **[2]**.

The arguments we developed above were largely known, the basic facts about properties of orthogonal polynomials having been discovered well before the 20th century. It is not uncommon in mathematics to have important results overlooked or forgotten over the years. The formulation of Shenitzer's question suggests that the close connection between Fourier and power series, provided by ultraspherical and Jacobi series, is not generally known now. Some classical analysts, on the other hand, including for example Szegő, **[6]**, were well aware of this useful relationship, and it has served as a source of interesting problems throughout the decades.

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