CHAPTER 8

SECOND-ORDER CIRCUITS

"Engineering is not only a learned profession, it is also a learning profession, one whose practitioners first become and then remain students throughout their active careers."

—William L. Everitt

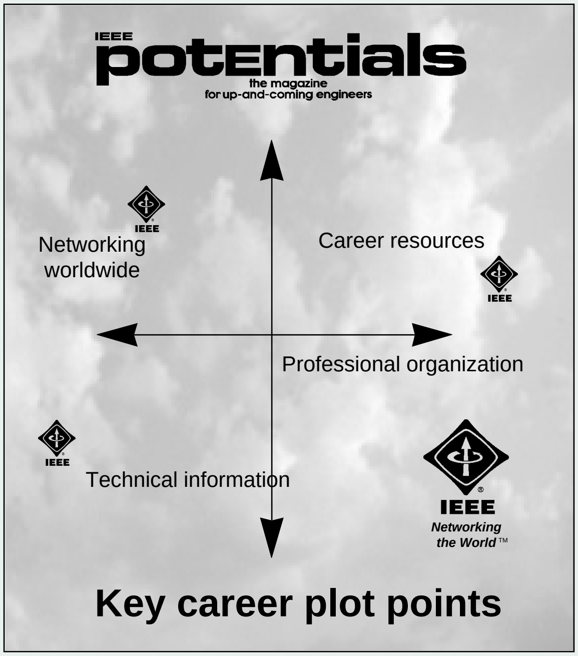
Enhancing Your Career

To increase your engineering career opportunities after graduation, develop a strong fundamental understanding in a broad set of engineering areas. When possible, this might best be accomplished by working toward a graduate degree immediately upon receiving your undergraduate degree.

Each degree in engineering represents certain skills the students acquire. At the Bachelor degree level, you learn the language of engineering and the fundamentals of engineering and design. At the Master's level, you acquire the ability to do advanced engineering projects and to communicate your work effectively both orally and in writing. The Ph.D. represents a thorough understanding of the fundamentals of electrical engineering and a mastery of the skills necessary both for working at the frontiers of an engineering area and for communicating one's effort to others.

If you have no idea what career you should pursue after graduation, a graduate degree program will enhance your ability to explore career options. Since your undergraduate degree will only provide you with the fundamentals of engineering, a Master's degree in engineering supplemented by business courses benefits more engineering students than does getting a Master's of Business Administration (MBA). The best time to get your MBA is after you have been a practicing engineer for some years and decide your career path would be enhanced by strengthening your business skills.

Engineers should constantly educate themselves, formally and informally, taking advantage of all means of education. Perhaps there is no better way to enhance your career than to join a professional society such as IEEE and be an active member.



Enhancing your career involves understanding your goals, adapting to changes, anticipating opportunities, and planning your own niche. (Courtesy of IEEE.)

8.1 INTRODUCTION

In the previous chapter we considered circuits with a single storage element (a capacitor or an inductor). Such circuits are first-order because the differential equations describing them are first-order. In this chapter we will consider circuits containing two storage elements. These are known as *second-order* circuits because their responses are described by differential equations that contain second derivatives.

Typical examples of second-order circuits are RLC circuits, in which the three kinds of passive elements are present. Examples of such circuits are shown in Fig. 8.1(a) and (b). Other examples are RC and RL circuits, as shown in Fig. 8.1(c) and (d). It is apparent from Fig. 8.1 that a second-order circuit may have two storage elements of different type or the same type (provided elements of the same type cannot be represented by an equivalent single element). An op amp circuit with two storage elements may also be a second-order circuit. As with first-order circuits, a second-order circuit may contain several resistors and dependent and independent sources.

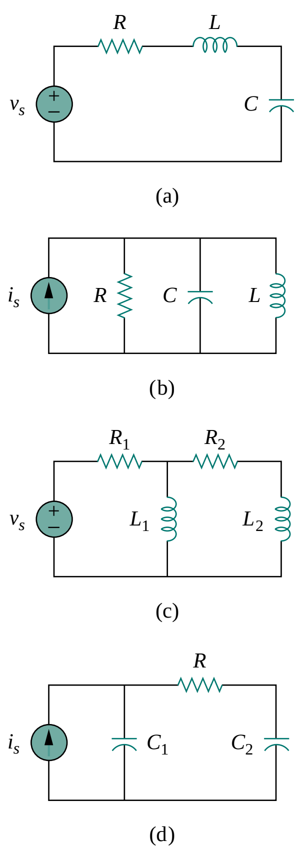


Figure 8.1 Typical examples of second-order circuits: (a) series RLC circuit, (b) parallel RLC circuit, (c) RL circuit, (d) RC circuit.

> A second-order circuit is characterized by a second-order differential equation. It consists of resistors and the equivalent of two energy storage elements.

Our analysis of second-order circuits will be similar to that used for first-order. We will first consider circuits that are excited by the initial conditions of the storage elements. Although these circuits may contain dependent sources, they are free of independent sources. These source-free circuits will give natural responses as expected. Later we will consider circuits that are excited by independent sources. These circuits will give both the natural response and the forced response. We consider only dc independent sources in this chapter. The case of sinusoidal and exponential sources is deferred to later chapters.

We begin by learning how to obtain the initial conditions for the circuit variables and their derivatives, as this is crucial to analyzing second-order circuits. Then we consider series and parallel RLC circuits such as shown in Fig. 8.1 for the two cases of excitation: by initial conditions of the energy storage elements and by step inputs. Later we examine other types of second-order circuits, including op amp circuits. We will consider *PSpice* analysis of second-order circuits. Finally, we will consider the automobile ignition system and smoothing circuits as typical applications of the circuits treated in this chapter. Other applications such as resonant circuits and filters will be covered in Chapter 14.

8.2 FINDING INITIAL AND FINAL VALUES

Perhaps the major problem students face in handling second-order circuits is finding the initial and final conditions on circuit variables. Students are usually comfortable getting the initial and final values of v and i but often have difficulty finding the initial values of their derivatives: dv/dt and di/dt. For this reason, this section is explicitly devoted to the subtleties of getting $v(0)$, $i(0)$, $dv(0)/dt$, $di(0)/dt$, $i(\infty)$, and $v(\infty)$. Unless otherwise stated in this chapter, v denotes capacitor voltage, while i is the inductor current.

There are two key points to keep in mind in determining the initial conditions.

First—as always in circuit analysis—we must carefully handle the polarity of voltage $v(t)$ across the capacitor and the direction of the current $i(t)$ through the inductor. Keep in mind that v and i are defined strictly according to the passive sign convention (see Figs. 6.3 and 6.23). One should carefully observe how these are defined and apply them accordingly.

Second, keep in mind that the capacitor voltage is always continuous so that

$$v(0^+) = v(0^-) \tag{8.1a}$$

and the inductor current is always continuous so that

$$i(0^+) = i(0^-) \tag{8.1b}$$

where $t = 0^-$ denotes the time just before a switching event and $t = 0^+$ is the time just after the switching event, assuming that the switching event takes place at $t = 0$.

Thus, in finding initial conditions, we first focus on those variables that cannot change abruptly, capacitor voltage and inductor current, by applying Eq. (8.1). The following examples illustrate these ideas.

EXAMPLE 8.1

The switch in Fig. 8.2 has been closed for a long time. It is open at $t = 0$. Find: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

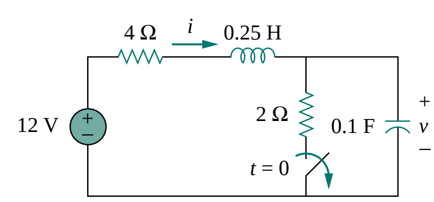


Figure 8.2 For Example 8.1.

Solution:

(a) If the switch is closed a long time before $t = 0$, it means that the circuit has reached dc steady state at $t = 0$. At dc steady state, the inductor acts like a short circuit, while the capacitor acts like an open circuit, so we have the circuit in Fig. 8.3(a) at $t = 0^-$. Thus,

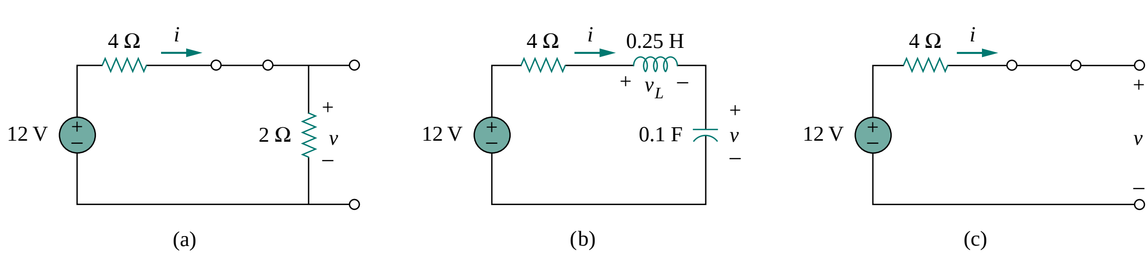


Figure 8.3 Equivalent circuit of that in Fig. 8.2 for: (a) $t = 0^-$, (b) $t = 0^+$, (c) $t \to \infty$.

$$i(0^-) = \frac{12}{4+2} = 2\text{ A}, \qquad v(0^-) = 2i(0^-) = 4\text{ V}$$

As the inductor current and the capacitor voltage cannot change abruptly,

$$i(0^+) = i(0^-) = 2\text{ A}, \qquad v(0^+) = v(0^-) = 4\text{ V}$$

(b) At $t = 0^+$, the switch is open; the equivalent circuit is as shown in Fig. 8.3(b). The same current flows through both the inductor and capacitor. Hence,

$$i_C(0^+) = i(0^+) = 2\text{ A}$$

Since $C\,dv/dt = i_C$, $dv/dt = i_C/C$, and

$$\frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{2}{0.1} = 20\text{ V/s}$$

Similarly, since $L\,di/dt = v_L$, $di/dt = v_L/L$. We now obtain v_L by applying KVL to the loop in Fig. 8.3(b). The result is

$$-12 + 4i(0^+) + v_L(0^+) + v(0^+) = 0$$

or

$$v_L(0^+) = 12 - 8 - 4 = 0$$

Thus,

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = \frac{0}{0.25} = 0\text{ A/s}$$

(c) For $t > 0$, the circuit undergoes transience. But as $t \to \infty$, the circuit reaches steady state again. The inductor acts like a short circuit and the capacitor like an open circuit, so that the circuit becomes that shown in Fig. 8.3(c), from which we have

$$i(\infty) = 0\text{ A}, \qquad v(\infty) = 12\text{ V}$$

PRACTICE PROBLEM 8.1

The switch in Fig. 8.4 was open for a long time but closed at $t = 0$. Determine: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

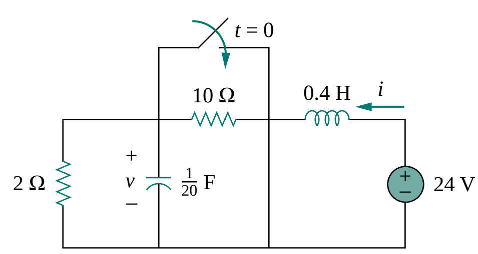


Figure 8.4 For Practice Prob. 8.1.

Answer: (a) 2 A, 4 V, (b) 50 A/s, 0 V/s, (c) 12 A, 24 V.

EXAMPLE 8.2

In the circuit of Fig. 8.5, calculate: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$, (b) $di_L(0^+)dt$, $dv_C(0^+)/dt$, $dv_R(0^+)/dt$, (c) $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$.

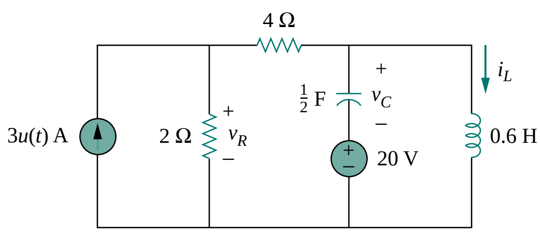


Figure 8.5 For Example 8.2.

Solution:

(a) For $t < 0$, $3u(t) = 0$. At $t = 0^-$, since the circuit has reached steady state, the inductor can be replaced by a short circuit, while the capacitor is replaced by an open circuit as shown in Fig. 8.6(a). From this figure we obtain

$$i_L(0^-) = 0, \qquad v_R(0^-) = 0, \qquad v_C(0^-) = -20 \text{ V} \tag{8.2.1}$$

Although the derivatives of these quantities at $t = 0^-$ are not required, it is evident that they are all zero, since the circuit has reached steady state and nothing changes.

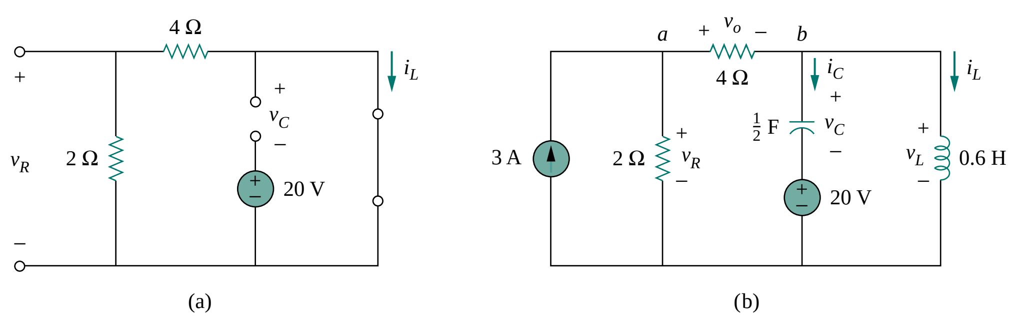


Figure 8.6 The circuit in Fig. 8.5 for: (a) $t = 0^-$, (b) $t = 0^+$.

For $t > 0$, $3u(t) = 3$, so that the circuit is now equivalent to that in Fig. 8.6(b). Since the inductor current and capacitor voltage cannot change abruptly,

$$i_L(0^+) = i_L(0^-) = 0, \qquad v_C(0^+) = v_C(0^-) = -20 \text{ V} \tag{8.2.2}$$

Although the voltage across the 4-Ω resistor is not required, we will use it to apply KVL and KCL; let it be called v_o. Applying KCL at node a in Fig. 8.6(b) gives

$$3 = \frac{v_R(0^+)}{2} + \frac{v_o(0^+)}{4} \tag{8.2.3}$$

Applying KVL to the middle mesh in Fig. 8.6(b) yields

$$-v_R(0^+) + v_o(0^+) + v_C(0^+) + 20 = 0 \tag{8.2.4}$$

Since $v_C(0+) = -20$ V from Eq. (8.2.2), Eq. (8.2.4) implies that

$$v_R(0^+) = v_o(0^+) \tag{8.2.5}$$

From Eqs. (8.2.3) and (8.2.5), we obtain

$$v_R(0^+) = v_o(0^+) = 4 \text{ V} \tag{8.2.6}$$

(b) Since $L\, di_L/dt = v_L$,

$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L}$$

But applying KVL to the right mesh in Fig. 8.6(b) gives

$$v_L(0^+) = v_C(0^+) + 20 = 0$$

Hence,

$$\frac{di_L(0^+)}{dt} = 0 \tag{8.2.7}$$

Similarly, since $C\, dv_C/dt = i_C$, then $dv_C/dt = i_C/C$. We apply KCL at node b in Fig. 8.6(b) to get i_C:

$$\frac{v_o(0^+)}{4} = i_C(0^+) + i_L(0^+) \tag{8.2.8}$$

Since $v_o(0^+) = 4$ and $i_L(0^+) = 0$, $i_C(0^+) = 4/4 = 1$ A. Then

$$\frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = \frac{1}{0.5} = 2 \text{ V/s} \tag{8.2.9}$$

To get $dv_R(0^+)/dt$, we apply KCL to node a and obtain

$$3 = \frac{v_R}{2} + \frac{v_o}{4}$$

Taking the derivative of each term and setting $t = 0^+$ gives

$$0 = 2\frac{dv_R(0^+)}{dt} + \frac{dv_o(0^+)}{dt} \tag{8.2.10}$$

We also apply KVL to the middle mesh in Fig. 8.6(b) and obtain

$$-v_R + v_C + 20 + v_o = 0$$

Again, taking the derivative of each term and setting $t = 0^+$ yields

$$-\frac{dv_R(0^+)}{dt} + \frac{dv_C(0^+)}{dt} + \frac{dv_o(0^+)}{dt} = 0$$

Substituting for $dv_C(0^+)/dt = 2$ gives

$$\frac{dv_R(0^+)}{dt} = 2 + \frac{dv_o(0^+)}{dt} \tag{8.2.11}$$

From Eqs. (8.2.10) and (8.2.11), we get

$$\frac{dv_R(0^+)}{dt} = \frac{2}{3} \text{ V/s}$$

We can find $di_R(0^+)/dt$ although it is not required. Since $v_R = 5i_R$,

$$\frac{di_R(0^+)}{dt} = \frac{1}{5}\frac{dv_R(0^+)}{dt} = \frac{1}{5}\frac{2}{3} = \frac{2}{15} \text{ A/s}$$

(c) As $t \to \infty$, the circuit reaches steady state. We have the equivalent circuit in Fig. 8.6(a) except that the 3-A current source is now operative. By current division principle,

$$\begin{aligned} i_L(\infty) &= \frac{2}{2+4} 3 \text{ A} = 1 \text{ A} \\ v_R(\infty) &= \frac{4}{2+4} 3 \text{ A} \times 2 = 4 \text{ V}, \qquad v_C(\infty) = -20 \text{ V} \end{aligned} \tag{8.2.12}$$

PRACTICE PROBLEM 8.2

For the circuit in Fig. 8.7, find: (a) $i_L(0^+)$, $v_C(0^+)$, $v_R(0^+)$, (b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, $dv_R(0^+)/dt$, (c) $i_L(\infty)$, $v_C(\infty)$, $v_R(\infty)$.

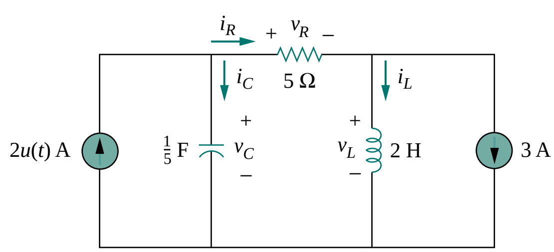


Figure 8.7 For Practice Prob. 8.2.

Answer: (a) −3 A, 0, 0, (b) 0, 10 V/s, 0, (c) −1 A, 10 V, 10 V.

8.3 THE SOURCE-FREE SERIES *RLC* CIRCUIT

Electronic Testing Tutorials

An understanding of the natural response of the series RLC circuit is a necessary background for future studies in filter design and communications networks.

Consider the series RLC circuit shown in Fig. 8.8. The circuit is being excited by the energy initially stored in the capacitor and inductor. The energy is represented by the initial capacitor voltage V_0 and initial inductor current I_0. Thus, at $t = 0$,

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i \, dt = V_0 \tag{8.2a}$$

$$i(0) = I_0 \tag{8.2b}$$

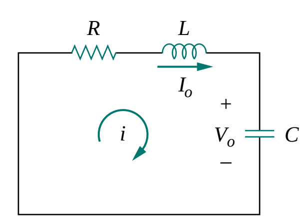


Figure 8.8 A source-free series RLC circuit.

Applying KVL around the loop in Fig. 8.8,

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i \, dt = 0 \tag{8.3}$$

To eliminate the integral, we differentiate with respect to t and rearrange terms. We get

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \tag{8.4}$$

This is a *second-order differential equation* and is the reason for calling the RLC circuits in this chapter second-order circuits. Our goal is to solve Eq. (8.4). To solve such a second-order differential equation requires that we have two initial conditions, such as the initial value of i and its first derivative or initial values of some i and v. The initial value of i is given in Eq. (8.2b). We get the initial value of the derivative of i from Eqs. (8.2a) and (8.3); that is,

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

or

$$\frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0) \tag{8.5}$$

With the two initial conditions in Eqs. (8.2b) and (8.5), we can now solve Eq. (8.4). Our experience in the preceding chapter on first-order circuits suggests that the solution is of exponential form. So we let

$$i = Ae^{st} \tag{8.6}$$

where A and s are constants to be determined. Substituting Eq. (8.6) into Eq. (8.4) and carrying out the necessary differentiations, we obtain

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

or

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0 \tag{8.7}$$

Since $i = Ae^{st}$ is the assumed solution we are trying to find, only the expression in parentheses can be zero:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \tag{8.8}$$

This quadratic equation is known as the *characteristic equation* of the differential Eq. (8.4), since the roots of the equation dictate the character of i. The two roots of Eq. (8.8) are

See Appendix C.1 for the formula to find the roots of a quadratic equation.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{8.9a}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{8.9b}$$

A more compact way of expressing the roots is

$$\boxed{s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}} \tag{8.10}$$

where

$$\boxed{\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}} \tag{8.11}$$

The roots s_1 and s_2 are called *natural frequencies*, measured in nepers per second (Np/s), because they are associated with the natural response of the circuit; ω_0 is known as the *resonant frequency* or strictly as the *undamped natural frequency*, expressed in radians per second (rad/s); and α is the *neper frequency* or the *damping factor*, expressed in nepers per second. In terms of α and ω_0, Eq. (8.8) can be written as

The *neper* (Np) is a dimensionless unit named after John Napier (1550–1617), a Scottish mathematician.

$$s^2 + 2\alpha s + \omega_0^2 = 0 \tag{8.8a}$$

The variables s and ω are important quantities we will be discussing throughout the rest of the text.

The ratio α/ω_0 is known as the *damping ratio* ζ.

The two values of s in Eq. (8.10) indicate that there are two possible solutions for i, each of which is of the form of the assumed solution in Eq. (8.6); that is,

$$i_1 = A_1 e^{s_1 t}, \qquad i_2 = A_2 e^{s_2 t} \tag{8.12}$$

Since Eq. (8.4) is a linear equation, any linear combination of the two distinct solutions i_1 and i_2 is also a solution of Eq. (8.4). A complete or total solution of Eq. (8.4) would therefore require a linear combination of i_1 and i_2. Thus, the natural response of the series RLC circuit is

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \tag{8.13}$$

where the constants A_1 and A_2 are determined from the initial values $i(0)$ and $di(0)/dt$ in Eqs. (8.2b) and (8.5).

From Eq. (8.10), we can infer that there are three types of solutions:

1. If $\alpha > \omega_0$, we have the *overdamped* case.
2. If $\alpha = \omega_0$, we have the *critically damped* case.
3. If $\alpha < \omega_0$, we have the *underdamped* case.

We will consider each of these cases separately.

The response is *overdamped* when the roots of the circuit's characteristic equation are unequal and real, *critically damped* when the roots are equal and real, and *underdamped* when the roots are complex.

Overdamped Case ($\alpha > \omega_0$)

From Eqs. (8.9) and (8.10), $\alpha > \omega_0$ when $C > 4L/R^2$. When this happens, both roots s_1 and s_2 are negative and real. The response is

$$\boxed{i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \tag{8.14}$$

which decays and approaches zero as t increases. Figure 8.9(a) illustrates a typical overdamped response.

Critically Damped Case ($\alpha = \omega_0$)

When $\alpha = \omega_0$, $C = 4L/R^2$ and

$$s_1 = s_2 = -\alpha = -\frac{R}{2L} \tag{8.15}$$

For this case, Eq. (8.13) yields

$$i(t) = A_1 e^{-\alpha t} + A_2 e^{-\alpha t} = A_3 e^{-\alpha t}$$

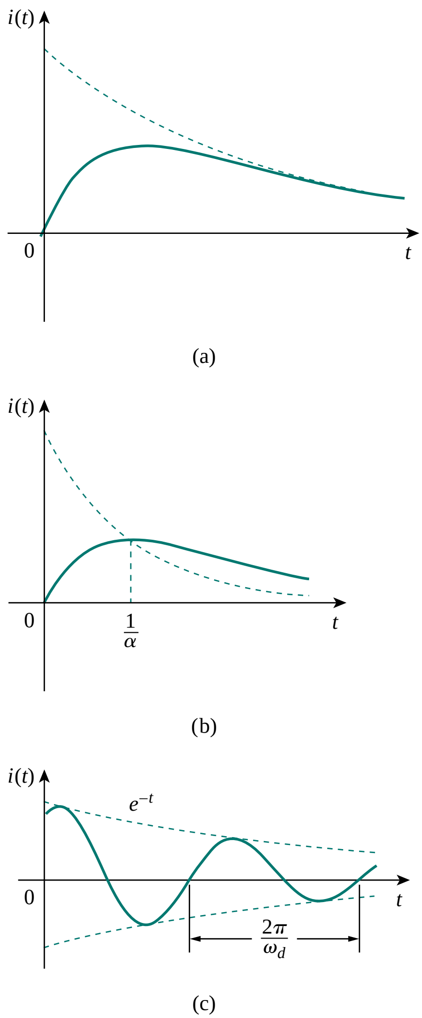


Figure 8.9 (a) Overdamped response, (b) critically damped response, (c) underdamped response.

where $A_3 = A_1 + A_2$. This cannot be the solution, because the two initial conditions cannot be satisfied with the single constant A_3. What then could be wrong? Our assumption of an exponential solution is incorrect for the special case of critical damping. Let us go back to Eq. (8.4). When $\alpha = \omega_0 = R/2L$, Eq. (8.4) becomes

$$\frac{d^2i}{dt^2} + 2\alpha\frac{di}{dt} + \alpha^2 i = 0$$

or

$$\frac{d}{dt}\left(\frac{di}{dt} + \alpha i\right) + \alpha\left(\frac{di}{dt} + \alpha i\right) = 0 \tag{8.16}$$

If we let

$$f = \frac{di}{dt} + \alpha i \tag{8.17}$$

then Eq. (8.16) becomes

$$\frac{df}{dt} + \alpha f = 0$$

which is a first-order differential equation with solution $f = A_1 e^{-\alpha t}$, where A_1 is a constant. Equation (8.17) then becomes

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

or

$$e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = A_1 \tag{8.18}$$

This can be written as

$$\frac{d}{dt}(e^{\alpha t} i) = A_1 \tag{8.19}$$

Integrating both sides yields

$$e^{\alpha t} i = A_1 t + A_2$$

or

$$i = (A_1 t + A_2)e^{\alpha t} \tag{8.20}$$

where A_2 is another constant. Hence, the natural response of the critically damped circuit is a sum of two terms: a negative exponential and a negative exponential multiplied by a linear term, or

$$\boxed{i(t) = (A_2 + A_1 t)e^{-\alpha t}} \tag{8.21}$$

A typical critically damped response is shown in Fig. 8.9(b). In fact, Fig. 8.9(b) is a sketch of $i(t) = te^{-\alpha t}$, which reaches a maximum value of e^{-1}/α at $t = 1/\alpha$, one time constant, and then decays all the way to zero.

Underdamped Case ($\alpha < \omega_0$)

For $\alpha < \omega_0$, $C < 4L/R^2$. The roots may be written as

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d \tag{8.22a}$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d \tag{8.22b}$$

where $j = \sqrt{-1}$ and $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the *damping frequency*. Both ω_0 and ω_d are natural frequencies because they help determine the natural response; while ω_0 is often called the *undamped natural frequency*, ω_d is called the *damped natural frequency*. The natural response is

$$\begin{aligned} i(t) &= A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \\ &= e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}) \end{aligned} \tag{8.23}$$

Using Euler's identities,

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta \tag{8.24}$$

we get

$$\begin{aligned} i(t) &= e^{-\alpha t}\left[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)\right] \\ &= e^{-\alpha t}\left[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t\right] \end{aligned} \tag{8.25}$$

Replacing constants $(A_1 + A_2)$ and $j(A_1 - A_2)$ with constants B_1 and B_2, we write

$$\boxed{i(t) = e^{-\alpha t}(B_1\cos\omega_d t + B_2\sin\omega_d t)} \tag{8.26}$$

With the presence of sine and cosine functions, it is clear that the natural response for this case is exponentially damped and oscillatory in nature. The response has a time constant of $1/\alpha$ and a period of $T = 2\pi/\omega_d$. Figure 8.9(c) depicts a typical underdamped response. [Figure 8.9 assumes for each case that $i(0) = 0$.]

Once the inductor current $i(t)$ is found for the RLC series circuit as shown above, other circuit quantities such as individual element voltages can easily be found. For example, the resistor voltage is $v_R = Ri$, and the inductor voltage is $v_L = L\,di/dt$. The inductor current $i(t)$ is selected as the key variable to be determined first in order to take advantage of Eq. (8.1b).

We conclude this section by noting the following interesting, peculiar properties of an RLC network:

1. The behavior of such a network is captured by the idea of *damping*, which is the gradual loss of the initial stored energy, as evidenced by the continuous decrease in the amplitude of the response. The damping effect is due to the presence of resistance R. The damping factor α determines the rate at which the response is damped. If $R = 0$, then $\alpha = 0$, and we have an LC circuit with $1/\sqrt{LC}$ as the undamped natural frequency. Since $\alpha < \omega_0$ in this case, the response is not only undamped but also oscillatory. The circuit is said to be *lossless*, because the dissipating or damping element (R) is absent. By adjusting the value of R, the response may be made undamped, overdamped, critically damped, or underdamped.
2. Oscillatory response is possible due to the presence of the two types of storage elements. Having both L and C allows the flow of energy back and forth between the two. The damped oscillation exhibited by the underdamped response is known as *ringing*. It stems from the ability of the storage elements L and C to transfer energy back and forth between them.

$R = 0$ produces a perfectly sinusoidal response. This response cannot be practically accomplished with L and C because of the inherent losses in them. See Figs. 6.8 and 6.26. An electronic device called an *oscillator* can produce a perfectly sinusoidal response.

Examples 8.5 and 8.7 demonstrate the effect of varying R.

The response of a second-order circuit with two storage elements of the same type, as in Fig. 8.1(c) and (d), cannot be oscillatory.

3. Observe from Fig. 8.9 that the waveforms of the responses differ. In general, it is difficult to tell from the waveforms the difference between the overdamped and critically damped responses. The critically damped case is the borderline between the underdamped and overdamped cases and it decays the fastest. With the same initial conditions, the overdamped case has the longest settling time, because it takes the longest time to dissipate the initial stored energy. If we desire the fastest response without oscillation or ringing, the critically damped circuit is the right choice.

What this means in most practical circuits is that we seek an overdamped circuit that is as close as possible to a critically damped circuit.

EXAMPLE 8.3

In Fig. 8.8, $R = 40\ \Omega$, $L = 4$ H, and $C = 1/4$ F. Calculate the characteristic roots of the circuit. Is the natural response overdamped, underdamped, or critically damped?

Solution:

We first calculate

$$\alpha = \frac{R}{2L} = \frac{40}{2(4)} = 5, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{4 \times \frac{1}{4}}} = 1$$

The roots are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -5 \pm \sqrt{25 - 1}$$

or

$$s_1 = -0.101, \qquad s_2 = -9.899$$

Since $\alpha > \omega_0$, we conclude that the response is overdamped. This is also evident from the fact that the roots are real and negative.

PRACTICE PROBLEM 8.3

If $R = 10\ \Omega$, $L = 5$ H, and $C = 2$ mF in Fig. 8.8, find α, ω_0, s_1, and s_2. What type of natural response will the circuit have?

Answer: 1, 10, $-1 \pm j9.95$, underdamped.

EXAMPLE 8.4

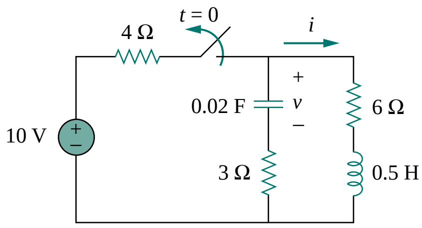


Figure 8.10 For Example 8.4.

Find $i(t)$ in the circuit in Fig. 8.10. Assume that the circuit has reached steady state at $t = 0^-$.

Solution:

For $t < 0$, the switch is closed. The capacitor acts like an open circuit while the inductor acts like a shunted circuit. The equivalent circuit is shown in Fig. 8.11(a). Thus, at $t = 0$,

$$i(0) = \frac{10}{4+6} = 1\ \text{A}, \qquad v(0) = 6i(0) = 6\ \text{V}$$

where $i(0)$ is the initial current through the inductor and $v(0)$ is the initial voltage across the capacitor.

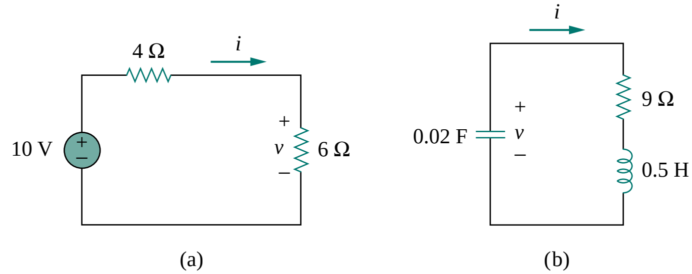


Figure 8.11 The circuit in Fig. 8.10: (a) for $t < 0$, (b) for $t > 0$.

For $t > 0$, the switch is opened and the voltage source is disconnected. The equivalent circuit is shown in Fig. 8.11(b), which is a source-free series RLC circuit. Notice that the 3-Ω and 6-Ω resistors, which are in series in Fig. 8.10 when the switch is opened, have been combined to give $R = 9\ \Omega$ in Fig. 8.11(b). The roots are calculated as follows:

$$\alpha = \frac{R}{2L} = \frac{9}{2\left(\frac{1}{2}\right)} = 9, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times \frac{1}{50}}} = 10$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm \sqrt{81 - 100}$$

or

$$s_{1,2} = -9 \pm j4.359$$

Hence, the response is underdamped ($\alpha < \omega$); that is,

$$i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) \tag{8.4.1}$$

We now obtain A_1 and A_2 using the initial conditions. At $t = 0$,

$$i(0) = 1 = A_1 \tag{8.4.2}$$

From Eq. (8.5),

$$\left.\frac{di}{dt}\right|_{t=0} = -\frac{1}{L}[Ri(0) + v(0)] = -2[9(1) - 6] = -6 \text{ A/s} \tag{8.4.3}$$

Note that $v(0) = V_0 = -6$ V is used, because the polarity of v in Fig. 8.11(b) is opposite that in Fig. 8.8. Taking the derivative of $i(t)$ in Eq. (8.4.1),

$$\frac{di}{dt} = -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t) + e^{-9t}(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359t)$$

Imposing the condition in Eq. (8.4.3) at $t = 0$ gives

$$-6 = -9(A_1 + 0) + 4.359(-0 + A_2)$$

But $A_1 = 1$ from Eq. (8.4.2). Then

$$-6 = -9 + 4.359A_2 \qquad \Longrightarrow \qquad A_2 = 0.6882$$

Substituting the values of A_1 and A_2 in Eq. (8.4.1) yields the complete solution as

$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t) \text{ A}$$

PRACTICE PROBLEM 8.4

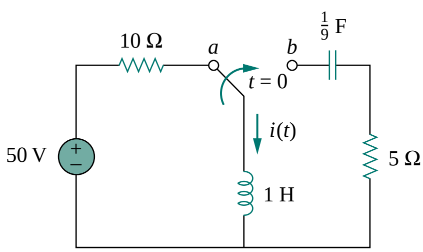


Figure 8.12 For Practice Prob. 8.4.

The circuit in Fig. 8.12 has reached steady state at $t = 0^-$. If the make-before-break switch moves to position b at $t = 0$, calculate $i(t)$ for $t > 0$.

Answer: $e^{-2.5t}(5 \cos 1.6583t - 7.5378 \sin 1.6583t)$ A.

Electronic Testing Tutorials

8.4 THE SOURCE-FREE PARALLEL *RLC* CIRCUIT

Parallel RLC circuits find many practical applications, notably in communications networks and filter designs.

Consider the parallel RLC circuit shown in Fig. 8.13. Assume initial inductor current I_0 and initial capacitor voltage V_0,

$$i(0) = I_0 = \frac{1}{L}\int_{\infty}^{0} v(t)\,dt \tag{8.27a}$$

$$v(0) = V_0 \tag{8.27b}$$

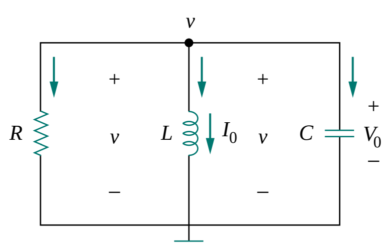


Figure 8.13 A source-free parallel RLC circuit.

Since the three elements are in parallel, they have the same voltage v across them. According to passive sign convention, the current is entering each element; that is, the current through each element is leaving the top node. Thus, applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v\,dt + C\frac{dv}{dt} = 0 \tag{8.28}$$

Taking the derivative with respect to t and dividing by C results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0 \tag{8.29}$$

We obtain the characteristic equation by replacing the first derivative by s and the second derivative by s^2. By following the same reasoning used in establishing Eqs. (8.4) through (8.8), the characteristic equation is obtained as

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0 \tag{8.30}$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

or

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \tag{8.31}$$

where

$$\boxed{\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}} \tag{8.32}$$

The names of these terms remain the same as in the preceding section, as they play the same role in the solution. Again, there are three possible solutions, depending on whether $\alpha > \omega_0$, $\alpha = \omega_0$, or $\alpha < \omega_0$. Let us consider these cases separately.

Overdamped Case ($\alpha > \omega_0$)

From Eq. (8.32), $\alpha > \omega_0$ when $L > 4R^2C$. The roots of the characteristic equation are real and negative. The response is

$$\boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}} \tag{8.33}$$

Critically Damped Case ($\alpha = \omega_0$)

For $\alpha = \omega$, $L = 4R^2C$. The roots are real and equal so that the response is

$$\boxed{v(t) = (A_1 + A_2 t)e^{-\alpha t}} \tag{8.34}$$

Underdamped Case ($\alpha < \omega_0$)

When $\alpha < \omega_0$, $L < 4R^2C$. In this case the roots are complex and may be expressed as

$$s_{1,2} = -\alpha \pm j\omega_d \tag{8.35}$$

where

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \tag{8.36}$$

The response is

$$\boxed{v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)} \tag{8.37}$$

The constants A_1 and A_2 in each case can be determined from the initial conditions. We need $v(0)$ and $dv(0)/dt$. The first term is known from Eq. (8.27b). We find the second term by combining Eqs. (8.27) and (8.28), as

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

or

$$\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC} \tag{8.38}$$

The voltage waveforms are similar to those shown in Fig. 8.9 and will depend on whether the circuit is overdamped, underdamped, or critically damped.

Having found the capacitor voltage $v(t)$ for the parallel RLC circuit as shown above, we can readily obtain other circuit quantities such as individual element currents. For example, the resistor current is $i_R = v/R$ and the capacitor voltage is $v_C = C\,dv/dt$. We have selected the capacitor voltage $v(t)$ as the key variable to be determined first in order to take advantage of Eq. (8.1a). Notice that we first found the inductor current $i(t)$ for the RLC series circuit, whereas we first found the capacitor voltage $v(t)$ for the parallel RLC circuit.

EXAMPLE 8.5

In the parallel circuit of Fig. 8.13, find $v(t)$ for $t > 0$, assuming $v(0) = 5$ V, $i(0) = 0$, $L = 1$ H, and $C = 10$ mF. Consider these cases: $R = 1.923\ \Omega$, $R = 5\ \Omega$, and $R = 6.25\ \Omega$.

Solution:

CASE 1 If $R = 1.923\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 1.923 \times 10 \times 10^{-3}} = 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

Since $\alpha > \omega_0$ in this case, the response is overdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

and the corresponding response is

$$v(t) = A_1 e^{-2t} + A_2 e^{-50t} \tag{8.5.1}$$

We now apply the initial conditions to get A_1 and A_2.

$$v(0) = 5 = A_1 + A_2 \tag{8.5.2}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{1.923 \times 10 \times 10^{-3}} = 260$$

But differentiating Eq. (8.5.1),

$$\frac{dv}{dt} = -2A_1 e^{-2t} - 50A_2 e^{-50t}$$

At $t = 0$,

$$260 = -2A_1 - 50A_2 \tag{8.5.3}$$

From Eqs. (8.5.2) and (8.5.3), we obtain $A_1 = 10.625$ and $A_2 = -5.625$. Substituting A_1 and A_2 in Eq. (8.5.1) yields

$$v(t) = 10.625e^{-2t} - 5.625e^{-50t} \tag{8.5.4}$$

CASE 2 When $R = 5\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 5 \times 10 \times 10^{-3}} = 10$$

while $\omega_0 = 10$ remains the same. Since $\alpha = \omega_0 = 10$, the response is critically damped. Hence, $s_1 = s_2 = -10$, and

$$v(t) = (A_1 + A_2 t)e^{-10t} \tag{8.5.5}$$

To get A_1 and A_2, we apply the initial conditions

$$v(0) = 5 = A_1 \tag{8.5.6}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5+0}{5 \times 10 \times 10^{-3}} = 100$$

But differentiating Eq. (8.5.5),

$$\frac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$$

At $t = 0$,

$$100 = -10A_1 + A_2 \tag{8.5.7}$$

From Eqs. (8.5.6) and (8.5.7), $A_1 = 5$ and $A_2 = 150$. Thus,

$$v(t) = (5 + 150t)e^{-10t} \text{ V} \tag{8.5.8}$$

CASE 3 When $R = 6.25\ \Omega$,

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10 \times 10^{-3}} = 8$$

while $\omega_0 = 10$ remains the same. As $\alpha < \omega_0$ in this case, the response is underdamped. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

Hence,

$$v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t} \tag{8.5.9}$$

We now obtain A_1 and A_2, as

$$v(0) = 5 = A_1 \tag{8.5.10}$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5+0}{6.25 \times 10 \times 10^{-3}} = 80$$

But differentiating Eq. (8.5.9),

$$\frac{dv}{dt} = (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

At $t = 0$,

$$80 = -8A_1 + 6A_2 \tag{8.5.11}$$

From Eqs. (8.5.10) and (8.5.11), $A_1 = 5$ and $A_2 = 20$. Thus,

$$v(t) = (5 \cos 6t + 20 \sin 6t)e^{-8t} \tag{8.5.12}$$

Notice that by increasing the value of R, the degree of damping decreases and the responses differ. Figure 8.14 plots the three cases.

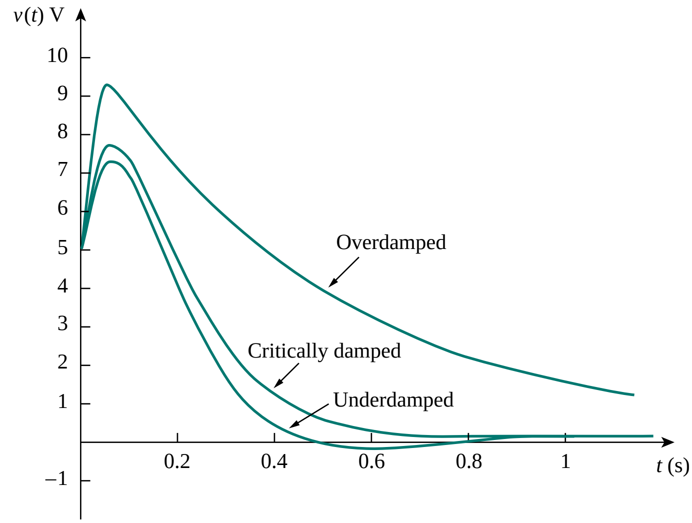


Figure 8.14 For Example 8.5: responses for three degrees of damping.

PRACTICE PROBLEM 8.5

In Fig. 8.13, let $R = 2\ \Omega$, $L = 0.4$ H, $C = 25$ mF, $v(0) = 0$, $i(0) = 3$ A. Find $v(t)$ for $t > 0$.

Answer: $-120te^{-10t}$ V.

EXAMPLE 8.6

Find $v(t)$ for $t > 0$ in the *RLC* circuit of Fig. 8.15.

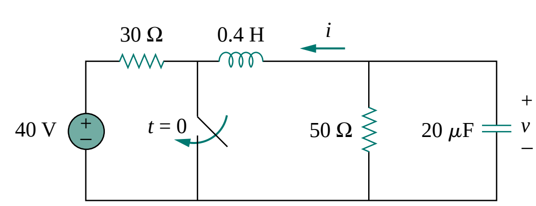


Figure 8.15 For Example 8.6.

Solution:

When $t < 0$, the switch is open; the inductor acts like a short circuit while the capacitor behaves like an open circuit. The initial voltage across the capacitor is the same as the voltage across the 50-Ω resistor; that is,

$$v(0) = \frac{50}{30 + 50}(40) = \frac{5}{8} \times 40 = 25\ \text{V} \tag{8.6.1}$$

The initial current through the inductor is

$$i(0) = -\frac{40}{30+50} = -0.5 \text{ A}$$

The direction of i is as indicated in Fig. 8.15 to conform with the direction of I_0 in Fig. 8.13, which is in agreement with the convention that current flows into the positive terminal of an inductor (see Fig. 6.23). We need to express this in terms of dv/dt, since we are looking for v.

$$\frac{dv(0)}{dt} = -\frac{v(0)+Ri(0)}{RC} = -\frac{25-50\times 0.5}{50\times 20\times 10^{-6}} = 0 \tag{8.6.2}$$

When $t > 0$, the switch is closed. The voltage source along with the 30-Ω resistor is separated from the rest of the circuit. The parallel RLC circuit acts independently of the voltage source, as illustrated in Fig. 8.16. Next, we determine that the roots of the characteristic equation are

$$\alpha = \frac{1}{2RC} = \frac{1}{2\times 50\times 20\times 10^{-6}} = 500$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.4\times 20\times 10^{-6}}} = 354$$

$$\begin{aligned} s_{1,2} &= -\alpha \pm \sqrt{\alpha^2-\omega_0^2} \\ &= -500 \pm \sqrt{250{,}000 - 124{,}997.6} = -500 \pm 354 \end{aligned}$$

or

$$s_1 = -854, \qquad s_2 = -146$$

Since $\alpha > \omega_0$, we have the overdamped response

$$v(t) = A_1 e^{-854t} + A_2 e^{-164t} \tag{8.6.3}$$

At $t = 0$, we impose the condition in Eq. (8.6.1),

$$v(0) = 25 = A_1 + A_2 \qquad \Longrightarrow \qquad A_2 = 25 - A_1 \tag{8.6.4}$$

Taking the derivative of $v(t)$ in Eq. (8.6.3),

$$\frac{dv}{dt} = -854 A_1 e^{-854t} - 164 A_2 e^{-164t}$$

Imposing the condition in Eq. (8.6.2),

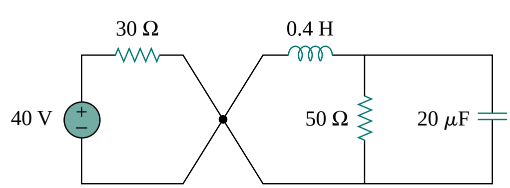


Figure 8.16 The circuit in Fig. 8.15 when $t > 0$. The parallel RLC circuit on the left-hand side acts independently of the circuit on the right-hand side of the junction.

$$\frac{dv(0)}{dt} = 0 = -854A_1 - 164A_2$$

or

$$0 = 854A_1 + 164A_2 \tag{8.6.5}$$

Solving Eqs. (8.6.4) and (8.6.5) gives

$$A_1 = -5.16, \qquad A_2 = 30.16$$

Thus, the complete solution in Eq. (8.6.3) becomes

$$v(t) = -5.16e^{-854t} + 30.16e^{-164t} \text{ V}$$

PRACTICE PROBLEM 8.6

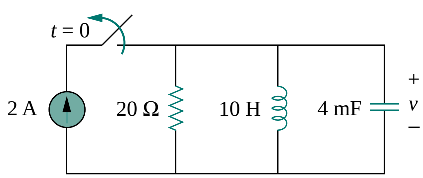


Figure 8.17 For Practice Prob. 8.6.

Refer to the circuit in Fig. 8.17. Find $v(t)$ for $t > 0$.

Answer: $66.67(e^{-10t} - e^{-2.5t})$ V.

Electronic Testing Tutorials

8.5 STEP RESPONSE OF A SERIES *RLC* CIRCUIT

As we learned in the preceding chapter, the step response is obtained by the sudden application of a dc source. Consider the series RLC circuit shown in Fig. 8.18. Applying KVL around the loop for $t > 0$,

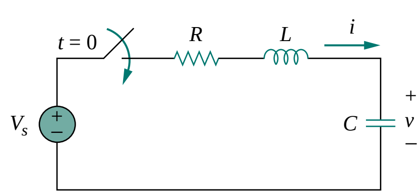


Figure 8.18 Step voltage applied to a series RLC circuit.

$$L\frac{di}{dt} + Ri + v = V_s \tag{8.39}$$

But

$$i = C\frac{dv}{dt}$$

Substituting for i in Eq. (8.39) and rearranging terms,

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC} \tag{8.40}$$

which has the same form as Eq. (8.4). More specifically, the coefficients are the same (and that is important in determining the frequency parameters) but the variable is different. (Likewise, see Eq. (8.47).) Hence, the characteristic equation for the series RLC circuit is not affected by the presence of the dc source.

The solution to Eq. (8.40) has two components: the natural response $v_n(t)$ and the forced response $v_f(t)$; that is,

$$v(t) = v_n(t) + v_f(t) \tag{8.41}$$

The natural response is the solution when we set $V_s = 0$ in Eq. (8.40) and is the same as the one obtained in Section 8.3. The natural response v_n for the overdamped, underdamped, and critically damped cases are:

$$v_n(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Overdamped)} \tag{8.42a}$$

$$v_n(t) = (A_1 + A_2 t)e^{-\alpha t} \quad \text{(Critically damped)} \tag{8.42b}$$

$$v_n(t) = (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \quad \text{(Underdamped)} \tag{8.42c}$$

The forced response is the steady state or final value of $v(t)$. In the circuit in Fig. 8.18, the final value of the capacitor voltage is the same as the source voltage V_s. Hence,

$$v_f(t) = v(\infty) = V_s \tag{8.43}$$

Thus, the complete solutions for the overdamped, underdamped, and critically damped cases are:

$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Overdamped)} \tag{8.44a}$$

$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t} \quad \text{(Critically damped)} \tag{8.44b}$$

$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \quad \text{(Underdamped)} \tag{8.44c}$$

The values of the constants A_1 and A_2 are obtained from the initial conditions: $v(0)$ and $dv(0)/dt$. Keep in mind that v and i are, respectively, the voltage across the capacitor and the current through the inductor. Therefore, Eq. (8.44) only applies for finding v. But once the capacitor voltage $v_C = v$ is known, we can determine $i = C\, dv/dt$, which is the same current through the capacitor, inductor, and resistor. Hence, the voltage across the resistor is $v_R = iR$, while the inductor voltage is $v_L = L\, di/dt$.

Alternatively, the complete response for any variable $x(t)$ can be found directly, because it has the general form

$$x(t) = x_f(t) + x_n(t) \tag{8.45}$$

where the $x_f = x(\infty)$ is the final value and $x_n(t)$ is the natural response. The final value is found as in Section 8.2. The natural response has the same form as in Eq. (8.42), and the associated constants are determined from Eq. (8.44) based on the values of $x(0)$ and $dx(0)/dt$.

EXAMPLE 8.7

For the circuit in Fig. 8.19, find $v(t)$ and $i(t)$ for $t > 0$. Consider these cases: $R = 5\ \Omega$, $R = 4\ \Omega$, and $R = 1\ \Omega$.

Solution:

CASE 1 When $R = 5\ \Omega$. For $t < 0$, the switch is closed. The capacitor behaves like an open circuit while the inductor acts like a short circuit. The initial current through the inductor is

$$i(0) = \frac{24}{5+1} = 4\text{ A}$$

and the initial voltage across the capacitor is the same as the voltage across the 1-Ω resistor; that is,

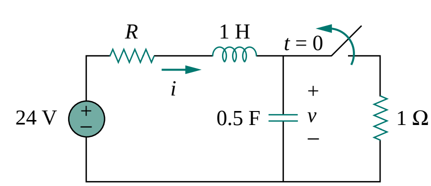


Figure 8.19 For Example 8.7.

$$v(0) = 1i(0) = 4 \text{ V}$$

For $t > 0$, the switch is opened, so that we have the 1-Ω resistor disconnected. What remains is the series RLC circuit with the voltage source. The characteristic roots are determined as follows.

$$\alpha = \frac{R}{2L} = \frac{5}{2 \times 1} = 2.5, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 0.25}} = 2$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -1, -4$$

Since $\alpha > \omega_0$, we have the overdamped natural response. The total response is therefore

$$v(t) = v_f + (A_1 e^{-t} + A_2 e^{-4t})$$

where v_f is the forced or steady-state response. It is the final value of the capacitor voltage. In Fig. 8.19, $v_f = 24$ V. Thus,

$$v(t) = 24 + (A_1 e^{-t} + A_2 e^{-4t}) \tag{8.7.1}$$

We now need to find A_1 and A_2 using the initial conditions.

$$v(0) = 4 = 24 + A_1 + A_2$$

or

$$-20 = A_1 + A_2 \tag{8.7.2}$$

The current through the inductor cannot change abruptly and is the same current through the capacitor at $t = 0^+$ because the inductor and capacitor are now in series. Hence,

$$i(0) = C\frac{dv(0)}{dt} = 4 \qquad \Longrightarrow \qquad \frac{dv(0)}{dt} = \frac{4}{C} = \frac{4}{0.25} = 16$$

Before we use this condition, we need to take the derivative of v in Eq. (8.7.1).

$$\frac{dv}{dt} = -A_1 e^{-t} - 4A_2 e^{-4t} \tag{8.7.3}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 16 = -A_1 - 4A_2 \tag{8.7.4}$$

From Eqs. (8.7.2) and (8.7.4), $A_1 = -64/3$ and $A_2 = 4/3$. Substituting A_1 and A_2 in Eq. (8.7.1), we get

$$v(t) = 24 + \frac{4}{3}(-16e^{-t} + e^{-4t}) \text{ V} \tag{8.7.5}$$

Since the inductor and capacitor are in series for $t > 0$, the inductor current is the same as the capacitor current. Hence,

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (8.7.3) by $C = 0.25$ and substituting the values of A_1 and A_2 gives

$$i(t) = \frac{4}{3}(4e^{-t} - e^{-4t})\ \text{A} \tag{8.7.6}$$

Note that $i(0) = 4$ A, as expected.

CASE 2 When $R = 4\ \Omega$. Again, the initial current through the inductor is

$$i(0) = \frac{24}{4+1} = 4.5\ \text{A}$$

and the initial capacitor voltage is

$$v(0) = 1i(0) = 4.5\ \text{V}$$

For the characteristic roots,

$$\alpha = \frac{R}{2L} = \frac{4}{2 \times 1} = 2$$

while $\omega_0 = 2$ remains the same. In this case, $s_1 = s_2 = -\alpha = -2$, and we have the critically damped natural response. The total response is therefore

$$v(t) = v_f + (A_1 + A_2 t)e^{-2t}$$

and, as $v_f = 24$ V,

$$v(t) = 24 + (A_1 + A_2 t)e^{-2t} \tag{8.7.7}$$

To find A_1 and A_2, we use the initial conditions. We write

$$v(0) = 4.5 = 24 + A_1 \qquad \Longrightarrow \qquad A_1 = -19.5 \tag{8.7.8}$$

Since $i(0) = C\,dv(0)/dt = 4.5$ or

$$\frac{dv(0)}{dt} = \frac{4.5}{C} = 18$$

From Eq. (8.7.7),

$$\frac{dv}{dt} = (-2A_1 - 2tA_2 + A_2)e^{-2t} \tag{8.7.9}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 18 = -2A_1 + A_2 \tag{8.7.10}$$

From Eqs. (8.7.8) and (8.7.10), $A_1 = -19.5$ and $A_2 = 57$. Thus, Eq. (8.7.7) becomes

$$v(t) = 24 + (-19.5 + 57t)e^{-2t}\ \text{V} \tag{8.7.11}$$

The inductor current is the same as the capacitor current, that is,

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (8.7.9) by $C = 0.25$ and substituting the values of A_1 and A_2 gives

$$i(t) = (4.5 - 28.5t)e^{-2t}\ \text{A} \tag{8.7.12}$$

Note that $i(0) = 4.5$ A, as expected.

CASE 3 When $R = 1\ \Omega$. The initial inductor current is

$$i(0) = \frac{24}{1+1} = 12 \text{ A}$$

and the initial voltage across the capacitor is the same as the voltage across the 1-Ω resistor,

$$v(0) = 1i(0) = 12 \text{ V}$$

$$\alpha = \frac{R}{2L} = \frac{1}{2 \times 1} = 0.5$$

Since $\alpha = 0.5 < \omega_0 = 2$, we have the underdamped response

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -0.5 \pm j1.936$$

The total response is therefore

$$v(t) = 24 + (A_1 \cos 1.936t + A_2 \sin 1.936t)e^{-0.5t} \tag{8.7.13}$$

We now determine A_1 and A_2. We write

$$v(0) = 12 = 24 + A_1 \quad\Longrightarrow\quad A_1 = -12 \tag{8.7.14}$$

Since $i(0) = C\,dv(0)/dt = 12$,

$$\frac{dv(0)}{dt} = \frac{12}{C} = 48 \tag{8.7.15}$$

But

$$\begin{aligned}\frac{dv}{dt} &= e^{-0.5t}(-1.936A_1 \sin 1.936t + 1.936A_2 \cos 1.936t)\\ &\quad - 0.5e^{-0.5t}(A_1 \cos 1.936t + A_2 \sin 1.936t)\end{aligned} \tag{8.7.16}$$

At $t = 0$,

$$\frac{dv(0)}{dt} = 48 = (-0 + 1.936A_2) - 0.5(A_1 + 0)$$

Substituting $A_1 = -12$ gives $A_2 = 21.694$, and Eq. (8.7.13) becomes

$$v(t) = 24 + (21.694 \sin 1.936t - 12 \cos 1.936t)e^{-0.5t} \text{ V} \tag{8.7.17}$$

The inductor current is

$$i(t) = C\frac{dv}{dt}$$

Multiplying Eq. (8.7.16) by $C = 0.25$ and substituting the values of A_1 and A_2 gives

$$i(t) = (3.1 \sin 1.936t + 12 \cos 1.936t)e^{-0.5t} \text{ A} \tag{8.7.18}$$

Note that $i(0) = 12$ A, as expected.

Figure 8.20 plots the responses for the three cases. From this figure, we observe that the critically damped response approaches the step input of 24 V the fastest.

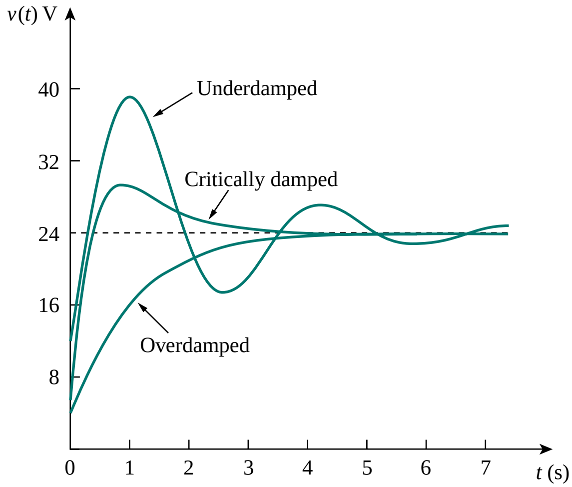


Figure 8.20 For Example 8.7: response for three degrees of damping.

PRACTICE PROBLEM 8.7

Having been in position a for a long time, the switch in Fig. 8.21 is moved to position b at $t = 0$. Find $v(t)$ and $v_R(t)$ for $t > 0$.

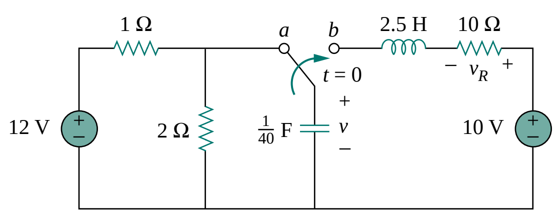


Figure 8.21 For Practice Prob. 8.7.

Answer: $10 - (1.1547 \sin 3.464t + 2 \cos 3.464t)e^{-2t}$ V, $2.31e^{-2t} \sin 3.464t$ V.

8.6 STEP RESPONSE OF A PARALLEL *RLC* CIRCUIT

Consider the parallel RLC circuit shown in Fig. 8.22. We want to find i due to a sudden application of a dc current. Applying KCL at the top node for $t > 0$,

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s \tag{8.46}$$

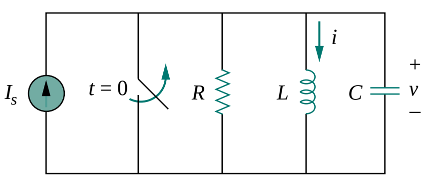


Figure 8.22 Parallel RLC circuit with an applied current.

But

$$v = L\frac{di}{dt}$$

Substituting for v in Eq. (8.46) and dividing by LC, we get

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC} \tag{8.47}$$

which has the same characteristic equation as Eq. (8.29).

The complete solution to Eq. (8.47) consists of the natural response $i_n(t)$ and the forced response i_f; that is,

$$i(t) = i_n(t) + i_f(t) \tag{8.48}$$

The natural response is the same as what we had in Section 8.3. The forced response is the steady state or final value of i. In the circuit in Fig. 8.22, the final value of the current through the inductor is the same as the source current I_s. Thus,

$$\boxed{\begin{aligned} i(t) &= I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} && \text{(Overdamped)} \\ i(t) &= I_s + (A_1 + A_2 t)e^{-\alpha t} && \text{(Critically damped)} \\ i(t) &= I_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} && \text{(Underdamped)} \end{aligned}} \tag{8.49}$$

The constants A_1 and A_2 in each case can be determined from the initial conditions for i and di/dt. Again, we should keep in mind that Eq. (8.49) only applies for finding the inductor current i. But once the inductor current $i_L = i$ is known, we can find $v = L\ di/dt$, which is the same voltage across inductor, capacitor, and resistor. Hence, the current through the resistor is $i_R = v/R$, while the capacitor current is $i_C = C\ dv/dt$. Alternatively, the complete response for any variable $x(t)$ may be found directly, using

$$x(t) = x_f(t) + x_n(t) \tag{8.50}$$

where x_f and x_n are its final value and natural response, respectively.

EXAMPLE 8.8

In the circuit in Fig. 8.23, find $i(t)$ and $i_R(t)$ for $t > 0$.

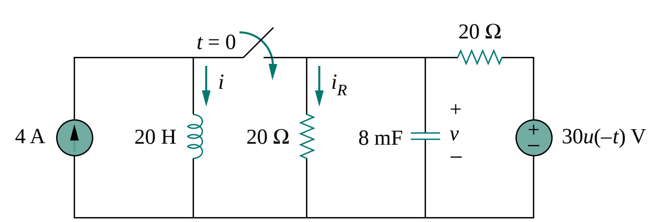


Figure 8.23 For Example 8.8.

Solution:

For $t < 0$, the switch is open, and the circuit is partitioned into two independent subcircuits. The 4-A current flows through the inductor, so that

$$i(0) = 4 \text{ A}$$

Since $30u(-t) = 30$ when $t < 0$ and 0 when $t > 0$, the voltage source is operative for $t < 0$ under consideration. The capacitor acts like an open circuit and the voltage across it is the same as the voltage across the 20-Ω resistor connected in parallel with it. By voltage division, the initial capacitor voltage is

$$v(0) = \frac{20}{20+20}(30) = 15 \text{ V}$$

For $t > 0$, the switch is closed, and we have a parallel RLC circuit with a current source. The voltage source is off or short-circuited. The two 20-Ω resistors are now in parallel. They are combined to give $R = 20 \parallel 20 = 10\ \Omega$. The characteristic roots are determined as follows:

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 10 \times 8 \times 10^{-3}} = 6.25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20 \times 8 \times 10^{-3}}} = 2.5$$

$$\begin{aligned} s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} &= -6.25 \pm \sqrt{39.0625 - 6.25} \\ &= -6.25 \pm 5.7282 \end{aligned}$$

or

$$s_1 = -11.978, \qquad s_2 = -0.5218$$

Since $\alpha > \omega_0$, we have the overdamped case. Hence,

$$i(t) = I_s + A_1 e^{-11.978t} + A_2 e^{-0.5218t} \tag{8.8.1}$$

where $I_s = 4$ is the final value of $i(t)$. We now use the initial conditions to determine A_1 and A_2. At $t = 0$,

$$i(0) = 4 = 4 + A_1 + A_2 \qquad \Longrightarrow \qquad A_2 = -A_1 \tag{8.8.2}$$

Taking the derivative of $i(t)$ in Eq. (8.8.1),

$$\frac{di}{dt} = -11.978 A_1 e^{-11.978t} - 0.5218 A_2 e^{-0.5218t}$$

so that at $t = 0$,

$$\frac{di(0)}{dt} = -11.978 A_1 - 0.5218 A_2 \tag{8.8.3}$$

But

$$L\frac{di(0)}{dt} = v(0) = 15 \qquad \Longrightarrow \qquad \frac{di(0)}{dt} = \frac{15}{L} = \frac{15}{20} = 0.75$$

Substituting this into Eq. (8.8.3) and incorporating Eq. (8.8.2), we get

$$0.75 = (11.978 - 0.5218)A_2 \qquad \Longrightarrow \qquad A_2 = 0.0655$$

Thus, $A_1 = -0.0655$ and $A_2 = 0.0655$. Inserting A_1 and A_2 in Eq. (8.8.1) gives the complete solution as

$$i(t) = 4 + 0.0655(e^{-0.5218t} - e^{-11.978t})\ \text{A}$$

From $i(t)$, we obtain $v(t) = L\,di/dt$ and

$$i_R(t) = \frac{v(t)}{20} = \frac{L}{20}\frac{di}{dt} = 0.785e^{-11.978t} - 0.0342e^{-0.5218t}\ \text{A}$$

PRACTICE PROBLEM 8.8

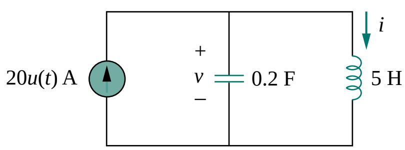


Figure 8.24 For Practice Prob. 8.8.

Find $i(t)$ and $v(t)$ for $t > 0$ in the circuit in Fig. 8.24.

Answer: $20(1 - \cos t)$ A, $100 \sin t$ V.

Electronic Testing Tutorials

8.7 GENERAL SECOND-ORDER CIRCUITS

Now that we have mastered series and parallel RLC circuits, we are prepared to apply the ideas to any second-order circuit. Although the series and parallel RLC circuits are the second-order circuits of greatest interest, other second-order circuits including op amps are also useful. Given a second-order circuit, we determine its step response $x(t)$ (which may be voltage or current) by taking the following four steps:

1. We first determine the initial conditions $x(0)$ and $dx(0)/dt$ and the final value $x(\infty)$, as discussed in Section 8.2.
2. We find the natural response $x_n(t)$ by turning off independent sources and applying KCL and KVL. Once a second-order differential equation is obtained, we determine its characteristic roots. Depending on whether the response is overdamped, critically damped, or underdamped, we obtain $x_n(t)$ with two unknown constants as we did in the previous sections.
3. We obtain the forced response as

$$x_f(t) = x(\infty) \tag{8.51}$$

   where $x(\infty)$ is the final value of x, obtained in step 1.
4. The total response is now found as the sum of the natural response and forced response

$$x(t) = x_n(t) + x_f(t) \tag{8.52}$$

   We finally determine the constants associated with the natural response by imposing the initial conditions $x(0)$ and $dx(0)/dt$, determined in step 1.

A circuit may look complicated at first. But once the sources are turned off in an attempt to find the natural response, it may be reducible to a first-order circuit, when the storage elements can be combined, or to a parallel/series *RLC* circuit. If it is reducible to a first-order circuit, the solution becomes simply what we had in Chapter 7. If it is reducible to a parallel or series *RLC* circuit, we apply the techniques of previous sections in this chapter.

We can apply this general procedure to find the step response of any second-order circuit, including those with op amps. The following examples illustrate the four steps.

EXAMPLE 8.9

Find the complete response v and then i for $t > 0$ in the circuit of Fig. 8.25.

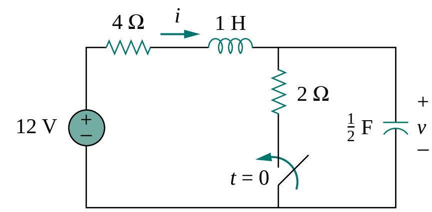


Figure 8.25 For Example 8.9.

Solution:

We first find the initial and final values. At $t = 0^-$, the circuit is at steady state. The switch is open, the equivalent circuit is shown in Fig. 8.26(a). It is evident from the figure that

$$v(0^-) = 12 \text{ V}, \qquad i(0^-) = 0$$

At $t = 0^+$, the switch is closed; the equivalent circuit is in Fig. 8.26(b). By the continuity of capacitor voltage and inductor current, we know that

$$v(0^+) = v(0^-) = 12 \text{ V}, \qquad i(0^+) = i(0^-) = 0 \tag{8.9.1}$$

To get $dv(0^+)/dt$, we use $C\, dv/dt = i_C$ or $dv/dt = i_C/C$. Applying KCL at node a in Fig. 8.26(b),

$$i(0^+) = i_C(0^+) + \frac{v(0^+)}{2}$$

$$0 = i_C(0^+) + \frac{12}{2} \qquad \Longrightarrow \qquad i_C(0^+) = -6 \text{ A}$$

Hence,

$$\frac{dv(0^+)}{dt} = \frac{-6}{0.5} = -12 \text{ V/s} \tag{8.9.2}$$

The final values are obtained when the inductor is replaced by a short circuit and the capacitor by an open circuit in Fig. 8.26(b), giving

$$i(\infty) = \frac{12}{4+2} = 2 \text{ A}, \qquad v(\infty) = 2i(\infty) = 4 \text{ V} \tag{8.9.3}$$

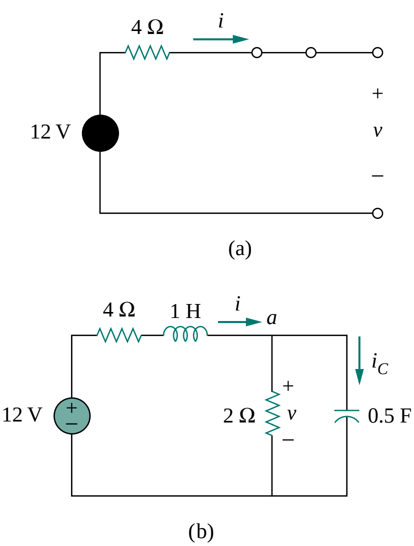


Figure 8.26 Equivalent circuit of the circuit in Fig. 8.25 for: (a) $t = 0$, (b) $t > 0$.

Next, we obtain the natural response for $t > 0$. By turning off the 12-V voltage source, we have the circuit in Fig. 8.27. Applying KCL at node a in Fig. 8.27 gives

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} \tag{8.9.4}$$

Applying KVL to the left mesh results in

$$4i + 1\frac{di}{dt} + v = 0 \tag{8.9.5}$$

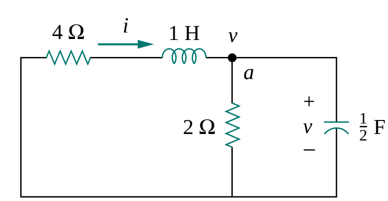


Figure 8.27 Obtaining the natural response for Example 8.9.

Since we are interested in v for the moment, we substitute i from Eq. (8.9.4) into Eq. (8.9.5). We obtain

$$2v + 2\frac{dv}{dt} + \frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2} + v = 0$$

or

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 0$$

From this, we obtain the characteristic equation as

$$s^2 + 5s + 6 = 0$$

with roots $s = -2$ and $s = -3$. Thus, the natural response is

$$v_n(t) = Ae^{-2t} + Be^{-3t} \tag{8.9.6}$$

where A and B are unknown constants to be determined later. The forced response is

$$v_f(t) = v(\infty) = 4 \tag{8.9.7}$$

The complete response is

$$v(t) = v_n + v_f = 4 + Ae^{-2t} + Be^{-3t} \tag{8.9.8}$$

We now determine A and B using the initial values. From Eq. (8.9.1), $v(0) = 12$. Substituting this into Eq. (8.9.8) at $t = 0$ gives

$$12 = 4 + A + B \qquad \Longrightarrow \qquad A + B = 8 \tag{8.9.9}$$

Taking the derivative of v in Eq. (8.9.8),

$$\frac{dv}{dt} = -2Ae^{-2t} - 3Be^{-3t} \tag{8.9.10}$$

Substituting Eq. (8.9.2) into Eq. (8.9.10) at $t = 0$ gives

$$-12 = -2A - 3B \qquad \Longrightarrow \qquad 2A + 3B = 12 \tag{8.9.11}$$

From Eqs. (8.9.9) and (8.9.11), we obtain

$$A = 12, \qquad B = -4$$

so that Eq. (8.9.8) becomes

$$v(t) = 4 + 12e^{-2t} - 4e^{-3t}\ \text{V}, \qquad t > 0 \tag{8.9.12}$$

From v, we can obtain other quantities of interest by referring to Fig. 8.26(b). To obtain i, for example,

$$\begin{aligned} i &= \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} = 2 + 6e^{-2t} - 2e^{-3t} - 12e^{-2t} + 6e^{-3t} \\ &= 2 - 6e^{-2t} + 4e^{-3t}\ \text{A}, \qquad t > 0 \end{aligned} \tag{8.9.13}$$

Notice that $i(0) = 0$, in agreement with Eq. (8.9.1).

PRACTICE PROBLEM 8.9

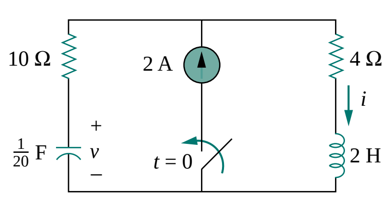


Figure 8.28 For Practice Prob. 8.9.

Determine v and i for $t > 0$ in the circuit of Fig. 8.28.

Answer: $8(1 - e^{-5t})$ V, $2(1 - e^{-5t})$ A.

EXAMPLE 8.10

Find $v_o(t)$ for $t > 0$ in the circuit of Fig. 8.29.

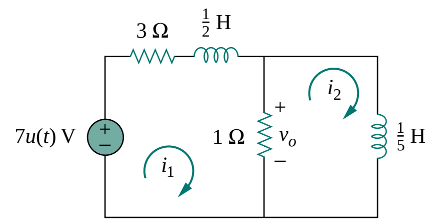


Figure 8.29 For Example 8.10.

Solution:

This is an example of a second-order circuit with two inductors. We first obtain the mesh currents i_1 and i_2, which happen to be the currents through the inductors. We need to obtain the initial and final values of these currents.

For $t < 0$, $7u(t) = 0$, so that $i_1(0^-) = 0 = i_2(0^-)$. For $t > 0$, $7u(t) = 7$, so that the equivalent circuit is as shown in Fig. 8.30(a). Due to the continuity of inductor current,

$$i_1(0^+) = i_1(0^-) = 0, \qquad i_2(0^+) = i_2(0^-) = 0 \tag{8.10.1}$$

$$v_{L2}(0^+) = v_o(0^+) = 1[(i_1(0^+) - i_2(0^+)] = 0 \tag{8.10.2}$$

Applying KVL to the left loop in Fig. 8.30(a) at $t = 0^+$,

$$7 = 3i_1(0^+) + v_{L1}(0+) + v_o(0^+)$$

or

$$v_{L1}(0^+) = 7 \text{ V}$$

Since $L_1\, di_1/dt = v_{L1}$,

$$\frac{di_1(0^+)}{dt} = \frac{v_{L1}}{L_1} = \frac{7}{\frac{1}{2}} = 14 \text{ V/s} \tag{8.10.3}$$

Similarly, since $L_2\, di_2/dt = v_{L2}$,

$$\frac{di_2(0^+)}{dt} = \frac{v_{L2}}{L_2} = 0 \tag{8.10.4}$$

As $t \to \infty$, the circuit reaches steady state, and the inductors can be replaced by short circuits, as shown in Fig. 8.30(b). From this figure,

$$i_1(\infty) = i_2(\infty) = \frac{7}{3} \text{ A} \tag{8.10.5}$$

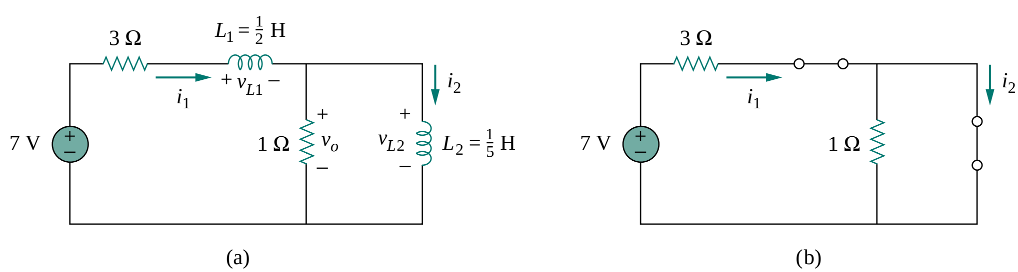


Figure 8.30 Equivalent circuit of that in Fig. 8.29 for: (a) $t > 0$, (b) $t \to \infty$.

Next, we obtain the natural responses by removing the voltage source, as shown in Fig. 8.31. Applying KVL to the two meshes yields

$$4i_1 - i_2 + \frac{1}{2}\frac{di_1}{dt} = 0 \tag{8.10.6}$$

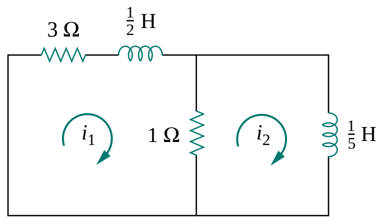


Figure 8.31 Obtaining the natural response for Example 8.10.

and

$$i_2 + \frac{1}{5}\frac{di_2}{dt} - i_1 = 0 \tag{8.10.7}$$

From Eq. (8.10.6),

$$i_2 = 4i_1 + \frac{1}{2}\frac{di_1}{dt} \tag{8.10.8}$$

Substituting Eq. (12.8.8) into Eq. (8.10.7) gives

$$4i_1 + \frac{1}{2}\frac{di_1}{dt} + \frac{4}{5}\frac{di_1}{dt} + \frac{1}{10}\frac{d^2i_1}{dt^2} - i_1 = 0$$
$$\frac{d^2i_1}{dt^2} + 13\frac{di_1}{dt} + 30i_1 = 0$$

From this we obtain the characteristic equation as

$$s^2 + 13s + 30 = 0$$

which has roots $s = -3$ and $s = -10$. Hence, the natural response is

$$i_{1n} = Ae^{-3t} + Be^{-10t} \tag{8.10.9}$$

where A and B are constants. The forced response is

$$i_{1f} = i_1(\infty) = \frac{7}{3}\text{ A} \tag{8.10.10}$$

From Eqs. (8.10.9) and (8.10.10), we obtain the complete response as

$$i_1(t) = \frac{7}{3} + Ae^{-3t} + Be^{-10t} \tag{8.10.11}$$

We finally obtain A and B from the initial values. From Eqs. (8.10.1) and (8.10.11),

$$0 = \frac{7}{3} + A + B \tag{8.10.12}$$

Taking the derivative of Eq. (8.10.11), setting $t = 0$ in the derivative, and enforcing Eq. (8.10.3), we obtain

$$14 = -3A - 10B \tag{8.10.13}$$

From Eqs. (8.10.12) and (8.10.13), $A = -4/3$ and $B = -1$. Thus,

$$i_1(t) = \frac{7}{3} - \frac{4}{3}e^{-3t} - e^{-10t} \tag{8.10.14}$$

We now obtain i_2 from i_1. Applying KVL to the left loop in Fig. 8.30(a) gives

$$7 = 4i_1 - i_2 + \frac{1}{2}\frac{di_1}{dt} \quad\Longrightarrow\quad i_2 = -7 + 4i_1 + \frac{1}{2}\frac{di_1}{dt}$$

Substituting for i_1 in Eq. (8.10.14) gives

$$\begin{aligned} i_2(t) &= -7 + \frac{28}{3} - \frac{16}{3}e^{-3t} - 4e^{-10t} + 2e^{-3t} + 5e^{-10t} \\ &= \frac{7}{3} - \frac{10}{3}e^{-3t} + e^{-10t} \end{aligned} \tag{8.10.15}$$

From Fig. 8.29,

$$v_o(t) = 1[i_1(t) - i_2(t)] \tag{8.10.16}$$

Substituting Eqs. (8.10.14) and (8.10.15) into Eq. (8.10.16) yields

$$v_o(t) = 2(e^{-3t} - e^{-10t}) \tag{8.10.17}$$

Note that $v_o(0) = 0$, as expected from Eq. (8.10.2).

@ Network Analysis

PRACTICE PROBLEM 8.10

For $t > 0$, obtain $v_o(t)$ in the circuit of Fig. 8.32. (*Hint:* First find v_1 and v_2.)

Answer: $2(e^{-t} - e^{-6t})$ V, $t > 0$.

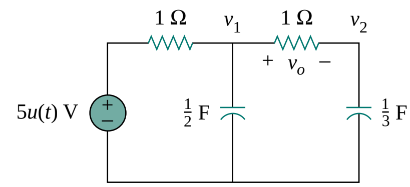


Figure 8.32 For Practice Prob. 8.10.

8.8 SECOND-ORDER OP AMP CIRCUITS

An op amp circuit with two storage elements that cannot be combined into a single equivalent element is second-order. Because inductors are bulky and heavy, they are rarely used in practical op amp circuits. For this reason, we will only consider RC second-order op amp circuits here. Such circuits find a wide range of applications in devices such as filters and oscillators.

The use of op amps in second-order circuits avoids the use of inductors, which are somewhat undesirable in some applications.

The analysis of a second-order op amp circuit follows the same four steps given and demonstrated in the previous section.

EXAMPLE 8.11

In the op amp circuit of Fig. 8.33, find $v_o(t)$ for $t > 0$ when $v_s = 10u(t)$ mV. Let $R_1 = R_2 = 10$ kΩ, $C_1 = 20\ \mu$F, and $C_2 = 100\ \mu$F.

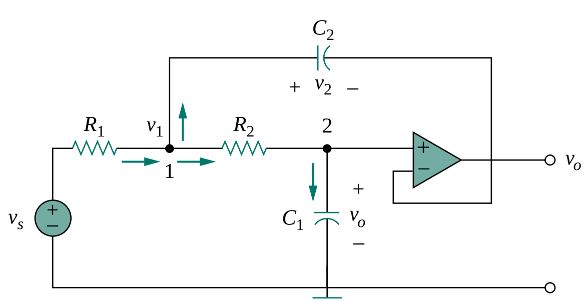


Figure 8.33 For Example 8.11.

Solution:

Although we could follow the same four steps given in the previous section to solve this problem, we will solve it a little differently. Due to the voltage follower configuration, the voltage across C_1 is v_o. Applying KCL at node 1,

$$\frac{v_s - v_1}{R_1} = C_2\frac{dv_2}{dt} + \frac{v_1 - v_o}{R_2} \tag{8.11.1}$$

At node 2, KCL gives

$$\frac{v_1 - v_o}{R_2} = C_1\frac{dv_o}{dt} \tag{8.11.2}$$

But

$$v_2 = v_1 - v_o \tag{8.11.3}$$

We now try to eliminate v_1 and v_2 in Eqs. (8.11.1) to (8.11.3). Substituting Eqs. (8.11.2) and (8.11.3) into Eq. (8.11.1) yields

$$\frac{v_s - v_1}{R_1} = C_2\frac{dv_1}{dt} - C_2\frac{dv_o}{dt} + C_1\frac{dv_o}{dt} \tag{8.11.4}$$

From Eq. (8.11.2),

$$v_1 = v_o + R_2C_1\frac{dv_o}{dt} \tag{8.11.5}$$

Substituting Eq. (8.11.5) into Eq. (8.11.4), we obtain

$$\frac{v_s}{R_1} = \frac{v_o}{R_1} + \frac{R_2C_1}{R_1}\frac{dv_o}{dt} + C_2\frac{dv_o}{dt} + R_2C_1C_2\frac{d^2v_o}{dt^2} - C_2\frac{dv_o}{dt} + C_1\frac{dv_o}{dt}$$

or

$$\frac{d^2v_o}{dt^2} + \left(\frac{1}{R_1C_2} + \frac{1}{R_2C_2}\right)\frac{dv_o}{dt} + \frac{v_o}{R_1R_2C_1C_2} = \frac{v_s}{R_1R_2C_1C_2} \tag{8.11.6}$$

With the given values of R_1, R_2, C_1, and C_2, Eq. (8.11.6) becomes

$$\frac{d^2v_o}{dt^2} + 2\frac{dv_o}{dt} + 5v_o = 5v_s \tag{8.11.7}$$

To obtain the natural response, set $v_s = 0$ in Eq. (8.11.7), which is the same as turning off the source. The characteristic equation is

$$s^2 + 2s + 5 = 0$$

which has complex roots $s_{1,2} = -1 \pm j2$. Hence, the natural response is

$$v_{on} = e^{-t}(A\cos 2t + B\sin 2t) \tag{8.11.8}$$

where A and B are unknown constants to be determined.

As $t \to \infty$, the circuit reaches the steady-state condition, and the capacitors can be replaced by open circuits. Since no current flows through C_1 and C_2 under steady-state conditions and no current can enter the input terminals of the ideal op amp, current does not flow through R_1 and R_2. Thus,

$$v_o(\infty) = v_1(\infty) = v_s$$

The forced response is then

$$v_{of} = v_o(\infty) = v_s = 10 \text{ mV}, \qquad t > 0 \tag{8.11.9}$$

The complete response is

$$v_o(t) = v_{on} + v_{of} = 10 + e^{-t}(A\cos 2t + B\sin 2t) \text{ mV} \tag{8.11.10}$$

To determine A and B, we need the initial conditions. For $t < 0$, $v_s = 0$, so that

$$v_o(0^-) = v_2(0^-) = 0$$

For $t > 0$, the source is operative. However, due to capacitor voltage continuity,

$$v_o(0^+) = v_2(0^+) = 0 \tag{8.11.11}$$

From Eq. (8.11.3),

$$v_1(0^+) = v_2(0^+) + v_o(0^+) = 0$$

and hence, from Eq. (8.11.2),

$$\frac{dv_o(0^+)}{dt} = \frac{v_1 - v_o}{R_2C_1} = 0 \tag{8.11.12}$$

We now impose Eq. (8.11.11) on the complete response in Eq. (8.11.10) at $t = 0$, for

$$0 = 10 + A \qquad \Longrightarrow \qquad A = -10 \tag{8.11.13}$$

Taking the derivative of Eq. (8.11.10),

$$\frac{dv_o}{dt} = e^{-t}(-A\cos 2t - B\sin 2t - 2A\sin 2t + 2B\cos 2t)$$

Setting $t = 0$ and incorporating Eq. (8.11.12), we obtain

$$0 = -A + 2B \tag{8.11.14}$$

From Eqs. (8.11.13) and (8.11.14), $A = -10$ and $B = -5$. Thus the step response becomes

$$v_o(t) = 10 - e^{-t}(10\cos 2t + 5\sin 2t) \text{ mV}, \qquad t > 0$$

PRACTICE PROBLEM 8.11

In the op amp circuit shown in Fig. 8.34, $v_s = 4u(t)$ V, find $v_o(t)$ for $t > 0$. Assume that $R_1 = R_2 = 10$ kΩ, $C_1 = 20\ \mu$F, and $C_2 = 100\ \mu$F.

Answer: $4 - 5e^{-t} + e^{-5t}$ V, $t > 0$.

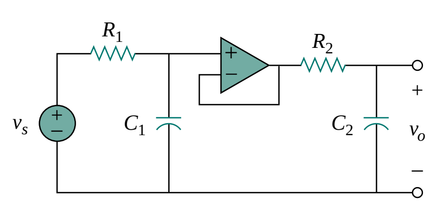


Figure 8.34 For Practice Prob. 8.11.

8.9 *PSPICE* ANALYSIS OF *RLC* CIRCUITS

RLC circuits can be analyzed with great ease using *PSpice*, just like the *RC* or *RL* circuits of Chapter 7. The following two examples will illustrate this. The reader may review Section D.4 in Appendix D on *PSpice* for transient analysis.

EXAMPLE 8.12

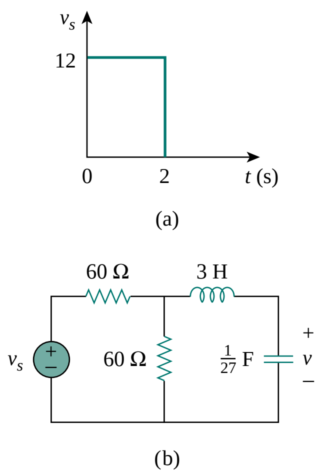


Figure 8.35 For Example 8.12.

The input voltage in Fig. 8.35(a) is applied to the circuit in Fig. 8.35(b). Use *PSpice* to plot $v(t)$ for $0 < t < 4$s.

Solution:

The given circuit is drawn using Schematics as in Fig. 8.36. The pulse is specified using VPWL voltage source, but VPULSE could be used instead. Using the piecewise linear function, we set the attributes of VPWL as T1 = 0, V1 = 0, T2 = 0.001, V2 = 12, and so forth, as shown in Fig. 8.36. Two voltage markers are inserted to plot the input and output voltages. Once the circuit is drawn and the attributes are set, we select **Analysis/Setup/Transient** to open up the *Transient Analysis* dialog box. As a parallel *RLC* circuit, the roots of the characteristic equation are −1 and −9. Thus, we may set *Final Time* as 4 s (four times the magnitude of the lower root). When the schematic is saved, we select **Analysis/Simulate** and obtain the plots for the input and output voltages under the Probe window as shown in Fig. 8.37.

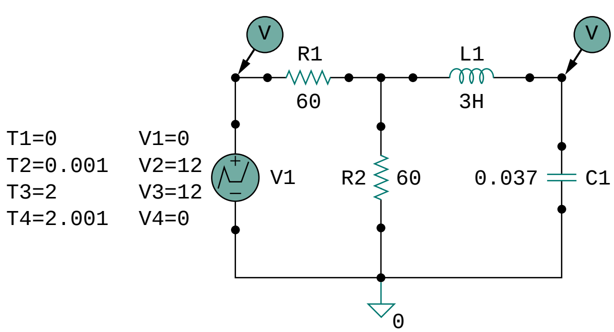


Figure 8.36 Schematic for the circuit in Fig. 8.35(b).

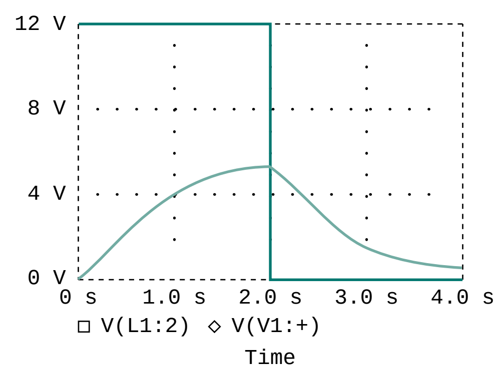


Figure 8.37 For Example 8.12: the input and output voltages.

PRACTICE PROBLEM 8.12

Find $i(t)$ using *PSpice* for $0 < t < 4$ s if the pulse voltage in Fig. 8.35(a) is applied to the circuit in Fig. 8.38.

Answer: See Fig. 8.39.

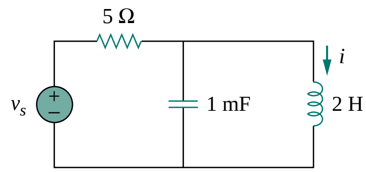


Figure 8.38 For Practice Prob. 8.12.

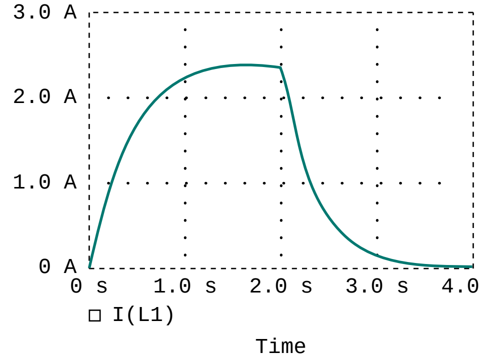


Figure 8.39 Plot of $i(t)$ for Practice Prob. 8.12.

EXAMPLE 8.13

For the circuit in Fig. 8.40, use *PSpice* to obtain $i(t)$ for $0 < t < 3$ s.

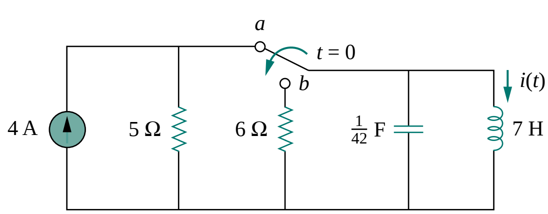


Figure 8.40 For Example 8.13.

Solution:

When the switch is in position a, the 6-Ω resistor is redundant. The schematic for this case is shown in Fig. 8.41(a). To ensure that current $i(t)$ enters pin 1, the inductor is rotated three times before it is placed in the circuit. The same applies for the capacitor. We insert pseudocomponents

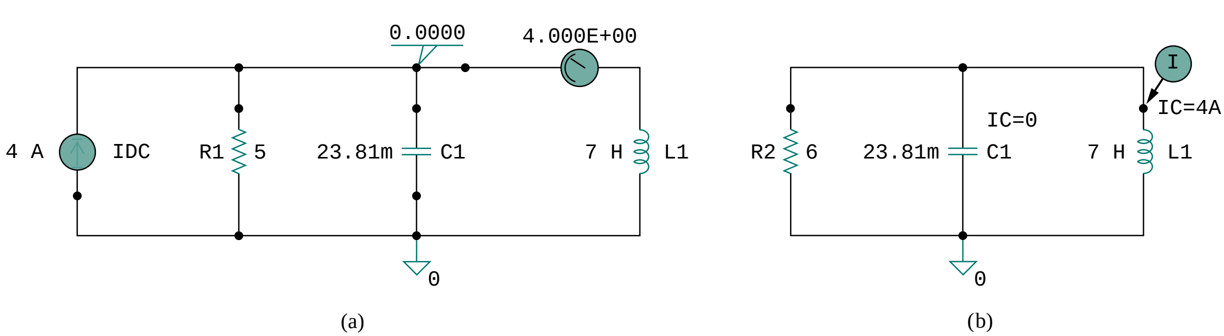


Figure 8.41 For Example 8.13: (a) for dc analysis, (b) for transient analysis.

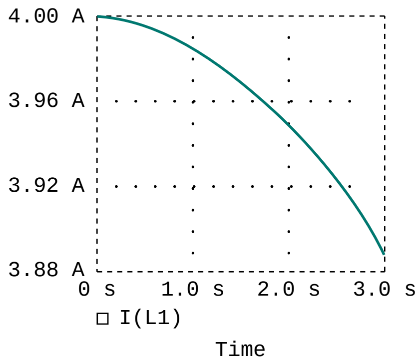


Figure 8.42 Plot of $i(t)$ for Example 8.13.

VIEWPOINT and IPROBE to determine the initial capacitor voltage and initial inductor current. We carry out a dc *PSpice* analysis by selecting **Analysis/Simulate**. As shown in Fig. 8.41(a), we obtain the initial capacitor voltage as 0 V and the initial inductor current $i(0)$ as 4 A from the dc analysis. These initial values will be used in the transient analysis.

When the switch is moved to position b, the circuit becomes a source-free parallel RLC circuit with the schematic in Fig. 8.41(b). We set the initial condition IC $=$ 0 for the capacitor and IC $=$ 4 A for the inductor. A current marker is inserted at pin 1 of the inductor. We select **Analysis/Setup/Transient** to open up the *Transient Analysis* dialog box and set *Final Time* to 3 s. After saving the schematic, we select **Analysis/Transient**. Figure 8.42 shows the plot of $i(t)$. The plot agrees with $i(t) = 4.8e^{-t} - 0.8e^{-6t}$ A, which is the solution by hand calculation.

PRACTICE PROBLEM 8.13

Refer to the circuit in Fig. 8.21 (see Practice Prob. 8.7). Use *PSpice* to obtain $v(t)$ for $0 < t < 2$.

Answer: See Fig. 8.43.

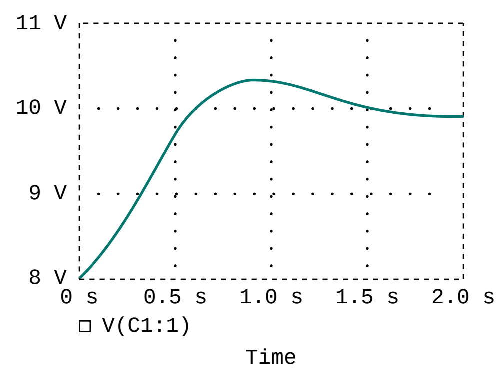


Figure 8.43 Plot of $v(t)$ for Practice Prob. 8.13.

†8.10 DUALITY

The concept of duality is a time-saving, effort-effective measure of solving circuit problems. Consider the similarity between Eq. (8.4) and Eq. (8.29). The two equations are the same, except that we must interchange the following quantities: (1) voltage and current, (2) resistance and conductance, (3) capacitance and inductance. Thus, it sometimes occurs in circuit analysis that two different circuits have the same equations and solutions, except that the roles of certain complementary elements are interchanged. This interchangeability is known as the principle of *duality*.

The duality principle asserts a parallelism between pairs of characterizing equations and theorems of electric circuits.

Dual pairs are shown in Table 8.1. Note that power does not appear in Table 8.1, because power has no dual. The reason for this is the principle of linearity; since power is not linear, duality does not apply. Also notice from Table 8.1 that the principle of duality extends to circuit elements, configurations, and theorems.

Two circuits that are described by equations of the same form, but in which the variables are interchanged, are said to be dual to each other.

Two circuits are said to be duals of one another if they are described by the same characterizing equations with dual quantities interchanged.

The usefulness of the duality principle is self-evident. Once we know the solution to one circuit, we automatically have the solution for the dual circuit. It is obvious that the circuits in Figs. 8.8 and 8.13 are dual. Consequently, the result in Eq. (8.32) is the dual of that in Eq. (8.11). We must keep in mind that the principle of duality is limited to planar circuits. Nonplanar circuits have no duals, as they cannot be described by a system of mesh equations.

To find the dual of a given circuit, we do not need to write down the mesh or node equations. We can use a graphical technique. Given a planar circuit, we construct the dual circuit by taking the following three steps:

1. Place a node at the center of each mesh of the given circuit. Place the reference node (the ground) of the dual circuit outside the given circuit.
2. Draw lines between the nodes such that each line crosses an element. Replace that element by its dual (see Table 8.1).
3. To determine the polarity of voltage sources and direction of current sources, follow this rule: A voltage source that produces a positive (clockwise) mesh current has as its dual a current source whose reference direction is from the ground to the nonreference node.

In case of doubt, one may verify the dual circuit by writing the nodal or mesh equations. The mesh (or nodal) equations of the original circuit are similar to the nodal (or mesh) equations of the dual circuit. The duality principle is illustrated with the following two examples.

TABLE 8.1 Dual pairs.

Resistance R	Conductance G
Inductance L	Capacitance C
Voltage v	Current i
Voltage source	Current source
Node	Mesh
Series path	Parallel path
Open circuit	Short circuit
KVL	KCL
Thevenin	Norton

Even when the principle of linearity applies, a circuit element or variable may not have a dual. For example, mutual inductance (to be covered in Chapter 13) has no dual.

EXAMPLE 8.14

Construct the dual of the circuit in Fig. 8.44.

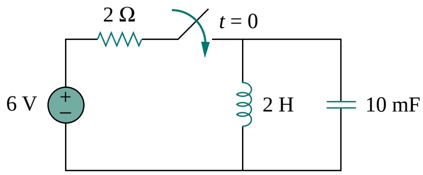


Figure 8.44 For Example 8.14.

Solution:

As shown in Fig. 8.45(a), we first locate nodes 1 and 2 in the two meshes and also the ground node 0 for the dual circuit. We draw a line between one node and another crossing an element. We replace the line joining the nodes by the duals of the elements which it crosses. For example, a line between nodes 1 and 2 crosses a 2-H inductor, and we place a 2-F capacitor (an inductor's dual) on the line. A line between nodes 1 and 0 crossing the 6-V voltage source will contain a 6-A current source. By drawing lines crossing all the elements, we construct the dual circuit on the given circuit as in Fig. 8.45(a). The dual circuit is redrawn in Fig. 8.45(b) for clarity.

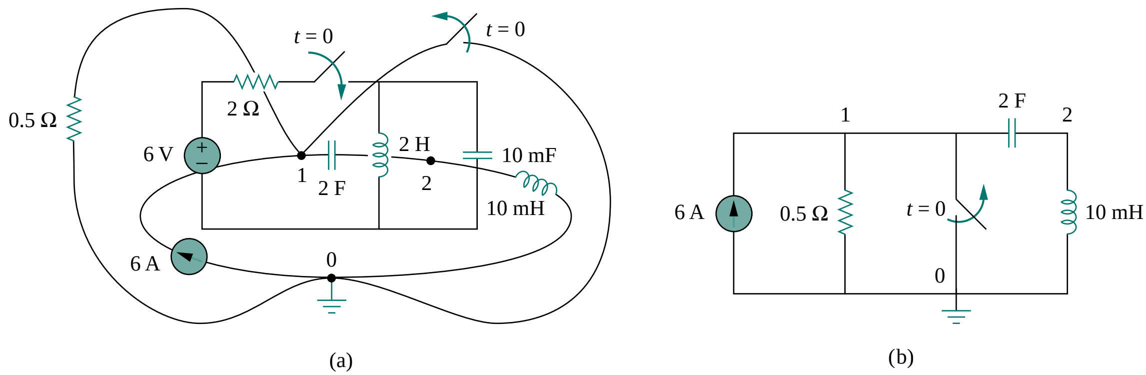


Figure 8.45 (a) Construction of the dual circuit of Fig. 8.44, (b) dual circuit redrawn.

PRACTICE PROBLEM 8.14

Draw the dual circuit of the one in Fig. 8.46.

Answer: See Fig. 8.47.

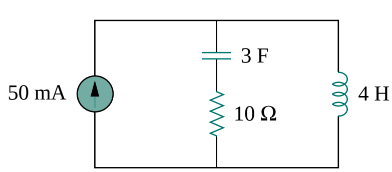


Figure 8.46 For Practice Prob. 8.14.

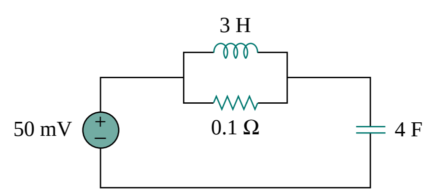


Figure 8.47 Dual of the circuit in Fig. 8.46.

EXAMPLE 8.15

Obtain the dual of the circuit in Fig. 8.48.

Solution:

The dual circuit is constructed on the original circuit as in Fig. 8.49(a). We first locate nodes 1 to 3 and the reference node 0. Joining nodes 1 and 2, we cross the 2-F capacitor, which is replaced by a 2-H inductor.

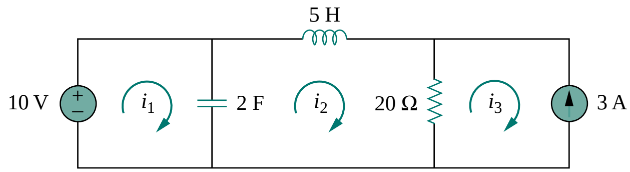


Figure 8.48 For Example 8.15.

Joining nodes 2 and 3, we cross the 20-Ω resistor, which is replaced by a 1/20-Ω resistor. We keep doing this until all the elements are crossed. The result is in Fig. 8.49(a). The dual circuit is redrawn in Fig. 8.49(b).

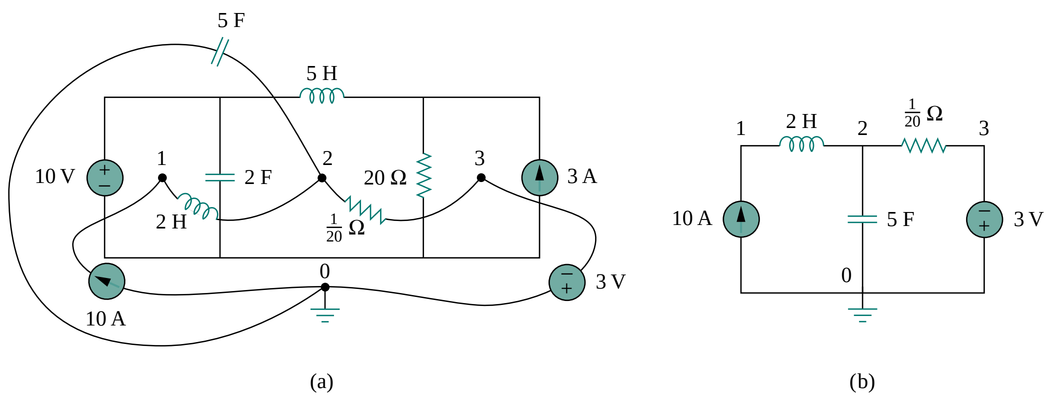


Figure 8.49 For Example 8.15: (a) construction of the dual circuit of Fig. 8.48, (b) dual circuit redrawn.

To verify the polarity of the voltage source and the direction of the current source, we may apply mesh currents i_1, i_2, and i_3 (all in the clockwise direction) in the original circuit in Fig. 8.48. The 10-V voltage source produces positive mesh current i_1, so that its dual is a 10-A current source directed from 0 to 1. Also, $i_3 = -3$ A in Fig. 8.48 has as its dual $v_3 = -3$ V in Fig. 8.49(b).

PRACTICE PROBLEM 8.15

For the circuit in Fig. 8.50, obtain the dual circuit.

Answer: See Fig. 8.51.

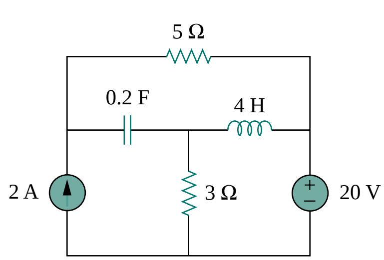


Figure 8.50 For Practice Prob. 8.15.

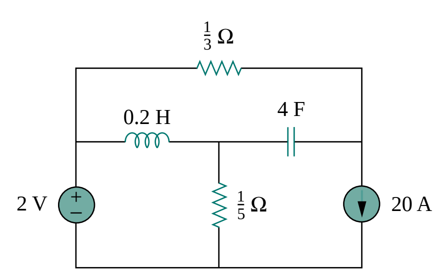


Figure 8.51 Dual of the circuit in Fig. 8.50.

†8.11 APPLICATIONS

Practical applications of RLC circuits are found in control and communications circuits such as ringing circuits, peaking circuits, resonant circuits, smoothing circuits, and filters. Most of the circuits cannot be covered until we treat ac sources. For now, we will limit ourselves to two simple applications: automobile ignition and smoothing circuits.

8.11.1 Automobile Ignition System

In Section 7.9.4, we considered the automobile ignition system as a charging system. That was only a part of the system. Here, we consider another part—the voltage generating system. The system is modeled by the circuit shown in Fig. 8.52. The 12-V source is due to the battery and alternator. The 4-Ω resistor represents the resistance of the wiring. The ignition coil is modeled by the 8-mH inductor. The 1-μF capacitor (known as the *condenser* to automechanics) is in parallel with the switch (known as the *breaking points* or *electronic ignition*). In the following example, we determine how the RLC circuit in Fig. 8.52 is used in generating high voltage.

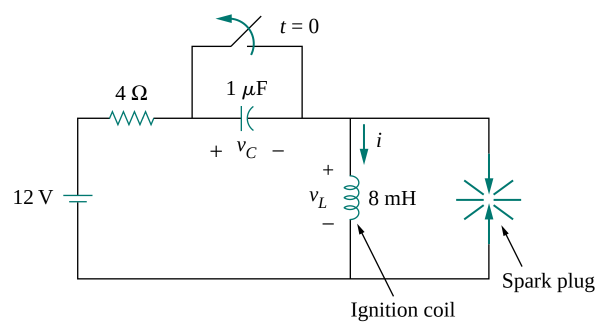


Figure 8.52 Automobile ignition circuit.

EXAMPLE 8.16

Assuming that the switch in Fig. 8.52 is closed prior to $t = 0^-$, find the inductor voltage v_L for $t > 0$.

Solution:

If the switch is closed prior to $t = 0^-$ and the circuit is in steady state, then

$$i(0^-) = \frac{12}{4} = 3\text{ A}, \qquad v_C(0^-) = 0$$

At $t = 0^+$, the switch is opened. The continuity conditions require that

$$i(0^+) = 3\text{ A}, \qquad v_C(0^+) = 0 \tag{8.16.1}$$

We obtain $di(0^+)/dt$ from $v_L(0^+)$. Applying KVL to the mesh at $t = 0^+$ yields

$$-12 + 4i(0^+) + v_L(0^+) + v_C(0^+) = 0$$
$$-12 + 4 \times 3 + v_L(0^+) + 0 = 0 \qquad \Longrightarrow \qquad v_L(0^+) = 0$$

Hence,

$$\frac{di(0^+)}{dt} = \frac{v_L(0^+)}{L} = 0 \tag{8.16.2}$$

As $t \to \infty$, the system reaches steady state, so that the capacitor acts like an open circuit. Then

$$i(\infty) = 0 \tag{8.16.3}$$

If we apply KVL to the mesh for $t > 0$, we obtain

$$12 = Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt + v_C(0)$$

Taking the derivative of each term yields

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \tag{8.16.4}$$

We obtain the natural response by following the procedure in Section 8.3. Substituting $R = 4\ \Omega$, $L = 8$ mH, and $C = 1\ \mu$F, we get

$$\alpha = \frac{R}{2L} = 250, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = 1.118 \times 10^4$$

Since $\alpha < \omega_0$, the response is underdamped. The damped natural frequency is

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \simeq \omega_0 = 1.118 \times 10^4$$

The natural response is

$$i_n(t) = e^{-\alpha}(A\cos\omega_d t + B\sin\omega_d t) \tag{8.16.5}$$

where A and B are constants. The forced response is

$$i_f(t) = i(\infty) = 0 \tag{8.16.6}$$

so that the complete response is

$$i(t) = i_n(t) + i_f(t) = e^{-250t}(A\cos 11{,}180t + B\sin 11{,}180t) \tag{8.16.7}$$

We now determine A and B.

$$i(0) = 3 = A + 0 \qquad \Longrightarrow \qquad A = 3$$

Taking the derivative of Eq. (8.16.7),

$$\begin{aligned}\frac{di}{dt} &= -250e^{-250t}(A\cos 11{,}180t + B\sin 11{,}180t)\\ &\quad + e^{-250t}(-11{,}180A\sin 11{,}180t + 11{,}180B\cos 11{,}180t)\end{aligned}$$

Setting $t = 0$ and incorporating Eq. (8.16.2),

$$0 = -250A + 11{,}180B \qquad \Longrightarrow \qquad B = 0.0671$$

Thus

$$i(t) = e^{-250t}(3\cos 11{,}180t + 0.0671\sin 11{,}180t) \tag{8.16.8}$$

The voltage across the inductor is then

$$v_L(t) = L\frac{di}{dt} = -268e^{-250t} \sin 11{,}180t \tag{8.16.9}$$

This has a maximum value when sine is unity, that is, at $11{,}180t_0 = \pi/2$ or $t_0 = 140.5\ \mu$s. At time $= t_0$, the inductor voltage reaches its peak, which is

$$v_L(t_0) = -268e^{-250t_0} = -259\text{ V} \tag{8.16.10}$$

Although this is far less than the voltage range of 6000 to 10,000 V required to fire the spark plug in a typical automobile, a device known as a *transformer* (to be discussed in Chapter 13) is used to step up the inductor voltage to the required level.

PRACTICE PROBLEM 8.16

In Fig. 8.52, find the capacitor voltage v_C for $t > 0$.

Answer: $12 - 12e^{-250t} \cos 11{,}180t + 267.7e^{-250t} \sin 11{,}180t$ V.

8.11.2 Smoothing Circuits

In a typical digital communication system, the signal to be transmitted is first sampled. Sampling refers to the procedure of selecting samples of a signal for processing, as opposed to processing the entire signal. Each sample is converted into a binary number represented by a series of pulses. The pulses are transmitted by a transmission line such as a coaxial cable, twisted pair, or optical fiber. At the receiving end, the signal is applied to a digital-to-analog (D/A) converter whose output is a "staircase" function, that is, constant at each time interval. In order to recover the transmitted analog signal, the output is smoothed by letting it pass through a "smoothing" circuit, as illustrated in Fig. 8.53. An RLC circuit may be used as the smoothing circuit.

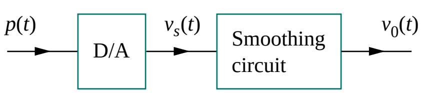


Figure 8.53 A series of pulses is applied to the digital-to-analog (D/A) converter, whose output is applied to the smoothing circuit.

EXAMPLE 8.17

The output of a D/A converter is shown in Fig. 8.54(a). If the RLC circuit in Fig. 8.54(b) is used as the smoothing circuit, determine the output voltage $v_o(t)$.

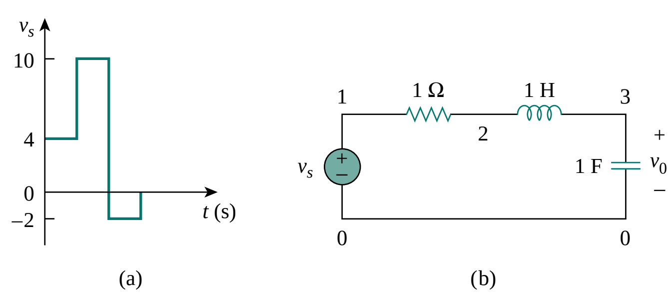
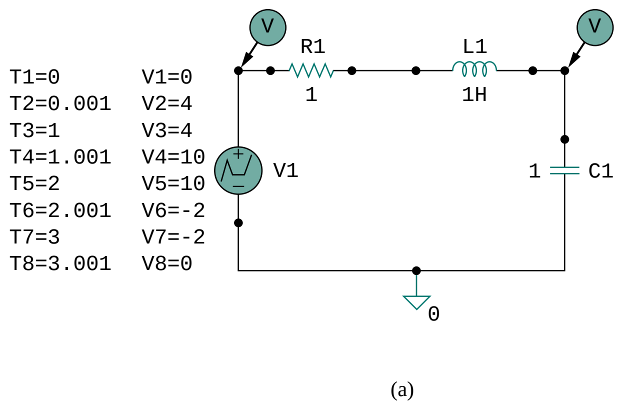


Figure 8.54 For Example 8.17: (a) output of a D/A converter, (b) an RLC smoothing circuit.

Solution:

This problem is best solved using *PSpice*. The schematic is shown in Fig. 8.55(a). The pulse in Fig. 8.54(a) is specified using the piecewise linear function. The attributes of V1 are set as T1 = 0, V1 = 0, T2 = 0.001, V2 = 4, T3 = 1, V3 = 4, and so on. To be able to plot both input and output voltages, we insert two voltage markers as shown. We select **Analysis/Setup/Transient** to open up the *Transient Analysis* dialog box and set *Final Time* as 6 s. Once the schematic is saved, we select **Analysis/Simulate** to run Probe and obtain the plots shown in Fig. 8.55(b).



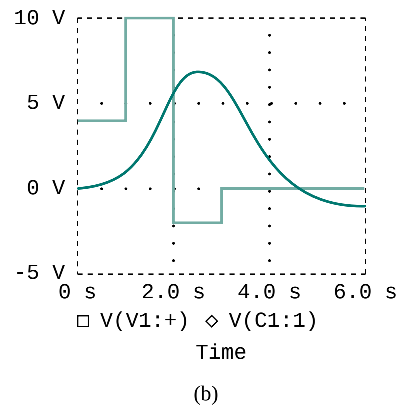


Figure 8.55 For Example 8.17: (a) schematic, (b) input and output voltages.

PRACTICE PROBLEM 8.17

Rework Example 8.17 if the output of the D/A converter is as shown in Fig. 8.56.

Answer: See Fig. 8.57.

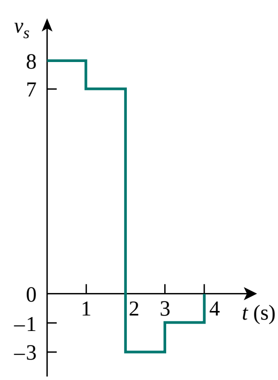


Figure 8.56 For Practice Prob. 8.17.

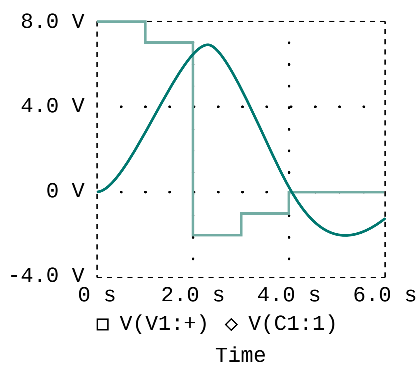


Figure 8.57 Result of Practice Prob. 8.17.

8.12 SUMMARY

1. The determination of the initial values $x(0)$ and $dx(0)/dt$ and final value $x(\infty)$ is crucial to analyzing second-order circuits.
2. The RLC circuit is second-order because it is described by a second-order differential equation. Its characteristic equation is $s^2 + 2\alpha s + \omega_0^2 = 0$, where α is the damping factor and ω_0 is the undamped natural frequency. For a series circuit, $\alpha = R/2L$, for a parallel circuit $\alpha = 1/2RC$, and for both cases $\omega_0 = 1/\sqrt{LC}$.
3. If there are no independent sources in the circuit after switching (or sudden change), we regard the circuit as source-free. The complete solution is the natural response.
4. The natural response of an RLC circuit is overdamped, underdamped, or critically damped, depending on the roots of the characteristic equation. The response is critically damped when the roots are equal ($s_1 = s_2$ or $\alpha = \omega_0$), overdamped when the roots are real and unequal ($s_1 \neq s_2$ or $\alpha > \omega_0$), or underdamped when the roots are complex conjugate ($s_1 = s_2^*$ or $\alpha < \omega_0$).
5. If independent sources are present in the circuit after switching, the complete response is the sum of the natural response and the forced or steady-state response.
6. *PSpice* is used to analyze RLC circuits in the same way as for RC or RL circuits.
7. Two circuits are dual if the mesh equations that describe one circuit have the same form as the nodal equations that describe the other. The analysis of one circuit gives the analysis of its dual circuit.
8. The automobile ignition circuit and the smoothing circuit are typical applications of the material covered in this chapter.

REVIEW QUESTIONS

8.1 For the circuit in Fig. 8.58, the capacitor voltage at $t = 0^-$ (just before the switch is closed) is:

(a) 0 V (b) 4 V (c) 8 V (d) 12 V

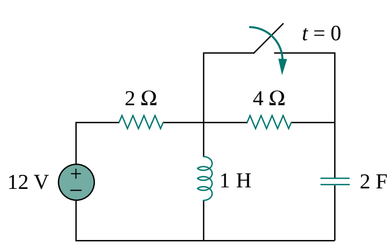


Figure 8.58 For Review Questions 8.1 and 8.2.

8.2 For the circuit in Fig. 8.58, the initial inductor current (at $t = 0$) is:

(a) 0 A (b) 2 A (c) 6 A (d) 12 A

8.3 When a step input is applied to a second-order circuit, the final values of the circuit variables are found by:

(a) Replacing capacitors with closed circuits and inductors with open circuits.
(b) Replacing capacitors with open circuits and inductors with closed circuits.
(c) Doing neither of the above.

8.4 If the roots of the characteristic equation of an RLC circuit are -2 and -3, the response is:

(a) $(A\cos 2t + B\sin 2t)e^{-3t}$
(b) $(A + 2Bt)e^{-3t}$
(c) $Ae^{-2t} + Bte^{-3t}$
(d) $Ae^{-2t} + Be^{-3t}$

where A and B are constants.

8.5 In a series RLC circuit, setting $R = 0$ will produce:

(a) an overdamped response

(b) a critically damped response
(c) an underdamped response
(d) an undamped response
(e) none of the above

8.6 A parallel RLC circuit has $L = 2$ H and $C = 0.25$ F. The value of R that will produce unity damping factor is:
(a) 0.5 Ω (b) 1 Ω (c) 2 Ω (d) 4 Ω

8.7 Refer to the series RLC circuit in Fig. 8.59. What kind of response will it produce?
(a) overdamped
(b) underdamped
(c) critically damped
(d) none of the above

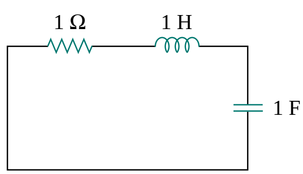


Figure 8.59 For Review Question 8.7.

8.8 Consider the parallel RLC circuit in Fig. 8.60. What type of response will it produce?
(a) overdamped
(b) underdamped
(c) critically damped
(d) none of the above

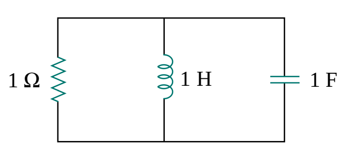


Figure 8.60 For Review Question 8.8.

8.9 Match the circuits in Fig. 8.61 with the following items:
(i) first-order circuit
(ii) second-order series circuit
(iii) second-order parallel circuit
(iv) none of the above

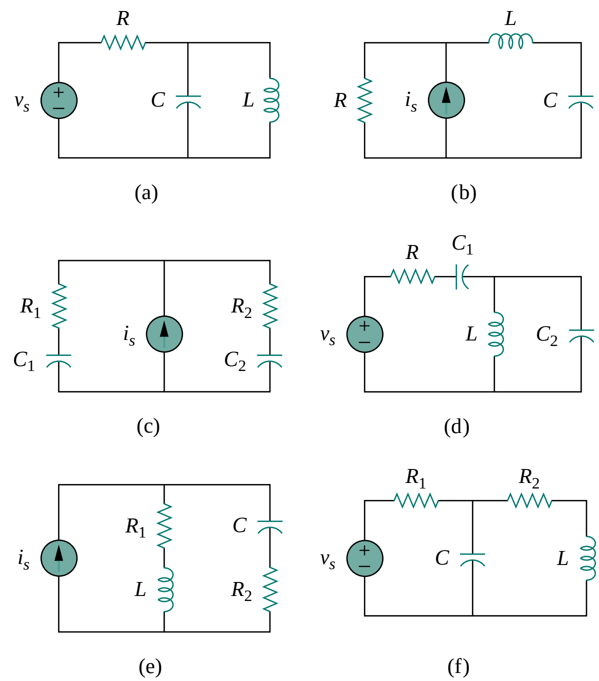


Figure 8.61 For Review Question 8.9.

8.10 In an electric circuit, the dual of resistance is:
(a) conductance (b) inductance
(c) capacitance (d) open circuit
(e) short circuit

Answers: 8.1a, 8.2c, 8.3b, 8.4d, 8.5d, 8.6c, 8.7b, 8.8b, 8.9 (i)-c, (ii)-b,e, (iii)-a, (iv)-d,f, 8.10a.

PROBLEMS

Section 8.2 Finding Initial and Final Values

8.1 For the circuit in Fig. 8.62, find:
(a) $i(0^+)$ and $v(0^+)$,
(b) $di(0^+)/dt$ and $dv(0^+)/dt$,
(c) $i(\infty)$ and $v(\infty)$.

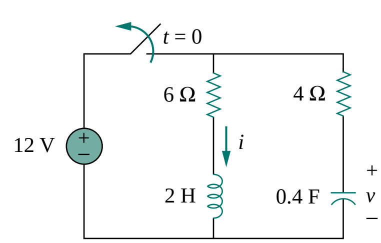


Figure 8.62 For Prob. 8.1.

8.2 In the circuit of Fig. 8.63, determine:
(a) $i_R(0^+)$, $i_L(0^+)$, and $i_C(0^+)$,
(b) $di_R(0^+)/dt$, $di_L(0^+)/dt$, and $di_C(0^+)/dt$,
(c) $i_R(\infty)$, $i_L(\infty)$, and $i_C(\infty)$.

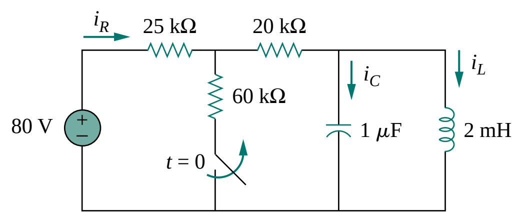


Figure 8.63 For Prob. 8.2.

8.3 Refer to the circuit shown in Fig. 8.64. Calculate:
(a) $i_L(0^+)$, $v_C(0^+)$, and $v_R(0^+)$,
(b) $di_L(0^+)/dt$, $dv_C(0^+)/dt$, and $dv_R(0^+)/dt$,
(c) $i_L(\infty)$, $v_C(\infty)$, and $v_R(\infty)$.

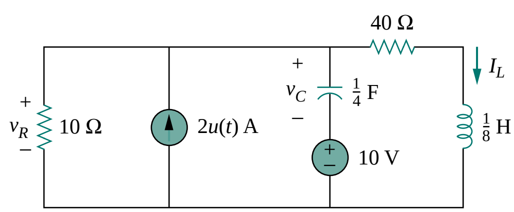


Figure 8.64 For Prob. 8.3.

8.4 In the circuit of Fig. 8.65, find:
(a) $v(0^+)$ and $i(0^+)$,
(b) $dv(0^+)/dt$ and $di(0^+)/dt$,
(c) $v(\infty)$ and $i(\infty)$.

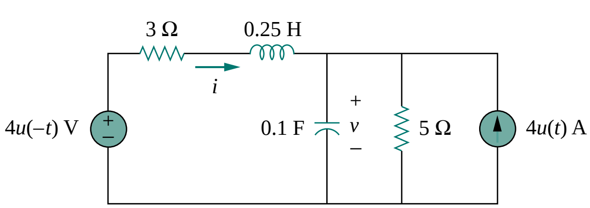


Figure 8.65 For Prob. 8.4.

8.5 Refer to the circuit in Fig. 8.66. Determine:
(a) $i(0^+)$ and $v(0^+)$,
(b) $di(0^+)/dt$ and $dv(0^+)/dt$,
(c) $i(\infty)$ and $v(\infty)$.

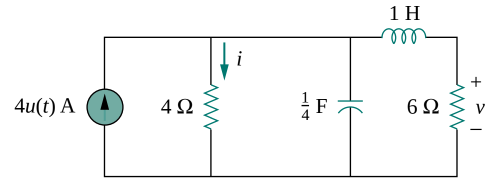


Figure 8.66 For Prob. 8.5.

8.6 In the circuit of Fig. 8.67, find:
(a) $v_R(0^+)$ and $v_L(0^+)$,
(b) $dv_R(0^+)/dt$ and $dv_L(0^+)/dt$,
(c) $v_R(\infty)$ and $v_L(\infty)$.

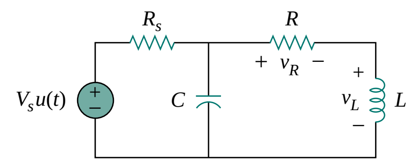


Figure 8.67 For Prob. 8.6.

Section 8.3 Source-Free Series *RLC* Circuit

8.7 The voltage in an RLC network is described by the differential equation

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 4v = 0$$

subject to the initial conditions $v(0) = 1$ and $dv(0)/dt = -1$. Determine the characteristic equation. Find $v(t)$ for $t > 0$.

8.8 The branch current in an RLC circuit is described by the differential equation

$$\frac{d^2i}{dt^2} + 6\frac{di}{dt} + 9i = 0$$

and the initial conditions are $i(0) = 0$, $di(0)/dt = 4$. Obtain the characteristic equation and determine $i(t)$ for $t > 0$.

8.9 The current in an RLC circuit is described by

$$\frac{d^2i}{dt^2} + 10\frac{di}{dt} + 25i = 0$$

If $i(0) = 10$ and $di(0)/dt = 0$, find $i(t)$ for $t > 0$.

8.10 The differential equation that describes the voltage in an RLC network is

$$\frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 4v = 0$$

Given that $v(0) = 0$, $dv(0)/dt = 10$, obtain $v(t)$.

8.11 The natural response of an RLC circuit is described by the differential equation

$$\frac{d^2v}{dt^2} + 2\frac{dv}{dt} + v = 0$$

for which the initial conditions are $v(0) = 10$ and $dv(0)/dt = 0$. Solve for $v(t)$.

8.12 If $R = 20\ \Omega$, $L = 0.6$ H, what value of C will make an RLC series circuit:
(a) overdamped, (b) critically damped,
(c) underdamped?

8.13 For the circuit in Fig. 8.68, calculate the value of R needed to have a critically damped response.

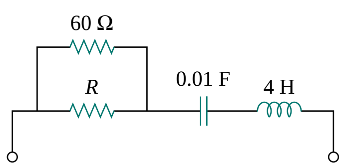


Figure 8.68 For Prob. 8.13.

8.14 Find $v(t)$ for $t > 0$ if $v(0) = 6$ V and $i(0) = 2$ A in the circuit shown in Fig. 8.69.

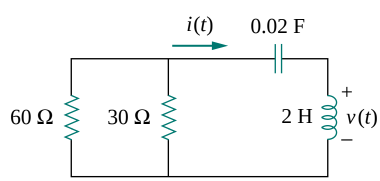


Figure 8.69 For Prob. 8.14.

8.15 The responses of a series RLC circuit are

$$v_C(t) = 30 - 10e^{-20t} + 30e^{-10t}\ \text{V}$$

$$i_L(t) = 40e^{-20t} - 60e^{-10t}\ \text{mA}$$

where v_C and i_L are the capacitor voltage and inductor current, respectively. Determine the values of R, L, and C.

8.16 Find $i(t)$ for $t > 0$ in the circuit of Fig. 8.70.

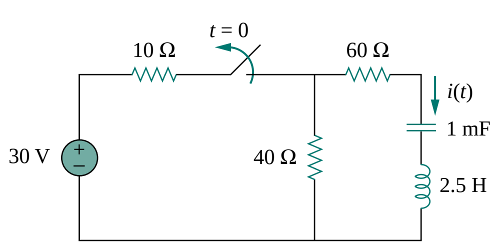


Figure 8.70 For Prob. 8.16.

8.17 Obtain $v(t)$ for $t > 0$ in the circuit of Fig. 8.71.

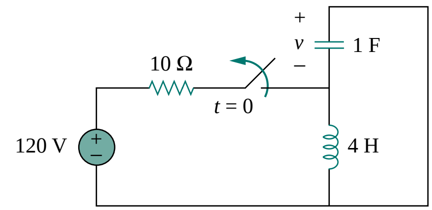


Figure 8.71 For Prob. 8.17.

8.18 The switch in the circuit of Fig. 8.72 has been closed for a long time but is opened at $t = 0$. Determine $i(t)$ for $t > 0$.

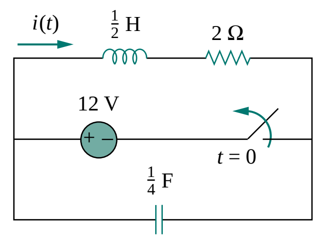


Figure 8.72 For Prob. 8.18.

*__8.19__ Calculate $v(t)$ for $t > 0$ in the circuit of Fig. 8.73.

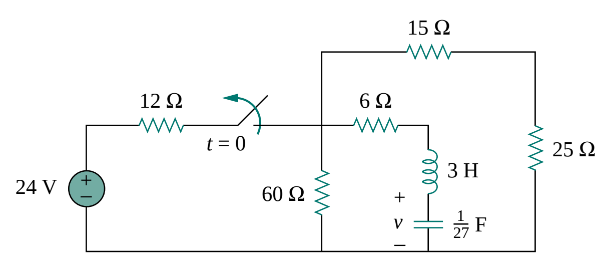


Figure 8.73 For Prob. 8.19.

Section 8.4 Source-Free Parallel *RLC* Circuit

8.20 For a parallel RLC circuit, the responses are

$$v_L(t) = 4e^{-20t}\cos 50t - 10e^{-20t}\sin 50t\ \text{V}$$

$$i_C(t) = -6.5e^{-20t}\cos 50t\ \text{mA}$$

where i_C and v_L are the capacitor current and inductor voltage, respectively. Determine the values of R, L, and C.

8.21 For the network in Fig. 8.74, what value of C is needed to make the response underdamped with unity damping factor ($\alpha = 1$)?

*An asterisk indicates a challenging problem.

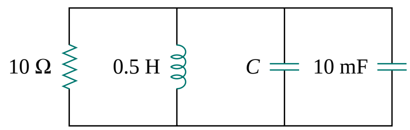


Figure 8.74 For Prob. 8.21.

8.22 Find $v(t)$ for $t > 0$ in the circuit in Fig. 8.75.

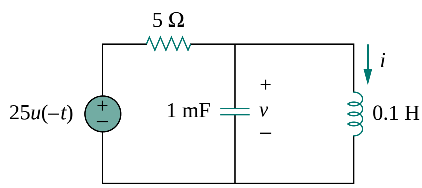


Figure 8.75 For Prob. 8.22.

8.23 In the circuit in Fig. 8.76, calculate $i_o(t)$ and $v_o(t)$ for $t > 0$.

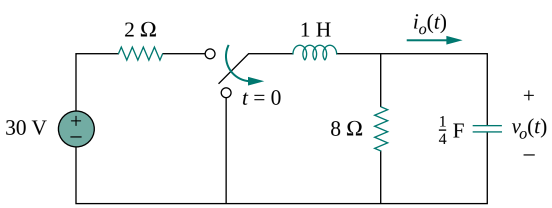


Figure 8.76 For Prob. 8.23.

Section 8.5 Step Response of a Series *RLC* Circuit

8.24 The step response of an RLC circuit is given by

$$\frac{d^2i}{dt^2} + 2\frac{di}{dt} + 5i = 10$$

Given that $i(0) = 2$ and $di(0)/dt = 4$, solve for $i(t)$.

8.25 A branch voltage in an RLC circuit is described by

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 8v = 24$$

If the initial conditions are $v(0) = 0 = dv(0)/dt$, find $v(t)$.

8.26 The current in an RLC network is governed by the differential equation

$$\frac{d^2i}{dt^2} + 3\frac{di}{dt} + 2i = 4$$

subject to $i(0) = 1$, $di(0)/dt = -1$. Solve for $i(t)$.

8.27 Solve the following differential equations subject to the specified initial conditions

(a) $d^2v/dt^2 + 4v = 12$, $v(0) = 0$, $dv(0)/dt = 2$

(b) $d^2i/dt^2 + 5\,di/dt + 4i = 8$, $i(0) = -1$, $di(0)/dt = 0$

(c) $d^2v/dt^2 + 2\,dv/dt + v = 3$, $v(0) = 5$, $dv(0)/dt = 1$

(d) $d^2i/dt^2 + 2\,di/dt + 5i = 10$, $i(0) = 4$, $di(0)/dt = -2$

8.28 Consider the circuit in Fig. 8.77. Find $v_L(0)$ and $v_C(0)$.

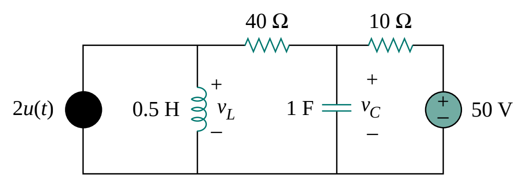


Figure 8.77 For Prob. 8.28.

8.29 For the circuit in Fig. 8.78, find $v(t)$ for $t > 0$.

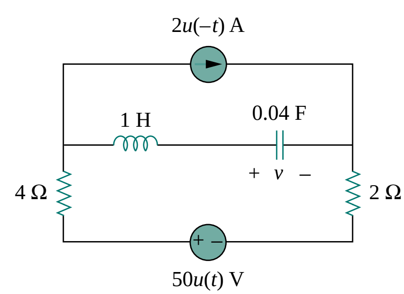


Figure 8.78 For Prob. 8.29.

8.30 Find $v(t)$ for $t > 0$ in the circuit in Fig. 8.79.

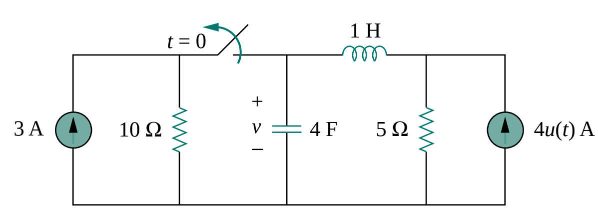


Figure 8.79 For Prob. 8.30.

8.31 Calculate $i(t)$ for $t > 0$ in the circuit in Fig. 8.80.

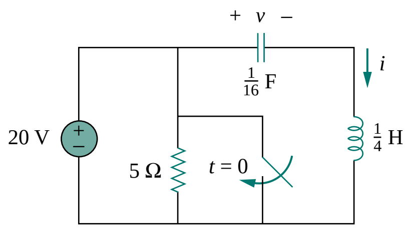


Figure 8.80 For Prob. 8.31.

8.32 Determine $v(t)$ for $t > 0$ in the circuit in Fig. 8.81.

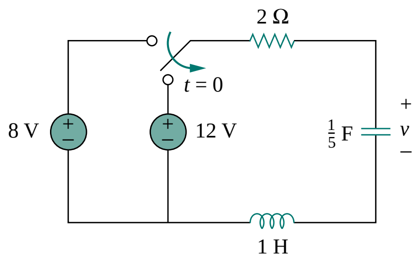


Figure 8.81 For Prob. 8.32.

8.33 Obtain $v(t)$ and $i(t)$ for $t > 0$ in the circuit in Fig. 8.82.

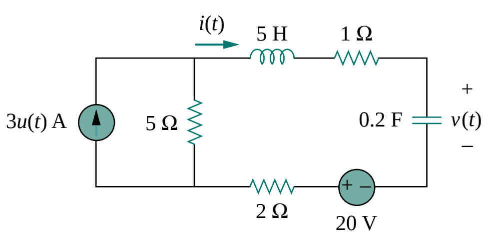


Figure 8.82 For Prob. 8.33.

*__8.34__ For the network in Fig. 8.83, solve for $i(t)$ for $t > 0$.

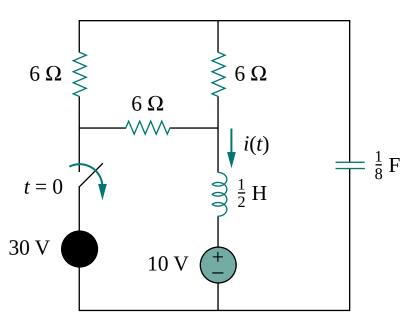


Figure 8.83 For Prob. 8.34.

8.35 Refer to the circuit in Fig. 8.84. Calculate $i(t)$ for $t > 0$.

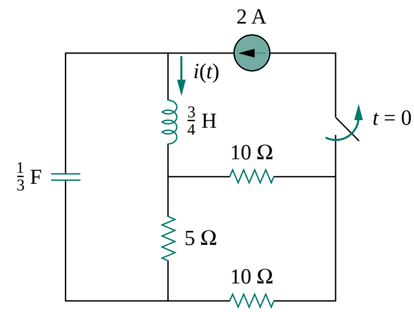


Figure 8.84 For Prob. 8.35.

8.36 Determine $v(t)$ for $t > 0$ in the circuit in Fig. 8.85.

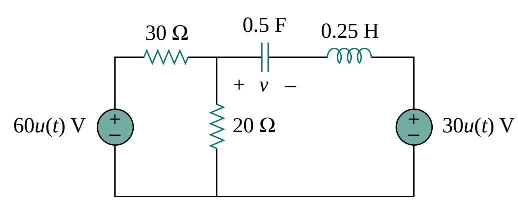


Figure 8.85 For Prob. 8.36.

8.37 The switch in the circuit of Fig. 8.86 is moved from position a to b at $t = 0$. Determine $i(t)$ for $t > 0$.

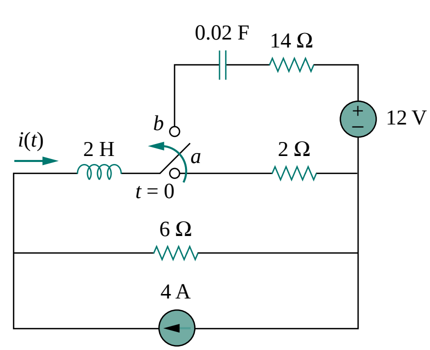


Figure 8.86 For Prob. 8.37.

*8.38 For the network in Fig. 8.87, find $i(t)$ for $t > 0$.

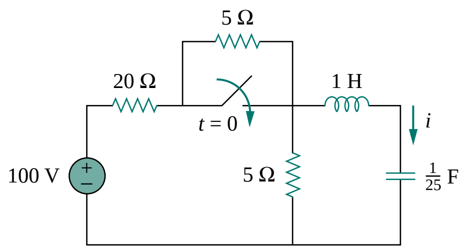


Figure 8.87 For Prob. 8.38.

*8.39 Given the network in Fig. 8.88, find $v(t)$ for $t > 0$.

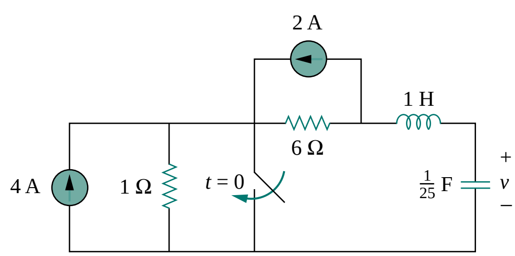


Figure 8.88 For Prob. 8.39.

Section 8.6 Step Response of a Parallel *RLC* Circuit

8.40 In the circuit of Fig. 8.89, find $v(t)$ and $i(t)$ for $t > 0$. Assume $v(0) = 0$ V and $i(0) = 1$ A.

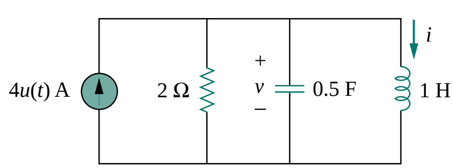


Figure 8.89 For Prob. 8.40.

8.41 Find $i(t)$ for $t > 0$ in the circuit in Fig. 8.90.

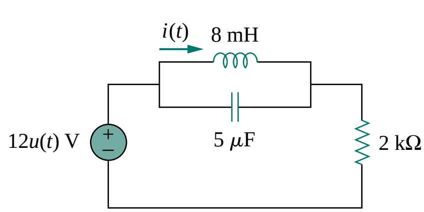


Figure 8.90 For Prob. 8.41.

8.42 Find the output voltage $v_o(t)$ in the circuit of Fig. 8.91.

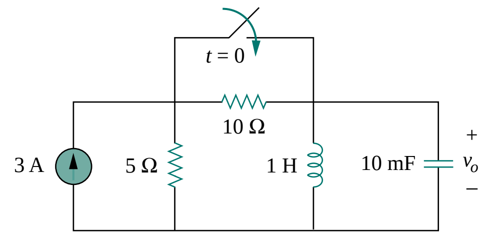


Figure 8.91 For Prob. 8.42.

8.43 Given the circuit in Fig. 8.92, find $i(t)$ and $v(t)$ for $t > 0$.

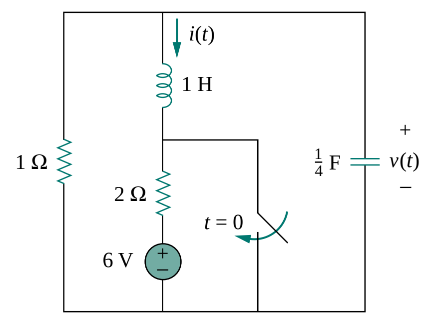


Figure 8.92 For Prob. 8.43.

8.44 Determine $i(t)$ for $t > 0$ in the circuit of Fig. 8.93.

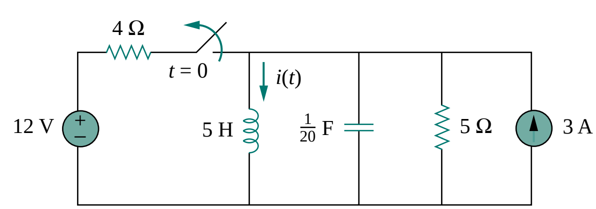


Figure 8.93 For Prob. 8.44.

8.45 For the circuit in Fig. 8.94, find $i(t)$ for $t > 0$.

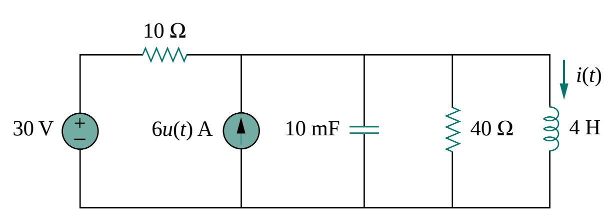


Figure 8.94 For Prob. 8.45.

8.46 Find $v(t)$ for $t > 0$ in the circuit in Fig. 8.95.

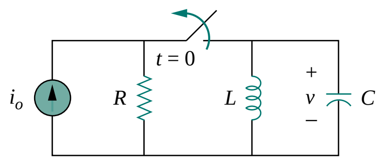


Figure 8.95 For Prob. 8.46.

Section 8.7 General Second-Order Circuits

8.47 Derive the second-order differential equation for v_o in the circuit of Fig. 8.96.

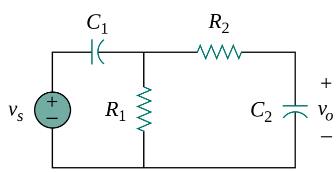


Figure 8.96 For Prob. 8.47.

8.48 Obtain the differential equation for v_o in the circuit in Fig. 8.97.

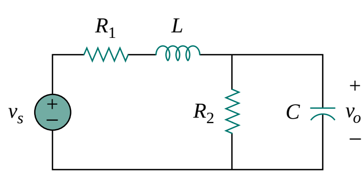


Figure 8.97 For Prob. 8.48.

8.49 For the circuit in Fig. 8.98, find $v(t)$ for $t > 0$. Assume that $v(0^+) = 4$ V and $i(0^+) = 2$ A.

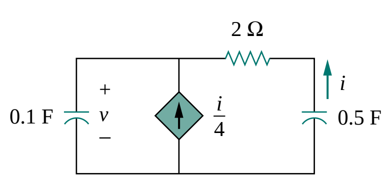


Figure 8.98 For Prob. 8.49.

8.50 In the circuit of Fig. 8.99, find $i(t)$ for $t > 0$.

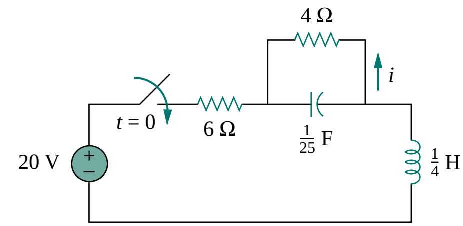


Figure 8.99 For Prob. 8.50.

8.51 If the switch in Fig. 8.100 has been closed for a long time before $t = 0$ but is opened at $t = 0$, determine:

(a) the characteristic equation of the circuit,

(b) i_x and v_R for $t > 0$.

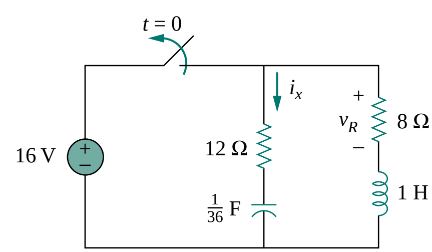


Figure 8.100 For Prob. 8.51.

8.52 Obtain i_1 and i_2 for $t > 0$ in the circuit of Fig. 8.101.

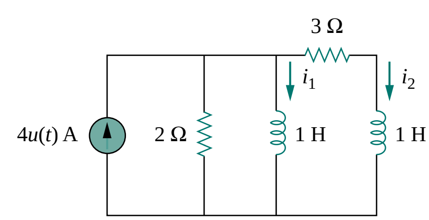


Figure 8.101 For Prob. 8.52.

8.53 For the circuit in Prob. 8.5, find i and v for $t > 0$.

8.54 Find the response $v_R(t)$ for $t > 0$ in the circuit in Fig. 8.102. Let $R = 3\ \Omega$, $L = 2$ H, and $C = 1/18$ F.

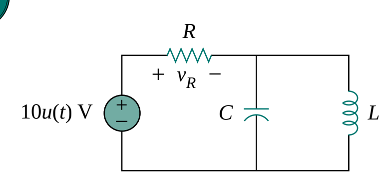


Figure 8.102 For Prob. 8.54.

Section 8.8 Second-Order Op Amp Circuits

8.55 Derive the differential equation relating v_o to v_s in the op amp circuit of Fig. 8.103.

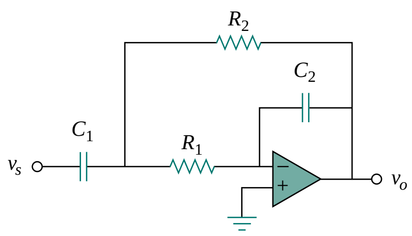


Figure 8.103 For Prob. 8.55.

8.56 Obtain the differential equation for $v_o(t)$ in the network of Fig. 8.104.

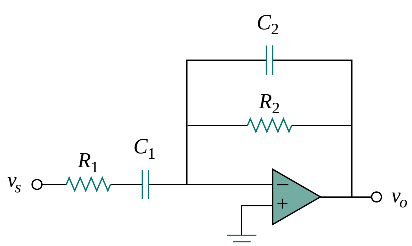


Figure 8.104 For Prob. 8.56.

8.57 Determine the differential equation for the op amp circuit in Fig. 8.105. If $v_1(0^+) = 2$ V and $v_2(0^+) = 0$ V, find v_o for $t > 0$. Let $R = 100$ kΩ and $C = 1\ \mu$F.

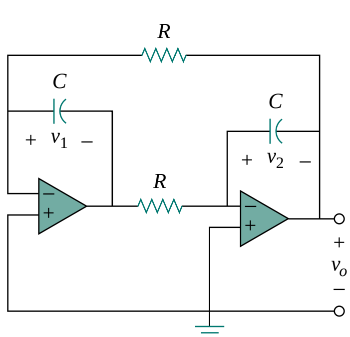
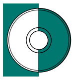


Figure 8.105 For Prob. 8.57.

8.58 Given that $v_s = 2u(t)$ V in the op amp circuit of Fig. 8.106, find $v_o(t)$ for $t > 0$. Let $R_1 = R_2 = 10$ kΩ, $R_3 = 20$ kΩ, $R_4 = 40$ kΩ, $C_1 = C_2 = 100\ \mu$F.



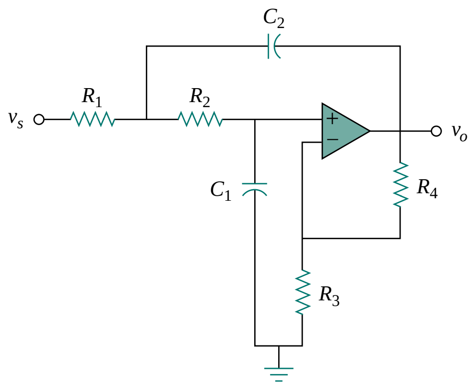
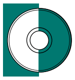


Figure 8.106 For Prob. 8.58.

*__8.59__ In the op amp circuit of Fig. 8.107, determine $v_o(t)$ for $t > 0$. Let $v_{in} = u(t)$ V, $R_1 = R_2 = 10$ kΩ, $C_1 = C_2 = 100\ \mu$F.



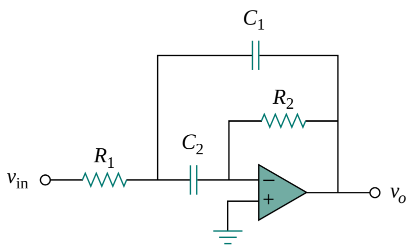


Figure 8.107 For Prob. 8.59.

Section 8.9 *PSpice* Analysis of *RLC* Circuit

8.60 For the step function $v_s = u(t)$, use *PSpice* to find the response $v(t)$ for $0 < t < 6$ s in the circuit of Fig. 8.108.

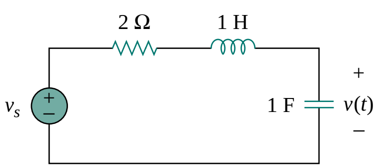


Figure 8.108 For Prob. 8.60.

8.61 Given the source-free circuit in Fig. 8.109, use *PSpice* to get $i(t)$ for $0 < t < 20$ s. Take $v(0) = 30$ V and $i(0) = 2$ A.

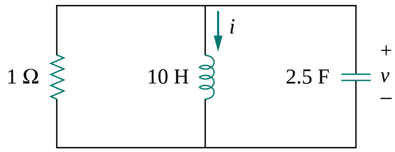


Figure 8.109 For Prob. 8.61.

8.62 Obtain $v(t)$ for $0 < t < 4$ s in the circuit of Fig. 8.110 using *PSpice*.

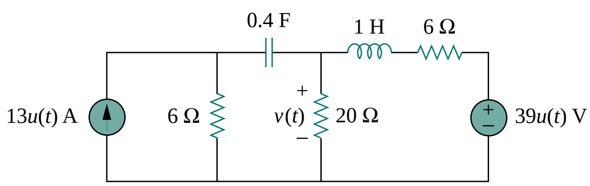


Figure 8.110 For Prob. 8.62.

8.63 Rework Prob. 8.23 using *PSpice*. Plot $v_o(t)$ for $0 < t < 4$ s.

Section 8.10 Duality

8.64 Draw the dual of the network in Fig. 8.111.

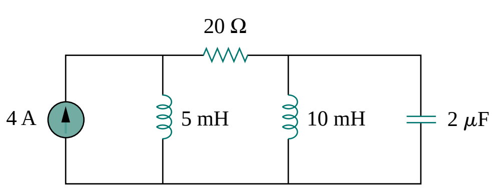


Figure 8.111 For Prob. 8.64.

8.65 Obtain the dual of the circuit in Fig. 8.112.

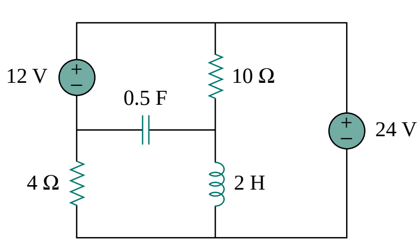


Figure 8.112 For Prob. 8.65.

8.66 Find the dual of the circuit in Fig. 8.113.

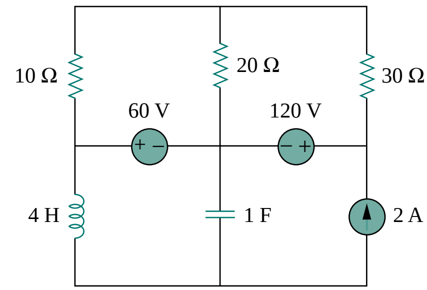


Figure 8.113 For Prob. 8.66.

8.67 Draw the dual of the circuit in Fig. 8.114.

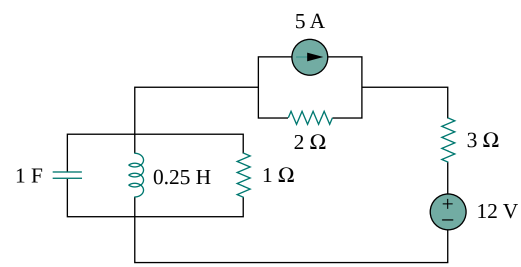


Figure 8.114 For Prob. 8.67.

Section 8.11 Applications

8.68 An automobile airbag igniter is modeled by the circuit in Fig. 8.115. Determine the time it takes the voltage across the igniter to reach its first peak after switching from A to B. Let $R = 3\ \Omega$, $C = 1/30$ F, and $L = 60$ mH.

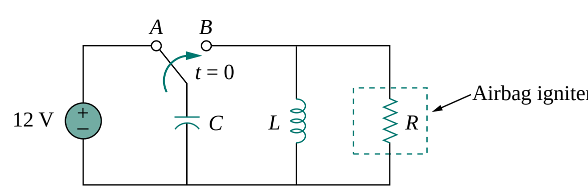


Figure 8.115 For Prob. 8.68.

8.69 A passive interface is to be designed to connect an electric motor to an ideal voltage source. If the motor is modeled as a 40-mH inductor in parallel with a 16-Ω resistor, design the interface circuit so that the overall circuit is critically damped at the natural frequency of 60 Hz.

COMPREHENSIVE PROBLEMS

8.70 A mechanical system is modeled by a series RLC circuit. It is desired to produce an overdamped response with time constants 0.1 ms and 0.5 ms. If a series 50-kΩ resistor is used, find the values of L and C.

8.71 An oscillogram can be adequately modeled by a second-order system in the form of a parallel RLC circuit. It is desired to give an underdamped voltage across a 200-Ω resistor. If the damping frequency is 4 kHz and the time constant of the envelope is 0.25 s, find the necessary values of L and C.

8.72 The circuit in Fig. 8.116 is the electrical analog of body functions used in medical schools to study convulsions. The analog is as follows:

C_1 = Volume of fluid in a drug

C_2 = Volume of blood stream in a specified region

R_1 = Resistance in the passage of the drug from the input to the blood stream

R_2 = Resistance of the excretion mechanism, such as kidney, etc.

v_0 = Initial concentration of the drug dosage

$v(t)$ = Percentage of the drug in the blood stream

Find $v(t)$ for $t > 0$ given that $C_1 = 0.5\ \mu$F, $C_2 = 5\ \mu$F, $R_1 = 5$ MΩ, $R_2 = 2.5$ MΩ, and $v_0 = 60u(t)$ V.

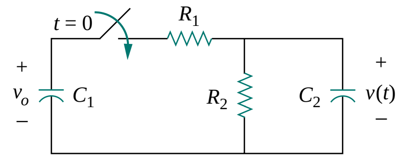


Figure 8.116 For Prob. 8.72.

8.73 Figure 8.117 shows a typical tunnel-diode oscillator circuit. The diode is modeled as a nonlinear resistor with $i_D = f(v_D)$, i.e., the diode current is a nonlinear function of the voltage across the diode. Derive the differential equation for the circuit in terms of v and i_D.

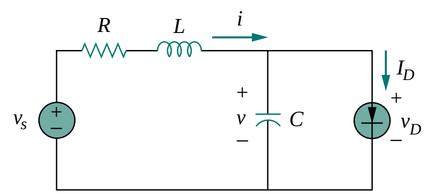


Figure 8.117 For Prob. 8.73.