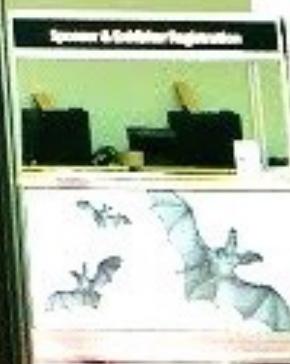


Computational Physics

Fourier Transforms

“Always code
as if the guy
who ends up
maintaining
your code will
be a violent
psychopath
who knows
where you live”

-Martin Golding



Fourier Transforms

- Functions can be represented in many different ways
- We normally use “real” space - $f(x)$
- Generally, arbitrarily many transforms exist to represent functions in different spaces - $F(y) = Af(x)$ for some matrix A and some new variable y .
Iff A is invertible, $f(x) = A^{-1}F(y)$
- One important basis nature has picked out is complex exponentials/sines and cosines. Fundamental across physics, particularly quantum mechanics.

Fundamental Definition

- $F(k) = \int f(x) * \exp(-2\pi i k x) dx$ (where $k = l/\omega$)
- Integral gets rid of x , replaces with k . New function has amplitude and phase as a function of k .
- Quantum mechanics - de Broglie says $p = \hbar k$. So, Fourier transform position to get momentum.
- Fourier transform electric field $E(t)$ to get frequency spectrum.
- Fourier transform to get fast correlations, convolutions, many other things.

DFT (Discrete FT)

- Computers don't do continuous. Not enough RAM for starters...
- Function exists over finite range in x at finite number of points.
- If input function has n points, output can only have n k's.
- Gives rise to discrete Fourier Transform (DFT)
- $F(k) = \sum f(x) \exp(-2\pi i k x / N)$ for N points and $0 \leq k < N$
- What would DFT of $f(0)=1$, otherwise $f(x)=0$ look like?
- What would DFT of $f(x)=1$ look like?
- DFTs have subtle behaviours not seen in continuous, infinite FTs.

Inverse

- One way to think about DFT is as a matrix multiply.
- $F(k) = Af$, $A_{mn} = \exp(-2\pi i mn/N)$
- But look: $A_{mn} = A_{nm}$, so matrix is symmetric.
- Also, columns are orthogonal under conjugation:
 $\sum \exp(-2\pi i kx) \exp(2\pi i k'x) = \sum \exp(2\pi i (k'-k)x).$ N if $k' = k$, otherwise 0.
- So, $A^{-1} = 1/N * \text{conj}(A)$. IFT = $1/N \sum F(k) \exp(2\pi i kx)$.
- Get back to where we started by just doing another DFT with a sign flip, then divide by # of data points.
- Alternative: divide by \sqrt{N} in both DFT and IFT, (not standard computationally)

Numpy Complex

```
import numpy
def exp_prod(m,n,N):
    #define imaginary unity
    J=numpy.complex(0,1)
    #now rest of code is just like for real numbers
    x=numpy.arange(0.0,N)*2*J*numpy.pi/N
    return numpy.sum(numpy.exp(-1*x*m)*numpy.exp(x*n))
if __name__=="__main__":
    print exp_prod(0,0,8)
    print exp_prod(2,4,8)
    print exp_prod(3,3,8)
    print exp_prod(0,7,8)
```

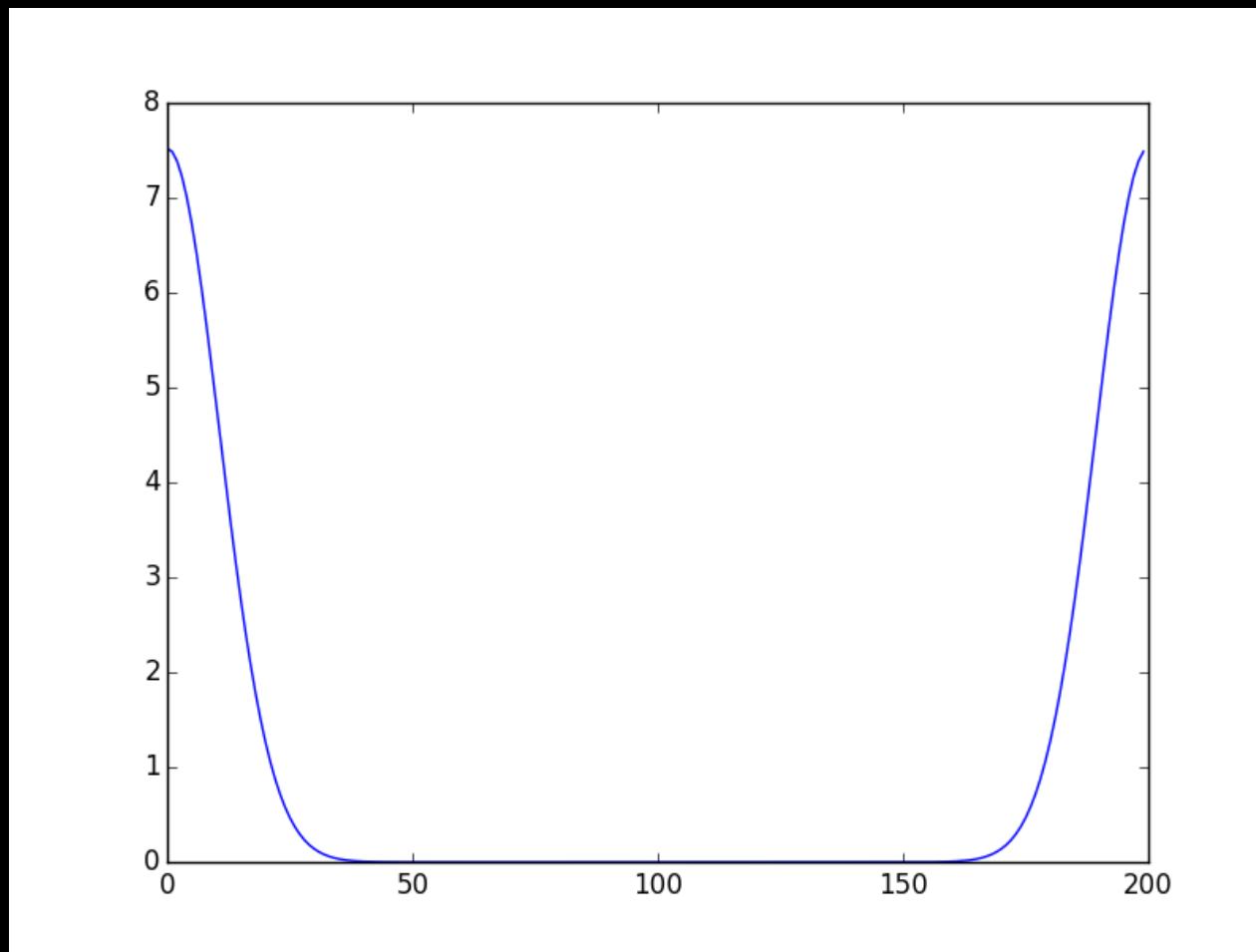
```
Jonathans-MacBook-Pro:lecture4 sievers$ python dft_columns.py
(8+0j)
(-4.28626379702e-16+4.4408920985e-16j)
(8+0j)
(3.44169137634e-15-1.11022302463e-15j)
Jonathans-MacBook-Pro:lecture4 sievers$
```

- Let's check orthogonality, need complex #'s.
- `numpy.complex(re,im)` will make a complex #
- `numpy` functions usually defined for complex #'s.

DFTs with Numpy

- Numpy has many Fourier Transform operations
- (for reasons to be seen) they are called *Fast Fourier Transforms* - FFT is one way of implementing DFTs.
- FFT's live in a submodule of numpy called FFT
- `xft=numpy.fft.fft(x)` takes DFT
- `x=numpy.fft.ifft(x)` takes inverse DFT
- Numpy normalizes such that `f==fft(ifft(f))==ifft(fft(f))`

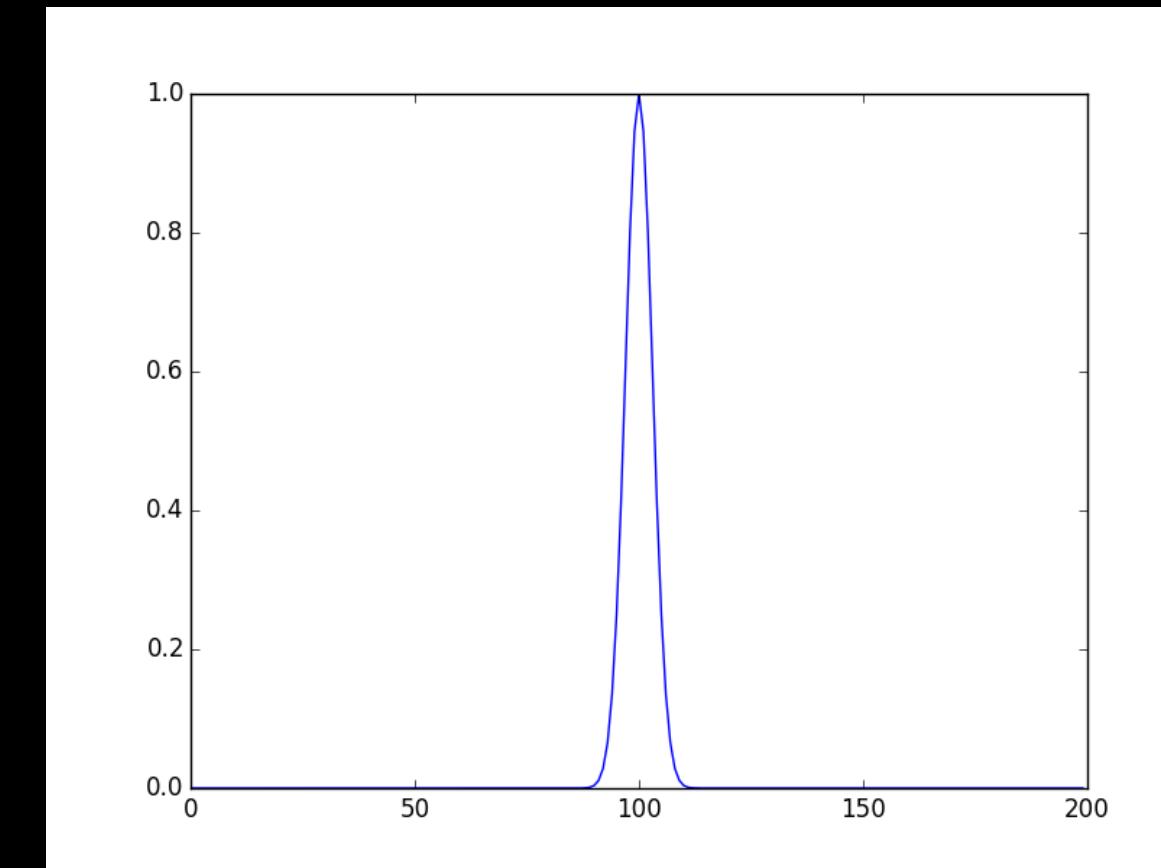
DFT in Action



- Right: input Gaussian
- Top: DFT of the Gaussian

```
import numpy
from matplotlib import pyplot as plt

x=numpy.arange(-10,10,0.1)
y=numpy.exp(-0.5*x**2/(0.3**2))
yft=numpy.fft.fft(y)
plt.plot(numpy.abs(yft))
plt.savefig('gauss_dft')
plt.show()
```



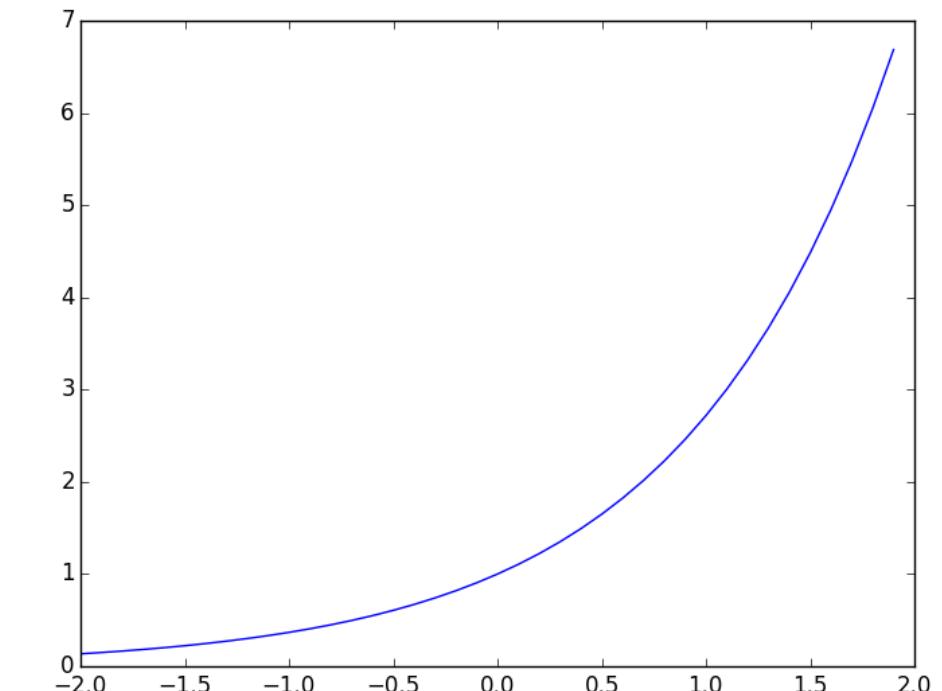
Periodicity

- $f(x) = \sum F(k) \exp(2\pi i kx/N)$
- What is $f(x+N)$? $\sum F(k) \exp(2\pi i k(x+N)/N)$
- $= \sum F(k) \exp(2\pi i k) \exp(2\pi i kx/N).$
- $\exp(2\pi i k) = 1$ for integer k , so $f(x+N) = f(x)$
- DFT's are periodic - they just repeat themselves ad infinitum.

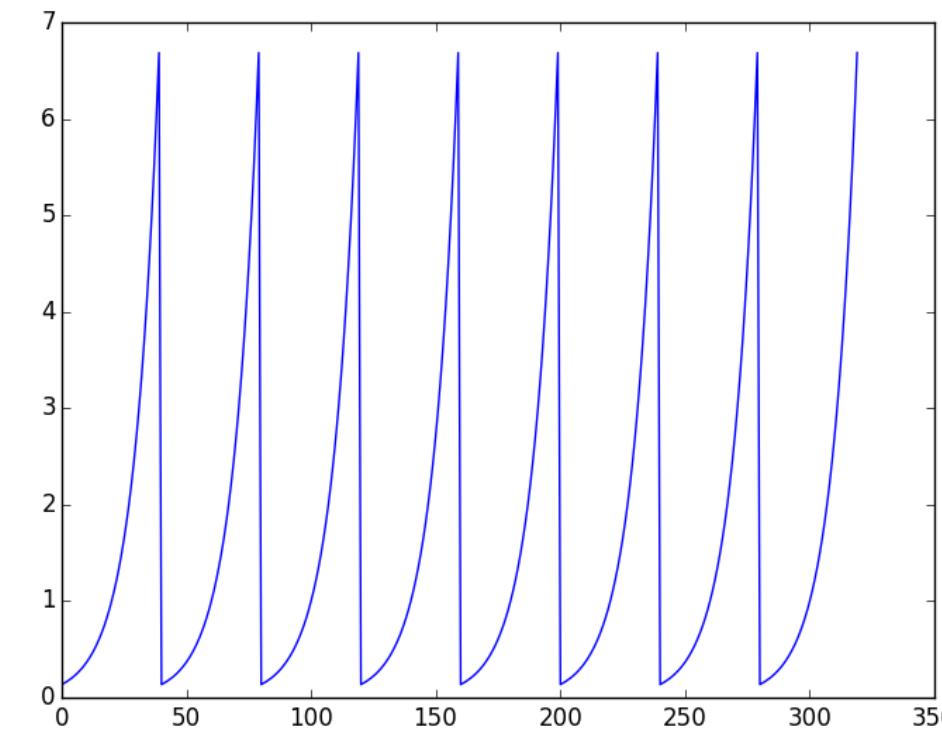
Periodicity

```
import numpy
from matplotlib import pyplot as plt

x=numpy.arange(-2,2,0.1)
y=numpy.exp(x)
plt.plot(x,y)
plt.savefig('fft_exp')
plt.show()
yy=numpy.concatenate((y,y))
yy=numpy.concatenate((yy,yy))
yy=numpy.concatenate((yy,yy))
plt.plot(yy)
plt.savefig('fft_exp_repeating')
plt.show()
```



- You may think you're taking top transform. You're not - you're taking the bottom one.
- In particular, jumps from right edge to left will strongly affect DFT



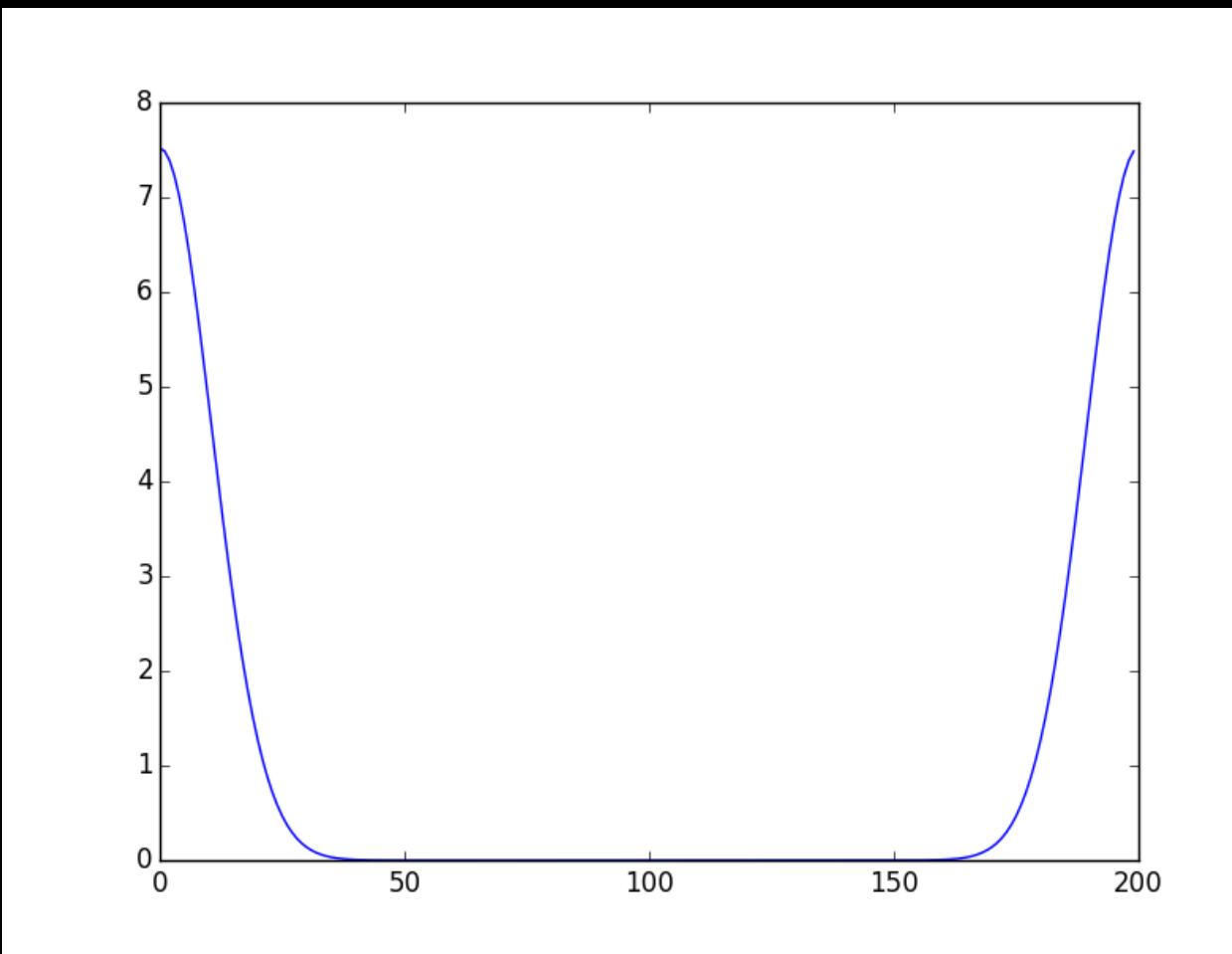
Aliasing

- $f(x) = \sum F(k) \exp(2\pi i kx/N)$
- What if I had higher frequency, $k > N$? let $k^* = k - N$ (i.e. k^* low freq.)
- $f(x) = \sum F(k) \exp(2\pi i (k^* + N)x/N) = \sum F(k) \exp(2\pi i x) \exp(2\pi i k^* x/N)$
- for x integer, middle term goes away: $\sum F(k^* + N) \exp(2\pi i k^* x/N)$
- High frequencies behave exactly like low frequencies - power has been *aliased* into main frequencies of DFT.
- Always keep this in mind! Make sure samples are fine enough to prevent aliasing.

Negative Frequencies

- All frequencies that are N apart behave identically
- DFT has frequencies up to $(N-1)$.
- Frequency $(N-1)$ equivalent to frequency (-1) . You will do better to think of DFT as giving frequencies $(-N/2, N/2)$ than frequencies $(0, N-1)$
- *Sampling theorem*: if function is band-limited - highest frequency is v - then I get full information if I sample twice per frequency, $dt=1/(2v)$. Factor of 2 comes from aliasing.

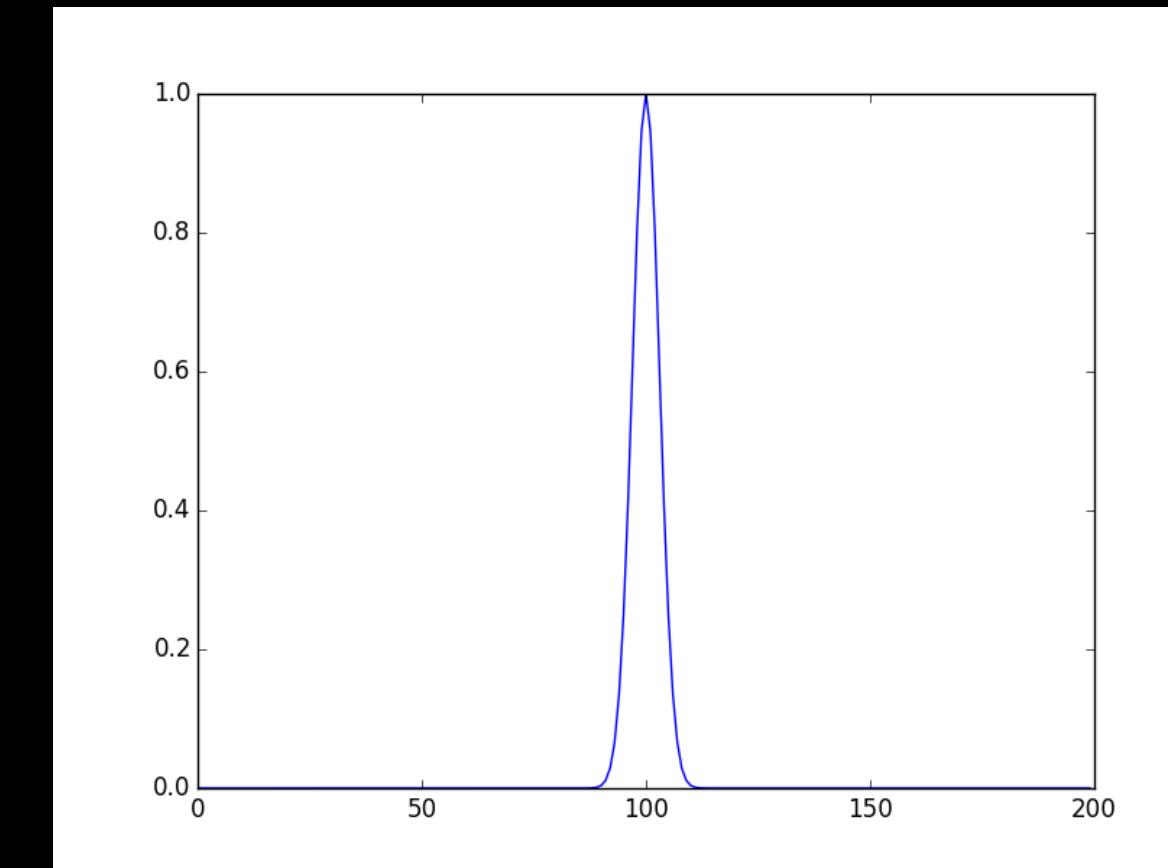
DFT in Action, Redux



- FFT makes more sense now - negative frequencies have been aliased to high frequency.

```
import numpy
from matplotlib import pyplot as plt

x=numpy.arange(-10,10,0.1)
y=numpy.exp(-0.5*x**2/(0.3**2))
yft=numpy.fft.fft(y)
plt.plot(numpy.abs(yft))
plt.savefig('gauss_dft')
plt.show()
```



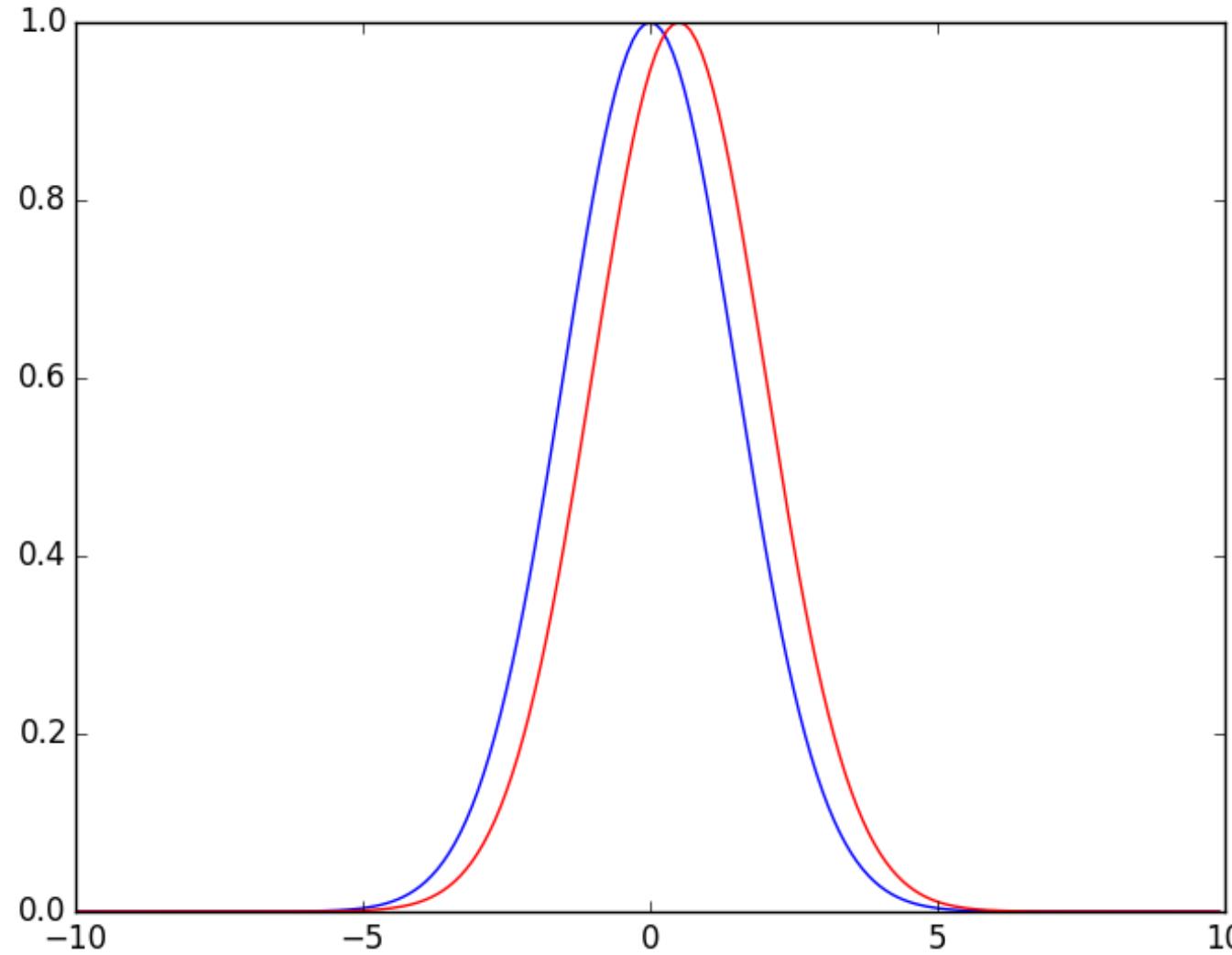
Flipping

- What is DFT of $f(-x)$?
- $\sum f(-x) \exp(-2\pi i k x / N)$, $x^* = -x$, $\sum f(x^*) \exp(-2\pi i k (-x^*) / N)$
- $DFT(f(-x)) = \sum f(x) \exp(2\pi i k x / N) = \text{conj}(F(k))$

Shifting

- What is $\text{FFT}(x+dx)$? $\sum f(x+dx) \exp(-2\pi i k x/N)$.
- $x^* = x + dx$: $F(k) = \sum f(x^*) \exp(-2\pi i k (x^* - dx)/N)$
- $F(k) = \exp(2\pi i k dx/N) \sum f(x^*) \exp(-2\pi i k x^*/N)$
- So, just apply a phase gradient to the DFT to shift in x

Shifting Example



```
import numpy
from matplotlib import pyplot as plt

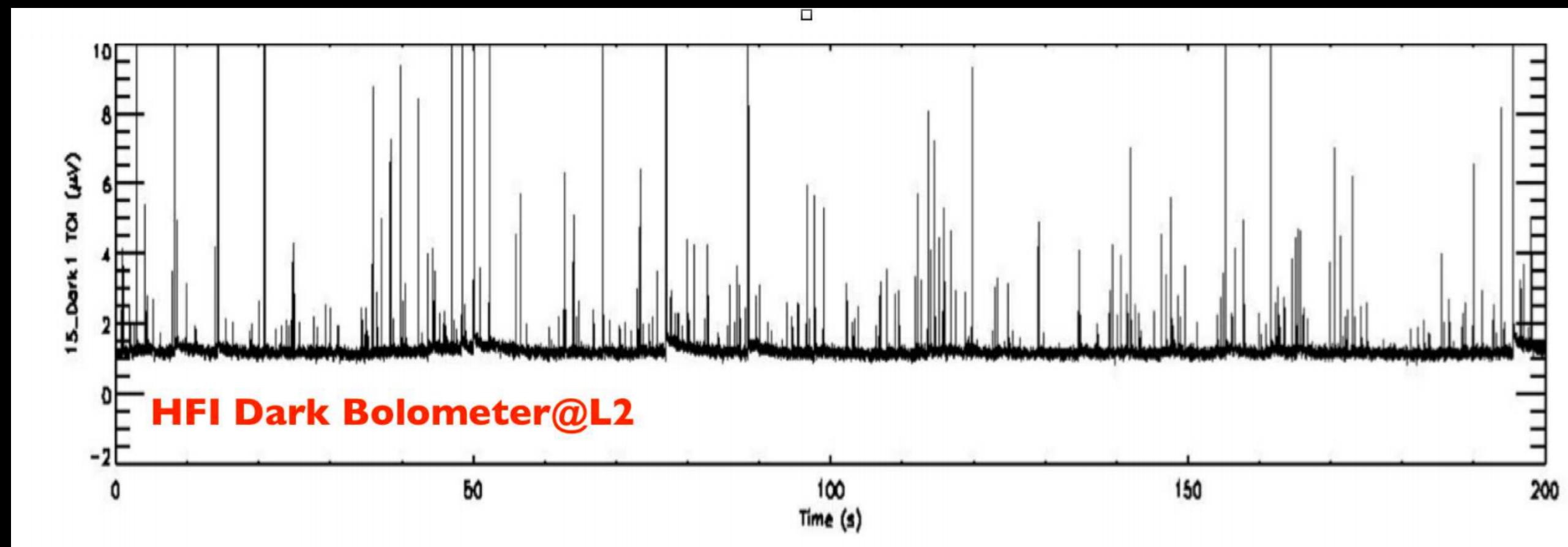
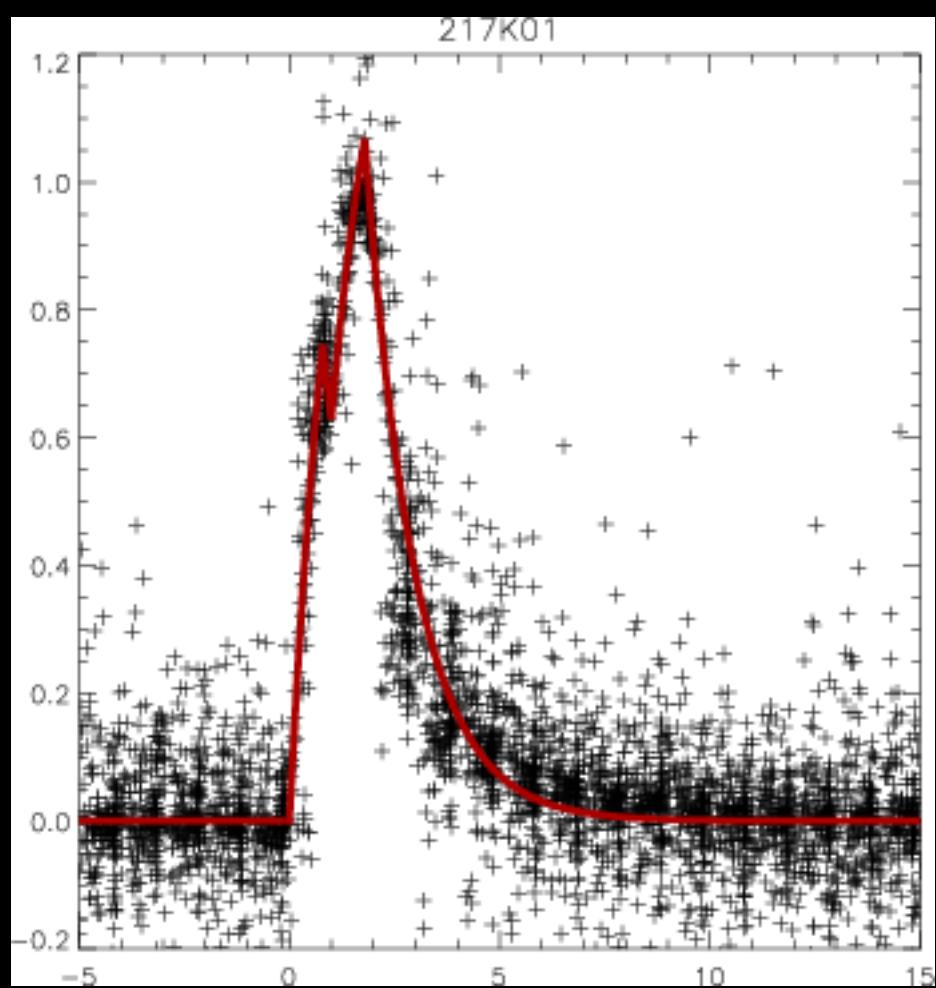
x=numpy.arange(-10,10,0.1)
y=numpy.exp(-0.5*x**2/(1.5**2))
N=x.size
kvec=numpy.arange(N)
yft=numpy.fft.fft(y)
J=numpy.complex(0,1)
dx=5.0;
yft_new=yft*numpy.exp(-2*numpy.pi*J*kvec*dx/N)
y_new=numpy.real(numpy.fft.ifft(yft_new))
plt.plot(x,y)
plt.plot(x,y_new,'r')
plt.savefig('shifted_gaussian')
plt.show()
```

Real Data Symmetry

- If I know $F(k)$, what is $F(N-k)$ if $f(x)$ is real?
- $F(N-k)=F(-k)$ (from alias theorem)
- $F(-k)=\sum f(x) \exp(-2\pi i (-k)x/N)$. let $x^*=-x$
- $F(-k)=\sum f(-x^*) \exp(2\pi i kx^*/N) = \text{conj}(F(k))$ by flipping
- So, if $f(x)$ is real, $F(k)=\text{conj}(F(N-k))$
- If N even, $F(N/2)=\text{conj}(F(N/2))$, so $F(N/2)$ must be real.

```
>>> x=numpy.random.randn(8)
>>> xft=numpy.fft.fft(x)
>>> for xx in xft:
...     print xx
...
(-4.53568815727+0j)
(-0.174046761579+2.08827239558j)
(2.15348308858+2.32162497273j)
(-0.423040513854-3.72126858798j)
(2.75685372591+0j)
(-0.423040513853+3.72126858798j)
(2.15348308858-2.32162497273j)
(-0.174046761579-2.08827239558j)
>>>
```

Cosmic Rays from Planck Satellite



Convolution Theorem

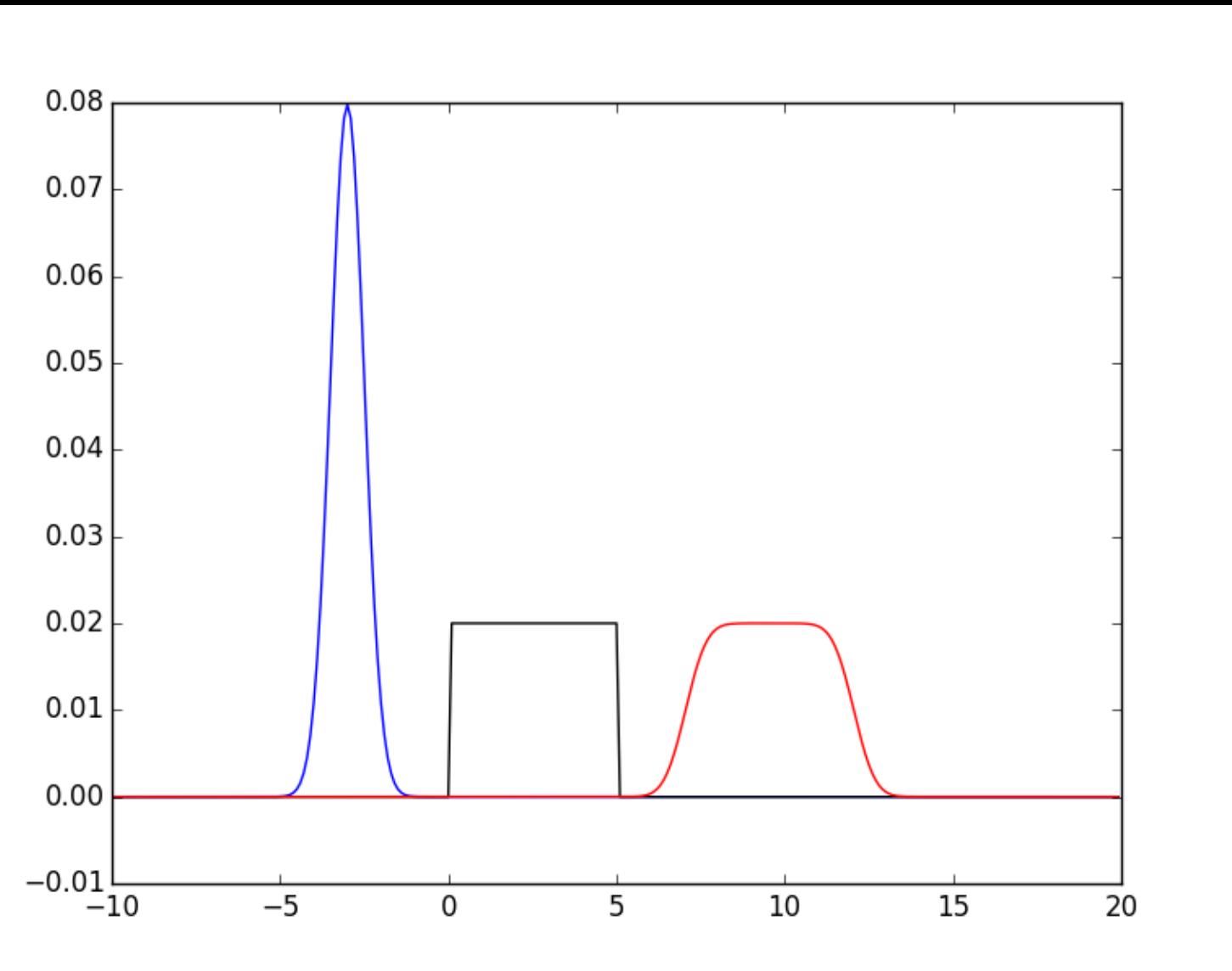
- Convolution defined to be $\text{conv}(y) = f \otimes g = \int f(x)g(y-x)dx$
- $\sum_x \sum F(k) \exp(2\pi i k x) \sum \text{conj}(G(k')) \exp(2\pi i k' x) \exp(-2\pi i k' y/N)$
- Reorder sum: $\sum \sum F(k) \text{conj}(G(k')) \exp(-2\pi i k' y/N) \sum_x \exp(2\pi i (k+k')x)$
- equals zero unless $k'=-k$. Cancels one sum, conjugates G
- $f \otimes g = \sum F(k) G(k) \exp(2\pi i ky/N) = \text{ift}(\text{dft}(f)^* \text{dft}(g))$
- So, to convolve two functions, multiply their DFTs and take the IFT

Convolution Example

```
from numpy import arange,exp,real
from numpy.fft import fft,ifft
from matplotlib import pyplot as plt
def conv(f,g):
    ft1=fft(f)
    ft2=fft(g)
    return real(ifft(ft1*ft2))

x=arange(-10,20,0.1)
f=exp(-0.5*(x+3)**2/0.5**2)
g=0*x;g[ (x>0)&(x<5) ]=1
g=g/g.sum()
f=f/f.sum()
h=conv(f,g)

plt.plot(x,f,'b')
plt.plot(x,g,'k')
plt.plot(x,h,'r')
plt.savefig('convolved')
plt.show()
```



Fast Fourier Transform

- How many operations does a DFT take?
- Have an N by N matrix operating on a vector of length N - clearly N^2 operations, right?
- Nope! Otherwise we'd never use them. What's actually going on?
- Note $DFT = \sum f(x) \exp(-2\pi i kx/N) = \sum f_{even}(x) \exp(-2\pi i k(2x)/N)$
 $+ \sum f_{odd}(x) \exp(-2\pi i k(2x+1)/N)$
- $= F_{even} + \exp(-2\pi i k/N) F_{odd}$. Let $W_k = \exp(-2\pi i k/N)$
- if $k > N/2$, then $k^* = k - N/2$ and $DFT = F_{even} + \exp(-2\pi i k^*/N + i\pi) F_{odd} = F_{even} - W_k F_{odd}$.

FFT cont'd

- So $F(k) = F_{\text{even}}(k) + W_k F_{\text{odd}}(k)$ ($k < N/2$) or $F_{\text{even}}(k) - W_k F_{\text{odd}}(k)$ ($k \geq N/2$)
- So, can get *all* the frequencies if I have 2 half-length FFTs.
- Well, just do the same thing again. FFT of a single element is itself.
- This algorithm works for arrays whose length is a power of 2
- Popularized by Cooley/Tukey in early computer days. Later found to go back to Gauss in 1805. Changes computational work from n^2 to $n \log n$.

Sample FFT

- Routine uses *recursion* - function calls itself. Recursion can be very powerful, but also easy to goof.
- `numpy.concatenate` will combine arrays - note that they have to be passed in as a tuple, hence the extra set of parenthesis
- Modern FFT routines deal with arbitrary length arrays. Fastest Fourier Transform in the West (FFTW) standard packaged - usually used by numpy.

```
from numpy import concatenate,exp,pi,arange,complex
def myfft(vec):
    n=vec.size
    #FFT of length 1 is itself, so quit
    if n==1:
        return vec
    #pull out even and odd parts of the data
    myeven=vec[0::2]
    myodd=vec[1::2]

    nn=n/2;
    j=complex(0,1)
    #get the phase factors
    twid=exp(-2*pi*j*arange(0,nn)/n)

    #get the dfts of the even and odd parts
    eft=myfft(myeven)
    oft=myfft(myodd)

    #Now that we have the partial dfts, combine them with
    #the phase factors to get the full DFT
    myans=concatenate((eft+twid*oft,eft-twid*oft))
    return myans
```

```
>>> import myft
>>> x=numpy.random.randn(32)
>>> xft1=numpy.fft.fft(x)
>>> xft2=myft.myfft(x)
>>> print numpy.sum(numpy.abs(xft1-xft2))
2.33937690259e-13
>>>
```

In Practice

- Should you write your own FFT code? (no)
- Should you understand what is going on under the hood? (yes)

```
import numpy
import time

n=2**16
x=numpy.random.randn(n)
t1=time.time();y=numpy.fft.fft(x);t2=time.time();t_ref=t2-t1
x=numpy.random.randn(n+1) #this is a prime
t1=time.time();y=numpy.fft.fft(x);t2=time.time();t_plus1=t2-t1
x=numpy.random.randn(n+2) #this is has largest factor 331
t1=time.time();y=numpy.fft.fft(x);t2=time.time();t_plus2=t2-t1
x=numpy.random.randn(n+14) #this is has largest factor 23
t1=time.time();y=numpy.fft.fft(x);t2=time.time();t_plus14=t2-t1
print 'Reference time was ',t_ref
print 'Extending by one increased time by a factor of ',t_plus1/t_ref
print 'Extending by two increased time by a factor of ',t_plus2/t_ref
print 'Extending by 14 increased time by a factor of ',t_plus14/t_ref
```

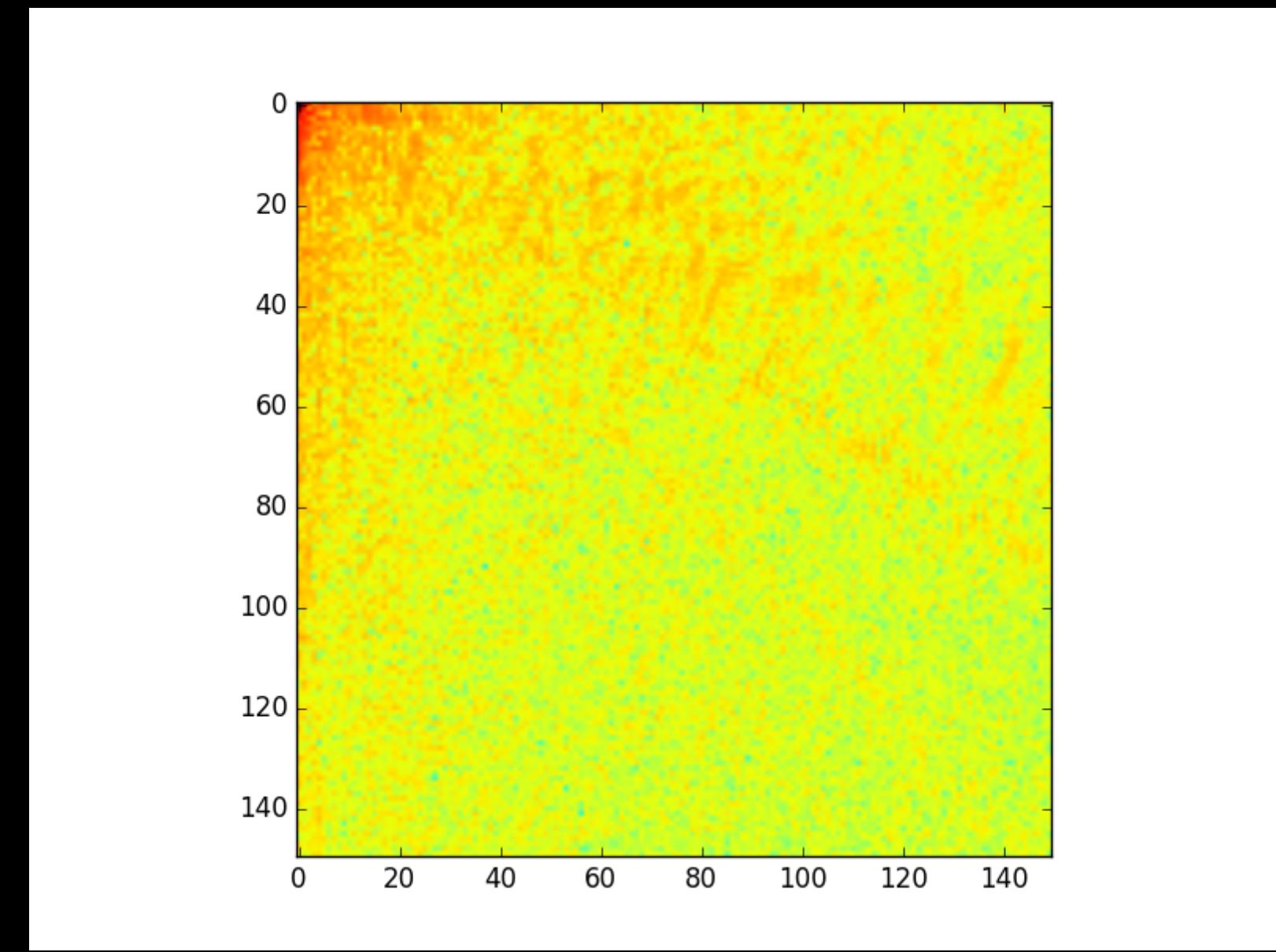
```
[>>> execfile('time_ffts.py')
Reference time was  0.00335788726807
Extending by one increased time by a factor of  2594.13178074
Extending by two increased time by a factor of  5.54963078671
Extending by 14 increased time by a factor of  1.8124112468
```

2D Fourier Transforms

- Fourier transform defined in 2 dimensions:
- $F(k,l) = \int \int f(x,y) \exp(-ikx) \exp(-ily) dx dy$
- 2D FT's extremely common in image processing.
- JPEGs in fact are based on picking out modes from image FT's.



numpy.fft.fft2



Smoothing Images

- Out-of-focus images are convolutions.
- Can defocus an image by convolving with a blurry kernel
- Let's fuzz out map by a Gaussian.

```

def get_fft_vec(n):
    vec=numpy.arange(n)
    vec[vec>n/2]=vec[vec>n/2]-n
    return vec
def smooth_map(map,npix,smooth=True):
    nx=map.shape[0]
    ny=map.shape[1]
    xind=get_fft_vec(nx)
    yind=get_fft_vec(ny)

    #make 2 1-d gaussians of the correct lengths
    sig=npix/numpy.sqrt(8*numpy.log(2))
    xvec=numpy.exp(-0.5*xind**2/sig**2)
    xvec=xvec/xvec.sum()
    yvec=numpy.exp(-0.5*yind**2/sig**2)
    yvec=yvec/yvec.sum()

    #make the 1-d gaussians into 2-d maps using numpy.repeat
    xmat=numpy.repeat([xvec],ny,axis=0).transpose()
    ymat=numpy.repeat([yvec],nx,axis=0)

    #if we didn't mess up, the kernel FT should be strictly real
    kernel=numpy.real(numpy.fft.fft2(xmat*ymat))

    #get the map Fourier transform
    mapft=numpy.fft.fft2(map)
    #multiply/divide by the kernel FT depending on what we're after
    if smooth:
        mapft=mapft*kernel
    else:
        mapft=mapft/kernel
    #now get back to the convolved map with the inverse FFT
    map_smooth=numpy.fft.ifft2(mapft)

    #since numpy gets imaginary parts from roundoff, return the real part
    return numpy.real(map_smooth)

```

smooth_map.py



```
import numpy
from matplotlib import pyplot as plt
import smooth_map

meerkat=plt.imread('meerkat_small.jpg')

smoothed_map=numpy.zeros(meerkat.shape)
smoothed_map=numpy.zeros(meerkat.shape)
npix_smooth=3.5
npix_restore=4
for i in range(3):
    tmp=numpy.squeeze(meerkat[:, :, i])
    tmp_smooth=smooth_map.smooth_map(tmp,npix_smooth)
    smoothed_map[:, :, i]=tmp_smooth
    tmp2=smooth_map.smooth_map(tmp_smooth,npix_restore,False)
    unsmoothed_map[:, :, i]=tmp2
```

Deconvolution

- Well, if I smear out by multiplying FT's, I can unsmear by dividing, right?
- If yes, worth billions and billions of your favo(u)rite currency. Save those fuzzy pictures...
- Maybe. Let's try smoothing image by 3.5 pixels, then unsmoothing by 4.
- What happened?

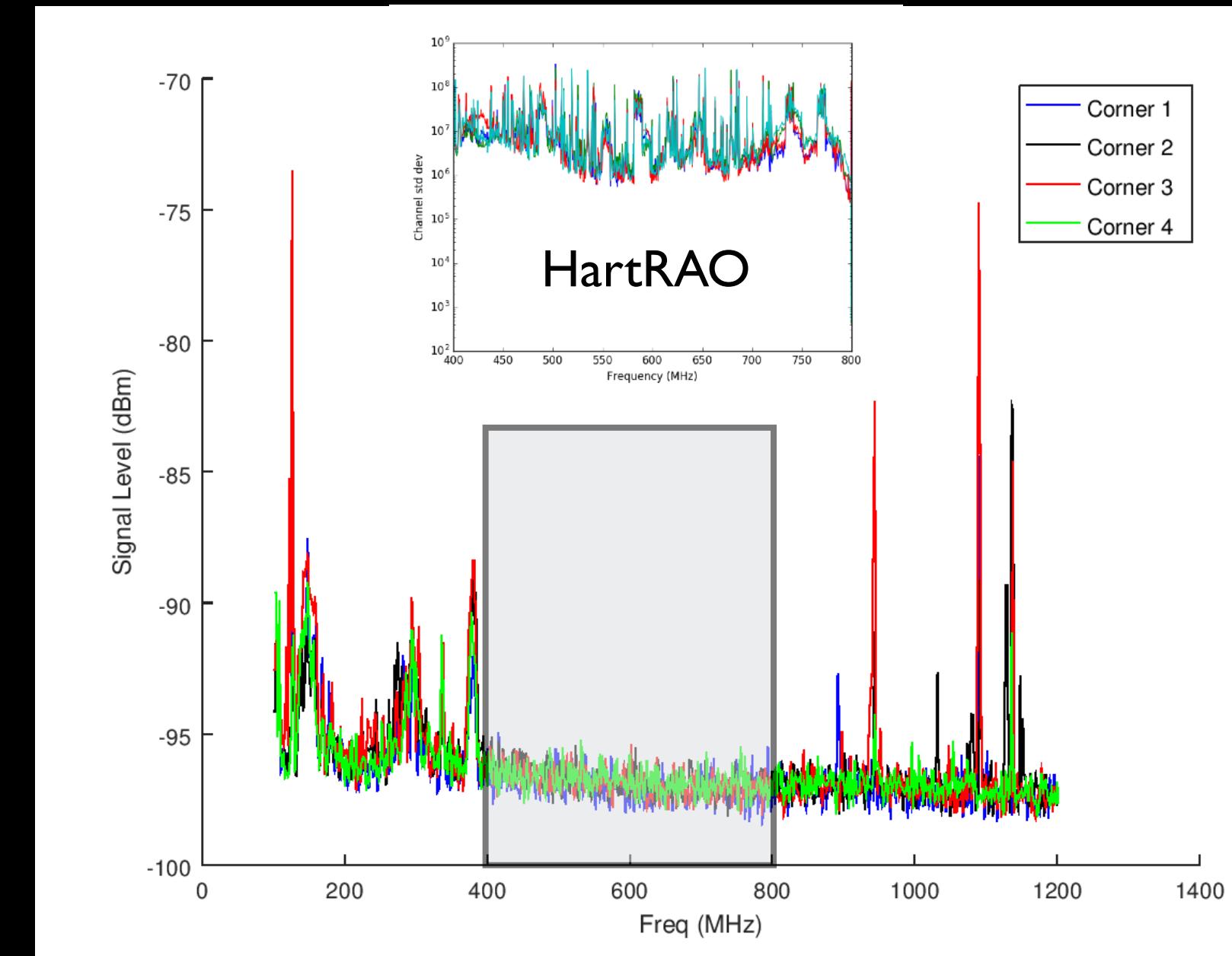
Deconvolution



- smoothing lowers high-frequency signal
- unsmoothing *must* raise it back up.
- if there's any noise, it gets amplified by unsmoothing.
- If I smooth, then write to jpg, I round to nearest integer. Equivalent to adding noise.
- So, think those fuzzy license plates in google maps are safe?

E field to Frequency

- Power as a function of frequency is just FT (squared) of E field
- True in ideal limit, but what happens in real life?
- We measure E field every so often, for some length of time
- Incoming field doesn't care about when we measured it. Our answer hopefully doesn't either...
- How to do this right hugely important question for any RF work (cell phones, TV, DSP, radio astronomy...)



Windowing

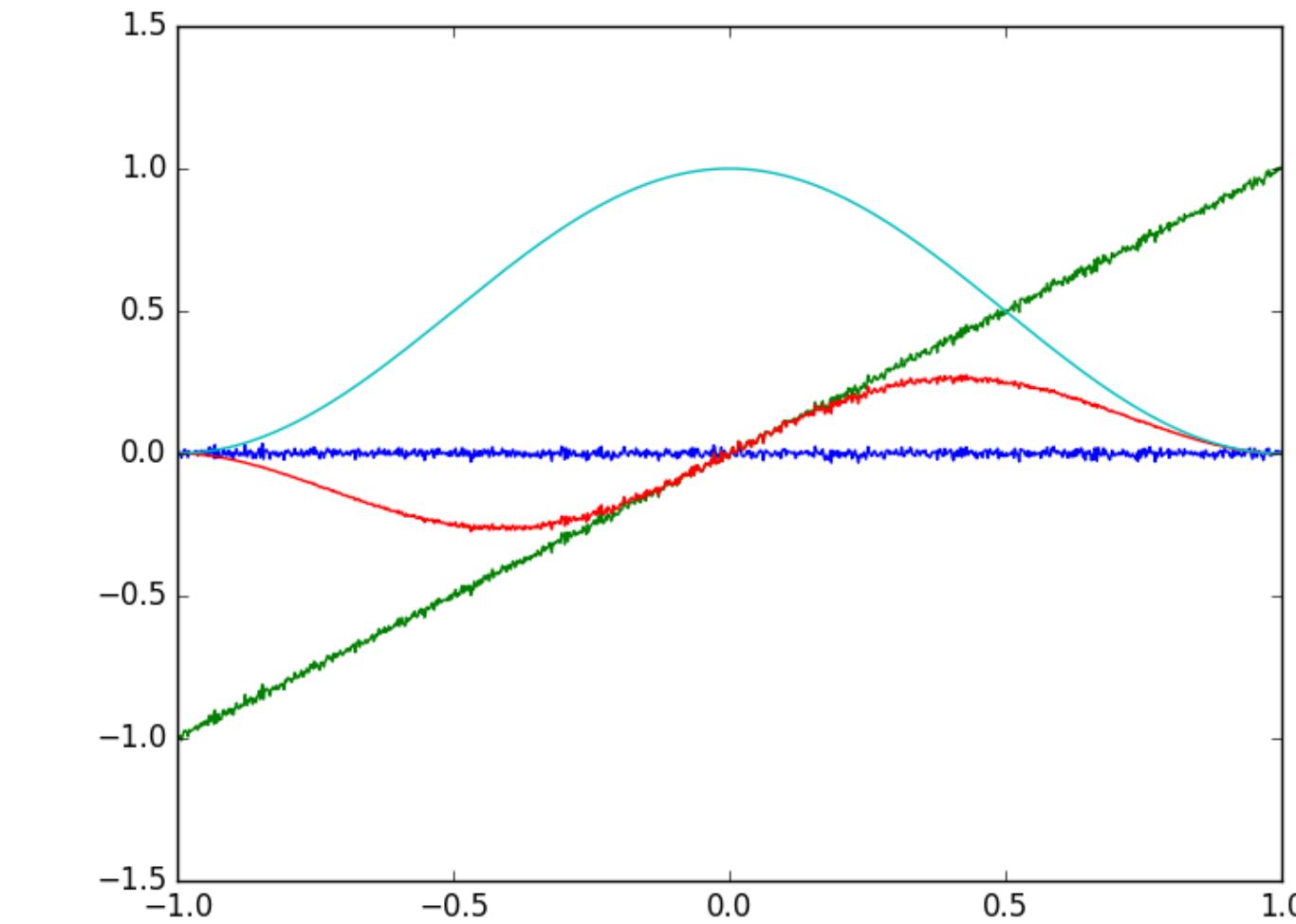
- Jumps around edges cause high frequency power in FFTs. This is a bad thing.
- Standard solution: multiply by a window that goes to zero (or some very small value) at edges.
- There are many possible windows, depending on what you want to do: Hamming, Hanning, cos... 28 listed on wikipedia page.
- If I multiply by window in real space, what have I done in Fourier space?

Window in Real Space

Use cos window that goes to zero on edges w/derivative zero.

If I take a piece of noisy data from a smooth long-term signal, smooth part may look like similar to a line.

What does FT look like?



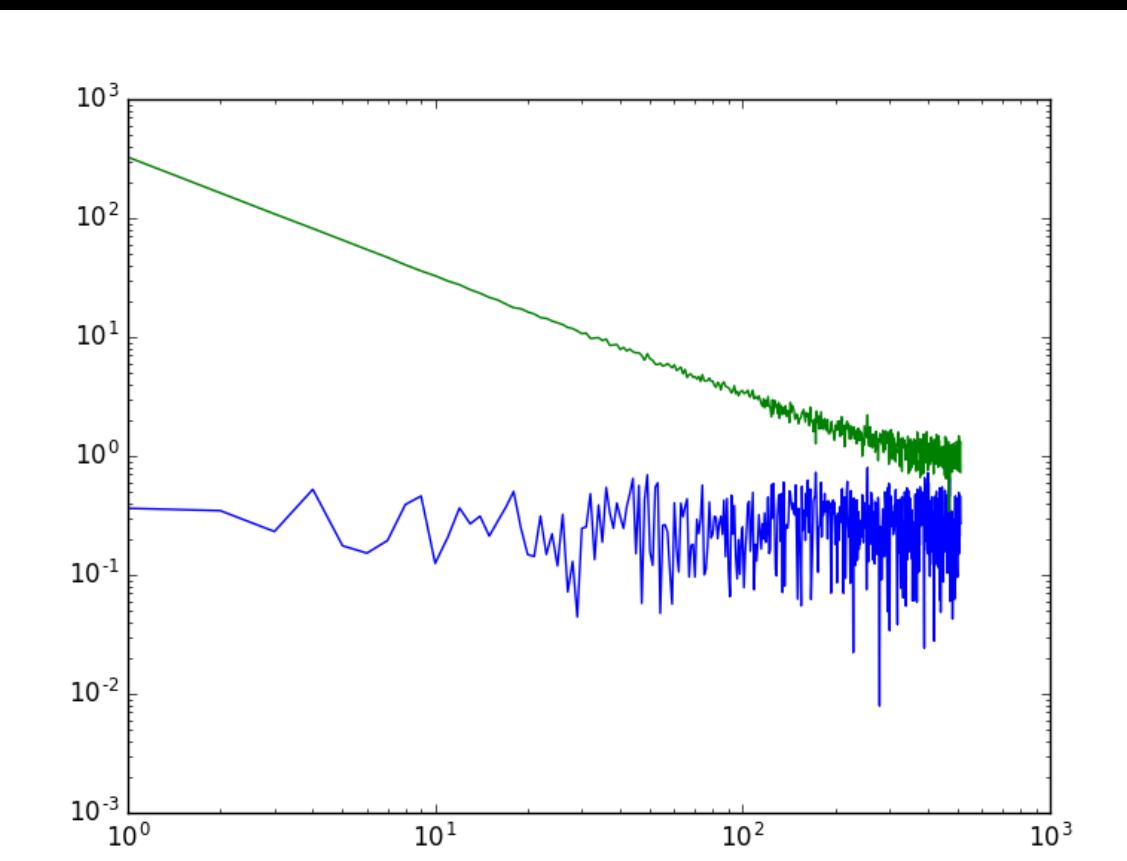
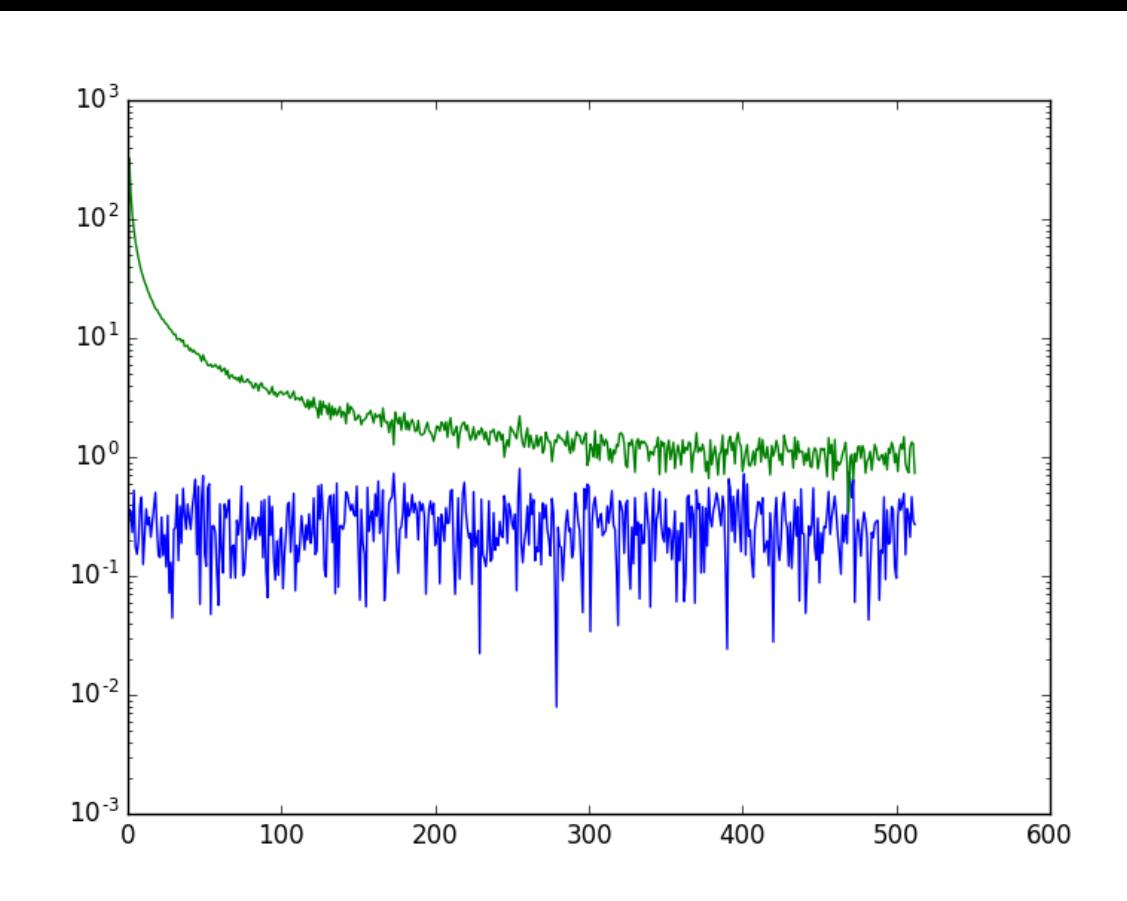
```
import numpy
from matplotlib import pyplot as plt
plt.ion();

x=numpy.arange(1024);
x=x-1.0*x.mean();x=x/x[-1]
y1=0.01*numpy.random.randn(x.size)
y2=y1+x
window=0.5*(1+numpy.cos(x*numpy.pi))
y3=y2*window
plt.clf();plt.plot(x,y1);plt.plot(x,y2);plt.plot(x,y3)
plt.plot(x,window);plt.savefig('raw_data.png')

y1ft=numpy.fft.rfft(y1)
y2ft=numpy.fft.rfft(y2)
```

Effects of Adding Slope

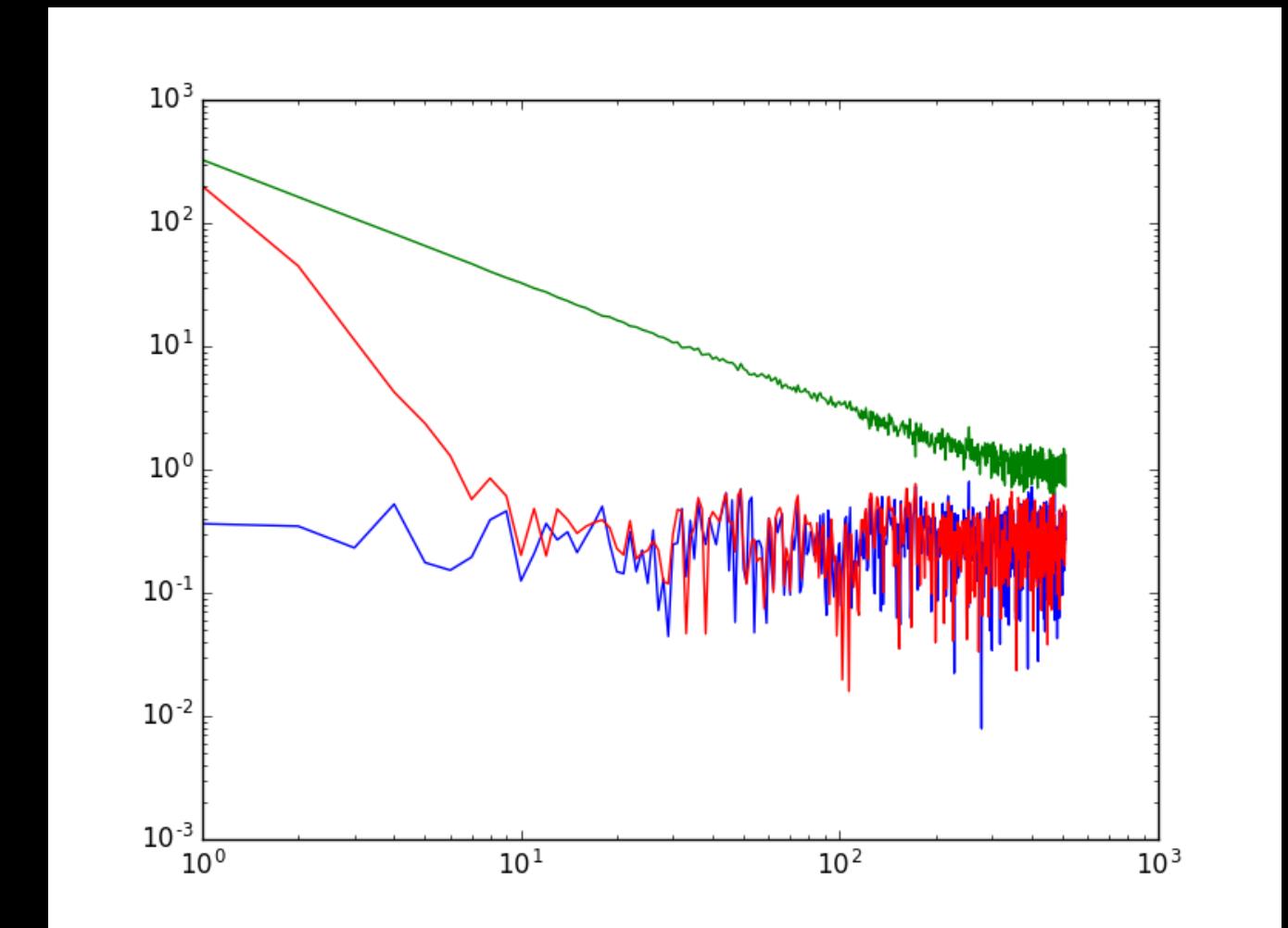
- Even though we know long-term signal is smooth, by taking piece we raise noise level in FT. This is a bad thing.
- Why does the FT look like a line in log-log space?



Adding Window

- Multiplying the data with a slope by the window makes the high-frequency power drop back down. This is usually considered a good thing.
- Low frequency power is still large - that's real, we do have a slope in our data.
- What am I doing with that normfac thing?

Parseval's theorem: FT is a unitary rotation, so length before/after must be the same.
Windowing removes power, so scale back up by average amount of windowing loss.



```
window=0.5*(1+numpy.cos(x*numpy.pi))
y3=y2*window
#why am I doing this normfac thing?
normfac=numpy.sqrt(numpy.mean(window**2))
y3ft=numpy.fft.rfft(y3)
plt.plot(numpy.abs(y3ft/normfac));
plt.savefig('window_log.png')
```

More FT asides

- What is the Fourier transform of a slope?
- What is the Fourier transform of a triangle ($f(x) = 1 - \text{abs}(x)$ for $-1 < x < 1$)?
- What is the (expected) Fourier transform of random noise?
- We will be looking at plots that show the amplitude of the Fourier transforms against wavelength. The variance of the FT is called the *power spectrum*, and is fundamental in many areas of electronics, physics, astronomy...

Stationary Noise

- Say we have noise where $\langle f(x)f(x+dx) \rangle = g(dx)$ (not of x)
- This is called stationary - noise at some point depends on noise we got at nearby points, but not on absolute time of when we measured
- Fourier space: $\langle F(k)F^*(k') \rangle = \langle \sum f(x) \exp(-2\pi i k x / N) \sum f(x') \exp(2\pi i k' x' / N) \rangle$
- Can swap x' for $x+dx$, since sum is over all points, can sum over dx :
- $\langle F(k)F^*(k') \rangle = \langle \sum f(x) \exp(-2\pi i k x / N) \sum f(x+dx) \exp(2\pi i k'(x+dx) / N) \rangle$
- Reorder sum over x then dx : $\langle \sum \exp(2\pi i k' dx / N) \sum f(x) f(x+dx) \exp(2\pi i x(k'-k) / N) \rangle$
- Now $\langle f(x)f(x+dx) \rangle = g(dx)$ (by assumption), can come out. $\sum \exp(2\pi i k' dx / N) g(dx) \sum \exp(2\pi i x(k'-k) / N)$
- Interior goes to N for $k'=k$, left with $N \sum g(dx) \exp(2\pi i k dx / N) \delta_{kk'}$, so Fourier transform of noise is diagonal.
- Further, variance of $F(k)$ given by Fourier transform of correlation function $g(dx)$.