Reminder: Advection

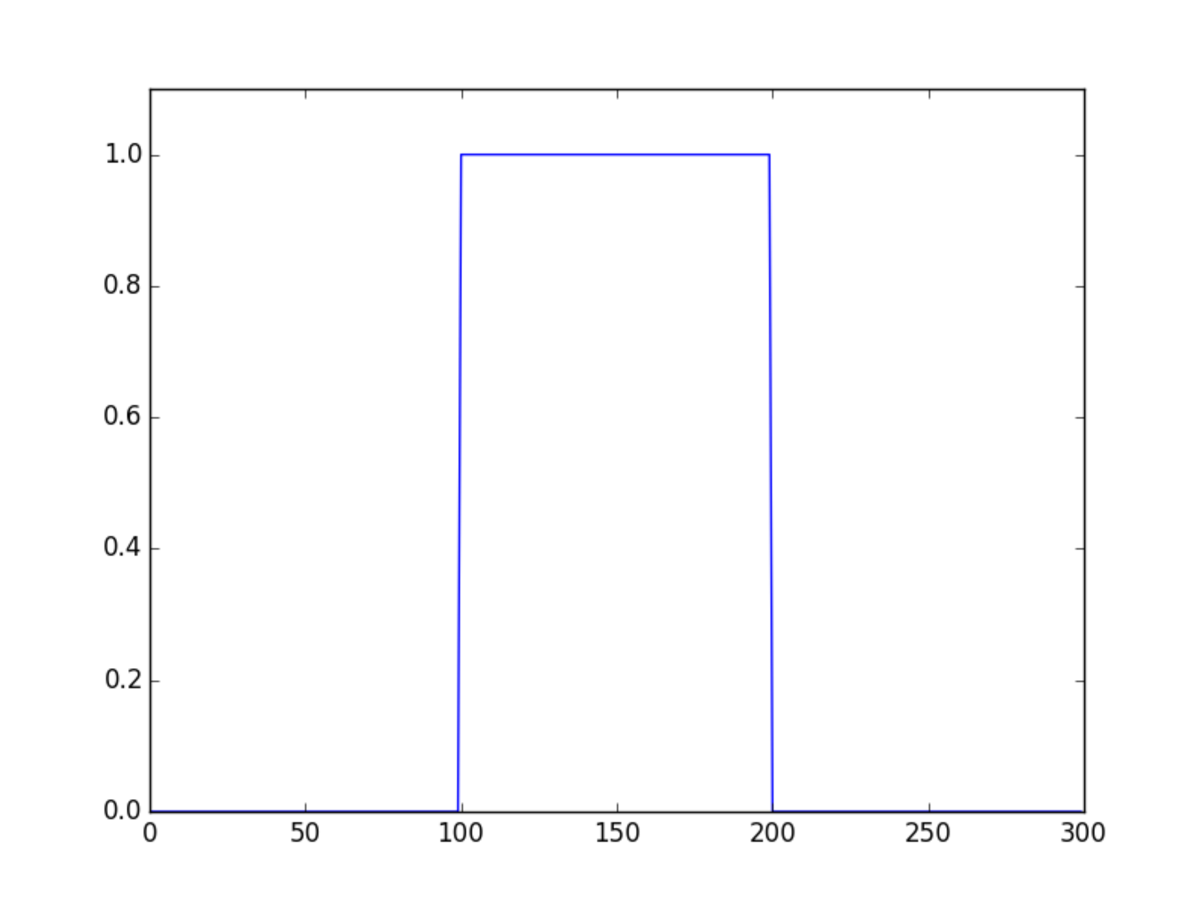
- Advection equation: $\partial f/\partial t+u\partial f/\partial x=0$
- Trial: $f=f(ut-x)$: then $uf' + u(-f')=0$. check
- So, any function of $(ut-x)$ will solve this equation.
- So, if we watch the spot in the function at x_0 when $t=0$, then at time=t, the position will be: $ut-x=0-x_0$, or $x=x_0+ut$. Information moves with velocity u.

Finite Volume Advection

```python
#simple_advect_finite_volume.py
import numpy
from matplotlib import pyplot as plt
n=300
rho=numpy.zeros(n)
rho[n/3:(2*n/3)]=1
v=1.0
dx=1.0
x=numpy.arange(n)*dx

plt.ion()
plt.clf()
plt.plot(x,rho)
```

Left: set up initial conditions. Density is 1 in the middle third of region, zero otherwise. Below left: initial density plotted. Bottom: advection code.




```python
dt=1.0
for step in range(0,50):
    #take the difference in densities
    drho=rho[1:]-rho[0:-1]
    #update density.  We haven't said what happens at
    #cell 0 (since cell -1 doesn't exist), ignore for now
    rho[1:]=rho[1:]-v*dt/dx*drho
    plt.clf()
    plt.plot(x,rho)
    plt.draw()
```

Reminder: Time Steps

- Smaller time step normally more accurate.
- Let's look at solution for some different time steps.
- What happened?
- Behaviour of sharp features often very important - in practice, run test problems with known solutions to verify behaviour.

```
#advect_finite_volume_timestep.py
dt=1.0
big_rho=numpy.zeros(n+1)
big_rho[1:]=rho
del rho  #we can delete the to save space
oversamp=10 #let's do finer timestamps
dt_use=dt/oversamp
for step in range(0,150):

    big_rho[0]=0
    for substep in range(0,oversamp):
        drho=big_rho[1:]-big_rho[0:-1]
        big_rho[1:]=big_rho[1:]-v*dt_use/dx*drho

    plt.clf()
    plt.axis([0,n,0,1.1])
    plt.plot(x,big_rho[1:])
    plt.draw()
```

Reminder: Stability

$$\rho_j^{new}=\rho_j-(\rho_j-\rho_{j-1})vdt/dx$$

- You can learn a lot by plugging in sine waves.
- If $\rho_j=\exp(ikj)$, ρ_j^{new}=what? define a=vdt/dx
- $\rho_j^{new}=\exp(ikj)-a(\exp(ikj)-\exp(ik(j-1)) = \exp(ikj)-a(\exp(ikj)-\exp(-ik)\exp(ikj))$
- $\rho_j^{new}=\exp(ikj)*[1-a(1-\exp(-ik))]$
- If quantity in [] gets bigger than unity, solution will grow with time. Our code would be *unstable* - this is bad!

Reminder: CFL Condition (*a*=vdt/dx)

- Look at 1-*a*(1-exp(-ik)). 1-exp(-ik) is bounded by (0,2)
- if 0, []=1, solution always stable.
- if 2, then []=1-2*a* can have magnitude >1 for sufficiently large *a*.
- By construction, *a* is positive, so can't get []>1. But can get []<-1: 1-2*a*<-1, 2<2*a*, or *a*>1.
- For stability, a<=1, or dt<dx/v. In words, dt has to be shorter than crossing time for cell.
- This is called the Courant–Friedrichs–Lewy (CFL) condition. vdt/dx is the Courant number.

Numerical Viscosity/Lax

- We saw setting df/dx with $(f_{x+1}-f_{x-1})/2dx$ led to unconditional instability in advection.
- However - take (f(x,t+dt)-f(x,t))/dt for time derivative to (f(x,d+dt)-(f(x+dx,t) +f(x-dx,t))/2) leads to stability. Can you guess criterion for stability?
- Rewrite: (f(x,t+dt)-f(x,t))/dt=-v(f(x+dx,t)-f(x-dx,t))/2dx + (f(x+dx),t-2f(x,t) +f(x-dx,t))/2dt.
- This is solving df/dt=-vdf/dx + (dx)^2/2dt ∇^2f. New term looks like diffusion/ viscosity equations - we're adding numerical viscosity to induce stability.

Conservation Equation

- If a quantity is conserved, time rate of change in a volume is equal to net flow into/out volume.
- If conserved quantity is ρ and velocity is u then flow out of region is $\rho_+ u_+$ and flow in is $\rho_- u_-$. Net flux is then $-\partial(\rho u)/\partial x$.
- Equation then become $\partial\rho/\partial t = - \partial(\rho u)/\partial x$, or $\partial\rho/\partial t+\partial(\rho u)/\partial x=0$
- If a quantity is created, then we pick up extra term for rate of creation:
- now $\partial\rho/\partial t=-\partial\rho/\partial x + q$, where q is the creation rate.

Euler Equations

- Now we're set to derive equations of fluid mechanics.
- The full fluid equations (Navier-Stokes) include forces from viscosity
- We will make approximation that viscosity is negligible
- Further, we will assume no energy flows between pieces of fluid (this is usually quite a good approximation)
- Leaves us with Euler equations. What equations should we have?

Mass Conservation

- Generally, no matter is created/destroyed, so mass is strictly conserved.
- Mass conservation becomes $\partial\rho/\partial t+\partial(u\rho)\partial x=0$
- Note that if you had source/sink of matter, it would appear as an extra term

Momentum

- Momentum is ρu. So conservation equation is $\partial(\rho u)/\partial t+\partial(\rho u^2)/\partial x=0$
- Velocity appears squared, so equation is nonlinear
- Fluid pressure will exert a force, so force term must be added.
- Force on right side of a packet is $-P_+$, force on left is $+P_-$, so total net force is difference, limit is $-\partial P/\partial x$. This force has to go into momentum equation.
- Momentum equation: $\partial(\rho u)/\partial t+\partial(\rho u^2)/\partial x=-\partial P/\partial x$
- Conservation form: rewrite as $p=\rho u$, get $\partial p/\partial t+\partial(pu+P)/\partial x=0$

Energy

- Two pieces of energy - internal thermal energy and bulk kinetic.
- Call total energy (thermal+kinetic) per unit mass E.
- Energy creation rate from pressure is power, or force * velocity
- Gives $\partial(\rho E)/\partial t+\partial(u\rho E)/\partial x=-\partial(uP)/\partial x$
- Rewrite into conservation form: $\partial(\rho E)/\partial t+\partial(u\rho E+uP)/\partial x=0$

Euler So Far

- $\partial\rho/\partial t+\partial(u\rho)\partial x=0$ $\partial p/\partial t+\partial(pu+P)/\partial x=0$ $\partial(\rho E)/\partial t+\partial(u\rho E+uP)/\partial x=0$
- Three equations, how many unknowns? Solution needs velocity, density, energy, and pressure.
- So, need one more equation. Normally done by specifying a relation between pressure and energy. This is called an *equation of state*.
- Classic EoS is gamma law, $P\sim\rho^{\gamma}$. For ideal gas, e=3/2 nkT, pressure is nKT, so $P=2/3\rho e$ (where $e=E-1/2\rho u^2$ is the thermal energy).

Derivation of γ

- Let's compress a volume of gas and see how energy changes.
- $dE=-PdV$. $E=aPV$ (where $a=3/2$ for ideal gas)
- $ad(PV)=-PdV$. $aVdP+aPdV=-PdV$
- $dP(aV)=-dV(P(1+a))$, $adP/P=-(1+a)dV/V$.
- $\log(P)\sim-(1+a)/a\log(V)$. $P\sim V^{-(1+a)/a}$. Density $\sim 1/V$, so $P\sim\rho^{1+1/a}$. The index is usually called γ (gamma). For ideal gas, a is 3/2, so $\gamma=1+2/3=5/3$.

Euler Equations with EoS

- We can now write down Euler equations in conservation form with EoS
- $E=1/2u^2+e$, $\rho e=P/(\gamma-1)$. So $P=\rho(\gamma-1)(E-1/2u^2)$
- $\partial Q/\partial t+\partial(f(Q))/\partial x=0$
- $Q=[\rho,\rho u,\rho E]$, $f(Q)=[\rho u, \rho u^2+P, \rho uE+uP]$
- using momentum $p=\rho u$: $Q=[\rho,p,\rho E]$, $f(Q)=[p,pu+P,pE+uP]$

System of PDE's

- Let's take a system of 2 equations with constant coefficients:
- $\partial f/\partial t + c_{11}\partial f/\partial x + c_{12}\partial g/\partial x = 0$ and $\partial g/\partial t + c_{21}\partial f/\partial x + c_{22}\partial g/\partial x = 0$
- Solution to 1-D advection was $h(x-ut)$, so let's guess solution is $f = v_1 h(ut-x)$, $g = v_2 h(ut-x)$
- Plug in: system is then $uv_1h' - c_{11}v_1h' - c_{12}v_2h' = 0$ and $uv_2h' - c_{21}v_1h' - c_{22}v_2h' = 0$
- h' drops out, and we're left with system: $uv = Cv$

PDE Systems, Ctd:

- System uv=Cv is just eigenvalue problem. Get a solution for each eigenvector/eigenvalue pair, where propagation speed is eigenvalue.
- When eigenvalues are real, system is called *hyperbolic*, solutions of form h(x-ut). Information propagates at finite speed.
- When eigenvalues are imaginary, system is *elliptical,* solutions of form h(x-iut). You might expect treatment in numerical solvers to be different.
- Do you think fluid equations should be elliptical or hyperbolic?

CFL Condition Revisited

- Euler equations give us a system of 3 coupled equations.
- This means 3 eigenvalues. For CFL condition, want time step stable for largest velocity eigenvalue.
- What do you think the three eigenvalues are? You should be able to guess from physical intuition. (recall that the speed of sound $c_s^2=\gamma P/\rho$)

Aside: Stiff Equations

- We get one eigenvalue for fluid velocity, and 2 for velocity ±speed of sound.
- If c_s >> u, then CFL means timestep has to be tiny compared to natural one from fluid velocity. When eigenvalues diverge like this, equations are called *stiff*. Different computational techniques required.
- Incompressible fluid mechanics - limit where c_s>>u. Fluid has time to move out of the way. Otherwise it would compress.
- Techniques to solve stiff equations are different. If you hit a stiff set, look them up. Always check if your system is stiff!

Structure of a Simple1-D Fluid Code

- First, do boundary conditions
- If we use density, momentum, total energy as variables (the conservation quantities) then need to calculate velocity
- Now need to calculate pressure
- Next calculate gradients - we use upwind 1st order scheme, where I flow with my velocity
- Calculate CFL timestep
- Finally, update density, momentum, Energy

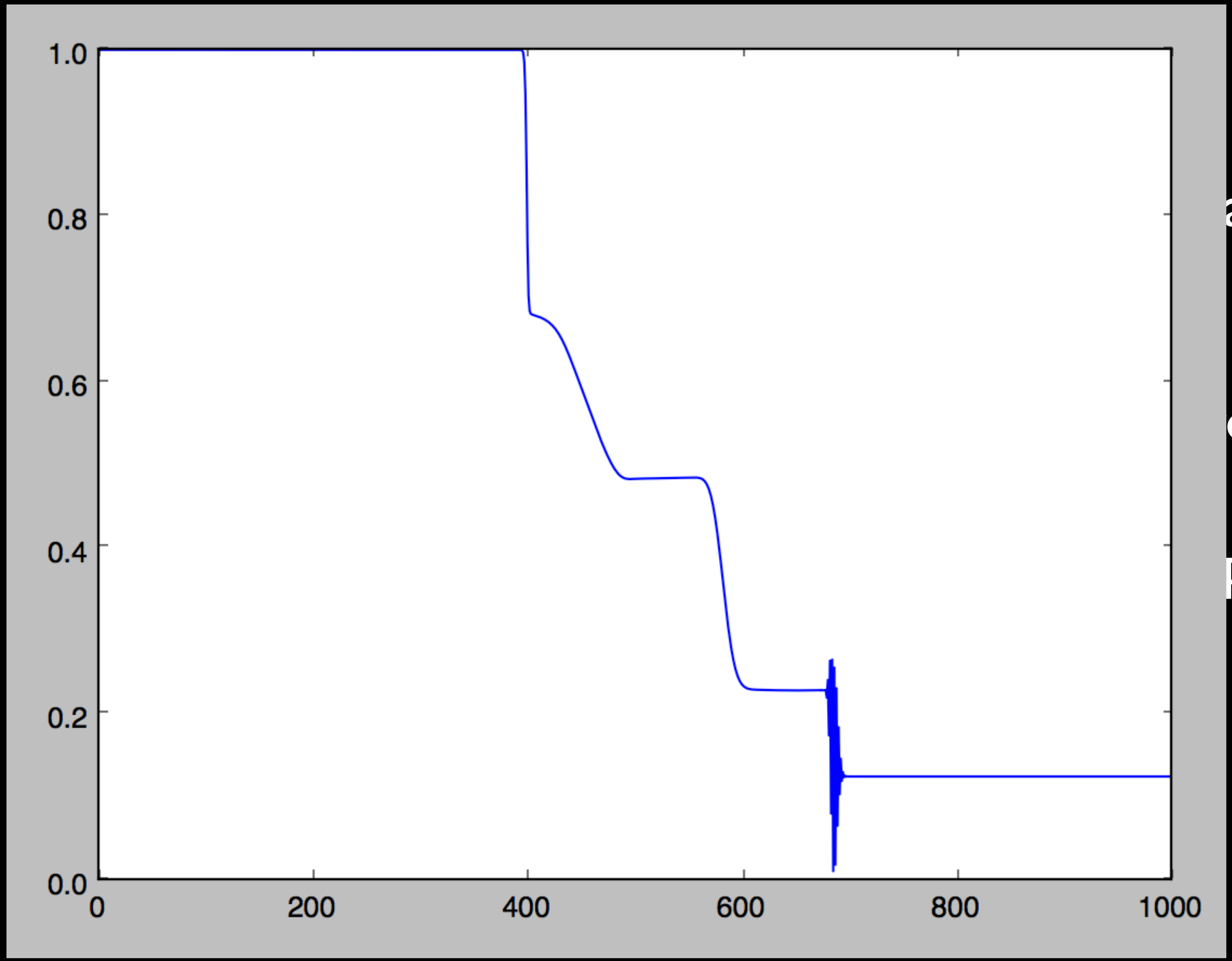
Primitive Equations

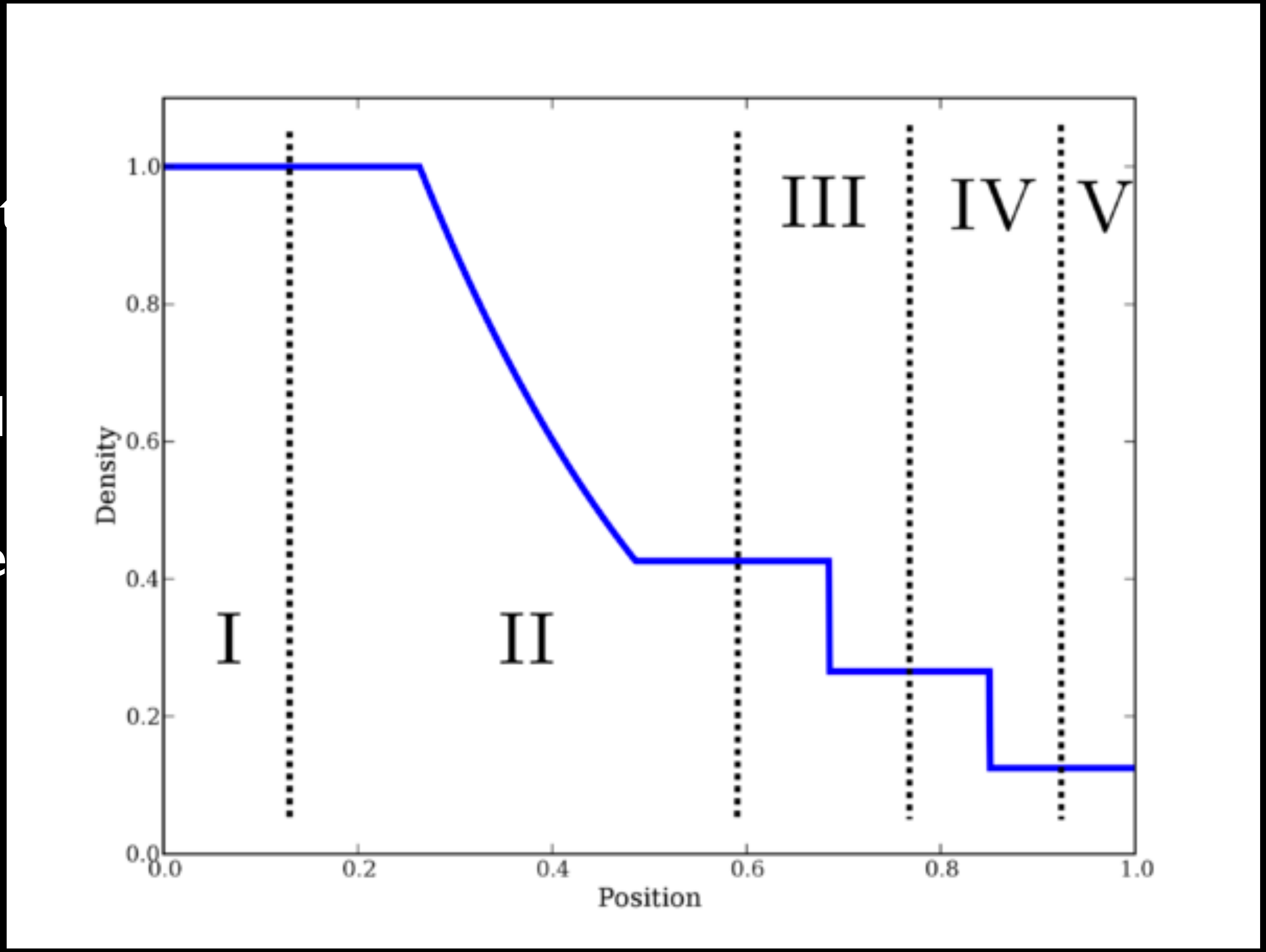
- Assuming $P \propto \rho^\gamma$, we can rewrite the Euler equations in *primitive* form. After math, we get:
- $\rho_t + u\rho_x + \rho u_x = 0$ where e.g. $\rho_t = \partial\rho/\partial t$
- $u_t + uu_x + 1/\rho \, P_x = 0$
- $P_t + uP_x + \gamma P u_x = 0$

Shock Tube

- Classic testing problem is a shock tube: start with a density/pressure jump in the middle, with velocity=0.
- What should this look like? let's run hydro1d.py
- What answer *should* look like from wikipedia:

Shock Tube


1.0
0.8
0.6
0.4
0.2
0.0
0
200
400
600
800
1000


I
II
III
IV
V
Density
Position
1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

Riemann problem/Godunov Solver

- If we're facing solving $u_t + Au_x = 0$, we rotate into the eigenspace of A. This gives us uncoupled equations that look like advection (when looking at short enough time).

- Finite volume can be mapped into Riemann problem - you have a discontinuity between cells. Know how to propagate eigenmodes

- Godunov solvers do this - evolve solution by solving Riemann problem.

- First order accurate, but can be built into more accurate solution.