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# How Does the Ledoit and Wolf Shrinkage Estimator Improve a Real Estate Portfolio?

***Executive Summary.*** *Portfolio theories are meant to provide a method for managing assets and constructing portfolios. Meanwhile, the mean-variance technique has been heavily criticized by some academics, and its application to real estate portfolio is questionable (Cheng and Liang, 2000). Indeed, the mean-variance analysis is quite sensitive to estimation errors, and traditional real estate databases can represent a huge pitfall to portfolio construction. Therefore, we look for a method to lessen the effect of inputs upon the optimization process; the shrinkage estimator appears to show some advantages in modeling real estate portfolios. This study employs the shrinkage estimator scheme of Ledoit and Wolf (2004) to examine the effect of a corner solution on the allocations. This process limits the impact of estimation errors on the optimization process, resulting in a distinctive investment strategy.*

*by* Eric Vu Anh Tuan*

A property asset can be held directly (the simple act of buying and selling properties) or indirectly (real estate investment trusts). There are multiple sectors, including residential, retail, industrial, office, and hotels (Gyourko, 2004); the main vehicle for investors remains the direct vehicle with its particular imperfections.

Indeed, the conventional real estate market is highly illiquid, having great disparity in market size and transaction costs, which makes real estate appraisal "challenging" due in part to the cyclical aspect of the property market. In a perfect world, real estate data would be available on a consistent basis; it would be realistically described and easily comparable, so the lengths of the series and data frequency would be reliable (Arthur, 2005). These conditions cannot be easily applied to property assets, as the different distribution of dwellings between countries reduces the values of the data. For instance, in Spain, the majority of the population owns their properties, whereas in Switzerland, most people choose to rent their homes. The technique used to gather data also tends to vary from one country to another, and urban regulation can vary a lot from city to city (Singh, 2009). Thus, real estate return series cannot be regarded as possessing the same characteristics as those of stocks or bonds with respect to a particular type of performance appraisal. Consequently, statistics are difficult to obtain, and they usually contain empty data fields. The outcomes are also hampered by the fact that transactions are infrequent and highly localized, as a small part of the population is incorporated in a property market's cross-series (Knight, Sirmans, Gelfand, and Ghosh, 1998).

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The asset allocation model based upon Markowitz's theory is driven by expected returns and covariance matrix estimates via a mean-variance procedure. These same identified inputs are usually derived from a historical return time series. The disadvantages associated with the use of historical data are widely documented in the literature. Whether or not one regards these as problematic, it can be supposed that the inputs used for mean-variance analysis from the sample mean and the sample covariance matrix are likely to be imprecise and, thus, exhibit estimation error (Michaud, 1989). As explained by Daszykowski, Kaczmarek, Vander Heyden, and Walczak (2007) and Welsch and Zhou (2007), the covariance matrix carries many variables to estimate, so two major problems may arise. First, the data probably contains outliers. Second, the numerous numbers to estimate affect the covariance matrix considerably. While the sample mean and covariance matrix are accepted as unbiased estimators under the supposition of multivariate normality, it can lead to extreme weight allocation and rapid weight shift when there is a minor change in the inputs (Michaud, 1989). In their investigation, Armonat and Pfnuer (2004) note that biased information produces spurious recommendations for property portfolios, leading to inadequate use of capital, with the main challenge coming from model estimates that import specified time series data. A large number of studies outside the real estate field have provided empirical results relating to the shortcomings of mean-variance analysis (e.g., Best and Grauer, 1991; Jorion, 1992; Chopra and Ziemba, 1993). Real estate investors are well aware of having inefficient information and, as such, tend also to use their private and microenvironment knowledge within their analyses (Armonat and Pfnuer, 2004).

Even though practitioners have adopted the method, the great sensitivity of the mean-variance inputs tends to make the technique unproductive in practical portfolio selection. Improving the mechanics of the mean-variance optimization with the emergence of new techniques can encourage investors to rely more on the mean-variance portfolio assumption (Michaud, 1989), particularly when portfolios are composed mainly of real estate assets. The solution to the problem lies in lessening estimation error (Michaud, 1989; Kaplan, 1998). The current study, therefore, aims to take the initiative in the investigation of the effectiveness of shrinkage estimators applied to the real estate portfolio context.

## The Markowitz Doctrine and its Limits

Historically, investors have allocated their money to assets that exhibit the highest return, where the volatility is usually stronger, allowing them to gain bargains on expected returns (Michaud, 1989). Assuming that investors are able to forecast market behavior and by using technical quantitative models, the assets with the most attractive returns could be selected for inclusion in the portfolio. However, it tends to be difficult to provide an exact forecast with any sort of asset, as volatility does exist and markets have collapsed in the past (e.g., the sub-prime lending and dot-com crises). Loss is a part of the "game," and investing in an asset that is doing well does not offer any guarantee of similar achievement in the future. As the adage, "Do not put all your eggs in one basket" says very well, portfolio theories provide a framework against the hazardous or marginal combination of assets. The most famous model is the well-known Modern Portfolio Theory (MPT). Markowitz introduced MPT in 1952. The economic sciences Nobel Prize winner explains that portfolios that seek to balance the volatility of their assets could perform better than standard portfolios (i.e., those constructed from securities with good opportunities). He details the mathematics of diversification and demonstrates how risk can be reduced through the efficient combination of assets such that a portfolio can be identified and will provide the highest return for a given level of risk or vice versa. As such, investors can simply select the portfolio that suits either their risk tolerance or targeted return. The procedure finds the optimal portfolio given knowledge of the correlations of each variable and can determine the entire set of an efficient portfolio based upon their preferences. Numerous papers and debates have surrounded Markowitz's theory, although, in general, many economists still consider it to be the centerpiece of financial theory (Jorion, 1992). The use of MPT stands out in real estate portfolio construction, as highlighted by Mr.

Bernstein during the introduction of the *Journal of Portfolio Management* (JPM) in 2005.

The heart of the procedure lies in the mean-variance analysis. Portfolio modeling usually concerns expected returns, risk, and covariance between assets, as confirmed by general economists (Lintner, 1965). The expected return of a portfolio is the weighted sum of the expected return of each of its component securities (i.e., the mean of their past returns).

The variance and standard deviation assess the risk of the portfolio; the larger these indicators are, the lower the probability that actual returns will match the expected returns. The covariance analysis is used to see how assets move relative to each other, as this is the fundamental principle of diversification. The covariance is the product of two different deviations; it describes the outcome of one asset against another. For example, when good performance by one stock is accompanied by a bad outcome for the other, the covariance would be negative.

The use of PC-based software allows the utilization of mean-variance analysis with few complications. However, the procedure has been criticized in a couple of studies because of the sensitivity of its input parameters. Michaud (1989) examines the available algorithm optimizers' labeled quadratic programming and finds that, upon either an exact or approximate basis, the solver works conveniently for the entire mean-variance efficient frontier. Misspecification of the utility function does not alter the portfolio optimization sequences, according to Chopra and Ziembra (1993), although limitations are exposed by Michaud (1989). According to Michaud (1989), prudent financial institutions would be opposed to MPT application in key decisions because of a lack of understanding regarding mean-variance analysis. He notes that most of the mean-variance optimization parameters come from the use of historical data; therefore, their sample mean may lead to uncertainty.

Michaud (1989) also states that several factors are absent from the mean-variance analysis, such as the liquidity and level of information associated with the various asset classes. Therefore, small changes in inputs may hamper the mean-variance analysis, so optimization may be heavily biased by statistical error, leading to estimation error (Michaud, 1989). Jorion (1992) illustrates the substantial effects of estimation error, which quantifies the inefficiency of the mean-variance approach through the derivation of the computed return and covariance matrix. The returns are initially drawn from historical returns and considered unbiased. The author repeats the procedure numerous times, with the weight being reallocated each time; thus, besides the original optimal portfolio (i.e., from the "true" estimates), the several iterations produce other optimal portfolios. Jorion (1992) captures the huge degree of dispersion between the observed portfolios and the original portfolio and finds that estimation error can dramatically change portfolio performance. Using a similar approach, Jobson and Korkie (1980) demonstrate the same issue. As Michaud (1989) outlines, the weights of securities with higher estimates are overweighted and have large estimation errors. Black and Litterman (1992) interpret this phenomenon as the corner solution. Forecasting variance and correlation precisely is quite difficult, and the high sensitivity of the optimal solution to such inputs tends to result in extreme allocation. Also, Jorion (1985) finds that the degree of estimation error depends on the selection criteria. A minimum variance portfolio could, thus, provide less dispersion than equivalent portfolios, as the variance and covariance matrix are measured with less inaccuracy. Best and Grauer (1991), while rating the holding portfolio as a function of the asset mean, report small variation in means but wide shifts in portfolio weights, particularly with correlated data. Chopra and Ziemba (1993) demonstrate that errors in means are extremely strong in comparison to the errors in variance and covariance, although the errors diminish with lower risk tolerance. Using cash equivalence (CEL), he demonstrates errors in means are notably superior to variances and covariances.

Bengtsson (2004) exposes the benefit of portfolio restriction and reports that there are large estimation errors in the means vectors compared to the covariance matrix. Jorion (1985) points out the better performance of the minimum mean-variance portfolio (MVP), in which the weight shifts less heavily than in the normal optimal portfolio. The MVP portfolios are more stable and diversified, notably against the risk-averse mean-variance portfolio and even against the Bayesian portfolio (e.g., Bayes-Stein).

Nevertheless, Cheng and Liang (2000) suggest that mean-variance analysis in real estate portfolios does not bring any benefits, finding that the mean-variance portfolio is not statistically better than a naïve diversified portfolio. As real estate investment relies on entrepreneurial parameters, Armonat and Pfnuer (2004) argue that capital market theory cannot be applied "blindly," as a small change in the parameters tends to be a huge pitfall when building a mean-variance real estate portfolio. The authors suggest that the capital market framework needs to be adapted to the real estate field, as Markowitz's theory was initially developed for financial markets and real estate assets cannot be considered to possess the same characteristics as the mainstream asset classes. Indeed, the insufficient market information associated with real estate markets leads to approximate inputs, so the results of the capital market framework can lead to inefficient outcomes for investors.

## Stability in Portfolio Analysis

According to Daszykowski, Kaczmarek, Vander Heyden, and Walczak (2007), the interpretation of data relies on the data mean, standard deviation, or variance metric. The mean is the most suitable estimate of a true value of a random variable $x$; nevertheless, as the outliers are themselves present in the figures, the data mean tends to lack reliability and it is said to be a non-robust estimator (Perret-Gentil and Victoria-Feser, 2003; Yu, 2005). The following example describes the outlier behavior:

Case 1: {5, 6, 7, 8, 9}   mean = 7
Case 2: {5, 68, 7, 8, 9}   mean = 19.4

The value of 68 in Case 2 can be considered an outlier since it is apart from the bulk of the remaining numbers, pulling the mean to 19.4.

The terms defined as robust or resistant are meant to describe measures that are not easily affected by outliers. Robust statistical estimation is not impacted by the error maximization issue and stands out from the main body of the data (Huber, 1981; Welsch and Zhou, 2007). This means that, if the outcomes are approximately computed, the robust estimator has a reasonable efficiency, having a bias tending toward 0 as the sample size tends toward infinity.

According to Daszykowski, Kaczmarek, Vander Heyden, and Walczak (2007), the quality aspect of robustness is intended to formulate the spread between two studied distributions; the smaller the distance is, the smaller the difference between the respective distributions will be. In general, estimators are listed as parametric or non-parametric estimators. The parametric type assumes the largest portion of the figures following the normal distribution, whereas the non-parametric type does not rely on knowledge of data distribution. For Yu (2005), the parametric estimator is designated to eliminate "contaminated" data. Indeed, when one point is "far" from the rest of the data, the data mean cannot reflect the true data. Parametric procedure is set to delete the outlying point, thereby adjusting the position of the mean. Conversely, semi-non-parametric estimators transform the data rather than eliminate outliers.

Perret-Gentil and Victoria-Feser (2003) evaluate the misspecification of an estimator via an influence function instrument under the name of "gross error sensitivity." The authors find that their robust portfolio allocation is a little sensitive to the outliers, being resistant to error maximization, as the influential data points are treated by their translated biweight S-estimator (TBS). They also demonstrate that, if the parameters model are misspecified, the resulting estimators are seriously biased, leading to sub-optimal portfolio selection. In this study, robustness relates to the stability of parameters from a given model.

## The Shrinkage Estimator

Prior to introducing the Ledoit and Wolf (2004) work, the work of Jorion (1985, 1986) is discussed. He exposes the benefit of a shrinkage estimator over conventional estimators in portfolio management in the form of:

$$E(r_{bs}) = w\bar{r} + (1 - w)r_p,$$

where:

$E(r_{bs})$ = The Bayes-Stein return;
$\bar{r}$ = The return based on the observed data (e.g., historic);
$r_p$ = The "prior" return, a sort of return reference (e.g., from a benchmark); and
$w$ = The specified shrinkage estimator.

To come close to true estimators, the means of the assets are shrunk toward a global mean, reducing the spread between marginal observations. The key point relates to the selection of the optimal value for the shrinkage coefficient and, hence, the intensity of the parameter. Jorion's shrinkage estimator is as follows:

$$\hat{w} = \frac{\hat{\lambda}}{T + \hat{\lambda}} \quad \text{with}$$

$$\hat{\lambda} = \frac{(N + 2)(T - 1)}{(\bar{r} - r_M 1)\ \Sigma^{-1}\ (\bar{r} - r_g)(T - N - 2)},$$

where:

$T$ = The number of time periods of data;
$N$ = The number of asset returns time series;
$\Sigma$ = The covariance matrix of return time series;
$r_g$ = The return of asset $g$; and
$r_M$ = The sort of return reference (e.g., a market return).

Stevenson (2000) investigates the Bayes-Stein shrinkage estimator method in a property portfolio. The author points out the cost of estimation error regarding the inputs. His results confirm the previous discussion (see estimation errors) and present the benefits of minimizing the risk portfolio, reducing the mean estimation error problem (Jorion, 1992). Although the Bayes-Stein approach reduces the weight shift in the asset allocations, with more committed allocation, the demonstration clearly shows the advantage of the minimum variance scheme in real estate portfolio optimization as having a better risk-return ratio. Also, both portfolios are still relatively undiversified, as their allocations shift sharply in some periods. The dataset issues are of less consequence in Stevenson's analysis, as indirect real estate securities are used. Indeed, typical property databases do not result in a similar achievement (see introduction). Similarly to the sample covariance matrix, most of the shrinkage estimators including Bayes-Stein are expensive in terms of data. In addition, property time series exhibit "infamous" issues, adding further constraints to input estimation, such as the covariance matrix.

The covariance matrix is computed with consequent errors when the proportion of data points is comparable or even inferior to the number of individual assets (Jobson and Korkie, 1980). This means that the marginal coefficients in the estimated matrix tend to exhibit extreme values. As such, the mean-variance optimization will target them and place substantial allocations on those coefficients since this is an error maximization (Michaud, 1989), one possible consequence of which is that asset managers can misrepresent their market/asset-picking abilities.

To solve the covariance matrix inaccuracy, Ledoit and Wolf (2004) introduce a shrinkage estimator in which the extremes are shrunk toward the center on a weighted average basis. Correctly implemented, this shrinkage will fix the problem of the sample covariance matrix (Daszykowski, Kaczmarek, Vander Heyden, and Walczak, 2007).

Ledoit and Wolf's (2004) quadratic loss function is as follows:

$$L(\delta) = \|\delta F + (1 - \delta)S - \Sigma\|^2,$$

where:

$F$ = The constant correlation matrix;
$S$ = The sample covariance matrix (i.e., *Cov* $(X;Y)$;
$\Sigma$ = The "true" covariance matrix; and
$\delta$ = The shrinkage estimator/shrinkage intensity.

The purpose of the weight $\delta$ ranging from zero to one is to control the extent to which structure is imposed upon the single index model; the heavier the weight is, the stronger the structure will be. The shrinkage target $F$ aims to forecast the correlation between pair-wise variables. It is formed by the sample correlation and variance means.

While empirical evidence shows that the average correlation outperforms diverse sophisticated models regarding the correlation estimation (Kwan, 2006), due to its structure, the nature of the observed assets/securities has to be in a comparable universe (Ledoit and Wolf, 2004).

In a "perfect world", the Ledoit and Wolf (2004) shrinkage estimator $\delta^*$ is described as a constant $\kappa$, $(\pi - p)/\gamma$, where $\pi$ is the sum of all the infinite variances of the sample covariance matrix (including the diagonal elements). $p$ is the asymptotic covariance, a matrix that has an infinite property, and $\gamma$ determines an incorrect description of the population shrinkage target. In truth, the constant $\kappa$ is unknown, and Ledoit and Wolf (2004) propose estimators $\hat{\pi}$, $\hat{p}$, and $\hat{\gamma}$ to compute the constant $\kappa$. Their elaborated mathematical demonstration ends with:

$$\hat{\pi} = \sum_{i=1}^{N} \sum_{j=1}^{N} \hat{\pi}_{ij} \quad \text{with}$$

$$\hat{\pi}_{ij} = \frac{1}{T} \sum_{t=1}^{T} \{(y_{it} - \bar{y}_i)(y_{jt} - \bar{y}_j) - s_{ij}\}^2$$

$$\hat{p} = \sum_{i=1}^{N} \hat{\pi}_{ii} + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\bar{r}}{2} \left( \sqrt{\frac{s_{jj}}{s_{ii}}}\, \vartheta_{ii,jj} + \sqrt{\frac{s_{ii}}{s_{jj}}}\, \vartheta_{jj,ii} \right)$$

$$\hat{\gamma} = \sum_{i=1}^{N} \sum_{j=1}^{N} (f_{ij} - s_{ij})^2,$$

where:

$y_{it}$ = The return of asset $i$;
$\bar{y}_i$ = The sample average of the performance for assets $i$ and $j$;
$S_{ij}$ = The sample correlation;
$\bar{r}$ = The average sample correlation; and
$f_{ij}$ = The sample constant correlation matrix.

Using a method called delta, Ledoit and Wolf (2004) obtain the asymptotic elements of $\hat{p}$ with $\vartheta_{ii,jj}$ being $AsyCov[\sqrt{T}s_{ii}, \sqrt{T}s_{ij}]$ and $\vartheta_{jj,ii}$ being $AsyCov[\sqrt{T}s_{jj}, \sqrt{T}s_{ij}]$. The *Asy* notation denotes the application by the researchers of the asymptotic approach, a method that allows "affranchising" from any distribution (e.g., normality). As a sequence of a variable goes to infinity, there is convergence in the distribution. The asymptotic distribution gives this cumulative distribution function of the statistical estimators. According to Schafer and Strimmer (2005), the major part of the proposed shrinkage estimators is commonly set at a specific underlying distribution and constrained by their dataset. Indeed, the number of series cannot exceed the observations to enable the empirical covariance to be inverted (e.g., the Bayes-Stein shrinkage estimator). These issues are smartly prevented by Ledoit and Wolf (2004). The researchers emphasize the consistency of the shrinkage estimator by calling upon the asymptotic property of its elements (i.e., the elements $\hat{\pi}$, $\hat{p}$, and $\hat{\gamma}$). Also, the squared error loss function is substituted by the squared Frobenius norm (Schafer and Strimmer, 2005). The Frobenius norm, $\|Z\|^2 = \Sigma_{i=1}^{N} \Sigma_{j=1}^{N} Z_{ij}^2$, allows the use of the shrinkage estimator without the inverse matrix. The eigenvectors act on certain vectors by changing only their magnitude and leaving their direction unchanged. The vector is the diagonal elements of the matrix and its symmetry $(N \times N)$. The Ledoit and Wolf loss functions settle the inverse covariance matrix problem, making the method intuitive.

Therefore, the shrinkage estimator *à la* Ledoit and Wolf (2004) allows the construction of an improved covariance estimator that is not only suitable for a small sample size and a large number of variables but one that is also inexpensive in terms of data, unlike the Bayes-Stein estimator.

## Empirical Results

### Data & Methodology

The data are annual total returns for 16 European office markets from the Investment Property Databank (IPD) database. IPD is a global information business dedicated to the objective measurement of commercial real estate performance. It operates in over 20 countries, including most of Europe. Its indices are the basis for the developing commercial property derivatives market and the most authoritative measures of real estate returns worldwide. To guarantee their integrity, they are not involved in investment markets and do not provide any investment consultancy. The European office property market is analyzed over a period ranging from

4 to 29 years. The numbers are examined on an annual basis, which seems acceptable, as real estate prices are not extremely volatile from month to month (Arthur, 2005). Exhibit 1 summarizes the dataset.

### A Simple Performance Analysis

Using historical IPD total return data for the main European countries, two types of minimum variance portfolio (MVP) are investigated. The first is the conventional portfolio. This is the MVP portfolio made with the sample covariance matrix (i.e., $Cov(X;Y)$). The second is the shrinkage portfolio. This is the minimum risk portfolio built with the Ledoit and Wolf (2004) shrinkage estimator. The optimal portfolios range from the lowest possible target returns to the maximum possible returns. In total, 110 portfolios are created that offer two complete efficient frontiers for the shrinkage and conventional models. The Sharpe ratio (1994) quickly depicts the arguments, $(R_p - R_f)/\vartheta_p$, with $R_p$ representing the observed portfolio return, $R_f$ denoting the risk-free returns, and $\vartheta_p$ being the risk of the specified portfolio. It estimates the relative performance of the two allocation strategies. It will gauge the risk management of the different methods. Exhibit 2 relates the efficient frontier and plots the Sharpe ratio.

The low portfolio returns constructed from the conventional portfolio outperform those of the shrinkage portfolio (Exhibit 2). Nonetheless, its market-picking ability is unequal regarding the portfolio target returns. The risk in the shrinkage portfolio is generally inferior to that of the conventional portfolio, being 300 bps less volatile, while some conventional MVPs do not show the same efficiency in several efficient markets, as the diversification tends to be highly specific for a certain level of returns. Investors may, thus, have to study a specific return area to capture an attractive efficient portfolio.

**Exhibit 1**
**Data Summary**

| Country/Market | Period | Mean | Std. Dev. |
|---|---|---|---|
| Denmark | 2000–2009 | 8.6 | 3.7 |
| Norway | 2000–2009 | 9.0 | 7.0 |
| Poland | 2005–2009 | 9.1 | 5.9 |
| Sweden | 1997–2009 | 8.5 | 8.2 |
| Switzerland | 2002–2009 | 5.5 | 0.8 |
| U.K. | 1981–2009 | 8.7 | 11.4 |
| Austria | 2004–2009 | 4.3 | 0.9 |
| Belgium | 2005–2009 | 5.2 | 1.9 |
| Finland | 1998–2009 | 7.3 | 2.7 |
| France | 1998–2009 | 10.4 | 7.2 |
| Germany | 1996–2009 | 3.0 | 2.0 |
| Ireland | 1984–2009 | 11.0 | 16.4 |
| Italy | 2003–2009 | 6.7 | 3.0 |
| Netherlands | 1995–2009 | 9.1 | 4.5 |
| Portugal | 2000–2009 | 6.8 | 2.1 |
| Spain | 2001–2009 | 6.4 | 7.4 |

Note: The source is IPD, 2009.

## The Dispersion

### Simulation with Error Magnitude

The out-of-sample performance of the Ledoit and Wolf (2004) shrinkage estimator is examined to determine whether it limits the impact of estimation error in a real estate portfolio. The conventional and the shrinkage portfolios are examined. The asset $i$ mean return $\bar{r}_i$ is $1/n \ \Sigma_{t=1}^{n} \ r_{it}(1 + k)$, where $r_{it}$ is the return of the asset $i$ at time $t$, $k \in [5\%; 50\%]$ starting with a shock level at 0.5% and moving gradually by 0.5%; thus, all the different series receive the same shocks. This is a simple growth on individual returns on an equal basis for every market. The corrections are not performed with respect to the variance and covariance. The initial shrinkage and conventional covariance matrix are employed and their efficient frontiers are then generated for each of the 100 shocks' magnitudes.

If estimation errors occur in the data, they would be leveled up with outliers. Thus, the degree of dispersion between the new allocations and the initial one can be captured at different shock magnitudes. The dispersion is the weights absolute differences from the different portfolio simulations. For each value of $k$, the spread between the allocations is computed with the standard deviation (i.e., $\sqrt{(\Sigma \ (x - \bar{x})^2)/n}$). Hence, when the portfolio is fairly resistant to estimation error, the allocation provided by the portfolio modeling will be resilient

**Exhibit 2**
***Shrinkage versus Conventional Portfolio Behavior***

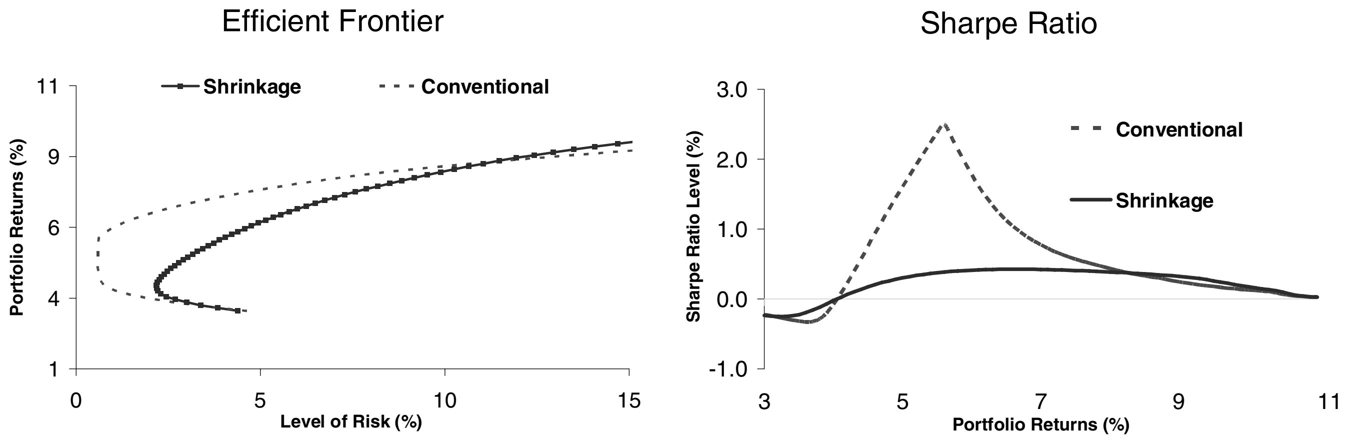



**Exhibit 3**
***The Dispersions with the Initial Allocation***

| | Initial Allocation | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Panel A: Shrinkage portfolio | | | | | | | | | | |
| Shocks | 5.0% | 10.0% | 15.0% | 20.0% | 25.0% | 30.0% | 35.0% | 40.0% | 45.0% | 50.0% |
| Dispersions[a] | 4.3% | 5.2% | 4.3% | 5.0% | 5.0% | 5.3% | 5.5% | 5.8% | 6.1% | 6.5% |
| Panel B: Conventional portfolio | | | | | | | | | | |
| Shocks | 5.0% | 10.0% | 15.0% | 20.0% | 25.0% | 30.0% | 35.0% | 40.0% | 45.0% | 50.0% |
| Dispersions[a] | 8.4% | 11.1% | 12.5% | 13.3% | 14.1% | 13.9% | 14.0% | 15.2% | 14.2% | 14.7% |

Note:
[a] Between the original portfolio and a specified allocation.

to growth in the inputs, and the corner solution effect may not be too apparent.

By definition, the lower the allocation shift is, the more structured the allocation will be, which implies that the optimization will not be significantly altered by the variation in its inputs. Therefore, optimization tends to follow a similar pace even if it is to a different degree. Regarding the entire shock impact, the gap between the initial allocation and another distribution is around 5% in the shrinkage portfolio. There are slightly higher dispersions following stronger shocks, but it seems quite flat, especially considering the observed dispersions in the other portfolio model. Indeed, note the stronger spread from the conventional allocation with 8.4% at only 5% magnitude. For the top magnitude, the level of the spread is clearly bigger with 14.2% at 45% magnitude, resulting in a more excessive target allocation in the successive error magnitudes. Thus, the corner solution may appear to be more serious, leading to inefficient outcomes for investors. Exhibit 3 demonstrates the benefits of the shrinkage estimator, as it successfully shows stability in its allocation; at 50% shocks, the dispersions are still not as important as in the conventional model at 5% shock, for instance.

In fact, the covariance matrix is heavily influenced by the presence of outliers (Jorion, 1986; Ledoit and Wolf, 2004). Exhibit 4 shows the dispersions of the covariance matrix for the specific input deviate from 15%, which is similar to the original covariance matrix results.

**Exhibit 4**
**Dispersions with the Initial Allocation at 50% Shock**

| | Initial Allocation | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Target returns | 5.0% | 5.5% | 6.0% | 6.5% | 7.0% | 7.5% | 8.0% | 8.5% | 9.0% | 9.5% | 10.0% | 10.5% | 11.0% |
| Panel A: The shrinkage covariance matrix | | | | | | | | | | | | | |
| Average weight dispersion[a] | 6.7% | 6.7% | 6.4% | 5.9% | 5.7% | 5.6% | 6.3% | 6.3% | 6.8% | 8.1% | 7.8% | 6.3% | 6.3% |
| Global Average | 6.5% | | | | | | | | | | | | |
| Panel B: The original covariance matrix | | | | | | | | | | | | | |
| Average weight dispersion[a] | 18.0% | 18.3% | 14.3% | 13.5% | 12.0% | 13.5% | 14.3% | 16.7% | 13.1% | 15.5% | 16.7% | 12.5% | 12.5% |
| Global Average | 14.7% | | | | | | | | | | | | |
| Panel C: The covariance matrix for the specific input | | | | | | | | | | | | | |
| Average weight dispersion[a] | 14.7% | 17.8% | 20.0% | 13.5% | 13.7% | 12.9% | 12.5% | 9.9% | 14.8% | 14.9% | 16.7% | 16.7% | 16.7% |
| Global Average | 15.0% | | | | | | | | | | | | |

Note:
[a] Between the original portfolio allocation and the specified portfolio allocation.

## The Corner Solution

The range of the target allocation is wider in the portfolio *à la* Ledoit and Wolf (2004) using, on average, 11 targets out of 16, whereas traditional optimization employs seven markets for its optimization. Indeed, there is a 70% chance of finding an allocated market in the same return portfolio at a different shock magnitude and even a 93% chance for certain portfolio returns. As evident in Exhibit 5, the shrinkage allocation gives the opportunity to distinguish an investment's strategy, although the allocation is slightly altered by new inputs and constraints, such as market size (i.e., liquidity) or transaction costs that may need to be added for a more practical analysis.

The opposite strategy provided by the traditionally made portfolio perfectly describes the corner solution. As the global returns for each series is exaggerated, the allocations shift sharply in Example 4, with Portugal disappearing from the allocation at circa 30% shock magnitude for no particular reason. Also, the allocation amplitude is very marginal for certain markets. The effect of outliers, as discussed above, probably hamper the optimization process, leading to a large corner solution and, hence, an unrealistic assumption.

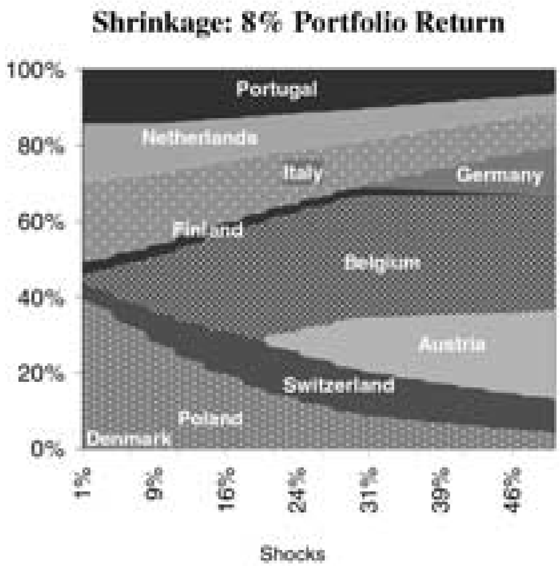
As Ledoit and Wolf (2004) explain, the shrinkage allocation outcome comes from the fact that the shrinkage estimator aims to diminish the impact of outliers on the optimization process by remodeling the heart of the mean-variance analysis, the covariance matrix. Decreasing the sensitivity of the portfolio optimization strengthens the allocation position and diminishes the corner solution effect, providing material for portfolio constructive qualitative analysis.

## The Outlier Test

In another experiment, extreme figures are added to the original data to evaluate the impact of outliers on the different portfolio optimizations. The series used for the portfolio construction is the same as before, consisting generally of around 9–10 years. In 2010, there were recoveries in the key core markets in response to the 2007–2009 financial crisis. The extreme figures for 2010 are added to the data series (Exhibit 6). These numbers are taken from the strongest historical total returns for each series, and as such, they probably increase the data volatility from approximately 4.8% and their total return mean from circa 7.5%. Thus, there are two case scenarios: Case 1 denotes portfolios based on actual figures and Case 2 is the portfolios constructed with the addition of the extreme 2010 numbers.

Assuming that the portfolio is built on historical data that ends in 2009, the two covariance method-

**Exhibit 5**
***The Allocation for the 8% Portfolio Return***



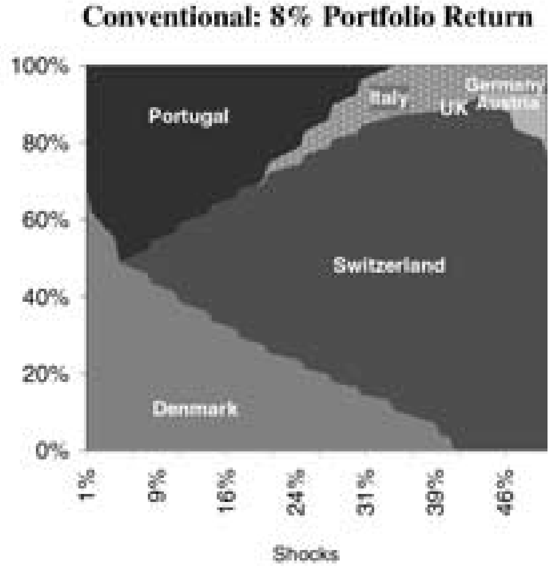
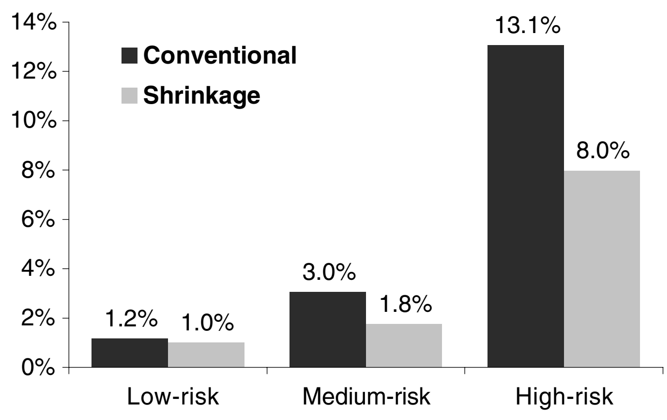




**Exhibit 6**
**The New Inputs**

| Countries | IPD 2010 Real Figures[a] | The New Inputs (2010 Horizon) |
|---|---|---|
| Average | 6.0% | 16.5% |

Notes: The source is IPD, April 2011.
[a] The IPD 2010 real numbers are not available for Poland, Austria, Belgium, and Italy.

**Exhibit 7**
***Dispersion between the Portfolios in Cases 1 and 2***




based portfolios are compared using former historical data and new inputs (i.e., adding 2010 figures into the data series). The dispersion between them is shown in Exhibit 7.

The optimal portfolios are classified into two groups according to their risk-adjusted return. The low-risk portfolios are the portfolios with lower returns and vice versa for the other segment. While it is logical to find dispersions between the initial allocations and the new ones, the level of dispersion is different between the two methods. There is a 42% variation between the two schemes in the middle-risk portfolio, which has higher dispersion in the traditional usage, and throughout the outputs, the shrinkage estimator provides a more stable allocation, as it better resists the presence of outliers in the data. Meanwhile, in the top return portfolio, both methods show signs of inaccuracy in their allocation, deviating from circa 29% from their original targets as the estimation error is maximized because of the outliers' impact.

Case 1 is compared with the portfolio based only upon the extreme 2010 total return figures. The same process is used to evaluate their allocation spreads (Exhibit 8). The allocations of risk-averse portfolios deviate similarly. Nevertheless, the shrinkage estimator method is more accurate than the conventional method for the top return portfolios. Their allocation is indeed close to the one

**Exhibit 8**
**Dispersion between the Portfolios in Case 1 and the 2010 Portfolio**

| | Average Dispersions | | |
|---|---|---|---|
| | Low-risk | Medium-risk | High-risk |
| Conventional | 14.6% | 16.7% | 16.9% |
| Shrinkage | 13.2% | 8.4% | 8.7% |

derived from the efficient 2010 portfolios. The shrinkage estimator lessens the effect of outliers and improves the precision of target allocation against shocks; thus, the optimization is more stable and less sensitive to estimation error (Ledoit and Wolf, 2003, 2004).

## Cost of Allocation

It is not surprising to find an office asset representing a fair share of the portfolio value. The transaction cost then becomes a significant variable in allocation distribution, as buying and selling assets leads to significant expenses for a portfolio manager. The cost of dispersion is examined by evaluating the transaction cost from one allocation to another; the higher the transaction charges are, the more expensive the allocation/reallocation will be. Therefore, maintaining an optimal portfolio at the same pace becomes more and more expensive because of allocation variations.

The following lemma illustrates the impact of transaction costs:

$$\sum_{k=0}^{n} C_{x_n} \left(1 + \left| \left(\frac{w_{x_n}}{w_{x_i}}\right) - 1 \right| \right),$$

where:

$w_{x_i}$ = The initial allocation for the market $X$;
$w_{x_n}$ = The allocation for the market $X$ after being shocked at $k$ magnitude; and
$C_{x_n}$ = The transaction cost for the market $X$.

The transaction cost is assumed to be global, this being the sum of taxes, stamp duties, broker fees, and other related charges. They are considered as a ratio of property value such as a likewise covenant (e.g., loan-to-value ratio), meaning that they represent a part of the equity allocation and, thus, a part of the portfolio equity. The relation can be written as $C_{x_n} = P \times w_{x_n} \times o_x$, with $P$ denoting the portfolio equity and $O_x$ being the transaction expenses out of the allocated equity in a property market. The comparison of transaction cost figures are based on a document provided by a property company (i.e., BNP Paribas Real Estate). The portfolio equity number is generated randomly. Indeed, the distance between costs does not change much, and with any variable, there would be a similar gap in costs. Exhibit 9 depicts the results.

By stabilizing allocations, the shrinkage model is less costly in reallocation and unstable allocation. In contrast, the conventional model does not match because of problems, such as the corner solution and sharp allocation shift across the experiments. The robustness of the shrinkage helps to lessen the transaction cost. Indeed, movements between allocations are more restrained, and their positions tend to alter proportionally to shocks rather than changing their allocation behavior completely. The shrinkage method allows for a reduction in the marginal cost of reallocation, which can be beneficial in an arbitrage context as feeding the portfolio to its maximum efficiency becomes more flexible.

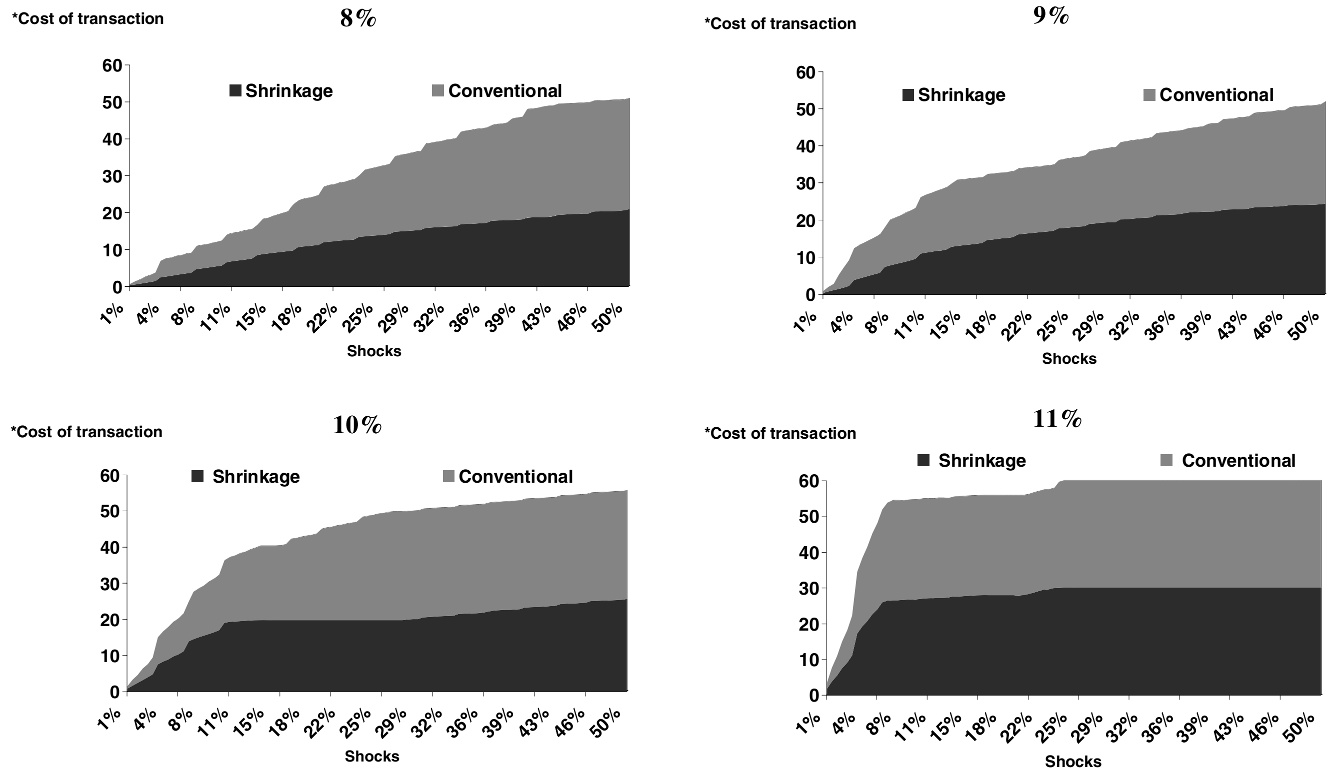
Even though the shrinkage estimator as in Ledoit and Wolf (2004) stabilizes the allocation distribution and offers a good opportunity for future research studies, in the meantime, the shrinkage MVP shows low Sharpe ratios compared to those of the traditional MVP, paying a heavy tribute in terms of the returns-risk ratio to obtain robust allocation. This price can be relativized if the transaction/dispersion costs yield superior marginal gain. This theoretical marginal gain can be reduced to a simple relation as follows:

$$\text{UTILITY GAIN} = \text{MARGINAL RETURN} - \text{MARGINAL COST}.$$

This study does not fully develop portfolio performance analysis, and in the future, it may be logical to examine the marginal gain of the two methods'

**Exhibit 9**

**Cost of Dispersion for Different Portfolio Target Returns**




*The portfolio equity = 150
*The transaction cost is 10% of the equity value

portfolios. Also, the topic may be interesting to explore in the case of arbitrage or gauging portfolio performance.

## Conclusion

While the mean-variance analysis provides a view of a possible portfolio strategy, its performance is hampered by the estimation error, which can transform the inputs into unreliable sources. Research studies have widely indicated the weaknesses of the Markowitz scheme in mean and covariance matrix computation. Because the optimization is very sensitive to its parameter estimations, the unbiased estimators are investigated to identify the observations more precisely. As stated by Welsch and Zhou (2007), there are a variety of robust estimators that can be used to estimate both the mean and covariance matrix. This paper suggests a plausible correction of the covariance matrix using the Ledoit and Wolf (2004) method and the optimization of the risk portfolio (i.e., the MVP). Indeed, the MVP has shown benefits over normal optimization and is widely acknowledged in the academic world.

The main part of the study concerns the modification of the covariance matrix into an asymptotic one thanks to the Ledoit and Wolf (2004) shrinkage estimator. The Ledoit and Wolf model is compared with the conventional MVP across different criteria: relative performance analysis and, principally, the dispersion of allocations. The various tests demonstrate the remarkable advantages of the portfolio over more traditional methods. Although the performance analysis seems exceptional compared to that of the conventional model, the results also expose great instability regarding its optimization, as the allocation is very dispersed and excessively sensitive to outliers, making it hard to distinguish a proper portfolio strategy. The constraints in the allocation may be a solution (Jorion, 1992).

These negative points are almost erased by the shrinkage estimator technique, which aims to illustrate the essential statistical properties of the data. This ensures a stable allocation and a less sensitive portfolio optimization.

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