

Generalized Instantaneous Reactive Power Theory for Three-Phase Power Systems

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Abstract—A generalized theory of instantaneous reactive power for three-phase power systems is proposed in this paper. This theory gives a generalized definition of instantaneous reactive power, which is valid for sinusoidal or nonsinusoidal, balanced or unbalanced, three-phase power systems with or without zero-sequence currents and/or voltages. The properties and physical meanings of the newly defined instantaneous reactive power are discussed in detail. A three-phase harmonic distorted power system with zero-sequence components is then used as an example to show reactive power measurement and compensation using the proposed theory.

I. INTRODUCTION

FOR single-phase power systems with sinusoidal voltages and sinusoidal currents, quantities such as active power, reactive power, active current, reactive current, power factor, etc., are based on the average concept. Many contributors have attempted to redefine these quantities to deal with three-phase systems with unbalanced and distorted currents and voltages [1]–[5].

Among them, Akagi *et al.* [1] have introduced an interesting concept of instantaneous reactive power. This concept gives an effective method to compensate for the instantaneous components of reactive power for three-phase systems without energy storage. However, this instantaneous reactive power theory still has a conceptual limitation as pointed out in [2], that is, the theory is only complete for three-phase systems without zero-sequence currents and voltages. To resolve this limitation and other problems, Willems proposed an attractive approach to define instantaneous active and reactive currents [2]. His approach, however, is to deal with the decomposition of currents into orthogonal components, rather than with power components.

In this paper, a generalized theory of instantaneous reactive power for three-phase power systems is proposed. The generalized theory is valid for sinusoidal or nonsinusoidal, balanced or unbalanced three-phase systems, with or without zero-sequence currents and/or voltages. Some interesting properties of the theory developed are shown.

II. DEFINITIONS OF INSTANTANEOUS REACTIVE COMPONENTS FOR THREE-PHASE SYSTEMS

For a three-phase power system shown in Fig. 1, instantaneous voltages, v_a, v_b, v_c , and instantaneous currents, i_a, i_b, i_c ,

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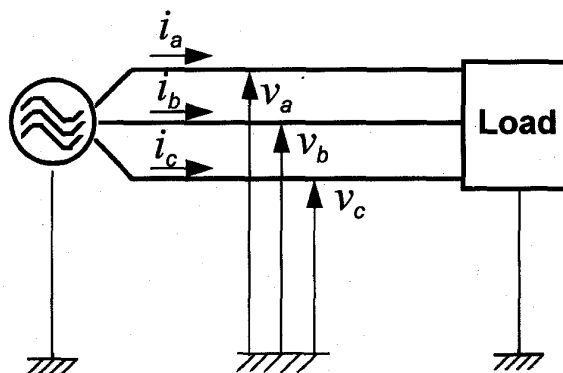


Fig. 1. Three-phase circuit structure.

are expressed as instantaneous space vectors, \mathbf{v} and \mathbf{i} , that is,

$$\mathbf{v} = \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix}, \quad \mathbf{i} = \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix}. \quad (1)$$

Fig. 2 shows the three-phase coordinates which are mutually orthogonal, representing phase 'a,' phase 'b,' and phase 'c,' respectively. The instantaneous active power of a three-phase circuit, p , can be given by

$$p = \mathbf{v} \cdot \mathbf{i} \quad (2)$$

where " \cdot " denotes the dot (internal) product, or scalar product of vectors. Equation (2) can also be expressed in the conventional definition

$$p = v_a i_a + v_b i_b + v_c i_c.$$

Here, we define a new instantaneous space vector \mathbf{q} as

$$\mathbf{q} \stackrel{\text{def}}{=} \mathbf{v} \times \mathbf{i} \quad (3)$$

where " \times " denotes the cross (exterior) product of vectors or vector product. Vector \mathbf{q} is designated as the instantaneous reactive (or nonactive) power vector of the three-phase circuit, and the magnitude or the length of \mathbf{q} , q , is designated as the instantaneous reactive power, that is,

$$q = \|\mathbf{q}\| = \|\mathbf{v} \times \mathbf{i}\|, \quad (4)$$

where " $\|\cdot\|$ " denotes the magnitude or the length of a vector. Equations (3) and (4) can be rewritten as

$$\mathbf{q} = \begin{bmatrix} q_a \\ q_b \\ q_c \end{bmatrix} = \begin{bmatrix} v_b i_c - v_c i_b \\ v_c i_a - v_a i_c \\ v_a i_b - v_b i_a \end{bmatrix}$$

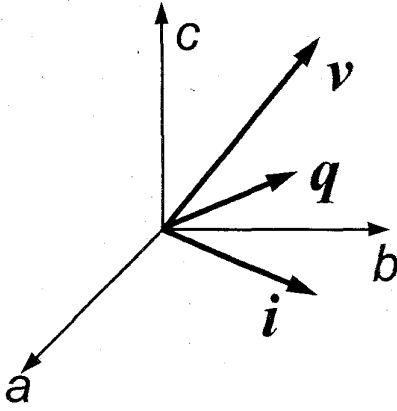


Fig. 2. Three-phase coordinates.

and

$$q = \|q\| = \sqrt{q_a^2 + q_b^2 + q_c^2},$$

respectively. In turn, we define the instantaneous active current vector, i_p , the instantaneous reactive current, i_q , the instantaneous apparent power, s , and the instantaneous power factor, λ , as

$$i_p = \begin{bmatrix} i_{ap} \\ i_{bp} \\ i_{cp} \end{bmatrix} \stackrel{\text{def}}{=} \frac{p}{v \cdot v} v, \quad (5)$$

$$i_q = \begin{bmatrix} i_{aq} \\ i_{bq} \\ i_{cq} \end{bmatrix} \stackrel{\text{def}}{=} \frac{q \times v}{v \cdot v}, \quad (6)$$

$$s \stackrel{\text{def}}{=} vi, \quad \text{and} \quad (7)$$

$$\lambda \stackrel{\text{def}}{=} \frac{p}{s}, \quad (8)$$

where $v = \|v\| = \sqrt{v_a^2 + v_b^2 + v_c^2}$ and $i = \|i\| = \sqrt{i_a^2 + i_b^2 + i_c^2}$ are the instantaneous magnitudes or norms of the three-phase voltage and current, respectively.

III. PROPERTIES AND PHYSICAL MEANINGS

A. Properties

The new reactive components defined above have some interesting properties, which are shown by the following theorems.

Theorem 1: The three-phase current vector, i , is always equal to the sum of the instantaneous active current vector, i_p , and the instantaneous reactive current vector, i_q , i.e., $i \equiv i_p + i_q$.

Proof: From the definitions expressed in (2), (3), (5), and (6), we have

$$\begin{aligned} i_p + i_q &= \frac{pv}{v \cdot v} + \frac{q \times v}{v \cdot v} \\ &= \frac{(v \cdot i)v + (v \times i) \times v}{v \cdot v}. \end{aligned}$$

Using the following formula of the vector product

$$(a \times b) \times c = -(b \cdot c)a + (a \cdot c)b, \quad (9)$$

we get

$$\begin{aligned} i_p + i_q &= \frac{(v \cdot i)v + [-(i \cdot v)v + (v \cdot v)i]}{v \cdot v} \\ &= \frac{(v \cdot v)i}{v \cdot v} \\ &= i. \end{aligned} \quad \square$$

Theorem 1 shows that any three-phase current vector, i , can always be decomposed into two components, i_p and i_q . The following theorem will show that i_p and i_q correspond with the instantaneous active power and reactive power, respectively.

Theorem 2: i_q is orthogonal to v , and i_p is parallel to v , namely, $v \cdot i_q \equiv 0$ and $v \times i_p \equiv 0$.

Proof:

$$\begin{aligned} v \cdot i_q &= v \cdot \left(\frac{q \times v}{v \cdot v} \right) \\ &= v \cdot \left(\frac{(v \times i) \times v}{v \cdot v} \right) \quad (\text{using the formula of (9)}) \\ &= v \cdot \frac{-(i \cdot v)v + (v \cdot v)i}{v \cdot v} \\ &= \frac{-(i \cdot v)(v \cdot v) + (v \cdot v)(i \cdot v)}{v \cdot v} \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} v \times i_p &= v \times \left(\frac{p}{v \cdot v} v \right) \\ &= 0. \end{aligned} \quad \square$$

From Theorem 2, we can see that i_p and i_q are mutually orthogonal, that is, $i_p \cdot i_q \equiv 0$. Therefore, we have the following theorem.

Theorem 3: $i^2 \equiv i_p^2 + i_q^2$, $s^2 \equiv p^2 + q^2$, and $i^2 \equiv \frac{p^2 + q^2}{v^2}$.

Proof: Because $i_p \cdot i_q \equiv 0$, it follows that

$$\begin{aligned} i^2 &= i \cdot i = (i_p + i_q) \cdot (i_p + i_q) \\ &= i_p \cdot i_p + i_q \cdot i_q + 2(i_p \cdot i_q) \\ &= i_p^2 + i_q^2, \end{aligned}$$

Similarly, we have

$$\begin{aligned} p^2 + q^2 &= (v \cdot i)^2 + \|v \times i\|^2 \\ &= (v \cdot i)(v \cdot i) + (v \times i) \cdot (v \times i) \\ &= (v \cdot i_p)^2 + (v \times i_q) \cdot (v \times i_q) \\ &= v^2 i_p^2 + v^2 i_q^2 = v^2 i^2 = s^2. \end{aligned}$$

From (7) and the above results, we get

$$i^2 = \frac{s^2}{v^2} = \frac{p^2 + q^2}{v^2}. \quad \square$$

Theorem 4: If $i_q = 0$, then the norm $\|i\|$ or i becomes minimal for transmitting the same instantaneous active power, and the maximal instantaneous power factor is achieved, namely $\lambda = 1$.

Proof: Since $i = \sqrt{i_p^2 + i_q^2} \geq i_p$, and $\lambda = \frac{p}{s} = \frac{p}{\sqrt{p^2 + q^2}} \leq 1$, it follows that if $i_q = 0$, then the norm $\|i\|$ or i becomes minimal, and $q = 0$, $\lambda = 1$ to transmit the active power, p . \square

From the above theorems, we can get the following conclusions:

- 1) The current vector \mathbf{i}_p is indispensable for the instantaneous active power transmission, whereas \mathbf{i}_q does not contribute to it, because $\mathbf{v} \cdot \mathbf{i}_q \equiv 0$, and $p = \mathbf{v} \cdot \mathbf{i} = \mathbf{v} \cdot \mathbf{i}_p$.
- 2) It needs no energy storage for a compensator to eliminate the instantaneous reactive power, q .
- 3) Using compensators without energy storage, the instantaneous active power cannot change, and hence the minimum line losses are obtained for zero instantaneous reactive power ($q = 0$).

B. Physical Meanings

In each phase the instantaneous active power can be split up into two parts

$$\begin{cases} p_a = v_a i_a = v_a (i_{ap} + i_{aq}) \stackrel{\text{def}}{=} p_{ap} + p_{aq} \\ p_b = v_b i_b = v_b (i_{bp} + i_{bq}) \stackrel{\text{def}}{=} p_{bp} + p_{bq} \\ p_c = v_c i_c = v_c (i_{cp} + i_{cq}) \stackrel{\text{def}}{=} p_{cp} + p_{cq} \end{cases} \quad (10)$$

where

$$\begin{cases} p_{ap} = v_a i_{ap} \\ p_{bp} = v_b i_{bp} \\ p_{cp} = v_c i_{cp} \end{cases} \quad \text{and} \quad \begin{cases} p_{aq} = v_a i_{aq} \\ p_{bq} = v_b i_{bq} \\ p_{cq} = v_c i_{cq} \end{cases} \quad (11)$$

Since $p = \mathbf{v} \cdot \mathbf{i} = \mathbf{v} \cdot \mathbf{i}_p$, $\mathbf{v} \cdot \mathbf{i}_q = 0$, and $\mathbf{q} = \mathbf{v} \times \mathbf{i}_q$, we can see that p_{ap} , p_{bp} and p_{cp} contribute to the total power, p , and they sum up to p , i.e., $p_{ap} + p_{bp} + p_{cp} = p$. Power components p_{aq} , p_{bq} and p_{cq} contribute to q and sum up to zero, i.e., $p_{aq} + p_{bq} + p_{cq} = 0$. Therefore, $p_{aq} + p_{bq} + p_{cq}$ corresponds to those powers that transfer or circulate between the three phases. The instantaneous reactive current, \mathbf{i}_q , does not convey any instantaneous active power from the source to the load (see Fig. 1), but indeed it increases the line losses and the norm of the three-phase current. If \mathbf{q} or \mathbf{i}_q is eliminated by a shunt compensator, then the norm of the source current will become minimum.

IV. DISCUSSION

A. Alternative Expressions

In Section II, the definitions of the instantaneous reactive components are all based on the direct quantities of three-phase voltages and currents: v_a, v_b, v_c , and i_a, i_b, i_c . If necessary, these newly defined quantities can be expressed in any other coordinates, e.g., $\alpha\beta 0$ coordinates. Here, let us express the defined quantities, $p, \mathbf{q}, \mathbf{i}_p, \mathbf{i}_q$, etc., in $\alpha\beta 0$ coordinates.

For three-phase voltages and currents, v_a, v_b, v_c , and i_a, i_b, i_c the α, β , and 0 components are expressed as

$$\begin{bmatrix} v_\alpha \\ v_\beta \\ v_0 \end{bmatrix} = [C] \begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} \quad (12)$$

and

$$\begin{bmatrix} i_\alpha \\ i_\beta \\ i_0 \end{bmatrix} = [C] \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} \quad (13)$$

where

$$[C] = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Since $[C]^{-1} = [C]^T$, we have

$$p = \mathbf{v}_{abc} \cdot \mathbf{i}_{abc} = \mathbf{v}_{\alpha\beta 0} \cdot \mathbf{i}_{\alpha\beta 0} \quad (14)$$

$$\begin{aligned} \mathbf{q}_{\alpha\beta 0} &= \mathbf{v}_{\alpha\beta 0} \times \mathbf{i}_{\alpha\beta 0} = ([C]\mathbf{v}_{abc}) \times ([C]\mathbf{i}_{abc}) \\ &= [C](\mathbf{v}_{abc} \times \mathbf{i}_{abc}) = [C]\mathbf{q}_{abc} \end{aligned} \quad (15)$$

where, suffixes “ abc ” and “ $\alpha\beta 0$ ” denote the corresponding coordinates, i.e., $\mathbf{v}_{abc} = [v_a \ v_b \ v_c]^T$, $\mathbf{v}_{\alpha\beta 0} = [v_\alpha \ v_\beta \ v_0]^T$, $\mathbf{q}_{abc} = [q_a \ q_b \ q_c]^T = \mathbf{v}_{abc} \times \mathbf{i}_{abc}$, and $\mathbf{q}_{\alpha\beta 0} = [q_\alpha \ q_\beta \ q_0]^T = \mathbf{v}_{\alpha\beta 0} \times \mathbf{i}_{\alpha\beta 0}$. From (15) and $\|[C]\| = 1$, we have

$$q_{\alpha\beta 0} = \|\mathbf{q}_{\alpha\beta 0}\| = q_{abc} = \|\mathbf{q}_{abc}\|. \quad (16)$$

Therefore, $\mathbf{q}_{\alpha\beta 0}$ is identical to \mathbf{q}_{abc} . Similarly, we can define the instantaneous active and reactive components in $\alpha\beta 0$ coordinates as

$$\mathbf{i}_{p\alpha\beta 0} = \frac{p}{\mathbf{v}_{\alpha\beta 0} \cdot \mathbf{v}_{\alpha\beta 0}} \mathbf{v}_{\alpha\beta 0}, \quad (17)$$

$$\mathbf{i}_{q\alpha\beta 0} = \frac{\mathbf{q}_{\alpha\beta 0} \times \mathbf{v}_{\alpha\beta 0}}{\mathbf{v}_{\alpha\beta 0} \cdot \mathbf{v}_{\alpha\beta 0}}. \quad (18)$$

The properties and physical meanings mentioned in Section III are valid and are independent of coordinates. For three-phase systems without zero-sequence components, i.e., v_0 and i_0 are equal to zero, the instantaneous active and reactive powers can be simplified as

$$p = \mathbf{v}_{\alpha\beta 0} \cdot \mathbf{i}_{\alpha\beta 0} = v_\alpha i_\alpha + v_\beta i_\beta, \quad (19)$$

$$\mathbf{q}_{\alpha\beta 0} = \mathbf{v}_{\alpha\beta 0} \times \mathbf{i}_{\alpha\beta 0} = \begin{bmatrix} 0 \\ 0 \\ v_\alpha i_\beta - v_\beta i_\alpha \end{bmatrix}, \quad (20)$$

and

$$q_{\alpha\beta 0} = \|\mathbf{q}_{\alpha\beta 0}\| = v_\alpha i_\beta - v_\beta i_\alpha. \quad (21)$$

Obviously, (19) and (21) are the definitions described in [1]. Therefore, the pq theory described in [1] is a special case of the generalized pq theory described in this paper.

B. A Practical Example

Here, we give a practical example to show how the proposed theory can be applied for measuring and compensating for the instantaneous reactive power of a three-phase four-wire system. Fig. 3 shows the configuration in which three single-phase rectifiers are connected to phases a, b , and c , respectively. A compensator consisting of a three-phase pulse width modulation (PWM) inverter is connected in parallel with the loads. The control circuit of the compensator is also shown in Fig. 3, which includes computational circuits for the instantaneous reactive power of the loads, \mathbf{q}_L , and the instantaneous reactive components of the load currents, \mathbf{i}_{Lq} ,

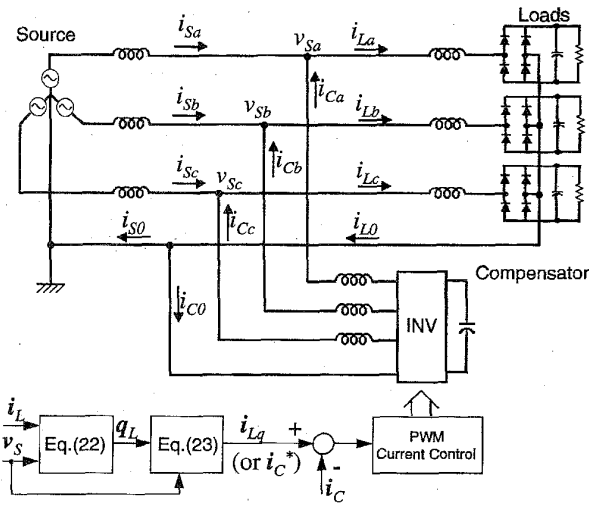


Fig. 3. System configuration of instantaneous reactive power compensation.

and the PWM control circuit for the inverter. Their relations can be expressed as

$$q_L = v_S \times i_L \quad (22)$$

and

$$i_{Lq} = \frac{q_L \times v_S}{v_S \cdot v_S} \quad (23)$$

The instantaneous reactive components of the load currents are used as the command current for the reactive compensator, i_C^* , i.e.,

$$i_C^* = i_{Lq} \quad (24)$$

For a three-phase four-wire power system, we may have three independent components for voltages and currents. The source voltage vector, v_S , the source current vector, i_S , the load current vector, i_L , and the compensator current vector, i_C , are expressed as

$$v_S = \begin{bmatrix} v_{Sa} \\ v_{Sb} \\ v_{Sc} \end{bmatrix}, \quad i_S = \begin{bmatrix} i_{Sa} \\ i_{Sb} \\ i_{Sc} \end{bmatrix}, \quad i_L = \begin{bmatrix} i_{La} \\ i_{Lb} \\ i_{Lc} \end{bmatrix},$$

and

$$i_C = \begin{bmatrix} i_{Ca} \\ i_{Cb} \\ i_{Cc} \end{bmatrix}.$$

The neutral current components are the sum of the three-phase currents of the source, the loads and the compensator, respectively, e.g., $i_{S0} = i_{Sa} + i_{Sb} + i_{Sc}$. Here, the reactive power compensator output current vector, i_C , is controlled by a PWM inverter to track the command current vector, i_C^* .

Fig. 4 shows waveforms of the system before and after reactive power compensation. In the figure, only the waveforms of phase a , v_{Sa} , i_{Sa} , $i_{Laq} (= i_{Ca})$, and the source side neutral current, i_{S0} , are shown. Before the compensator was started, $i_S = i_L$ and $i_C = 0$. After the compensator was started, i_S became in phase with the source voltage immediately, and i_{S0} became zero without any time delay. This indicates that the

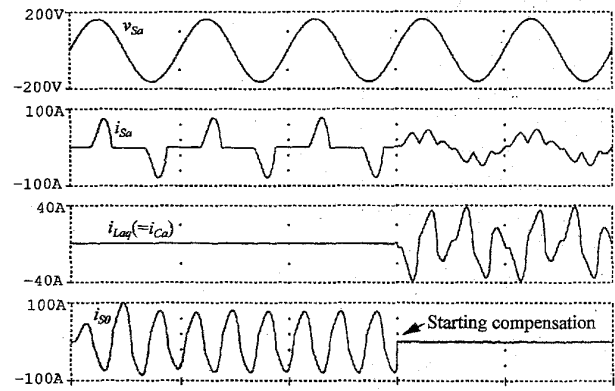


Fig. 4. Waveforms of the compensation system.

zero-sequence current of the loads, i_{L0} , only contributes to the instantaneous reactive power, q_L . The above example also indicates that the proposed instantaneous reactive power theory for a three-phase power system can deal with the following cases:

- sinusoidal and nonsinusoidal waves,
- balanced and unbalanced systems, and
- with or without zero-sequence components.

The pq theory of [1], however, is limited to a system without zero-sequence components only and cannot deal with the above example.

V. CONCLUSION

In this paper, a generalized instantaneous reactive power theory has been proposed. Not only clear definitions for the instantaneous active and reactive components such as active power, reactive power, active current, reactive current, power factor, etc., have been given, but also their properties, relationships, and physical meanings of these instantaneous quantities have been described in detail. The proposed theory is valid for sinusoidal or nonsinusoidal, balanced or unbalanced three-phase power systems with or without zero-sequence components. A power system with a reactive compensator has been used as an application example for the proposed theory. This generalized reactive power theory discloses an important algorithm for instantaneous reactive power measurement and compensation applications.

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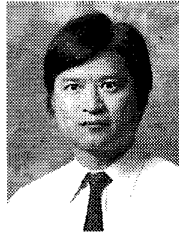


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