Name: Philip Pham

AMATH 515 Homework Set 4

Due: Wednesday, March 18th, on Canvas.

(1) Prove the following identity for $\alpha \in \mathbb{R}$:

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2.$$

Proof. This follows from the fact that $\|u - v\|^2 = \langle u, u\rangle + \langle v, v\rangle - 2\langle u, v\rangle$.

With some algebra,

$$\begin{aligned}
\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 &= \alpha^2\langle x, x\rangle + (1-\alpha)^2\langle y, y\rangle + 2\alpha(1-\alpha)\langle x, y\rangle \\
&\quad + \alpha(1-\alpha)\langle x, x\rangle + \alpha(1-\alpha)\langle y, y\rangle - 2\alpha(1-\alpha)\langle x, y\rangle \\
&= \alpha\langle x, x\rangle + (1-\alpha)\left[1 - \alpha + \alpha\right]\langle y, y\rangle \\
&= \alpha\|x\|^2 + (1-\alpha)\|y\|^2.
\end{aligned}$$

□

(2) An operator T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all (x, y). For any such nonexpansive operator T, define

$$T_\lambda = (1-\lambda)I + \lambda T.$$

(a) Show that T_λ and T have the same fixed points.

Proof. Suppose that $Tx = x$. Then, it's easy to see that

$$\begin{aligned}
T_\lambda x &= \left((1-\lambda)I + \lambda T\right)x \\
&= (1-\lambda)Ix + \lambda Tx \\
&= (1-\lambda)x + \lambda x = x.
\end{aligned}$$

Similarly, suppose $T_\lambda y = y$. If $\lambda \neq 0$, then $T = \lambda^{-1}\left(T_\lambda - (1-\lambda)I\right)$, and

$$\begin{aligned}
Ty &= \lambda^{-1}\left(T_\lambda - (1-\lambda)I\right)y \\
&= \lambda^{-1}\left(T_\lambda y - (1-\lambda)Iy\right) \\
&= \lambda^{-1}\left(y - (1-\lambda)y\right) \\
&= \lambda^{-1}(\lambda y) = y.
\end{aligned}$$

Thus, when $\lambda \neq 0$, the set of fixed points are the same. When $\lambda = 0$, the set of fixed points of T is a subset of the fixed points of $T_\lambda = I$. □

(b) Use problem 1 to show

$$\|T_\lambda z - \overline{z}\|^2 \leq \|z - \overline{z}\|^2 - \lambda(1-\lambda)\|z - Tz\|^2.$$

where $\overline{z}$ is any fixed point of T, i.e. $T\overline{z} = \overline{z}$.

Proof. We first apply the definition of T_λ. Next, we use the identity in problem 1. Then, we use that $\overline{z}$ is a fixed point of T. Finally, we use the nonexpansive property of T for the inequality.

$$\begin{aligned}
\|T_\lambda z - \overline{z}\|^2 &= \|(1-\lambda)z + \lambda Tz - \lambda\overline{z} - (1-\lambda)\overline{z}\|^2 \\
&= \|(1-\lambda)(z-\overline{z}) + \lambda(Tz - \overline{z})\|^2 \\
&= \|(1-\lambda)(z-\overline{z}) + \lambda(Tz - \overline{z})\|^2 + \lambda(1-\lambda)(z - Tz) - \lambda(1-\lambda)(z-Tz) \\
&= (1-\lambda)\|z-\overline{z}\|^2 + \lambda\|Tz - \overline{z}\|^2 - \lambda(1-\lambda)(z-Tz) \\
&= (1-\lambda)\|z-\overline{z}\|^2 + \lambda\|Tz - T\overline{z}\|^2 - \lambda(1-\lambda)(z-Tz) \\
&\leq (1-\lambda)\|z-\overline{z}\|^2 + \lambda\|z - \overline{z}\|^2 - \lambda(1-\lambda)(z-Tz) \\
&= \|z-\overline{z}\|^2 - \lambda(1-\lambda)(z-Tz).
\end{aligned}$$

□

(3) An operator T is *firmly nonexpansive* when it satisfies

$$\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2.$$

(a) Show T is firmly nonexpansive if and only if

$$\langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2.$$

Proof. "$\Rightarrow$": Suppose $\langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2$.

"$\Leftarrow$": Suppose T is firmly nonexpansive.

Suppose T is firmly nonexpansive.

$$\begin{aligned}
&\|Tx-Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2 \\
&\Leftrightarrow \|Tx-Ty\|^2 + \|(x-y) - (Tx-Ty)\|^2 \leq \|x-y\|^2 \\
&\Leftrightarrow \|Tx-Ty\|^2 + \|(x-y) - (Tx-Ty)\|^2 \leq \|x-y\|^2 \\
&\Leftrightarrow \|Tx-Ty\|^2 + \|x-y\|^2 + \|Tx-Ty)\|^2 - 2\langle x-y, Tx-Ty\rangle \leq \|x-y\|^2 \\
&\Leftrightarrow 2\|Tx-Ty\|^2 \leq 2\langle x-y, Tx-Ty\rangle \\
&\Leftrightarrow 2\|Tx-Ty\|^2 \leq 2\langle x-y, Tx-Ty\rangle \\
&\Leftrightarrow \langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2,
\end{aligned}$$

which is the desired result. □

(b) Show T is firmly nonexpansive if and only if

$$\langle Tx - Ty, (I-T)x - (I-T)y \rangle \geq 0.$$

Proof. This follows from the previous result in part (a) and the bilinearity of the inner product:

$$\begin{aligned}
T \text{ is firmly nonexpansive} &\Leftrightarrow \langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2 \\
&\Leftrightarrow \langle x - y, Tx - Ty \rangle - \|Tx - Ty\|^2 \geq 0 \\
&\Leftrightarrow \langle x - y, Tx - Ty \rangle - \langle Tx - Ty, Tx - Ty \rangle \geq 0 \\
&\Leftrightarrow \langle x - y - (Tx - Ty), Tx - Ty \rangle \geq 0 \\
&\Leftrightarrow \langle (I-T)x - (I-T)y, Tx - Ty \rangle \geq 0 \\
&\Leftrightarrow \langle Tx - Ty, (I-T)x - (I-T)y \rangle \geq 0,
\end{aligned}$$

which is the desired result. □

(c) Suppose that $S = 2T - I$. Let

$$\mu = \|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 - \|x - y\|^2$$

and let

$$\nu = \|Sx - Sy\|^2 - \|x - y\|^2.$$

Show that $2\mu = \nu$ (you may find it helpful to use problem (1)). Conclude that T is firmly nonexpansive exactly when S is nonexpansive.

Proof. To prove $\nu = 2\mu$, there's a lot of algebra:

$$\begin{aligned}
\nu &= \|Sx - Sy\|^2 - \|x - y\|^2 = \|(Tx - Ty) - [(I-T)x - (I-T)y]\|^2 - \|x - y\|^2 \\
&= \|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 - 2\langle Tx - Ty, (I-T)x - (I-T)y \rangle - \|x - y\|^2.
\end{aligned}$$

We focus on the $-2\langle Tx - Ty, (I-T)x - (I-T)y \rangle$ term.

$$\begin{aligned}
&-2\langle Tx - Ty, (I-T)x - (I-T)y \rangle \\
&= \langle Tx - Ty, Tx - Ty \rangle + \langle Tx - Ty, Tx - Ty \rangle - 2\langle Tx - Ty, x - y \rangle \\
&= \langle Tx - Ty, Tx - Ty \rangle + \langle Tx - Ty, Tx - Ty \rangle - 2\langle Tx - Ty, x - y \rangle + \langle x - y, x - y \rangle - \langle x - y, x - y \rangle \\
&= \|Tx - Ty\|^2 + \langle Tx - Ty, Tx - Ty \rangle - 2\langle Tx - Ty, x - y \rangle + \langle x - y, x - y \rangle - \langle x - y, x - y \rangle \\
&= \|Tx - Ty\|^2 + \|(Tx - Ty) - (x - y)\|^2 - \|x - y\|^2 \\
&= \|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 - \|x - y\|^2
\end{aligned}$$

where we have completed the square to get the $\|(Tx - Ty) - (x - y)\|^2$ term.

Substituting back into the first equation, we have that

$$\begin{aligned}
\nu &= \|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \\
&\quad + \left[\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 - \|x-y\|^2\right] - \|x-y\|^2 \\
&= 2\|Tx - Ty\|^2 + 2\|(I-T)x - (I-T)y\|^2 - 2\|x-y\|^2 \\
&= 2\left[\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 - \|x-y\|^2\right] \\
&= 2\mu
\end{aligned}$$

as desired.

Using the definitions of expansive and firmly nonexpansive,

S is nonexpansive $\Leftrightarrow \nu \leq 0 \Leftrightarrow \mu \leq 0 \Leftrightarrow T$ is firmly nonexpansive.

$\square$

Coding Assignment

Please download `515Hw4_Coding.ipynb` and `solvers.py` to complete problem (4) and (5).

(4) Implement an interior point method to solve the problem

$$\min_x \frac{1}{2}\|Ax - b\|^2 \quad \text{s.t.} \quad Cx \leq d.$$

Let the user input A, b, C, and d. Test your algorithm using a box constrained problem (where you can apply the prox-gradient method).

(5) Implement a Chambolle-Pock method to solve

$$\min_x \|Ax - b\|_1 + \|x\|_1.$$