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AMATH 515 Homework Set 4

Due: Wednesday, March 18th, on Canvas.

(1) Prove the following identity for $\alpha \in \mathbb{R}$:

$$\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2.$$

Proof. This follows from the fact that $\|u-v\|^2 = \langle u, u\rangle + \langle v, v\rangle - 2\langle u, v\rangle$.

With some algebra,

$$\begin{aligned}\|\alpha x + (1-\alpha)y\|^2 + \alpha(1-\alpha)\|x-y\|^2 &= \alpha^2\langle x, x\rangle + (1-\alpha)^2\langle y, y\rangle + 2\alpha(1-\alpha)\langle x, y\rangle \\ &\quad + \alpha(1-\alpha)\langle x, x\rangle + \alpha(1-\alpha)\langle y, y\rangle - 2\alpha(1-\alpha)\langle x, y\rangle \\ &= \alpha\langle x, x\rangle + (1-\alpha)\left[1-\alpha+\alpha\right]\langle y, y\rangle \\ &= \alpha\|x\|^2 + (1-\alpha)\|y\|^2.\end{aligned}$$

□

(2) An operator T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all (x, y). For any such nonexpansive operator T, define

$$T_\lambda = (1-\lambda)I + \lambda T.$$

(a) Show that T_λ and T have the same fixed points.

(b) Use problem 1 to show

$$\|T_\lambda z - \overline{z}\|^2 \leq \|z - \overline{z}\|^2 - \lambda(1-\lambda)\|z - Tz\|^2.$$

where $\overline{z}$ is any fixed point of T, i.e. $T\overline{z} = \overline{z}$.

(3) An operator T is *firmly nonexpansive* when it satisfies

$$\|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 \leq \|x-y\|^2.$$

(a) Show T is firmly nonexpansive if and only if

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2.$$

(b) Show T is firmly nonexpansive if and only if

$$\langle Tx - Ty, (I-T)x - (I-T)y\rangle \geq 0.$$

(c) Suppose that $S = 2T - I$. Let

$$\mu = \|Tx - Ty\|^2 + \|(I-T)x - (I-T)y\|^2 - \|x - y\|^2$$

and let

$$\nu = \|Sx - Sy\|^2 - \|x - y\|^2.$$

Show that $2\mu = \nu$ (you may find it helpful to use problem (1)). Conclude that T is firmly nonexpansive exactly when S is nonexpansive.

Coding Assignment

Please download `515Hw4_Coding.ipynb` and `solvers.py` to complete problem (4) and (5).

(4) Implement an interior point method to solve the problem

$$\min_x \frac{1}{2}\|Ax - b\|^2 \quad \text{s.t.} \quad Cx \leq d.$$

Let the user input A, b, C, and d. Test your algorithm using a box constrained problem (where you can apply the prox-gradient method).

(5) Implement a Chambolle-Pock method to solve

$$\min_x \|Ax - b\|_1 + \|x\|_1.$$