

Name: Philip Pham

AMATH 515

Homework Set 4

Due: Wednesday, March 18th, on Canvas.

- (1) Prove the following identity for $\alpha \in \mathbb{R}$:

$$\|\alpha x + (1 - \alpha)y\|^2 + \alpha(1 - \alpha)\|x - y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2.$$

Proof. This follows from the fact that $\|u - v\|^2 = \langle u, u \rangle + \langle v, v \rangle - 2\langle u, v \rangle$.

With some algebra,

$$\begin{aligned} \|\alpha x + (1 - \alpha)y\|^2 + \alpha(1 - \alpha)\|x - y\|^2 &= \alpha^2\langle x, x \rangle + (1 - \alpha)^2\langle y, y \rangle + 2\alpha(1 - \alpha)\langle x, y \rangle \\ &\quad + \alpha(1 - \alpha)\langle x, x \rangle + \alpha(1 - \alpha)\langle y, y \rangle - 2\alpha(1 - \alpha)\langle x, y \rangle \\ &= \alpha\langle x, x \rangle + (1 - \alpha)[1 - \alpha + \alpha]\langle y, y \rangle \\ &= \alpha\|x\|^2 + (1 - \alpha)\|y\|^2. \end{aligned}$$

□

- (2) An operator T is *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all (x, y) . For any such nonexpansive operator T , define

$$T_\lambda = (1 - \lambda)I + \lambda T.$$

- (a) Show that T_λ and T have the same fixed points.

Proof. Suppose that $Tx = x$. Then, it's easy to see that

$$\begin{aligned} T_\lambda x &= ((1 - \lambda)I + \lambda T)x \\ &= (1 - \lambda)Ix + \lambda Tx \\ &= (1 - \lambda)x + \lambda x = x. \end{aligned}$$

Similarly, suppose $T_\lambda y = y$. If $\lambda \neq 0$, then $T = \lambda^{-1}(T_\lambda - (1 - \lambda)I)$, and

$$\begin{aligned} Ty &= \lambda^{-1}(T_\lambda - (1 - \lambda)I)y \\ &= \lambda^{-1}(T_\lambda y - (1 - \lambda)Iy) \\ &= \lambda^{-1}(y - (1 - \lambda)y) \\ &= \lambda^{-1}(\lambda y) = y. \end{aligned}$$

Thus, when $\lambda \neq 0$, the set of fixed points are the same. When $\lambda = 0$, the set of fixed points of T is a subset of the fixed points of $T_\lambda = I$. □

(b) Use problem 1 to show

$$\|T_\lambda z - \bar{z}\|^2 \leq \|z - \bar{z}\|^2 - \lambda(1 - \lambda)\|z - Tz\|^2.$$

where \bar{z} is any fixed point of T , i.e. $T\bar{z} = \bar{z}$.

Proof. We first apply the definition of T_λ . Next, we use the identity in problem 1. Then, we use that \bar{z} is a fixed point of T . Finally, we use the nonexpansive property of T for the inequality.

$$\begin{aligned} \|T_\lambda z - \bar{z}\|^2 &= \|(1 - \lambda)z + \lambda Tz - \lambda \bar{z} - (1 - \lambda)\bar{z}\|^2 \\ &= \|(1 - \lambda)(z - \bar{z}) + \lambda(Tz - \bar{z})\|^2 \\ &= \|(1 - \lambda)(z - \bar{z}) + \lambda(Tz - \bar{z})\|^2 + \lambda(1 - \lambda)(z - Tz) - \lambda(1 - \lambda)(z - Tz) \\ &= (1 - \lambda)\|z - \bar{z}\|^2 + \lambda\|Tz - \bar{z}\|^2 - \lambda(1 - \lambda)(z - Tz) \\ &= (1 - \lambda)\|z - \bar{z}\|^2 + \lambda\|Tz - T\bar{z}\|^2 - \lambda(1 - \lambda)(z - Tz) \\ &\leq (1 - \lambda)\|z - \bar{z}\|^2 + \lambda\|z - \bar{z}\|^2 - \lambda(1 - \lambda)(z - Tz) \\ &= \|z - \bar{z}\|^2 - \lambda(1 - \lambda)(z - Tz). \end{aligned}$$

Note that the inequality is only valid when $\lambda \geq 0$. □

(3) An operator T is *firmly nonexpansive* when it satisfies

$$\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2.$$

(a) Show T is firmly nonexpansive if and only if

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2.$$

Proof. “ \Rightarrow ”: Suppose $\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2$.

“ \Leftarrow ”: Suppose T is firmly nonexpansive.

Suppose T is firmly nonexpansive.

$$\begin{aligned} \|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 &\leq \|x - y\|^2 \\ \Leftrightarrow \|Tx - Ty\|^2 + \|(x - y) - (Tx - Ty)\|^2 &\leq \|x - y\|^2 \\ \Leftrightarrow \|Tx - Ty\|^2 + \|(x - y) - (Tx - Ty)\|^2 &\leq \|x - y\|^2 \\ \Leftrightarrow \|Tx - Ty\|^2 + \|x - y\|^2 + \|Tx - Ty\|^2 - 2\langle x - y, Tx - Ty \rangle &\leq \|x - y\|^2 \\ \Leftrightarrow 2\|Tx - Ty\|^2 &\leq 2\langle x - y, Tx - Ty \rangle \\ \Leftrightarrow 2\|Tx - Ty\|^2 &\leq 2\langle x - y, Tx - Ty \rangle \\ \Leftrightarrow \langle x - y, Tx - Ty \rangle &\geq \|Tx - Ty\|^2, \end{aligned}$$

which is the desired result. □

(b) Show T is firmly nonexpansive if and only if

$$\langle Tx - Ty, (I - T)x - (I - T)y \rangle \geq 0.$$

Proof. This follows from the previous result in part (a) and the bilinearity of the inner product:

$$\begin{aligned} T \text{ is firmly nonexpansive} &\Leftrightarrow \langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2 \\ &\Leftrightarrow \langle x - y, Tx - Ty \rangle - \|Tx - Ty\|^2 \geq 0 \\ &\Leftrightarrow \langle x - y, Tx - Ty \rangle - \langle Tx - Ty, Tx - Ty \rangle \geq 0 \\ &\Leftrightarrow \langle x - y - (Tx - Ty), Tx - Ty \rangle \geq 0 \\ &\Leftrightarrow \langle (I - T)x - (I - T)y, Tx - Ty \rangle \geq 0 \\ &\Leftrightarrow \langle Tx - Ty, (I - T)x - (I - T)y \rangle \geq 0, \end{aligned}$$

which is the desired result. \square

(c) Suppose that $S = 2T - I$. Let

$$\mu = \|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 - \|x - y\|^2$$

and let

$$\nu = \|Sx - Sy\|^2 - \|x - y\|^2.$$

Show that $2\mu = \nu$ (you may find it helpful to use problem (1)). Conclude that T is firmly nonexpansive exactly when S is nonexpansive.

Proof. To prove $\nu = 2\mu$, there's a lot of algebra:

$$\begin{aligned} \nu &= \|Sx - Sy\|^2 - \|x - y\|^2 = \|(Tx - Ty) - [(I - T)x - (I - T)y]\|^2 - \|x - y\|^2 \\ &= \|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 - 2\langle Tx - Ty, (I - T)x - (I - T)y \rangle - \|x - y\|^2. \end{aligned}$$

We focus on the $-2\langle Tx - Ty, (I - T)x - (I - T)y \rangle$ term.

$$\begin{aligned} &-2\langle Tx - Ty, (I - T)x - (I - T)y \rangle \\ &= \langle Tx - Ty, Tx - Ty \rangle + \langle Tx - Ty, Tx - Ty \rangle - 2\langle Tx - Ty, x - y \rangle \\ &= \langle Tx - Ty, Tx - Ty \rangle + \langle Tx - Ty, Tx - Ty \rangle - 2\langle Tx - Ty, x - y \rangle + \langle x - y, x - y \rangle - \langle x - y, x - y \rangle \\ &= \|Tx - Ty\|^2 + \langle Tx - Ty, Tx - Ty \rangle - 2\langle Tx - Ty, x - y \rangle + \langle x - y, x - y \rangle - \langle x - y, x - y \rangle \\ &= \|Tx - Ty\|^2 + \|(Tx - Ty) - (x - y)\|^2 - \|x - y\|^2 \\ &= \|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 - \|x - y\|^2 \end{aligned}$$

where we have completed the square to get the $\|(Tx - Ty) - (x - y)\|^2$ term.

Substituting back into the first equation, we have that

$$\begin{aligned}
\nu &= \|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 \\
&\quad + [\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 - \|x - y\|^2] - \|x - y\|^2 \\
&= 2\|Tx - Ty\|^2 + 2\|(I - T)x - (I - T)y\|^2 - 2\|x - y\|^2 \\
&= 2[\|Tx - Ty\|^2 + \|(I - T)x - (I - T)y\|^2 - \|x - y\|^2] \\
&= 2\mu
\end{aligned}$$

as desired.

Using the definitions of expansive and firmly nonexpansive,

S is nonexpansive $\Leftrightarrow \nu \leq 0 \Leftrightarrow \mu \leq 0 \Leftrightarrow T$ is firmly nonexpansive.

□

Coding Assignment

Please download `515Hw4_Coding.ipynb` and `solvers.py` to complete problem (4) and (5).

- (4) Implement an interior point method to solve the problem

$$\min_x \frac{1}{2} \|Ax - b\|^2 \quad \text{s.t.} \quad Cx \leq d.$$

Let the user input A , b , C , and d . Test your algorithm using a box constrained problem (where you can apply the prox-gradient method).

- (5) Implement a Chambolle-Pock method to solve

$$\min_x \|Ax - b\|_1 + \|x\|_1.$$