STAT 527: Assignment #1

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Problem 1

(a) temp

Proof. This follows from the definition of variance after some algebra.

$$
\begin{aligned}
\mathrm{E}\left[\left(\hat{f}(x) - f(x)\right)^2\right] &= \mathrm{E}\left[\left[\left(\hat{f}(x) - \mathrm{E}\left[\hat{f}(x)\right]\right) + \left(\mathrm{E}\left[\hat{f}(x)\right] - f(x)\right)\right]^2\right] \\
&= \mathrm{E}\left[\left(\hat{f}(x) - \mathrm{E}\left[\hat{f}(x)\right]\right)^2\right] + 2\,\mathrm{E}\left[\left(\hat{f}(x) - \mathrm{E}\left[\hat{f}(x)\right]\right)\left(\mathrm{E}\left[\hat{f}(x)\right] - f(x)\right)\right] + \mathrm{E}\left[\left(\mathrm{E}\left[\hat{f}(x)\right] - f(x)\right)^2\right] \\
&= \mathrm{var}\left(\hat{f}(x)\right) + \left(\mathrm{E}\left[\hat{f}(x)\right] - f(x)\right)^2 + 2\left(\mathrm{E}\left[\hat{f}(x)\right] - f(x)\right)\mathrm{E}\left[\left(\hat{f}(x) - \mathrm{E}\left[\hat{f}(x)\right]\right)\right] \\
&= \left(\mathrm{E}\left[\hat{f}(x)\right] - f(x)\right)^2 + \mathrm{var}\left(\hat{f}(x)\right).
\end{aligned}
$$

$\square$

(b) temp

Proof. The prediction error is

$$
\begin{aligned}
\mathrm{E}\left[\left(y_{\text{new}} - \hat{f}\left(x_{\text{new}}\right)\right)^2\right] &= \mathrm{E}\left[\left(f\left(x_{\text{new}}\right) + \epsilon_{\text{new}} - \hat{f}\left(x_{\text{new}}\right)\right)^2\right] \\
&= \mathrm{E}\left[\left(\epsilon_{\text{new}} + \left(f\left(x_{\text{new}}\right) - \hat{f}\left(x_{\text{new}}\right)\right)\right)^2\right] \\
&= \mathrm{E}\left[\epsilon_{\text{new}}^2\right] + 2\,\mathrm{E}\left[\epsilon_{\text{new}}\left(f\left(x_{\text{new}}\right) - \hat{f}\left(x_{\text{new}}\right)\right)\right] + \left(\mathrm{E}\left[\hat{f}(x_{\text{new}})\right] - f(x_{\text{new}})\right)^2 + \mathrm{var}\left(\hat{f}\left(x_{\text{new}}\right)\right) \\
&= \sigma^2 + \left(\mathrm{E}\left[\hat{f}(x_{\text{new}})\right] - f(x_{\text{new}})\right)^2 + \mathrm{var}\left(\hat{f}\left(x_{\text{new}}\right)\right).
\end{aligned}
$$

There is an additional σ^2 term to account for the variance of our new observation. $\square$

(c) temp

Proof.

$$\begin{aligned} a \geq \mathrm{E}\left[L\right] &= \int_0^\infty t \; \mathrm{d}L(t) \\ &= \int_0^{a/\epsilon} t \; \mathrm{d}L(t) + \int_{a/\epsilon}^\infty t \; \mathrm{d}L(t) \\ &\geq \int_0^{a/\epsilon} t \; \mathrm{d}L(t) + \frac{a}{\epsilon}\int_{a/\epsilon}^\infty \mathrm{d}L(t) \\ &\geq \frac{a}{\epsilon}\int_{a/\epsilon}^\infty \mathrm{d}L(t) = \frac{a}{\epsilon}\,\mathrm{P}\left(L > \frac{a}{\epsilon}\right). \end{aligned}$$

Thus, we have that

$$\frac{a}{\epsilon}\,\mathrm{P}\left(L > \frac{a}{\epsilon}\right) \leq a \Leftrightarrow \mathrm{P}\left(L > \frac{a}{\epsilon}\right) \leq \epsilon \Leftrightarrow \mathrm{P}\left(\frac{L}{a} > \frac{1}{\epsilon}\right) \leq \epsilon.$$

□