# Coursework 4: STAT 570

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1. Consider the so-called Neyman-Scott problem in which $Y_{ij} \mid \mu_i, \sigma_2 \sim_{\text{ind}} \mathcal{N}\left(\mu_i, \sigma^2\right), i = 1, \ldots, n, j = 1, 2.$

   (a) Obtain the MLE of $\sigma^2$ and show that it is inconsistent. Why does this inconsistency arise in this example?

   **Solution:** The likelihood is

$$\begin{aligned} L\left(\mu, \sigma\right) &= \prod_{i=1}^{n}\prod_{j=1}^{2} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(Y_{ij} - \mu_i\right)^2\right) \\ &= \prod_{i=1}^{n} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\left[\left(Y_{i1} - \mu_i\right)^2 + \left(Y_{i2} - \mu_i\right)^2\right]\right), \end{aligned} \tag{1}$$

   so the log-likelikehood is

$$l\left(\mu, \sigma\right) = -n \log\left(2\pi\right) - n\log\left(\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[\left(Y_{i1} - \mu_i\right)^2 + \left(Y_{i2} - \mu_i\right)^2\right]. \tag{2}$$

   Taking the derivative with respect to $\sigma^2$, we have

$$\frac{\partial}{\partial\sigma^2} l\left(\mu, \sigma^2\right) = -\frac{n}{\sigma^2} + \frac{1}{2\left(\sigma^2\right)^2}\sum_{i=1}^{n}\left[\left(Y_{i1} - \mu_i\right)^2 + \left(Y_{i2} - \mu_i\right)^2\right]. \tag{3}$$

   Solving Equation 3, where $\frac{\partial}{\partial\sigma^2} l\left(\hat{\mu}, \hat{\sigma}^2\right) = 0$, we have

$$\hat{\sigma}^2 = \frac{1}{2n}\sum_{i=1}^{n}\left[\left(Y_{i1} - \hat{\mu}_i\right)^2 + \left(Y_{i2} - \hat{\mu}_i\right)^2\right]. \tag{4}$$

   Taking the derivative of Equation 2 with respect to $\mu_i$, we have

$$\frac{\partial}{\partial\mu_i} l\left(\mu, \sigma^2\right) = \frac{1}{\sigma^2}\left(Y_{i1} + Y_{i2} - 2\mu_i\right). \tag{5}$$

   Solving Equation 5, where $\frac{\partial}{\partial\mu_i} l\left(\hat{\mu}, \hat{\sigma}^2\right) = 0$, we have

$$\hat{\mu}_i = \frac{Y_{i1} + Y_{i2}}{2}. \tag{6}$$

   Substituting Equation 6 into Equation 4, we have

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{Y_{i1} - Y_{i2}}{2}\right)^2. \tag{7}$$

Taking the expected value of Equation 7, we have

$$\begin{aligned}\mathbb{E}\left[\hat{\sigma}^2\right] &= \frac{1}{4n}\sum_{i=1}^{n}\left(\mathbb{E}\left[Y_{i1}^2\right] + \mathbb{E}\left[Y_{i2}^2\right] - 2\mathbb{E}\left[Y_{i1}Y_{i2}\right]\right)\\ &= \frac{1}{4n}\sum_{i=1}^{n}\left(\left(\sigma^2+\mu_i^2\right)+\left(\sigma^2+\mu_i^2\right)-2\mu_i^2\right)\\ &= \frac{\sigma^2}{2}. \end{aligned} \tag{8}$$

Clearly, $\mathbb{E}\left[\hat{\sigma}^2\right] = \sigma^2/2 \nrightarrow \sigma^2$, so the estimator is not consistent.
This is because the MLE estimate of $\sigma^2$ depends on $\mu_1, \ldots, \mu_n$, so the number of parameters being estimated increases with $n$. Thus, the model is not well-defined.

(b) Derive the posterior distribution corresponding to the prior

$$\pi\left(\mu_1, \ldots, \mu_n, \sigma^2\right) \propto \sigma^{-n-2} \tag{9}$$

and show that

$$\mathbb{E}\left[\sigma^2 \mid Y\right] = \frac{1}{2(n-1)}\sum_{i=1}^{n}\frac{\left(Y_{i1}-Y_{i2}\right)^2}{2}. \tag{10}$$

**Solution:** Using the likelihood in Equation 1 and the prior in Equation 9. We have that

$$p\left(\mu, \sigma^2 \mid Y\right) \propto L\left(\mu, \sigma^2\right)\pi\left(\mu_1, \ldots, \mu_n, \sigma^2\right). \tag{11}$$

We have that

$$\begin{aligned} p\left(Y\right) &= \int_0^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} L\left(\mu,\sigma^2\right)\pi\left(\mu_1,\ldots,\mu_n,\sigma^2\right)\,\mathrm{d}\mu_1\cdots\mathrm{d}\mu_n\,\mathrm{d}\sigma^2\\ &= \int_0^{\infty}\frac{1}{2^n\pi^n\left(\sigma^2\right)^{(3n+2)/2}}\left(\pi\sigma^2\right)^{n/2}\prod_{i=1}^{n}\exp\left(-\frac{1}{4\sigma^2}\left(Y_{i1}-Y_{i2}\right)^2\right)\,\mathrm{d}\sigma^2\\ &= \int_0^{\infty}\frac{1}{2^n\pi^{n/2}\left(\sigma^2\right)^{n+1}}\exp\left(-\frac{1}{4\sigma^2}\sum_{i=1}^{n}\left(Y_{i1}-Y_{i2}\right)^2\right)\,\mathrm{d}\sigma^2\\ &= -\frac{2^n}{\pi^{n/2}}\left(\sum_{i=1}^{n}\left(Y_{i1}-Y_{i2}\right)^2\right)^{-n}\int_{\infty}^{0}u^{n-1}\exp\left(-u\right)\,\mathrm{d}u\\ &= \frac{1}{\pi^{n/2}}\left(\sum_{i=1}^{n}\frac{\left(Y_{i1}-Y_{i2}\right)^2}{2}\right)^{-n}\Gamma(n). \end{aligned} \tag{12}$$

Normalizing Equation 11 with the evidence Equation 12, we have the posterior

$$p\left(\mu,\sigma^2 \mid Y\right) = \frac{\left(\sigma^2\right)^{-(3n+2)/2}}{2^n\pi^{n/2}\Gamma(n)}\left(\sum_{i=1}^{n}\frac{\left(Y_{i1}-Y_{i2}\right)^2}{2}\right)^{n}\prod_{i=1}^{n}\prod_{j=1}^{2}\exp\left(-\frac{1}{2\sigma^2}\left(Y_{ij}-\mu_i\right)^2\right). \tag{13}$$

Marginalizing $\mu$ in Equation 13, we get

$$\begin{aligned} p\left(\sigma^2 \mid Y\right) &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p\left(\mu,\sigma^2 \mid Y\right)\,\mathrm{d}\mu_1\cdots\mathrm{d}\mu_n \qquad (14)\\ &= \frac{\left(\sigma^2\right)^{-n-1}}{2^n\Gamma(n)}\left(\sum_{i=1}^{n}\frac{\left(Y_{i1}-Y_{i2}\right)^2}{2}\right)^{n}\exp\left(-\frac{1}{4\sigma^2}\sum_{i=1}^{n}\left(Y_{i1}-Y_{i2}\right)^2\right). \end{aligned}$$

Taking the expectation over the distribution in Equation 14, we have that

$$\begin{aligned}
\mathbb{E}\left[\sigma^2 \mid Y\right] &= \int_0^\infty \sigma^2 p\left(\sigma^2 \mid Y\right) \, \mathrm{d}\sigma^2 \\
&= \frac{1}{\Gamma(n)} \int_0^\infty \left(\frac{1}{4\sigma^2} \sum_{i=1}^n \left(Y_{i1} - Y_{i2}\right)^2\right)^n \exp\left(-\frac{1}{4\sigma^2} \sum_{i=1}^n \left(Y_{i1} - Y_{i2}\right)^2\right) \\
&= \frac{\sum_{i=1}^n \left(Y_{i1} - Y_{i2}\right)^2}{4\Gamma(n)} \int_0^\infty u^{n-1-1} \exp\left(u\right) \, \mathrm{d}u \\
&= \frac{\Gamma(n-1)}{2\Gamma(n)} \sum_{i=1}^n \frac{\left(Y_{i1} - Y_{i2}\right)^2}{2} \\
&= \frac{1}{2(n-1)} \sum_{i=1}^n \frac{\left(Y_{i1} - Y_{i2}\right)^2}{2}, \qquad (15)
\end{aligned}$$

which is the desired result.

(c) Hence, using Equation 15, show that $\mathbb{E}\left[\sigma^2 \mid Y\right] \to \sigma^2/2$ as $n \to \infty$, so that the posterior mean is inconsistent.

**Solution:** From Equation 15, we have that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[\sigma^2 \mid Y\right] &= \lim_{n\to\infty} \frac{1}{2(n-1)} \sum_{i=1}^n \frac{\mathbb{E}\left[\left(Y_{i1} - Y_{i2}\right)^2\right]}{2} \\
&= \lim_{n\to\infty} \frac{1}{2(n-1)} \sum_{i=1}^n \frac{\mathrm{Var}\left(Y_{i1} - Y_{i2}\right)}{2} \\
&= \lim_{n\to\infty} \frac{n\sigma^2}{2(n-1)} \\
&= \frac{\sigma^2}{2} \neq \sigma^2, \qquad (16)
\end{aligned}$$

so the posterior mean is inconsistent.

(d) Examine the posterior distribution corresponding to the prior

$$\pi\left(\mu_1, \ldots, \mu_n \sigma^2\right) \propto \sigma^{-2}. \qquad (17)$$

**Solution:** If we use Equation 17, Equation 12 becomes

$$\begin{aligned}
p\left(Y\right) &= \int_0^\infty \frac{1}{2^n \pi^{n/2} \left(\sigma^2\right)^{n/2+1}} \exp\left(-\frac{1}{4\sigma^2} \sum_{i=1}^n \left(Y_{i1} - Y_{i2}\right)^2\right) \mathrm{d}\sigma^2 \\
&= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{n/2}} \left(\sum_{i=1}^n \left(Y_{i1} - Y_{i2}\right)^2\right)^{-n/2}. \qquad (18)
\end{aligned}$$

With Equation 18, the posterior becomes

$$p\left(\mu, \sigma^2 \mid Y\right) = \frac{\left(\sigma^2\right)^{-n-1}}{2^n \pi^{n/2} \Gamma(n/2)} \left(\sum_{i=1}^n \frac{\left(Y_{i1} - Y_{i2}\right)^2}{2}\right)^n \prod_{i=1}^n \prod_{j=1}^2 \exp\left(-\frac{1}{2\sigma^2}\left(Y_{ij} - \mu_i\right)^2\right). \qquad (19)$$

Marginalizing Equation 19 over $\mu$, we have

$$p\left(\sigma^2 \mid Y\right) = \frac{1}{\sigma^2 \Gamma\left(n/2\right)} \left(\frac{\sum_{i=1}^n \left(Y_{i1} - Y_{i2}\right)^2}{4\sigma^2}\right)^{n/2} \exp\left(-\frac{\sum_{i=1}^n \left(Y_{i1} - Y_{i2}\right)^2}{4\sigma^2}\right). \qquad (20)$$

Equation 20 is quite similar to Equation 14, but with $n$ replaced by $n/2$ in the gamma function and the exponent of the sum of squares.

(e) Is the posterior mean for $\sigma^2$ consistent in this case?

**Solution:** Yes. Taking the expectation with Equation 20, we have

$$\begin{aligned}
\mathbb{E}\left[\sigma^2 \mid Y\right] &= \int_0^\infty p\left(\sigma^2 \mid Y\right) \, \mathrm{d}\sigma^2 \\
&= \frac{1}{4\Gamma\left(n/2\right)} \sum_{i=1}^{n} \left(Y_{i1} - Y_{i2}\right)^2 \int_0^\infty u^{n/2-1-1} \exp\left(-u\right) \, \mathrm{d}u \\
&= \frac{\Gamma\left(n/2-1\right)}{4\Gamma\left(n/2\right)} \sum_{i=1}^{n} \left(Y_{i1} - Y_{i2}\right)^2 \\
&= \frac{1}{2\left(n/2-1\right)} \sum_{i=1}^{n} \frac{\left(Y_{i1} - Y_{i2}\right)^2}{2} \\
&= \frac{1}{n-2} \sum_{i=1}^{n} \frac{\left(Y_{i1} - Y_{i2}\right)^2}{2}. \qquad (21)
\end{aligned}$$

Taking the limit of Equation 21, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[\sigma^2 \mid Y\right] &= \lim_{n\to\infty} \frac{1}{n-2} \sum_{i=1}^{n} \frac{\mathbb{E}\left[\left(Y_{i1} - Y_{i2}\right)^2\right]}{2} \\
&= \lim_{n\to\infty} \frac{n}{(n-2)} \frac{1}{n} \sum_{i=1}^{n} \frac{\operatorname{Var}\left(Y_{i1} - Y_{i2}\right)}{2} \\
&= \lim_{n\to\infty} \frac{n}{(n-2)} \frac{1}{n} \sum_{i=1}^{n} \frac{2\sigma^2}{2} \\
&= \lim_{n\to\infty} \frac{n}{(n-2)} \sigma^2 \\
&= \sigma^2, \qquad (22)
\end{aligned}$$

so the posterior mean is consistent when the prior doesn't depend on $n$.