Coursework 2: STAT 570

Philip Pham

October 5, 2018

1. Consider the simple linear regression model

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i,\ i = 1, \ldots, n,$$

where the error terms ϵ_i are such that $\mathbb{E}[\epsilon_i] = 0$, $\operatorname{Var}(\epsilon_i) = \sigma^2$, and $\operatorname{Cov}(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$.

In the following you will consider $x_i \sim_{\text{iid}} \mathcal{N}(20, 3^2)$, with $\beta_0 = 2$ and $\beta_1 = -2.5$ and $n = 15, 30$.

Consider the model in Equation 1 with the error terms ϵ_i, independent and identically distributed, from the distributions:

- The normal distribution with mean 0 and variance 2^2.
- The uniform distribution on the range $(-r, r)$ for $r = 2$.
- A skew normal distribution with $\alpha = 5$, $\omega = 1$, and ξ chosen to given mean 0.

(a) What is the theoretical bias for $\hat{\beta}$ if the errors are of the form specified?

Solution: The theoretical bias for $\hat{\beta}$ is 0. Let

$$X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix} \tag{1}$$

If we use the least squares estimate, we have

$$\begin{aligned} \hat{\beta} &= \left(X^\intercal X\right)^{-1} X^\intercal y \\ &= \left(X^\intercal X\right)^{-1} X^\intercal \left(X\beta + \epsilon\right) \\ &= \beta + \left(X^\intercal X\right)^{-1} X^\intercal \epsilon, \end{aligned} \tag{2}$$

Thus, using Equation 2 and linearity of expectations, we have

$$\boxed{\operatorname{bias}\left(\hat{\beta}\right) = \mathbb{E}\left[\hat{\beta}\right] - \beta = \beta + \left(X^\intercal X\right)^{-1} X^\intercal \mathbb{E}[\epsilon] - \beta = 0.} \tag{3}$$