Coursework 4: STAT 570

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1. Consider the so-called Neyman-Scott problem in which $Y_{ij} \mid \mu_i, \sigma_2 \sim_{\text{ind}} \mathcal{N}\left(\mu_i, \sigma^2\right), i = 1, \ldots, n, j = 1, 2$.

   (a) Obtain the MLE of σ^2 and show that it is inconsistent. Why does this inconsistency arise in this example?

   Solution: The likelihood is

$$\begin{aligned} L\left(\mu, \sigma\right) &= \prod_{i=1}^n \prod_{j=1}^2 \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}\left(Y_{ij} - \mu_i\right)^2\right) \\ &= \prod_{i=1}^n \frac{1}{2\pi\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\left[\left(Y_{i1} - \mu_i\right)^2 + \left(Y_{i2} - \mu_i\right)^2\right]\right), \end{aligned} \tag{1}$$

so the log-likelikehood is

$$l\left(\mu, \sigma\right) = -n\log\left(2\pi\right) - n\log\left(\sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^n\left[\left(Y_{i1} - \mu_i\right)^2 + \left(Y_{i2} - \mu_i\right)^2\right]. \tag{2}$$

Taking the derivative with respect to σ^2, we have

$$\frac{\partial}{\partial\sigma^2} l\left(\mu, \sigma^2\right) = -\frac{n}{\sigma^2} + \frac{1}{2\left(\sigma^2\right)^2}\sum_{i=1}^n\left[\left(Y_{i1} - \mu_i\right)^2 + \left(Y_{i2} - \mu_i\right)^2\right]. \tag{3}$$

Solving Equation 3, where $\frac{\partial}{\partial\sigma^2} l\left(\hat{\mu}, \hat{\sigma}^2\right) = 0$, we have

$$\hat{\sigma}^2 = \frac{1}{2n}\sum_{i=1}^n\left[\left(Y_{i1} - \hat{\mu}_i\right)^2 + \left(Y_{i2} - \hat{\mu}_i\right)^2\right]. \tag{4}$$

Taking the derivative of Equation 2 with respect to μ_i, we have

$$\frac{\partial}{\partial\mu_i} l\left(\mu, \sigma^2\right) = \frac{1}{\sigma^2}\left(Y_{i1} + Y_{i2} - 2\mu_i\right). \tag{5}$$

Solving Equation 5, where $\frac{\partial}{\partial\mu_i} l\left(\hat{\mu}, \hat{\sigma}^2\right) = 0$, we have

$$\hat{\mu}_i = \frac{Y_{i1} + Y_{i2}}{2}. \tag{6}$$

Substituting Equation 6 into Equation 4, we have

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n\left(\frac{Y_{i1} - Y_{i2}}{2}\right)^2. \tag{7}$$

Taking the expected value of Equation 7, we have

$$\begin{aligned}\mathbb{E}\left[\hat{\sigma}^2\right] &= \frac{1}{4n}\sum_{i=1}^{n}\left(\mathbb{E}\left[Y_{i1}^2\right] + \mathbb{E}\left[Y_{i2}^2\right] - 2\mathbb{E}\left[Y_{i1}Y_{i2}\right]\right) \\ &= \frac{1}{4n}\sum_{i=1}^{n}\left(\left(\sigma^2 + \mu_i^2\right) + \left(\sigma^2 + \mu_i^2\right) - 2\mu_i^2\right) \\ &= \frac{\sigma^2}{2}. \end{aligned} \tag{8}$$

Clearly, $\mathbb{E}\left[\hat{\sigma}^2\right] = \sigma^2/2 \nrightarrow \sigma^2$, so the estimator is not consistent. This is because the MLE estimate of σ^2 depends on $\mu_1, \ldots, \mu_n$, so the number of parameters being estimated increases with n.

(b)