UNIVERSITY of BRADFORD
Academic Skills Advice

Complex Numbers Summary

What does a complex number mean?

A complex number has a 'real' part and an 'imaginary' part (the imaginary part involves the square root of a negative number).

We use Z to denote a complex number:

e.g. $Z = \underbrace{x}_{\text{Real}} + \underbrace{iy}_{\text{Imaginary}}$

Example: Z = 4 + 3i
Re(Z) = 4 Im(Z) = 3

You might see the i before or after it's number - it doesn't matter which.

Sometimes (especially in engineering) a j is used instead of i – they mean the same thing.

Powers of i

i stands for $\sqrt{-1}$ so: $i^2 = \left(\sqrt{-1}\right)^2 = -1$
$i^4 = (i^2)^2 = (-1)^2 = 1$

Summary:

$$\boldsymbol{i = \sqrt{-1}}$$
$$\boldsymbol{i^2 = -1 \qquad i^{-2} = -1}$$
$$\boldsymbol{i^4 = 1 \qquad i^{-4} = 1}$$

For any power of i take out as many i^4's and i^2's as possible and they will all end up as $\pm i$ or ± 1.

Example: $i^{11} = (i^4)^2 i^2 i = 1^2 \times -1 \times i = \underline{\underline{-i}}$

OR: just take out i^2's if you find it easier to remember.

Example: $i^{33} = (i^2)^{16} i = (-1)^{16} i = i$

Adding & Subtracting

This is easy – just add or subtract the real part and add or subtract the imaginary parts:

Examples: $(4 + 3i) + (2 + 6i) = \underline{(6 + 9i)}$
$(3 + 7i) - (1 - 3i) = \underline{(2 + 10i)}$

Multiplying

Multiply out the 2 brackets.

Example: $(3 + 5i)(4 - 2i) = 12 - 6i + 20i - 10i^2 = 12 + 14i - 10(-1) = \underline{\underline{22 + 14i}}$

Complex Conjugate

The conjugate is exactly the same as the complex number but with the opposite sign in the middle. When multiplied together they always produce a real number because the middle terms disappear (like the difference of 2 squares with quadratics).

Example: $(4+6i)(4-6i) = 16 - 24i + 24i - 36i^2 = 16 - 36(-1) = 16 + 36 = \underline{\underline{52}}$

Dividing

Dividing by a real number: divide the real part and divide the imaginary part.

Dividing by a complex number: Multiply top and bottom of the fraction by the complex conjugate of the denominator so that it becomes real, then do as above.

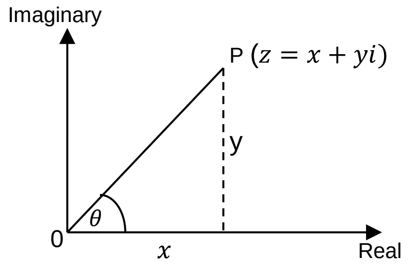
Examples: $\frac{3+4i}{2} = \frac{3}{2} + \frac{4}{2}i = \underline{\underline{1.5 + 2i}}$

$$\frac{4-5i}{3+2i} = \frac{4-5i}{3+2i} \times \frac{3-2i}{3-2i} = \frac{12-8i-15i+10i^2}{9-6i+6i-4i^2} = \frac{12-23i+10(-1)}{9-4(-1)} = \frac{2-23i}{13} = \underline{\underline{\boldsymbol{\frac{2}{13} - \frac{23}{13}i}}}$$

Graphical Representation

A complex number can be represented on an **Argand diagram** by plotting the real part on the x-axis and the imaginary part on the y-axis.

Example:



Nb Tan(θ) = $\frac{y}{x}$ so $\boldsymbol{\theta = tan^{-1}\left(\frac{y}{x}\right)}$

Modulus: is written as $|z|$ and is the length of OP, therefore $|z| = \sqrt{x^2 + y^2}$

Argument: is the angle **θ** that is made with the horizontal axis (denoted by ∠).

Polar & Exponential Form

As well as the basic form $(z = x + iy)$ there are 2 more ways of writing a complex number:

Polar:	$\boldsymbol{z = r(cos\theta + isin\theta)}$
Exponential:	$\boldsymbol{z = re^{i\theta}}$

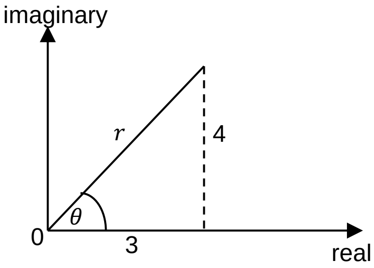
Where r is the length of the line and θ is the angle it makes with the x-axis *(this should be in radians for the exponential form).*

Remember:
To find the modulus (length), r: use Pythagoras
To find the argument (angle), θ: use $tan^{-1}\left(\frac{y}{x}\right)$

Converting between the different forms:

Basic $\longrightarrow$ Polar or Exponential		Need to find r and θ $r = \sqrt{x^2 + y^2}$ $\theta = tan^{-1}\left(\frac{y}{x}\right)$
Polar or Exponential $\longrightarrow$ Basic		Need to find x and y $x = rcos\theta$ $y = rsin\theta$

Example: Express $z = 3 + 4i$ in polar and exponential form



Modulus: $r = \sqrt{3^2 + 4^2} = \sqrt{16+9} = \sqrt{25} = 5$

Argument: $\theta = tan^{-1}\left(\frac{4}{3}\right) = 53.1^o$ (in radians $\theta = 0.927_r$)

Polar form: $z = 5(\cos(53.1) + \ isin(53.1))$

Exp form: $z = 5e^{0.927i}$

Nb always do a quick sketch of the complex number and if it's in a different quadrant adjust the angle as necessary.

Example: Express $z = 7e^{i\frac{\pi}{3}}$ in basic form

$x = rcos\theta \quad \therefore x = 7\cos\left(\frac{\pi}{3}\right) = 3.5$

$y = rsin\theta \quad \therefore y = 7\sin\left(\frac{\pi}{3}\right) = 6.1$

Basic form: $z = 3.5 + 6.1i$

A reminder of the 3 forms:

Basic	**Polar**	**Exponential**
$\boldsymbol{z = x + iy}$	$\boldsymbol{z = r(cos\theta + isin\theta)}$	$\boldsymbol{z = re^{i\theta}}$

Conversions: $x = rcos\theta$ $\qquad r = \sqrt{x^2 + y^2}$

$y = rsin\theta$ $\qquad \theta = tan^{-1}\left(\frac{y}{x}\right)$

Multiplying with Polar or Exponential form

Let $z_1 = z_2\, z_3$

Then $|z_1| = |z_2| \times |z_3|$
And $\angle z_1 = \angle z_2 + \angle z_3$

This means, when multiplying 2 complex numbers:

<u>Multiply</u> the $r's$
<u>Add</u> the angles ($\theta's$)

Example: If $z_1 = 5e^{\frac{\pi}{2}i}$ and $z_2 = 3e^{\frac{\pi}{3}i}$ find $z_1 z_2$

New modulus: $5 \times 3 = 15$
New angle: $\frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

$\therefore\ z_1 z_2 = 15e^{\frac{5\pi}{6}i}$

Dividing with Polar or Exponential form

Let $z_1 = \dfrac{z_2}{z_3}$

Then $|z_1| = \frac{|z_2|}{|z_3|}$
And $\angle z_1 = \angle z_2 - \angle z_3$

This means, when dividing 2 complex numbers:

<u>Divide</u> the $r's$
<u>Subtract</u> the angles ($\theta's$)

Example: if $z_1 = 5\left(cos\left(\frac{\pi}{2}\right) + isin\left(\frac{\pi}{2}\right)\right)$ and $z_2 = 3\left(cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)\right)$ find $\frac{z_1}{z_2}$

New modulus: $5 \div 3 = \frac{5}{3}$

New angle: $\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$

$\therefore\ \frac{z_1}{z_2} = \frac{5}{3}\left(cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right)$

De Moivres Theorem:

Is used for raising a complex number to a power.

Think: Raise r to the power of n and multiply the angle by n.

$$\boxed{\boldsymbol{z^n = r^n(cos(n\theta) + isin(n\theta))}}$$

e.g If $z = 3\left(cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)\right)$

then $z^5 = 3^5\left(cos\frac{5\pi}{3} + isin\frac{5\pi}{3}\right)$

e.g.2: $(1+i)^{100}$

We could use De Moivres or:
$(1+i)^{100} = ((1+i)^2)^{50}$

The same method can be used for a root (e.g. $z^{\frac{1}{n}}$). However, there will be n answers, all with the same modulus but with different arguments. To find the arguments you need to keep adding $\frac{2\pi}{n}$ to your previous answer.