Notes in representation theory

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Contents

Chapter 1

Introduction

Chapters 1 and 2 give a brief introduction and a review of some of the basic algebra required later. A more thorough grounding may be achieved by reading the cited works listed in §1.2: *Notes and References*.

Section 1.1 (upon which later chapters do not depend) attempts to provide a sketch overview of topics in the representation theory of finite dimensional algebras. In order to bootstrap this process, we use some terms without prior definition. We assume you know what a vector space is, and what a ring is (else see Section 2.1.1). For the rest, either you know them already, or you must intuit their meaning and wait for precise definitions until after the overview.

1.1 Representation theory preamble

Let $M_{m,n}(R)$ denote the additive group of $m \times n$ matrices over a ring R, with additive identity $0_{m,n}$. Let $M_n(R)$ denote the ring of $n \times n$ matrices over R. Define a block diagonal composition (matrix direct sum)

$$\begin{aligned} \oplus : M_m(R) \times M_n(R) &\to M_{m+n}(R) \\ (A, A') &\mapsto A \oplus A' = \begin{pmatrix} A & 0_{m,n} \\ 0_{n,m} & A' \end{pmatrix} \end{aligned}$$

Define Kronecker product

$$\otimes : M_{a,b}(R) \times M_{m,n}(R) \to M_{am,bn}(R) \tag{1.1}$$

$$(A, B) \mapsto \begin{pmatrix} a_{11}B & a_{12}B & \dots \\ a_{21}B & a_{22}B & \dots \\ \vdots & & \end{pmatrix} \tag{1.2}$$

In general $A \otimes B \neq B \otimes A$, but for each pair A, B there exists a pair of permutation matrices S, T such that $S(A \otimes B) = (B \otimes A)T$ (if A, B square then $T = S$ — the *intertwiner* of $A \otimes B$ ans $B \otimes A$).

(**1.1.1**) A matrix representation of a group G over a commutative ring R is a map

$$\rho : G \to M_n(R) \tag{1.3}$$

such that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$. In other words it is a map from the group to a different system, which nonetheless respects the extra structure (of multiplication) in some way. The study of representations — models of the group and its structure — is a way to study the group itself.

(**1.1.2**) The map ρ above is an example of the notion of representation that generalises greatly. A mild generalisation is the representation theory of R-algebras that we shall discuss, but one could go further. Physics consists in various attempts to model or represent the observable world. In a model, Physical entities are abstracted, and their behaviour has an image in the behaviour of the model. We say we understand something when we have a model or representation of it mapping to something we understand (better), which does not wash out too much of the detailed behaviour.

(**1.1.3**) Representation theory itself seeks to classify and construct representations (of groups, or other systems). Let us try to be more explicit about this.

(I) Suppose ρ is as above, and let S be an arbitrary invertible element of $M_n(R)$. Then one immediately verifies that

$$\rho_S : G \to M_n(R) \tag{1.4}$$
$$g \mapsto S\rho(g)S^{-1} \tag{1.5}$$

is again a representation.
(II) If ρ' is another representation (by $m \times m$ matrices, say) then

$$\rho \oplus \rho' : G \to M_{m+n}(R) \tag{1.6}$$
$$g \mapsto \rho(g) \oplus \rho'(g) \tag{1.7}$$

is yet another representation.
(III) For a finite group G let $\{g_i \ : \ i = 1, ..., |G|\}$ be an ordering of the group elements. Each element g acts on G, written out as this list $\{g_i\}$, by multiplication from the left (say), to permute the list. That is, there is a permutation $\sigma(g)$ such that $gg_i = g_{\sigma(g)(i)}$. This permutation can be recorded as a matrix,

$$\rho_{Reg}(g) = \sum_{i=1}^{|G|} \epsilon_{i\ \sigma(g)(i)}$$

(where $\epsilon_{ij} \in M_{|G|}(R)$ is the i, j-elementary matrix) and one can check that these matrices form a representation, called the *regular representation*.

Clearly, then, there are unboundedly many representations of any group. However, these constructions also carry the seeds for an organisational scheme...

(**1.1.4**) Firstly, in light of the ρ_S construction, we only seek to classify representations *up to isomorphism* (i.e. up to equivalences of the form $\rho \leftrightarrow \rho_S$).

Secondly, we can go further (in the same general direction), and give a cruder classification, by *character*. (While cruder, this classification is still organisationally very useful.) We can briefly explain this as follows.

Let c_G denote the set of classes of group G. A *class function* on G is a function that factors through the natural set map from G to the set c_G. Thus an R-valued class function is completely specified by a c_G-tuple of elements of R (that is, an element of the set of maps from c_G to R, denoted R^{c_G}). For each representation ρ define a *character* map from G to R

$$\begin{aligned} \chi_\rho : G &\rightarrow R && (1.8)\\ g &\mapsto \mathrm{Tr}(\rho(g)) && (1.9) \end{aligned}$$

(matrix trace). Note that this map is fixed up to isomorphism. Note also that this map is a class function. Fixing G and varying ρ, therefore, we may regard the character map instead as a map χ_- from the collection of representations to the set of c_G-tuples of elements of R.

Note that pointwise addition equips R^{c_G} with the structure of abelian group. Thus, for example, the character of a sum of representations isomorphic to ρ lies in the subgroup generated by the character of ρ; and $\chi_{\rho\oplus\rho'} = \chi_\rho + \chi_{\rho'}$ and so on.

We can ask if there is a small set of representations whose characters '$\mathbb{N}_0$-span' the image of the collection of representations in R^{c_G}. (We could even ask if such a set provides an R-basis for R^{c_G} (in case R a field, or in a suitably corresponding sense — see later). Note that $|c_G|$ provides an upper bound on the size of such a set.)

(**1.1.5**) Next, conversely to the direct sum result, suppose $R_1 : G \rightarrow M_m(R)$, $R_2 : G \rightarrow M_n(R)$, and $V : G \rightarrow M_{m,n}(R)$ are set maps, and that a set map $\rho_{12} : G \rightarrow M_{m+n}(R)$ takes the form

$$\rho_{12}(g) = \begin{pmatrix} R_1(g) & V(g) \\ 0 & R_2(g) \end{pmatrix} \tag{1.10}$$

(a matrix of matrices). Then ρ_{12} a representation of G implies that both R_1 and R_2 are representations. Further, $\chi_{\rho_{12}} = \chi_{R_1} + \chi_{R_2}$ (i.e. the character of ρ_{12} lies in the span of the characters of the smaller representations). Accordingly, if the isomorphism class of a representation contains an element that can be written in this way, we call the representation *reducible*.

(**1.1.6**) For a finite group over $R = \mathbb{C}$ (say) we shall see later that there are only a finite set of 'irreducible' representations needed (up to equivalences of the form $\rho \leftrightarrow \rho_S$) such that every representation can be built (again up to equivalence) as a direct sum of these; and that all of these irreducible representations appear as direct summands in the regular representation.

We have done a couple of things to simplify here. Passing to a field means that we can think of our matrices as recording linear transformations on a space with respect to some basis. To say that ρ is equivalent to a representation of the form ρ_{12} above is to say that this space has a G-subspace (R_1 is the representation associated to the subspace). A representation is irreducible if there is no such proper decomposition (up to equivalence). A representation is *completely reducible* if for every decomposition $\rho_{12}(g)$ there is an equivalent identical to it except that $V(g) = 0$ — the direct sum.

Theorem [Mashke] Let ρ be a representation of a finite group G over a field K. If the characteristic of K does not divide the order of G, then ρ is completely reducible.

Corollary Every complex irreducible representation of G is a direct summand of the regular representation.

Representation theory is more complicated in general than it is in the cases to which Mashke's Theorem applies, but the notion of irreducible representations as fundamental building blocks survives in a fair degree of generality. Thus the question arises:

Over a given R, what are the irreducible representations of G (up to $\rho \leftrightarrow \rho_S$ equivalence)?

There are other questions, but as far as physical applications (for example) are concerned, this is the main interesting question.

(**1.1.7**) Examples: In this sense, of constructing irreducible representations, the representation theory of the symmetric groups S_n over $\mathbb{C}$ is completely understood! (We shall review it.) On the other hand, over other fields we do not have even so much as a conjecture as to how to organise the statement of a conjecture! So there is work to be done.

(**1.1.8**) Remark: When working with R a *field* it is natural to view $M_n(R)$ as the ring of linear transformations of vector space R^n expressed with respect to a given ordered basis. The equivalence $\rho \leftrightarrow \rho_S$ corresponds to a change of basis, and so working up to equivalence corresponds to demoting the matrices themselves in favour of the underlying linear transformations (on R^n). In this setting it is common to refer to the linear transformations by which G acts on R^n as the representation (and to spell out that the matrices are a *matrix* representation, regarded as arising from a choice of ordered basis).

Such an action of a group G on a set makes the set a G-set.[1] However, given that it is a set with extra structure (in this case, a vector space), it is a small step to want to try to take advantage of the extra structure. For example, we can define RG to be the R-vector space with basis G, and define a multiplication on RG by

$$\left(\sum_i r_i g_i\right)\left(\sum_j r'_j g_j\right) = \sum_{ij}(r_i r'_j)(g_i g_j)$$

which makes RG a ring. One can quickly check that

$$\rho : RG \to M_n(R) \tag{1.11}$$

$$\sum_i r_i g_i \mapsto \sum_i r_i \rho(g_i) \tag{1.12}$$

extends a representation ρ of G to a representation of RG in the obvious sense. Superficially this construction is extending the use we already made of the multiplicative structure on $M_n(R)$, to make use not only of the additive structure, but also of the particular structure of 'scalar' multiplication (multiplication by an element of the centre), which plays no role in representing the group multiplication *per se*. The construction *also* makes sense at the G-set/vector space level, since linear transformations support the same extra structure.

The same formal construction of RG works when R is an arbitrary commutative ring, except that RG is not then a vector space. Instead it is called (in respect of the vector-space-like aspect of its structure) a free R-module with basis G. The idea of matrix representation goes through unchanged. If one wants a generalisation of the notion of G-set for RG to act on, the additive structure is forced from the outset. This is called a (left) RG-module. This is, then, an abelian group $(M, +)$ with a suitable action of RG defined on it: $r(x + y) = rx + ry$, $(r + s)x = rx + sx$, $(rs)x = r(sx)$, $1x = x$ $(r, s \in RG,\ x, y \in M)$, just as the original vector space R^n was. What is new at this level is that such a structure may not have a basis (a *free* module has a basis), and so may not correspond to any class of matrix representations.

[1] For a set S, a map $\psi : G \times S \to S$ (written $\psi(g, s) = gs$) such that $(gg')s = g(g's)$, equips S with the property of left G-set.

From this point the study of representation theory may be considered to include the study of both matrix representations and modules.

(**1.1.9**) What other kinds of systems can we consider representation theory for?
A natural place to start studying representation theory is in Physical modeling. Unfortunately we don't have time for this in the present work, but we will generalise from groups at least as far as rings and algebras. The generalisation from groups to group algebras RG over a commutative ring R is quite natural as we have seen. The most general setting within the ring-theory context would be the study of arbitrary ring homomorphisms from a given ring. However, if one wants to study this ring by studying its modules (the obvious generalisation of the RG-modules introduced above) then the parallel of the matrix representation theory above is the study of modules that are also free modules over the centre, or some subring of the centre. (For many rings this accesses only a very small part of their structure, but for many others it captures the main features. *Every* module over a commutative ring is free if and only if the ring is a field, so this is our most accessible case. We shall motivate the restriction shortly.) This leads us to the study of algebras.

To introduce the general notion of an algebra, we first write $\mathrm{cen}\,(A)$ for the centre of a ring A

$$\mathrm{cen}\,A = \{a \in A \mid ab = ba \;\forall b \in A\}$$

(**1.1.10**) An algebra A (over a commutative ring R), or an R-algebra, is a ring A together with a homomorphism $\psi : R \to \mathrm{cen}\,(A)$, such that $\psi(1_R) = 1_A$.

Examples: Any ring is a $\mathbb{Z}$-algebra. Any ring is an algebra over its centre. The group ring RG is an R-algebra by $r \mapsto r1_G$. The ring $M_n(R)$ is an R-algebra.

Let $\psi : R \to \mathrm{cen}\,(A)$ be a homomorphism as above. We have a composition $R \times A \to A$:

$$(r, a) = ra = \psi(r)a$$

so that A is a left R-module with

$$r(ab) = (ra)b = a(rb) \tag{1.13}$$

Conversely any ring which is a left R-module with this property is an R-algebra.

(**1.1.11**) An R-representation of A is a homomorphism of R-algebras

$$\rho : A \to M_n(R)$$

(**1.1.12**) The study of RG depends heavily on R as well as G. The study of such R-algebras takes a relatively simple form when R is an algebraically closed field; and particularly so when that field is $\mathbb{C}$. We shall aim to focus on these cases. However there are significant technical advantages, even for such cases, in starting by considering the more general situation. Accordingly we shall need to know a little ring theory, even though general ring theory is not the object of our study.

Further, as we have said, neither applications nor aesthetics restrict attention to the study of representations of groups and their algebras. One is also interested in the representation theory of more general algebras.

(**1.1.13**) TO DO: A left R-module is *simple* if it has no non-trivial submodules...

(**1.1.14**) TO DO:

Finish overview of modules
Grothendieck group
Tensor product
induction
—

(1.1.15) Operators acting on a space; their eigenvectors and eigenvalues.
Here we remark very briefly and generally on the kind of Physical problem that can lead us into representation theory.
A typical Physical problem has a linear operator Ω acting on a space H, with that action given by the action of the operator on a (spanning) subset of the space. One wants to find the eigenvalues of Ω.
The eigenvalue problem may be thought of as the problem of finding the one-dimensional subspaces of H as an $\langle\Omega\rangle$-module, where $\langle\Omega\rangle$ is the (complex) algebra generated by Ω. That is, we want to find elements h_i in H such that:

$$\Omega h_i = \lambda_i h_i$$

— noting only that, usually, the object of primary physical interest is λ_i rather than h_i. If H is finite dimensional then (the complex algebra generated by) Ω will obey a relation of the form

$$\prod_i (\Omega - \lambda_i)^{m_i} = 0$$

Of course the details of this form are *ab initio* unknown to us. But, proceeding formally for a moment, if any $m_i > 1$ (necessarily) here, so that $S = \prod_i (\Omega - \lambda_i) \neq 0$, then S generates a non-vanishing nilpotent ideal (we say, the algebra has a radical). Obviously any such nilpotent object has 0-spectrum, so two operators differing by such an object have the same spectrum. In other words, the image of Ω in the quotient algebra by the radical has the same spectrum $\{\lambda_i\}$. An algebra with vanishing radical (such as the quotient of a complex algebra by its radical) has a particularly simple structural form, so this is a potentially useful step.

However, gaining *access* to this form may require enormously greater arithmetic complexity than the original algebra. In practice, a balance of techniques is most effective, even when motivated by physical ends. This balance can often be made by analysing the regular module (in which every eigenvalue is manifested), and thus subquotients of projective modules, but not more exotic modules. (Of course Mathematically other modules may well also be interesting — but this is a matter of aesthetic judgement rather than application.)
It may also be necessary to find the subspaces of H as a module for an algebra generated by a set of operators $\langle\Omega_i\rangle$. A similar analysis pertains.

A particularly nice (and Physically manifested) situation is one in which the operators Ω_i (whose unknown spectrum we seek to determine) are known to take the form of the representation matrices of elements of an abstract algebra A in some representation:

$$\Omega_i = \rho(\omega_i)$$

Of course any reduction of Ω_i in the form of (1.10) reduces the problem to finding the spectrum of $R_1(\omega_i)$ and $R_2(\omega_i)$. Thus the reduction of ρ to a (not necessarily direct) sum of irreducibles:

$$\rho(\omega_i) \cong +_\alpha \, \rho_\alpha(\omega_i)$$

reduces the spectrum problem in kind. In this way, Physics drives us to study the representation theory of the abstract algebra A.

1.2 Notes and references

The following texts are recommended reading: Jacobson[25, 26], Bass[4], Maclane and Birkoff[31], Green[22], Curtis and Reiner[16, 17], Cohn[12], Anderson and Fuller[3], Adamson[2], Cassels[9], and references therein. .

Chapter 2

Basic definitions, notations and examples

2.1 Preliminaries

2.1.1 Definition summary

There follows a list of definitions in the form
Algebraic system $A = (A \text{ a set, } n\text{−ary operations})$, axioms.
(The selection of a special element $u \in A$, say, counts as a 0-ary operation.)
Extended examples are postponed to the relevant sections.

Semigroup	$S = (S, \square)$, $\square$ a closed associative binary operation on S.
Monoid	$M = (M, \square, u)$, $(M, \square)$ a semigroup, $u \in M$ a *unit element* (i.e. $au = a = ua \;\; \forall a \in M$). Example: $(\mathbb{N}_0, +, 0)$.
Group	$G = (G, ., u)$, G a monoid, $\forall a \in G \exists a'$ such that $aa' = u = a'a$.
Abelian group	$G = (G, +, 0)$, G a group, $a + b = b + a$.
Ring	$R = (R, +, ., 1, 0)$, $(R, +, 0)$ an abelian group, $(R, ., 1)$ a monoid, $a(b + c) = ab + ac$, $(a + b)c = ac + bc$.
Division ring	D, D a ring, every non–zero element has a multiplicative inverse.
Local ring	A, A a ring, sum of two nonunits is a nonunit (a a nonunit means there does not exist b such that $ab = ba = 1$). [1]
Domain	K, K a ring, $0 \neq 1$, $mn = 0$ implies either $m = 0$ or $n = 0$.
Integral domain	K, K a ring, . commutative, $0 \neq 1$, $mn = 0$ implies either $m = 0$ or $n = 0$. (I.e. an integral domain is a commutative domain.)
Principal ideal domain	K, K an integral domain, every ideal $J \subseteq K$ is principal (i.e. $\exists$ $a \in K$ such that $J = aK$).
Field	F, F an integral domain, every $a \neq 0$ has a multiplicative inverse.

Our other core definitions are, for S a semigroup, R a ring as above:

S–IDEAL J: $J \subset S$ and $rj, jr \in J$ for all $r \in S$, $j \in J$.
R–IDEAL J: $J \subset R$ and $rj, jr \in J$ for all $r \in R$, $j \in J$.

(LEFT) R–MODULE M: M an abelian group with map $R \times M \to M$ such that $r(x+y) = rx+ry$, $(r+s)x = rx+sx$, $(rs)x = r(sx)$, $1x = x$ ($r \in R$, $x, y \in M$).
Right modules defined similarly.

(LEFT) R–MODULE HOMOMORPHISM : Ψ from left R-module M to N is a map $\Psi : M \to N$ such that $\Psi(x+y) = \Psi(x) + \Psi(y)$, $\Psi(rx) = r\Psi(x)$ for $x, y \in M$ and $r \in R$.

(**2.1.1**) EXERCISE. $\mathbb{Z}$ is a ring. Form examples of as many of the other structures as possible from this one. (And some non-examples.)

In the following table k is a field and $\mathbb{H}$ is the ring of real quaternions (see §2.3.2).

	$DivR$	LR	ID	PID
$\mathbb{Z}$	$\times$	$\times$	$\surd$	$\surd$
$\mathbb{Z}[x]$	$\times$	$\times$	$\surd$	$\times$
$k[x]$	$\times$	$\times$	$\surd$	$\surd$
$k[x,y]$	$\times$	$\times$	$\surd$	$\times$
$\mathbb{H}$	$\surd$	$\surd$	$\times$	$\times$

(**2.1.2**) For more on semigroups see for example Howie [**?**].

2.1.2 Glossary

$GL(N)$ general linear group on $\mathbb{C}^N$
Λ set of integer partitions
Λ_n set of integer partitions of n
$O(N)$ orthogonal group on $\mathbb{C}^N$
P_S partitions of a set S
J_S pair partitions of a set S
$P(S)$ power set (lattice) of a set S

2.2 Elementary set theory notations and constructions

As in Green [22], let
$$\underline{n} \;:=\; \{1, 2, .., n\}$$
Similarly here $\underline{n'} \;:=\; \{1', 2', .., n'\}$ (and so on).

(**2.2.1**) For S a set, let $P(S)$ be the lattice of subsets of S. For S, T sets, let $U_{S,T}$ be the set of relations on S to T. That is,
$$U_{S,T} = P(S \times T).$$
Set $U_S = U_{S,S}$, and
$$T^S \;:=\; \hom(S,T) \;\subset\; U_{S,T}$$

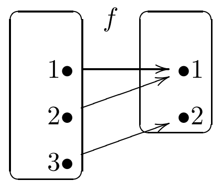
For example

$$\underline{2}^{\underline{2}} = \{\{(1,1),(2,1)\},\{(1,1),(2,2)\},\{(1,2),(2,1)\},\{(1,2),(2,2)\}\}$$

(**2.2.2**) It will be useful to have in mind the *mapping diagram* realisation of such functions. For example

$$f = \{(1,1),(2,1),(3,2)\} \in \underline{2}^{\underline{3}}$$

is



(**2.2.3**) If T, S finite it will be clear that any total order on each of T and S puts T^S in bijection with $\underline{|T|}^{\underline{|S|}}$. We may represent the elements of T^S as T-ordered lists of elements from S. Thus

$$\underline{2}^{\underline{2}} = \{11, 12, 21, 22\}$$

(for example $22(1) = 2$, since the first entry in 22 is the image of 1).

(**2.2.4**) A *composition* of n is a finite sequence λ in $\mathbb{N}_0$ that sums to n. We write $\lambda \vDash n$.

We define the *shape* of an element f of $\underline{m}^{\underline{n}}$ as the composition of n given by

$$\lambda(f)_i = |f^{-1}(i)|$$

Example: for $111432525 \in \underline{6}^{\underline{9}}$ we have $\lambda(111432525) = (3,2,1,1,2,0)$.

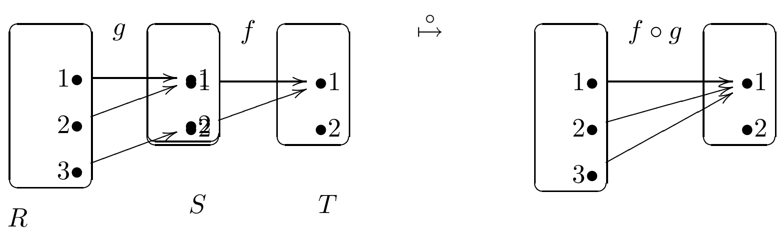
If $\lambda \vDash n$ we write $|\lambda| = n$.

(**2.2.5**) Of course, composition of functions defines a map

$$\hom(S,T) \times \hom(R,S) \quad \to \quad \hom(R,T) \tag{2.1}$$
$$(f,g) \mapsto f \circ g \tag{2.2}$$

where as usual $(f \circ g)(x) = f(g(x))$. For example $11 \circ 22 = 11$ (since $11(22(1)) = 11(2) = 1$; and so on).

The *mapping diagram realisation* of composition is to first juxtapose the two functions so that the two instances of the set S coincide, then define a direct path from R to T for each path of length 2 so formed:



(2.3)

(**2.2.6**) If the image $f(S)$ of a map $f : S \to T$ is of finite order we shall say that f has order $|f(S)|$ (otherwise it has infinite order). We have the *bottleneck principle*

$$|(f \circ g)(R)| \;\leq\; \min(|f(S)|, |g(R)|)$$

(**2.2.7**) Proposition. *(i) For S a set, $S^S = \hom(S,S)$ is a monoid under composition of functions. (ii) For each $d \in \mathbb{N}$ then set $\{f \in S^S \mid |f(S)| < d\}$ is an ideal (hence a sub-semigroup) of S^S.*

2.2.1 Set partitions

(**2.2.8**) Let E_S be the set of equivalence relations on set S, and let P_S be the set of partitions of S. Note the natural bijection

$$E_S \leftrightarrow P_S.$$

We have $E_S \subset U_S$. For $\rho \in U_S$ let $\bar{\rho} \in U_S$ be the smallest transitive relation containing ρ — this is called the *transitive closure* of ρ.

(**2.2.9**) For a, b equivalence relations on any two finite sets let ab be the transitive closure of $a \cup b$ (an equivalence relation on the union of the two finite sets).

(**2.2.10**) Let $J_S \subset P_S$ be the set of pair-partitions of S. Let $P_{n,m} = P_{\underline{n} \cup \underline{m'}}$ and

$$J_{n,m} = J_{\underline{n} \cup \underline{m'}} \;\subset P_{n,m}$$

(**2.2.11**) For $a \in P_{n,m}$ let a' be the partition of $\underline{n'} \cup \underline{m''}$ obtained by adding a prime to each object in every part. We may define a map

$$\circ : P_{l,m} \times P_{m,n} \to P_{l,n}$$

as follows. For $a \in P_{l,m}$, $b \in P_{m,n}$ partitions (and hence equivalence relations) note that ab' is an equivalence relation on $\underline{l} \cup \underline{m'} \cup \underline{n''}$. Restricting to $\underline{l} \cup \underline{n''}$ this equivalence relation is again a partition, call it $r(ab')$ (indeed if a, b are pair-partitions then so is $r(ab')$). For $x \in \underline{l} \cup \underline{n''}$ let $u(x) \in \underline{l} \cup \underline{n'}$ be the image under the action of replacing double primes with single. Let $a \circ b = u(r(ab')) \in P_{l,n}$ be the image under the obvious application of this map.

Set $P_n = P_{n,n}$ and $J_n = J_{n,n}$.

(**2.2.12**) Proposition. *For each $n \in \mathbb{N}$ the map $\circ : (a,b) \mapsto u(r(ab'))$ defines an associative unital product on P_n, making it a monoid. The construction also restricts to make J_n a monoid.*

2.3 Initial examples in representation theory

2.3.1 The monoid $\hom(\underline{2}, \underline{2})$

Two matrices A, B are conformable to a product AB if (i) the number of rows of A equals the number of columns of B; (ii) they have entries in the same ring R. By convention, if R is a

K-algebra, then a matrix over K is considered a matrix over R by the homomorphism ψ (see (1.1.10)), taking elements of K to scalar multiples of 1_R.

(**2.3.1**) Consider the monoid $M = \underline{2}^{\underline{2}}$, and the free $\mathbb{Z}$-module $\mathbb{Z}M$ with basis M. This is a $\mathbb{Z}$-algebra (by virtue of the monoid multiplication). Totally ordering this (or any other) basis we may encode $x \in \mathbb{Z}M$ by

$$x = \begin{pmatrix} x_{11} & x_{12} & x_{21} & x_{22} \end{pmatrix} \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix}$$

(here we have used the shorthand for monoid elements give in (2.2.3)). This organisational scheme yields a generalisation of the regular representation construction mentioned in the Introduction. Indeed there is both a left and a right regular construction. We shall consider both.

(**2.3.2**) Firstly consider the encoding of multiplication by

$$\begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} * (11,\ 12,\ 21,\ 22) \ = \begin{pmatrix} 11 \circ 11 & 11 \circ 12 & 11 \circ 21 & 11 \circ 22 \\ 12 \circ 11 & 12 \circ 12 & 12 \circ 21 & 12 \circ 22 \\ 21 \circ 11 & 21 \circ 12 & 21 \circ 21 & 21 \circ 22 \\ 22 \circ 11 & 22 \circ 12 & 22 \circ 21 & 22 \circ 22 \end{pmatrix}$$

(we put the $*$ in on the left, to emphasise that this is matrix multiplication over a non-commutative ring) and hence

$$\begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} * m = \begin{pmatrix} 11 \circ m \\ 12 \circ m \\ 21 \circ m \\ 22 \circ m \end{pmatrix} \qquad m \in \underline{2}^{\underline{2}}$$

That is

$$\begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} * 11 = \begin{pmatrix} 11 \circ 11 \\ 12 \circ 11 \\ 21 \circ 11 \\ 22 \circ 11 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \\ 22 \\ 22 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix}$$

$$\begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} * 12 = \begin{pmatrix} 11 \circ 12 \\ 12 \circ 12 \\ 21 \circ 12 \\ 22 \circ 12 \end{pmatrix} = \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix}$$

and so on. By this (general) construction we have a map $R_r : M \to M_4(\mathbb{Z})$

$$R_r(11) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ R_r(12) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ R_r(21) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ R_r(22) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

These matrices give a representation.

(**2.3.3**) We do not yet have the tools for a systematic analysis of representations of a monoid, but a couple of observations are in order. This representation is, up to a reordering of the basis, in the form of (1.10):

$$R_{r'}(11)=\left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),\ R_{r'}(21)=\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

This corresponds to the fact that the free $\mathbb{Z}$-submodule of $\mathbb{Z}M$ with basis $\{11, 22\}$ is also invariant under this action of M from the right.

This representation does not have a manifest direct sum decomposition, but we can ask if such a decomposition can be manifested by basis change. However the possibilities for basis change beyond reordering depend on the choice of ring.

(**2.3.4**) *Provided we pass to a ring in which 2 is invertible*, another basis is $\{-11 + 12 + 21 - 22, 11, 11 - 22, 12 - 21\}$. (Questions: Where did this come from?! How did the restriction arise?) Using this basis we get another representation:

$$R'_r(11)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ R'_r(12)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ R'_r(21)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ R'_r(22)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

In other words there is a direct sum decomposition:

$$R' = R_1 \oplus R_{1'} \oplus R_2$$

Note that R_2 is not irreducible, but it is not amenable to a direct sum decomposition in any basis over any ring. It is, however, of the form in Equation (1.10). In this sense it 'contains' two one-dimensional (hence irreducible) representations:

$$R_2 = R_{1'} + R_{1''}$$

(**2.3.5**) Alternatively, we may encode multiplication by

$$m * \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} = \begin{pmatrix} m \circ 11 \\ m \circ 12 \\ m \circ 21 \\ m \circ 22 \end{pmatrix} \qquad m \in \underline{2}^{\underline{2}}$$

That is

$$11 * \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} = \begin{pmatrix} 11*11 \\ 11*12 \\ 11*21 \\ 11*22 \end{pmatrix} = \begin{pmatrix} 11 \\ 11 \\ 11 \\ 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix}$$

$$12 * \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} = \begin{pmatrix} 12*11 \\ 12*12 \\ 12*21 \\ 12*22 \end{pmatrix} = \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 11 \\ 12 \\ 21 \\ 22 \end{pmatrix}$$

and so on. By this construction we have another map $R^r : M \to M_4(\mathbb{Z})$

$$R^r(11)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ R^r(12)=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ R^r(21)=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},\ R^r(22)=\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

These matrices give an antirepresentation. That is

$$R^r(a)R^r(b) = R^r(ba)$$

This becomes a representation once we compose with the matrix transpose map. Note however that there is no similarity transformation between R_r and $(R^r)^t$ (the map from the algebra to the ring defined for each representation by matrix trace is not changed by similarity, and differs between the two), so they are not equivalent representations.

Quite generally, if a representation can be expressed in the form of Equation (1.10):

$$\rho_{12}(g) = \begin{pmatrix} \rho_1(g) & V(g) \\ 0 & \rho_2(g) \end{pmatrix} \tag{2.4}$$

then

$$\mathrm{Tr}(\rho_{12}(g)) = \mathrm{Tr}(\rho_1(g)) + \mathrm{Tr}(\rho_2(g)) \qquad \text{(any } g\text{)}$$

If we assume that $(R^r)^t$ is a (not necessarily direct) sum of the irreducible representations we have already seen, then we can deduce immediately that this sum contains two copies of $R_{1''}$, since $\mathrm{Tr}(R_{1''}(21)) = -1$ and $\mathrm{Tr}(R_1(21)) = \mathrm{Tr}(R_{1'}(21)) = 1$, and so this is the only way to get $\mathrm{Tr}(R^r(21)) = 0$. Considering $\mathrm{Tr}(R^r(11)) = 1$ we then see that

$$(R^r)^t = R_1 + R_{1'} + R_{1''} + R_{1''}$$

(under the stated assumption). In other words $(R^r)^t$ does not even have quite the same irreducible summands as R_r — at least the multiplicities are different.

That was Too much linear algebra! How can we be more slick? We shall shortly begin to address this question.

(**2.3.6**) For K a given commutative ring, and M a left K-module write $End(M)$ for the set of linear transformations of M. For any subset $S \in End(M)$ we define $End_S(M)$ as the subset of linear transformations that commute with every element of S.

(**2.3.7**) EXERCISE. Consider R'_r as a subset of $End(\mathbb{Z}\underline{2}^{\underline{2}})$. What is $End_{R'}(\mathbb{Z}\underline{2}^{\underline{2}})$?

2.3.2 quaternions

Set

$$\mathbf{i} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \mathbf{j} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

We define $\mathbb{H}$ as the subring of $M_4(\mathbb{R})$ generated as an $\mathbb{R}$-algebra by these matrices (as an $\mathbb{R}$-vector space it is spanned by $\{\mathbf{i}, \mathbf{j}, \mathbf{ii}, \mathbf{ij}\}$). This is a noncommutative division ring.

2.4 Basic tools: topology

(**2.4.1**) A *sigma-algebra* over a set S is a subset Σ of the power set $P(S)$ which includes S and $\emptyset$ and is closed under countable unions, and complementation in S.

Any subset S' of $P(S)$ defines a sigma-algebra — the smallest sigma-algebra generated by S'. For example $\{\{1\}\} \subset P(\{1,2,3\})$ generates $\Sigma = \{\emptyset, \{1\}, \{2,3\}, \{1,2,3\}\}$.

(**2.4.2**) A *topological space* is a set S together with a subset T of the power set $P(S)$ which includes S and $\emptyset$ and is closed under unions and finite intersections.

The set T is called a *topology* on S. The elements of T are called the *open sets* of this topology. A set is *closed* if it is the complement in S of an open set. A function between topological spaces is *continuous* if the inverse image of every open set is open. Two spaces are *homeomorphic* if there is a bijection between them, continuous in both directions.

The restriction of T to $S' \subset S$ is a topology on S', called the *subspace topology*.

A subset S' of a topological space (S,T) is *irreducible* if $S' = S_1 \cup S_2$ with S_1 closed implies S_2 not closed.

(**2.4.3**) Let k be a field. A polynomial $p \in k[x_1, ..., x_r]$ determines a map from k^r to k by evaluation. For $P = \{p_i\}_i \subset k[x_1, ..., x_r]$ define

$$Z(\{p_i\}_i) = \{x \in k^r \ : \ p_i(x) = 0 \ \forall \ i\}$$

An *affine algebraic set* is any such set, in case k algebraically closed. An *affine variety* is any such set, that cannot be written as the union of two proper such subsets. (See for example, Hartshorne [23, I.1].)

The set of affine varieties in k^r satisfy the axioms for closed sets in a topology. This is called the *Zariski topology*. The Zariski topology on an affine variety is simply the corresponding subspace topology.

The set $I(P) \in k[x_1, ..., x_r]$ of all functions vanishing on $Z(P)$ is the ideal in $k[x_1, ..., x_r]$ generated by P. We call

$$k_P = k[x_1, ..., x_r]/I(P)$$

the *coordinate ring* of $Z(P)$.

(**2.4.4**) Let Z be an affine variety in k^r and $f : Z \to k$. We say f is *regular* at $z \in Z$ if there is an open set containing z, and $p_1, p_2 \in k[x_1, ..., x_r]$, such that f agrees with p_1/p_2 on this set.

(**2.4.5**) A morphism of varieties is a Zariski continuous map $f : Z \to Z'$ such that if V is open in Z' and $g : V \to k$ is regular then $g \circ f : f^{-1}(V) \to k$ is regular.

(**2.4.6**) Given affine varieties X, Y then $X \times Y$ may be made in to an algebraic variety in the obvious way.

(**2.4.7**) An *algebraic group* G is a group that is an affine variety such that inversion is a morphism of algebraic varieties; and multiplication is a morphism of algebraic varieties from $G \times G$ to G.

Chapter 3

More basic tools

3.1 reflection groups and geometry

Here we largely follow Humphreys [24].

(**3.1.1**) Examples of reflections acting on a Euclidean space:

$$(ij) : \mathbb{R}^n \quad \to \quad \mathbb{R}^n \tag{3.1}$$
$$(x_1, x_2, ..., x_i, ..., x_j, ...x_n) \quad \mapsto \quad (x_1, x_2, ..., x_j, ..., x_i, ...x_n) \tag{3.2}$$

with reflection hyperplane

$$H_{(ij)} = \{x \in \mathbb{R}^n \mid x_i = x_j\}$$

$$(i)_- : \mathbb{R}^n \quad \to \quad \mathbb{R}^n \tag{3.3}$$
$$(x_1, x_2, ..., x_i, ..., x_j, ...x_n) \quad \mapsto \quad (x_1, x_2, ..., -x_i, ..., x_j, ...x_n) \tag{3.4}$$

with reflection hyperplane

$$H_{(i)_-} = \{x \in \mathbb{R}^n \mid x_i = 0\}$$

In these examples the hyperplane contains the origin of coordinates, i.e. they are subspaces. A hyperplane not containing the origin is an *affine* hyperplane. Any non-affine hyperplane may be characterised as the subspace of vectors perpendicular to a given non-zero vector. For example $H_{(i)_-}$ is perpendicular to e_i.

(**3.1.2**) A Coxeter system is a pair (W, S) consisting of a group W, and a set of generators $S \subset W$, together with a symmetric matrix M indexed by S such that $M_{s,s} = 1$ and $M_{s,s'} \geq 2$ (with every entry in $\mathbb{N} \cup \{\infty\}$), defining relations by

$$(ss')^{M_{s,s'}} = 1$$

(here $M_{s,s'} = \infty$ means no relation).
For example every $(W, \{s\})$ is a group of order 2. A *parabolic subsystem* of (W, S) is a system generated by $I \subset S$.
Coxeter systems are closely related to reflection groups. Indeed (see [24, §6.4])

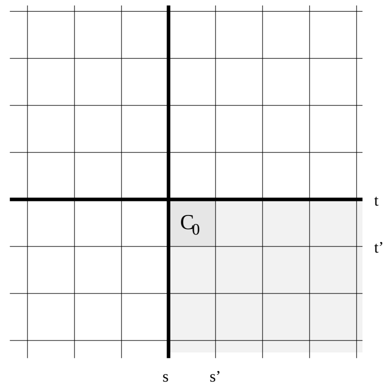


Figure 3.1: Chamber geometry of type $\tilde{A}_1 \times \tilde{A}_1$ with $S = \{s, s', t, t'\}$, and a fundamental region for the $W_{\{s,t\}}$ parabolic.

(**3.1.3**) Theorem. Coxeter group W is finite if and only if it is a finite reflection group.

And for our present purposes their main role is as an organisational device for the study of reflection groups.

(**3.1.4**) Let V be a Euclidean space, and (W, S) a Coxeter system with an action generated by reflections on V. For example, any single hyperplane in V generates a reflection group of order 2.

Starting with this example, one may add in a second hyperplane and ask about the group generated by reflections in the two hyperplanes. The nature of this group depends on the relationship of the two hyperplanes; and the possibilities for this relationship depend on the dimension of V. In dimension 2 or greater reflection in one hyperplane may fix the other; or they may be parallel; or they may generate a finite number of other hyperplanes; or (if the second hyperplane is in generic position with respect to the first) an infinite number. Here we are interested in all but the generic position case.

In dimension 1 of course, any two distinct hyperplanes (i.e. points) are necessarily parallel. The associated system is necessarily

$$M = \begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix}$$

3.1.1 Chamber geometry

(**3.1.5**) Let H_s be the reflection hyperplane of $s \in S$, or indeed of any reflection $s \in W$ generated by these. For T any subset of S let $[T]$ be the set of reflections generated by T. Set

$$\mathbb{H}_T = \bigcup_{t \in [S] \setminus [T]} H_t$$

A chamber is a maximal connected component of $V \setminus \mathbb{H}_\emptyset$. Write $\mathcal{C}_W$ for the set of chambers.

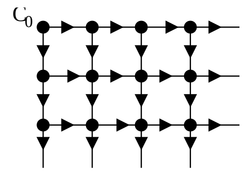


Figure 3.2: The digraph G_a for $W' = W_{\{s,s',t,t'\}}$ and $W = W_{\{s,t\}}$ from Figure 3.1.

In our example above the chambers are simply a collection of disjoint open intervals of the line. For an example generalising this we may think of the group generated by two parallel lines in $\mathbb{R}^2$, together with two other parallel lines, perpendicular to the first two:

$$M = \begin{pmatrix} 1 & \infty & 2 & 2 \\ \infty & 1 & 2 & 2 \\ 2 & 2 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix}$$

The chambers here are a collection of disjoint open rectangles (which may be arranged to be squares without loss of generality here). See Figure 3.1.

(**3.1.6**) The set $H_t \setminus \mathbb{H}_{\{t\}}$ (the subset of hyperplane H_t that intersects no other hyperplane) may similarly be broken up into connected components. At most one of these components intersects any given chamber closure $\overline{C}$. If H_t intersects $\overline{C}$ in this way it is called a wall of C. For any given C, the set of ts that make up its walls functions as a choice of S in W.

The choice of a prefered chamber C_0 corresponds to the choice of a simple system in V, and the associated reflections are simple reflections. (Given a non-commuting pair of these, the conjugate of one by the other is also a reflection, but not 'simple' in this choice.)

A reflection s in W is simple for chamber B if its hyperplane H_s makes a wall of B (NB simple for B is not the same as simple, unless $B = C_0$). For our purposes it will be convenient to think specifically of the intersection of the hyperplane with the chamber closure (i.e. this facet) as the wall (thus we distinguish the walls of distinct chambers in general, even if they come from the same hyperplane).

(**3.1.7**) The reflection action of W acts to permute $\mathcal{C}_W$. This action is transitive and indeed regular (simply transitive). See for example [24, §1.12].
Note that W does *not* act transitively on V, or specifically, on the set of walls. The walls of C_0 are representatives for the W orbits of the set of all walls.

Regularity says that we may identify $\mathcal{C}_W$ with W, and the action of W with the left-action on itself. In particular write

$$A = w_A C_0 \tag{3.5}$$

(so we may indentify C_0 with 1).

Note that it follows from this identification that there is another commuting action of W on $\mathcal{C}_W$, corresponding to the right-action of W on itself.

Define a length function on $\mathcal{C}_W$: $l_W(A)$ is the number of hyperplanes separating A from C_0. (If W is clear from context we shall write simply $l = l_W$.)

(**3.1.8**) We define a digraph $G(W,S)$ with vertex set $\mathcal{C}_W$ by (A,B) an edge if $B = tA$ with t simple for A and $l(B) = l(A)+1$.

We call t the left-action label of edge (A,tA).

By (3.5) the edge (A,tA) may also be written $(w_A C_0, tw_A C_0)$. The image under w_A of a particular 'initial' edge (C_0, sC_0) $(s \in S)$ is

$$(w_A C_0, w_A s C_0) = (w_A C_0, w_A s w_A^{-1} w_A C_0) = (A, w_A s w_A^{-1} A)$$

Using the right-action this can be expressed as

$$(w_A C_0, w_A s C_0) = (w_A C_0, w_A C_0 s) = (A, As)$$

We call this s the right-action label of the edge. (With this label the graph is essentially the right Cayley graph $\Gamma(W,S)$, and s is the 'colour' label.)

Evidently $G(W,S)$ is a rooted acyclic digraph, with root C_0.

(**3.1.9**) Let $v \in C_0$, and let Wv be the W-orbit of v in V. In the same way as above we may associate a graph to this orbit. It will be evident that this graph is isomorphic to $G(W,S)$, for any such v.

3.1.2 Alcove geometry

Let (W',S') be a system containing (W,S) as a parabolic subsystem, with both acting on V. With regard to this pairing, the chambers of W' are then called alcoves. Thus the alcoves are a further subdivision of the chambers of W. Write $\mathcal{A} = \mathcal{C}_{W'}$ for the set of alcoves, and X^+ for the set of alcoves lying in C_0. Thus X^+ is a representative set for the W-orbits of $\mathcal{A}$. (In this setting we will call any $v \in C_0$ *dominant*.)

Choose C' a prefered alcove in C_0. As before, the hyperplanes bounding C' determine S' (a superset of S, by the inclusion in C_0).

The digraph $G(W',S')$ has vertex set $\mathcal{A}$, and (A,B) an edge if $B = sA$ with s simple for A and $l(B) = l(A)+1$. This is evidently a rooted acyclic digraph, with root C'. The edges are in correspondence with the set of walls, and may thus be partitioned into W'-orbits, labelled by the walls of C'.

(**3.1.10**) Let $G_a = G_a(W',W)$ denote the full subgraph of $G(W',S')$ with vertex set X^+. This is still rooted. Thus any alcove $A \in X^+$ may be reached from C' by a sequence of simple reflections, always remaining in X^+.
We shall denote the poset defined by the acyclic digraph G_a as $(X^+, <)$.
See Figure 3.2 for an example.

3.2 Partial orders, lattices and graphs

(**3.2.1**) Let G be a digraph with a countable vertex set. The adjacency matrix $M^G = A(G)$ is such that M^G_{ij} is the number of edges from i to j in G.

(**3.2.2**) Let $(S,>)$ be a poset, and $s,t \in S$. We say s *covers* t if $s > t$ and there does not exist $s > u > t$.

(**3.2.3**) The notion of cover leads to the notion of *Hasse diagram*, as for example in [**?**].

(**3.2.4**) A poset satisfies ACC (is *Noetherian*) if every ascending chain terminates.

For example, the poset of ideals, ordered by inclusion, of the ring $\mathbb{Z}$ satisfies ACC.

(**3.2.5**) We say a graph is *connected* if for any pair of vertices there is a finite chain of edges connecting them.

3.3 Young graph combinatorics

3.3.1 Nearest-neighbour graphs on $\mathbb{Z}^n$

(**3.3.1**) We define $\mathbb{Z}^n$-graph as the graph with vertex set $\mathbb{Z}^n = \mathbb{Z}\{e_1, e_2, ..., e_n\}$ and an edge whenever

$$x - x' = \pm e_i$$

This is also known as the simple hypercubical lattice in n-dimensions. Note that this is a bipartite connected graph.

(**3.3.2**) Note that the symmetric group S_n action on $\mathbb{R}^n$ given by (3.1) induces an action on $\mathbb{Z}^n$-graph. The generator (ij) acts by swapping the corresponding coordinates. Note that if there is an edge (x, x') then there is an edge $((ij)x, (ij)x')$. Hence this is a graph automorphism.
(Indeed the set of points in the linear interval between x and x' is taken to the set of points between $(ij)x$ and $(ij)x'$.)

(**3.3.3**) The orbits of the S_n action may be represented by the set of (not necessarily strictly) descending sequences.

Alternatively we may partition $\mathbb{Z}^n$ by the hyperplanes of the reflection group action $\langle (ij) \rangle_{ij}$ on $\mathbb{R}^n$. The hyperplanes partition $\mathbb{R}^n$ into chambers and facets. The strictly descending sequences lie in a single 'dominant' chamber, and all descending sequences lie in the closure of that chamber.

For example, with $n = 4$, $(1, 0, 0, 0)$ and $(0, 0, 0, -1)$ lie in the closure of the dominant chamber.

(**3.3.4**) A sequence in $\mathbb{Z}^n$ is called *regular* (or A-regular) if it is not fixed by any group element. If a sequence is regular then so is every element of its orbit. The regular orbits may be represented by the set of strictly descending sequences.

For example $(4, 3, -3, -6)$ is a representative element of a regular S_4 orbit in $\mathbb{Z}^4$.

(**3.3.5**) We define $\mathbb{Z}^n$-graph^A as the following graph. The vertices are the orbits of the S_n action on $\mathbb{Z}^n$-graph (or equivalently the integer sequences in the closure of the dominant chamber); and there is an edge $([x], [x'])$ in $\mathbb{Z}^n$-graph^A if there are representatives x, x' connected by an edge in $\mathbb{Z}^n$-graph.

(**3.3.6**) We define $\mathbb{Z}^n$-graph$^A_{reg}$ as the full subgraph of $\mathbb{Z}^n$-graph^A on the regular orbits (or equivalently on the integer sequences in the open dominant chamber).

For example $((4, 3, -3, -6), (4, 3, -2, -6))$ represents an edge in $\mathbb{Z}^n$-graph$^A_{reg}$.

We CLAIM that $\mathbb{Z}^n$-graph$^A_{reg}$ is isomorphic to the full subgraph of $\mathbb{Z}^n$-graph on vertices in the open dominant chamber.

(**3.3.7**) A walk on $\mathbb{Z}^n$-graph is A-regular if it visits only A-regular vertices.

For example $(4, 3, 2, 0) - (4, 3, 2, -1) - (5, 3, 2, -1)$ is A-regular.

3.3.2 Large n limits

(**3.3.8**) We define the 'natural' inclusion of $\mathbb{Z}^n \hookrightarrow \mathbb{Z}^{n+1}$ by $(x_1, x_2, ..., x_n) \mapsto (x_1, x_2, ..., x_n, 0)$. We extend this to a natural inclusion of $\mathbb{Z}^n$-graph.

(**3.3.9**) We define $\mathbb{Z}^f$-graph as the natural inverse limit of the graphs $\mathbb{Z}^n$-graph. This 'finitary' graph is still bipartite connected.

We define $\mathbb{Z}^{\mathbb{N}}$-graph as the 'infinite' version. For this the vertex set is all integer sequences. Note that this has infinitely many connected components. For example the vertices $(0, 0, 0, ...)$ and $(1, 1, 1., , ,)$ are not in the same component.

The connected component of $\mathbb{Z}^{\mathbb{N}}$-graph containing $\mathbb{Z}^f$-graph is the component containing $(0, 0, 0, ...)$. Note that every other connected component is isomorphic to this one.

(**3.3.10**) Remark. Consider generalising the partition of $\mathbb{Z}^n$ by orbits/hyperplanes to the case $\mathbb{Z}^{\mathbb{N}}$. Here an example of a descending sequence is $-(0, 1, 2, 3, ...)$. A sequence formally in the same orbit is $-(1, 0, 3, 2, ...)$ (transpose in pairs). Is $-(1, 2, 3, ...)$ in the same orbit (formally, an infinite cyclic shift)? There are some choices to be made about how one defines the orbits of the reflection group action (one could restrict orbits to sequences related by reflection group elements of finite length, say, so that $-(0, 1, 2, 3, ...)$ and $-(1, 0, 3, 2, ...)$ are not in the same orbit).

On the other hand, the partition into chambers generalises relatively usefully.

We note that two sequences are only in the same connected component of the graph if they are different in finitely many places. This means that if they are in the same connected component *and* in the same orbit then they are necessarily related by a *hyperfinite* group element (i.e. an element in the hyperfinite subgroup of the infinite reflection group — the inductive limit of finite group inclusions). This will be sufficient for our purposes.

(**3.3.11**) Remark. We might like to try to define $\mathbb{Z}^{\mathbb{N}}$-graph^A similarly to $\mathbb{Z}^n$-graph^A. Vertices would be integral points in the closure of the dominant chamber.

What about edges?:
The points $(0, 0, 0, ...)$ and $(-1, 0, 0, 0, ...)$ are connected in the underlying graph, but $(-1, 0, 0, 0, ...)$ does not lie in the closure of the dominant chamber. The image (y, say) of $(-1, 0, 0, 0, ...)$ in the closure of the dominant chamber is not a finitary sequence. That is, $(0, 0, 0, ...)-y$ is not polynomial. Further y is not adjacent to $(0, 0, 0, ...)$. That is, there is no $i \in \mathbb{N}$ such that $(0, 0, 0, ...) - y = \pm e_i$.

We shall not need to resolve this obstruction.

(**3.3.12**) We define $\mathbb{Z}^{\mathbb{N}}$-graph$^A_{reg}$ (similarly to $\mathbb{Z}^n$-graph$^A_{reg}$) as the full subgraph of $\mathbb{Z}^{\mathbb{N}}$-graph on vertices in the open dominant chamber.

(**3.3.13**) CLAIM: Let $2w = (1, 1, 1, ...)$ and $\rho = (0, 1, 2, ...)$.
(i) Every sequence of form $\rho_\delta = \delta w - \rho$ is regular.
(ii) The slowest integral descent (in the obvious sense) from any initial integer is of the form ρ_δ.
(iii) No two distinct ρ_δs are in the same connected component.
(iv) The connected component of $\mathbb{Z}^{\mathbb{N}}$-graph$^A_{reg}$ containing ρ_δ is isomorphic to the Young graph (as defined in (3.3.15)) for each δ. The isomorphic image of the Young graph is given by $x \mapsto x + \rho_\delta$ for each vertex x in the Young graph.

Proof. (i-iii) are clear. (iv): note that every such $x + \rho_\delta$ is strictly descending and in the connected component. On the other hand every strictly descending element in the component must differ from ρ_δ by a finitary (non-strictly) descending sequence, and hence by an integer partition. □

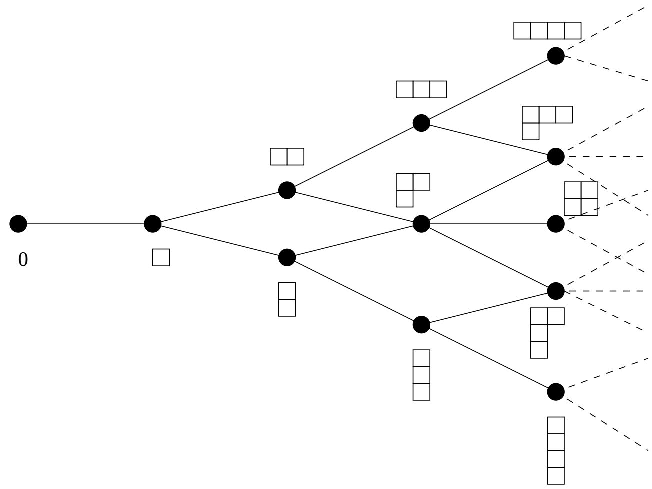


Figure 3.3: The Young graph (covering DAG of the Young lattice, increasing from left to right).

3.3.3 The Young lattice

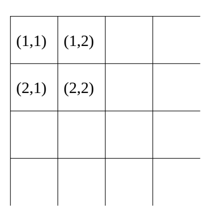
(**3.3.14**) Recall that Λ is the set of all integer partitions. The subpartition order $(\Lambda, \supset)$ is a lattice (the *Young lattice* — meet is partition intersection and join is union).
In this case λ covers μ (in the sense of (3.2.2)) if λ/μ is a single box. See Figure 8.1.

(**3.3.15**) The *Young directed graph* $\mathcal{Y}^+$ is the Hasse graph of the Young lattice. The *Young graph* $\mathcal{Y}$ is the underlying (undirected) graph of $\mathcal{Y}^+$.

(**3.3.16**) The *Young matrix* is the (semiinfinite) adjacency matrix of the underlying (undirected) graph of the Hasse graph of the Young lattice. We have

$$A(\mathcal{Y}) = A(\mathcal{Y}^+) + A(\mathcal{Y}^+)^t$$

(**3.3.17**) As usual we visualise integer partitions as Young diagrams. One first pictures the lower right quadrant of the plane partitioned into unit boxes (it is useful to label the boxes by their row and column position in this arrangement — in matrix labelling, as it were; so that the top-left box has label $(1,1)$):



The set of all these boxes is thus in bijection with $\mathbb{N}^2$.

(**3.3.18**) We associates to each such box b the rectangle of boxes $r(b)$ between it and the top-left box. We define a *light-cone order* on the set of boxes by $b \geq b'$ if $b' \in r(b)$. If $b \geq b'$ we say b *pins* b'.

For B a set of boxes, define

$$r(B) = \bigcup_{b \in B} r(b)$$

A Young diagram is a subset of boxes such that if a box is included, then every box in its rectangle is included. That is, B is a Young diagram if $B = r(B)$.

We identify a partition λ with the diagram whose i-th row has length λ_i.

(**3.3.19**) For any box b there is a minimal $\lambda \in \Lambda$ containing this box, and this coincides with $r(b)$. Given a partition μ and a box, there is a minimal $\lambda \in \Lambda$ containing both. Given a partition μ and a box b, and hence a container λ, the skew λ/μ is called the skew over μ *pinned* by b.

(**3.3.20**) Lemma. Fix a diagram μ. For a set of boxes γ to be a skew λ/μ it must not intersect μ, and must not pin any box outside $\gamma \cup \mu$.

(**3.3.21**) For each pair of Young diagrams λ, μ there is a skew diagram

$$\lambda \setminus \mu := \lambda/(\lambda \cap \mu)$$

(i.e. such that a box is in $\lambda \setminus \mu$ if it occurs in λ but not in μ); and a skew diagram

$$\lambda \Delta \mu := (\lambda \setminus \mu) \cup (\mu \setminus \lambda) = \lambda \cup \mu/(\lambda \cap \mu)$$

Chapter 4

Categories

4.1 Categories I

(**4.1.1**) A Category is a triple

$$A = (\mathrm{Ob}\,A,\ \hom_A(-,-),\ \circ)$$

(one sometimes writes simply A for $\mathrm{Ob}\,A$ and $A(-,-)$ for $\hom_A(-,-)$) where
(i) $\mathrm{Ob}\,A$ is a collection of 'objects'; [1,2]
(ii) for each ordered pair (M, N) of objects $A(M, N)$ is a set of 'morphisms';
(iii) $\circ$ is an associative composition

$$A(M, N) \times A(L, M) \to A(L, N)$$

such that for each object M there is an identity $1_M \in A(M, M)$.
(**4.1.2**) Remark. It is not uncommon to find a variant of (iii) used instead:

$$A(N, M) \times A(M, L) \to A(N, L)$$

This is ultimately just a matter of organisation (we have reversed the order of writing of all the pairs). The first formulation is natural in some settings (examples coming up), and the second formulation in others.

(**4.1.3**) Examples: Let **Set** be the collection of all sets, and for $M, N \in \mathbf{Set}$ let $\mathbf{Set}(M, N)$ be the set of maps from M to N. The usual composition of maps is associative and has identities, so this is a category.

Let **Ab** be the collection of all abelian groups and $\mathbf{Ab}(M, N)$ the set of group homomorphisms from M to N. This is a category.

Grp is the obvious extension of **Ab** to arbitrary groups.

[1](the possible failure of this collection to be a set will not concern us here [8])
[2]The notation $\mathrm{Ob}\,A$ is used, for example, in [26]; the notation $\hom_A(-,-)$ is used, for example, in [26] and in [31].

Let A be a category. Consider a triple A' consisting of any subclass of $\mathrm{Ob}\, A$; and a subset of $A(M,N)$ for each pair M,N in the subclass, such that $1_M \in A'(M,M)$, and the composition from A closes on these subsets; and the composition from A. This is a category — a *subcategory* of A. A subcategory is *full* if every $A'(M,N) = A(M,N)$.

Let $\mathbf{Set}^f$ be the full subcategory of $\mathbf{Set}$ consisting of finite sets.

For R a ring let $\mathbf{Mat}_R$ be the category of R-valued matrices. That is $\mathrm{Ob}\, \mathbf{Mat}_R = \mathbb{N}$, and $\hom_{\mathbf{Mat}_R}(m,n)$ is the set of $m \times n$ matrices, and composition is matrix multiplication. [1]

For R a ring R-mod is the category of left R-modules and their homomorphisms.

(**4.1.4**) Proposition. *Let A be a category and N an object in A. Then the full subcategory of A induced on the single object N consists essentially in the set $A(N,N)$ with its unital associative composition, and hence is a monoid.*

Conversely any monoid is a category on (essentially) any single object.

(**4.1.5**) Example. The monoid $\mathbf{Set}(\underline{2},\underline{2}) = \underline{2}^{\underline{2}}$ is the one studied in Section 2.3.1.

(**4.1.6**) Given a category A there is a Dual category (or opposite category) A^o which has the same objects, and $A^o(M,N) = A(N,M)$, and composition is reversed.

Note that a category and its dual can be very different. For example, $\mathbf{Set}(S,\emptyset)$ is empty unless $S = \emptyset$, while $\mathbf{Set}(\emptyset,S) = \mathbf{Set}^\circ(S,\emptyset)$ contains precisely one element for each S (the appropriate empty relation).

(**4.1.7**) Remarks: In a given category we may write

$$M \xrightarrow{\theta} N$$

for $\theta \in A(M,N)$. Thus a (small) category is a directed graph with some extra data. To say that a triangle of such homs/arrows

$$\begin{tikzcd} L \arrow[rr, "\mu"'] \arrow[dr, "\phi"'] & & P \\ & M \arrow[ur, "\pi"'] & \end{tikzcd}$$

commutes is to say that the long arrow (in the obvious sense) is the composite of the shorter ones. Then associativity says that commutativity of any three of the triangles here:

$$\begin{tikzcd} L \arrow[rrr, "\mu"'] \arrow[dr, "\phi"'] \arrow[drr] & & & P \\ & M \arrow[r, "\theta"'] \arrow[urr] & N \arrow[ur, "\psi"'] & \end{tikzcd}$$

implies commutativity of the fourth.

(**4.1.8**) A morphism $f \in A(M,N)$ is an Isomorphism if there exists $g \in A(N,M)$ such that $gf = 1_M$ and $fg = 1_N$.

Example: The isomorphisms in $\hom_{\mathbf{Set}}(\underline{n},\underline{n})$ form a submonoid which is a subgroup — the symmetric group S_n.

(**4.1.9**) Given a set $\{C_i\}_{i\in I}$ of categories, the triple $\times_i C_i$ consisting of object class $\times_i \mathrm{Ob}\, C_i$, I-tuples of morphisms, and the corresponding pointwise composition, is a category.

(**4.1.10**) Exercise. Show that

$$\mathbf{Br} = (\mathbb{N}, \mathbf{Br}(-,-), \circ)$$

where $\mathbf{Br}(m,n) = J_{m,n}$ (from (2.2.10)), is a category.

4.1.1 Functors

(**4.1.11**) Concrete Category. If there is a map und : $A \to \mathbf{Set}$ from the object collection of category A to $\mathbf{Set}$, such that $A(M,N) \subseteq \mathbf{Set}(\mathrm{und}(M), \mathrm{und}(N))$ for each $M, N \in A$, with $1_M = 1_{\mathrm{und}(M)}$, and composition is the usual composition of maps, then A is a concrete category.

Examples: **Ab**, **Grp** are a concrete categories for which 'und' is simply inclusion. **Br** is not a concrete category by inclusion (indeed its objects are not sets).

(**4.1.12**) For A, B categories, a (covariant) Functor $F : A \to B$ is a map on objects together with a map on morphisms which preserves composition and identities.

A Contravariant functor from A to B is a functor from A^o to B (examples later).

Examples: As noted, **Grp** is concrete. Thus there is an 'und' functor $U : \mathbf{Grp} \to \mathbf{Set}$. Indeed for each $n \in \mathbb{N}$ there is a 'pointwise' functor $D_n : \mathbf{Grp} \to \mathbf{Set}$ given by

$$D_n(A \xrightarrow{f} B) = A^n \xrightarrow{f^n} B^n$$

(**4.1.13**) More examples: (The following simple examples will come up again later, when we develop the notions of *natural transformation* and of *adjoint pairs* of functors.) Let S be a set. Then there is a functor $F_S : \mathbf{Set} \to \mathbf{Set}$ given by $F_S(T) = T \times S$ and

$$F_S(T \xrightarrow{f} T')(t,s) = (f(t), s)$$

And a functor $F^S : \mathbf{Set} \to \mathbf{Set}$ given by $F^S(T) = \hom(S,T)$ and $F^S(f) : g \mapsto f \circ g$.

(**4.1.14**) Exercise. Since $F^S(T) = \hom(S,T)$ is a set as well as a hom set, we may consider the hom set $\hom(U, F^S T) = \hom(U, \hom(S,T))$ in **Set**. Let U', T' be two further sets. A pair of maps $u' : U' \to U$ and $t : T \to T'$ define a map from $\hom(U, F^S T)$ to $\hom(U', F^S T')$ by

$$g \mapsto (u, F^S t)(g) = (F^S t) \circ g \circ u'$$

(note the direction of the map u'!).
Show that this gives rise to a functor

$$\hom(-, F^S -) : \mathbf{Set}^o \times \mathbf{Set} \to \mathbf{Set}$$

(cf. (4.1.9)), and construct an analogous functor

$$\hom(F_S -, -) : \mathbf{Set}^o \times \mathbf{Set} \to \mathbf{Set}$$

(Remark: Functors from products are sometimes called bifunctors.)
Answer to last part: the map from $\hom(F_S U, T)$ to $\hom(F_S U', T')$ is given by $f \mapsto (F_S u', t)(f) = t \circ f \circ F_S u'$.

(**4.1.15**) A *forgetful functor* is a functor to a category whose objects have some structure (binary operation; inverses; etc) from a category whose objects have this and additional structure. The functor simply forgets the additional structure.

Our 'und' functors are examples of forgetful functors. Another example would be the functor from **Fld** (the category of fields) to the category of integral domains and injective ring maps (call it C), inside the category **Rng** of rings.
(The restriction to injective maps is just because every field homomorphism is injective.)

4.1.2 Natural transformations

(**4.1.16**) Let A, B be categories, and T, S be functors from A to B. A 'natural transformation' $a : T \to S$ is a family $a = (a_M)_{M \in A}$ of B–morphisms

$$a_M : TM \to SM$$

such that for each $f \in A(M, N)$ we have $Sfa_M = a_N Tf$.

Example: The functors from A to B are the objects of a category B^A with morphisms the (set of!) natural transformations.

(**4.1.17**) As we have noted, a group is an example of an algebraic system — one with a binary operation. That is, for each group, group multiplication is a function

$$\kappa_G : G \times G \to G$$

The collection of all group multiplications $\kappa = (\kappa_G)$ is thus a candidate to be a natural transformation κ from D_2 to U (the functors **Grp** $\to$ **Set** defined in (4.1.12)):

$$\begin{tikzcd} G \times G \arrow[r, "\kappa_G"'] \arrow[d, "f^2"] & G \arrow[d, "f"'] \\ G' \times G' \arrow[r, "\kappa_{G'}"'] & G' \end{tikzcd}$$

and the commutativity condition $Sfa_M = a_N Tf$ is $Uf\kappa_G(a, b) = \kappa_{G'}(D_2 f)(a, b)$ which is simply

$$f(ab) = f(a)f(b)$$

That is, group multiplication (collectively) is a natural transformation.
Other operations in categories of algebraic systems are viewable as natural transformations similarly.

(**4.1.18**) Example. Recall the functors F_S, F^S from (4.1.13), and $\hom(-, F^S-)$, $\hom(F_S-, -)$ from (4.1.14). Let $x \in \hom(F_S V, U)$. For each such we can define an element $\psi x \in \hom(V, F^S U) = \hom(V, \hom(S, U))$ by $(\psi x)(v)(s) = x((v, s)) \in U$. On the other hand, for $y \in \hom(V, F^S U)$ we define $\psi' y \in \hom(F_S V, U) = \hom(V \times S, U)$ by $(\psi' y)(v, s) = (y(v))(s)$.

Comparing with (4.1.14) one finds that ψ and ψ' are natural transformations between the functors $\hom(-, F^S-), \hom(F_S-, -) : \mathbf{Set}^o \times \mathbf{Set} \to \mathbf{Set}$. For example, for each object (V, U) in $\mathbf{Set}^o \times \mathbf{Set}$ we have $\psi_{U,V}$ such that the diagram

$$\begin{tikzcd} \hom(F_S V, U) \arrow[r, "\psi_{U,V}"] \arrow[d] & \hom(V, F^S U) \arrow[d] \\ \hom(F_S V', U') \arrow[r, "\psi_{U',V'}"] & \hom(V', F^S U') \end{tikzcd}$$

commutes for vertical maps built from any $(f, g) = (V' \xrightarrow{f} V, U \xrightarrow{g} U') \in \hom_{\mathbf{Set}^o \times \mathbf{Set}}((V, U), (V', U')))$. To see this note that going to the right first we have

$$((\hom(-, F^S-)(f, g))\psi_{U,V})(V \times S \xrightarrow{x} U) = (\hom(-, F^S-)(f, g))(V \xrightarrow{\psi x} \hom(S, U))$$
$$= (V' \xrightarrow{f} V \xrightarrow{\psi x} \hom(S, U) \xrightarrow{F^S g} \hom(S, U'))$$

so this way round the image of x is a map in which $v' \in V'$ is taken to a map which takes s in S to $g(x(f(v'), s))$. The other way round

$$(\psi_{U',V'}(\hom(F_S-, -)(f, g)))(V \times S \xrightarrow{x} U) = (\psi_{U',V'})((V' \times S \xrightarrow{f \otimes 1} V \times S \xrightarrow{x} U \xrightarrow{g} U')$$

which eventually gives the same thing.

4.2 R-linear and ab-categories

(**4.2.1**) Let R be a commutative ring. An R-linear category is a category in which each hom set is an R-module, and the composition map is bilinear.

A basis for an R-linear category C is a subset $\hom^o_C$ of $\hom_C$ such that

$$\hom^o_C(m, n) = \hom^o_C \cap \hom_C(m, n)$$

is a basis for $\hom_C(m, n)$.

Any category C extends R-linearly to an R-linear category RC.

(**4.2.2**) If C is an R-linear category then each $\hom_C(m, m)$ is an R-algebra.

(**4.2.3**) Remark. A good working aim for this course is to compute the dimensions of the irreducible modules for the $\mathbb{C}$-algebras contained in $\mathbb{C}\mathbf{Br}$ (as defined in Exercise (4.1.10)).

(**4.2.4**) A category C is called an *ab-category* if there is a + operation on each $\hom_C(A, B)$ making it an abelian group; and morphism composition distributes over +:

$$f(g + h) = fg + fh \qquad \text{and} \qquad (g + h)f = gf + hf$$

(**4.2.5**) Example: We can define a + for any $\hom_{\mathbf{Ab}}(A, B)$ pointwise:

$$(g + h)(a) = g(a) + h(a)$$

(this defines an element of $\hom_{\mathbf{Set}}(A, B)$, but $(g + h)(a + b) = g(a + b) + h(a + b) = g(a) + g(b) + h(a) + h(b) = (g + h)(a) + (g + h)(b)$, so $(g + h) \in \hom_{\mathbf{Ab}}(A, B)$ as required).
Thus **Ab** is an ab-category.

(**4.2.6**) A functor $F : A \to B$ between ab-categories is *additive* if for $f, g \in \hom_A(X, Y)$:

$$F(f + g) = F(f) + F(g)$$

(**4.2.7**) If there is an object 0 in a category C such that $|\hom_C(0, A)| = |\hom_C(A, 0)| = 1$ for all A then 0 is called a *zero object*.

(**4.2.8**) Consider L, M, N objects in an ab-category C. If there are morphisms $a : L \to N$, $a' : N \to L$, $b : M \to N$, $b' : N \to M$ such that $a'a = 1_L$, $b'b = 1_M$ and

$$aa' + bb' = 1_N$$

then we write $N \cong L \oplus M$.
If there is an object $N \cong L \oplus M$ for any two objects L, M we say C *has direct sums*.

(**4.2.9**) An *additive category* is an ab-category with direct sums and zero object.

Example: **Ab** with the trivial group as zero object.

4.2.1 Abelian categories

See for example Freyd's 1964 book [21]. Abelian categories can be regarded as abstractions of the class of module categories, and so are useful in representation theory.

(**4.2.10**) An additive category A is an *abelian category* if
(I) every $f \in \hom_A(M, N)$ has a kernel and a cokernel.
(II) every monomorphism is a kernel; every epimorphism is a cokernel.

4.3 Categories II

(**4.3.1**) A functor $F : A \to B$ is:
full (respectively *faithful*) if all hom set maps

$$F : \hom_A(S, T) \to \hom_B(FS, FT)$$

are surjective (respectively injective);
isomorphism dense if for every object T in B there is an object S in A such that $F(S)$ is isomorphic to T.

(**4.3.2**) A skeleton for a category is a full isomorphism dense subcategory in which no two objects are isomorphic.

(**4.3.3**) Example. The assembly of sets in $\mathbf{Set}^f$ into cardinality classes induces a corresponding set of isomorphisms between hom sets

$$\begin{aligned} f_S : S &\xrightarrow{\sim} S' && (4.1)\\ f : \hom(S,T) &\to \hom(S',T') && (4.2)\\ g &\mapsto f_T \circ g \circ f_S^{-1} && (4.3)\end{aligned}$$

Associate a representative element of each class to each cardinality. We may then construct a category $C_{\mathbb{N}}$ whose objects are the set $\mathbb{N}$ of finite cardinalities, and with $\hom_{C_{\mathbb{N}}}(m,n) = \hom(\underline{m},\underline{n})$. The functor

$$F : C_{\mathbb{N}} \to \mathbf{Set}^f$$

which takes object n to object $\underline{n}$ and identifies the corresponding hom sets is isomorphism dense and full. This $C_{\mathbb{N}}$ is thus a subcategory of $\mathbf{Set}^f$, from which the rest of $\mathbf{Set}^f$ can easily be constructed. We have:

(**4.3.4**) Proposition. *This $C_{\mathbb{N}}$ is a skeleton for $\mathbf{Set}^f$.*

(**4.3.5**) Note that the set of isomorphisms in an end set form a group. The set of isomorphisms in $\hom(\underline{n},\underline{n})$ form the symmetric group S_n.

(**4.3.6**) A *congruence relation* I on a category C is an equivalence relation on each hom set such that $f' \in [f]_I$ and $g' \in [g]_I$ implies $f'g' \in [fg]_I$ (compositions of morphisms).
The quotient category C/I has the same object class as C but $\hom_{C/I}(F,G) = \hom_C(F,G)/I$, with the obvious composition well-defined by congruence.

4.3.1 Adjunctions

(**4.3.7**) An *adjunction* between categories A, B is a pair of functors $F : A \to B$ and $G : B \to A$ such that for all objects (U,V) in $A \times B$ there is a bijection

$$\psi_{U,V} : \hom_A(GV,U) \to \hom_B(V,FU)$$

such that

$$\psi : \hom_A(G-,-) \to \hom_B(-,F-)$$

is a natural isomorphism of bifunctors.

That is, we have

$$\begin{tikzcd} \hom_A(GV,U) \arrow[r, "\psi_{U,V}"] \arrow[d] & \hom_B(V,FU) \arrow[d] \\ \hom_A(GV',U') \arrow[r, "\psi_{U',V'}"] & \hom_B(V',FU') \end{tikzcd}$$

commutative for each $f \in \hom_{A^o\times B}(V,U)$ (and hence each pair of vertical maps, cf. (4.1.16)).

(**4.3.8**) Example. Recall the functors F_S, F^S from (4.1.13). Let $x \in \hom(F_S V, U)$. For each such we can define an element $\psi x \in \hom(V, F^S U) = \hom(V, \hom(S,U))$ by $(\psi x)(v)(s) = x((v,s)) \in U$.

On the other hand, for $y \in \hom(V, F^S U)$ we define $\psi' y \in \hom(F_S V, U) = \hom(V \times S, U)$ by $(\psi' y)(v, s) = (y(v))(s)$.

Comparing with (4.1.14) one checks that ψ and ψ' are natural transformations (the diagram above commutes for vertical maps built from $\hom_{\mathbf{Set}^o \times \mathbf{Set}}(V, U)$) and hence isomorphisms. Thus (F_S, F^S) is an adjunction.

(**4.3.9**) The left adjoint to a forgetful functor is usually something interesting!

4.4 Categories III

4.4.1 Tensor/monoidal categories

See Section 9.4. See also Joyal–Street [**?**], Kassel [30], Reshetikhin–Turaev [37].

Let A be a category and $A \times A$ the product category as in (4.1.9). Whenever we have a functor $F : A \times A \to A$ we have in particular an association of an object $F(m, n)$ to each pair of objects. If this binary operation is associative and unital (so that an object set becomes a monoid) then (A, F) is a *strict tensor category*. If the binary operation is associative and unital up to (certain suitable) natural isomorphisms

$$a_{LMN} : F(F(L, M), N) \to F(L, F(M, N))$$

$$l_M : F(1, M) \to M$$

$$r_M : F(M, 1) \to M$$

(see later for axioms) then $(A, F) = (A, F, 1, a, l, r)$ is a *tensor category*.

Suppose there are additional natural isomorphisms

$$g_{LM} : F(L, M) \to F(M, L)$$

Then we can reorder and move brackets in any expression of form $F(M_1, F(F(M_2, F(M_3, M_4)), M_5))$ by applying suitable a_{LMN} and g_{LM}s. Suppose we associate such a manipulation to an element of the braid group by associating each $g_{M_i M_j}$ to a braiding in that position. If the manipulation morphism depends only on the associated braiding, then the tensor category A together with (the collection) g is a *braided tensor category*.

A natural example is the category of modules of a finite group algebra, where $F(M, N) = M \otimes N$. (Indeed later we will write $F(M, N)$ as $M \otimes N$ quite generally.)

Chapter 5

Rings

5.1 Rings I

(**5.1.1**) Recall a ring R is a set with two laws of composition such that
$(R, +, 0)$ is a commutative group with identity 0
$(R, ., 1)$ is a monoid with identity 1
$r(s+t) = rs + rt, \qquad (s+t)r = sr + tr$

For example, $\mathbb{Z}$ is a ring.

(**5.1.2**) Example. For each abelian group A the set $\hom_{\mathbf{Ab}}(A, A)$ of endomorphisms is of course a monoid. But defining pointwise addition of endomorphisms by

$$(f+g)(a) = f(a) + g(a)$$

(a special case of (4.2.5)) one finds that it is also an abelian group under addition. It is straightforward to check that distributivity holds, so $\hom_{\mathbf{Ab}}(A, A)$ becomes a ring.

(**5.1.3**) Example. If R is a ring then

$$M_2(R) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in R\}$$

is a ring. Similarly $M_n(R)$ and

$$\mathrm{Tri}_2(R) = \{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in R\} \qquad \text{and} \qquad \mathrm{Tri}'_2(R) = \{ t_{a,b} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in R\}$$

(**5.1.4**) Note that $t_{a,b}t_{c,d} = t_{ac,ad+bc}$ in $\mathrm{Tri}'_2(\mathbb{C})$. Thus $t_{a,b}$ is a unit in $\mathrm{Tri}'_2(\mathbb{C})$ if and only if $a \neq 0$. Thus every nonunit takes the form $t_{0,b}$, and since $t_{0,b} + t_{0,c} = t_{0,b+c}$ we have that $\mathrm{Tri}'_2(\mathbb{C})$ is a local ring. On the other hand $\mathrm{Tri}_2(R)$ is not a local ring for any R (since the idempotent elementary matrices are nonunits that sum to 1).

(**5.1.5**) Next we consider examples based broadly on the idea of a pair of rings related by an injective ring homomorphism: $\psi : R \hookrightarrow S$. Then S can be seen as an 'extension' of the image of R. One can, for example, ask for the smallest subring of S that contains R and some $s \in S$.

(**5.1.6**) Example. For R a ring, the set $R[X]$ is the subset of $R^{\mathbb{N}_0}$ of elements $r = (r_0, r_1, r_2, ...)$ such that only finitely many of the r_i are non-zero. This set closes under pointwise addition and 'Cauchy' multiplication, and indeed these operations make this set a ring. A convenient representation of this is

$$r = \sum_{i=0}^{n_r} r_i X^i$$

(where n_r is the index on the last nonzero entry) whereupon the ring operations become 'polynomial' arithmetic, that is, X is treated as if a central extension of R.
For this reason $R[X]$ is often called the ring of polynomials in indeterminate X over R.

Another way to think about this is that X acts as a shift operator $X : R[X] \to R[X]$: $(X(r))_0 = 0$ and otherwise

$$(X(r))_i = r_{i-1}$$

Most of the time we shall be interested in the case where R is commutative, so that $R[X]$ is also commutative. See for example (5.1.8).

(**5.1.7**) For R a ring, the ring $R[X_1, ..., X_n] := R[X_1, ..., X_{n-1}][X_n]$. Similarly to the above, this is thought of as the ring of polynomials in n (commuting) indeterminates. By this construction each X_i is a central extension, and in particular all the X_is commute with each other. (One could also consider extensions that do not commute with each other. For example one could consider combining two extensions, each separately central, of a commutative ring, but where the extensions do not commute. This non-commutation could be 'free'; or determined, such as in extensions by X and d/dX. However we shall not introduce notation for this here.)

(**5.1.8**) For S commutative and $x_1, ..., x_n \in S \supseteq R$ we write $R[x_1, ..., x_n]$ for the ring of polynomials in $x_1, ..., x_n$.

5.1.1 Properties of elements of a ring

(**5.1.9**) By $\epsilon_{ij} \in M_n(R)$ we mean the *elementary matrix* which is zero in every position except $(\epsilon_{ij})_{ij} = 1$.

(**5.1.10**) Idempotents. An element $e \in R$ is an *idempotent* if $ee = e$.

For example $\epsilon_{11}\epsilon_{11} = \epsilon_{11}$ in $M_n(R)$ (any R). Indeed in $M_2(R)$ we have

$$\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} \qquad \text{(any } r \in R)$$

$$\begin{pmatrix} 0 & r' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & r' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & r' \\ 0 & 1 \end{pmatrix} \qquad \text{(any } r \in R)$$

(**5.1.11**) $x \in R$ is:
a *unit* if it has an inverse.
irreducible if R is commutative and x has no proper factorisation into non-units.

(**5.1.12**) Let R be commutative, then $x \in R \supset S$ is:
algebraic if there exists $p(X) \in S[X]^*$ such that $p(x) = 0$.
integral if there exists p as above, monic.

(**5.1.13**) Lemma. Let R be commutative. Ring $R[x] \cong R[X]$ if x is *transcendental* (i.e. not algebraic).

(**5.1.14**) Let us unpack the notion of integrality a little. Let $A \subset B$ be rings and $x \in B$. If there exists a subset $\{a_0, ..., a_n\}$ of A such that

$$x^{n+1} = \sum_i a_i x^i$$

we say x is *integral over* A. This is to say, there is a monic polynomial $f = X^{n+1} - \sum_i a_i X^i$ in the polynomial ring $A[X]$ such that the evaluation image $f(x) = 0$ in B.
If there is a not-necessarily monic polynomial $g \in A[X]$ such that $g(x) = 0$ we say x is *algebraic* over A. (If x is not algebraic it is *transcendental*.)
Example: $a \in A$ is integral over A, since $a = aa^0$.
Example: $\sqrt{2} \in \mathbb{R}$ is integral over $\mathbb{Z}$ since $\sqrt{2}^2 = 2\sqrt{2}^0$.
Example: with $n, m \in \mathbb{N}$, then $\sqrt{n} + m \in \mathbb{R}$ is integral over $\mathbb{Z}$ since $(\sqrt{n} + m)^2 = 2m(\sqrt{n} + m)^1 + (n - m^2)(\sqrt{n} + m)^0$.

(**5.1.15**) Suppose x is integral over A, via polynomial f as above. Then $\{x^0, x, ..., x^n\}$ contains a basis of $A[x]$, so the ring $A[x]$ is a finite A-module.
In fact the converse holds (see e.g. [**?**, V,1,255]): if $A[x]$ is a finite A-module then x is integral over A.

(**5.1.16**) Suppose r, s integral over A. Then there are appropriately vanishing monic polynomials, f_r, f_s say. Using (5.1.15) we can show that $r + s$ is also integral over A.

The set of elements of B that are integral over A is a ring, called the *integral closure* of A in B. If every $b \in B$ is integral over A then say B is *integrally dependent* on A.

(**5.1.17**) Remark. See [**?**, §1A] for the case of B an A-algebra (so A is commutative but B is not necessarily so).

5.2 Ideals

(**5.2.1**) A subgroup I of R is an *ideal* if $rI \subseteq I$ and $Ir \subseteq I$ for all $r \in R$.

More generally, a (left/right) ideal I of a ring R is an additive subgroup closed under (left/right) multiplication by R.

For example, RrR is an ideal for each $r \in R$. Further

$$RrsR \subseteq RrR \subseteq R$$

$(r, s \in R)$ is a nest of ideals. In particular $2\mathbb{Z}$ is an ideal of $\mathbb{Z}$, and indeed mZ is an ideal for each $m \in Z$.

Note that the left ideal Rr can only be proper if r a nonunit (else $Rr \supseteq Rr^{-1}r = R1 = R$), and that every element of a proper left ideal is a nonunit. However the nonunits do not form an ideal in general, since the sum of two nonunits may be a unit. Since this does not happen in a local ring, the set of nonunits is a proper left ideal in a local ring. This left ideal is clearly maximal (the only remaining elements of the ring are units) and indeed contains every other left ideal. Thus a local ring has a unique maximal left ideal. (Or right ideal, by the same argument.)

(**5.2.2**) Note that

$$U_2(R) = \{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R \}$$

is a ring without 1. It is an ideal of $\mathrm{Tri}_2(R)$ contained in $\mathrm{Tri}'_2(R)$, and hence also an ideal of $\mathrm{Tri}'_2(R)$.

Neither $\mathrm{Tri}_2(R)$ nor $U_2(R)$ is an ideal of $M_2(R)$. The set

$$C_2(R) = \{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b, c \in R \}$$

is a left ideal of $M_2(R)$ but not an ideal or right ideal.

(**5.2.3**) Lemma. For R a ring, the only ideals of $M_n(R)$ are the subsets $M_n(I)$ for I an ideal of R.

(**5.2.4**) A ring homomorphism is a map

$$\phi : R \to S$$

such that $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in R$.

(**5.2.5**) Lemma. For $\phi : R \to S$ a ring homomorphism, $\ker(\phi)$ is an ideal of R.

(**5.2.6**) Let I be an ideal of R. Then the set R/I of cosets of I in R form a ring, and

$$\phi_I : x \mapsto I + x$$

is a ring epimorphism

$$\phi_I : R \to R/I$$

with $\ker(\phi_I) = I$.

(**5.2.7**) Example. We have

$$\mathrm{Tri}_2(\mathbb{Q})/U_2(\mathbb{Q}) \cong D_2(\mathbb{Q})$$

the ring of diagonal matrices.

5.2.1 Posets

See also §3.2. Recall:

(**5.2.8**) A poset satisfies ACC (is *Noetherian*) if every ascending chain terminates.

For example, the poset of ideals, ordered by inclusion, of the ring $\mathbb{Z}$ satisfies ACC.

5.2.2 Properties of ideals: Artinian and Noetherian rings

(**5.2.9**) A proper ideal I of a ring R is *maximal* if there is no proper ideal $J \supset I$.

For example, $p\mathbb{Z}$ is maximal iff p is prime.
Similarly $U_2(\mathbb{C})$ is a maximal ideal of $T_2'(\mathbb{C})$. (Any extension of $U_2(\mathbb{C})$ would have to be by an element with non-zero diagonal, but this would then be the whole of $T_2'(\mathbb{C})$.) Indeed $U_2(\mathbb{C})$ is the unique maximal ideal of $T_2'(\mathbb{C})$.

(**5.2.10**) Lemma. Every commutative ring with $1 \neq 0$ has a maximal ideal.

Proof. Consider the poset P of proper ideals in R. Consider a chain T in P, and let I be the union of all ideals in T. This is (evidently) an ideal, and does not contain 1, so $I \in P$. *Zorn's Lemma* states that a poset in which every chain has an upper bound ($u \in P$ such that $u \geq t$ for all $t \in T$) has a maximal element ($m \in P$ such that $\nexists x \in P$, $x > m$). Thus our P has a maximal element.

(**5.2.11**) A ring is *left-Noetherian* if the poset of left ideals satisfies ACC.
(Similarly right-Noetherian / Noetherian.)
A ring is *left-Artinian* if the poset of left ideals satisfies DCC. (Similarly right-Artinian / Artinian.)

Thus any field is Noetherian; $\mathbb{Z}$ is Neotherian; $\mathbb{Z}[x]$ is Neotherian.
$\mathbb{Z}[x_1, x_2, ...]$ is not Neotherian, since $\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset ...$

The set F of continuous functions $f : \mathbb{R} \to \mathbb{R}$ is a ring by pointwise operations. For each subset $S \subseteq \mathbb{R}$ the subset F_S of functions with $\ker f = S$ is an ideal. Each chain $S_1 \supset S_2 \supset S_3 \supset ...$ induces a chain of ideals $F_{S_1} \subset F_{S_2} \subset F_{S_3} \subset ...$ (ascending), so F is not Noetherian.

(**5.2.12**) Theorem. [Hopkins-Levitzki] A left(right)-Artinian ring R is left(right)-Noetherian. (Indeed if R is Artinian then any finitely generated R-module is Artinian and Noetherian.)

5.2.3 Properties of ideals: Integral domains

(**5.2.13**) An ideal I in R is *prime* if $xRy \subset I$ implies that at least one of x, y lies in I.
This is the same as to say that an ideal I is prime if, whenever it contains a product of ideals HJ then at least one of H, J is contained in I.

Example: (i) $p\mathbb{Z}$ is prime iff p is prime.
(ii) $I = 2x\mathbb{Z}[x] = 2\mathbb{Z}[x]x$ is not prime, since neither of $2, x \in I$.

(**5.2.14**) The *field of fractions* of an integral domain R is the quotient of $R \times (R \setminus \{0\})$ by $(r, s) \sim (t, u)$ if $ru = st$. Addition is represented by $(r, s) + (t, u) = (ru + ts, su)$ and multiplication by $(r, s)(t, u) = (rt, su)$.

Example: The field of fractions of $\mathbb{Z}$ is $\mathbb{Q}$.

(**5.2.15**) Suppose I is an ideal of integral domain R, and $\mathbb{Q}_R$ the field of fractions. Embedding $R \hookrightarrow \mathbb{Q}_R$ by $r \mapsto (r, 1)$, we write

$$I^{-1} := \{q \in \mathbb{Q}_R \mid qI \subseteq R\}$$

Example: $(p\mathbb{Z})^{-1} = \{(r, p^i) \mid r \in R;\ i \in \{0, 1\}\}$
We say I is invertible if $II^{-1} = R \subset \mathbb{Q}_R$.

(**5.2.16**) Let R be an integral domain and I a prime ideal. Then the subset of the field of fractions with representatives of form (r, s) with $s \notin I$ is a subring.
This subring, denoted R_I, is the *localisation* of R at I.

(**5.2.17**) An integral domain in which every ideal can be written as a product of prime ideals (with those prime factors unique up to ordering) is a *Dedekind domain.*

(**5.2.18**) Proposition. *An integral domain is a Dedekind domain iff every ideal is invertible.*

(**5.2.19**) An integral domain in which every non-unit has a finite irreducible factorisation, that is unique up to order and units, is a UFD.
An integral domain with a division algorithm is a Euclidean domain.

(**5.2.20**) We can summarize the commutative part of the story so far:
commutative rings $\supset$ integral domains $\supset$ UFDs/Dedekind domains $\supset$ PIDs $\supset$ Euclidean domains $\supset$ fields.

5.3 Rings II

In this Section we want to construct rings with certain special properties that will be useful later. We start with the notion of valuation on a ring, from which we build absolute value. This gives a metric, and hence a notion of Cauchy sequences, and hence of completion (generalising the completion of $\mathbb{Q}$ to $\mathbb{R}$).

5.3.1 Order and valuation

Let $(S, \geq)$ be a poset. A lower bound of $T \subset S$ is an element $b \in S$ such that $b \leq t \; \forall t \in T$. A greatest lower bound of $T \subset S$ is an element $b \in S$ such that b is a lower bound and for each lower bound c we have $b \geq c$.

(**5.3.1**) Ordered ring (respectively domain, field) R, R a ring, $0 \neq 1$, $\exists$ nonempty $P \subset R$ such that $a, b \in P$ implies $a + b, ab \in P$, $0 \notin P$, $\forall a \in P$ either $a \in P$ or $(-a) \in P$.
In an ordered ring we write $a > b$ if $a + (-b) \in P$ ('positive').

(**5.3.2**) Complete ordered domain D, D an ordered domain, every nonempty $S \subset P$ has GLB in D.
(Counter)Example: $\mathbb{Q}$ is an ordered field with $x \in P$ if $x > 0$, but $\mathbb{Q}$ is not complete. (Consider any sequence of rational approximations to $\sqrt{2}$ which approaches from above — any *rational* $r \leq \sqrt{2}$ would be a LB. But a GLB g would have to be rational and equal to $\sqrt{2}$ — a contradiction! — else a rational $O < \epsilon < \sqrt{2} - g$ could be added to g contradicting 'greatest'.)

(**5.3.3**) Valuation. For D an integral domain and $(\Gamma, +, 0)$ a totally ordered group a *valuation* on D is a map $v : D \to \Gamma \cup \{\infty\}$ (here $\infty + g = \infty$ and $\infty > g$ for all $g \in \Gamma$) such that $v(0) = \infty$, $v(a) < \infty$ if $a \neq 0$, $v(ab) = v(a) + v(b)$, $v(a + b) \geq \min(v(a), v(b))$.

(**5.3.4**) The set $R = \{x \in D | v(x) \geq 0\}$ is a subring of D called the Valuation ring.
Note in particular that $v(1) = v(1.1) = v(1) + v(1)$, so $v(1) = 0$.
Also, if $d \in D$ is invertible then $v(d) + v(d^{-1}) = v(d.d^{-1}) = v(1) = 0$, so either $v(d) = v(d^{-1}) = 0$ or precisely *one* of d, d^{-1} lies in the valuation ring.

(**5.3.5**) Examples: (i) $\mathbb{Q} \ni x = \pm p^{\mu} \frac{r}{s}$ with p, r, s coprime, p prime, $\mu(0) = \infty$, defines $\mu = \mu(x)$ uniquely. This is the p–ADIC VALUATION on $\mathbb{Q}$.
Check: Consider $p = 5$, so $\mu(99/100) = -2$ and $\mu(15/27) = 1$. We have $\mu(\frac{m}{n} + \frac{r}{s}) = \mu(\frac{ms+nr}{ns})$ and with $m = 99$, $n = 100$, $r = 15$, $s = 27$ we have $\mu(\frac{4173}{2700}) = -2$ which obeys $-2 \geq \min(-2, 1)$.
(N.B., The feature illustrated in this example is generally true: if $\mu(a) \neq \mu(b)$ then $\mu(a+b) = \min(\mu(a), \mu(b))$.)
(ii) Let K be a field and $K(X)$ the field of fractions of the polynomial ring $K[X]$. Then $K(X) \ni x = p(X)^{\mu} \frac{f(X)}{g(X)}$ with f, g, p coprime and p irreducible, and $\mu(0) = \infty$, defines $\mu = \mu(x)$ uniquely.
(iii) (More generally) Let K be the field of fractions of a Dedekind domain R, and P a maximal ideal of R. For $x \in K$ there is a factorisation of Rx into a product of prime ideal powers (some possibly negative in the sense of (5.2.15); some possibly repeated). Let $\mu(x)$ be the power to which P appears, and $\mu(0) = \infty$.

(**5.3.6**) PROPOSITION. *A subring R of a field F is a valuation ring in F iff $x \in F \setminus \{0\}$ implies [either $x \in R$ or $x^{-1} \in R$].*

In particular a field F with a valuation v is an ordered field (as in (5.3.1)) with set P equated to the valuation ring.
If $\Gamma = \mathbb{R}$ a valuation is REAL VALUED, and if $\Gamma = \mathbb{Z}$ a valuation is PRINCIPAL or DISCRETE RANK 1.

(**5.3.7**) NON-ARCHIMEDEAN ABSOLUTE VALUE. For R a commutative ring a non–archimedian absolute value on R is a real–valued function $x \mapsto |x|$ such that $|x+y| \leq \max(|x|, |y|)$ (ultrametric inequality), $|xy| = |x|.|y|$ and $|x| \geq 0$ (saturated iff $x = 0$).
Example: Let K be a field with real–valued valuation v then

$$|x| = 2^{-v(x)}$$

defines a non–archimedean absolute value on K.
Check: $|xy| = 2^{-v(xy)} = 2^{-(v(x)+v(y))} = 2^{-v(x)} 2^{-v(y)} = |x||y|$;
$|x+y| = 2^{-v(x+y)} = 2^{\leq -\min(v(x),v(y))} \leq \max(2^{-v(x)}, 2^{-v(y)})$.

(**5.3.8**) An *Archimedean* absolute value (or simply an absolute value) obeys the weaker relation $|x+y| \leq |x| + |y|$ in place of the ultrametric inequality.

(**5.3.9**) Let R be a ring with (non–archimedean) absolute value $||$. Then $d(x, y) = |x - y|$ is a metric on R. The ring operations are continuous so R becomes a TOPOLOGICAL RING with the metric topology.

(**5.3.10**) Suppose that K is some non-archimedean valued field. For $r > 0 \in \mathbb{R}$ and $a \in K$ we set $B_a(r) = \{x \in K : |x-a| \leq r\}$ and $B_a(r^-) = \{x \in K : |x-a| < r\}$. The sets $B_0(1)$ and $B_0(1^-)$ are called the CLOSED (respectively OPEN) UNIT DISCS. Note that each is both open and closed however!

Now $B_0(1)$ is a subring of K, and $B_0(1^-)$ is an ideal in $B_0(1)$. Indeed since every non-zero $d \in K$ has an inverse, and every d with $|d| = 1$ is in $B_0(1)$, every d with $|d| = 1$ has an inverse in $B_0(1)$; thus $B_0(1^-)$ is a maximal ideal, since any larger ideal contains some d with $|d| = 1$ and this d generates everything: $B_0(1)d^{-1}d = B_0(1)$.

It is clear that every non-zero element of the quotient ring $B_0(1)/B_0(1^-)$ takes the form $d + B_0(1^-)$, with $|d| = 1$, and has an inverse. Thus $B_0(1)/B_0(1^-)$ is a field, which we call the RESIDUE CLASS FIELD k of K.

(**5.3.11**) Now suppose that there is a discrete valuation μ on K. In this case we have $B_0(1) = \{x \in K \;:\; \mu(x) \geq 0\}$. An element $p \in B_0(1)$ is a *primal* element if $\mu(p)$ generates the additive group $\mu(K \setminus \{0\})$. We might as well assume that $\mu(K \setminus \{0\}) = (\mathbb{Z}, +)$ and that $\mu(p) = 1$. Note that $\mu(p^n) = n$.

Suppose I is a proper ideal of $B_0(1)$. Since μ is discrete, there is some minimum $\mu(a)$ among the $a \in I$, call it n_I. Let $a \in I$ such that $\mu(a) = n_I$. If $\mu(b) \geq \mu(a)$ then b/a (computed in K) lies in $B_0(1)$, so $b = a(b/a)$ lies in I. Thus in particular I contains p^{n_I} and indeed every element of K of this or greater valuation. We have $p^{n_I}B_0(1) \subset I \subset p^{n_I}B_0(1)$. Thus every ideal I of $B_0(1)$ is a principal ideal.

5.3.2 Complete discrete valuation ring

(**5.3.12**) Let R be a ring with absolute value $||$ as in (5.3.7). For $\{c_i\} \in R^{\mathbb{N}}$, write $\lim_{i\to\infty} |c_i| = c$ (or $|c_i| \to c$ as $i \to \infty$) if given any real $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|c_i - c| < \epsilon \; \forall i \geq N$.

A sequence $\{c_i\}$ in R *converges* to $c \in R$ if $|c_i - c| \to 0$ as $i \to \infty$.
A CAUCHY SEQUENCE is a sequence such that $|c_i - c_j| \to 0$ as $i, j \to \infty$.
In particular this does *not* require that $\{c_i\}$ converges to any $c \in R$.

(**5.3.13**) We say a ring R is COMPLETE with respect to $||$ if every Cauchy sequence is convergent. This is equivalent to (5.3.2).

(**5.3.14**) PROPOSITION. *For R a commutative ring with absolute value $||$ there exists a complete ring $\check{R}$ unique up to metric isomorphism (?) and a dense embedding $R \to \check{R}$ preserving $||$.*

Proof: Firstly we construct a candidate for $\check{R}$. Take the set C of all Cauchy sequences over R and endow with componentwise addition and multiplication. The set of sequences of the form $\{c\} = \{c_i = c \forall i\}$ form a subring isomorphic to R (identify this with R in C). Then $(C, +, ., \{0\}, \{1\})$ is a commutative ring. Sequences convergent to 0 form an ideal $\mathfrak{n}$ in C (need that $\{|c_i|\}$ is bounded for any Cauchy sequence). Note that $R \cap \mathfrak{n} = \{\{0\}\}$. Take $\check{R} = C/\mathfrak{n}$. We can again identify R with a subring of $\check{R}$, namely by $a \mapsto \{a\} + \mathfrak{n}$.

For $\{c_i\} \in C$ then $\{|c_i|\}$ is a Cauchy sequence of real numbers, so $\lim |c_i| \in \mathbb{R}$ by completeness of $\mathbb{R}$ (!). If $\{c_i'\} \in \{c_i\} + \mathfrak{n}$ then $\lim |c_i'| = \lim |c_i|$ so we can define

$$|\{c_i\} + \mathfrak{n}| = \lim |c_i|$$

giving a map of $\check{R}$ into $\mathbb{R}$ (indeed into $\mathbb{R}_{\geq 0}$). Eventually (!) we can verify that this $||$ is an absolute value on $\check{R}$. In particular

$$|(\{c_i\} + \mathfrak{n})(\{d_i\} + \mathfrak{n})| = |(\{c_i d_i\} + \mathfrak{n})| = \lim |c_i d_i| = \lim |c_i||d_i|$$

$$= \lim |c_i| \lim |d_i| = |(\{c_i\} + \mathfrak{n})||(\{d_i\} + \mathfrak{n})|$$

Denseness requires to show that any element a' of $\check{R}$ is a limit of a sequence of elements of R. But if $a' = \{c_i\} + \mathfrak{n}$ then $\lim_{i\to\infty} c_i = a'$ (for $\{c_i\}$ is a Cauchy sequence...) so we are done.

Let $A = \{a_j'\}$ be a Cauchy sequence of elements of $\check{R}$. This is a sequence of sequences, since $a' = \{a_i\}$ (we may write $A = \{\{a_{i_j}\}\}$, neglecting $\mathfrak{n}$). That is to say,

$$\lim_{k\to\infty, j\to\infty} |a_k' - a_j'| = \lim_{k,j} |\{a_{i_k}\} - \{a_{i_j}\}| = \lim_{k,j} |\{a_{i_k} - a_{i_j}\}| = \lim_{k,j} \lim_{i} |a_{i_k} - a_{i_j}| = 0. \tag{5.1}$$

Since this must hold however $i, j, k \to \infty$ it must hold if $i = k$, hence

$$\lim_j \lim_i |a_{i_i} - a_{i_j}| = 0. \tag{5.2}$$

Since each a'_j is a Cauchy sequence in $(R, ||)$ we also have

$$\lim_{i,k} |a_{i_j} - a_{k_j}| = 0 \tag{5.3}$$

for each j. In particular this holds in lim_j, and then if limits are taken with $k = j$, giving

$$\lim_{i,j} |a_{i_j} - a_{j_j}| = 0 \tag{5.4}$$

We require to show that the sequence A converges to an element of $\check{R}$, and we will consider the element $\{c_i = a_{i_i}\}$ (i.e. the i^{th} element is the i^{th} element of a'_i). Why is this a Cauchy sequence? — RTS

$$\lim_{i,j} |a_{i_i} - a_{j_j}| = 0.$$

By the ultrametic inequality we have

$$|\{a_{i_i} - a_{j_j}\}| \leq |\{a_{i_i} - a_{i_j}\}| + |\{a_{i_j} - a_{j_j}\}|$$

but on applying $\lim_{ij}$ the RHS becomes zero using equation (5.2) and equation (5.4).

Thus we RTS

$$\lim_{k\to\infty} |\{a'_k - \{a_{i_i}\}\}| = \lim_k |\{a_{i_k} - a_{i_i}\}| = \lim_k \lim_i |a_{i_k} - a_{i_i}| = 0$$

Now note that

$$|\{a_{i_k} - a_{i_i}\}| \leq |\{a_{i_k} - a_{i_j}\}| + |\{a_{i_i} - a_{i_j}\}|$$

by the ultrametric inequality. This holds for any j, and hence in $\lim_j$ on the RHS. Applying $\lim_{k,i}$ the first term on the RHS then becomes $\lim_{ijk} |\{a_{i_k} - a_{i_j}\}|$ which vanishes by equation (5.1), and the last term vanishes by equation (5.2). (Still have to show denseness and uniqueness.) □

5.3.3 p–adic numbers

(**5.3.15**) Fix a prime p. The set of p-ADIC INTEGERS $\mathbb{Z}_p$ is the set of all infinite sequences $\cdots, a_2, a_1, a_0$ where each $a_i \in \{0, \ldots, p-1\}$. Given $n \in \mathbb{N}$, we can write $n = \sum a_i p^i$ with each $a_i \in \{0, \ldots, p-1\}$. In this way we identify $\mathbb{N}$ with the subset of $\mathbb{Z}_p$ where almost every a_i is zero.

Addition and multiplication are defined in $\mathbb{Z}_p$ so as to extend the usual definition on $\mathbb{N}$. However, every element also has an additive inverse. For example in $\mathbb{Z}_2$:

$$(...1111) + (...0001) = (...000) = 0$$

(where we use ellipsis when a pattern is established sufficient to determine all terms). Further

$$(...1111)(...00011) = (...1111)((...00010) + (...0001)) = (...11110) + (...1111) = (...11101)$$

and

$$(...10101011)(...00011) = (...101010110) + (...10101011) = (...0001) = 1$$

Indeed, $\mathbb{Z}_p$ is a commutative ring with subring $\mathbb{Z}$.

It is clear that, beside our earlier identification of $\mathbb{N}$, the set $-\mathbb{N}$ can be identified with the subset of $\mathbb{Z}_p$ where almost every a_i is $p-1$.

So far we have not use the primality of p. However, if p is composite then there exist $x, y \neq 0$ such that $xy = 0$. For p prime the ring $\mathbb{Z}_p$ is an integral domain, and an element of $\mathbb{Z}_p$ has an inverse if and only if $a_0 \neq 0$.

(**5.3.16**) Given a non-zero p-adic integer a, we let $v(a)$ be the least m for which a_m is non-zero. It is easy to see that for a non-zero element a of $\mathbb{Z}_p$, we have $a = p^{v(a)}b$ where b is invertible in $\mathbb{Z}_p$. Thus, to find a field containing $\mathbb{Z}_p$, it is enough to find an inverse for p.

(**5.3.17**) Inspired by decimal notation, we define the p-ADIC NUMBERS $\mathbb{Q}_p$ to be the set of all infinite sequences $\cdots, a_2, a_1, a_0, a_{-1}, a_{-2}, \cdots$ where each $a_i \in \{0, \ldots, p-1\}$ and $a_{-n} = 0$ for all $n >> 0$. We identify $\mathbb{Z}_p$ with the subset of sequences where $a_{-n} = 0$ for all $n > 0$. Addition and multiplication are extended from $\mathbb{Z}_p$ to $\mathbb{Q}_p$ in the obvious way.

It is easy to show that $\mathbb{Q}_p$ is a field containing $\mathbb{Q}$ as a subfield and $\mathbb{Z}_p$ as a subring. In fact, $\mathbb{Q}_p$ is the quotient field of $\mathbb{Z}_p$. Under our identifications, the elements of $\mathbb{Q}$ in $\mathbb{Q}_p$ are precisely those sequences which are periodic — that is those for which there exists $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $a_i = a_{i+n}$ for all $i \geq m$.

(**5.3.18**) For $0 \neq a \in \mathbb{Q}_p$ we define (as before) $v(a)$ to be the least m for which a_m is non-zero. Then the p-ADIC VALUE of a is

$$|a|_p = \begin{cases} p^{-v(a)} & a \neq 0 \\ 0 & a = 0 \end{cases}$$

A routine exercise shows that v is a valuation on $\mathbb{Q}_p$, and that the closed unit disc in Q_p with respect to $|.|_p$ is $\mathbb{Z}_p$.

(**5.3.19**) For convenience, we now summarise some of the main topological properties of $\mathbb{Z}_p$ and $\mathbb{Q}_p$. First, $\mathbb{Z}_p$ is compact (and hence complete), containing $\mathbb{Z}$ as a dense subset.

Similarly, $\mathbb{Q}$ is dense in $\mathbb{Q}_p$, which is locally compact (and hence complete and separable). Thus we could also have defined $\mathbb{Q}_p$ to be the completion of $\mathbb{Q}$ with respect to $|.|_p$.

(**5.3.20**) Note that $\mathbb{Q}_p$ is very different from $\mathbb{R}$; for example, $\mathbb{Q}_p$ is totally disconnected. However, like $\mathbb{R}$ it is complete but not algebraically closed. We denote the completion of $\mathbb{Q}_p$ by $\mathbb{C}_p$.

Proposition 5.1. *The field $\mathbb{C}_p$ is algebraically closed. As a $\mathbb{Q}_p$-vector space it is infinite dimensional. It is separable, but not locally compact. The residue class field of $\mathbb{C}_p$ is the algebraic closure of $\mathbb{F}_p$ (the field of p elements).*

(**5.3.21**) Perhaps the most surprising result concerning $\mathbb{C}_p$ (and the one that shall be the key in our applications) is

Theorem 5.2. *As fields we have $\mathbb{C}_p \cong \mathbb{C}$.*

(**5.3.22**) SUMMARY.

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{Z} & \hookrightarrow & \mathbb{Q} & \hookrightarrow & \mathbb{C} \\
 & & | & & \downarrow & & \downarrow \\
 & & | & & \mathbb{Q}_p & \overset{c}{\hookrightarrow} & \mathbb{C}_p \\
 & & \downarrow & \overset{a}{\nearrow} & & & \downarrow \\
\mathbb{N} & \hookrightarrow & \mathbb{Z}_p & & & & 0 \\
 & & & \searrow & & & \\
 & & & & \mathbb{F}_p & & \\
n & \mapsto & \sum_{i\geq 0} n_i p^i & \mapsto & n_0 & &
\end{array}
$$

vertical sequences exact, a is an inclusion.

5.3.4 Idempotents over the p–adics

(**5.3.23**) Why might we be interested in the p-adic numbers, from a representation theory perspective? The reason is that they provide a concrete example of the following general setup. (See [17, 5] for more details.)

Let $\mathcal{O}$ be a complete discrete valuation ring with field of fractions K of characteristic zero, maximal ideal $\mathfrak{p}$, and quotient field $k = \mathcal{O}/\mathfrak{p}$. Then if $p \in \mathcal{O}$ generates $\mathfrak{p}$ we call the triple $(K, \mathcal{O}, k)$ a p-MODULAR SYSTEM.

By the various results collected together in the previous section, one can check that $(\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p)$ is an example of such a system. (The only fact we have not explicitly seen is the identification of k with $\mathbb{F}_p$ in this case — but this follows from the description of the invertible elements in $\mathbb{Z}_p$.)

(**5.3.24**) Let Λ be an algebra over $\mathcal{O}$ that as an $\mathcal{O}$-module is free of finite rank. We set $\bar{\Lambda} = k \otimes_{\mathcal{O}} \Lambda$, and extend this notation to elements of the tensor product in the obvious way. The importance of p-modular systems is that we can lift idempotents from algebras defined over the quotient field by the following theorem.

(**5.3.25**) THEOREM. (i) Let e be an idempotent in $\bar{\Lambda}$. Then there is an idempotent f in Λ with $e = \bar{f}$. If e_1 is conjugate to e_2 in $\bar{\Lambda}$, with $\bar{f}_1 = e_1$ and $\bar{f}_2 = e_2$, then f_1 is conjugate to f_2 in Λ.

(ii) By this lifting process we may lift a decomposition of 1 into a sum of primitive orthogonal idempotents from $\bar{\Lambda}$ to Λ.

(iii) Suppose that reduction modulo $\mathfrak{p}$ is a surjective map between the respective centres of Λ and $\bar{\Lambda}$. Then we may lift a decomposition of 1 into a sum of primitive central idempotents from $\bar{\Lambda}$ to Λ.

We shall see later that these results on idempotents provide a paradigm for passing representation theoretic information between algebras constructed by different base changes.

(**5.3.26**) Very briefly, this is exemplified as follows: In the group algebra $\mathbb{F}_2 S_n$ $(n > 2)$ we have $e = (1 + (123) + (321))$ obeying

$$(1 + (123) + (321))^2 = (1 + (123) + (321)) \qquad (mod.2)$$

so we should be able to lift this to $\mathbb{Z}_2 S_n$. Considering $f = (\ldots 10101011)(1+(123)+(321)) \in \mathbb{Z}_2 S_n$ we see that this reduces to e mod.2; and that

$$(1+(123)+(321))^2 = (\ldots 0011)(1+(123)+(321))$$

and $(\ldots 0011)(\ldots 10101011) = (\ldots 001)$, so

$$f^2 = (\ldots 10101011)^2(\ldots 0011)(1+(123)+(321)) = f$$

Of course the inclusion of $\mathbb{N}$, and the inverses of odd numbers, in $\mathbb{Z}_2$ means that $(\ldots 0011) = 3$ and $(\ldots 10101011) = \frac{1}{3}$. Since $\mathbb{Q}$ (which includes in $\mathbb{Q}_2$) is a splitting field for S_n we anticipate that we can decompose this idempotent further over $\mathbb{Q}S_n$. But any further decomposition requires an inverse of 2, which we do not have *here* over $\mathbb{Z}_2$. Over $\mathbb{Q}$ we have

$$f = e_{(3)} + e_{(1^3)}$$

(each term on the RHS requiring coefficients of form $\frac{1}{6}$).

Chapter 6

Ring–modules

6.1 Ring–modules

Here R will be a ring.

(**6.1.1**) (Left) R–Module M: M an abelian group with map $R\times M \to M$ (written $(r,m)\mapsto rm$) such that $r(x+y)=rx+ry$, $(r+s)x=rx+sx$,

$$(rs)x = r(sx),$$

$1x = x$ ($r \in R$, $x, y \in M$).
Right modules defined similarly (but instead $(rs)x = s(rx)$; or equivalently we can change the notation to $(r,m)\mapsto mr$ giving $x(rs)=(xr)s$).

(**6.1.2**) Examples:
R is both a left and a right R–module by the ring multiplication (on the left and on the right respectively).
Every left R–module is a right R^{op} module, where R^{op} is the opposite ring.
Let M be a left module and $m \in M$, then

$$Rm = \{rm \mid r \in R\}$$

is a submodule.
The set $R\times R$ is an abelian group by $(a,b)+(c,d)=(a+c,b+d)$ and an R-module by the action $r(a,b)=(ra,rb)$.

(**6.1.3**) For any module M the set of submodules partially ordered by inclusion forms a lattice. (Meet and join are intersection and sum respectively.)

Example: Consider the R-module $R\times R$ defined above, and suppose for simplicity that R is a field.

Then the submodule lattice looks, in part, like

$$\begin{tikzcd}[column sep=small]
 & & R\times R \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{r(1,0)\mid r\in R\} & \{r(0,1)\mid r\in R\} & \{r(1,1)\mid r\in R\} & \cdots \\
 & & \{(0,0)\} \arrow[ull, dash] \arrow[ul, dash] \arrow[u, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Setting $R_{ab} = \{r(a,b) \mid r \in R\}$ note that

$$(R_{10} + R_{01}) \cap R_{11} = R_{11}$$

$$(R_{10} \cap R_{11}) + (R_{01} \cap R_{11}) = R_{00}$$

so the distributive law does *not* hold in this lattice.
(Later we will consider finite dimensional modules for algebras over fields. There we will see that the distributive law holds iff the module has no section isomorphic to a direct sum of isomorphic simple modules. Note that our example illustrates the only-if part of this.)

6.2 R-homomorphisms and the category R-mod

(**6.2.1**) For M, N left R-modules, an R-module homomorphism, or R-homomorphism, is an element

$$f : M \to N$$

of $\hom_{\mathbf{Ab}}(M,N)$ such that $f(rm) = rf(m)$ $(r \in R,\, m \in M)$.

Note that such an f is, in particular, an abelian group homomorphism, so that $\ker f$ is defined.

We write $\mathrm{Hom}_R(M,N)$ for the set of these homomorphisms. We write $\mathrm{End}_R(M)$ for $\mathrm{Hom}_R(M,M)$.

(**6.2.2**) The class $R-\mathrm{mod}$ of left R–modules is a category, with morphisms the R–module homomorphisms. ($\mathrm{mod} - R$ defined similarly for right modules.)

6.2.1 quotients

(**6.2.3**) If M' is a submodule of $M \in R - \mathrm{mod}$ then there is an action of R on the cosets of M' in M given by

$$r(m + M') = rm + M'$$

making the collection of cosets M/M' a left R-module.

For example, in (6.3.2) we have $I_d/I_{d-1} \cong I_2$ for all $d > 1$.

(**6.2.4**) If $f \in \mathrm{Hom}_R(M,N)$ then $\ker f$ is an R-submodule of M; $f(M) = im\, f$ is an R-submodule of N; and $f(M)$ and $M/(\ker f)$ are isomorphic R-modules.

(**6.2.5**) Let $f \in \mathrm{Hom}_R(M,N)$. For $n \in N$ define $f^{-1}n$ (as for any map $f : M \to N$) as the set of elements m of M such that $f(m) = n$. For $S \subset N$ then $f^{-1}S$ is the subset of M whose images lie in S.

Note that f^{-1} is not an R-homomorphism (it is not even a set map). However:

(**6.2.6**) Proposition. *If N' is a submodule of N then $f^{-1}N'$ is a submodule of M. If M' is a submodule of M then fM' is a submodule of N.*

Proof. If $a, b \in f^{-1}N'$ then $f(a), f(b) \in N'$ so $f(a) + f(b) \in N'$ so $f(a) + f(b) = f(a+b) \in N'$, so $a + b \in f^{-1}N'$; and if $r \in R$ then $f(ra) = rf(a) \in N'$ so $ra \in f^{-1}N'$. The proof of the other claim is similar. □

(**6.2.7**) Remark. For more on quotients see for example Zariski–Samuel [40, §III.3].

6.2.2 Direct sums and simple modules

(**6.2.8**) If M, N are R-modules then the external direct sum $M \dot{+} N$ is $M \times N$ with componentwise addition and

$$r(m, n) = (rm, rn)$$

This is an R-module. Further

$$M' = \{(m, 0) \mid m \in M\}$$

is a submodule (as is N' defined similarly).

If M_1, M_2 submodules of R-module M then we write $M_1 + M_2$ for the obvious subset of M. This is another submodule.

(**6.2.9**) If M_1, M_2 submodules of R-module M we write

$$M_1 + M_2 = M_1 \oplus M_2$$

if $m_1 + m_2 = 0$ ($m_i \in M_i$) implies that each $m_i = 0$.

The module $M_1 \oplus M_2$ is the *(internal) direct sum* of M_1 and M_2. This extends to $\oplus_i M_i$.

Refering back to the external direct sum we have:

$$M \dot{+} N = M' \oplus N'$$

(**6.2.10**) A left R-module M is *irreducible* (or *simple*) if $M' \subset M$ implies $M' = \{0\}$.

(**6.2.11**) Proposition. [SCHUR'S LEMMA] *Let S be a simple R–module. Then $End_R(S)$ is a division ring.*

Proof. Let $f \in \mathrm{End}_R(S)$ be non–zero. The kernel of f is a submodule of S, so it is empty. Thus f is an injection. Similarly the image of f is S, so f is a surjection and hence a bijection, and so has an inverse. □

(**6.2.12**) Semisimple module M, M is a module which is a direct sum of simple modules.

(**6.2.13**) A non-zero left R-module M is *indecomposable* if it cannot be expressed as a direct sum of two non-zero submodules.

Example: The ring $T_2'(\mathbb{C})$ is indecomposable as a left-module for itself. It is not irreducible, since $U_2(\mathbb{C})$ is a submodule, but the only other nonzero submodule is $T_2'(\mathbb{C})$ itself, so there is evidently no direct sum decomposition.

(**6.2.14**) A diagram $L \xrightarrow{f} N \xrightarrow{g} M$ in R-mod is *exact at* N if $im(f) = ker(g)$.
A finite sequence of maps in R-mod is an *exact sequence* if it is exact at every step.
An exact sequence of form

$$0 \to L \xrightarrow{f} N \xrightarrow{g} M \to 0$$

is called a *short exact sequence*.
If such a sequence has a reverse (there is an $f' : N \to L$ with $f'f = 1_L$), it is *split*.

For example, the natural sequence

$$0 \to L \to L \oplus M \to M \to 0$$

is split.

(**6.2.15**) Note that R-mod is an additive category (as in 4.2.9), with the category direct sum given by module direct sum. In particular, for every split short exact sequence

$$\begin{tikzcd} 0 \arrow[r, shift left] & L \arrow[l, shift left] \arrow[r, shift left, "a"] & N \arrow[l, shift left, "a'"] \arrow[r, shift left, "b'"] & M \arrow[l, shift left, "b"] \arrow[r, shift left] & 0 \arrow[l, shift left] \end{tikzcd}$$

there is an idempotent decomposition of $1_N \in \mathrm{End}_R(N)$:

$$aa' + bb' = 1_N$$

6.2.3 Free modules

(**6.2.16**) A set $\{m_i\}$ of elements of an R-module M is called R-*free* if the only solution to

$$\sum_i r_i m_i = 0$$

is $r_i = 0$ for all i.

(**6.2.17**) A set $\{m_i\}$ of elements of R-module M is called a *set of generators* of M if every $m \in M$ can be expressed in the form

$$m = \sum_i r_i(m) m_i \qquad r_i(m) \in R$$

an R-linear combination of a finite number of the $\{m_i\}$.

(**6.2.18**) A set of generators of M that is R-free is called a basis.

A (left) R-module with a basis is called a free (left) R-module.

(**6.2.19**) Suppose that $\rho : A \to M_n(R)$ is a representation of an R-algebra A. Let $\{b_1, .., b_n\}$ be a set of symbols, and let M be the free R-module with this set as basis. Then the action of A on M given by $ab_i = \sum_j \rho(a)_{ij} b_j$ makes M an A-module.
Note however that this M is not a free A-module in general.

6.2.4 Matrices over R and free module basis change

Here we shall take R to be commutative.

(**6.2.20**) A matrix $Y \in M_n(R)$ is *unimodular* if there exists Y' such that $YY' = Y'Y = 1_n$. Equivalently Y is unimodular if $\det(Y)$ is a unit in R.

(**6.2.21**) Matrices $S, T \in M_{m,n}(R)$ are *equivalent* if $S = Y_1 T Y_2$ with Y_i unimodular. We write $S \sim T$.

(**6.2.22**) Let M be a free R-module with ordered basis $m = (m_1, m_2, .., m_n)$. Let $Y \in M_n(R)$. Then $m' = mY$ is a basis iff Y unimodular.

(**6.2.23**) Example. From (2.3.1)

$$(-11+12+21-22, 11, 11-22, 12-21) = (11, 12, 21, 22)\begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

and det $Y = -(-1+(-1)) = 2$.

(**6.2.24**) Theorem. Suppose R a PID. Let $S \in M_{m,n}(R)$. Then

$$S \sim \mathrm{diag}(d_1, d_2, .., d_l, 0, 0, .., 0)$$

where $d_i | d_{i+1} \in R$. The elements $\{d_i\}$ are the *invariant factors* of S (defined up to units in R). See e.g. [**?**] for a proof.

(**6.2.25**) Theorem. Let R be a PID. If M is a free R-module of rank n and N is a submodule, then N is free of rank $n' \leq n$. (See e.g. [**?**, §16.1].) Further, there is an ordered basis $(b_1, b_2, ..., b_n)$ of M, and elements $r_1 | r_2 | ... | r_{n'} \in R^*$ such that $(r_1 b_1, r_2 b_2, ..., r_{n'} b_{n'})$ is an ordered basis for N. The r_i are uniquely determined up to units. (See e.g. [**?**, §16.8].)

6.3 Finiteness issues

(**6.3.1**) A module M is *finitely generated* if there is a finite set of generators.

Example: $R = R1$ is finitely generated. (Next we construct a non-fg module.)

(**6.3.2**) Fix a prime p. Consider the abelian additive group I of rational numbers of form $\frac{q}{p^l}$, modulo the integers. Thus writing simply x for $x + \mathbb{Z}$:

$$\frac{80}{3^4} + \frac{79}{3^4} \equiv \frac{26}{3^3}$$

Regard I as a $\mathbb{Z}$-module. The subset $I_d = \{0, \frac{q}{p^l} \mid d > l > 0\}$ is a finite subgroup and a submodule for any $d \in \mathbb{N}$. For example, $I_3 = \{\frac{q}{p^2} \mid q = 0, 1, 2, ..., p^2 - 1\}$.

We have

$$\{0\} = I_1 \subset I_2 \subset I_3 ...$$

I_2 is generated by $\frac{1}{p}$, and I_d by $\frac{1}{p^{d-1}}$, but I is not finitely generated as a $\mathbb{Z}$-module.

(**6.3.3**) The submodules of an R-module M satisfy ACC if every chain

$$M_1 \subset M_2 \subset M_3 \subset ...$$

terminates (i.e. there is an index i such that the chain cannot be extended on the right beyond M_i except by $M_{i+1} = M_i$).

DCC means, analogously, that every descending chain terminates.

(**6.3.4**) Theorem. TFAE:
(i) The submodules of R-module M satisfy ACC
(ii) Every submodule of M is finitely generated
(iii) Every collection of submodules of M contains a maximal element.

(**6.3.5**) Our example (6.3.2) above satisfies DCC but not ACC.

(**6.3.6**) Theorem. Submodules of left R-module M satisfy DCC iff every set $\{M_i \mid M_i \subset M\}$ has a minimal element.

(**6.3.7**) If R is a commutative integral domain in which every ideal is principle (PID), then the left ideals of R satisfy ACC.

(**6.3.8**) A ring whose left ideals satisfy DCC (i.e. a left-Artinian ring) is sometimes called a ring with (left) minimum condition (MC).

All fd algebras over fields are rings with MC. On the other hand the ring $\mathbb{Z}/p_1p_2\mathbb{Z}$ is finite, with p_1p_2 elements, and so has MC, but is not an algebra over a field (since this field would be finite, and any vector space over it would be finite and have prime power elements).

$\mathbb{Z}$ itself does not satisfy MC.

(**6.3.9**) The *socle* of a left R-module M is the sum of its irreducible submodules.

Of course R may not have any irreducible submodules, if it does not have MC.

(**6.3.10**) The *radical* of a module is the intersection of its maximal submodules. (Set $\mathbf{rad} M = M$ if there are no maximal submodules.)

By Theorem (6.3.4) a nonzero fg R-module has

$$\mathbf{rad}\, M \subset M$$

(**6.3.11**) If module M satisfies DCC then $\mathbf{rad}(M) = 0$ iff M is a finite direct sum of irreducible modules.

(**6.3.12**) The *Jacobson radical* of a ring R is the radical of R as a left-module for itself.

Example: $\mathbf{rad}\, \mathbb{Z} = \{0\}$.
If R is a ring with Jacobson radical J then

$$\mathbf{rad}\, M_n(R) = M_n(J)$$

Let $\mathrm{Tri}_n(R)$ denote the ring of upper-triangular $n \times n$ matrices over R, and $U_n(R)$ the ideal of strictly upper-triangular matrices (0s on the diagonal). Then $\mathbf{rad}\, \mathrm{Tri}_n(K) = U_n(K)$ for any field K.

(**6.3.13**) NIL IDEAL I of ring R is an ideal such that for each $x \in I$ there is a natural number n such that $x^n = 0$.

(**6.3.14**) NILPOTENT IDEAL I of ring R is an ideal such that there is a natural number n such that $\prod_{i=1}^n x_i = 0$ for every n-tuple $x \in I^n$.

(**6.3.15**) A ring is said to be *semisimple* if the Jacobson radical $J = 0$.

(**6.3.16**) For a ring R with MC the Jacobson radical coincides with the (two-sided) ideal which is the sum of all nilpotent left ideals

(**6.3.17**) LEMMA. If R has MC and is semisimple then every R-module is a direct sum of irreducible modules.
Conversely, if R has MC and is a direct sum of irreducible left modules (as a left module for itself) then it is a semisimple ring.

(**6.3.18**) THEOREM. [Wedderburn-Artin] If ring R is semisimple and has MC then it is isomorphic to a direct sum of a uniquely determined set of matrix rings over division rings $\{M_{d_i}(D_i) \mid i = 1, ..., r\}$. This index set $i = 1, ..., r$ also indexes the isomorphism classes of simple modules.

6.3.1 Composition series

(**6.3.19**) Let Γ be a set of R-modules. An R-module M has a Γ-*filtration* if there is a chain of modules $M = M_1 \supset M_2 \supset ... \supset M_l \supset M_{l+1} = \{0\}$ such that every factor M_k/M_{k+1} is isomorphic to some element of Γ.

(**6.3.20**) A chain of modules

$$M = M_1 \supset M_2 \supset ... \supset M_l \supset M_{l+1} = \{0\}$$

is a *composition series* for M if the factors M_k/M_{k+1} are irreducible.

(**6.3.21**) A left R-module M has a composition series iff it satisfies ACC and DCC.

(**6.3.22**) THEOREM. [Jordan-Holder] Any two composition series for a left R-module are equivalent (i.e. of the same length, and the sequence of factors is the same up to order and R-isomorphism).

(**6.3.23**) If S is an irreducible R-module we write $[M : S]$ for the multiplicity of S as a composition factor in M up to isomorphism.

(**6.3.24**) THEOREM. If submodules of a left R-module M satisfy DCC then M can be expressed as a direct sum of finitely many indecomposable modules.

(**6.3.25**) THEOREM. [Krull-Schmidt] If M is a left R-module satisfying ACC and DCC then and two decompositions into a direct sum of indecomposables have the same length, and an ordering bringing the summands into pairwise isomorphism.

(**6.3.26**) In summary, we have seen that modules satisfying ACC and DCC are characterised in large part by the list of their simple factors, together with the possible orderings of these factors. We are interested in fd algebras over fields, for which modules satisfying ACC and DCC are available. We will see that the possible orderings are determined by the radical. So interest turns naturally to the construction and 'detection' of simple modules.

6.3.2 More on chains of modules and composition series

Suppose we have a chain of R-modules $M = M_0 \supset M_1 \supset ... \supset M_l = 0$ and not every section is simple. In particular, suppose $M_i/M_{i+1} = N = N_0 \supset N_1 \supset N_2 = 0$. Can we refine the first chain using the second? I.e. can we insert M_i' in $M_i \supset M_{i+1}$ so that $M_i/M_i' \cong N_0/N_1$ and $M_i'/M_{i+1} \cong N_1/N_2 = N_1$?

(**6.3.27**) Proposition. *(Cf. [40, §III.4 Th.4]) Let $M_0 \supset M_1$ be R-modules, and $f : M_0 \to M_0/M_1$ be the quotient map; i.e. there is a short exact sequence*

$$0 \to M_1 \to M_0 \xrightarrow{f} M_0/M_1 \to 0$$

Every submodule $L' \subset M_0/M_1$ can be expressed as M_0'/M_1 where R-module $M_0' = f^{-1}L'$ obeys $M_0 \supset M_0' \supset M_1$.

Proof. Note that $f^{-1}L' \subset M_0$ (it is the set of elements of M_0 taken to elements of L' by f). We need to show (i) that it is an R-module; and (ii) that $M_0'/M_1 \cong L'$. For (i) we note Proposition 6.2.6. For (ii) we note that $f(f^{-1}L') = fM_0' = L'$. □

(**6.3.28**) Corollary: Any R-module chain that is not a composition series can be refined.

PROOF: Consider $M_i \supset M_{i+1}$ with M_i/M_{i+1} not simple. Then there is a proper submodule $L' \subset M_i/M_{i+1}$. Then for $f : M_i \to M_i/M_{i+1}$ we have $f^{-1}L' \subset M_i$, refining the chain between M_i and M_{i+1}.

6.4 Tensor product

(**6.4.1**) Given a pair of rings $R \subset S$ we have a functor

$$\mathrm{Res}_R^S : S - \mathrm{mod} \;\to\; R - \mathrm{mod}$$

where $\mathrm{Res}_R^S M = M$, an R-module via inclusion in the S-action ($r \in R \subset S$).

Indeed, given any ring homomorphism $\psi : R \to S$ each $M \in S - \mathrm{mod}$ becomes an R-module via ψ; and this extends to a functor Res_ψ. (This functor is exact — see e.g. [3, §16 Ex.1].)

(**6.4.2**) The *restriction functor* $\mathrm{Res}_R^S -$ introduced in (6.4.1) will be very useful (see e.g. §6.8.1), but it would also be useful to have a functor going the other way. This requires a new technical device — the tensor product.

(**6.4.3**) Let R be a ring; M a right R-module; N a left R-module; and Q an additive abelian group. A *balanced map*

$$g : M \times N \to Q$$

is a bilinear map such that $g(m, rn) = g(mr, n)$.

(**6.4.4**) Let M, N as above. Let S_{MN} be the subgroup of $\mathbb{Z}(M \times N)$ generated by all formal sums of form $(m + m', n) - (m, n) - (m', n)$, $(m, n + n') - (m, n) - (m, n')$, $(m, rn) - (mr, n)$. Let $z : M \times N \to \mathbb{Z}(M \times N)/S_{MN}$ via

$$z(m, n) = (m, n) + S_{MN}$$

Then z is a balanced map.

(**6.4.5**) We define the tensor product

$$M \otimes_R N = \mathbb{Z}(M \times N)/S_{MN}$$

We write $m \otimes n$ for the image under z of (m, n).

Remark: A general element of $M \otimes_R N$ is of form $\sum_i m_i \otimes n_i$. It is not uncommon to find this abbreviated to $a \otimes b$ (i.e. a sum, and a suitable unpacking of a and b, is understood).

(**6.4.6**) Example. Consider $\mathbb{Q} \supset \mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$ as $\mathbb{Z}$-modules. Then $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = 0$, since $x \otimes y = xpp^{-1} \otimes y = xp^{-1} \otimes py = 0$. Meanwhile $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$.

(**6.4.7**) Exercise. Regard $\mathbb{Z}S_3$ as a right $\mathbb{Z}S_2$-module by restriction. Compute the abelian group $(\mathbb{Z}S_3)_{\mathbb{Z}S_2} \otimes_{\mathbb{Z}S_2} M_0$ where M_0 is the trivial $\mathbb{Z}S_2$-module.
Hints: Consider

$$S_3 = \langle \sigma_1 = (12),\ \sigma_2 = (23) \rangle \quad \supset \quad S_2 = \langle \sigma_1 \rangle.$$

We have $\mathbb{Z}S_3 = \mathbb{Z}\{1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$. Now $\sigma_1 \otimes m = 1 * \sigma_1 \otimes m = 1 \otimes \sigma_1 m = 1 \otimes m$ and so on, so $(\mathbb{Z}S_3)_{\mathbb{Z}S_2} \otimes_{\mathbb{Z}S_2} M_0 = \mathbb{Z}\{1 \otimes m, \sigma_2 \otimes m, \sigma_1\sigma_2 \otimes m\}$.

(**6.4.8**) Suppose M' a submodule of M as above. Then another balanced map $z' : M' \times N \to M' \otimes_R N$ arises immediately. In this way there are two possible meanings for $m' \otimes n$. Unfortunately it is not generally possible to embed $M' \otimes_R N$ in $M \otimes N$ (as comparison of the two examples in (6.4.6) shows), so care is needed with this notation.

(**6.4.9**) Theorem. Let $f \in \mathrm{Hom}(M, M')_R$ (right modules) and $f' \in \mathrm{Hom}_R(N, N')$. Then the map $(m, n) \mapsto f(m) \otimes f'(n)$ is balanced; and there is a unique map

$$f \otimes f' : M \otimes_R N \to M' \otimes_R N'$$

such that

$$(f \otimes f')(m \otimes n) = f(m) \otimes f'(n)$$

(**6.4.10**) Exercise. Consider the case in which the ring is a field K, and M, N finite dimensional vector spaces, with bases B_M and B_N respectively. Construct a basis for $M \otimes_K N$, regarded as a vector space via the obvious left action of K on M.
Elevate this whole picture to describe homomorphisms of based tensor spaces (matrices) constructed from homomorphisms of their tensor factor spaces (again realised as matrices).

(**6.4.11**) MORE TO GO HERE!

6.4.1 R-lattices etc

(**6.4.12**) For R a Dedekind domain (e.g. a PID) the torsion submodule of a module M is $\tau(M) = \{m \mid rm = 0, \ \text{ some } r \in R^*\}$.

(**6.4.13**) An R-lattice for a Dedekind domain R is a f.g. torsion-free R-module.
Let R^0 be the field of fractions of R. Then $V = R^0 \otimes_R M$ is a vector space. The dimension of this space is the R-rank of M. (See [**?**, §4D].)

6.5 Functors on categories of modules

(**6.5.1**) A, B–BIMODULE ${}_AM_B$ is a left A–module and right B–module such that $a(mb) = (am)b$ for all $a \in A$, $b \in B$, $m \in M$.

(**6.5.2**) REMARK. Jacobson [**?**, §9], for example, uses the term *left A, right B-module* for such a bimodule.

(**6.5.3**) A *left A– left B–bimodule* $M = {}_{A,B}M$ is a left A–module and a right B–module such that $a(bm) = b(am)$ for all $a \in A$, $b \in B$, $m \in M$.
(Again Jacobson omits the 'bi'.)

(**6.5.4**) Let ${}_AM_B$ be a left A– right B–bimodule. Then for each $b \in B$ we may define an element $b' \in \mathrm{Hom}_A({}_AM_B, {}_AM_B)$ by $b'(m) = mb$. Note that

$$(bc)'(m) = m(bc) = (mb)c = c'(b'(m)). \tag{6.1}$$

6.5.1 Hom functors

(**6.5.5**) For each $M \in R-\mathrm{mod}$ there is a (covariant additive) functor

$$\hom_R(M, -) : \ R{-}\mathrm{mod} \ \to \ \mathbf{Ab}$$

with object map given by $X \mapsto \hom_R(M, X)$. The action on maps is $f \in \hom_R(X, Y)$ goes to $f_* : \hom_R(M, X) \to \hom_R(M, Y)$ given by $f_*a = fa$, $a \in \hom_R(M, X)$.

(**6.5.6**) Fix M as above and consider any $A' \xrightarrow{f} A \ \in \ \hom_R(A', A)$. What can we say about $\ker f_*$? Unpacking we have

$$\hom_R(M, A') \stackrel{\hom_R(M,f)}{\to} \hom_R(M, A)$$

$$M \xrightarrow{g} A' \qquad \mapsto \qquad \begin{tikzcd} M \arrow[rr, "f\circ g"] \arrow[dr, "g"'] & & A \\ & A' \arrow[ur, "f"'] & \end{tikzcd}$$

We have $\ker(\hom_R(M, f)) = \{M \xrightarrow{g} A' \mid f \circ g(M) = 0\} = \{M \xrightarrow{g} A' \mid g(M) \in \ker f\}$ so

$$\ker(\hom_R(M, f)) = \Psi_{A'}(\hom_R(M, \ker f))$$

— the isomorphic image of $\hom_R(M, \ker f\} \hookrightarrow \hom_R(M, A')$ got by simply enlarging the codomain to A'. Thus for $0 \to A' \xrightarrow{f} A \xrightarrow{h} A''$ exact ($\ker f = 0$; $\mathrm{im}\ f = \ker h$) we have $\ker f_* = 0$ and $\ker h_* = \Psi_A(\hom_R(M, \ker h)) = \Psi_A(\hom_R(M, \mathrm{im}\ f)) = \Psi_A(\hom_R(M, f(A'))) = \Psi_A(\hom_R(M, A'))$ by the injectivity of f. On the other hand $\mathrm{im}\ f_* = \mathrm{im}\ (\hom_R(M, f)) \cong \hom_R(M, A')$. Thus

$$\ker h_* = \mathrm{im}\ f_*$$

(Note that this is not true in general for the image of a sequence exact at A — we have used the short-exactness on the left.)

(**6.5.7**) There is similarly a contravariant functor $\hom_R(-, M)$. It is contravariant because the construction takes $g \in \hom_R(Y, X)$ and builds an element g^* in $\hom_{\mathbf{Ab}}(\hom_R(X, M), \hom_R(Y, M))$ mapping $a \in \hom_R(X, M)$ to $g^*(a) = a \circ g \in \hom_R(Y, M)$.

(**6.5.8**) We may go further. Taking $M = R$, the right action of R on R (by the ring multiplication), commutes with the left action we are using, and hence survives to equip

$$X^* := \hom_R(X, R)$$

with the property of right R–module. The functor is then from $R-\text{mod}$ to $\text{mod}-R$ (and is called *duality*). In particular the image of a sequence of modules under duality is a sequence in the other direction:

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\hom_R(-,R)"'] & M \arrow[r] \arrow[d, "\hom_R(-,R)"'] & M'' \arrow[r] \arrow[d, "\hom_R(-,R)"'] & 0 \\
0 & (M')^* \arrow[l] & M^* \arrow[l] & (M'')^* \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

(**6.5.9**) Next we elevate the target of the $\hom_R(M, -)$ functor from $\mathbf{Ab}$ to a module category, in case M a bimodule.
We can make $\mathrm{Hom}_A({}_AM_B, {}_AL)$ an element of $B-\text{mod}$ (not $\text{mod}-B$, note), as follows. Applying $\mathrm{Hom}_A(-, L)$ to $b' \in \mathrm{End}_A(M)$ (from (6.5.4)) we obtain

$$\begin{aligned} b'^* : \mathrm{Hom}_A(M, L) &\to \mathrm{Hom}_A(M, L) \\ \beta &\mapsto b'^*\beta \end{aligned} \tag{6.2}$$

given by

$$b'^*\beta(m) = \beta(b'(m)).$$

Then we define the (left) action of B on $\mathrm{Hom}_A({}_AM_B, {}_AL)$ by

$$b.\beta(m) = b'^*\beta(m) \tag{6.3}$$

Finally we verify for $b, c \in B$, $\beta \in \mathrm{Hom}_A(M, L)$ that $(bc).\beta = b.(c.\beta)$ (i.e. that it *is* a left action):

$$((bc).\beta)(m) = ((bc)'^*\beta)(m) = (\{c'b'\}^*\beta)(m) = \beta(c'b'(m)) = (b'^*(c'^*\beta))(m) = (b.\{c.\beta\})(m).$$

(We leave it as an exercise to check that image maps are also maps in $B-\text{mod}$.)

(**6.5.10**) Example. By the action in (6.3) $\mathrm{Hom}_R(R, M)$ is, in particular, a left R-module, for any ring R and R-module M. In fact, the map:

$$\psi : M \to \mathrm{Hom}_R(R, M) \tag{6.4}$$

$$m \mapsto r \overset{\psi(m)}{\mapsto} rm \tag{6.5}$$

is an isomorphism of left R-modules.
Proof: First note that each $\psi(m)$ is indeed in $\mathrm{Hom}_R(R, M)$ (and not just some arbitrary set map). Secondly note that ψ is R-linear. Since $\psi(m) = 0$ implies $m = \psi(m)(1) = 0$, ψ is an injection. Finally, for any $\gamma \in \mathrm{Hom}_R(R, M)$ choosing $m = \gamma(1)$ gives $\psi(m)(r) = \psi(\gamma(1))(r) = r\gamma(1)$. We have $\gamma(r) = r\gamma(1)$, so ψ is surjective. □

(**6.5.11**) Of course this says that $\mathrm{Hom}_R(R,M)$ is nonzero for any nonzero M. Thus there is a nonzero map from R to M for any such M, and in particular a surjection from R to M for any simple M. For a ring with MC, this says that every simple module appears as a composition factor[1] of R (regarded as a left module for itself); and indeed that any simple module may be chosen as the last composition factor.

(**6.5.12**) Proposition. *Let $\{m_i\}$ a basis for $N \in R-mod$. A set $\{\phi_j\} \subset N^*$ is a basis iff matrix $(\phi_j(m_i))_{ij}$ is full rank.*

Proof: Define $\theta_j \in N^* = \mathrm{Hom}_R(N,R)$ by $\theta_j(m_i) = \delta_{ij}$. Note that $\{\theta_j\}$ is a basis of N^*. Now use linearity. □

6.5.2 Tensor functors, Adjointness and Exactness

(**6.5.13**) For ${}_RM \in R-\mathrm{mod}$ define (covariant) functor

$$- \otimes_R {}_RM : \mathrm{mod} - R \to \mathbf{Ab}$$

by $- \otimes_R {}_RM : X_R \mapsto X_R \otimes_R {}_RM$ and for $a \in \hom_R(X,Y)$, $- \otimes_R {}_RM(a) = a \otimes 1$.

(**6.5.14**) For ${}_SM_R$ a bimodule as indicated, define (covariant) functor

$${}_SM_R \otimes_R - : R-\mathrm{mod} \ \to \ S-\mathrm{mod}$$

by ${}_SM_R \otimes_R - : {}_RX \mapsto {}_SM_R \otimes_R {}_RX$ and for $a \in \hom_R(X,Y)$, $({}_SM_R \otimes_R -)(a) = 1 \otimes a$.

(**6.5.15**) Adjointness.
Let R,S be rings with modules ${}_RL, {}_SM_R, {}_SN$. There is an isomorphism of additive groups

$$\gamma : \hom_R(L, \hom_S(M,N)) \cong \hom_S(M \otimes_R L, N) \tag{6.6}$$

given by

$$(\gamma f)(m \otimes l) = f_l(m)$$

where $f_l \in \hom_S(M,N)$ is the image of l under f.
That is to say, the pair of functors $(M \otimes_R -,\ \hom_S(M,-))$ form an adjunction, in the sense of (4.3.7), between categories $R-\mathrm{mod}$ and $S-\mathrm{mod}$.

$$R-\mathrm{mod} \underset{\hom_S(M,-)}{\overset{M\otimes_R -}{\rightleftarrows}} S-\mathrm{mod}$$

Outline Proof: $\hom_S(M,N)$ is a left R–module, and $M \otimes_R L \in\ S-\mathrm{mod}$, and γf is well defined. It may be shown that γ has an inverse μ defined as follows. For each $g \in \hom_S(M \otimes_R L, N)$ let $\mu g \in \hom_R(L, \hom_S(M,N))$ be given by $\{(\mu g)_l\} m = g(m \otimes l)$, where $(\mu g)_l$ is the image of l under μg. Done.

We have that $M \otimes_R -$ is *left adjoint* to $\hom_S(M,-)$.

[1] Hopkins Theorem says that a ring with MC has a composition series, as a left module. (See for example [17, (54.1)].)

(**6.5.16**) The canonical example is FROBENIUS RECIPROCITY. This is where we take $M = S$ and R a subring of S:

$$\hom_R({}_RL, {}_RN) \cong \hom_S({}_SS_R \otimes_R {}_RL, {}_SN)$$

Here ${}_RN$ is the obvious *restriction* to R, as is S_R, and we have used $\mathrm{Hom}_S(S, N) \cong N$ from (6.5.10).

The functor ${}_SS_R \otimes_R -$ in this case is called *induction* from R to S.

(**6.5.17**) In particular suppose that S is an algebra over a field k, and consider that $L = k$ and N is a simple S-module. Then this is

$$\hom_k(k, {}_kN) \cong \hom_S(S, {}_SN)$$

an isomorphism of k-vector spaces of dimension $\dim N$.

(**6.5.18**) A functor F between module categories is EXACT if it takes an exact sequence

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N$$

to an exact sequence

$$F(L) \xrightarrow{F(\lambda)} F(M) \xrightarrow{F(\mu)} F(N).$$

A functor F between module categories is LEFT EXACT (respectively right exact) if it takes a short exact sequence

$$0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0 \tag{6.7}$$

to a sequence

$$0 \longrightarrow F(L) \xrightarrow{F(\lambda)} F(M) \xrightarrow{F(\mu)} F(N) \longrightarrow 0$$

that is exact at $F(L)$ and at $F(M)$ (respectively at $F(M)$ and at $F(N)$).

A functor which is left and right exact is exact.

(**6.5.19**) THEOREM. If functors F, G form an adjunction between module categories

$$\begin{tikzcd} C_{\mathcal{A}} \arrow[r, shift left, "F"] & C_{\mathcal{B}} \arrow[l, shift left, "G"] \end{tikzcd}$$

then the left adjoint F is right exact and the right adjoint G is left exact.

Proof. Applying the adjunction isomorphism (we write it out in the form (6.6), but the general form will be clear) to a short exact sequence

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$$

in the 'N position', and any L, we get

$$\begin{tikzcd} 0 \arrow[r] & \hom_R(L, GA') \arrow[r] \arrow[d, "\cong"'] & \hom_R(L, GA) \arrow[r] \arrow[d, "\cong"'] & \hom_R(L, GA'') \arrow[d, "\cong"'] \\ 0 \arrow[r] & \hom_S(FL, A') \arrow[r] & \hom_S(FL, A) \arrow[r] & \hom_S(FL, A'') \end{tikzcd}$$

[2] The bottom row is exact since the functor $\hom_S(FL,-)$ is left exact (see (6.5.6)). Therefore the top row is exact. Note that the top row is the image under $\hom(L,-)$ of another sequence — the image of the original sequence under G. Since L can be chosen freely, if the preimage (the image under G) were not exact it would pass to an inexact sequence for some choice (exercise), so it is exact. Thus G is left exact. A similar argument with all the arrows reversed shows F right exact. □

In particular:

(**6.5.20**) Proposition. *Let ${}_AV_B$ be a left $A-$ right $B-$ bimodule. The hom functor $Hom_A({}_AV_B,-)$ from $A-mod$ to $B-mod$ is left exact.*
The tensor functor F_V given by ${}_AV_B\otimes_B -$ from $B-mod$ to $A-mod$ is right exact. That is, if (6.7) is a short exact sequence in $B-mod$ then

$${}_AV_B\otimes_B L \xrightarrow{F_V(\lambda)} {}_AV_B\otimes_B M \xrightarrow{F_V(\mu)} {}_AV_B\otimes_B N \longrightarrow 0$$

is exact in $A-mod$. □

To address the question of when such functors are properly exact, it is useful to consider *projective modules.*

6.6 Simple modules, idempotents and projective modules

(**6.6.1**) Two idempotents e_1, e_2 in a ring R are *orthogonal* if $e_1e_2 = e_2e_1 = 0$.
For example the elementary matrices $\epsilon_{11}\epsilon_{22} = 0$; and

$$\begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -r \\ 0 & 1 \end{pmatrix} = 0.$$

An idempotent e is *primitive* if it has no proper decomposition $e = e_1 + e_2$ into orthogonal idempotents.
If $1 = \sum_i e_i$ is a decomposition into orthogonal idempotents in R then

$$R = \bigoplus_i Re_i$$

as a left module. Here Re_i is indecomposable iff e_i is primitive.

(**6.6.2**) Suppose $M \in R-$mod and $e^2 = e \in R$. Then $eM \subset M$ is the abelian group $\{em \mid m \in M\}$. (Note that eM is also an eRe-module.) Note that an R-module morphism $\psi : M' \to M$ defines an

[2]In the hom/tensor case this is:

$$\begin{tikzcd}
0 \arrow[r] & \hom_R(L,\hom_S(M,A')) \arrow[r] \arrow[d, "\cong"] & \hom_R(L,\hom_S(M,A)) \arrow[r] \arrow[d, "\cong"] & \hom_R(L,\hom_S(M,A'')) \arrow[d, "\cong"] \\
0 \arrow[r] & \hom_S(M\otimes_R L, A') \arrow[r] & \hom_S(M\otimes_R L, A) \arrow[r] & \hom_S(M\otimes_R L, A'')
\end{tikzcd}$$

abelian group/eRe-module morphism $eM' \to eM$, by restriction ($em \mapsto \psi(em) = e\psi(em) \in eM$). Thus $M \mapsto eM$ defines a functor from R-mod to **Ab** (or to eRe-mod).

Now consider a short exact sequence $0 \to M' \to M \to M'' \to 0$. We *claim* that $0 \to eM' \to eM \to eM'' \to 0$ is short exact (i.e. that the functor is exact).

Proof. Certainly $eM' \hookrightarrow eM$. We have $eM/eM' = \{em + eM' \mid m \in M\}$ and $e(M/M') = \{e(n + M') \mid n \in M\}$. But $e(n + M') = en + eM'$. □

(**6.6.3**) Proposition. *If M is an R-module and e an idempotent then*

$$\begin{aligned} f : eM &\to Hom_R(Re, M) && (6.8) \\ em &\mapsto f(em) : ae \mapsto aem && (6.9) \end{aligned}$$

is an isomorphism of abelian groups, with inverse $g(\gamma) = \gamma(e)$.

Proof. $f(em)(e) = em$ and $f(\gamma(e))(ae) = f(e\gamma(e))(ae) = ae\gamma(e) = \gamma(ae)$. □

(**6.6.4**) Proposition. *If $e^2 = e \in R$ then the functor $Hom_R(Re, -)$ is exact.*

Proof. Consider a short exact sequence in R-mod: $0 \to M' \to M \to M'' \to 0$. Applying the functor $\mathrm{Hom}_R(Re, -)$ we obtain a sequence of abelian groups. We *claim* that this sequence is also exact (i.e. that the functor is exact).
(Since hom-functors are only left-exact in general (see (6.5.20)), there is something to show here.)
Now the sequence $0 \to eM' \to eM \to eM'' \to 0$ is exact (by (6.6.2)). Applying (6.6.3) to this sequence we see that the $\mathrm{Hom}_R(Re, -)$ image of the original sequence is exact. □

(**6.6.5**) Let $e \in R$ be an idempotent and consider $f \in \mathrm{End}_R(Re)$. We have $f(x) = f(xe) = xf(e)$, so f may be realised by right multiplication by $f(e)$.

(**6.6.6**) Proposition. *$End_R(Re) \cong (eRe)^{op}$*

Proof. Right multiplication by $ef(e)e$ would also realise $f \in \mathrm{End}_R(Re)$, as in (6.6.5), so there is a map, and $f(e) = 0$ only if $f = 0$, so it is an injection. Indeed the map $x \to xeae$ is in $\mathrm{End}_R(Re)$ for any a, so the map is surjective.

(**6.6.7**) Proposition. *A left R–module M is indecomposable if and only if $End_R(M)$ has no idempotents except 1.*

For example, if $eRe \cong \mathbb{Z}$ then Re is indecomposable and e is primitive. N.B. the converse does not hold in general.

(**6.6.8**) Proposition. *eRe a division ring implies Re indecomposable.*

Proof. For a contradiction suppose Re has two non–zero direct summands, and that p_1 is the projection map onto the first. Then $p_1 \in \mathrm{End}_R(Re)$ and not invertible. Thus $\mathrm{End}_R(Re)$ is not a division ring, and by proposition 6.6.6 neither is eRe. □

(**6.6.9**) Proposition. *A left ideal J of R is a direct summand of R (as a left-module for itself) iff*

$$J = Re$$

for some idempotent $e \in R$; whereupon

$$R = Re \oplus R(1-e)$$

as a left-module.

(**6.6.10**) FREE MODULE M, M is a module with a basis.
For S any set we may write RS for the free R–module with basis S.

(**6.6.11**) PROPOSITION. *For any R–module M there is a short exact sequence*

$$0 \longrightarrow G \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is free.

6.6.1 Projective modules

There are a number of equivalent conditions for an R-module to be projective. Which one is the 'definition' depends on ones perspective. For us (for now):

(**6.6.12**) PROJECTIVE MODULE P, P is an R-module and the functor $\hom_R({}_RP_S, -) : R-\mathrm{mod} \to S-\mathrm{mod}$ (as in (6.5.9)) is exact for each bi-module structure ${}_RP_S$ on P.
Example: If $e \in R$ is idempotent then Re is projective.

(**6.6.13**) PROPOSITION. *The following are equivalent:*

1. *P is projective;*
2. *For every R-module surjection $M \xrightarrow{f} M'' \to 0$ and homomorphism $P \xrightarrow{a} M''$ there is a homomorphism $P \xrightarrow{a'} M$ such that $fa' = a$.*
3. *P is a direct summand of a free module;*
4. *Every short exact sequence of the form*

$$0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} P \longrightarrow 0 \tag{6.10}$$

   splits.

Proof. (1) implies (4) since given equation (6.10) (1) says its image under $\hom_R(P, -)$ is exact, but $1_P \in \hom_R(P, P)$ and so in particular there is a ν in $\hom_R(P, M)$ such that $\nu\mu = 1_P$, splitting equation (6.10).

Now, (4) implies (3) since by proposition 6.6.11 there is an F free such that

$$0 \longrightarrow \ker\mu \longrightarrow F \xrightarrow{\mu} P \longrightarrow 0$$

and (4) says this splits.

Next we set off to prove that (3) implies (1).

(**6.6.14**) PROPOSITION. *Let $\{M_i\}$ be a set of right R-modules. Then for any left R-module N*

$$(\oplus_i M_i) \otimes_R N \xrightarrow{\sim} \oplus_i (M_i \otimes_R N)$$

(See for example Jacobson[26, p.154] for a proof.)

(**6.6.15**) A right R-module M is *flat* if $M \otimes_R -$ is exact.

(**6.6.16**) LEMMA. A right R-module $M = \oplus_i M_i$ is flat iff each M_i is flat.

(**6.6.17**) PROPOSITION. *If M_R is projective in sense (3) of (6.6.13) then the functor $M_R \otimes_R -$ is exact.*

Proof. $R \otimes_R -$ takes any sequence to an isomorphic sequence, so is flat. Thus by Lemma (6.6.16) any free R-module F is flat. For any projective P, for some such F we have $F = P \oplus P'$, by Prop. (6.6.13)(3). Thus $P \otimes_R -$ is exact by Lemma (6.6.16) again. □
(See also Hilton–Stambach[**?**, p.111], Anderson–Fuller[3, p.227].)

(**6.6.18**) EXERCISE. Show that (3) implies (1) in Prop. (6.6.13).
We omit (2) from the loop for now. □

6.6.2 Idempotent refinement

We see that idempotents are important structural tools in ring theory. We also see that if I is a nilpotent ideal in ring R then it contains no idempotent. Thus the idempotents of R and R/I are related.

Before we start, note that ideal $I^2 \subseteq I$ in R (indeed $I^2 \subset I$ if I is nilpotent). Suppose $e \in R/I$. Then $e = r_e + I$ for some $r_e \in R$. Similarly $f \in R/I^2$ is $f = r_f + I^2$ for some $r_f \in R$. Thus $f + I = r_f + I^2 + I = r_f + I \in R/I$. In other words $f + I$ makes sense, because I is, roughly speaking, a 'cruder', bigger thing than I^2.

(**6.6.19**) LEMMA. If I is a nilpotent ideal in a ring R and $ee = e \in R/I$ then there is an $ff = f \in R/I^2$ such that $e = f + I$.

Proof. Let $r \in R/I^2$ such that $e = r + I$. Then

$$0 = e(e-1) = (r+I)(r-1+I) = r(r-1) + rI + I(r-1+I)$$

so $r(r-1) \in I$ and so $r^2(r-1)^2 = 0$ in R/I^2. Note that $e_2 := (1+2(1-r))r^2$ obeys $e_2 = r + I$ and

$$e_2(e_2-1) = (1+2(1-r))r^2((1+2(1-r))r^2-1) = (1+2(1-r))r^2(-(1+2r))(r-1)^2 = 0$$

Thus we can take $f = e_2$. □

(**6.6.20**) THEOREM. [Idempotent refinement] (i) If I is a nilpotent ideal in a ring R and $ee = e \in R/I$ then there is an $ff = f \in R$ such that $e = f + I$.
(ii) If $1 = \sum_i e_i$ is a primitive orthogonal idempotent decomposition in R/I then there is a corresponding decomposition $1 = \sum_i f_i$ in R with $f_i + I = e_i$. Further if $(R/I)e_i \cong (R/I)e_j$ then $Rf_i \cong Rf_j$.

Proof. (i) Since I is nilpotent, $I \supset I^2 \supset ... \supset I^i \supset I^{i+1} \supset ...$ until some $I^n = 0$. Let $f' \in R/I^2$ be idempotent passing to e as in Lemma 6.6.19. Of course $I^4 = (I^2)^2$ so there is an idempotent $f'' \in R/I^4$ that passes to f' by the same Lemma. Iterating we shall eventually reach an idempotent f in R/I^m with $m \geq n$, so that $R/I^m = R$.
(ii) Exercise.

6.7 Structure of an Artinian ring

Hereafter let us suppose that R is a ring with MC.

(**6.7.1**) For an Artinian ring the set of nil ideals coincides with the set of nilpotent ideals.

(**6.7.2**) RADICAL J of Artinian ring R: J is the maximal nilpotent ideal of R.
(Recall from (6.3.16) that the Jacobson radical of an Artinian ring is the sum of all nilpotent left ideals, and so coincides with J.)

(**6.7.3**) RADICAL FILTRATION of an R–module M.
The HEAD (or TOP) of a module is M/JM, a semisimple module. We have $M \supset JM \supset J^2M \supset \cdots \supset 0$, and each section is semisimple. (The term *head* is used, for example, by Benson [5].)

(**6.7.4**) SOCLE FILTRATION of an R–module M.
The SOCLE (TAIL) of a module M is the maximal semisimple submodule $\mathbf{Soc}(M)$, i.e. the sum of all simple submodules. There is a sequence of submodules $M \supset M_1 \supset M_2 \supset \cdots \supset \mathbf{Soc}(M)$ unique up to isomorphism such that each section M_i/M_{i+1} is a maximal semisimple submodule of M/M_{i+1}.

(**6.7.5**) If a module M is a direct sum of d copies of a module N we may write simply

$$M = dN$$

Since ring R has MC here, and $R/J(R)$ is semisimple and has MC, then by the Artin-Wedderburn Theorem (6.3.18) we have

$$R/J(R) \cong \bigoplus_i M_{d_i}(D_i) \cong \bigoplus_i d_i L_i$$

(where the first isomorphism is as a ring and the second is) as a left module, where L_i is the simple module. (That is, the multiplicity of a given simple module in the left regular module for a semisimple ring is given by the dimension of that simple (over the opposite of the associated division ring).) Let $1 = \sum_i e_i$ be the corresponding orthogonal idempotent decomposition in $R/J(R)$; and $1 = \sum_i f_i$ the associated decomposition in R (as in Theorem (6.6.20)). Then

$$R \cong \bigoplus_i d_i P_i$$

as a left module. That is, the multiplicity of an indecomposable projective module in R (as a left module for itself) is given by the dimension (in the same sense as before) of the corresponding simple module $L_i = P_i/(J(R)P_i)$.

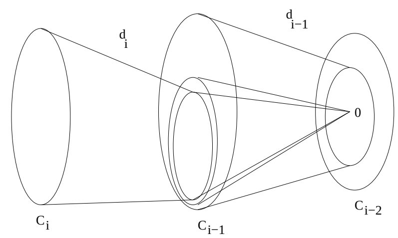
6.7.1 Homology

See for example Jacobson II [26].

(**6.7.6**) A chain of R-module (or indeed abelian group) homomorphisms $d_i : C_i \to C_{i-1}$ ($i \in \mathbb{Z}$) is a *complex* (C,d) if $d_i d_{i+1} = 0$ for all i. That is, in a complex (C,d) we have

$$C_i \stackrel{d_i}{\to} C_{i-1} \stackrel{d_{i-1}}{\to} C_{i-2} \qquad (i \in \mathbb{Z})$$

and $d_{i-1}(d_i(c)) = 0$ for all $c \in C_i$ (for all i):



(**6.7.7**) A *chain homomorphism* $a : (C,d) \to (C',d')$ is a set of homomorphisms $a_i : C_i \to C'_i$ such that $a_{i-1}d_i = d'_i a_i$ for all i.

(**6.7.8**) If (C,d) is a complex and $C_i = 0$ for all $i < 0$ then (C,d) is a *positive complex.*

(**6.7.9**) Example. (I) For M an R module then $C_i = M$ and $d_i = 0 : M \to 0$ is a complex.
(II) A short exact sequence

$$0 \to M' \stackrel{\alpha}{\to} M \stackrel{\alpha'}{\to} M'' \to 0$$

defines a complex with $C_3, C_2, C_1 = M', M, M''$ and all other $C_i = 0$ and $d_2, d_3 = \alpha', \alpha$ and all other $d_i = 0$.

(**6.7.10**) Let (C,d) be a complex and note that $\ker d_i$ is a submodule of C_i; and that the image $d_{i+1}C_{i+1}$ is a submodule of $\ker d_i$. Define the i-the *homology module* of (C,d) as

$$H_i(C) = \ker d_i / d_{i+1}C_{i+1}$$

Note that the complex is exact at C_i iff $H_i(C) = 0$.

(**6.7.11**) For M an R-module, a *chain over* M is a positive complex (C,d) together with a homomorphism $e : C_0 \to M$ such that $ed_1 = 0$.

A complex over M is a *resolution* of M if the extended chain including

$$... \stackrel{d_2}{\to} C_1 \to C_0 \to M \to 0$$

is exact. It is a *projective* complex if every C_i is projective.

(**6.7.12**) Theorem. If (C,e) is a projective complex over M and (C',e') is a resolution of M', and $u : M \to M'$ a homomorphism, then there is a chain homomorphism $a : C \to C'$ such that $ue = e'a_0$.

(**6.7.13**) Let M be an R-module and

$$... \stackrel{d_2}{\to} C_1 \stackrel{d_1}{\to} C_0 \stackrel{e}{\to} M \to 0$$

a projective resolution of M. Now suppose we apply some functor G (from R-mod to **Ab**):

$$... \stackrel{Gd_2}{\to} GC_1 \stackrel{Gd_1}{\to} GC_0 \stackrel{Ge}{\to} GM \to G0$$

If G is multiplicative then $(GC, \{Gd, Ge\})$ is a positive complex over GM. If G is exact then the image sequence is exact, but this is not true in general, so $H_i(GC)$ may not vanish. Define

$$L_i GM \ := \ H_i(GC)$$

This object map may be extended (for each i) to another functor from R-mod to **Ab**, called the i-th *left derived functor* of G. Note that it depends on the choice of resolution, but that the notation omits explicit reference to this (and in fact the dependence is, in a suitable sense, almost negligible — see e.g. [26, §6.6]).

(**6.7.14**) What is handy about $L_i G$ is that if G is, say, right exact but not exact we can use it to develop exact sequences from the G-image of short exact sequences:

$$0 \to A \to B \to C \to 0$$

$$... \to L_1 GC \to L_0 GA \to L_0 GB \to L_0 GC \to 0$$

(**6.7.15**) Exercise. Prove it.

6.8 More on tensor products

Recall from the construction of the tensor product in (6.4) (or see for example Curtis–Reiner I [16, §12]):

(**6.8.1**) Theorem. For left R-module M and right R-module L, and abelian group A, every balanced map $\mu_0 : L \times M \to A$ factors through $L_R \otimes_R M$. That is, every balanced map μ_0 induces a map $\mu : l \otimes m \mapsto \mu_0(l, m)$.

(**6.8.2**) Suppose that left R-module M and right R-module L are in fact a principle left and right ideal respectively: $M = Rm_0$, $L = l_0 R$. Note that the restriction of the ring multiplication to $L \times M$, i.e. $(l, m) \mapsto lm$, is balanced. Then by Theorem (6.8.1), associated to the tensor product $L_R \otimes_R M$ is a *multiplication map* given by

$$\mu : L_R \otimes_R M \to l_0 R m_0$$

$$l \otimes m \mapsto lm$$

(Actually we don't need the restriction to principle ideals — the main point is that multiplication makes sense.) Every element of $l_0 R m_0$ is clearly hit; but is this also an injection? Suppose that one of the generators is idempotent (say l_0, WLOG), then define

$$\nu : l_0 R m_0 \to L_R \otimes_R M$$

$$r \mapsto l_0 \otimes r$$

Note that in this case ν is inverse to μ.

(**6.8.3**) Example. We return to Example (6.4.7). The module M_0 is isomorphic to an ideal in $\mathbb{Z}S_2$: $M_0 \cong \mathbb{Z}(1+\sigma_1)$. This allows us to include $M_0 \subset \mathbb{Z}S_2 \subset \mathbb{Z}S_3$, and thus for M_0 to act on $\mathbb{Z}S_3$ directly by the mutliplication in this algebra. Thus we have a balanced map $\mu_0 : \mathbb{Z}S_3 \times M_0 \to \mathbb{Z}S_3$ (given by $(a,b) \mapsto ab$); and a multiplication map μ taking $\mathbb{Z}S_3 \otimes M_0 \to \mathbb{Z}S_3(1+\sigma_1)$. The image is $\mathbb{Z}\{e', \sigma_2 e', \sigma_1\sigma_2 e'\}$ where $e' = 1+\sigma_1$ (it is easy to see that these elements span; and they are linearly independent in $\mathbb{Z}S_3$). That is, the image is free of rank 3.

Preimages of the basis elements are, for example $\{1 \otimes e', \sigma_2 \otimes e', \sigma_1\sigma_2 \otimes e'\}$. We have already noted that these elements span the tensor product. We now see that they must also be linearly independent in the tensor product, since if some combination $\sum_i c_i a_i \otimes e' = 0$ then $\mu(\sum_i c_i a_i \otimes e') = \sum_i \mu(c_i a_i \otimes e') = \sum_i c_i a_i e' = 0$ so their images would be linearly dependent.

(**6.8.4**) Theorem. Tensor product is associative, i.e. for $L_R, {}_R M_S, {}_S N$ modules as indicated:

$$L_R \otimes_R ({}_R M_S \otimes_S {}_S N) \cong (L_R \otimes_R {}_R M_S) \otimes_S {}_S N$$

(**6.8.5**) Theorem.

$$(L \oplus M)_R \otimes_R N \cong (L_R \otimes_R N) \dotplus (M_R \otimes_R N)$$

6.8.1 Induction and restriction functors

We now return to our original motivation for introducing tensor products — the construction of an 'adjoint' to the restriction functor.

Recall that for a pair of rings with a homomorphism $\phi : R \to S$ we have a functor Res_ϕ : $S - \mathrm{mod} \to R - \mathrm{mod}$ given on objects by $\mathrm{Res}_\phi M = M$ and $rm = \phi(r)m$.

We already considered the case where ϕ is injective. See (6.5.16).

6.8.2 Globalisation and localisation functors

(**6.8.6**) Let ${}_S M_R$ and ${}_R N_S$ be bimodules as indicated. Suppose

$${}_S M_R \otimes_R {}_R N_S \cong S$$

as S-bimodule. Then the functor $G = {}_R N_S \otimes_S -$ is called a *globalisation*; and the functor $F = {}_S M_R \otimes_R -$ is called a localisation. We have

$$F(G(A)) = {}_S M_R \otimes_R {}_R N_S \otimes_S A \;\cong S \otimes_S A \cong A$$

so that F is a kind of left inverse to G.
[3]

[3] This DOES NOT WORK!!! Need something more like $S = eRe$...

(**6.8.7**) With this setup, supposing also that F is exact, if L a simple S-module and B a proper submodule of $G(L)$, then $F(B) = 0$.

Proof. $F(B) \subseteq L$ by construction, but L is simple, so either $F(B) = 0$ or $F(B) = L$.

Chapter 7

Algebras

7.1 Algebras and A-modules

Here R is a commutative ring. We start by recalling the (second) definition from (1.1.10).

(**7.1.1**) R–ALGEBRA A: A a ring and an R–module such that $ax.y = x.ay = a(xy) \quad \forall x, y \in A$, and $a \in R$.
Examples:
(i) Any ring K is a $\mathbb{Z}$–algebra with $na = a + a + \cdots + a$ (n summands).
(ii) Let G be a finite group or monoid, R a commutative ring and RG the free R–module with basis G. Then R–linear extension of the group multiplication equips RG with the property of R–algebra.
(iii) Let A' be the free abelian monoid generated by 1 and a symbol x, and let RA' be the monoid algebra as above. Let RA be the quotient of this algebra by the ideal generated by $x^2 - 2$. If $R = \mathbb{Q}$ then $\{1, x\}$ is a basis for RA.

(**7.1.2**) REMARK. Let A be an R-algebra. It is interesting to recast the definition in terms of commutative diagrams. The ring operations of A itself obey $(x + y)z = xz + yz$ and $z(x + y) - zx - zy = 0$ of course, and the commutativity of R means that the interaction with the R-module structure can be written $(ax)y = (xa)y = x(ay)$ $(a \in R)$. Thus the multiplication factors through a balanced map (compare (6.4.3)), and may be considered as an R-linear map $\nabla : A \otimes_R A \to A$. Similarly the multiplicative identity induces an R-linear map $\eta : R \to A$. In these terms associativity becomes commutativity of

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\nabla\otimes id"] \arrow[d, "id\otimes\nabla"] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"'] & A
\end{tikzcd}$$

where we use the already asserted associativity of tensor product (Theorem 6.8.4) to give the two interpretations of $A \otimes A \otimes A$ needed. Meanwhile

$$\begin{tikzcd} A \otimes R \cong A \cong R \otimes A \arrow[r, "\eta\otimes id"] \arrow[d, "id\otimes\eta"'] \arrow[dr, "id"] & A \otimes A \arrow[d, "\nabla"'] \\ A \otimes A \arrow[r, "\nabla"'] & A \end{tikzcd}$$

also commutes.

(**7.1.3**) If an R-algebra A is free of finite rank as an R-module then that rank is its *rank* as an algebra. In particular if R is a field and A is a finite dimensional R-vector space, then that is its *dimension* as an algebra over the field.

(**7.1.4**) Suppose A is an algebra over a commutative ring $\mathcal{Z}$. For k a ring that is a $\mathcal{Z}$-algebra define k-algebra

$$A^k = k_{\mathcal{Z}} \otimes_{\mathcal{Z}} A$$

If the map $A \to A^k$ given by $a \mapsto 1 \otimes a$ is an injection we may write simply a for $1 \otimes a$.

(**7.1.5**) Suppose A as above, $x \in A$ obeys $x^2 \neq 0$ and

$$xAx \subseteq \mathcal{Z}x$$

Then $x^2 = c_x x$ for some $c_x \in \mathcal{Z}$. Suppose further that k is such that $1 \otimes c_x \neq 0$. Then $(1 \otimes x)A^k(1 \otimes x) = k(1 \otimes x)$. This implies $(1 \otimes x)$ is proportional to an idempotent e_x (say) in A^k, and hence that $e_x A^k e_x$ is a field. Thus by Prop.(6.6.8) we have that $A^k(1 \otimes x)$ is indecomposable. Since the functor $k_{\mathcal{Z}} \otimes_{\mathbb{Z}} -$ preserves direct sums this implies that Ax is indecomposable.

Now suppose there is a k' such that $1 \otimes_{k'} c_x = 0$. Is $A^{k'}(1 \otimes_{k'} x)$ indecomposable?

7.2 Finite dimensional algebras over fields

In the remainder of this section k is a field, and A a finite dimensional algebra over k.

(**7.2.1**) Theorem. [Burnside's Theorem] Let A be a f.d. algebra over k and M a simple left A-module. If $\mathrm{End}_A(M) = k.1_M$ then every k-endomorphism of M is realised by left by multiplication by some element of A.

7.2.1 Dependence on the field

(**7.2.2**) Proposition. *Let M be a simple left A-module. If k is algebraically closed then $End_A(M) = k.1_M$.*

(**7.2.3**) Absolutely simple module. A left A-module M is absolutely simple if $k' \otimes_k M$ is simple for every extension field $k' \supseteq k$.

(**7.2.4**) Example: The $\mathbb{Q}$-algebra generated by x obeying $x^2 = 2$ defined in 7.1.1(iii) is simple as a left module for itself, but not absolutely simple. To see this we note that if there is a submodule then it must be spanned by an element of the form $\alpha 1 + \beta x$. We have $x(\alpha 1 + \beta x) = 2\beta 1 + \alpha x = c(\alpha 1 + \beta x)$

for some scalar c. Elimination gives $c^2 = 2$ and $\frac{\alpha}{\beta} = c$. This has no solution in $\mathbb{Q}$, so there is no submodule. However it has a solution over $\mathbb{Q}[\sqrt{2}]$ (indeed $x + \sqrt{2}.1$ spans a submodule), so the algebra is not absolutely simple as a module for itself.

(**7.2.5**) THEOREM. A simple left A-module obeys $\mathrm{End}_A(M) = k.1_M$ iff it is absolutely simple.

In our example consider the map $\phi \in \mathrm{End}_A(M = A)$ given by $\phi : m \mapsto xm$, which takes $1 \mapsto x$ and $x \mapsto 2$.

(**7.2.6**) SPLITTING FIELD. For A a f.d. algebra over k, an extension $k' \supseteq k$ is a splitting field for A over k if every simple left $k' \otimes_k A$-module is absolutely simple.

Note that every algebraically closed field containing k is splitting for A over k.

7.2.2 Representation theory preliminaries

(**7.2.7**) Suppose R is a field and A and R-algebra. The dual R–module $X^* = \hom_R(X, R)$ of $X \in A - \mathrm{mod}$ is equipped with the property of right A–module via: $\psi \in X^*$, $a \in A$,

$$(\psi a)(x) = \psi(ax).$$

To see that X^* is a *right* module, we need to check that $m(ab) = (ma)b$ ($m \in X^*$, $a, b \in A$):

$$(\psi(ab))(x) \;=\; \psi(abx) \;=\; (\psi a)(bx) \;=\; ((\psi a)b)(x).$$

(**7.2.8**) An algebra A is *primitive* if it has a faithful irreducible representation and *semi–primitive* if there is a faithful direct sum of irreducibles.

The radical is the intersection of the kernels of the irreducible representations of an algebra A.

(**7.2.9**) An algebra A which is finite dimensional over a field k obeys the structure and representation theory of an *Artinian ring* (see [27, §4.5]), as described in §6.7. Then **rad** A is a nilpotent ideal containing every nilpotent one sided ideal of A.

If k is algebraically closed, or even a splitting field for A, then each division ring D_i appearing in the Artin-Wedderburn Theorem for $A/\mathbf{rad}\, A$ is in fact k.

(**7.2.10**) A free module for an algebra is a direct sum of copies of the algebra as a module for itself.

7.2.3 Structure of a finite dimensional algebra over a field

Let A be a finite dimensional algebra over a splitting field. Then every simple module appears as a composition factor of the left regular module. Thus there are finitely many isomorphism classes of simple modules. The enumeration of these isomorphism classes, or more generally the construction of an indexing set, is thus a fundamental problem in representation theory.

(**7.2.11**) The regular module will not split up into a direct sum of simple modules in general (unless the algebra is semisimple); but the indecomposable summands of the regular module will be projective modules P_i:

$$A \cong \bigoplus_i d_i P_i$$

as left modules. Each such indecomposable projective module has a simple head, and the collection of these heads contains a complete set of simple modules (indeed two such indecomposable projective modules are isomorphic if and only if they have isomorphic heads, and the set of heads of a complete set of class representatives is a complete set of simple modules). Another fundamental problem is thus to compute the multiplicity of each simple module L_j as a composition factor in each P_i.

However, in general neither the simple nor projective modules are amenable to direct construction. An intermediate problem is to compute the blocks of A.

(**7.2.12**) The *blocks* of algebra A are the parts in the partition of the index set for simples given by the closure of the relation $i \sim j$ if L_j appears as a composition factor in P_i.

Matters are somewhat more straightforward under certain special circumstances, some examples of which we collect in the next few sections.

7.3 Cartan invariants (Draft)

(**7.3.1**) We will now encounter several algebras that are defined integrally (i.e. over a ring $\mathcal{Z}$ amenable to multiple distinct localisations), and that are free of finite rank as $\mathcal{Z}$-modules. For each of these, A say, we will want to study the representation theory of various f.d. algebras over fields obtained from it by base change:

$$A_k = k_{\mathcal{Z}} \otimes_{\mathcal{Z}} A$$

Among these may be semisimple cases, and others more complicated. Our next task is to consider how we might pass information (on the structure of modules) *between* these various cases, using the integral case as a conduit.

Essentially we consider two kinds of field under $\mathcal{Z}$: extensions (sometimes written K), such as the field of fractions of $\mathcal{Z}$ an integral domain; and quotients (written k), by a maximal ideal. (These may turn out to be splitting for A or not, but we will largely be able to assume that they are splitting.) The main issue will be the different methods for passing between A-modules and A_K-modules (integral basis for special modules), and between A-modules and A_k-modules (idempotent lifting).

(**7.3.2**) Let $\mathcal{Z}$ be a complete rank 1 discrete valuation ring. Let K be its field of fractions, I a maximal ideal, and $k = \mathcal{Z}/I$ the quotient field.

(For example see §5.3.4.)

(**7.3.3**) Under suitable circumstances (for example, that k and $\mathcal{Z}$ are as above, but see also later) there is, for each idempotent in A_k an idempotent in A which reduces to it. Indeed a primitive orthogonal idempotent decomposition of 1 corresponding to the decompostion of A_k (as a left module for itself) into indecomposable projectives lifts to a primitive orthogonal idempotent decomposition of 1 in A, and hence to a certain indecomposable projective decomposition of A. We write Π_i for the A-module in this decomposition corresponding to k-projective P_i. That is

$$P_i = k \otimes \Pi_i.$$

We will write S_i for the simple head of P_i.

7.3.1 Examples

(**7.3.4**) We start with some examples from the group algebras of symmetric groups. The field $\mathbb{Q}$ is a splitting field in every case here (see later), and it is known how to construct at least one primitive idempotent decomposition of 1 in $\mathbb{Q}S_n$ for each n (see e.g. [29]). Working with $(\mathbb{Q}_p, \mathbb{Z}_p, \mathbb{F}_p)$ for various choices of prime number p, the field $\mathbb{Q}_p$ of p-adic numbers always contains $\mathbb{Q}$, so the idempotents in $\mathbb{Q}S_n$ provide useful hints for constructing idempotents over the other rings. (We do not claim a systematic procedure here — just some useful low rank hints.)

(**7.3.5**) In $\mathbb{Q}S_2$, set $e'_{(2)} = 1 + (12)$ and $e_{(2)} = \frac{1}{2}e'_{(2)}$. Then $1 = e_{(2)} + (1 - e_{(2)})$ is a primitive idempotent decomposition (in this case, the only one). A corresponding decomposition will be possible over some given ring if and only if (the image of) 2 has an inverse.

In $\mathbb{F}_3S_n$ then, setting $e = 2(1 + (12))$ we have $ee = e$:

$$2(1 + (12)) \,.\, 2(1 + (12)) \;=\; 2(1 + (12)) \qquad mod.3$$

In $\mathbb{Z}_3$ we have (...0002)(...1112)=(...0001) so in $\mathbb{Z}_3S_n$

$$f \;=\; (...1112)(1 + (12))$$

obeys $f.f = f$ and reduces to e over $\mathbb{F}_3$.

In $\mathbb{F}_2S_n$ no such decomposition is possible, i.e. the idempotent 1 itself remains primitive.

(**7.3.6**) In $\mathbb{Q}S_3$ the primitive decomposition is not uniquely defined. Set $e'_{(3)} = 1 + (12) + (23) + (13) + (123) + (132)$; $e'_{(1^3)} = 1 - (12) - (23) - (13) + (123) + (132)$; and $f' = (1 + (12))(1 - (13)) = 1 + (12) - (13) - (123)$. Then $e_{(3)} = \frac{1}{6}e'_{(3)}$, $e_{(1^3)} = \frac{1}{6}e'_{(1^3)}$, and $f = \frac{1}{3}f'$ are idempotents obeying $fe_{(3)} = fe_{(1^3)} = e_{(3)}e_{(1^3)} = 0$. Setting $g = 1 - f - e_{(1^3)} - e_{(3)}$ then

$$1 = f + g + e_{(1^3)} + e_{(1^3)}$$

is a primitive idempotent decomposition.

Alternatively, $1 = e_{(2)} + (1 - e_{(2)})$ remains an orthogonal (but not primitive) decomposition. We have

$$1 \;=\; e_{(3)} \;+\; (e_{(2)} - e_{(3)}) \;+\; e_{(1^3)} \;+\; (e_{(1^2)} - e_{(1^3)})$$

as a primitive refinement of this. Indeed since $e_{(3)}$ and $e_{(1^3)}$ must appear in every primitive decomposition, this is the only decomposition refining the S_2 decomposition.

Neither decomposition above can be mirrored over another ring unless 2 and 3 have inverses. If 2 or 3 a nonunit then we can try to find a partial decomposition.

Over $\mathbb{F}_2$ we define idempotent $f_2 = (1+(12))(1-(13))$ from f, and $e_2 = 1+(123)+(132)$ from $e_{(3)} + e_{(1^3)}$; and obtain decomposition

$$1 = f_2 + e_2 + (1 - f_2 - e_2)$$

as primitive idempotent decomposition. It is easy to see that f_2 and $1 - f_2 - e_2$ are conjugate, so $\mathbb{F}_2S_3f_2$ and $\mathbb{F}_2S_3e_2$ are a complete set of inequivalent indecomposable projectives. We computed the lift of e_2 earlier: $e_I \;=\; (...10101011)(1 + (123) + (132))$. The lift of f_2 is

$$f_I \;=\; (...10101011)(1 + (12))(1 - (13)).$$

Thus we claim $\mathbb{Z}_2 S_3 f_I$ and $\mathbb{Z}_2 S_3 e_I$ are indecomposable projectives. Finally we can base-change these modules to $\mathbb{Q}_p$, where they will be projective (indeed trivially so, since the algebra is semisimple) but not necessarily indecomposable.

On the other hand, over $\mathbb{F}_3$ we can proceed at least as far as the S_2 decomposition, but then no further. Thus we get a different pair of indecomposables over $\mathbb{F}_3$, with different 'lifts' to $\mathbb{Z}_3$, and then different base-changes to $\mathbb{Q}_3$ (and hence to $\mathbb{Q}$, since the extension of $\mathbb{Q}$ to $\mathbb{Q}_3$ is not relevant to the module structure).

7.3.2 Idempotent lifting revisited

What are some other examples of $(K, \mathcal{Z}, k = \mathcal{Z}/I)$-systems where we can build connections between A and A_k as above? We shall be interested later in algebras that can be defined over $\mathbb{Z}[\delta]$, where we want to do representation theory over the field $\mathbb{C}$ (made a $\mathbb{Z}[\delta]$-algebra by mapping δ to some $\delta_c \in \mathbb{C}$). We can think of this as follows.

First embed $\mathbb{Z}[\delta] \subset \mathbb{C}[\delta]$ (this adds in inverses to all primes, and hence *precludes* passing to a field of finite chractersitic — see later for this). Then consider ideal $I = \mathbb{C}[\delta](\delta - \delta_c)$.

(**7.3.7**) Exercise. Show that $\mathbb{C}[\delta]$ is a PID and hence a Dedekind domain.

(**7.3.8**) Recall from (**??**)(ii) that there is a valuation on $\mathbb{C}(\delta)$, and hence an absolute value. Thus by Prop.**??** there is a complete ring containing it.

Indeed from (**??**)(iii) *et seq.* there is such a construction for any Dedekind domain.

(**7.3.9**) Exercise. Prove the idemotent lifting claim from (7.3.3).

We start with the idempotent refinement theorem, (6.6.20). Note that I^{i-1} is nilpotent in $\mathcal{Z}/I^i$, so $I^{i-1}A$ is nilpotent in A/I^iA. Thus we can associate idempotents in A/I^i with idempotents in A/I^{i-1}. The idea is to take a kind of limit (of idempotents in A/I^n as n gets large). We need to consider the 'suitable circumstances' provided by the ring $\mathcal{Z}$, and convince ourselves that we can reach an idempotent in A by this kind of limit.

Let us have in mind the example of $I = (x-3)\mathcal{Z}$, with $\mathcal{Z} \supseteq \mathbb{Z}[x]$ (as well as, say, $I = 3\mathbb{Z}$).

...

7.3.3 Brauer reciprocity

(**7.3.10**) Continuing from (7.3.3), we next suppose that there are A-modules $\{\Delta_i\}_i$, free of finite rank over $\mathcal{Z}$, such that $\{K \otimes \Delta_i\}_i$ are a complete set of simple modules for the split semisimple algebra A_K. In other words

$$A_K = \bigoplus_i d_i (K \otimes \Delta_i)$$

(d_i will be the rank of Δ_i, by the Artin-Wedderburn Theorem (6.3.18)), and there is a corresponding primitive orthogonal idempotent decomposition of 1 in A_K. Of course *this* decomposition will not have a lift to A in general.
(For example the idempotent $e_{(n)} = (1/n!) \sum_{w \in S_n} w \ \in \mathbb{Q}S_n$ has no lift to $\mathbb{Z}S_n$.)
However this just says that Δ_i is not projective in A. We can still use Δ_i to investigate the structure of projectives in A_k.

(**7.3.11**) We define

$$D_{ij} = [k \otimes \Delta_i : S_j]$$

— the multiplicity of simple A_k-module S_j as a composition factor in $k \otimes \Delta_i$.

This means that there is a composition series

$$k \otimes \Delta_i = M_0 \supset M_1 \supset ... \supset M_l = 0$$

and that there are D_{ij} instances of $M_a/M_{a+1} \cong S_j$ (as a varies). Consider the image under the exact functor $\mathrm{Hom}_{A_k}(P_j, -)$ of the short exact sequence $0 \to M_{a+1} \to M_a \to S_{i(a)} \to 0$:

$$0 \to \mathrm{Hom}_{A_k}(P_j, M_{a+1}) \to \mathrm{Hom}_{A_k}(P_j, M_a) \to \mathrm{Hom}_{A_k}(P_j, S_{i(a)}) \to 0$$

Note that every time $i(a) = j$, the last term had dimension $+1$, and so the dimension of the middle term is $+1$ greater than that of the first term. Thus

$$\dim_k \mathrm{Hom}_{A_k}(P_j, M) \;=\; [M : S_j]$$

(**7.3.12**) Proposition. *Under the conditions of this section (in particular that there is a projective Π_j such that $k \otimes \Pi_j = P_j$)*

$$D_{ij} = [K \otimes \Pi_j : K \otimes \Delta_i]$$

(Note that since A_K is semisimple this just counts the number of direct summands in $K \otimes \Pi_j$ isomorphic to simple module $K \otimes \Delta_i$.
Aside: This data 'lifts' to a statement about characters over A; and hence passes to one over A_k. If $\{k \otimes \Delta_i\}_i$ is a basis for the Grothendieck group of $A_k - mod$ and P_j has a filtration by the Δ-modules, then this data would determine the filtration multiplicities.)

Proof. (Outline) Since S_i lies uniquely in the head of $P_i = k \otimes \Pi_j$ and P_i is projective, we have $[M : S_j] = \dim_k \hom_{A_k}(k \otimes \Pi_j, M)$ for any A_k-module M, and in particular

$$[k \otimes \Delta_i : S_j] = \dim_k \hom_{A_k}(k \otimes \Pi_j, k \otimes \Delta_i) \tag{7.1}$$

Since Π_i is idempotently generated it is projective, so $\hom_A(\Pi_j, \Delta_i)$ has a basis. Any such basis passes to a basis for $\hom_{A_k}(k \otimes \Pi_j, k \otimes \Delta_i)$, and so must have order $[k \otimes \Delta_i : S_j]$ by (7.1). It also works as a basis for $\hom_{A_K}(K \otimes \Pi_j, K \otimes \Delta_i)$. But the natural basis for this is the set of projections onto each of the copies of $K \otimes \Delta_i$ in $K \otimes \Pi_j$, of which there are $[K \otimes \Pi_j : K \otimes \Delta_i]$. Thus $[K \otimes \Pi_j : K \otimes \Delta_i] = [k \otimes \Delta_i : S_j]$. □

We have no particular interest in (and quite possibly no way of gaining access to) $K \otimes \Pi_j$. Rather we are interested in $k \otimes \Pi_j$. The point is that if P_i has a filtration by $\{k \otimes \Delta_i\}_i$, and this set is a basis for the Grothendieck group, then the filtration multiplicities will be given by $[K \otimes \Pi_j : K \otimes \Delta_i]$, so that

$$[P_i, S_j] = \sum_k D_{kj} D_{ki}$$

In fact this holds true regardless of the filtration and basis properties, simply by considering simple characters throughout.

This says that we can completely determine these fundamental invariants of the representation theory of A_k by studying either the simple content of $\{k \otimes \Delta_i\}_i$, or (if the filtration and basis conditions *do* hold) the $\{k \otimes \Delta_i\}_i$ content, as it were, of the indecomposable projectives $\{P_i\}_i$.

(**7.3.13**) Note that, for fixed k, the index sets for $\{P_i\}_i$ and $\{S_i\}_i$ are the same, but are not necessarily the same as for $\{\Delta_i\}_i$.

We will give some rather detailed examples of the use of this machinery later.

7.4 More axiomatic frameworks

7.4.1 Summary of Donkin on finite dimensional algebras

We summarize from Cline–Parshall-Scott [], Dlab–Ringel [] and Donkin [**?**, Appendix].

First we assemble some generalities about finite dimensional algebras. Later we extend these in the quasi-hereditary case.

Here k is a field and A a finite dimensional k-algebra. $\{L(\lambda) \mid \lambda \in \Lambda^+\}$ is a complete set of simple A-modules up to isomorphism. $P(\lambda)$ and $I(\lambda)$ are the corresponding projective cover and injective envelope of $L(\lambda)$.

We shall routinely notationally confuse $L(\lambda)$ with λ (where no ambiguity can arise). For π any subset of Λ^+, we say A-module V belongs to π if $[V : L(\lambda)] \neq 0$ implies $\lambda \in \pi$. Fixing π, there is a unique maximal submodule of $V \in A-\text{mod}$ denoted $O_\pi(V)$; and a unique minimal submodule $O^\pi(V)$ such that $V/O^\pi(V)$ belongs to π.

Fixing a partial order $\leq$ on Λ^+ we define $\pi(\lambda)$ as the $\leq$-ideal below λ. Define $M(\lambda)$ as the unique maximal submodule of $P(\lambda)$; $K(\lambda) = O^{\pi(\lambda)}(M(\lambda))$ and $\Delta^{\leq}(\lambda) = P(\lambda)/K(\lambda)$.

Note that whatever the partial order, $\Delta^{\leq}(\lambda)$ has simple head $L(\lambda)$, and $L(\lambda)$ is *not* a composition factor of the maximal submodule.

(**7.4.1**) Lemma. For all $\lambda \in \Lambda^+$, $\text{End}_A(\Delta^{\leq}(\lambda)) = k$.

Let us try to prove the more general assertion:

(**7.4.2**) Lemma. SURELY FALSE CLAIM: Let module M have simple head L and no other composition factor L. Then $\text{End}(M) = k$.

7.4.2 Quasi-hereditary algebras

See Dlab–Ringel [18], Cline–Parshall–Scott [10] or ...

7.4.3 Cellular algebras

See Graham—Lehrer [**?**].

7.5 Forms, module morphisms and Gram matrices (Draft)

(**7.5.1**) A group algebra is isomorphic to its opposite (defined as for opposite ring) since $g \mapsto g^{-1}$ defines a group antiautomorphism (an isomorphism $G \cong G^{op}$).

There may be other isomorphisms. For example, a suitable group of matrices may be mapped to its opposite by $g \mapsto g^{tr}$ (transpose matrix). Here, when considering any algebra isomorphic to its opposite, we will generally fix a given involutive antiautomorphism.

(**7.5.2**) Let R be a commutative ring and let A be any R-algebra with an involutive antiautomorphism (generally denoted $g \mapsto g^t$). It follows that for each $M \in A-\text{mod}$ there is another left module M^o obtained from M^* by this isomorphism (regarding M^* as a left module for the opposite), with the property that $R-\text{mod}$ is invariant under taking to its dual combined with taking all $M \mapsto M^o$. (I.e. if defined, $(\text{Head } M)^o \cong \text{Soc } M^o$, and so on.)

We will call the map $M \mapsto M^o$ *contravariant duality* (see e.g. [22]). We have $(M^o)^o = M$.

(**7.5.3**) Exercise. Curtis–Reiner [**?**]:
1. They claim (10.29) that 'contragredient' of projective RG-module is projective. How do we square this with being false for general algebra A with t?
(For the symmetric group we will see in (7.5.13) that the regular module is contravariant self-dual for any R. Thus the collection of indecomposable summands over R a field must be fixed under duality, which verifies the claim in this case. However the regular module is not always self-dual for an algebra A with t (we shall have an example from the Temperley–Lieb algebras shortly).)
2. Why are duals of lattices done differently p.89 cf. p.245?

(**7.5.4**) Exercise. Claim: Suppose A is in fact a finite group algebra over R. Let $x \in A$ be mapped to x^o by the opposite isomorphism (and regard x^o as an element of A). Then $M = Ax$ implies $M^o = Ax^o$.
Prove this, or provide a counter-example.

(**7.5.5**) Proposition. *Suppose that R is a field, and A is an R-algebra with a given involutive antiautomorphism. If left A-module M has simple head L, and this composition factor has multiplicity one in M, and $L \cong L^o$, then* $\dim Hom_A(M, M^o) = 1$, *and* $\psi \in Hom_A(M, M^o)$ *has rank* $\dim(L)$.

Proof: NB, every simple factor in M^o is extended by L below it. There *is* a map $\psi \in \mathrm{Hom}_A(M, M^o)$ — that which kills the unique maximal proper submodule M_o and so makes the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & M_o \arrow[r] & M \arrow[r] \arrow[d, "\psi"'] & L \arrow[r] \arrow[d, "\cong"'] & 0 \\
0 & (M_o)^o \arrow[l] & M^o \arrow[l] & L^o \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

(or any scalar multiple thereof). No reduction is possible in the kernel, since this would require factors appearing in the image below L, which M^o does not have, as already noted. No enlargement of the image is possible since this, correspondingly, requires factors above L in M. □
(NB, the converse does not hold in general.)

(**7.5.6**) Bilinear form:
A bilinear function

$$<,>: M \times N \to R$$

on $M, N \in R-\text{mod}$. (cf. Perlis [36], Maclane-Birkoff [31, §X.1])
Suppose M, N free. Let $B_M = \{b_1, b_2, \ldots, b_k\}$ be an ordered basis of M and $B_N = \{c_1, c_2, \ldots, c_l\}$ of N. Then a matrix $B(M, N)$ of form

$$(B(M,N))_{ij} \ = < b_i, c_j >$$

determines $<,>$. That is, in an obvious notation,

$$B(M,N) = \begin{pmatrix} \langle b_1| \\ \langle b_2| \\ \vdots \\ \langle b_k| \end{pmatrix} \begin{pmatrix} \ |c_1\rangle, \ |c_2\rangle, \cdots, \ |c_l\rangle \ \end{pmatrix}$$

By linearity, the effect of basis changes $B'_M = B_M U$, $B'_N = B_N W$ (U, W unimodular) is

$$B'(M,N) = U^t \begin{pmatrix} \langle b_1| \\ \langle b_2| \\ \vdots \\ \langle b_k| \end{pmatrix} \begin{pmatrix} |c_1\rangle, & |c_2\rangle, \cdots, & |c_l\rangle \end{pmatrix} W \;=\; U^t B(M,N) W$$

Recall that if R is a PID then among the possible such basis changes would be a pair that bring $B'(M,N)$ into Smith normal form.

(**7.5.7**) Recall that $N^* = \mathrm{Hom}_R(N,R)$ has the structure of R-module. Note that $<,>$ defines an R-module homomorphism $\psi : M \to N^*$ by $\psi(m)(n) =< m, n >$. From this perspective we can think of $B'(M,N)$ as characterising the image of M in N^*.

If R is a field, then the rank of matrix $B(M,N)$ is independent of the specific choice of bases (to see this note that it is unchanged on replacing b_1 by a linear combination with nonzero component of b_1). Accordingly

$$\text{rank } <,> \;:=\; \text{rank } (B(M,N)).$$

(**7.5.8**) For I an ideal of R define $L^I_{<,>}$ as the subset of M such that $< m, n > \in\; I$ for every $m \in L^I_{<,>}$ and $n \in N$. By linearity $L^I_{<,>}$ is a submodule.
The Left radical $L_{<,>} = L^0_{<,>}$ of $<,>$ is the submodule M' of M such that $< m, n > =\; 0$ for every $m \in M'$ and $n \in N$.

If R is a field then

$$\text{rank } \;<,>\; = \; \dim M - \dim L_{<,>}.$$

(**7.5.9**) Contravariant form.
Let A be an R–algebra with t as above. For $M, N \in A-$mod, an R–bilinear form $<,>: M \times N \to R$ is *contravariant* if $< am, n > = < m, a^t n >$ for all $a \in A$, $m \in M$, $n \in N$.

(**7.5.10**) Proposition. *Let A be an R-algebra with involutive antiautomorphism t, and $M, N \in A -$ mod. If $<,>: M \times N \to R$ is a contravariant form then the subset $S = L^I_{<,>} \subseteq M$ (such that $< s, N > \in I$ for all $s \in S$) is an A-submodule of M.*

Proof. For $a \in A$, $s \in S$, $n \in N$ we have $< as, n >=< s, a^t n > \in I$ since $a^t n \in N$, so $as \in S$. □

(**7.5.11**) Remark. Note that if $I' \subset I$ then $L^{I'}_{<,>} \subseteq L^I_{<,>}$. Of course if R is a field then the only possibility is $I = 0$.
Note that if we start with R a commutative ring and compute $S = L^0_{<,>}$; then base change to some $A^k = k \otimes_R A$ and write S^k for the corresponding submodule computed here, then S^k may be bigger than $k \otimes_R S$.

(**7.5.12**) Proposition. *Let R be a commutative ring and $A = RG$ for some group G (or else an R-algebra with involutive antiautomorphism).*
(I) To each contravariant form $<,>: M \times N \to R$ we may associate an element $\psi \in \hom_A(M, N^o)$ given by $\psi(m)(n) = <m, n>$.
(II) This association defines a bijective correspondence between forms and morphisms.
(III) If R is a field and $M = N$ satisfies the assumptions in Proposition (7.5.5) then there is a unique form up to scalars, and the form is non–singular iff the associated ψ is an isomorphism (in particular it is non–singular if $M = N$ is simple). (cf. [22, §2.7].)

Proof. (I) We first need to show that ψ defined in this way is a homomorphism of left A–modules, i.e. that $a\psi(m) = \psi(am)$. Putting aside the way A acts on it for a moment we have $N^o = N^* = \mathrm{Hom}_R(N, R)$, so $\psi(m) \in \mathrm{Hom}_R(N, R)$. By construction we have that $\psi(am) \in \mathrm{Hom}_R(N, R)$ is given by:

$$\psi(am)(n) \; = < am, n > = < m, a^t n > = \; \psi(m)(a^t n)$$

Meanwhile for $a\psi(m)(n)$ the action of a on the left is achieved by the action of a^t on the right of N^*, which we recall is given by $(\phi a^t)(n) = \phi(a^t n)$ for any $\phi \in N^*$. Thus $(a \circ \psi(m))(n)) = (\psi(m)a^t)(n) = \psi(m)(a^t n)$ as required.

(II) Note that for given $\psi \in \hom_A(M, N^o)$ we can define a form by $<m, n>_\psi \; = \; \psi(m)(n)$.

(III) Finally observe (cf. proposition 6.5.12, noting that the difference between N^o and N^* is not relevant, since the algebra action will not be used) that the rank of the image under ψ is rank $<,>$. $\square$

7.5.1 Examples

(**7.5.13**) Example. The symmetric group S_3 acts on the set of sequences $\{211, 121, 112\}$ by place permutation:

$$g_2(g_1 211) = g_2 121 = 112 = (g_2 g_1)211$$

These sequences thus form a basis for a left $\mathbb{Z}S_3$-module, M.

The bilinear form on M given by $< t, t' > \; = \; \delta_{t,t'}$ is contravariant. We can see this as follows. If $gt = t'$ then $g^{-1}t' = t$ so $< gt, t' > = < t, g^{-1}t' >$.

Note that the rank of a form depends, in general, on the ground field. However in our case there is clearly no such dependence. Since this form is of full rank it defines an isomorphism between M and M^o. (Of course M does not satisfy the conditions of proposition 7.5.5.)

(**7.5.14**) We can restrict our form to a form on a submodule S. For example, consider the element of $\mathbb{Z}S_3$ given by $V_{13} = 1 - (13)$ and the submodule of M generated by

$$e_{112} \; := \; V_{13}112 \; = 112 - 211$$

This is spanned by e_{112} and $e_{121} = 121 - 211$. (This submodule will turn out to satisfy the conditions of proposition 7.5.5.) The restricted form has Gram matrix

$$\begin{pmatrix} \langle e_{112}| \\ \langle e_{121}| \end{pmatrix} \begin{pmatrix} |e_{112}\rangle, & |e_{121}\rangle \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(in the obvious notation), for which

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \tag{7.2}$$

is an equivalent, giving Gram determinant 3. So the Gram matrix has full rank over $\mathbb{Q}$, but not so over certain other fields. As we shall see, over a field F this submodule S has a simple head which may or may not be the whole thing, depending on F. If it is not the whole thing then the submodule is not a direct summand of the original module M (since this is contravariant self-dual).

We note that the unimodular transformations applied in (7.2) give

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \langle e_{112}| \\ \langle e_{121}| \end{pmatrix} \begin{pmatrix} |e_{112}\rangle, & |e_{121}\rangle \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \langle e_{112} + e_{121}| \\ \langle e_{121}| \end{pmatrix} \begin{pmatrix} |2e_{112} - e_{121}\rangle, & |-e_{112} + e_{121}\rangle \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

(**7.5.15**) Fixing a field F, let S' be the set of vectors in M orthogonal to the above submodule S with respect to the original form ($112 + 121 + 211 \in S'$ for example, over any field). Note that S' is another submodule. Note that, depending on F, this new submodule is either non-intersecting of S, in which case the restricted form is of full rank, or intersects S in a submodule of dimension given by the discrepancy between the full rank and the actual rank of the form on S. (In our case $S \ni 2e_{112} - e_{121} = 2.112 - 121 - 211 \equiv -(112 + 121 + 211)$ over a field of characteristic 3, so the new module, over such a field, intersects S.) We will see that the quotient of S by $S \cap S'$ is simple (indeed absolutely irreducible). This is because of the following key result.
Evidently $V_{13}121 = 0$, so for any $m \in M$ we have $V_{13}m \in Fe_{112}$.
Suppose m is in some submodule T of M, so either $V_{13}m = 0$ or $V_{13}m \neq 0$. In the latter case $e_{112} \in T$ so $S \hookrightarrow T$. In the former case

$$0 =< V_{13}m, 112 >=< m, V_{13}112 >=< m, e_{112} >$$

so $T \hookrightarrow S'$. Now suppose in particular that T is a submodule of S in M. Then either it is the whole of S, or it is also in S', and hence in $S \cap S'$. This shows that $S/S \cap S'$ is irreducible.
This is the same as to say that its Gram matrix is non-singular over F.

(**7.5.16**) There are several further exampled later. See (8.5.1), (**??**) ...

(**7.5.17**) JOBS: FIX AND FINISH THE ABOVE!

Chapter 8

Representations of the symmetric group

Recall that S_n is the group of permutations of a set of n objects.

8.1 Introduction

We write (ij) for the elementary transposition in the symmetric group S_n. Note that S_n is generated by $\{(i\,i{+}1) \mid i = 1, 2, ..., n-1\}$.

The trivial representation of S_n is given by $R_{(n)}((12)) = 1$; and the alternating representation by $R_{(1^n)}((12)) = -1$.

(**8.1.1**) We write $(i_1, i_2, ..., i_k) \in S_n$ for the cyclic permutation of the listed distinct elements of $\{1, 2, ..., n\}$.

Any two such cyclic elements of S_n commute if they have no list element in common. Thus a set partition of $\{1, 2, ..., n\}$ and an ordering of the elements in each part defines an element w of S_n. The *cycle structure* of w is the integer partition associated to the set partition. Every element of S_n may be expressed in this form. Two elements are in the same class if and only if they have the same cycle structure. We have shown the following.

(**8.1.2**) Theorem. The classes of S_n may be indexed by the integer partitions of n.

8.2 Young diagrams and the Young lattice

Integer partitions and their Young diagrams play a useful role in S_n representation theory. Here we review some relevant properties (revisiting the topics of §3.3).

(**8.2.1**) Let $(S, >)$ be a poset, and $s, t \in S$. We say *s covers t* if $s > t$ and there does not exist $s > u > t$.

(**8.2.2**) Recall that Λ is the set of all integer partitions. The subpartition order $(\Lambda, \supset)$ is a lattice (the *Young lattice* — meet is partition intersection and join is union).
In this case λ covers μ if λ/μ is a single box. See Figure 8.1.

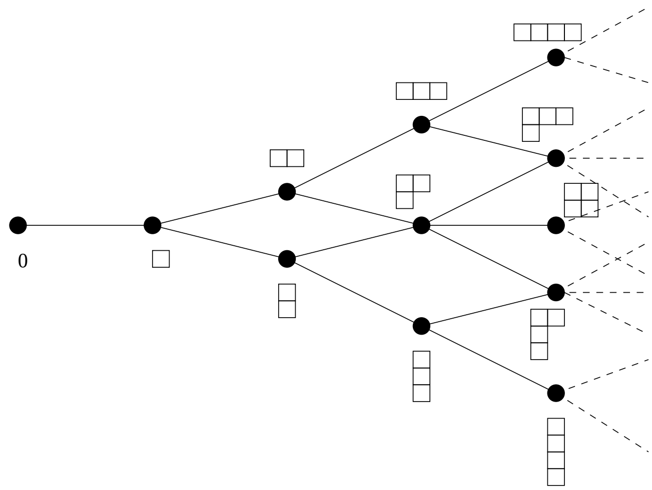
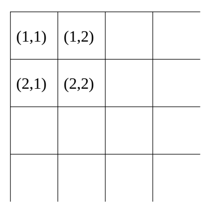


Figure 8.1: The Young graph (covering DAG of the Young lattice, increasing from left to right).

(**8.2.3**) As usual we visualise integer partitions as Young diagrams. One first pictures the lower right quadrant of the plane partitioned into unit boxes (it is useful to label the boxes by their row and column position in this arrangement — in matrix labelling, as it were; so that the top-left box has label $(1,1)$):



We associates to each such box b the rectangle of boxes $r(b)$ between it and the top-left box. We define a *light-cone order* on the set of boxes by $b \geq b'$ if $b' \in r(b)$. If $b \geq b'$ we say b *pins* b'.

For B a set of boxes, define

$$r(B) = \bigcup_{b \in B} r(b)$$

A Young diagram is a subset of boxes such that if a box is included, then every box in its rectangle is included. That is, B is a Young diagram if $B = r(B)$.

We identify a partition λ with the diagram whose i-th row has length λ_i.

(**8.2.4**) For any box b there is a minimal $\lambda \in \Lambda$ containing this box, and this coincides with $r(b)$. Given a partition μ and a box, there is a minimal $\lambda \in \Lambda$ containing both. Given a partition μ and a box b, and hence a container λ, the skew λ/μ is called the skew over μ *pinned* by b.

(**8.2.5**) Lemma. Fix a diagram μ. For a set of boxes γ to be a skew λ/μ it must not intersect μ, and must not pin any box outside $\gamma \cup \mu$.

(**8.2.6**) The *Young matrix* is the (semiinfinite) adjacency matrix of the underlying (undirected) graph of the Hasse graph of the Young lattice.

(**8.2.7**) For each pair of Young diagrams λ, μ there is a skew diagram

$$\lambda \setminus \mu := \lambda/(\lambda \cap \mu)$$

(i.e. such that a box is in $\lambda \setminus \mu$ if it occurs in λ but not in μ). We also define a skew diagram

$$\lambda \Delta \mu := (\lambda \setminus \mu) \cup (\mu \setminus \lambda) = \lambda \cup \mu/(\lambda \cap \mu)$$

8.3 Representations of S_n from Set

Recall the category **Set** from §4.1. For convenience define $\mathbf{Set}(m,n) = \mathbf{Set}(\underline{m},\underline{n})$. Recall that the category composition equips $\mathbf{Set}(m,m)$ with the property of monoid; and $\mathbf{Set}(m,n)$ with the property of left $\mathbf{Set}(m,m)$-set; and right $\mathbf{Set}(n,n)$-set. For any commutative ring R these sets extend R-linearly to modules. Since $S_m \subset \mathbf{Set}(m,m)$ we can also build a left S_m-module by restriction (respectively a right S_n-module).

What does $\mathbf{Set}(3,2)$ look like as a left $\mathbf{Set}(3,3)$- or left S_3-module? In our notation the (ordered) basis is $\{111, 112, 121, 122, 211, 212, 221, 222\}$ and we have

$$R_{3,2}((12)) = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 0 && 1 &&& \\ &&& 0 && 1 && \\ && 1 && 0 &&& \\ &&& 1 && 0 && \\ &&&&&& 1 & \\ &&&&&&& 1 \end{pmatrix}$$

We return to consider the general case shortly.

8.3.1 Connection with Schur's work and Schur functors

In this subsection we follow Green[22, §3,§6] closely. (Note that §6.2 of Green is also focal to §9.7.)

(**8.3.1**) Note that Green uses $I(n,r)$ for $\mathbf{Set}(r,n)$, and writes $i \in I(n,r)$, that is $i : \underline{r} \to \underline{n}$, as a vector or multi-index: $i = (i_1, ..., i_r)$.

(**8.3.2**) Let $G = GL_n(K)$ be the group of invertible matrices over an infinite field K. We regard $\mathbf{Set}(G,K)$ as a commutative K-algebra by $ab(g) = a(g)b(g)$; $(a+b)(g) = a(g) + b(g)$.

The identity element is given by $1(g) = 1_K$ for all g.

For each $s \in G$ we define $L_s \in \mathrm{End}_K(\mathbf{Set}(G,K))$ by $L_s f(g) = f(sg)$ (and define $R_s f(g) = f(gs)$ similarly).

Proposition. The map $R(s) = R_s$ is a representation of G (and L gives an antirepresentation). Thus $\mathbf{Set}(G,K)$ is a left KG-module:

$$sf = R_s f \qquad \text{(similarly } fs = L_s f\text{)}$$

The two actions commute.

(**8.3.3**) Let $c_{ij} : G \to K$ be given by $c_{ij}(g) = g_{ij}$. Write A for the K-subalgebra of $\mathbf{Set}(G, K)$ generated by all the c_{ij}s.

For each r we write $A(n, r)$ for the subspace of A of polynomials homogeneous of degree r in the c_{ij}s. For example $A(n, 1) = K\{c_{11}, c_{12}, ..., c_{1n}, c_{21}, c_{22}, ..., c_{2n}, ..., c_{nn}\} = K\{c_{ij}\}_{ij}$ (a K-space of dimension n^2). Thus A has grading

$$A = \sum_r A(n, r)$$

as a K-algebra.

For $i, j \in \mathbf{Set}(r, n)$

$$c_{ij} \;:=\; c_{i_1 j_1} ... c_{i_r j_r} \tag{8.1}$$

Then $A(n, r)$ is spanned by these monomials. Note that the RHS of (8.1) does not determine i, j.

(**8.3.4**) Define

$$S_K(n, r) = \hom_K(A(n, r), K)$$

This has basis $\{\xi_{ij} \mid i, j \in \mathbf{Set}(r, n)\}$ with ξ_{ij} given by

$$\xi_{ij}(c_{kl}) = \begin{cases} 1 & if \quad (i, j) \sim (k, l) \\ 0 & o/w \end{cases}$$

where $\sim$ means the orbit of the diagonal action of S_r.

Algebra $A(n, r)$ is a coalgebra, so the dual $S_K(n, r)$ is an associative algebra. We have (see Schur [38], Green [22, p.21], or Martin–Woodcock [**?**])

$$\xi_{ij}\xi_{kl} \;=\; \sum_{p,q} Z(i, j, k, l, p, q).1_K \; \xi_{pq} \tag{8.2}$$

where the sum is over a transversal of $\sim$; and

$$Z(i, j, k, l, p, q) = |\{s \in \mathbf{Set}(r, n) \mid (i, j) \sim (p, s),\ (k, l) \sim (s, q)\}|$$

(**8.3.5**) For example $\xi_{ii}^2 = \xi_{ii}$ and $\xi_{ii}\xi_{jj} = 0$ if i, j not in the same orbit of the S_r-action (i.e. if $\xi_{ii} \neq \xi_{jj}$). Indeed

$$1_{S_K(n,r)} = \sum_i \xi_{ii}$$

where the sum is over the distinct elements.

(**8.3.6**) Note from (8.2) that the $\mathbb{Z}$-submodule $S_{\mathbb{Z}}(n, r)$ of $S_{\mathbb{Q}}(n, r)$ generated by the ξ_{ij} is multiplicatively closed. That is, it is a $\mathbb{Z}$-order in $S_{\mathbb{Q}}(n, r)$. For any field K there is an isomorphism of K-algebras $S_{\mathbb{Z}}(n, r) \otimes_{\mathbb{Z}} K \cong S_K(n, r)$. In this sense, for fixed n, r, the 'scheme' or family of algebras $S_K(n, r)$ is 'defined over $\mathbb{Z}$'.

(**8.3.7**) Let $\Lambda(n, r)$ be the set of S_r-orbits in $\mathbf{Set}(r, n)$ (the set of 'weights'). For example, for $r \leq n$ there exist functions $i \in \mathbf{Set}(r, n)$ of form $i = (s(1), s(2), ..., s(r))$, where $s \in S_r$. The weight w of any such i is

$$w = (1, 1, ..., 1, 0, 0, ..., 0) \in \mathbb{Z}^n$$

(r nonzero entries).

Consider the commuting S_n action on $\mathbf{Set}(r,n)$. Each S_n-orbit contains one dominant weight. Write $\Lambda^+(n,r)$ for the set of dominant weights.

(**8.3.8**) If $i \in \mathbf{Set}(r,n)$ belongs to $a \in \Lambda(n,r)$ we may write ξ_a for ξ_{ii}.

(**8.3.9**) Proposition. There is an isomorphism of K-algebras

$$\xi_w S_K(n,r)\xi_w \;\cong\; KS_r$$

which takes $\xi_{us,u}$ to s for all $s \in S_r$.

(**8.3.10**) This allows us to construct a 'Schur' functor relating the representation theory of the Schur algebra $S_K(n,r)$ (and hence part of the representation theory of the general linear group) to the symmetric group S_r:

$$F : S_K(n,r) - \text{mod} \;\to\; KS_r - \text{mod} \tag{8.3}$$

where $FM = \xi_w M$.

(**8.3.11**) (TO CONTINUE we should summarize [22, §3.2].)

(**8.3.12**) A closely related idea is that, CLAIM:

$$S_K(n,r) \cong \text{End}_{KS_r}((K^n)^{\otimes r})$$

8.3.2 Idempotents and other elements in $\mathbb{Z}S_n$

(**8.3.13**) For $x \subset \underline{n}$ we write S_x for the subgroup of S_n in which only the elements in x may be permuted nontrivially. If $p = \{p_1, p_2, ...\}$ is a set partition of $\underline{n}$ we write S_p for the Young subgroup

$$S_p = S_{p_1} S_{p_2} ...$$

(**8.3.14**) A *composition* of n (into m parts) is a sequence of elements of $\mathbb{N}_0$ (of length m) that sums to n. We may associate a composition to each ordered set partition of $p = \{p_1, p_2, ...\}$ by

$$\lambda_i = |p_i|$$

It will be evident that

$$S_p \cong \times_i S_{\lambda_i}$$

For each integer partition $\lambda \vdash n$ we associate a set partition by

$$p(\lambda) = \{\{1, 2, .., \lambda_1\}, \{\lambda_1 + 1, .., \lambda_1 + \lambda_2\}, ...\}$$

We write $S_\lambda = S_{p(\lambda)}$. When the factor groups are simply arranged side-by-side in this way we write $\otimes_i w_i$ for the image of $(w_1, w_2, ...) \in \times_i S_{\lambda_i}$ in S_n.

(**8.3.15**) Define elements of $\mathbb{Z}S_n$ by

$$e'_{(n)} = \sum_{w \in S_n} R_{(n)}(w)\; w$$

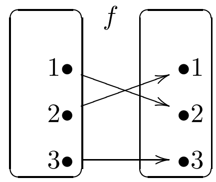


Figure 8.2: Mapping diagram of $(12) \in S_3$.

$$e'_{(1^n)} = \sum_{w \in S_n} R_{(1^n)}(w)\, w$$

Note that

$$e'_{(n)} e'_{(1^m)} = 0 \qquad \text{for } m, n > 1 \tag{8.4}$$

Using the side-by-side notation define

$$e'_\lambda = \otimes_i e'_{(\lambda_i)}$$

$$f'_\lambda = \otimes_i e'_{(1^{\lambda'_i})}$$

where λ' is the transpose partition to λ.

(**8.3.16**) Define a map C_L from S_n to set partitions of $\underline{n}$ as follows. Draw the mapping diagram of $w \in S_n$ (as illustrated in Figure 8.2), and put i, j $(i < j)$ in the same part of $C_L(w)$ if the lines from i, j cross, i.e. if $i < j$ and $w(i) > w(j)$. Define $C_R(w) = C_L(w^{-1})$.

Example: $C_L((12)) = \{\{1, 2\}, \{3\}, ...\}$.

(**8.3.17**) For any set partition $p \in P_{\underline{n}}$ we call a $w \in S_n$ left p-noncrossing if $i \sim^p j$ implies $i \not\sim^{C_L(w)} j$.
The idea here is that if we look at any of the parts of p regarded as a subset of $\underline{n}$, we will find that the corresponding strings (as labelled on the left) do not cross each other in w.
For example, if p is the partition into singletons then every $w \in S_n$ is p-noncrossing. If $p = \{\underline{n}\}$ then only $1 \in S_n$ is p-noncrossing.

Define right p-noncrossing similarly. Write $S_n(p, q)$ for the set of left p-noncrossing right q-noncrossing elements of S_n. Write $S'_n(p, q)$ for the subset of elements such that no two strings in the same part at p are in the same part at q.

(**8.3.18**) Proposition. *There is a $w_\lambda \in S_n$ such that $e'_\lambda w_\lambda f'_\lambda \neq 0$ and*

$$e'_\lambda w f'_\lambda \;=\; \epsilon_\lambda(w)\; e'_\lambda w_\lambda f'_\lambda$$

with $\epsilon_\lambda(w) \in \{0, \pm 1\}$, for all $w \in S_n$.

Proof. A conceptual/diagramatic proof is effective here. Draw a box on the left for each factor in e'_λ and one on the right for each factor in f'_λ. By (8.4) if w connects any box on the left to any box on the right by more than one line, then $e'_\lambda w f'_\lambda = 0$. Also, if two lines coming out of any box cross this crossing can be removed (at cost a factor -1 if it is a box on the right). Since λ'_i is the number of boxes on the left with at least i lines, one sees that there is precisely one w producing a non-vanishing product and with no removable crossings. This is w_λ. One readily confirms that $e'_\lambda w_\lambda f'_\lambda = w_\lambda + ...$ is non-vanishing.

8.3.3 Young modules

Recall from §4.2 and 4.3 that for k a ring, $k\mathbf{Set}^f$ is the k-linear category over $\mathbf{Set}^f$, and $kC_{\mathbb{N}}$ is a corresponding skeleton. As noted there, S_n is the group of isomorphisms in $\hom(n,n)$, so $k\hom(n,m)$ is a left kS_n right kS_m bimodule by restriction. Recall that the basis $\hom(n,m)$ of functions $f : \underline{n} \to \underline{m}$ may be written as words $f(1)f(2)...f(n)$ in $\underline{m}$. Then the action of S_n is to permute the n entries in this list; while the action of S_m is to permute the set of symbols. For example

$$(23)1244 = 1424$$

$$1244(23) = 1344$$

(**8.3.19**) We may associate a composition λ of n to each function f in $\hom(n,m)$ by

$$\lambda_i = \#\{j \mid f(j) = i\}$$

For example $\lambda(12444) = (1,1,0,3)$.
(Note that the notation $12444 \in \hom(n,m)$ tells us that $n = 5$, but only that $m > 3$. In the above we have assumed $12444 \in \hom(5,4)$. If we regard $12444 \in \hom(5,6)$ then we have $\lambda(12444) = (1,1,0,3,0,0)$.)

For $\lambda \in \mathbb{N}_0^m$, we write $\hom(n,\lambda)$ for the subset of $\hom(n,m)$ of functions of fixed λ. For example

$$\hom(4,(3,1)) = \{1112, 1121, 1211, 2111\}$$

(For the moment we leave this simply as an abuse of notation, as far as the categorical context is concerned.)

Note that the left action of S_n on $\hom(n,m)$ fixes this composition, so we may decompose $k\hom(n,m)$ as a direct sum of left modules indexed by compositions:

$$k\hom(n,m) \cong \bigoplus_{\lambda \in \mathbb{N}_0^m} k\hom(n,\lambda)$$

Example:

$$k\hom(3,2) \cong k\{111\} \oplus k\{112, 121, 211\} \oplus k\{122, 212, 221\} \oplus k\{222\}$$

Any two such left submodules are isomorphic if their compositions are related by a reordering of the terms in the sequence λ (since the invertible right action by S_m achieves all such reorderings). We will take integer partitions to be representative elements of the orbits of compositions under this action.

(**8.3.20**) We now consider the integral, or ring independent, decomposition of the regular left $\hom(n,n)$-module $k\hom(n,n)$. We are interested in this as a left kS_n-module by restriction.

We have seen that formally ignoring the difference in codomain gives an inclusion of sets $\hom(n,m) \hookrightarrow \hom(n,m+1)$. This gives an example of an injection $k\hom(n,m) \hookrightarrow k\hom(n,m+1)$, and indeed this injection is split. The sections contain sums of the $k\,\mathrm{hom}(n,\lambda)$'s with precicely $m+1$ parts.

It follows that we can find all the simple left modules for kS_n by looking in the $k\,\mathrm{hom}(n,\lambda)$'s with $\lambda \vdash n$.
(From now on we shall mean $\lambda \vdash n$ by λ, unless otherwise stated.)
(Indeed the claim follows directly on noting that $k\,\mathrm{hom}(n,(1^n))$ is isomorphic to kS_n as a left module. But we shall make use of the others too.)

(**8.3.21**) In particular if k is a field, all simple left kS_n-modules will appear as composition factors for this collection of modules — $\{k\,\mathrm{hom}(n,\lambda) \mid \lambda \vdash n\}$ — which we shall call the *Young modules* of kS_n. However it will be evident that these modules are not themselves simple in general. For example

$$e'_{(n)} k\,\mathrm{hom}(n,\lambda) \neq \{0\}$$

for any λ, so $e'_{(n)} k\,\mathrm{hom}(n,\lambda)$ is a proper submodule of $k\,\mathrm{hom}(n,\lambda)$ for any $\lambda \neq (n)$.

On the other hand

$$e'_{(1^n)} k\,\mathrm{hom}(n,\lambda) = \{0\} \qquad \text{for any } \lambda \neq (1^n).$$

This gives us a clue as to how to extract useful submodules from the Young modules more systematically.

(**8.3.22**) Proposition. *(I) Let $\lambda \vdash n$. The left kS_n module $kS_n e'_\lambda$ is k-free with basis the elements of form we'_λ with $w \in S_n$ such that no two lines cross if they meet the same 'symmetriser' factor $e'_{(\lambda_i)}$ in e'_λ. (II) There is a bijection between this basis and* $\mathrm{hom}(n,\lambda)$ *obtained by modifying w to an element of* $\mathrm{hom}(n,\lambda'_1)$ *by making each of these subsets of λ_i noncrossing lines meet at a point i on the target side. (III) This bijection extends k-linearly to an isomorphism of left modules:*

$$kS_n e'_\lambda \cong k\,\mathrm{hom}(n,\lambda)$$

Proof. (I) A spanning set for the LHS is elements of form we'_λ with $w \in S_n$ such that no two lines cross if they meet the same 'symmetriser' factor in e'_λ. One can check that we'_λ contains the group element w with coefficient 1, and no other such group element, so the set is k-free and a basis. (II,III) The map described is evidently a set bijection and hence an isomorphism of free k-modules, but it also commutes with the S_n action (indeed the computation of this action is essentially the same computation on each side). □

8.3.4 Specht modules

See for examples James' Lecture Notes [28]. Here we give a quick summary; with more details in the next section.

(**8.3.23**) Comparing Prop. (8.3.22) with Prop. (8.3.18) it follows that $f'_\lambda k\,\mathrm{hom}(n,\lambda)$ is a rank-1 k-module.

QUESTION/CAVEAT: How do we know that the non-vanishing claim in Prop. (8.3.18) does not fail in finite characteristic?

(**8.3.24**) The kS_n-modules of form

$$\mathcal{S}(\lambda) \;:=\; S_n f'_\lambda k\,\mathrm{hom}(n,\lambda) \;\; \cong S_n f'_\lambda k S_n e'_\lambda$$

are called *Specht* modules after [39].

These are free modules of finite rank over $\mathbb{Z}$, and hence the rank is not affected by base change to k. (For this reason the dependence of $\mathcal{S}(\lambda)$ on k is often left implicit. However some properties do depend on k.)

(**8.3.25**) PROPOSITION. [James] For field k of characteristic $p \neq 2$ this $\mathcal{S}(\lambda)$ is an indecomposable kS_n submodule of $k\,\hom(n, \lambda)$ (cf. (7.1.4)).

However for $p = 2$ the Specht module with $\lambda = (5, 1, 1)$ is decomposable (and Murphy shows that there are infinitely many others such).

(**8.3.26**) An integer partition λ is *p-regular* if no part is repeated p or more times.

(**8.3.27**) Let k be a field of characteristic $p > 0$. James has shown that if λ is p-regular then $\mathcal{S}(\lambda)$ has simple head over k, and that

$$\{L^k(\lambda) = \text{head}\,{}^k\mathcal{S}(\lambda) \;\mid\; \lambda \;\; p\text{-regular}\}$$

is a complete set of kS_n-modules.

(**8.3.28**) Basis / restriction rules — see below.

The notation $\mathcal{S}(\lambda)^\perp$ for the next Theorem is explained in the following section (where the module $S^\lambda \cong \mathcal{S}(\lambda)$).

(**8.3.29**) Let $\triangleleft$ denote the dominance order. Note that dictionary order is a total order refining the dominance order.

For k be a field of char. p, let

$$D_n^p = [\mathcal{S}(\lambda) : L^k(\mu)]_{\lambda,\mu}$$

be the S_n Specht decomposition matrix over k; where for the rows p-regular partitions and then other partitions (and for the columns, p-regular partitions) are written out in the dictionary order.

(**8.3.30**) THEOREM. [James78 12.2] Fix k a field of char. p, and any n. Then $L^k(\lambda) = \mathcal{S}(\lambda)/(\mathcal{S}(\lambda) \cap \mathcal{S}(\lambda)^\perp)$ if λ is p-regular; and $[\mathcal{S}(\lambda) \cap \mathcal{S}(\lambda)^\perp : L^k(\mu)] \neq 0$ implies $\mu \triangleright \lambda$ for all λ; and $[\mathcal{S}(\lambda) : L^k(\mu)] \neq 0$ implies $\mu \triangleright \lambda$ for λ non-p-regular.

That is, D_n^p is lower unitriangular.

(**8.3.31**) In e.g. Hemmer [**?**] (see also Green [22, §6.3]) it is asserted that the Schur functor from (8.3) takes Weyl and coWeyl modules to Specht and dual Specht modules respectively; and takes projective modules to Young permutation modules.

The Schur functor is exact, and one then has (again from Hemmer) that *Young permutation modules have Specht and dual-Specht filtrations.* In particular the regular module has such filtrations. Distinct such filtrations do not necessarily have the same multiplicities, unless the characteristic is $p > 3$.

Note that filtration of the regular module does not necessarily imply that an indecomposable projective module has a filtration, since a Specht module may not be indecomposable. But this is only an issue for $p = 2$.

8.4 Characteristic p, Nakayama and the James abacus

See for example James–Kerber [29].

(**8.4.1**) A *rim-hook* of a Young diagram d is a skew-subdiagram of d whose dual graph is a chain. Define *the rim* of d as the maximal rim-hook.
Note that a rim-hook is not necessarily a hook

For $p \in \mathbb{N}$, a *p-hook* of diagram d is a rim-hook of length p. We define a partial order on Young diagrams by $d' \overset{p}{<} d$ if $d' \subset d$ and the skew is a p-hook. A *p-core* of d is a minimal element in any chain containing d.

FACT: All p-cores of d coincide. (This follows from (8.4.5) below.)

(**8.4.2**) Theorem. [Nakayama conjecture] Let k be a field of prime characteristic p. Then Specht module $\mathcal{S}(\lambda)$ lies in the same block as $\mathcal{S}(\lambda')$ iff their diagrams have the same p-core.

Proof. See e.g. James–Kerber [29, p.245].

(**8.4.3**) Remark. Note that we have not shown that the block of $\mathcal{S}(\lambda)$ is well-defined if $p = 2$.

(**8.4.4**) Abacus: see James–Kerber [29, p.77-78]. A q-abacus is an abacus with q (vertical) runners, with the upper frame fixed and the lower frame very far away. The abacus may be considered to be filled with equal sized beads, almost all of which are 'empty' beads. The bead positions are numbered from 0 in reading order (i.e. left to right then top to bottom). A *bead configuration* records the position of the non-empty beads.

To each bead configuration there corresponds a Young diagram as follows. Associate to each non-empty bead the number of empty beads encountered in reading to that point. This gives a non-decreasing sequence from $\mathbb{N}_0$. Thus writing the sequence in reverse order and ignoring any 0s we get a non-increasing sequence in $\mathbb{N}$, and hence a Young diagram.

A useful fact is the following.

(**8.4.5**) CLAIM: Suppose that removing a rim q-hook from d gives d'. Then replacing any bead configuration for d with a bead configuration in which one bead has been moved one space up, gives a bead configuration for d'.

(**8.4.6**) Example. (Omitting the vertical runners etc for reasons of laziness)

$$\begin{pmatrix} \circ & \circ & \circ & - & \circ \\ \circ & - & \circ & - & \circ \\ \circ & \circ & \circ & - & \circ \end{pmatrix} \begin{pmatrix} \circ & \circ & \circ & - & \circ \\ \circ & \circ & \circ & - & \circ \\ \circ & - & \circ & - & \circ \end{pmatrix}$$

The first abacus gives 00011233334 and hence $43^4 21^2$. The second gives 00011112234 and hence $432^2 1^4$. The difference is a rim 5-hook:

$$\begin{pmatrix} x & x & x & x \\ x & x & x & \\ x & x & x' & \\ x & x & x' & \\ x & x' & x' & \\ x & x' & & \\ x & & & \\ x & & & \end{pmatrix}$$

8.5 James–Murphy theory

This section is a summary of standard symmetric group results, cast in a form following [29, Ch.7]. The objective is to give a definite (if not canonical) construction for symmetric group Specht module Gram matrices.

We use various notations for S_n elements. If $f \in S_n$ then $(f(1), f(2), ..., f(n))$ is a permutation. On the other hand in *cycle* notation, if $S = \{i, j, ..., k\} \subseteq \underline{n}$ then by $(ij..k) \in S_n$ we mean $f(i) = j$,..., $f(k) = i$ and $f(l) = l$ for $l \notin S$. Thus

$$(13).(12) = (123)$$

is an example of group multiplication.
(Then again, the common *diagram algebra* notation effectively composes permutations backwards, i.e. as in the opposite group. This is not a major issue, since the groups are isomorphic.)

A *tableau* is an ordering of the boxes in a Young diagram, usually given by writing the counting numbers in the boxes. A β-tableau is a tableau for diagram β. If $\beta \vdash n$ then $\pi \in S_n$ acts on tableau in the obvious way:

$$(12)\,\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$$

We have

$$(13)(12)\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} = (13)\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} = (123)\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

so this is a 'left' action, as written.

A β-tabloid is an equivalence class of tableaux under permutations within rows. (Abusing notation somewhat) One writes $\{t\}$ for the equivalence class of t. Thus for example

$$\{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\} = \{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\}$$

Note that $\pi\{t\} = \{\pi t\}$ is well-defined.

There is a sequence notation for tabloids, in which one writes $s(t)_i = j$ if the number i appears in row j in a tableau $u \in \{t\}$. (Note that this does not depend on the choice of class representative u.) Thus for example

$$s(t) = 112$$

for the tabloid above.

Let F be an arbitrary field. Define S_n module

$$M^\beta = F\{\{t\}\}_{t \in \beta\text{-tableau}}$$

Note that $M^\beta = FS_n\{t\}$ for any suitable t. Examples: $M^{(2,1)}$ has basis

$$\{\{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\},\ \{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\},\ \{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\}\}$$

$M^{(1^3)}$ has basis

$$\left\{\{\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\},\ \{\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 2 \\ \hline\end{array}\},\ \{\begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 3 \\ \hline\end{array}\},\ \{\begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array}\},\ \{\begin{array}{|c|}\hline 3 \\ \hline 1 \\ \hline 2 \\ \hline\end{array}\},\ \{\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}\}\right\}$$

Note that the different treatment of rows and colums is a source of non-canonicalness. This is unavoidable.

Let $V(t) \in \mathbb{Z}S_n$ be the unnormalised column antisymmetriser associated to tableau t.
Examples:

$$V\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 4 & 3 \\ \cline{1-2}\end{array}\right) = (1-(14))(1-(23))$$

and

$$V\left(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\right) = 1-(12)-(23)-(13)+(123)+(321)$$

Note that for all $\pi \in S_n$

$$\pi V(t) = V(\pi t)\pi.$$

Then

$$e_t \ := \ V(t)\{t\}$$

is a β-polytabloid in M^β.
Examples:

$$e_{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}} = (1-(13))\{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\} = \{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\} - \{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\}$$

$$e_{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}} = (1-(12))\{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\} = \{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\} - \{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\}$$

$$e_{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}} = (1-(12))\{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\} = \{\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array}\} - \{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\}$$

The Specht module S^β is the subspace of M^β spanned by β-polytabloids:

$$\pi e_t = \pi V(t)\{t\} = V(\pi t)\pi\{t\} = V(\pi t)\{\pi t\} = e_{\pi t}.$$

(**8.5.1**) Define a bilinear form on M^β by

$$\Phi(\{t\},\{t'\}) = \begin{cases} 1 & \{t\} = \{t'\} \\ 0 & o/w \end{cases}$$

Define $Gram(\beta)$ as the matrix with entries $\Phi(e_t, e_{t'})$, with t, t' varying over standard β-tabloids in the lexicographic total order.
Examples: $Gram((1^3)) = (6)$; $Gram((3)) = (1)$;

$$Gram((2,1)) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

Define $S^{\beta\perp}$ as the submodule of M^β spanned by elements orthogonal to β-polytabloids.
Theorem 8.1. *Let F be any field. Suppose $A \hookrightarrow M^\beta$ is an inclusion of FS_n-modules. Then either $S^\beta \hookrightarrow A$ or $A \hookrightarrow S^{\beta\perp}$.*
Proof: Suppose $a \in A$, and t a β-tabloid. Then it is easy to see that $V(t)a \in Fe_t$. Thus if $V(t)a \neq 0$ then $e_t \in A$, so $S^\beta = FS_n e_t \hookrightarrow A$. If $V(t)a = 0$ then

$$0 = \Phi(V(t)a, \{t\}) = \Phi(a, V(t)\{t\}) = \Phi(a, e)$$

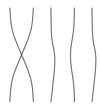
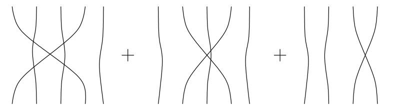
so $A \hookrightarrow S^{\beta\perp}$. $\square$
Theorem 8.2. *Let F be any field. Then $S^\beta/S^\beta \cap S^{\beta\perp}$ is either an absolutely irreducible FS_n-module (i.e. remains irreducible under any field extension) or zero.*
Proof: If A is a submodule of S^β (and hence of M^β) then by Theorem 8.1 it is either S^β or else in $S^\beta \cap S^{\beta\perp}$. Thus $S^\beta/S^\beta \cap S^{\beta\perp}$ has no submodule, so it is irreducible. Absolute irreducibility follows from a consideration of the rank of the Gram matrix over the prime subfield. $\square$

8.5.1 Murphy elements

For $(i,j) \in S_n$ the pair permutation define Murphy elements of S_n:

$$\mathcal{M}_m = \sum_{i=1}^{m-1} (i,m)$$

Example: In S_5 we have

$\mathcal{M}_2 =$  , $\mathcal{M}_4 =$ 

8.6 Outer product representations of S_n

See e.g. Robinson [?], Hamermesh [?], Hoefsmit [?], Martin–Woodcock–Levy [?].

8.6.1 Multipartitions and their tableaux

Recall that Λ is the set of integer partitions. For $d \in \mathbb{N}$ define the set of *d-component multipartitions*

$$\Gamma^d = \Lambda^{\times d}$$

and Γ^d_n as the subset of multipartitions of total degree n (we take this notation from Martin–Woodcock–Levy [?]).

We write

$$\mu = (\mu^1, \mu_2, ..., \mu^d) \; \in \; \Gamma^d_n$$

The *shape* of μ is the composition of n whose i-th component is $|\mu^i|$.

(**8.6.1**) The natural inclusion of $\Gamma^d \hookrightarrow \Gamma^{d+1}$ is given by appending an empty component. We define Γ^{fin} as the inverse limit of these inclusions.

(**8.6.2**) We define a digraph on vertex set Γ^{fin} such that there is an edge from μ to ν if, regarded as multi(-Young)diagrams, they differ by the addition of a single box.

Note that this digraph is rooted, simple, loop-free and non-tree.

(**8.6.3**) A *tableau* of shape $\mu \in \Gamma^d_n$ is an arrangement of the symbols $1, 2, ..., n$ in the n boxes of μ regarded as a multi-Young diagram.

A tableau of shape μ is *standard* if each component tableau μ^i is standard. We write T^μ for the set of standard tableau of shape μ. Assuming a given total order on this set, we write

$$T^\mu = \{T^\mu_1, T^\mu_2, ..., T^\mu_t\}$$

A tableau in T^μ determines a walk on the digraph from $\emptyset$ to μ.

8.6.2 Actions of S_n

(**8.6.4**) We define an action of S_n on tableau of shape $\mu \vdash n$ so that $\sigma_i(T^\mu_p)$ is obtained by interchanging symbols i and $i+1$ in T^μ_p. Note that this does not necessarily take standard to standard. Over all (not necessarily standard) tableau this is evidently a basis for the regular module.

We want to describe various other actions of S_n.

(**8.6.5**) For the moment suppose that $\mu = ((1),(1),...,(1))$ (that is, an n-tuple of single boxes), and let $p(i)$ denote the position in which i appears in T^μ_p. Let x be an n-tuple of complex numbers. Define $\sigma_i T^\mu_p$ by

$$\sigma_i \begin{pmatrix} T^\mu_p \\ \sigma_i(T^\mu_p) \end{pmatrix} = \frac{1}{h} \begin{pmatrix} -1 & -(h-1) \\ -(h+1) & 1 \end{pmatrix} \begin{pmatrix} T^\mu_p \\ \sigma_i(T^\mu_p) \end{pmatrix}$$

where $h = x_{p(i)} - x_{p(i+1)}$.

In this way σ_i acts by a linear transformation on the vector space with tableaux basis, and hence as a particular matrix on the tableau basis. Let us write R^x for this map from Coxeter generators to matrices (for x indeterminate, and otherwise when x such that this map is well-defined per se).

(**8.6.6**) Note that x may be chosen so that all h are large magnitude (complex). In the large magnitude limit, then σ_i- coincides with the regular representation $\sigma_i(-)$.

(**8.6.7**) We claim that R^x extends to a representation.

To verify this we need to show:

(1) $\sigma_i\sigma_i - 1 \stackrel{R^x}{=} 0$:

This is clear since every matrix $R(\sigma_i)$ falls into 2x2 blocks as above, each of which is traceless and has det$=-1$.

(2) $\sigma_i\sigma_{i+1}\sigma_i - \sigma_{i+1}\sigma_i\sigma_{i+1} \stackrel{R^x}{=} 0$:

This follows by looking at σ_i and σ_{i+1} on a typical block of tableaux. Here $h = x_{p(i)} - x_{p(i+1)}$, $h_2 = x_{p(i)} - x_{p(i+2)}$ and $h_1 = x_{p(i+1)} - x_{p(i+2)}$:

$$\sigma_i \begin{pmatrix} T^\mu_p \\ \sigma_i(T^\mu_p) \\ \sigma_{i+1}(T^\mu_p) \\ \sigma_i(\sigma_{i+1}(T^\mu_p)) \\ \sigma_{i+1}(\sigma_i(T^\mu_p)) \\ \sigma_i(\sigma_{i+1}(\sigma_i(T^\mu_p))) \end{pmatrix} = \begin{pmatrix} \frac{-1}{h} & \frac{1-h}{h} & 0 & & & \\ \frac{-h-1}{h} & \frac{1}{h} & 0 & & & \\ 0 & 0 & \frac{-1}{h_2} & \frac{1-h_2}{h_2} & 0 & 0 \\ 0 & 0 & \frac{-h_2-1}{h_2} & \frac{1}{h_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{h_1} & \frac{1-h_1}{h_1} \\ 0 & 0 & 0 & 0 & \frac{-h_1-1}{h_1} & \frac{1}{h_1} \end{pmatrix} \begin{pmatrix} T^\mu_p \\ \sigma_i(T^\mu_p) \\ \sigma_{i+1}(T^\mu_p) \\ \sigma_i(\sigma_{i+1}(T^\mu_p)) \\ \sigma_{i+1}(\sigma_i(T^\mu_p)) \\ \sigma_i(\sigma_{i+1}(\sigma_i(T^\mu_p))) \end{pmatrix}$$

On the same part of the basis we have:

$$R_{part}(\sigma_{i+1}) = \begin{pmatrix} \frac{-1}{h_1} & 0 & \frac{1-h_1}{h_1} & 0 & & \\ 0 & \frac{-1}{h_2} & 0 & 0 & \frac{1-h_2}{h_2} & 0 \\ \frac{-h_1-1}{h_1} & 0 & \frac{1}{h_1} & 0 & & \\ 0 & 0 & 0 & \frac{-1}{h} & 0 & \frac{1-h}{h} \\ 0 & \frac{-h_2-1}{h_2} & 0 & 0 & \frac{1}{h_2} & 0 \\ 0 & 0 & 0 & \frac{-h-1}{h} & 0 & \frac{1}{h} \end{pmatrix}$$

One can then verify the relation by brute force[1].

(3) distant commutation...

(**8.6.8**) Consider now the case $x = (3, 2, 1)$. We have $h = h_1 = 1$ and $h_2 = 2$.

Note that the tableau 123 (in the obvious notation) spans a submodule.

(**8.6.9**) If we set $h = x_1 - x_2 = -1$ and $h_2 = x_1 - x_3 = 1$ then $h_1 = x_2 - x_3 = 2$.

8.6.3 Generalised hook lengths and geometry

(**8.6.10**) Hook length: ...

(**8.6.11**) We define $h_{ij}(T_p^\mu)$ as the hook length between the boxes containing i and j in T_p^μ, with $h_{ij}(T_p^\mu) = \infty$ if i, j in different parts. We define $h^0_{ij}(T_p^\mu)$ as the hook length between the boxes containing i and j in T_p^μ, with all Young diagrams overlaying so that the $(1, 1)$-box is in the same position.

For x a d-tuple of complex numbers we define

$$h^x_{ij}(T_p^\mu) \ := \ h^0_{ij} + x_{\#i} - x_{\#j}$$

where $\#i$ is the position in the tuple corresponding to the Young diagram containing i in T_p^μ.

8.7 Outer products

(**8.7.1**) Let G, G' be groups and $R_G, R_{G'}$ representations. Then the Kronecker product $R_G \otimes R_{G'}$ is a representation of $G \otimes G'$.

8.7.1 Outer products over Young subgroups

(**8.7.2**) If $\lambda \vdash n$ then

$$S_\lambda := \otimes_i S_{\lambda_i}$$

is a subgroup of S_n (each factor acts nontrivially on a corresponding subset of $\underline{n}$).

Let $\mu \in \Gamma^d_n$ of shape λ. Fix a representation R_ν for each Specht module $\Delta(\nu)$. Then we have a representation $\otimes_i R_{\mu^i}$ of S_λ. Fix a commutative ring K and write M'_μ for the corresponding left KS_λ-module. We define a left KS_n-module by

$$M_\mu \ := \ KS_n \otimes_{KS_\lambda} M'_\mu$$

This M_μ is called an *outer product* representation.

Our next task is to construct an explicit basis and action for each M_μ.

8.7.2 Outer products over wreath subgroups

Whenever a partition occurs more than once in an outer product representation there is an automorphism acting permuting the identical factors. We can use this to decompose the representation.

[1]For example, *Maxima* [**?**] the open source algebraic computation package does this. See my Maxima file "defns.mac".

8.7.3 The Leduc–Ram–Wenzl representations

As kS_n-modules the Leduc–Ram modules are particular examples of mixtures of the previous two types of modules. (They are of interest via their extensions to Brauer algebra representations, but we shall start by considering them as S_n representations.)

For each n and m such that $n = 2m + k$, and $\lambda \vdash k$ we have the subgroup $S_2 \wr S_m \subset S_{2m} \subset S_n$.

Chapter 9

The Temperley–Lieb algebra

9.1 Ordinary Hecke algebras in brief

(**9.1.1**) Define a group $\mathfrak{B}_n$ with generators $g_1, g_2, \ldots g_{n-1}$ and relations $g_i g_j = g_j g_i$ if $|i-j| > 1$, $g_i g_j g_i = g_j g_i g_j$ if $|i-j| = 1$.

(**9.1.2**) We work over $\mathbb{C}$ for now. Fix $q \in \mathbb{C}$ and $n \in \mathbb{N}$. Let $H_n = H_n(q)$ denote the *Hecke algebra* over $\mathbb{C}$, the $\mathbb{C}$-algebra that is the quotient of $\mathbb{C}\mathfrak{B}_n$ by

$$(g_i - 1)(g_i + q^2) = 0$$

(This parameterisation is used by Martin in [32]. There are other choices in common use.)

9.2 Representations

(**9.2.1**) For $d \in \mathbb{N}$, denote by Γ_n^d the set of all d-tuples $\lambda = (\lambda^1, \lambda^2, ..., \lambda^d)$ of Young diagrams with $|\lambda| = \sum_{i=1}^d |\lambda^i| = n$.

Fix $\lambda \in \Gamma_n^d$. A *tableau of shape* λ is any arrangement of symbols $1, 2, .., n$ in the n boxes of λ.

A tableau is *standard* if each component tableau λ^i is standard. Denote the set of all standard tableaux of shape λ by T^λ.

(**9.2.2**) Number the rows of multipartition λ by placing the whole of the component diagram λ^{i+1} under λ^i for all i, and numbering the rows from top to bottom. Then define a total order $<$ on standard tableaux of shape λ by setting $T < U$ if the highest number which appears in different rows of T and U is in an earlier row in U.

(**9.2.3**) Let T be a tableau. For $i \in \{1, 2, \ldots, n-1\}$, let $\sigma_i = (i\ i+1) \in \Sigma_n$, the symmetric group of degree n. We define $\sigma_i(T)$ to be the tableau obtained by interchanging i and $i+1$. In this way we get an action of Σ_n on the set of all tableaux of shape λ. Note that this action does not necessarily take a standard tableau to a standard tableau.

(**9.2.4**) Let $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. For $i, j \in \{1, 2, \ldots, n-1\}$ and $T \in \Gamma^d_n$ let $h^x_{ij} = h^x_{ij}(T)$ denote the *generalised hook length* between the symbols i and j in T. Thus h^x_{ij} is given by:

$$h^x_{ij} = h^0_{ij} + x_{\#i} - x_{\#j},$$

where h^0_{ij} is the usual hook length obtained by superimposing the component tableaux of T containing i and j, and $\#i$ is the number of the component containing i in T. See [34] (note that there is a typographical error in this paper at the relevant point) and also [32, p.244].

(**9.2.5**) Proposition. [34] *Let $\lambda \in \Gamma^d_n$. Then the set T^λ is a basis for a left H_n–module R^λ. For $i \in \{1, 2, \ldots, n-1\}$ and $T \in T^\lambda$, the action is as follows:*
(a) If $i, i+1$ lie in the same row of T then $g_i T = T$.
(b) If $i, i+1$ lie in the same column of T then $g_i T = -q^2 T$.
(c) If neither (a) nor (b) hold, then $\sigma_i(T)$ is also standard. Let $h = h^x_{i,i+1}$. Then the action is given by

$$g_i \begin{pmatrix} T^\lambda_p \\ \sigma_i(T^\lambda_p) \end{pmatrix} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{q}{[h]} \begin{pmatrix} [h+1] & [h-1] \\ [h+1] & [h-1] \end{pmatrix} \right) \begin{pmatrix} T^\lambda_p \\ \sigma_i(T^\lambda_p) \end{pmatrix},$$

provided $T < \sigma_i(T)$.

(Young's orthogonal form (see e.g. [**?**, IV.6]) involves an action via symmetric matrices related to those above via conjugation).

We remark that the action is only well-defined provided $[h]$ never vanishes.

9.3 Temperley–Lieb algebras

(**9.3.1**) We now restrict attention to the situation in which λ has exactly two components, each consisting of exactly one row. We can represent $T \in T^\lambda$ by an n-tuple $(a_1, a_2, \ldots, a_n)$ with entries in $\{1, 2\}$, defined by the condition that $i \in \lambda^{a_i}$ for all $i \in \{1, 2, \ldots, n\}$. Such a tuple can be regarded as a walk of length n in $\mathbb{Z}^2$ starting at the origin. The ith step of the walk consists of adding the vector $(1, 1)$ if $a_i = 1$ or adding the vector $(1, -1)$ if $a_i = 2$.

(**9.3.2**) For example, if $n = 4$ and each component of λ is a row of length 2, the elements of T^λ and the corresponding tuples and walks are as shown in Figure 9.1.

(**9.3.3**) We note that in the walk realisation of a standard tableau T, σ_i swaps a pair of steps $(1, 2)$ with the pair $(2, 1)$, i.e. a local maximum is swapped with a local minimum or vice versa. Thus, in order for there to be mixing between two basis elements as in Proposition 9.2.5(c), the corresponding walks must agree in all but their ith and $(i+1)$st steps, and in each diagram separately the second coordinate (or *height*) after $i-1$ steps and after $i+1$ steps must coincide. In fact, it is not hard to show that the height coincides with the usual hook length $h^0_{i,i+1}$. If $T < \sigma_i(T)$ then $h^x_{i,i+1}$ is equal to the sum of $x_1 - x_2$ and the height of the walk after $i-1$ steps.

(**9.3.4**) If this value is 1, it follows from the description of the action in case (c) that the elements are not actually mixed. It follows that, if we choose x so that $x_1 - x_2 = 1$, there is an action of H_n on the set of (standard tableaux corresponding to) walks which do not go below the horizontal

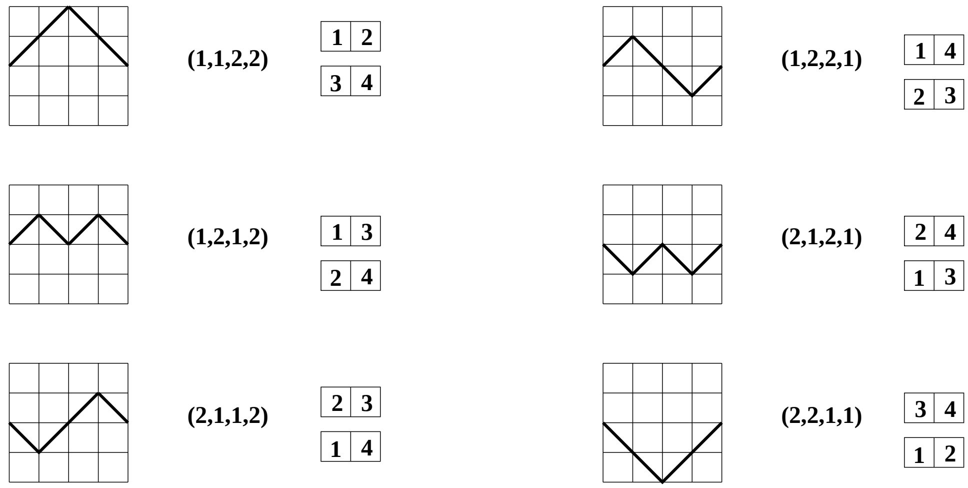


Figure 9.1: Standard tableaux of shape $((2),(2))$ and the corresponding walks.

axis given by the formulas in Proposition 9.2.5. In fact, in this case, the action cannot be extended to the whole of T^λ since the action is not defined for hook length zero.

(**9.3.5**) Similarly, if we set $x_1 - x_2 = 2$, only the walk $(2,2,1,1)$ is decoupled from the rest. In other words, changing the value of x allows us to define a module for H_n with basis elements corresponding to walks which do not go below a certain "exclusion" line.

(**9.3.6**) The walks we have been considering can be regarded as walks on $\mathbb{Z}$ (with edges joining integers with difference 1), by projecting onto the first coordinate. Thus, in summary, we have extracted an H_n-module with a basis of walks on $\mathbb{Z}$ which only visit vertices on a certain subgraph, from the formal closure of a Zariski-open set of modules (that is, actions depending on a parameter) whose bases consist of walks on a larger subgraph. The decoupling of the subgraph, in this sense, is determined by the structure of the graph.

(**9.3.7**) The case $x_1 - x_2 = 1$ is special in that the decoupled module is irreducible for generic values of q. It is an analogue of the boundary of the dominant region in the Weyl group construction in Lie theory. The most interesting step, however, is the next one. We now fix $x_1 - x_2 = 1$, and also specialise q to be an lth root of unity, so that $[l] = 0$. In this situation, there is a further decoupling: we obtain a module whose basis corresponds to walks whose height is bounded above by $l-1$. In other words, we now only include walks that lie between two 'walls': the lines given by setting the second coordinate to 0 and $l-1$. It can be shown that this module is simple in this specialisation. Such simple modules are otherwise very hard to extract, but here their combinatorics is manifested relatively simply.

9.4 Diagram categories

A diagram is, humanistically, a collection of open and (possibly filled) closed lines on the page. I.e. a subset of $\mathbb{R}^2$ with some topological structure. This collection may well represent a 3D geometrical shape, so it may be, in part, an embedding of lines in 3D onto 2D. A simple version of this is the embedding of knots as braids. For this reason, one source of categories with a form of diagram calculus is braided tensor categories (with duals). A major source of such categories is the representation theory of quasi-triangular bialgebras, such as quantum groups.

9.4.1 Relation to quantum groups

See for example Joseph [?], Kassel [30], [?], [?].

Fix a field R, and recall the definition of R-algebra from Section 7.1. Recall that an R-bialgebra $(A, \nabla, \nu, \Delta, \epsilon)$ is an R-algebra (A, ∇, ν) with coassociative comultiplication Δ and counit $\epsilon : A \to R$ (that are morphisms of algebras). A bialgebra with

$$\Delta(a) = \sum_i l(a)_i \otimes r(a)_i$$

is cocommutative if the opposite coproduct

$$\Delta^o(a) = \sum_i r(a)_i \otimes l(a)_i$$

coincides with the coproduct. For example a group algebra is cocommutative, since $\Delta(a) = a \otimes a$.

(**9.4.1**) A bialgebra is *quasicocommutative* if there is an element R of $A \otimes A$ such that

$$\mathcal{R}\Delta^o(a) = \Delta(a)\mathcal{R}$$

This is like the intertwiner of Kronecker product matrices, except that it works at the algebra level. On this basis such an R is sometimes called a *universal $\mathcal{R}$-matrix.*

(**9.4.2**) A *quasitriangular bialgebra* (or braided bialgebra) is a quasicocommutative bialgebra such that, writing $\mathcal{R} = \sum_i L_i \otimes R_i$, we have

$$\sum_i \sum_j l(L_i)_j \otimes r(L_i)_j \otimes R_i = (\sum_m L_m \otimes 1 \otimes R_m)(\sum_n 1 \otimes L_n \otimes R_n) = \mathcal{R}_{13}\mathcal{R}_{23}$$

and similarly 'reversed': $(Id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$.

This formulation looks clunky, but we have

(**9.4.3**) THEOREM. If A is a quasitriangular bialgebra then

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

Given $\mathcal{R}$ and any two A-modules M, N we have

$$\begin{aligned} \mathcal{R}_{MN} : M \otimes N &\to N \otimes M && (9.1)\\ m \otimes n &\mapsto \sum_i r_i n \otimes l_i m && (9.2) \end{aligned}$$

This map can be shown to be an isomorphism of A-modules. It makes the category $A-\mathrm{mod}$ into a braided tensor category (as defined in Section 4.4.1).

9.5 Temperley–Lieb diagram algebras

9.5.1 TL diagram notations

Let $T^o(n,m)$ be the set of (n,m)-Temperley–Lieb diagrams, up to boundary order preserving isotopy. Let $T^o(n,l,m)$ be the subset of Temperley–Lieb diagrams with l propagating lines.

For example there is a single element $1_1 \in T^o(1,1)$, $u \in T^o(2,0)$, $u^* \in T^o(0,2)$, and $U \in T^o(2,0,2)$. There is also a single element $1_0 \in T^o(0,0)$.

(**9.5.1**) We draw TL diagrams with their vertices on the N and S edges of the frame. We define

$$\boxtimes : T^o(n,m) \times T^o(n',m') \to T^o(n+n', m+m')$$

by side-by-side concatenation. Note that this is well-defined and associative. We define $U_1 \in T^o(n,n)$ by $U_1 = U \boxtimes 1_1 \boxtimes 1_1 \boxtimes ... \boxtimes 1_1$.

(**9.5.2**) For $d \in T^o(l,m)$ and $d' \in T^o(m,n)$ we may pick representatives and then form a new 'diagram' by *vertical* concatenation. Note that there are representatives such that the m vertices match up in *vertical* concatenation. The combined 'diagram' then *determines* a (representative of a) TL diagram $c(d,d')$ in $T^o(l,n)$. Note that this TL diagram is independent (up to isotopy) of any further details of the choice of representatives of d and d' (so the notation $c(d,d')$ is well-defined). The combined diagram *may not be a TL diagram per se*, since closed loops may appear, but the *number* $n(d,d')$ of closed loops appearing is also independent of the choice of representatives.

(**9.5.3**) Now fix a ring R and $\delta \in R$ and for $n \in \mathbb{N}_0$ let $T_n = T_n(\delta)$ be the TL algebra over R with basis $T^o(n,n)$ and multiplication given by

$$dd' \ = \ \delta^{n(d,d')} \, c(d,d')$$

For n also given, we may write A for the TL algebra T_n.

9.6 Representations of Temperley–Lieb diagram algebras

9.6.1 Tower approach: Preparation of small examples

(**9.6.1**) Note that both T_0 and T_1 are isomorphic to the ground ring R.

(**9.6.2**) Suppose $n > 1$ and consider the left ideal $T_n U_1$. This ideal has a basis of diagrams (in bijection, for example, with the diagram basis of T_{n-1}) over any ring. Diagrams in this ideal have at most $n-2$ propagating lines. There are also diagrams with fewer propagating lines (provided that $n > 3$), and these span a submodule. Thus there is a module in A-mod which is the quotient of AU_1 by this submodule. This quotient, denoted $\Delta_{n-2}(n)$, has basis the set of diagrams in AU_1 with precisely $n-2$ propagating lines. The action of the algebra on this basis is rather like the action on these diagrams if they were regarded as elements of A, except that if the nominal outcome is a diagram with fewer propagating lines then the action gives zero. We may similarly construct a module $\Delta_{n-4}(n)$ from AU_1U_3, and so on.

For $0 \leq 2m \leq n$ we have defined

$$\Delta_{n-2m}(n) \ = \ T_n U_1 U_3 ... U_{2m-1} / \ T_n T^o(n, n-2m-2, n)$$

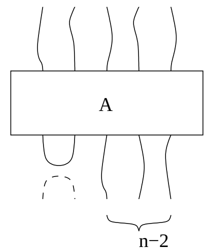


Figure 9.2: Schematic for T_nU_1.

(by convention M/N means $M/M \cap N$).

(**9.6.3**) Note that T_nU_1 becomes a left T_n right T_{n-2} bimodule, by allowing T_{n-2} to act on the last $n-2$ strings on the right. (In our convention the schematic for this is as in figure 9.2, where the legs on which T_{n-2} is to act by diagram composition are indicated.) This means that we can construct a module $AU_1 \otimes_{T_{n-2}} \Delta_{n-4}(n-2)$ and study it. This module has elements of form $a \otimes b$, and is spanned by the subset of these where a is taken from the diagram basis for AU_1 and b is taken from the nominal diagram basis for $\Delta_{n-4}(n-2)$.

(**9.6.4**) Proposition. *For* $n \geq 4$

$$T_nU_1 \otimes_{T_{n-2}} \Delta_{n-4}(n-2) \;\cong\; \Delta_{n-4}(n)$$

Proof. We compare the given spanning set on the left with the basis on the right.

First suppose a to be in the subset of AU_1 diagrams with less than $n-4$ propagating lines. Then a can be constructed as cd where c is another element of the basis of AU_1 and d is an element of the T_{n-2} which acts on AU_1, and which has less than $n-4$ propagating lines. We can thus see that $a \otimes b$ can be written in the form $cd \otimes b \sim c \otimes db$. Thus $a \otimes b \sim 0$.

If we consider the subset of AU_1 diagrams with $n-2$ propagating lines we can see that $a \otimes b$ can be written in the form $a \otimes fg \sim af \otimes g$, where g is another element of the basis of Δ and f is an element of T_{n-2} which has $n-4$ propagating lines. Thus af has $n-4$ propagating lines. Thus the module is spanned by objects $a \otimes b$ in which a has $n-4$ propagating lines and b is a given element. Without loss of generality this element can be chosen to have the last $n-4$ lines propagating (in the nominal sense). Then again $a \otimes b \sim 0$ unless the last $n-4$ lines (on the side of the right action) in a are propagating. We can now construct a surjective homomorphism from this module onto $\Delta_{n-4}(n)$ by simply concatenating $a \otimes b \mapsto ab$, where ab is to be understood in $\Delta_{n-4}(n)$. But the degree of what remains of the basis of AU_1 from our reduction to a smaller spanning set coincides with the rank of $\Delta_{n-4}(n)$, so this is an isomorphism. □

9.7 Idempotent subalgebras, F and G functors

Proposition 9.6.4 illustrates that the functor $T_nU_1 \otimes_{T_{n-2}} -$ gives a natural connection of certain modules of T_n with modules of T_{n-2}. We can use this to determine the representation theory of T_n largely from lower rank cases. We examine aspects of this strategy next. (In fact the strategy is well described in the literature, such as in Green[22] — at least in case $\delta \neq 0$, where U_1 can be replaced by an idempotent — so here we also focus on its limitations!)

(**9.7.1**) Let A be an algebra over a field k, and $e \in A$ idempotent. Then Ae is a left A- right eAe-bimodule. Define functors

$$\begin{aligned} F : A - \text{mod} &\to eAe - \text{mod} && (9.3)\\ M &\mapsto eM && (9.4) \end{aligned}$$

$$\begin{aligned} G : eAe - \text{mod} &\to A - \text{mod} && (9.5)\\ N &\mapsto Ae \otimes_{eAe} N && (9.6) \end{aligned}$$

Proposition 9.1. *The functor F is an exact functor.*
(I.e. it takes a short exact sequence to a short exact sequence.)
Proposition 9.2. *The functor G is a right inverse to F.*
(I.e. $F(G(N)) \cong N$.)

9.7.1 Non-exactness of G

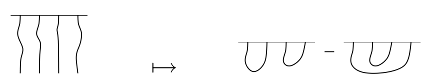
Note that G is right exact by Proposition 6.5.20. An example illustrating its failure to be left exact in general is as follows.

(**9.7.2**) Example. We work with $A = T_6(1)$, the Temperley–Lieb algebra (over $\mathbb{C}$) with $\delta = 1$, and $e = U_1$. We use the notation $T_{n_1} \boxtimes T_{n_2} \hookrightarrow T_{n_1+n_2}$ for the usual 'parabolic' subalgebra. We note that $e \in T_2$ and that $T_4 \to e \boxtimes T_4 \subset T_6$ defines an isomorphism of T_4 and eAe.

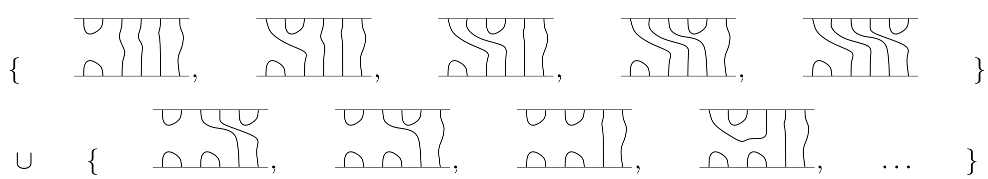
First note that in the $n = 4$ Temperley–Lieb algebra with $\delta = 1$ there is a sequence of standard modules

$$0 \to \Delta_4(4) \xrightarrow{\psi} \Delta_0(4)$$

exact at $\Delta_4(4) = T_4/T_4U_1T_4$, given in diagram shorthand by

ψ : 

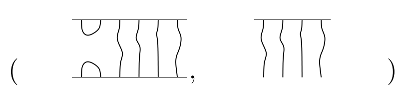
(note that the action of the algebra on the diagram on the left involves a quotient of the ordinary algebra action on diagrams). However we can show that the image under G is not an injection. The image of $\Delta_4(4)$ in $eAe -$mod can be represented as $\Delta_4(4)$ itself, noting that e acts like 1. The bimodule Ae includes elements such as



(9.7)

(9.8)

(and many with fewer than four propagating lines). The image $G(\Delta_4(4)) = Ae \otimes_{eAe} \Delta_4(4)$ thus contains elements like

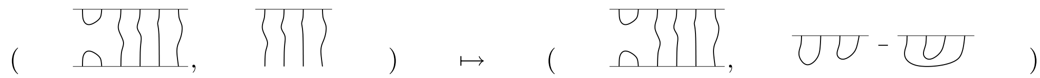


In particular the set of such objects built using the elements from the list in (9.7) on the left-hand side is spanning, since elements of Ae with fewer propagating lines kill $\Delta_4(4)$. Indeed this set is a basis, since $G(\Delta_4(4)) = \Delta_4(6)$.

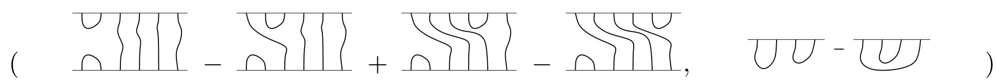
The image $G(\Delta_0(4)) = Ae \otimes_{eAe} \Delta_0(4)$ contains elements like

and is mapped isomorphically to $\Delta_0(6)$ by the multiplication map.

Finally we have

$G(\psi):$ 

and so on. The image on the right is to be understood to lie in $G(\Delta_0(4)) = \Delta_0(6)$ (NB, the expression on the right would mean something else as the image of the submodule) so again there is a multiplication map (as in $a \otimes d \mapsto ad$), which is an isomorphism. It is easy to check that

 $\mapsto 0$

under this isomorphism, thus the preimage indicated by the sum on the left-hand side of this construct lies in the kernel of $G(\psi)$.

9.7.2 More fun with F and G

We continue here with the assumptions of §9.7. See also Green[22, §6.2].

(**9.7.3**) Lemma. (Suppose G left-adjoint to F.) Let $N \in eAe-\mathrm{mod}$. Then $\mathrm{End}_{eAe}(N) \cong \mathrm{End}_A(G(N))$.

Proof. By left-adjointness

$$\mathrm{Hom}_A(G(N), M) \cong \mathrm{Hom}_{eAe}(N, F(M))$$

Take $M = G(N)$:

$$\mathrm{Hom}_A(G(N), G(N)) \cong \mathrm{Hom}_{eAe}(N, F(G(N)))$$

but $F(G(N)) \cong N$. □

On the other hand, a finite dimensional module is indecomposable if and only if it has a local endomorphism ring. Thus N is indecomposable if and only if $G(N)$ is indecomposable.

(**9.7.4**) Lemma. If $N \in eAe-\mathrm{mod}$ simple and M a proper submodule of $G(N)$, then $F(M) = 0$.

Proof. Else $F(M) \cong N$ by simplicity, and $M \supseteq AeM = A(e \otimes N) = G(N)$, a contradiction. □

(**9.7.5**) For $M \in A-\mathrm{mod}$ let M_e be the largest submodule of M contained in $(1-e)M$. Thus $F(M_e) = 0$.

Proposition 9.3. *If $N \in eAe-mod$ simple then $G(N)$ has simple head. The unique maximal proper submodule is $G(N)_e$.*
(I.e. if L is a simple factor below the head it is also a simple A/AeA-module.)

Proof. We have $F(G(N)_e) = 0$, so by proposition 9.1 the exactness of

$$0 \to G(N)_e \to G(N) \to G(N)/G(N)_e \to 0$$

implies exactness of

$$0 \to 0 \to F(G(N)) \to F(G(N)/G(N)_e) \to 0, \tag{9.9}$$

i.e.

$$F(G(N)/G(N)_e) \cong F(G(N)) \cong N$$

by proposition 9.2. Thus $G(N)/G(N)_e$ is not zero, and so $G(N)_e$ is a *proper* submodule. But by lemma 9.7.4, every proper submodule is contained in $G(N)_e$. □

Proposition 9.4. *If $M \in A - mod$ simple, and $eM \neq 0$, then $F(M) = eM$ is simple.*

Proof. If $M' = eM'$ is a nonzero eAe-submodule of eM then it is also a k-submodule of M. Since M is simple as an A-module we have $AM' = AeM' = M$. Thus $eM = eAeM' = M'$. That is, eM does not have a proper eAe-submodule. □

(**9.7.6**) Let $M \in A-$mod and $eM \neq 0$. There is an A-module map from $G(F(M))$ to M given by

$$a \otimes em \mapsto aem.$$

If M is simple, then $AeM = M$ and this map is surjective. Since this is a surjective map onto a simple module, the kernel is the maximal proper submodule of $G(F(M))$. But since $F(M)$ is simple this is $G(F(M))_e$, by proposition 9.3. Thus

Proposition 9.5. *If $M \in A - mod$ simple, and $eM \neq 0$, then*

$$\textit{head}\ (G(F(M))) \cong M.$$

Thus $F(M) \cong F(M')$ with M, M' simple implies $M = M'$.

We have

Theorem 9.6. *Let $\{L(\lambda) \mid \lambda \in \Lambda^e\}$ be a full set of simples in $eAe - mod$, and $\{L_A(\lambda) \mid \lambda \in \Lambda^0\}$ a full set of simples in $A/AeA - mod$. Then the disjoint union $\{L_A(\lambda) = \textit{head}\ (G(L(\lambda))) \mid \lambda \in \Lambda^e\} \cup \{L_A(\lambda) \mid \lambda \in \Lambda^0\}$ is a full set of simples in $A - mod$.*

9.7.3 Decomposition numbers

Let $\{L_x \mid x \in \Lambda\}$ be a complete set of simple A-modules. Let $e \in A$ be an idempotent and Λ' the set of $x \in \Lambda$ such that $eL_x \neq 0$. Note that eL_x is simple.

(**9.7.7**) Lemma. For any A-module M a composition series

$$M = M_0 \supset M_1 \supset ... \supset M_l = 0$$

passes to a series

$$eM = eM_0 \supseteq eM_1 \supseteq ... \supseteq eM_l = 0$$

where, as eAe-modules:

$$e(M_{j-1}/M_j) \cong eM_{j-1}/eM_j \tag{9.10}$$

Removing a term whenever $eM_{j-1} = eM_j$ (which is whenever the section in the M-series is killed by e) we have a composition series for eM; and

$$[eM : eL_x] = [M : L_x]$$

Proof. Note that each section in the initial eM series is simple or zero (Prop.9.4). The only other non-trivial step is (9.10), but this follows by the exactness of F. □

—

(**9.7.8**) Remark. We shall continue to investigate the properties of the setup described in this section in §14.1.1.

Chapter 10

Representations of the partition algebra

For each commutative ring k the partition algebras over k may be constructed as the end-algebras of a certain k-linear category [33]. This category is useful in representation theory, so we construct it first.

10.1 The partition category

We can use a diagram calculus to describe the partition category. But it is convenient to start more formally.

Recall the following notations from §2.2.1. For S a set, P_S is the set of partitions of S, and $P_{n,m} := P_{\underline{n} \cup \underline{m}'}$. Also E_S is the set of equivalence relations on S (and we may apply the bijection $E_S \leftrightarrow P_S$ without further comment). For a, b equivalence relations, we define ab as the transitive closure of the relation $a \cup b$. We then define a product $P_{n,l} \times P_{l,m} \to P_{n,m}$ by $a \circ b = u(r(ab'))$.

(**10.1.1**) Theorem. The triple $P = (\mathbb{N}_0, P_{n,m}, \circ)$ is a category.

Proof. We will treat this as a special case of the following Theorem.

(**10.1.2**) If $p \in P_S$ and T a set, then $\#^T(p)$ denotes the (cardinal) number of parts of p which contain only elements of T.

(**10.1.3**) Now consider a categorical triple $P^o = (\mathbb{N}_0, P_{n,m} \times \mathbb{N}_0, *)$ where $*$ is defined as follows. Consider $a * b$ for $a = (a_1, a_2)$, $b = (b_1, b_2)$, with $a_1 \in P_{n,l}$ and $b_1 \in P_{l,m}$. Then

$$(a_1, a_2) * (b_1, b_2) \;=\; (a_1 \circ b_1,\; a_2 + b_2 + \#^{\underline{l}'}(a_1 b_1'))$$

We call the $\#(-)$ in this setting the *vacuum number*.

(**10.1.4**) Theorem. The triple P^o is a category.

Proof. The main issue is associativity. Diagrams provide a convenient way to give the argument, so we discuss these next.

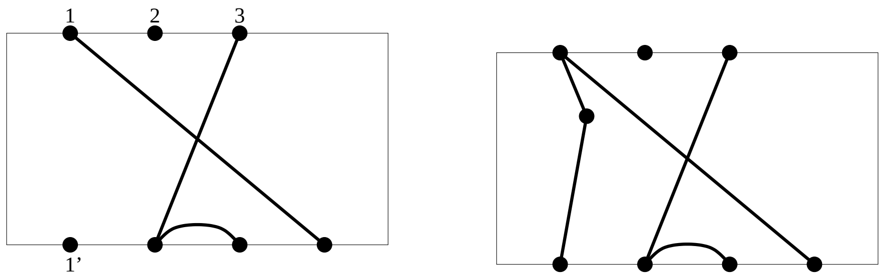


Figure 10.1: Partition diagrams

10.1.1 Partition diagrams

(**10.1.5**) A shorthand for elements of $P_{n,m}$ is as follows. Draw a rectangular frame, and draw n vertices on the northern edge and m vertices on the southern edge. Any graph on these vertices, with edges drawn embedded in the rectangle (as in figure 10.1), describes a partition: two vertices are in the same part if there is an edge (or chain of edges) between them.

Of course the map from graphs to partitions is surjective but not injective in general. If d is a graph we shall write $[d]$ for the corresponding partition.

Indeed a graph G on any collection of vertices together with a map from $\underline{n} \cup \underline{m}'$ to the vertex set of G defines an element of $P_{n,m}$ similarly. We have in mind something slightly less general however — the frame construction as above with some extra vertices in the interior of the rectangle. It will be convenient to *exclude* graphs with interior components disconnected from the exterior components as representatives of partitions, although $[d]$ is still well-defined for such graphs.

(**10.1.6**) The composition $a * b$, for the triple (n, l, m), can then be envisaged as follows. Consider a d such that $[d] = a$ and a d' such that $[d'] = b$. We call these *juxtaposable* if they can be juxtaposed in such a way that the l vertices on the l-vertex edges meet up (and become 'interior' vertices of the resultant graph). It will be evident that juxtaposable representatives exist, so take any such pair d, d' and form their juxtaposition. Call this new graph dd'. Comparing the constructions for $[-]$ and dd' with the definition of $\circ$ we see (1) that

$$[dd'] = [d] \circ [d']$$

and (2) that there may be some components of the graph disconnected from the top and bottom edges; and the number of these is the vacuum number.

10.1.2 Partition categories

(**10.1.7**) *proof of Theorem 10.1.4:* Note that $(dd')d'' = d(d'd'')$ for any suitably juxtaposable triple of graphs. Now note that the computation of $a * (b * c)$ and of $(a * b) * c$ can be done by drawing the same diagram $dd'd''$ (where $[d''] = c$). This verifies associativity of $*$.

□

(**10.1.8**) Next we formally extend P^o to a k-linear category, where k is some commutative ring. For each $\delta \in k$ the relation

$$(a_1, b_1 + 1) \;\sim\; \delta(a_1, b_1)$$

defines a congruence, and hence a quotient category P_k^δ.

Note that the end-sets, $\hom_{P_k^\delta}(n, n)$, in this category are k-algebras. For any given k, the n-th case has basis $P_{n,n}$ and is the partition algebra $P_n(\delta)$ over k.

(**10.1.9**) In our construction we forced the top and bottom vertex sets of $P_n(\delta)$ to be disjoint. However this is not necessary. We define algebra $P_{n+}(\delta)$ similarly, except that the basis is the subset of elements of $P_{n+1,n+1}$ is which $n+1$ and $n+1'$ are identified (or equivalently are always in the same part). Thus P_{n+} has a basis of partitions of a set of $2n+1$ objects.

(**10.1.10**) There are a large number of other interesting generalisations and subcategories of the partition category. For example, the product closes on the span of partitions of at most two parts. Mazorchuk calls the corresponding algebra the *rook Brauer algebra*, so have the 'rook Brauer category'. We shall write $RB_{n,m}$ for the corresponding subset of $P_{n,m}$, and so on.

10.1.3 Properties of partition categories

(**10.1.11**) We make

$$P \;:=\; \cup_{n,m} P_{n,m}$$

a monoid by lateral composition.

(**10.1.12**) Note that there is a unique element in $P_{1,0}$. Let us define u, u^* as the unique elements of $P_{1,0}$ and $P_{0,1}$ respectively. Write

$$U \;:=\; u \otimes u^* \quad \in P_{1,1}.$$

(**10.1.13**) Note

$$P_n \subset P_{n+} \subset P_{n+1}$$

is a sequence of unital algebra injections.

Write 1_n for the identity element in P_n. If we write $U \in P_n$ we shall mean $U \otimes 1_{n-1}$. We shall extend this notation in the obvious way to other elements.

(**10.1.14**) Define $P^l_{n,m} := P_{n,l} \circ P_{l,m} \;\subset P_{n,m}$ Note that this is the subset of partitions with at most l propagating lines. Define

$$P^{=l}_{n,m} = P^l_{n,m} \setminus P^{l-1}_{n,m}$$

Automatically then, we have that

$$kP_{n,n} \supset kP^{n-1}_{n,n} \supset kP^{n-2}_{n,n} \supset ... \supset kP^0_{n,n}$$

is a chain of two-sided ideals. The l-th ideal is generated by $U^{\otimes l}$. The l-th section (counting from the right) has basis $P^{=l}_{n,n}$. Let us write $P^{l/}_{n,n}$ for this section.

Note that a bimodule $P^{l/}_{n,m}$ may be defined similarly. In particular $P^{l/}_{n,l}$ has the nice property that its basis consists of all diagrams in $P^l_{n,l}$ such that each vertex on the bottom edge is in a distinct propagating part.

Note that $(P^{=n}_{n,n}, \circ)$ gives a copy of the symmetric group S_n.

(**10.1.15**) Note that there is a unique element in $P^{=1}_{1,1}$, denoted 1_1. There are two elements in $P^{=2}_{2,2}$, one of which is $1_2 := 1_1 \otimes 1_1$. Write (12) for the other element.

(**10.1.16**) We may consider the parts of $p \in P_{n,m}$ that meet the top set of vertices to be totally ordered by the natural order of their lowest numbered elements (from the top set). We may define a corresponding order for parts that meet the bottom set of vertices. We say that p is *non-permuting* if the subset of propagating parts has the same order from the top and from the bottom.

(**10.1.17**) Now consider the section $P^{l/}_{n,n}$ as a left-module. We have

$$P^{l/}_{n,n} \cong \bigoplus_{w \in P^{||}_{l,n}} P^{l/}_{n,l} w$$

as a left-module. Here $P^{||}_{l,n}$ is the subset of $P^{=1}_{l,n}$ of non-permuting partitions. Every summand is isomorphic to $P^{l/}_{n,l}$. We have that $P^{=l}_{n,l}$ is a basis for $P^{l/}_{n,l}$, and

$$P^{=l}_{n,l} = P^{||}_{n,l} \circ P^{l}_{l,l} \tag{10.1}$$

where $P^{l}_{l,l} = S_l$.

10.1.4 Δ-modules

(**10.1.18**) Note that $P^{l/}_{n,l}$ is also a free left kS_n-module in a natural way. For each $\lambda \vdash l$ let us choose an element $f_\lambda \in kS_l$ such that

$$\Delta_\lambda = kS_l f_\lambda$$

is the corresponding Specht module. Then define

$$\Delta_n(\lambda) := P^{l/}_{n,l} f_\lambda$$

(including $f_\lambda \in kS_l$ in P_l in the obvious way allows us to draw a picture for this).

CLARIFY THIS!

(**10.1.19**) Proposition. *For each basis b_λ of Δ_λ there is a basis $P^{||}_{n,l} \times b_\lambda$ of $\Delta_n(\lambda)$.*

Proof. Note that the module is spanned by elements of form abf_λ where $a \in P^{||}_{n,l}$ and $b \in S_l$ (consider (10.1)). □

(**10.1.20**) Note that the regular module has a filtration by Δ-modules, and indeed a filtration in which all the modules labelled with partitions of a given degree are consecutive. Indeed, if $k = \mathbb{C}$ (or at least contains $\mathbb{Q}$ so that kS_l is semisimple) they do not extend each other (so can be arranged, among themselves, in any order in the filtration).

(**10.1.21**) Note that provided δ is invertible then

$$P_n \cong U P_{n+1} U$$

so that the functor $G := P_{n+1}U \otimes_{P_n} -$ fully embeds P_n-mod in P_{n+1}-mod.

(**10.1.22**) PROPOSITION. *Provided δ invertible (or $n > 2$) we have*

$$G\Delta_n(\lambda) = \Delta_{n+1}(\lambda)$$

Proof. ...

10.2 Set partitions and diagrams

In this section we discuss enumerations of the sets $P_{\underline{n}}$ of set partitions (noting that these sets are the bases of our various partition algebras). One idea, for example, is to parallel the Robinson–Schensted correspondence [**?**], regarded as an enumeration of S_n (noting that the Young graph facilitates an enumeration of Young tableau). There are various ways to do this. We describe one that is natural from a representation theory perspective.

(**10.2.1**) First note that a partition p of S can be described by giving the restriction to some subset S', denoted $p|_{S'}$; the restriction to the complement of S' in S; and the details of the connections between the parts thus described. These connection details must give the list of parts in $p|_{S'}$ that are connected; the equinumerate list of parts for the complement; and a bijection between these lists. If a canonical list order is fixed (for any such list), then the bijection may be represented by an element of the symmetric group. Via the RS correspondence this element may be represented as an ordered pair of Young tableaux. We may think of giving the first tableau to S' and the second to the complement. In this way, p is split into two 'halves'.

(**10.2.2**) A *half-partition* is an ordered partition of a set partition into two parts, called *non-propagating* and propagating respectively.

(**10.2.3**) Consider the graph G shown in figure 10.2 (figure taken from Marsh–Martin[**?**]).

(**10.2.4**) A vertex in G is labelled by a pair consisting of a natural number n (or $n+$) and a Young diagram. The vertex labelled (n, λ) consists in the set of ordered pairs where the first element is a half-partition of $\underline{n}$ with $|\lambda|$ propagating parts; and the second part is a Young tableau of shape λ. If the label is $(n+, \lambda)$ then one takes instead a half-partition of $\underline{n+1}$, but requires that $n+1$ itself lies in a propagating part.

(**10.2.5**) We shall use shortly the following construction (again see [**?**]). Let S_ν be an S_{l-1}-Specht module, with a basis of standard tableau of shape ν, and $\mathbb{C}S_l \otimes_{\mathbb{C}S_{l-1}} S_\nu$ the induced module. This has a basis of elements of form $\pi_k \otimes T$ ($k \in \{0, 1, 2, ..., l-1\}$) where, in cycle notation, $\pi_k := (l\ l-1\ ...\ l-k)$ is a coset representative for $S_{l-1} \subset S_l$ and T is a tableau from the standard tableau basis above. The induced module also has a basis of standard tableau of various shapes corresponding to its Specht content (i.e. shapes of form $\nu + e_i$). Thus one may choose a bijection between these bases. Such a bijection gives a correspondence between the subset of standard tableau of shape $\lambda = \nu + e_i$ and some subset of the elements of form $\pi_k \otimes T$.

(**10.2.6**) We now introduce maps describing how to construct each vertex set of G from those in the layer above in G. (Firstly we shall define some maps, and then later we shall show that they do the job.)

There is a map from (n, λ) to $(n+, \lambda)$ adding a propagating part $\{n+1\}$ and keeping the same tableau. (Note that no permutation of the part involving $n+1$ is possible if $n+1$ is common to both halves.)

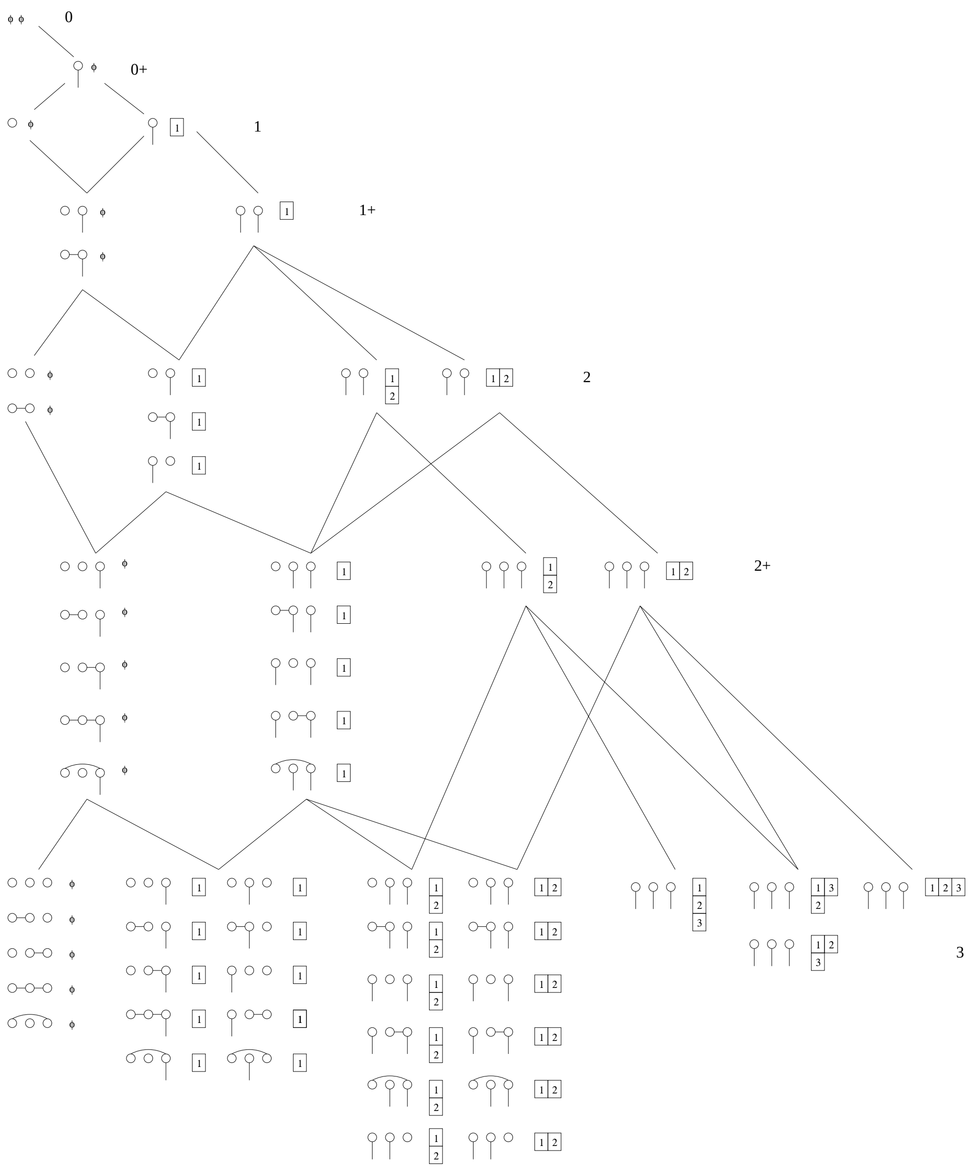


Figure 10.2: The array of ‘half-partitions’.

There is a map from (n, λ) to $(n+, \lambda - e_i)$ [1] as follows. Let (h, T') be the pair to be mapped. Then T' has shape λ and our bijection in (**??**) above takes this to some element $\pi_k \otimes T$. In this case add $n+1$ to the k-th propagating part of h, and replace T' with T.

There is a map from $(n+, \lambda)$ to $(n+1, \lambda)$ that makes the part containing $n+1$ non-propagating, and leaves the tableau unchanged.

There is a map from $(n+, \lambda)$ to $(n+1, \lambda + e_i)$ that leaves the half-partition unchanged and inserts $|\lambda| + 1$ into the new box in the Young diagram, leaving the rest unchanged.

(**10.2.7**) THEOREM. This is everything.

Proof. ...

10.3 Representation theory

This section is based on [33]

10.4 Representation theory via Schur algebras

This section is designed as a companion to Martin–Woodcock [**?**]. This is an approach to partition algebra representation theory using generalised Schur algebras, motivated by certain n-stability properties of tensor product rules for symmetric group representations. We start by recalling some notations used in [**?**].

Here Λ is the set of all compositions:

$$\Lambda \ = \ \{\lambda : \mathbb{N} \to \mathbb{N}_0 \mid supp(\lambda)\ finite\}$$

$$\Lambda_0 \ = \ \{\lambda : \mathbb{N}_0 \to \mathbb{N}_0 \mid supp(\lambda)\ finite\}$$

The elements of Λ and Λ_0 are called *weights*. If Γ is any set of weights and $Q \in \mathbb{R}$ then $\Gamma(Q)$ is the subset of weights of degree Q.

The *dominant weights* Λ^+ (and Λ_0^+) are the set of not-strictly descending weights. There is a natural action of the symmetric group $S_{\mathbb{N}}$ (resp. $S_{\mathbb{N}_0}$) on Λ, for which Λ^+ is a fundamental region.

For $\lambda \in \Lambda$ and $Q \in \mathbb{N}_0$ (with $Q \geq |\lambda|$) define $\lambda^{(Q)} \in \Lambda_0(Q)$ by

$$\lambda^{(Q)} := (Q - |\lambda|, \lambda_1, \lambda_2, ...)$$

(**10.4.1**) Define

$$\mathcal{A} \ = \ \{g \in \mathbb{Q}[v] \mid g(\mathbb{Z}) \subset \mathbb{Z}\}$$

Note that this is a subring — the ring of 'numerical polynomials'. For example,

$$\binom{v}{i} = \frac{v(v-1)...(v-i)}{i!} \ \in \mathcal{A}$$

Indeed

$$\mathcal{A} \ = \ \bigoplus_{i \geq 0} \mathbb{Z}\binom{v}{i} \ = \ \mathbb{Z} \ \oplus \ \mathbb{Z}v \ \oplus \mathbb{Z}\frac{v(v-1)}{2} \oplus ...$$

[1] WHICH IS FIDDLY — CHECK IT!

10.4.1 The Schur algebras

Here $n \in \mathbb{N}$ and K is an infinite field. Following Green [22] we define $I(n,r) = \hom(\underline{r}, \underline{n})$; an action of S_r on the right on $I(n,r)$ by place permutation; and an equivalence relation on $I(n,r)^2$ by $ij \sim kl$ if there is a $w \in S_r$ such that $k = iw$ and $l = jw$. We write $I(n,r)^{2'}$ for a fixed but arbitrary transversal of $I(n,r)^2/\sim$.

Function $c_{ij} : GL_n(K) \to K$ takes $g \in GL_n(K)$ to its i,j-entry. The set of all functions $f : GL_n(K) \to K$ is an algebra via $(f+f')(g) = f(g) + f'(g)$ and $(ff')(g) = f(g)f'(g)$. Following Green [22] we define $A_K(n)$ as the subalgebra generated by the c_{ij}s. The subspace of elements expressible as homogeneous polynomials of degree r in the c_{ij}s is denoted $A_K(n,r)$. In fact $A_K(n,r)$ has a coalgebra structure.

(**10.4.2**) We then define the *Schur algebra*

$$S_K(n,r) = \mathrm{Hom}_K(A_K(n,r), K)$$

This has K-basis $\{\zeta_{ij} \mid i,j \in I(n,r)^{2'}\}$. The multiplication is

$$\zeta_{ij}\zeta_{kl} = \sum_{p,q} Z(i,j,k,l,p,q)\zeta_{pq} \tag{10.2}$$

where ...

Put

$$E_k = \oplus_{i \geq 0} k e_i$$

(**10.4.3**) The *global Schur algebra of degree Q* is

$$S_k(Q) = \mathrm{End}^{fin}_{kS_Q}(E_k^{\otimes Q})$$

There are sub-k-modules of $E_k^{\otimes Q}$ of form

$$M_k(\lambda) = k\{e_i \mid wt_0(i) = \lambda\}$$

so that

$$E_k^{\otimes Q} = \bigoplus_{\lambda \in \Lambda_0(Q)} M_k(\lambda)$$

is a decomposition of kS_Q-modules.

Let $\xi_\lambda \in S_k(Q)$ be the idempotent projecting onto $M_k(\lambda)$. For $n \in \mathbb{N}$ let

$$\xi = \sum_{\lambda \,:\, |\lambda| = Q;\ \lambda_i = 0 \ for\ i \geq n} \xi_\lambda.$$

Then

$$S_k(n,Q) = \xi S_k(Q) \xi \tag{10.3}$$

(**10.4.4**) If $n \geq Q$ then (10.3) defines a Morita equivalence of $S_k(Q)$ with $S_k(n,Q)$.

(**10.4.5**) For $i, j \in I(\mathbb{N}_0, Q)$ write $(i,j) \sim (k,l)$ if the pairs are conjugate under the right S_Q-action on $I(\mathbb{N}_0, Q)^2$. Let $\xi_{ij} \in S_k(Q)$ be

$$\xi_{ij} : e_m \mapsto \sum_{(i,j)\sim(l,m)} e_l$$

Multiplication of these elements is essentially the same as for the ζ_{ij} in (10.2).

(**10.4.6**) We now take a kind of inverse limit of large Q.
Let $\mathcal{T_A}$ be the free $\mathcal{A}$-module with basis $\{\xi_{ij} \mid (i,j) \in I(\mathbb{N}_0, \mathbb{N})^2/\sim\}$.

(**10.4.7**) Proposition. *There are unique elements $\hat{Z}(i,j,l,m,p,q)$ such that*

$$\xi_{ij}\xi_{lm} \;=\; \sum_{(p,q)} \hat{Z}(i,j,l,m,p,q)\xi_{pq}$$

(where the sum is over a transversal of $I(\mathbb{N}_0, \mathbb{N})^2/\sim$)) makes $\mathcal{T_A}$ an associative $\mathcal{A}$-algebra without identity.

(**10.4.8**) Example. An $i \in I(\mathbb{N}_0, \mathbb{N})$ is an infinite list of integers, almost all zero. In writing them we may omit trailing zeros. Thus 11=11000, 11111, 101, 0011 are all examples. We write elements ξ_{ij} as bracketed pairs in this notation, with i over j, such as

$$\xi_{ij} \;=\; \begin{bmatrix} 11000 \\ 11111 \end{bmatrix} \;=\; \begin{bmatrix} 10100 \\ 11111 \end{bmatrix}$$

Then for example

$$\begin{bmatrix} 11000 \\ 11111 \end{bmatrix}\begin{bmatrix} 11111 \\ 11000 \end{bmatrix} \;=\; \ldots$$

may be computed by considering a 'general Q' case of the finite problem. This has a given pq on the right only if there is an s such that $(11000, 11111) \sim (p, s)$ and $(s, q) \sim (11111, 11000)$ (we continue to omit trailing zeros even in the general-finite case). Note that any such s must have five 1s, but there are potentially many possible distributions, depending on p, q. We may fix $p = 11000$ in the transversal. There are then various possibilities for q.

Clearly there are solutions when $p = q = 11000$. This requires that s has five 1s, with the first two in the first two positions, so there are (as it were) $(v-2)(v-3)(v-4)/6!$ possibilities.

Another possibility for q is then $q = 101$. Here s must start 111, but the remaining two 1s can go anywhere: $(v-3)(v-4)/2$ possibilities.

The last possibility in the transversal is $q = 0011$. Here s must start 1111, but the remaining 1 can go anywhere: $(v-4)$ possibilities.

Altogether we have

$$\begin{bmatrix} 11000 \\ 11111 \end{bmatrix}\begin{bmatrix} 11111 \\ 11000 \end{bmatrix} \;=\; \binom{v-2}{3}\begin{bmatrix} 11 \\ 11 \end{bmatrix} + \binom{v-3}{2}\begin{bmatrix} 110 \\ 101 \end{bmatrix} + \binom{v-4}{1}\begin{bmatrix} 1100 \\ 0011 \end{bmatrix}$$

(**10.4.9**) For k a commutative ring and $Q \in \mathbb{Z}$ we write $k^{(Q)}$ for k made into an $\mathcal{A}$-algebra via evaluation of polynomials at Q.

When $Q \in \mathbb{N}$ there is an isomorphism between suitable finite pieces of $\mathcal{T}_{k(Q)}$ and $S_k(Q)$:

$$\left(\sum_{\lambda\in\Lambda[Q/2]} \xi_\lambda\right) \mathcal{T}_{k(Q)} \left(\sum_{\lambda\in\Lambda[Q/2]} \xi_\lambda\right) \;\cong\; S_k(\Gamma, Q)$$

where

$$\Gamma \;=$$

10.4.2 The global partition algebra as a localisation

The idea is to identify the Potts module U_k (for fixed Q), viewed as a right S_Q-module, with a summand of the defining module $E_k^{\otimes Q}$ of $S_k(Q)$ and then to "take limits".

10.4.3 Representation theory

(**10.4.10**) Here F is a commutative ring that we shall specify shortly. For any such F, and $Q \in \mathbb{Z}$, we write $F^{(Q)}$ for F made into a $\mathcal{A}$-algebra or $\mathbb{Z}[v]$-algebra (say) by evaluating polynomials at $v = Q$ (see [**?**, (3.2),(3.8)]).

Now let $R \in \mathbb{N}_0$ and $F = \mathbb{F}_p^{(R)}$ for some characteristic $p > 0$ (precise choice of which will eventually not matter). Our first objective is to say something about the modules of the global Schur algebra $\mathcal{T}_k$, where k is an $\mathcal{A}$-algebra which is a field of char.0 in which element v maps to $R \in \mathbb{N}_0$. Under suitable circumstances, simple modules for $\mathcal{T}_F$ are obtained by reduction mod.p of those for $\mathcal{T}_k$. We can thus study $\mathcal{T}_k$ (at the level of characters, say) by studying $\mathcal{T}_F$. But $\mathcal{T}_F$ in turn can be studied by studying a suitable collection of ordinary Schur algebras, and hence via the representation theory of the general linear groups.

For $\nu \in \Lambda_0^+$ with support at most in positions 0 through n (note that for each ν this just sets a lower bound for n), let $\Delta_F(\nu)$ denote the Weyl module for the F-group scheme GL_{n+1} (rows and columns of matrices indexed from 0). Let $\Delta_F^l(\nu)$ denote the l-th term in the Jantzen filtration [**?**, II.8] of $\Delta_F(\nu)$. Write $e_0, e_1, ..., e_n$ for the standard ordered basis in the weight lattice $\mathbb{Z}^{n+1}$.

Set

$$Q = R + p.$$

Now fix n (some $n >> 0$, say) and set $\rho = \rho_n = (n, n-1, ..., 0)$. If $\nu = \lambda^{(Q)}$ then the Jantzen sum formula [**?**, II.8.19] gives:

$$\sum_{l>0} ch\ \Delta_F^l(\nu) \;=\; \sum_{\substack{1 \le j \le n \\ \langle e_0 - e_j, \nu + \rho_n\rangle > p}} \chi(\nu(j)) \tag{10.4}$$

where $\chi(\mu) = ch\ \Delta_F(\mu)$ if μ dominant and $\chi((ij).\mu) = -\chi(\mu)$; and

$$\begin{aligned}\nu(j) \;&=\; (0j).\nu + p(e_0 - e_j) \\ &=\; (\lambda_j - j, \lambda_1, ..., \lambda_{j-1}, Q - |\lambda| + j, \lambda_{j+1}, ...) + (p, 0, ..., 0, -p, 0, ...)\end{aligned}$$

Here

$$w.\lambda \;:=\; w(\lambda + \rho_n) - \rho_n$$

(**10.4.11**) Remarks. (1) The dot action is used here so that the nominal index scheme for modules is the natural scheme for GL. One could work with ρ-shifted weights from the start, whereupon the dot action would be replaced by ordinary reflections.
(2) The $+(p,0,...,-p,0,...)$ in $\nu(j)$ makes it the image of ν in an affine wall. However, $\lambda^{(Q)} = \lambda^{(R+p)}$ has a p in the 0-th term, so $(0j).\nu$ has a p in the j-th term, which is then just moved back to the first term by $+(p,0,...,-p,0,...)$.

(**10.4.12**) Let us examine the sum on the RHS in (10.4). We have

$$\langle e_0 - e_j, \nu+\rho\rangle \;=\; (Q-|\lambda|+n)-(\lambda_j+n-j)$$

so there is a j-term in the sum on the RHS in (10.4) iff

$$R+j>|\lambda|+\lambda_j$$

If there is no such j then $\Delta_F(\nu)$ is simple. If there is such a j, let i be the least such. Then $\nu(i)$ fails to be dominant iff

$$R+i-1=|\lambda|+\lambda_{i-1}$$

If this holds, then not only is $\nu(i)$ non-dominant, but it lies on the $(i-1,i)$-reflection wall:

$$\nu(i) \;=(...,\lambda_{i-1},\lambda_{i-1}+1,...)$$

so $\chi(\nu(i))=0$. It follows that $\chi(\nu(j))=0$ for all $j>i$ too, so again $\Delta_F(\nu)$ is simple.

On the other hand if $\nu(i)$ is dominant then, noting that $R+i>|\lambda|+\lambda_i$ implies $R+j>|\lambda|+\lambda_j$ for all $j>i$, we have

$$\sum_{l>0} ch\;\Delta_F^l(\nu) \;=\; ch\;\Delta_F(\nu(i)) \;+\; \sum_{j>i}\chi(\nu(j))$$

Is $\nu(i+1)$, say, dominant? We can bypass this question.

The sum formula for $\Delta_F(\nu(i))$ in this case involves:

$$\langle e_0-e_i,\nu(i)+\rho\rangle \;=\; (\lambda_i-i+p)\;-\;(R+i-|\lambda|)+i\;<\;p$$

$$\langle e_0-e_{i+1},\nu(i)+\rho\rangle \;=\; (\lambda_i-i+p)\;-\;(\lambda_{i+1}-(i+1))\;>\;p$$

so j gives a contribution iff $j>i$:

$$\sum_{l>0} ch\;\Delta_F^l(\nu(i)) \;=\; \sum_{j>i}\chi((\nu(i))(j))$$

In this case

$$(\nu(i))(j) \;=\; (\lambda_j-j+p,...,Q+i-|\lambda|,...,\lambda_i-i+j,...)$$

(displaying positions $1,i,j$). In fact

$$(\nu(i))(j) \;=\; (ij).\nu(j)$$

Thus

$$\sum_{l>0} ch\;\Delta_F^l(\nu) \;=\; ch\;\Delta_F(\nu(i)) \;-\; \sum_{l>0} ch\;\Delta_F^l(\nu(i)) \;=\; ch\;L_F(\nu(i)) \;-\; \sum_{l>1} ch\;\Delta_F^l(\nu(i))$$

Since the LHS is a non-negative sum of simple characters the nominally negative part must vanish, and we have

$$\Delta^1_F(\nu) = L_F(\nu(i))$$

(**10.4.13**) JOB. Recast all this in the P-natural ρ-shift setting.

(**10.4.14**) EXAMPLES. The simplest examples is $\lambda = \emptyset$, $R = 0$. In principle we need to choose n and p. We note (a) that this can always be done; and (b) that the choice plays no subsequent role. Indeed the primeness of p is a vestige of the Schur algebra 'finesse' that allows us to use the Jantzen sum formula. With this in mind, we shall take n large and shift so that the first p-affine reflection wall parallel to $(0i)$ (each i) is drawn as if the 'non-affine' wall. This corresponds, combinatorially, to setting $p = 0$ — we must then remember that the $(0i)$ wall drawn is not at the boundary of the 'dominant region'. We will also apply (an n-independent version of) the ρ-shift derived above to weights at the outset, so we can replace the dot action of the 'Weyl group' by the ordinary action. After the ρ-shift we have the embedding:

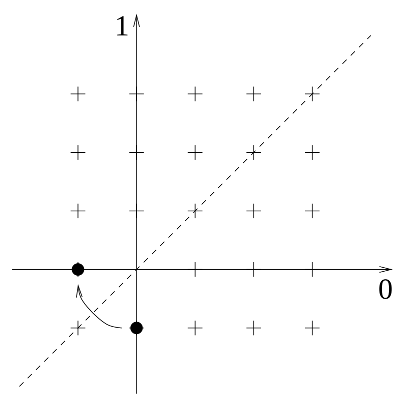
$$\Lambda \to \mathbb{Z}^{\mathbb{N}_0}$$

$$\mathfrak{e}_R : \lambda \mapsto (R - |\lambda|, \lambda_1, \lambda_2, ...) + (0, -1, -2, -3, ...)$$

so

$$\emptyset^{(0)} \;\mapsto\; (0, -1, -2, -3, ...)$$

We can represent this diagramatically by projecting onto some i, j-subspace, such as the $0, 1$-subspace:



The figure also shows

$$(01)(0, -1, -2, -3, ...) \;=\; (-1, 0, -2, -3, ...)$$

Note that since we are working with the ρ-shifted weight we use the simple reflection, not the dot action. Note that $\mathfrak{e}_R$ is invertible:

$$\mathfrak{e}_o^{-1}(-1, 0, -2, -3, ...) = \;(1)$$

Let us consider the images of some other weights:

λ	$\mathfrak{e}_0(\lambda)$	$\mathfrak{e}_1(\lambda)$	$\mathfrak{e}_2(\lambda)$
0	$(0,-1,-2,-3,...)$	$(1,-1,-2,-3,...)$	$(2,-1,-2,-3,...)$
(1)	$(-1,0,-2,-3,...)$	$(0,0,-2,-3,...)$	$(1,0,-2,-3,...)$
(2)	$(-2,1,-2,-3,...)$	$(-1,1,-2,-3,...)$	$(0,1,-2,-3,...)$
(1^2)	$(-2,0,-1,-3,...)$	$(-1,0,-1,-3,...)$	$(0,0,-1,-3,...)$

Note that some of these weights do not look 'dominant', but this is because we have omitted $+p$ from the 0-th term. Note that $\mathfrak{e}_0(2)$ lies on the (02)-wall. Recall that this is an affine wall in the GL setting (with large p):

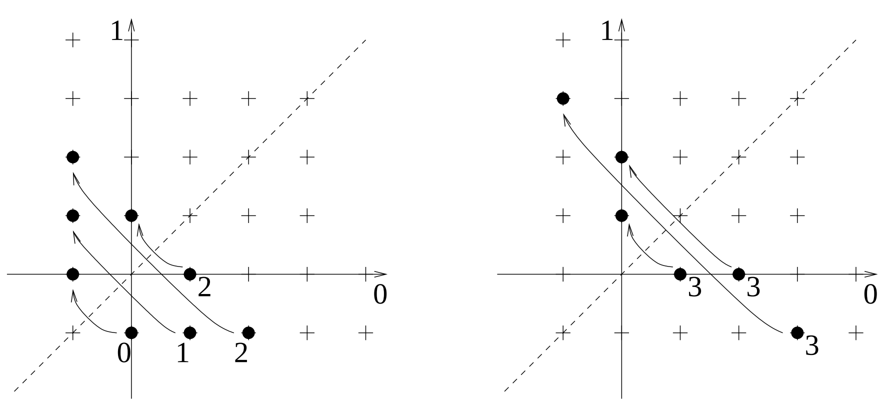
$$(2) \mapsto (p-2, 1, -2, -3, ...)$$

so does not imply that $\Delta_F(2)$ has vanishing character. However, it follows that all images under reflections of form $(0i)$ lie on an (ij)-wall. This implies that $\Delta_F(2)$ has vanishing radical. Note indeed that every λ lies on a wall — the (first affine) $(0\ |\lambda|)$-wall, unless it takes the form $\lambda = (1^m)$ for some m.

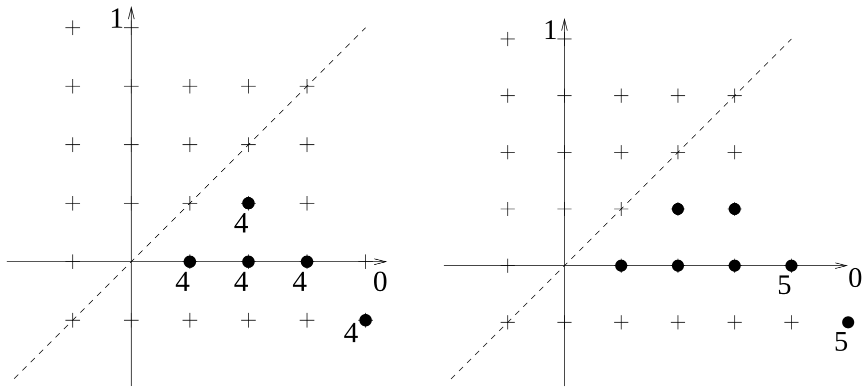
Similarly both $\mathfrak{e}_1(1)$ and $\mathfrak{e}_1(1^2)$ give simple modules.

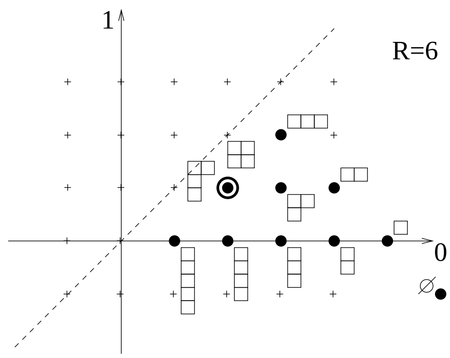
(**10.4.15**) Lemma. (I) each block consists either of a singleton, or of a chain of weights.
(II) for given R each chain block begins with a weight which is a partition of R with the first row removed.

The first two elements in each chain for $R = 0, 1, 2, 3$ are here:



The first elements in each chain for $R = 4, 5, 6$ are here:





In the latter figure ($R = 6$) we recall the original labels for these elements. Note that position in the 0,1-projection is no longer sufficient to distinguish all the weights depicted here.

10.4.4 Alcove geometric charaterisation

If we consider the $S_{\mathbb{N}}$ parabolic in $S_{\mathbb{N}_0}$ then the weights $\cup_R \mathfrak{e}_R(\Lambda)$ (a disjoint union???) all lie in the 'dominant' fundamental chamber. If we maintain our convention of considering $p = 0$ then dominance with respect to the 0-th position is not imposed.

(1) The weights that label the various chain blocks are the fully dominant weights — the weights in the fundamental alcove (in the sense of the Coxeter group/parabolic above).

(2) The other weights in the chain blocks are images of these weights in the various $(0, j)$-walls (which I guess are not walls intersecting facets of the fundamental alcove), which can also be realised via simple reflection chains of the form $(i\ i+1)...(23)(12)(01)$.

(3) The weights in the fundamental chamber that lie on a wall (i.e. on a $0, j$-wall) are singletons. Because of the $\mathfrak{e}_R$ embedding we use for $\mathcal{T}_k$ and the partition algebra (cf. that used for the Brauer algebra, say), there are many of these. If we make n large compared to R then almost every block is a singleton.

10.4.5 More

What about walks and so on? Can we embed the walks on the multiplicity-free Bratelli diagram in the same setup?

(1) can we embed the index set for the odd-partition algebras in the same setup?

10.5 Notes and references

The partition algebra first appears in the context of the partition vector formalism for Potts models in computational Statistical Mechanics [**?**, 32] ('partition vector' refers to a vector of partition functions, not to set partitions). In this setting it is a quotient of a case of the so-called graph Temperley–Lieb algebras. It appears as a focus for study in its own right in the 1992 Yale preprints YCTP-P33-92 and YCTP-P34-92 [**?**, **?**].

10.5.1 Notes on the Yale papers on the partition algebra

A set of subsets of a set M *covers* M if its union is M.

First we focus on the partition category in the form introduced in [**?**, §7]. There the set P_M of partitions of a set M is denoted S_M. Further, for q a covering set of subsets, $\mathcal{Q}(q)$ is defined as the transitive closure.

(**10.5.1**) Fix a field k. For sets $N \subseteq M$ define

$$In_N : S_M \to k(Q)S_N$$

by $In_N(p) = Q^{f_N(p)} p|_N$, where $p|_N$ is the restriction of p to N and $f_N(p)$ is the number of parts of p not intersecting N.

(**10.5.2**) With $N \subset M \cup M'$ define $Ag : S_M \times S_{M'} \to S_{M\cup M'}$ by $Ag(A,B) = \mathcal{Q}(A \cup B)$; and composition $\mathcal{P}_N$ by commutativity of

$$\begin{tikzcd} S_M \times S_{M'} \arrow[rr, "\mathcal{P}_N"] \arrow[dr, "Ag"'] & & k(Q)S_N \\ & S_{M\cup M'} \arrow[ur, "In"'] & \end{tikzcd}$$

(**10.5.3**) It will be clear how to use this composition to define a composition on, say, $S_{\underline{m}\cup\underline{n'}} \times S_{\underline{n}\cup\underline{l'}} \to k(Q)S_{\underline{m}\cup\underline{l'}}$ (by mapping $\underline{n} \cup \underline{l'} \to \underline{n'} \cup \underline{l''}$ and then using $\mathcal{P}_{\underline{n'}}$ directly). This then extends $k(Q)$-linearly to define composition in the partition category. .

Chapter 11

On representations of the Brauer algebra

The Brauer algebra [6] provides a good testing ground for some of the technniques we met in Chapters 1-7. An exposition of Brauer algebras over the complex field is given in the sequence of papers of Cox, De Visscher, Martin [13, 14, 15], and references therein.

In this Chapter we first note some motivations for studying the Brauer algebra. Then we summarize a general approach to Brauer algebra representation theory. Finally we define the algebra and associated 'diagram categories' in terms of bases of Brauer diagrams; and determine some useful properties of these diagrams. In Chapter 12 we study general aspects of the representation theory of the Brauer algebra, such as the construction and properties of a special class of 'integral' modules. In Chapter 13 we study the representation theory over $\mathbb{C}$.

11.1 Context of the Brauer algebra

The Brauer algebra is a device proposed by Brauer to study invariant theory of the orthogonal groups, generalising Schur–Weyl duality.

11.2 Introduction to Brauer algebra representations

For each field k, natural number n and parameter $\delta \in k$, the Brauer algebra $B_n(\delta)$ is a finite dimensional algebra, with a basis of pair partitions of the set $\{1, 2, ..., 2n\}$. (It is a certain deformation (from $\delta = 1$) of the k-linearisation of our example in (4.1.10). We describe it shortly.)

11.2.1 Reductive and Brauer-modular representation theory

A finite dimensional algebra over a field presents us with the following tasks in representation theory.

Firstly we have the 'reductive' representation theory — the aspects of representation theory concerned with the extraction of simple characters from the study of the regular module:

(1) There are finitely many isomorphism classes of simple modules — index these.
(2) Describe the blocks (the RST closure of the relation on the index set for simple modules given by $\lambda \sim \mu$ if L_λ and L_μ are composition factors of the same indecomposable projective module).
(3) Describe the composition multiplicities of indecomposable projective modules.

Then there are various more 'constructive' tasks such as:
(4) Describe the composition series of indecomposable projective modules.
(5) give explicit constructions for simple modules.
And then there are more esoteric tasks which we shall pass over here.

Accordingly $B_n(\delta)$ presents us with these tasks.

(**11.2.1**) A finite dimensional algebra over a field has a finite collection of (isomorphism classes of) simple modules $L = L(\lambda)$ and corresponding indecomposable projective modules $P = P(\lambda)$.
(If the radical $J = 0$ then $L = P$. Otherwise, the correspondence is either characterised by $L = P/JP$; or equivalently by taking a minimal projective cover of L.
EXERCISE: verify this equivalence.)
For the Brauer algebra B_n (for given k, δ) in particular we write $L_n(\lambda)$ and $P_n(\lambda)$ if we need to emphasise n.

The Brauer algebras also have *Brauer-Specht modules* $\Delta_n^k(\lambda)$ (sometimes we just call these Specht modules). This means the following.
For each n there is
(A) a $\mathbb{Z}[v]$-algebra $B_n^{\mathbb{Z}}$, free of finite rank as a $\mathbb{Z}[v]$-module, that passes to each Brauer algebra by base change (making k a $\mathbb{Z}[v]$-algebra by $v \mapsto \delta$); and
(B) a collection of modules $\Delta_n = \{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$ for this algebra (here Λ^n is the set of integer partitions of $n, n-2, ..., 0/1$, as we shall see) that are $\mathbb{Z}[\delta]$-free modules of known rank, thus defining

$$\Delta_n^k(\lambda) \;=\; k \otimes_{\mathbb{Z}[v]} \Delta_n(\lambda)$$

for each k and $\delta \in k$ (note that the notation assumes δ is given); and that there is a choice of field k ($k = k^o$, say) extending $\mathbb{Z}[v]$ for which $\{\Delta_n^k(\lambda)\}_{\lambda \in \Lambda^n}$ is a complete set of simple modules.

(**11.2.2**) The above conditions do not determine the Δ-modules completely (but see e.g. Benson [**?**]). We give a construction shortly.

The idea is firstly the generalisation of Brauer's approach to the modular representation theory of a finite group G (N.B., the ubiquity of Brauer's name here is mathematically coincidental). This essentially starts with the irreducible representations of G over $\mathbb{C}$ (or some smaller extension of $\mathbb{Q}$), the *ordinary* irreducibles, and uses integral lattices in these in the role for which we shall use Δ-modules. That is, it uses them to construct p-modular systems.

Secondly the idea is to generalise in particular the Specht modules of symmetric group representation theory (ordinary irreducibles which are defined over $\mathbb{Q}$, and can even be constructed by base change from modules over $\mathbb{Z}$).

(**11.2.3**) Hereafter we may abuse notation slightly by writing $\mathbb{Z}[\delta]$ for $\mathbb{Z}[v]$ when k is to be a $\mathbb{Z}[v]$-algebra by $v \mapsto \delta \in k$.

(**11.2.4**) CLAIM: The index set Λ^n for the Δ-modules of B_n contains an index set for simple modules over any k and δ (indeed for each k, δ, a certain subset of heads of suitably constructed Brauer-Specht modules is a complete set of simple modules, and the labels for simple modules may be obtained in this way).

This general k assertion is not verified here. The idea is that it should work similarly to the kS_n case.

(**11.2.5**) As we shall see, when $k = \mathbb{C}$ as a field (note that this does not fix the action of δ *per se*) the claim holds for each δ, and indeed each of these simple index sets, here denoted $\Lambda^{n,\delta}$, coincides with Λ^n, unless $\delta = 0$, in which case the index set for simples has one less element when n is even: we can take $\Lambda^{n,0} := \Lambda^n \setminus \Lambda^0$. Either way, note the inclusions $\Lambda^{m,\delta} \subset \Lambda^{m+2,\delta}$.

(**11.2.6**) As in (7.3.11), or [22, §6.6], we define the Brauer-Specht module decomposition matrix for $B_n(\delta)$ over k (note that k is a field with a structure of $\mathbb{Z}[v]$-algebra) by

$$d_{ij} \;=\; [k \otimes_{\mathbb{Z}[v]} \Delta_n(i) : L_n(j)]$$

(we shall also use $D_{ji} = d_{ij}$), the composition multiplicity, where $\{L_n(j)\}_{j \in \Lambda^{n,\delta}}$ is the set of simple $B_n(\delta)$ modules over k.

As we shall see, while the matrix D is not necessarily square, it is lower unitriangular in an order that respects the inclusions $\Lambda^{m,\delta} \subset \Lambda^{m+2,\delta}$.

As usual we define the Cartan decomposition matrix C (for given k, δ) by

$$C_{ij} \;=\; [P_n(i) : L_n(j)]$$

Recall from the general machinery of §**??** that there is a $B_n^{\mathbb{Z}}$-module Π_i — strictly speaking a module over some k, δ-dependent $\mathfrak{p}$-adic extension of $\mathbb{Z}[v]$ — that passes to each $P_n(i)$ by base change (note that Π_i depends on k, δ, but we are here holding these fixed), and hence a corresponding module over the field k^o of fractions, Π_i^o. We have from this, in principle, another natural collection of invariants D^o associated to k, δ, given by:

$$D^o_{ij} \;:=\; [\Pi_i^o : \Delta_n^o(j)]$$

However, as we know on general grounds $D = D^o$.

11.2.2 Globalisation and towers of recollement

(**11.2.7**) Another important organisational property is that there is an idempotent $e \in B_n$ (any ground ring, at least for sufficiently large n) such that $eB_ne \cong B_{n-2}$, so that (cf. [22, §6.6]) we can construct an inverse limit for the sets $\{\Lambda^{n,\delta}\}_n$, and a corresponding limit for C and D, so that any fixed n case can be extracted by projection. The set limit is simply Λ (i.e. corresponding to the obvious inclusions), and the projections are correspondingly transparent. By the unitriangular property it is just a truncation (noting, as we shall see, that $C = DD^T$).

We have from Green §6.6 that e defines an inclusion of $\Lambda^m \subset \Lambda^{m+2}$ and $\Lambda^{m,\delta} \subset \Lambda^{m+2,\delta}$ (any m), such that d_{ij} does not depend on n whenever i, j belong in the appropriate set. Since $D_{ji} = d_{ij}$ we have a corresponding stability property of D, and hence of C.

In fact we claim that we can prove this stability of D directly, and strengthen this to a statement about filtration multiplicities in a filtration of each $P_n(\lambda)$ by Δ-modules.

(**11.2.8**) Fix k, δ. By general properties of P-modular systems we can 'lift' each $P_n(i)$ to a projective module over a suitable ground ring and hence define a corresponding module over the field that makes B_n semisimple. This module obviously has a well-defined decomposition into Δ-modules.

(At least if all fields have char.0) This defines a set of invariants of $P_n(i)$ that will provide a character formula in terms of Δ-characters. Let us write

$$(P_n(i) : \Delta_n(j))_{\mathbb{Z}}$$

for these invariants. In the theory of quasihereditary algebras one has a similar notation: $(P_n(i) : \Delta_n(j))$. It will be helpful to compare these.

Separately from the P-modular and q.h. settings, $P_n(i)$, or indeed any module, might have a 'weak' Δ-filtration (i.e. a filtration without necessarily well-defined multiplicities). If this is in fact a ('strong') Δ-filtration (i.e. a filtration with well-defined multiplicities), then this would define another set of invariants. (Note that if the Δ_n are a basis for the Grothendieck group then well-definedness is forced, since any filtration determines a character. This is usually, but not always, the case for us.)

Separately from this again, Δ_n, *or some subset*, will be a basis for the Grothendieck group (for now we are still working in char.0). The choice of a given subset gives another character formula. In Lie theory/q.h. there is a rather analogous setup in which the Δ_n are always a basis for the Grothendieck group. In this case there can only be one Δ-character formula for any module, so the various formal invariants considered above would all coincide. In this case, if M has a Δ-filtration it is automatically strong and $(M : \Delta_n(j))$ is then *defined* by

$$[M] \;=\; \sum_j (M : \Delta_n(j))\, [\Delta_n(j)]$$

in the Grothendieck group (see e.g. Donkin (Appendix) [**?**]).

(**11.2.9**) An interesting question arises when the basis for the Grothendieck group is smaller (in our case with $k = \mathbb{C}$ this is just the case $\delta = 0$), so that the two sets of invariants differ. The question is: are the filtration multiplicities well-defined? Are they forced to agree with the well-defined numbers $(- : -)_{\mathbb{Z}}$?

An extra ingredient that we have available is knowledge about the *order* in which Δ-modules must appear in a filtration. Can we use this?

We shall return to this later.

(**11.2.10**) Anyway, our approach to task (3) is to compute the $(P_n(i) : \Delta_n(j))$, by computing each number in the first n-level in which it appears. We make heavy use of the solution to task (2) (via the study of a fortuitous choice of central elements of B_n); and the properties of Δ-modules under ordinary induction (noting that induction takes projectives to projectives).

11.2.3 Overview of the Chapter

In (11.3.11) we define the Brauer algebra $B_n = B_n(\delta)$ (initially over $\mathbb{Z}[\delta]$ with δ indeterminate, and hence for each choice of commutative ground ring k and $\delta \in k$ by base change). In §11.4 we prove some Lemmas on Brauer diagram manipulation that we use later.

In §11.5 we construct some useful idempotents and idempotent subalgebras. In particular we note an idempotent $e \in B_n$ $(n > 2)$ such that

$$B_{n-2} \cong eB_ne \qquad (n > 2)$$

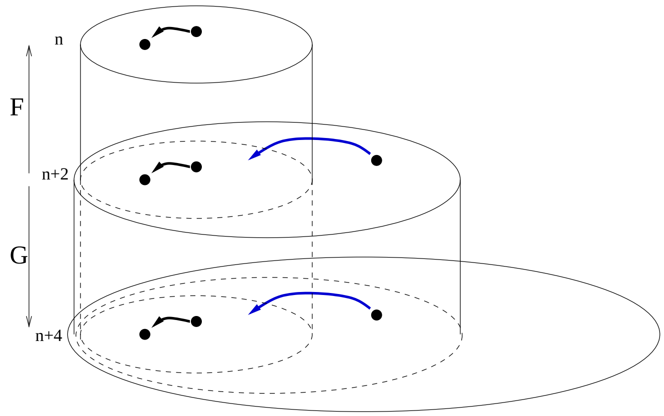


Figure 11.1: Schematic for embeddings of B_n-module categories.

while

$$B_n/B_neB_n \quad \cong kS_n$$

It follows that an index set for simple B_n-modules may be obtained as the disjoint union of an index set for B_{n-2} and one for S_n (e.g Λ_n if $k = \mathbb{C}$). We label simple modules accordingly. It follows from the first isomorphism that $M \mapsto eM$ defines an exact functor from B_n-mod to B_{n-2}-mod. Then for all simple modules $L_\lambda = L^n_\lambda$ such that $eL_\lambda \neq 0$ we have that eL_λ is simple and

$$(M : L_\lambda) \;=\; (eM : eL_\lambda)$$

In (12.1.14) we construct a certain special set of B_n-modules called Δ-modules. In fact we give three constructions, showing isomorphism via the contruction of a basis in each case.

In (12.1.30) we show that there is an extension of the ring $k = \mathbb{Z}[\delta]$ over which these Δ-modules are a complete set of simple modules, and B_n is semisimple, for every n. (From this we note that the simple decomposition matrices (D_{ij}) of these modules over a suitable choice of k, δ determine the Cartan decomposition matrices over that k, δ.)

We show more generally that these modules provide a basis for the Grothendieck group — the character ring — of each algebra. We also show that they filter projective modules. (Note that the Δ-content of projectives is reciprocal data to the simple content of Δ-modules.)

We construct functors $F : B_{n+2} - \text{mod} \to B_n - \text{mod}$ and $G : B_n - \text{mod} \to B_{n+2} - \text{mod}$ that allow us to embed the category B_n-mod in B_{n+2}-mod. These functors thus define another class of modules — the G-images of B_n-modules in B_{n+2}-mod. We show that (with one manageable exception over $\mathbb{C}$) this embedding takes the Δ-module $\Delta_n(\lambda)$ to the Δ-module $\Delta_{n+2}(\lambda)$; and takes indecomposable projectives to indecomposable projectives similarly. Thus the Δ-content of projective modules is, once defined, stable with n. We determine each new such datum by an induction on n, using also the induction functor associated to the natural inclusion $B_n \hookrightarrow B_{n+1}$ (see later); known restriction rules for Δ-modules; and projection onto blocks (which we also determine).

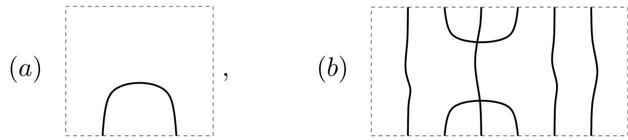


Figure 11.2: (a) A $(0,2)$-Brauer diagram; (b) a $(6,6)$-Brauer diagram, representing $U_{24} \in J_{6,6}$.

11.3 Brauer diagrams and diagram categories

We recall some notation from §2.2. For $n \in \mathbb{N}$ define $\underline{n} = \{1, 2, ..., n\}$ and $\underline{n'} = \{1', 2', ..., n'\}$. For S a set, $P(S)$ is the power set; P_S the set of partitions of S; J_S the set of pair partitions of S; and $J_{n,m} = J_{\underline{n} \cup \underline{m'}}$.

We shall use $J_{n,n}$ as a basis for $B_n(\delta)$, but we also introduce a useful diagram calculus.

(**11.3.1**) For given $n > 1$, we define some particular pair partitions in $J_{n,n}$:

$$U_{ij} = \{\{1,1'\}, \{2,2'\}, ..., \{i,j\}, \{i',j'\}, ..., \{n,n'\}\} \tag{11.1}$$

$$(ij) = \{\{1,1'\}, \{2,2'\}, ..., \{i,j'\}, \{i',j\}, ..., \{n,n'\}\}$$

We shall call pair partitions of this form *generators*.

(**11.3.2**) An (n,m)-*Brauer diagram* is a representation of a pair partition of a row of n and a row of m vertices, arranged on the top and bottom edges (respectively) of a rectangular frame. Each part is drawn as a line, joining the corresponding pair of vertices, in the rectangular interval. See Figure 11.2 for an example.

Of course there are many different ways of representing a given partition in this way. However, it will be clear that an (n,m)-Brauer diagram can be used to represent an element of $J_{n,m}$. Again see Figure 11.2 for an example.

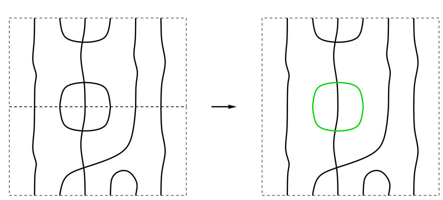
For d such a diagram, write $[d]$ for the corresponding pair partition.

(**11.3.3**) Recall the *bare* Brauer category

$$\mathbf{Br} = (\mathbb{N}_0, J_{n,m}, \circ)$$

(as in (4.1.10)). Brauer diagrams can be used to describe the homs in this category. They can also be used to compute compositions.

(**11.3.4**) Suppose d, d' are diagrams such that $[d] \in J_{n,m}$ and $[d'] \in J_{m,l}$. If the vertices are suitably spaced, then d and d' may be juxtaposed so as to make each of the m lower vertices in d coincide with the corresponding upper vertex in d', as illustrated on the left here:



Write dd' for this juxtaposition. Note that a pair partition of the vertices on the exterior of the combined frame may be read off from dd', in direct analogy to (2.3). Write $[dd']$ for this partition, regarded as an element of $J_{n,l}$.

(**11.3.5**) LEMMA. Suppose d, d' are diagrams as above.
(1) The partition $[dd']$ depends only on $[d]$ and $[d']$.
(2) Composition of partitions in the category **Br** may be computed by juxtaposition of diagrams. That is, $[d] \circ [d'] = [dd']$.
(3) Another feature of dd' that depends only on $[d]$ and $[d']$ is the number $\#(dd')$ of internal closed loops formed (such as the one in the example). □

(**11.3.6**) Fix a ring k, and $\delta \in k$. Noting Lemma (11.3.5)(3), we define a deformation of the k-linear extension of **Br** by modifying the category composition to:

$$[d] * [d'] = \delta^{\#(dd')}[dd']$$

(and extending k-linearly). We denote this category

$$\mathbf{Br}_\delta^k = (\mathbb{N}_0, kJ_{n,m}, *),$$

or just $\mathbf{Br}_\delta$ if k is fixed.

(**11.3.7**) We define a category $\mathbf{Br}^* = (\mathbb{N}, J_{n,m} \times \mathbb{N}_0, \bullet)$ by the composition

$$([d], i) \bullet ([d'], i') = ([dd'], i + i' + \#(dd'))$$

(a diagram calculus for this uses Brauer diagrams with closed loops, sometimes called Brauer pseudodiagrams).

(**11.3.8**) LEMMA. For any ring k and $\delta \in k$ the category $\mathbf{Br}_\delta^k$ is a quotient of the k-linear extension of $\mathbf{Br}^*$, given by

$$([d], i) \mapsto \delta^i [d]$$

(**11.3.9**) We write $\mathbf{Br}(m, n)$ for $J_{m,n}$ realised as the set of (m, n)-Brauer diagrams up to equivalence. Hereafter a Brauer 'diagram' generally means a diagram up to equivalence. Thus the hom-set in the k-linear category $\mathbf{Br}_\delta^k$ can equally well be considered to be $k\mathbf{Br}(m, n)$.

(**11.3.10**) EXAMPLE. The sets $\mathbf{Br}(0, 0)$, $\mathbf{Br}(1, 1)$, $\mathbf{Br}(2, 0)$ and $\mathbf{Br}(0, 2)$ each have a single element, here denoted 1_0, 1_1, u and u' respectively.

(**11.3.11**) The Brauer algebra $B_n(\delta)$ over k is the free k-module with basis $\mathbf{Br}(n, n)$ and the $\mathbf{Br}_\delta^k$ category composition (i.e. replacing each closed loop formed in composition by a factor δ).

11.3.1 Remarks on the ground ring and Cartan matrices

(**11.3.12**) REMARK. The basic, 'integral' version of the Brauer algebra is the case over the ring $k = \mathbb{Z}[\delta]$. From here there are thus two aspects to the base change to an algebra over a field: the choice of field k and the choice of δ. More precisely this is the choice of field k equipped with the structure of $\mathbb{Z}[\delta]$-algebra. Thus we have possible intermediate steps: base change to ring $k[\delta]$ (k some field); base change to $\mathbb{Z}$ (a $\mathbb{Z}[\delta]$-algebra by fixing $\delta = d \in \mathbb{Z}$).

Each of these ground rings is a PID and hence a Dedekind domain, and hence amenable to a P-modular treatment (in the sense of Brauer's general approach to 'modular' representation theory of finite dimensional algebras, described variously in Benson [5], Curtis–Reiner [16], Brauer [7] and so on). This means that the Cartan matrix C can be computed via a Δ-decomposition matrix D. See §7.3.2 for a description of the general setting.

(**11.3.13**) Roughly speaking, the setup is as follows. Let B be an algebra over $\mathbb{Z}[\delta]$ (we are thinking of one of the Brauer algebras).
(I) There is a valuation on $k[\delta]$ (any field k — we are thinking of $k = \mathbb{C}$, and will use this example hereafter); and hence an absolute value. It then follows that there is a completion of $\mathbb{C}[\delta]$ to a complete DVR, call it R. (Since R simply extends $\mathbb{Z}[\delta]$ we have a version of B over R, which we could call B_R.)
(NB According to Curtis–Reiner [16] the various composition multiplicities we shall compute do not depend on the completion — it is needed only to satisfy certain existence requirements in the proof.)
(II) Now pick $\delta_c \in \mathbb{C}$ and consider an element $(\delta - \delta_c)$ in $\mathbb{C}[\delta]$ (and hence in R) and consider the quotient ring by the ideal I so generated. Since the ideal is maximal in R (see **??**) the quotient is a field, call it $\bar{R}$.
(III) Next suppose that there is an extension $\hat{R}$ of R (typically just the field of fractions), such that $B_{\hat{R}}$ is semisimple (with simple modules $\hat{\Delta}_i = \hat{R} \otimes_R \Delta_i$ for some set of R-free B-modules Δ_i).

Let us write L_j for a complete set of simple $B_{\bar{R}}$-modules (so far unknown). And $\bar{\Delta}_i = \bar{R} \otimes_R \Delta_i = R/I \otimes_R \Delta_i$. (NB we have $\bar{\Delta}_i = \Delta_i / I\Delta_i$, right!?) We have some well-defined (but unknown) multiplicities

$$d_{ij} = [\bar{\Delta}_i : L_j]$$

(NB This notation d_{ij} is exactly as used by Benson [5]. We shall mainly adhere to a slightly different notation — $D_{ji} = d_{ij}$.) Now consider an indecomposable projective $B_{\bar{R}}$-module P.

(**11.3.14**) CLAIM: Let $ee = e \in B_{\bar{R}}$. Then there is an idempotent e' in B whose image in $B_{\bar{R}}$ is e.

ASIDE: The point of this observation is that there is a projective $B_{\bar{R}}$-module $B_{\bar{R}}e$, and a projective B-module Be' that, by the observation, reduces to it. On the other hand, Be' is also a lattice inside a $B_{\hat{R}}$-module M, say. Since $B_{\hat{R}}$ is semisimple, *this* module decomposes as a sum of simple $B_{\hat{R}}$-modules. The multiplicity of $\hat{\Delta}_i$ in M can be expressed as

$$[M : \hat{\Delta}_i] \;=\; \dim_{\hat{R}} \hom_{B_{\hat{R}}}(M, \hat{\Delta}_i)$$

On the other hand:
CLAIM: $\hom_B(Be', \Delta_i)$ is an R-form for $\hom_{B_{\hat{R}}}(M, \hat{\Delta}_i)$, so its rank agrees with the dimension.
(Certainly $\hom_B(Be', \Delta_i)$ sits inside. But is $\hom_B(Be', \Delta_i)$ even a free R-module? We have (from earlier) $\hom_B(Be', \Delta_i) \cong e'\Delta_i$. Does this help?)

11.4 Properties of the diagram basis

11.4.1 Manipulation of Brauer diagrams: lateral composition

(**11.4.1**) Extending the equivalence of diagrams from (11.3.9), there is a set bijection between $\mathbf{Br}(m, n)$ and a set of pair partitions similarly drawn on a disk with marked boundary point. The

$m+n$ vertices are drawn around the disk clockwise from the marked point. For any $i \in \mathbb{Z}$ such that $m-i, n+i \in \mathbb{N}_0$ this induces a bijection $\mathcal{R}_i : \mathbf{Br}(m,n) \to \mathbf{Br}(m-i,n+i)$. For example, if $i=1$ this moves a single vertex ('ambient isotopically') from the top edge clockwise to the bottom edge.

(**11.4.2**) For $m,n,r,s \in \mathbb{N}_0$, define a product

$$\boxtimes : \mathbf{Br}(m,n) \times \mathbf{Br}(r,s) \ \to \mathbf{Br}(m+r,n+s)$$

by placing diagrams side by side. Hence define an injection adding propagating lines $\{\{m+1,n+1'\},...,\{m+r,n+r'\}\}$:

$$i_{m+1,m+r} : \mathbf{Br}(m,n) \hookrightarrow \mathbf{Br}(m+r,n+r)$$
$$D \ \mapsto \ \ D \boxtimes 1_r$$

(**11.4.3**) The map $d \mapsto d \boxtimes 1_1$ defines an inclusion of algebras $B_{n-1}(\delta) \hookrightarrow B_n(\delta)$ $(n \geq 1)$. By this restriction, any B_n-module (such as $k\mathbf{Br}(n,m)$) is also a B_{n-1}-module.

(**11.4.4**) We may define an injection of $\mathbf{Br}(n,n) \hookrightarrow \mathbf{Br}(n,n+2)$ by $d \mapsto \mathcal{R}_1(d \boxtimes 1_1)$. This is the same as $d \mapsto d \boxtimes u'$.

(**11.4.5**) A non-crossing Brauer diagram is called a Temperley–Lieb diagram. Using both the category and the lateral composition the elements 1_1, u and u' generate all such diagrams. There are three elements in $\mathbf{Br}(2,2)$, including a crossing diagram. If we write x for this, then:

(**11.4.6**) Proposition. *Any Brauer diagram may be realised using the two compositions $\circ$ and $\boxtimes$ (or indeed $*$ and $\boxtimes$) on the diagrams $\{1_1, u, u', x\}$.*

Proof. Note first that the realisation of any TL diagram with $\{1_1, u, u'\}$ is easy. Note next that $\{1_1, x\}$ generate all the S_n subgroups. Using these it is easy to see that the number of crossings in a diagram can be reduced to zero. This reduces to the TL case. □

11.4.2 Ket-bra diagram decomposition

(**11.4.7**) We write $\mathbf{Br}^{\leq l}(m,n)$ for the subset of $\mathbf{Br}(m,n)$ with $\leq l$ propagating lines.

(**11.4.8**) Lemma. Using the bare composition

$$\mathbf{Br}^{\leq l}(m,n) = \mathbf{Br}(m,l) \circ \mathbf{Br}(l,n)$$

Thus

$$k\mathbf{Br}^{\leq l}(m,n) = k\mathbf{Br}(m,l) * k\mathbf{Br}(l,n)$$

unless $m,n=0$, $l \geq 2$, in which case

$$k\mathbf{Br}(m,l) * k\mathbf{Br}(l,n) = \delta k\mathbf{Br}^{\leq l}(0,0)$$

(**11.4.9**) We write $\mathbf{Br}^{l}(m,n)$ for the subset of $\mathbf{Br}(m,n)$ with l propagating lines; and $\mathbf{Br}^{1_l}(m,n)$ for the subset of these in which none of the l propagating lines cross.

(**11.4.10**) Lemma. The identity diagram in $(\mathbf{Br}(l,l),\circ)$ (and hence in $(k\mathbf{Br}(l,l),*)$ for any k,δ) is denoted 1_l. The pair $(\mathbf{Br}^l(l,l),*)$ is isomorphic to the symmetric group S_l.

(**11.4.11**) For this reason we sometimes call elements of $\mathbf{Br}^l(l,l)$ (any l) *permutations*.

(**11.4.12**) Since an element of $\mathbf{Br}(m,n)$ is specified by the list of pair partitions it depicts, it is specified in particular by the decomposition of this list into (i) pairs on the top row; (ii) pair on the bottom row; (iii) pairs from top to bottom. We have immediately the following Lemma.

(**11.4.13**) Lemma. (*Bra-ket Lemma*) Let $m,n \geq l$. The category composition defines bijections:

$$\mathbf{Br}^{1_l}(m,l) \,\times\, \mathbf{Br}^l(l,l) \,\xrightarrow{\sim}\, \mathbf{Br}^l(m,l) \tag{11.2}$$

$$\mathbf{Br}^{1_l}(m,l) \,\times\, \mathbf{Br}^l(l,l) \,\times\, \mathbf{Br}^{1_l}(l,n) \xrightarrow{\sim} \mathbf{Br}^l(m,n) \tag{11.3}$$

We call this *ket-bra decomposition*. $\square$

(**11.4.14**) The *ket* $|d\rangle$ of $d \in \mathbf{Br}(m,n)$ is the projection of its preimage (in (11.3)) into $\mathbf{Br}^{1_l}(m,l)$.

(**11.4.15**) Remark. Note the opposite equivalence of our various categories. By this equivalence, every 'left' result has a right version. In general we shall only state one version explicitly.

(**11.4.16**) Lemma. Fix $m,n \geq l$.
(I) For any $d,d' \in \mathbf{Br}^l(m,l)$, there exists a permutation $w \in \mathbf{Br}^m(m,m)$ such that $wd = d'$ (hereafter we may omit the composition symbol, where no ambiguity arises).
(II) For any $d \in \mathbf{Br}^l(m,n)$ and $d' \in \mathbf{Br}^{\leq l}(m,n)$ there exist diagrams $w \in B_m$ and $w' \in B_n$ such that $d' = wdw'$.

Proof. (I): This follows from (11.4.12). (II): This follows similarly. Note first that the case of $d' \in \mathbf{Br}^l(m,n)$ is a direct extension of (I). If $d' \in \mathbf{Br}^{\leq l-2}(m,n)$ then d has at least two propagating lines, and any two such can be replaced by a 'cup and cap', with all other pairs unchanged, by multiplying by a suitable diagram d''. That is, $dd'' \in \mathbf{Br}^{l-2}(m,n)$. Now use the first part again, at level $l-2$. $\square$

11.5 Idempotent diagrams and subalgebras in $B_n(\delta)$

We shall have a convention of using capital letters for elements of Brauer algebras defined by generators (and hence making sense in any sufficiently large Brauer algebra); and using lower-case for elements defined as specific diagrams in specific algebras.

In this section the category $\mathbf{Br}_\delta$, and each B_n, is defined over an arbitrary commutative ring k with $\delta \in k$.

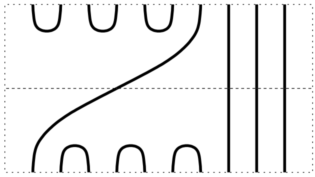
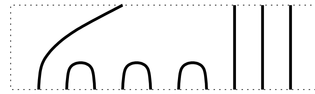
(**11.5.1**) The product $E_1 = U_{12}U_{23}$ in B_n defines an element in $\mathbf{Br}^{n-2}(n,n)$ for any $n \geq 3$. For $m \in \mathbb{N}$, the product

$$\overline{E}_m := \; U_{12}\; U_{34}\; ...\; U_{2m-1\;2m}$$

defines an element in $\mathbf{Br}^{n-2m}(n,n)$ for any $n \geq 2m$;

$$E_m := (U_{12}U_{34}...U_{2m-1\;2m})(U_{23}U_{45}...U_{2m\;2m+1}) = U_{2m-1\;2m}E_{m-1}U_{2m\;2m+1}$$

defines an element in $\mathbf{Br}^{n-2m}(n,n)$ for any $n \geq 2m+1$. (See Figure 11.3(a) for an example.)

$E_3 \;=$ , $\quad (1_1 \boxtimes (u')^{\boxtimes 3}) \boxtimes 1_3 \;=$ 

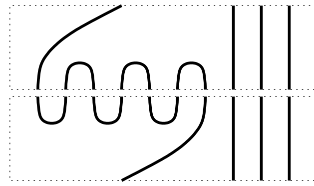
$((1_1 \boxtimes (u')^{\boxtimes 3})(u^{\boxtimes 3} \boxtimes 1_1)) \boxtimes 1_3 \;=$ 

Figure 11.3: (a), (b), (c).

(**11.5.2**) Lemma. All the elements E_m are idempotent.

Proof. Draw a diagram. □

(**11.5.3**) One finds, for example, that $E_m = (u^{\boxtimes m} \boxtimes 1_1 \boxtimes (u')^{\boxtimes m}) \boxtimes 1_r \in B_n$ for a suitable choice of r (such that $n = 2m+1+r$). In particular

$$e_3^1 := \; u \boxtimes 1_1 \boxtimes u' = U_{12}U_{23}.$$

in $\mathbf{Br}^1(3,3)$. Note that

$$(1_1 \boxtimes (u')^{\boxtimes m})(u^{\boxtimes m} \boxtimes 1_1) \;=\; 1_1 \tag{11.4}$$

(**11.5.4**) Lemma. Any 'bra' in $\mathbf{Br}^l(l,n)$ with $l > 0$, may be completed to an idempotent.

Proof. Apply Lemma (11.4.16).

(**11.5.5**) Lemma. Fix k and δ. For $n, m > 0$, consider $E_m \in B_{n+2m}$ (note that the condition $n > 0$ is needed for this to make sense). Then
(I) $u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m) \subset \mathbf{Br}(n+2m, n+2m)$ is a k-basis of $E_m B_{n+2m}$. Thus
(II) there is an isomorphism of left B_n right B_{n+2m}-bimodules

$$k\mathbf{Br}(n, n+2m) \;\; \stackrel{\sim}{\to} \;\; E_m B_{n+2m} \tag{11.5}$$
$$d \;\; \mapsto \;\; u^{\boxtimes m} \boxtimes d \tag{11.6}$$

(Note that the left action of B_n on $E_m B_{n+2m}$ is defined by this map. It corresponds to a certain specific inclusion of B_n into B_{n+2m}.)

Proof. (I): By construction $u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m)$ is the set of all elements of $\mathbf{Br}(n+2m, n+2m)$ that have $u^{\boxtimes m}$ in the top-left-hand corner. Since E_m has this property, the right ideal $E_m B_{n+2m}$ is spanned by diagrams with this property, and so any such spanning set is contained in $u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m)$. On the other hand, by (11.4), $E_m d = d$ for every $d \in u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m) \subset \mathbf{Br}(n+2m, n+2m)$. Thus $E_m(u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m)) = u^{\boxtimes m} \boxtimes \mathbf{Br}(n, n+2m)$.

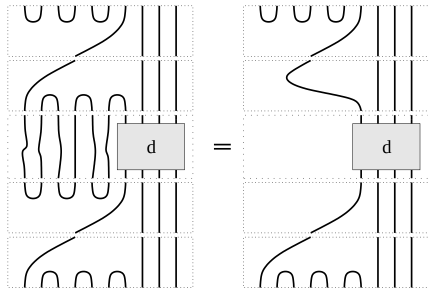


Figure 11.4: Image of $d \in B_4$ in $E_3 B_{10} E_3$ under the map in Lemma 11.5.7.

(II): The bimodule structure on $k\mathbf{Br}(n, n+2m)$ is clear. On the other side, the right action is clear. The easiest way to see the left action is to note that the algebra $E_m B_{n+2m} E_m$ acts, and that we shall establish an isomorphism with B_n shortly. (A direct proof is also possible.) □

(**11.5.6**) Note that various other idempotents will also work in place of E_m, in general (both here and subsequently).

(**11.5.7**) Lemma. Fix commutative ring k and $\delta \in k$. For $n, m > 0$ the map defined by

$$\Psi_E : B_n \xrightarrow{\sim} E_m B_{n+2m} E_m \quad (11.7)$$

$$d \mapsto E_m(1_{2m} \boxtimes d) E_m \quad (11.8)$$

is an isomorphism of k-algebras. Every diagram on the right has $u^{\boxtimes m}$ in the top left-hand corner, and $(u')^{\boxtimes m}$ starting from the second position in the bottom left-hand corner. Removing these cups and caps from a diagram leaves a diagram in B_n. This process defines the inverse map.

Proof. The proof of isomorphism as k-modules is analogous to (11.5.5)(I). See Figures 11.4 and 11.5 for illustrations of the key calculation in verifying the algebra structure.

(**11.5.8**) Lemma. (I) For $m, n > 0$,

$$k\mathbf{Br}^{\leq n}(n+2m, n+2m) = B_{n+2m} E_m B_{n+2m}$$

(II) For $m > 0$, there is an idempotent in $k\mathbf{Br}^0(2m, 2m)$ if and only if δ is a unit in k.

Proof. (I): We have $E_m \in \mathbf{Br}^n(n+2m, n+2m)$. Now use Lemma (11.4.16). (II): Every diagram composition from $\mathbf{Br}^0(2m, 2m)$ creates a closed loop. □

Examples

(**11.5.9**) We define the 'cup' map $c : \mathbf{Br}(m, n) \hookrightarrow \mathbf{Br}(m+2, n)$ by $d \mapsto u \boxtimes d$, and extend this k-linearly to a morphism of free k-modules. Observe that $\mathbf{Br}(m, n)$ is a right-B_n-module; and that $c(k\mathbf{Br}(m, n))$ is a right-B_n-submodule of $k\mathbf{Br}(m+2, n)$ that is isomorphic to $\mathbf{Br}(m, n)$.

We define the 'herniation' map $h : \mathbf{Br}(1, 1) \hookrightarrow \mathbf{Br}(1, 3) \times \mathbf{Br}(3, 1)$ by $1_1 \mapsto (1_1 \boxtimes u', u \boxtimes 1_1)$ (see figure 11.6). Note that this map is inverted by composition: $\circ(h(1_1)) = 1_1$. This may be applied

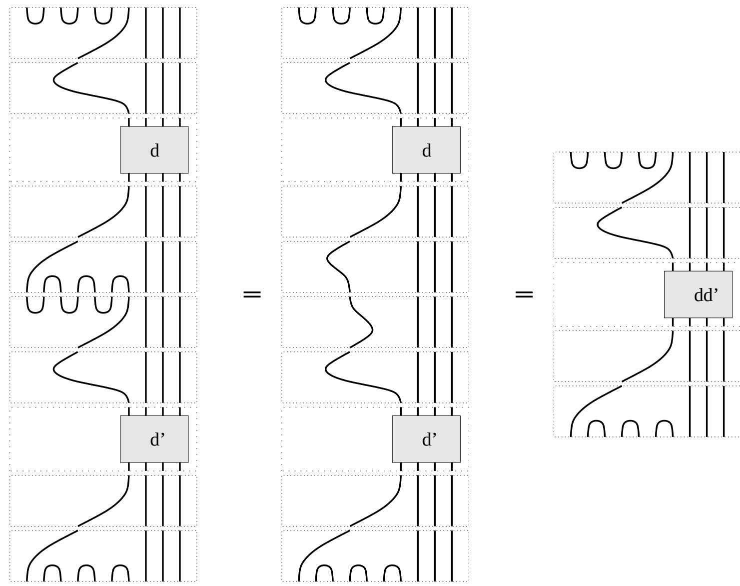


Figure 11.5: Schematic for proof of algebra isomorphism $B_4 \cong E_3 B_{10} E_3$.

more widely: for example, note that there is a trivial bijection $\mathbf{Br}(1,3) \to \mathbf{Br}(1,1) \times \mathbf{Br}(1,3)$; applying $(h, 1)$ to this, and then applying c (i.e applying $(c, 1, 1)$) we have figure 11.6 (where the shaded region represents any element of $\mathbf{Br}(1,3)$). Multiplying out, this shows an injection of $\mathbf{Br}(1,3) \hookrightarrow U_{12}U_{23}B_3$. But the steps are reversible, so we see that, as a right B_3-module we have

$$U_{12}U_{23}B_3 \;\cong\; k\mathbf{Br}(1,3) \tag{11.9}$$

(pictorially, the right action corresponds to acting with diagrams from $\mathbf{Br}(3,3)$ from *below*).

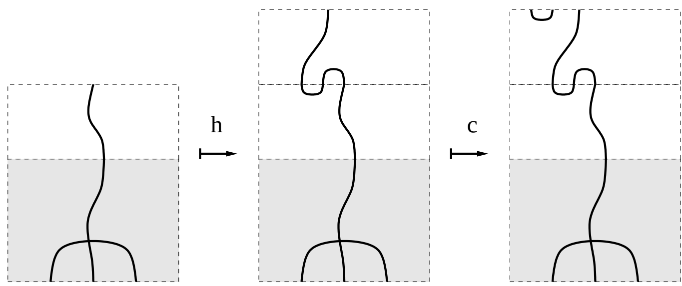


Figure 11.6: Illustrating the isomorphism $U_{12}U_{23}B_3 \cong k\mathbf{Br}(1,3)$.

Chapter 12

General representation theory of the Brauer algebra

12.1 Brauer Δ-modules

12.1.1 Filtration of the left regular module

Here k is an arbitrary commutative ring, and $\delta \in k$. For $n \in \mathbb{N}$ we set $n_0 = 0$ if n even and $n_0 = 1$ if n odd.

(**12.1.1**) Proposition. (I) For any ring k and $\delta \in k$, we have a sequence of $B_n(\delta)$-bimodules:

$$k\mathbf{Br}(n,n) = k\mathbf{Br}^{\leq n}(n,n) \supset k\mathbf{Br}^{\leq n-2}(n,n) \supset k\mathbf{Br}^{\leq n-4}(n,n) \supset ... \supset k\mathbf{Br}^{n_0}(n,n) \qquad (12.1)$$

(II) There is a corresponding sequence of left-B_n-right-B_m-bimodules for any $k\mathbf{Br}(n,m)$. □
We call (12.1) the *p-sequence* (p for 'pure') of B_n-ideals.

(**12.1.2**) For any l, n define quotient algebra

$$B_n^l := B_n / k\mathbf{Br}^{<l}(n,n)$$

In particular the first section in (12.1) is a quotient algebra, and obeys

$$B_n^n = k\mathbf{Br}^{\leq n}(n,n)/k\mathbf{Br}^{\leq n-2}(n,n) \cong kS_n$$

as a k-algebra. Thus each kS_n-module restricts to a B_n-module identical to it as a k-module, where the action of any diagram with fewer than n propagating lines is by 0.

(**12.1.3**) Note that the i-th section of the sequence (12.1) has basis $\mathbf{Br}^{n-2i}(n,n)$. Hereafter we shall understand $k\mathbf{Br}^{n-2i}(n,n)$ to be a B_n-bimodule by identification with this section. Similarly any $k\mathbf{Br}^l(n,l)$ ($l \leq n$) is a left B_n- right B_l-bimodule.

Where k is clear we may simply write $\mathfrak{B}^l(n,l)$ for $k\mathbf{Br}^l(n,l)$ understood as a bimodule in this way.

(**12.1.4**) Remark. The section $k\mathbf{Br}^l(n,n)$ is also an ideal of the quotient algebra B_n^l. (Example: $k\mathbf{Br}^n(n,n) = B_n^n$.)

(**12.1.5**) LEMMA. For $n-2i=l$ we have a decomposition of this i-th section *as a left B_n-module*:

$$k\mathbf{Br}^l(n,n) \;\cong \bigoplus_{w\in\mathbf{Br}^{1_l}(l,n)} k\mathbf{Br}^l(n,l)\, w \tag{12.2}$$

All the summands are isomorphic to $k\mathbf{Br}^l(n,l)$. □

(**12.1.6**) By the restriction $kS_l \subset B_l$, it will be evident that $\mathbf{Br}^l(n,l)$ is a basis for a left-$B_n(\delta)$ right-kS_l bimodule, where the action on the left is via the category composition, quotienting by $k\mathbf{Br}^{\leq l-2}(n,l)$ as before, and on the right by the natural diagram composition.

One again, where the intention is clear, we may abbreviate $k\mathbf{Br}^l(n,l)$ regarded as a bimodule in this way to $\mathfrak{B}^l(n,l)$.

(**12.1.7**) PROPOSITION. *Fix any ring k, and $n \geq l \in \mathbb{N}_0$. The free k-module $\mathfrak{B}^l(n,l) = k\mathbf{Br}^l(n,l)$, which is a left $B_n(\delta)$ right kS_l-bimodule, is a projective right kS_l-module. Hence the functor*

$$\Phi^n \;:\; kS_l - mod \quad\to\quad B_n(\delta) - mod \tag{12.3}$$

$$M \quad\mapsto\quad \mathfrak{B}^l(n,l) \otimes_{kS_l} M \tag{12.4}$$

is exact. Similarly $\mathfrak{B}^l(l,n) = k\mathbf{Br}^l(l,n)$ is a right B_n-module and a projective left kS_l-module. The functor

$$\mathfrak{B}^l(l,n) \otimes_{B_n} - \;:\; B_n(\delta) - mod \quad\to\quad kS_l - mod$$

is right exact.

Proof. Noting that arcs on the n-vertex edge of a diagram play no role in the right S_l-action, we see that $\mathfrak{B}^l(n,l)$ is a direct sum of copies of the regular right kS_l-module. Now use Prop. (6.6.17) for exactness. The other case is standard. □ [1]

[1]

(**12.1.8**) EXAMPLE. Let us consider the result of composing the functors in the above. In case $l=0$, $m=2$, i.e. $k\mathbf{Br}^0(0,2)$, we have basis $\mathbf{Br}^0(0,2) = \{u'\}$. This is a left kS_0-module, where S_0 is the trivial group, and a right $B_2(\delta)$-module. Let us examine the composite functor

$$\underbrace{{}_{kS_0} k\mathbf{Br}^0(0,2) \otimes_{B_2} k\mathbf{Br}^0(2,0)} \otimes_{kS_0} -$$

The first thing is to examine the underbraced factor as a kS_0-bimodule, i.e. as a k-module. Since each factor is a free k-module with singleton basis, the module is spanned by $u \otimes_{B_2} u'$. That is, it is spanned by the equivalence class of the pair (u,u'). To construct this class we need the set of elements (b,m) of $B_n \times k\mathbf{Br}^0(2,0)$ such that $bm = u'$ (then $(u,u') \sim (ub,m)$). Note that $m = cu'$ for some scalar c, since $\{u'\}$ is a basis.

Let us consider the case in which $k\delta = 0$. Then only scalar multiples of 1 act suitably on any such m. The class is simply elements of form $(cu, c^{-1}u')$ (c an invertible scalar).

Note that there is a multiplication map defined here: $(u,u') \mapsto uu' \in k\mathbf{Br}(0,0)$. In the case $k\delta = 0$ we have $uu' = 0$, so this is not a surjective map onto $k\mathbf{Br}(0,0)$. In other cases this map defines a bijection, so long as δ is invertible.

(**12.1.9**) According to [3, 19.10 Th.] tensor functors like this preserve direct sums.

12.1.2 Specht modules

(**12.1.10**) Set $\Lambda_n = \{\lambda \vdash n\}$ and

$$\Lambda^n \ := \Lambda_n \cup \Lambda_{n-2} \cup \ldots \cup \Lambda_{0/1}$$

(**12.1.11**) For $\lambda \in \Lambda_n$ we write $\mathcal{S}(\lambda)$ for the $\mathbb{Z}S_n$ Specht module (as in Section 8.3.4, or for example [29]).

Recall from (8.3.23), or [29, Lem.7.1.4], that for each λ we may choose an element $v_\lambda \in kS_l$ such that

$$\mathcal{S}(\lambda) \cong kS_l v_\lambda.$$

and $v_\lambda \mathcal{S}(\nu) = 0$ unless $\lambda = \nu$.

(**12.1.12**) If k contains $\mathbb{Q}$ then v_λ may be chosen a primitive idempotent.

(**12.1.13**) Note that if $\lambda \vdash l$ then $\Phi^l(\mathcal{S}(\lambda))$ is the restriction noted in (12.1.2).

12.1.3 Δ-module constructions

(**12.1.14**) For $\lambda \in \Lambda^n$ define B_n-module

$$\Delta_n(\lambda) \ := \ \Phi^n(\mathcal{S}(\lambda))$$

(**12.1.15**) More explicitly we have $\Delta_n(\lambda) = \ \mathfrak{B}^l(n,l) \otimes_{kS_l} \mathcal{S}(\lambda)$.

(**12.1.16**) Lemma. For a a Brauer diagram let $\#(a)$ be the number of propagating lines. Then

$$a\Delta_n(\lambda) = 0 \qquad if \qquad \#(a) < |\lambda|$$

and indeed by construction $E_m \Delta_n(\lambda) = 0$ iff $\#(E_m) = n - 2m < |\lambda|$.

Proof. By construction. We have $a\mathfrak{B}^l(n,l) = 0$. □

(**12.1.17**) Now recall the natural inclusion $kS_l \hookrightarrow B_l$. [2] Either regarding $k\mathbf{Br}^l(n,l)$ as a right B_l- or a right kS_l-module, we have for each $v \in kS_l$ a multiplication map $\circ$ giving $k(\circ(\mathbf{Br}^l(n,l) \times kS_l v)) = k\mathbf{Br}^l(n,l)v \subset k\mathbf{Br}^l(n,l)$. Note that $k\mathbf{Br}^l(n,l)v$ is a left B_n-submodule of $k\mathbf{Br}^l(n,l)$. For each λ, with v_λ as in (12.1.11), define

$$D_n(\lambda) \ := \ k\mathbf{Br}^l(n,l)v_\lambda \tag{12.5}$$

(**12.1.18**) Lemma. Let $\lambda \vdash l$ and let $b(\lambda)$ be a basis in $kS_l v_\lambda$. Include this in B_l in the natural way, as above. Then $\circ$ (as above) restricts to an injective map on $\mathbf{Br}^{1_l}(n,l) \times b(\lambda)$, and the image

$$b_{D_n(\lambda)} \ := \ \circ(\mathbf{Br}^{1_l}(n,l) \times b(\lambda))$$

is a basis of $D_n(\lambda)$. □

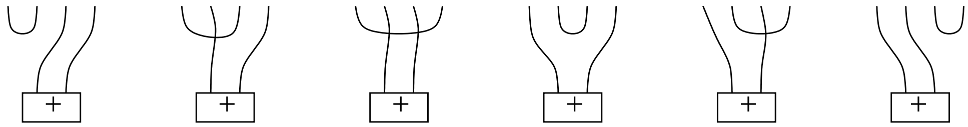


Figure 12.1: Basis for $D_4(2)$ (and $\Delta_4(2)$).

Proof. An element in the image of the restriction of $\circ$, $\circ(d,x)$ say, is a linear combination of diagrams all with their non-propagating arcs in the same positions. These positions determine d, and removing all these arcs determines x, thus $\circ$ is reversible on such elements. Using (11.2), we have that $\circ(\mathbf{Br}^{1_l}(n,l) \times b(\lambda))$ spans $k\mathbf{Br}^l(n,l)v_\lambda$. □

(**12.1.19**) Example. A basis for $D_4(2)$ is given in Figure 12.1, where $+$ denotes the S_2 symmetrizer $1 + (12)$. We shall see next that this also serves as a basis for $\Delta_4(2)$.

(**12.1.20**) Proposition. *Let $\lambda \vdash l$ and let $b(\lambda)$ be a basis for $\mathcal{S}(\lambda)$. Then*

$$b_{\Delta_n(\lambda)} = \{a \otimes_{kS_l} b \ : \ (a,b) \in \mathbf{Br}^{1_l}(n,l) \times b(\lambda)\}$$

is a basis for $\Delta_n(\lambda)$.

Proof. The set $b_{\Delta_n(\lambda)}$ is a set of generators for $\Delta_n(\lambda)$ by (11.2), since factors in $\mathbf{Br}^l(l,l)$ may be moved to the right of the tensor product and 'absorbed' by $b(\lambda)$. By the same argument as for (12.1.18) the given set passes to a basis (of the image) under the multiplication map. It follows that this set is independent, so the module is k-free with this set as basis. □

(**12.1.21**) Proposition. *Choose $b(\lambda) \subset kS_l \subset B_l$ as in (12.1.18). Then*
(i) $\Delta_n(\lambda) \cong D_n(\lambda)$.
(ii) The map defined on diagrams by $d \mapsto d \boxtimes (u')^{\boxtimes m}$ (with $m = (n-l)/2$), and extended k-linearly, takes $k\mathbf{Br}(n,l)$ to a left ideal in B_n. This map takes $k(\circ(\mathbf{Br}^{1_l}(n,l) \times b(\lambda)))$ to a left ideal of B_n^l isomorphic to $\Delta_n(\lambda)$ (as a left B_n^l-module).

Proof. Follows immediately. □

12.1.4 Δ-module examples

(**12.1.22**) Example. The reader will readily confirm that when $\delta = 2$ there is a submodule of $\Delta_4(2)$ with basis given in Figure 12.2, where $-$ denotes the S_2 antisymmetrizer $1 - (12)$. This submodule is isomorphic to $\Delta_4(2^2)$, that is there is a map

$$\Delta_4(2^2) \stackrel{\delta=2}{\rightarrow} \Delta_4(2) \tag{12.6}$$

[2]We have a multiplication map $\circ$ giving $k(\circ(\mathbf{Br}(n,l) \times kS_l v_\lambda)) = k\mathbf{Br}(n,l)v_\lambda \subset k\mathbf{Br}(n,l)$. The image $k\mathbf{Br}(n,l)v_\lambda$ is also a left B_n-module. Indeed the subset $k\mathbf{Br}^{\leq l-2}(n,l)v_\lambda$ is also a left B_n-module. We may define a left B_n-module $k\mathbf{Br}(n,l)v_\lambda / k\mathbf{Br}^{\leq l-2}(n,l)v_\lambda$. Similarly,

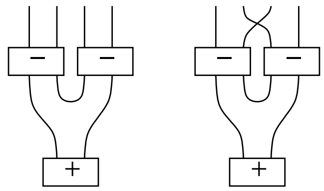


Figure 12.2: Basis for submodule of $\Delta_4(2)$ when $\delta = 2$.

12.1.5 Simple head conditions for Δ-modules

(**12.1.23**) For $l > 0$, $v \in B_l$ and m such that $n = l + 2m$, define

$$E_m^v = (1_{2m} \boxtimes v)E_m \quad \in B_{l+2m}$$

See the top-lefthand corner of figure 12.3 for a picture of this.

(**12.1.24**) Let us write ψ_l for the natural ring homomorphism $\psi_l : B_n \to B_n^l$. Thus we have an exact functor $\mathrm{Res}_{\psi_l} : B_n^l-\mathrm{mod} \;\to\; B_n-\mathrm{mod}$ that takes module $M \mapsto M$.

We also have $\mathrm{Ind}_{\psi_l} : B_n-\mathrm{mod} \;\to\; B_n^l-\mathrm{mod}$ given by $N \mapsto B_n^l \otimes_{B_n} N$. (The behaviour of modules under this functor is less easy to predict.)

(**12.1.25**) Lemma. For $\lambda \vdash l > 0$ and $m = (n-l)/2$ as above, $\Delta_n(\lambda)$ is also a B_n^l-module. Indeed

$$\Delta_n(\lambda) \cong \; B_n^l E_m^{v_\lambda} \tag{12.7}$$

Meanwhile $\Delta_{2m}(\emptyset) \cong \; B_{2m}\overline{E}_m$.

(**12.1.26**) Exercise. What can we say about $B_n^{l'} E_m^{v_\lambda}$ for $l' < l$?

Regarded as B_n-modules we see that these modules are a succession of quotients of the (not necessarily indecomposable) module $B_n E_m^{v_\lambda}$; and that this latter module is projective over $\mathbb{Q}[\delta]$ (and hence over $\mathbb{C}$).

(**12.1.27**) If $\lambda \neq \nu \vdash l$ then $E_m^{v_\lambda}\Delta_n(\nu) = 0$. (To see this consider (12.7) and a corresponding mild variation of figure 12.3, where the vs above and below differ.)

(**12.1.28**) Note from Figure 12.3 that for any $v \in B_l$ ($l > 0$) there is a k-module isomorphism

$$E_m^v \; B_{l+2m} \; E_m^v \;\mapsto\; vB_l v$$

Thus if v is idempotent, then (I) so is E_m^v; (II) $E_m^v B_{l+2m} E_m^v$ and $vB_l v$ are isomorphic k-algebras.

Suppose v is idempotent in kS_l. The ideals spanned by elements with support in diagrams with fewer than l propagating lines (i.e. by $E_m^v \; \mathbf{Br}^{l-2}(l+2m, l+2m) \; E_m^v$ and $v\mathbf{Br}^{l-2}(l,l)v$) are also isomorphic; and hence so are the corresponding quotient algebras, $E_m^v B_{l+2m}^l E_m^v$ and $vB_l^l v$. If $vB_l^l v$ is rank 1, then so is $E_m^v B_{l+2m}^l E_m^v$. Thus E_m^v is primitive in B_{l+2m}^l when v is primitive in $B_l^l = kS_l$.

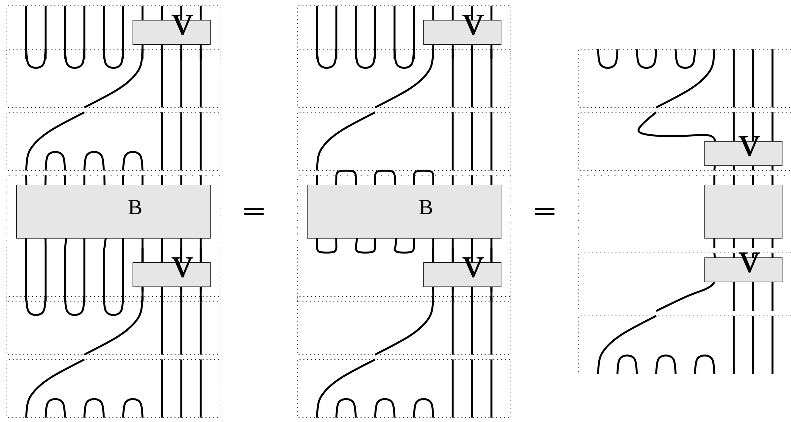


Figure 12.3: Diagramatic realisation of $E_m^v B_{l+2m} E_m^v$

(**12.1.29**) LEMMA. (I) Suppose $\lambda \vdash l$, and v_λ in (12.5) may be chosen a primitive idempotent of kS_l (as if $k = \mathbb{C}$ or $k = \mathbb{Q}[\delta]$, for example). Then for $l > 0$, or $l = 0$ and δ a unit, $\Delta_n(\lambda)$ is indecomposable projective as a B_n^l-module, and hence has simple head.
(II) If $n > 1$, $l = 0$ and $\delta = 0$ we have a surjection

$$\Delta_n((2)) \to \Delta_n(\emptyset) \to 0$$

so that when $\Delta_n((2))$ has simple head, $\Delta_n(\emptyset)$ also has simple head.

Proof. (I) Consider (12.1.28). Now note (12.7).
The $l = 0$, δ a unit case is similar.
(II) Note that the basis of $\Delta_n((2))$ contains elements of the form $dv_{(2)}$ where $d \in \mathbf{Br}^2(4,2)$ (and $v_{(2)} = 1_2 + (12)$ in $k\mathbf{Br}^2(2,2)$). The map is given by $dv_{(2)} \mapsto du$. One readily checks that this is well-defined, and a surjection. □

We shall see shortly that:

(**12.1.30**) LEMMA. In particular, over the field of fractions $\mathbb{Q}(\delta)$, (I) every module $\Delta_n(\lambda)$ is simple; and (II) B_n is semisimple.

Proof. See (12.3.15).

12.1.6 The base cases

(**12.1.31**) We have $B_0(\delta) \cong B_1(\delta) \cong k$. For $B_2(\delta)$ we have $\Delta_2(\emptyset)$, $\Delta_2(2)$, $\Delta_2(1^2)$. These are each of rank 1, but are non-isomorphic if $k = \mathbb{C}$ as a field (or indeed if 2 is a unit in k), unless $\delta = 0$.

For $\delta = 0$ we have, over $\mathbb{C}$,

$$\Delta_2(2) \xrightarrow{\sim} \Delta_2(\emptyset)$$

Thus we may regard $\Delta_2(2)$, $\Delta_2(1^2)$ as the inequivalent simple $B_2(0)$-modules. In this case $P_2(2)$ is the self-extension of $\Delta_2(2)$, while $P_2(1^2) = \Delta_2(1^2)$.

12.1.7 The case $k \supseteq \mathbb{Q}$

Note that all copies of $\Delta_n(\lambda)$ occur in a single p-section of B_n.

12.2 Δ-Filtration of projective modules

12.2.1 Some character formulae

(**12.2.1**) Recall (e.g. from Lang [**?**, §III.8]) that if A is a finite dimensional algebra over a field k then associated to A-mod is the *Grothendieck group* $K(A-\text{mod})$ (or just $K(A)$) — the quotient of the free abelian group generated by isomorphism classes $[M]$ of A-modules by $[X]+[Z]-[Y]=0$ whenever $0 \to X \to Y \to Z \to 0$ is a short exact sequence.

(**12.2.2**) Write χ_M for the image of module M in the Grothendieck group of B_n-mod. By (12.1.5)

$$\chi_{B_n} = \sum_{l=n_0}^{n} d_n^l \chi_{k\mathbf{Br}^l(n,l)}$$

for $d_n^l = |\mathbf{Br}^{1_l}(n,l)|$; and

$$\chi_{k\mathbf{Br}^l(n,l)} = \sum_{\lambda \vdash l} d_\lambda \chi_n(\lambda)$$

where $d_\lambda = rank_k(\mathcal{S}(\lambda))$ and $\chi_n(\lambda) = \chi_{\Delta_n(\lambda)}$. Altogether

$$\chi_{B_n} = \sum_{\lambda \in \Lambda^n} d_n^\lambda \chi_n(\lambda)$$

where $d_n^\lambda = rank_k(\Delta_n(\lambda))$.

12.2.2 General preliminaries

(**12.2.3**) Lemma. Suppose $B = P \oplus Q$ is a decomposition of some R-module B. Then by (6.2.15) there is an idempotent e in $\text{End}_R(B)$ that projects onto P. That is, $P = eB$. For D an indecomposable submodule of B we see that the decomposition of D by $1_B = e + (1-e)$ must be trivial, hence D is a submodule either of P or Q.

Now suppose that D' is an indecomposable submodule of B/D. By the same argument this submodule also lies on one 'side' of $B/D = P/D \oplus Q/D$ (here we understand $Q/D = Q/(Q \cap D)$, noting that one of $P \cap D$ and $Q \cap D$ is zero). Thus we can iterate to pass a filtration of B by a set of indecomposable modules to a filtration of P (one simply omits some sections from the filtration — the omitted sections being the sections that go to form a filtration of Q).

(**12.2.4**) Proposition. *Suppose that A is an algebra and Γ, Γ' are sets of A-modules; and that*

$$N = N_0 \supset N_1 \supset \ldots \supset N_l = 0$$

is a chain of A-modules such that every section N_i/N_{i+1} is isomorphic to some $M \in \Gamma$. That is, N has a Γ-filtration. Now suppose also that every $M \in \Gamma$ has a Γ'-filtration. Then N has a Γ'-filtration.

Proof. Suppose that $N_i/N_{i+1} \cong M_i = M_0^i \supset M_1^i \supset ... \supset M_r^i = 0$ is a chain corresponding to a Γ'-filtration of N_i/N_{i+1}. We write $f : N_i \to N_i/N_{i+1}$ for the quotient map. By Prop.6.3.27 there is a chain

$$N_{i-1} \supset N_i = f^{-1}M_0^i \supset f^{-1}M_1^i \supset ... \supset N_{i+1}$$

that is a (local) refinement of the initial chain, and that gives (locally) a Γ'-filtration. □

12.2.3 Δ-filtration

(**12.2.5**) PROPOSITION. (I) *Suppose k is such that ${}_{kS_l}kS_l$ is filtered by $\{\mathcal{S}(\lambda)\}_{\lambda\in\Lambda_l}$ for all $l \leq n$. Then, for any δ, the left regular module ${}_{B_n}B_n$ is filtered by $\{\Delta_n(\lambda)\}_{\lambda\in\Lambda^n}$.*
(II) *In case $k = \mathbb{C}$, for any δ, projective B_n-modules are filtered by $\{\Delta_n(\lambda)\}_{\lambda\in\Lambda^n}$.*

Proof. (I) Note first that if a module M is filtered by a set $\{N_i\}_i$, and these modules are all filtered by a set $\{N'_j\}_j$, then M is filtered by $\{N'_j\}_j$. By (12.1.1) the set $\{k\mathbf{Br}^l(n,l)\}_l$ gives (via the action therein) a left-B_n filtration of B_n. By Prop. 12.1.7 the functor Φ^n is exact, so each factor itself has a filtration by Δs under the stated condition. Specifically, if

$${}_{kS_l}kS_l \;\cong\; //_i\; \mathcal{S}(\lambda(i))$$

is a filtration (in the notation of (12.4.11)) then

$$\Phi^n{}_{kS_l}kS_l \;\cong\; //_i\; \Phi^n\mathcal{S}(\lambda(i))$$

is a filtration of $k\mathbf{Br}^l(n,l)$.
Remark: See (8.3.31) for conditions on k for Specht filtrations of kS_l. In short, there is a filtration for any field k, but the multiplicities may not be unique unless charactersitic $p > 3$.

For (II), first note that $\mathbb{C}S_l$ is semisimple and hence filtered by $\{\mathcal{S}(\lambda)\}_{\lambda\in\Lambda_l}$. Secondly, the set $\{\Delta_n(\lambda)\}_{\lambda\in\Lambda^n}$ are all indecomposable by the simple-head conditions (12.1.29)(I) and (12.1.29)(II) (whose hypotheses are met by $k = \mathbb{C}$).

Since $P_n(\lambda)$ is a direct summand of ${}_{B_n}B_n$, we may use Lemma 12.2.3 to deduce [3] that eventually $P_n(\lambda)$ will have a Δ-filtration inherited from that of B_n. □

(**12.2.6**) Consider further the case $k = \mathbb{C}$ (any δ). Note from () that in this case, regarded as a left-B_m-module, $k\mathbf{Br}^l(m,l)$ is a direct sum

$${}_{B_m}k\mathbf{Br}^l(m,l) = \oplus_{\lambda\vdash l} d_\lambda \Delta_m(\lambda)$$

(with multiplicities d_λ as indicated). Thus we have a filtration of form

$${}_{B_m}B_m \;=\; \oplus_{\lambda\vdash m} d_m^m d_\lambda \Delta_m(\lambda) \;//\; \oplus_{\lambda\vdash m-2}\, d_m^{m-2} d_\lambda \Delta_m(\lambda) \;//...$$

Note that this stretches to a filtration by Δ-modules, by any ordering of the summand modules in a section above. Since Δ-modules are indecomposable (Lemma 12.1.29) it follows that there is a filtration of each summand of ${}_{B_m}B_m$ obtained by deleting terms from such a stretched filtration.

[3]— there is an idempotent e in $\mathrm{End}_{B_n}(B_n) \cong B_n$ that projects onto $P_n(\lambda)$. Now let Δ be a submodule of B_n. If e splits Δ then it is not indecomposable, so any indecomposable such Δ is a submodule either of $P_n(\lambda)$ or of its complement in B_n. Now consider B_n/Δ, which either contains $P_n(\lambda)$ or $P_n(\lambda)/\Delta$ as a direct summand. Let Δ' be an indecomposable submodule of B_n/Δ. We can iterate this process.

...um, not quite, since the splitting idempotent is not in B_n... BUT, either way, there is a projection, e' say, —

In particular every indecomposable projective module P must have such a filtration. Thus for each P there is a λ such that $\Delta_m(\lambda)$ is the first term (with head agreeing with P); and subsequent terms $\Delta_m(\mu)$ having $|\mu| \leq |\lambda|$. This filtration would proceed next with any Δ-modules with $|\mu| = |\lambda|$; however no $\Delta_m(\mu)$ with $|\mu| = |\lambda|$ can appear in this filtration (else one such is next to $\Delta_m(\lambda)$ and there would be another such filtration in which they are reordered, (eventually) contradicting the simple head condition).
[4]

12.2.4 On simple modules, labelling and Brauer reciprocity

We have a useful set of Δ-modules indexed by Λ^n. What can we say about simple modules (and indecomposable projective modules) so far? And can we invoke Brauer reciprocity (essentially the $C = D^T D$ property of π-modular systems, as in §**??**) to relates these data?

(**12.2.8**) REMARK: we would like to assume that Δ-modules have simple head, but we have basically only shown this here for $k = \mathbb{C}$ (10.4.22 **??**) — and it is not always true (for example, Specht modules are special cases of Δ-modules, and these do not always have simple head, for example when the characteristic of k is $p = 2$ they are not always indecomposable). We might also be able to show it when the inflated Specht module has simple head. But I'm not sure we know when this happens (apart from p-regularly). And anyway we don't know about Specht filtrations in general... — One thing we could do in what follows is fix $k = \mathbb{C}$ and use δ in the notation instead of k... —

(**12.2.9**) Fixing k, δ, define

$$L_n^k(\lambda) = \text{ head}_k\, \Delta_n(\lambda)$$

(Note that the dependence on δ is left implicit in this notation. However if in particular $k = \mathbb{C}$, we may write instead $L^\delta(\lambda)$.) Note that $L_n^k(\lambda)$ is not necessarily simple.

(**12.2.10**) If $k = \mathbb{C}$ then by Prop.12.2.5 the head of every indecomposable projective is also the head of a Δ-module. Accordingly we have

(**12.2.11**) PROPOSITION. *If $k = \mathbb{C}$, the set $\{L_n^k(\lambda)\}_\lambda$ contains a complete set of simple modules for $B_n(\delta)$.* □

In fact we shall see that if $k = \mathbb{C}$ then this set *is* a complete set.

(**12.2.12**) REMARK. In fact the Proposition is true for much more general choice of k. We shall return to this point later.

(**12.2.13**) PROPOSITION. *CLAIM: Let k, δ, and λ be such that $\Delta_n(\lambda)$ is an (indecomposable) top factor in a Δ-filtration of some projective $B_n(\delta)$-module P. Let $P_n(\lambda)$ be an indecomposable*

[4]IGNORE: (**12.2.7**) Can we now complete the proof of generic semisimplicity asserted in (12.1.30)? NO!: Since each indecomposable projective contains precisely one copy of the defining Δ-module we deduce by Brauer reciprocity — (NO! NO! NO! ARGUMENT STILL CIRCULAR! DEMOTE!) — that the Δ-module contains precisely one copy of the head simple module. The following argument now works.
(I): There is a non-degenerate form in each case, and hence an isomorphism from $\Delta_n(\lambda)$ to its contravariant dual. Thus a simple module with the same character as the simple head (call the head $L_n(\lambda)$, simple by (12.1.29), (12.1.29)) also gives the simple socle. One can show (using (**??**) and the π-modular reciprocity [5]) that such a character can only occur with multiplicity 1, thus the head and socle coincide. (II): The sum of squares of these module dimensions is the dimension of the algebra, thus the radical has dimension 0. □

summand of P containing this factor. (For now $P_n(\lambda)$ is just a local name. Later we will see that it is well-defined generally. Note that, up to isomorphism, only one such summand has the property of containing this factor.)
Then (i) every Δ-filtration of $P_n(\lambda)$ starts with $\Delta_n(\lambda)$ (up to isomorphism), and (ii) if $\Delta_n(\mu)$ lies below $\Delta_n(\mu')$ in all filtrations then $|\mu| < |\mu'|$.

Proof. By construction of $\Delta_n(\lambda)$ and the ${}_{B_n}B_n$ filtration.

??!!

(i)/(ii) idea is somehow to look at the original ideal chain for B_n as a chain of left-modules (and the refined chain with Δs in it). ...!

(**12.2.14**) Corollary: Since every other Δ-factor lies below the top $\Delta_n(\lambda)$ in $P_n(\lambda)$, none is $\Delta_n(\lambda)$, except in case $\delta = 0$, $n = 2$ where $\Delta_2(2) \cong \Delta_2(\emptyset)$.

12.3 Globalisation functors

(**12.3.1**) Remark. Fixing commutative ring k and $\delta \in k$, we want a way to treat the representation theory of B_n for each n as far as possible simultaneously — *globalisation.*
We also want a way to regard Δ-modules (or something similar) as canonical, at least up to simple multiplicities — providing a basis for the Grothendieck group of B_n over any field k (and choice of δ); and so that projective modules have good filtration properties.

These wants are relatable. A good paradigm for this is Green [22, §6.6]. This treats a single pair of algebras B and eBe — but Brauer algebras provide a chain of such pairings. (Another similar, but more restricted, approach is Cox–Martin–Parker–Xi [**?**].)

The 'canonical module' want is related to the role of Specht modules (for S_n for fixed n) as 'generic irreducible modules'. In general this means: modules for an algebra B defined over a DVR K (in this S_n case containing $\mathbb{Z}$) that pass to simple modules over a suitable extension field K_0 of K. One then considers projective modules over some quotient field k of K that is of interest — these have preimages over K by *idempotent lifting* and hence have positive Specht characters by 'passing everything' to K_0. The Cartan decomposition matrix for B over k is then determined by, say, the Specht module decomposition numbers.

Now what about varying n?: Nominally there are a number of B_n-modules in level n that are 'images' of some level $l < n$ Specht module. We can inflate fully with Φ^n, or partially with some Φ (perhaps not even changing l) then continue with various G-functor-type steps up to n.

How are these modules related? What properties do they have?

12.3.1 Preliminaries

(**12.3.2**) Lemma. For $n \geq m \geq l$ the multiplication map defines a bimodule isomorphism

$$k\mathbf{Br}(n,m) \otimes_{B_m} k\mathbf{Br}(m,l) \xrightarrow{\sim} k\mathbf{Br}(n,l)$$

unless δ is a non-unit and $l = 0$.

Proof. The case $l = m$ is trivial, so suppose $l < m$. In case $l > 0$ we have the bimodule isomorphism $k\mathbf{Br}(m,l) \cong B_m E_x$ from Lemma (11.5.5), where $x = (m-l)/2$. (In case δ a unit we may use

$k\mathbf{Br}(m,0) \cong B_m E'_x$ with $E'_x = \prod_{i\ odd} \delta^{-1} U_i$ instead.) Thus

$$k\mathbf{Br}(n,m) \otimes_{B_m} k\mathbf{Br}(m,l) \cong k\mathbf{Br}(n,m) \otimes_{B_m} B_m E_x = k\mathbf{Br}(n,m) E_x \otimes_{B_m} E_x.$$

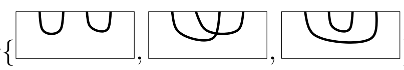
using that $E_x E_x = E_x$. The projection from here to $k\mathbf{Br}(n,m)E_x$ is clearly invertible and hence an isomorphism. Using Lemma 11.5.5 again we have $k\mathbf{Br}(n,m) \cong k\mathbf{Br}(n,n)E_{x'}$ $(x' = (n-m)/2)$. Recall that this is a right B_m-module via the isomorphism $B_m \cong E_{x'} B_n E_{x'}$ that takes $d \mapsto E_{x'}(1_{2x'} \boxtimes d)E_{x'}$. In particular it takes $E_x \in B_m$ to $E_{x'}(1_{2x'} \boxtimes E_x)E_{x'}$. But one readily confirms (plugging in a suitable E_x for d in Figure 11.4 for example) that

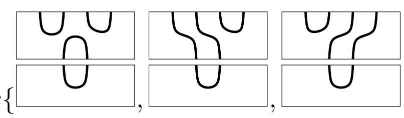
$$E_{x+x'} = E_{x'}(1_{2x'} \boxtimes E_x)E_{x'}$$

Thus $k\mathbf{Br}(n,m)E_x \cong k\mathbf{Br}(n,n)E_{x+x'}$. Finally [6] $k\mathbf{Br}(n,n)E_{x+x'} \cong k\mathbf{Br}(m,l)$ by using Lemma 11.5.5 once again. □

(**12.3.3**) Remark. CLAIM: This can be strengthened to exlude only the case $\delta = 0$, $m = 2$ and $l = 0$ (see the following example).

(**12.3.4**) Example. Case $n = 4$, $m = 2$, $l = 0$:

$k\mathbf{Br}(4,0) = k\{$  $\}$

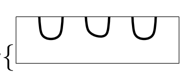
$k\mathbf{Br}(4,2) \otimes_{B_2} k\mathbf{Br}(2,0) = k\{$  $, \ldots\}$

If δ is invertible then the first three diagrams as drawn here are not independent: write the first one as $a \otimes u$ and the second as $a' \otimes u$, say, then

$$a \otimes u = a'U_1 \otimes u = a' \otimes U_1 u = \delta a' \otimes u$$

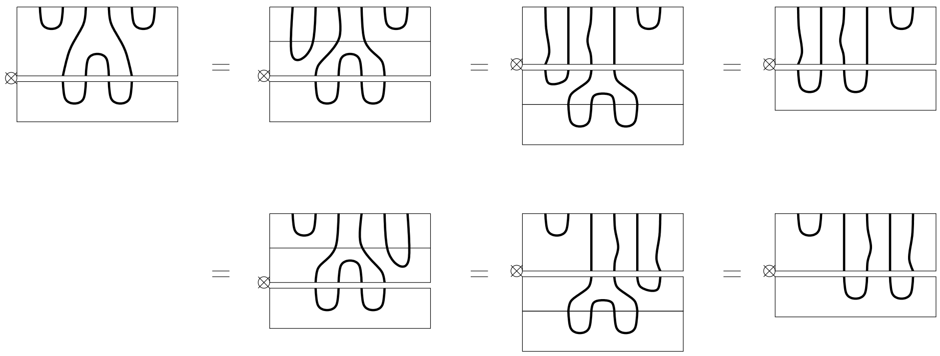
To construct the isomorphism one eliminates two of them, then maps the remaining one to the first diagram above; and so on.
If $\delta = 0$ then the second and third diagrams as drawn here are independent, and the isomorphism fails.

Case $n = 6$, $m = 4$, $l = 0$:

$k\mathbf{Br}(6,0) = k\{$  $, \ldots\}$

[6] (RIGHT!!?!?!?!),

while $k\mathbf{Br}(6,4) \otimes_{B_4} k\mathbf{Br}(4,0)$ has elements of form



In this case one finds, by similar manipulations to those shown, that all the elements that would map to the first element of $\mathbf{Br}(6,0)$ drawn above if $\otimes$ were replaced by the multiplication map *are* equal (without any intermediate δ-factors appearing, and hence independently of δ). Thus in this case the isomorphism holds for all δ.

12.3.2 G-functors

(**12.3.5**) For $n+m$ even the k-space $k\mathbf{Br}(n,m)$ is a bimodule, so there is a functor

$$\begin{aligned} G_m^n \; : \; B_m\text{-mod} \quad &\to \quad B_n\text{-mod} && (12.8) \\ M \quad &\mapsto \quad k\mathbf{Br}(n,m) \otimes_{B_m} M && (12.9) \end{aligned}$$

For given n, let us simply write F for the functor $G_n^{n-2} = k\mathbf{Br}(n-2,n) \otimes_{B_n} -$; and G for the functor $G_{n-2}^n = k\mathbf{Br}(n,n-2) \otimes_{B_{n-2}} -$.

(**12.3.6**) CLAIM: For $n \geq l$ and $\lambda \vdash l$,

$$\Delta_n(\lambda) \;\cong\; G_l^n \Phi^l(\mathcal{S}(\lambda))$$

That is,

$$\begin{aligned} \Delta_n(\lambda) \quad &\cong \quad k\mathbf{Br}(n,l) \otimes_{B_l} \Phi^l(\mathcal{S}(\lambda)) && (12.10) \\ &= \quad k\mathbf{Br}(n,l) \otimes_{B_l} k\mathbf{Br}^l(l,l) \otimes_{kS_l} \mathcal{S}(\lambda) && (12.11) \end{aligned}$$

Proof. Our strategy is to construct a basis for $G_l^n \Phi^l(\mathcal{S}(\lambda))$ and compare it with the basis $b_{\Delta_n(\lambda)}$ for $\Delta_n(\lambda)$.

Let $b(\lambda)$ be as in (12.1.20). The B_l-module $\Phi^l(\mathcal{S}(\lambda))$ may be identified with $\mathcal{S}(\lambda)$ as a k-module. Thus the image of $\mathbf{Br}(n,l) \times b(\lambda)$ spans $k\mathbf{Br}(n,l) \otimes_{B_l} \Phi^l(\mathcal{S}(\lambda))$ by construction (cf. the basis $b_{\Delta_n(\lambda)}$, which is in bijection with $\mathbf{Br}^{1_l}(n,l) \times b(\lambda)$). If $l = 0$ or 1 this set is a basis and we are done. We return to $l = 2$ shortly. If $l > 2$ we may proceed as follows. By Lemma 11.4.16, diagrams in $\mathbf{Br}^{<l}(n,l)$ can be expressed in the form $dE_1 d'$ for $d \in \mathbf{Br}^l(n,l)$ and $E_1, d' \in B_l$ (note that this requires $l \geq 3$). But $dE_1 d' \otimes_{B_l} \Phi^l(\mathcal{S}(\lambda)) = d \otimes_{B_l} E_1 d' \Phi^l(\mathcal{S}(\lambda)) = 0$ since $E_1 d' \Phi^l(\mathcal{S}(\lambda)) =$

$E_1\Phi^l(\mathcal{S}(\lambda)) = 0$. Thus $\mathbf{Br}^l(n,l) \times b(\lambda)$ spans, and hence $\mathbf{Br}^{1_l}(n,l) \times b(\lambda)$, spans. Independence of the image under the multiplication map (as in (12.1.20)) ensures that this set is a basis.

Finally note that the given basis may be identified as a set with that for $\Delta_n(\lambda)$, and that the actions on the 'corresponding' basis elements are the same.

The case $l = 2$ is similar. One sees that elements of $\mathbf{Br}^0(n,2)$ can be expressed in the form dU_{12} where $d \in \mathbf{Br}(n,2)$ (note that this does not require any idempotency property), whereupon $dU_{12} \otimes_{B_2} \Phi^l(\mathcal{S}(\lambda)) = d \otimes_{B_2} U_{12}\Phi^l(\mathcal{S}(\lambda)) = 0$. One then argues as before. □

(**12.3.7**) Remark. CAUTIONARY TAIL: If δ is not a unit then

$$G_2^4 G_0^2 \not\cong G_0^4$$

i.e. the underlying bimodules are not isomorphic (see (12.3.4)). In particular

$$G_2^4 \Delta_2(\emptyset) \not\cong \Delta_4(\emptyset)$$

A quotient of B_n by an ideal contained in the radical is filtered by the Δ's excluding $\Delta_n(\emptyset)$, so the others span the Grothendieck group without $\Delta_n(\emptyset)$. [7] On the other hand, the others do not provide a filtration of projectives (see (**??**)).

(**12.3.8**) Proposition. *For $\lambda \vdash l$ (and making $\mathcal{S}(\lambda)$ a B_l-module as in (12.1.2)), we have*

$$\Delta_{2m+l}(\lambda) \cong \; G^{\circ m}\Phi^l(\mathcal{S}(\lambda))$$

unless $l = 0$, $m = 2$ and $\delta = 0$.

Proof. Apply Lemma(12.3.2) to Lemma(12.3.6). □

12.3.3 Idempotent globalisation

(**12.3.9**) Proposition. *Suppose either $n > 2$, or $n \geq 2$ and δ invertible in commutative ring k. Then*
(I) the free k-module $k\mathbf{Br}(n-2,n)$ is projective as a right B_n-module; and indeed there is an idempotent $e \in k\mathbf{Br}(n,n)$ such that

$$k\mathbf{Br}(n-2,n) \;\cong\; e\,(k\mathbf{Br}(n,n))$$

as a right B_n-module (for $n > 2$ take $e = U_{12}U_{23}$; for $n = 2$ take $e = \delta^{-1}U_{12}$).
(II) Functor $F : B_n-mod \to B_{n-2}-mod$ is exact; G is a right-exact left-adjoint/right-inverse to F.

Proof. (I) follows from Lemma 11.5.5. [8] For (II) use Prop. 6.6.17. □

---

[7] Exercise: EXPLAIN THIS.

[8] (We prove a left-handed version. The right-handed follows immediately.)
As a left module

$$B_3U_{12}U_{23} \cong B_3U_{23}U_{12} \cong k\mathbf{Br}(3,1)$$

(see also (11.9)), but $U_{12}U_{23}$ is idempotent, so $k\mathbf{Br}(3,1)$ is projective. The injection $i_{4,n} : \hom(3,1) \hookrightarrow \hom(n,n-2)$ $(n > 2)$ allows us to induce to $k\hom(n,n)\; i_4(\hom(3,1))$, which is therefore also left projective. This is a submodule of $k\hom(n,n-2)$ by construction; but considering for example the 'herniated' form of a diagram in

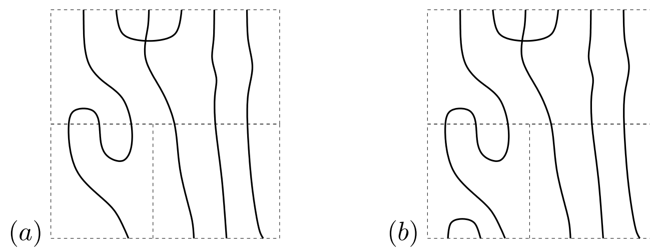


Figure 12.4: Schematic for mapping $k\mathbf{Br}(6,4)$ into $k\mathbf{Br}(6,6)E_1$.

9

(**12.3.10**) In particular (unless $n = 2$ and $\delta = 0$) $B_{n-2}-$mod fully embeds in B_n-mod under G, and this embedding takes $\Delta_{n-2}(\lambda)$ to $\Delta_n(\lambda)$.

The embedding allows us to consider a formal limit module category (we take n odd and even together), from which all B_n-mod may be obtained by 'localisation' (action of the functor F).

---

hom$(n, n-2)$ as in figure 12.4(a), we deduce that every diagram appears in the submodule and hence

$$k\,\mathrm{hom}(n, n-2) \quad = \quad k\,\mathrm{hom}(n,n)\;\; i_{4,n}(\mathrm{hom}(3,1))$$

is left projective. The (left-handed version of the) claimed isomorphism is indicated in the passage to figure (b) above (in particular this shows that a suitable choice for e in case $n > 2$ is $e = U_{23}U_{12}$). In case δ invertible in k one sees directly that $k\,\mathrm{hom}(2,0)$ is left projective.

9

Lemma 12.1. *CLAIM: For $n-2>0$ and $i \in \mathbb{N}_0$*

$$k\mathbf{Br}(n-2,n) \otimes_{B_n} k\mathbf{Br}(n, n+2i) \cong k\mathbf{Br}(n-2, n+2i)$$

Proof. The case $i = 0$ is trivial. We do $i = 1$ and leave the rest as an exercise. By repeated use of (12.3.9)(I)

$$k\mathbf{Br}(n-2,n) \otimes_{B_n} k\mathbf{Br}(n,n+2) \cong ek\mathbf{Br}(n,n) \otimes_{B_n} e'k\mathbf{Br}(n+2,n+2)$$

$$\cong e \otimes_{B_n} ee'k\mathbf{Br}(n+2,n+2) \cong ee'k\mathbf{Br}(n+2,n+2) \cong k\mathbf{Br}(n-2,n+2)$$

(the last step uses a mild generalisation of (12.3.9)(I)).

12.3.4 Simple head conditions revisited using G-functors

We have already seen in (12.1.29) that $\Delta_n(\lambda)$ has simple head if v_λ may be chosen a primitive idempotent of kS_l ($\lambda \vdash l$). We now give a different approach, using the G-functors (following Green[22, §6.2]).

(**12.3.11**) REMARK. Note that $\mathcal{S}(\lambda)$ is, in general, one of various non-isomorphic integral lattices in the corresponding $\mathbb{Q}S_l$-module (the contravariant dual is not isomorphic in general, for example). In this Section we always pass, eventually, to a ring containing $\mathbb{Q}$, where these forms become isomorphic. However if we wanted to pass to char.$p > 0$ our choice of $\mathcal{S}(\lambda)$ as starting point for 'growing' by the G-functor would be potentially arbitrary, and would merit review. (Although these modules do often provide a complete set of simple heads for kS_l.)

(**12.3.12**) LEMMA. Let $n > l \geq 0$, with $n - l$ even.. If either $l > 0$ or $\delta \neq 0$ then functor $F' := G^l_n$ (the *direct* functor as in (12.3.5)) is right adjoint to G^n_l, and $F'G^n_l(M) \cong M$ for any B_l-module M.

Proof. We shall bundle the proof in with the following.

(**12.3.13**) LEMMA. Let $\lambda \in \Lambda^n$, $l = |\lambda|$. Suppose k a field such that Specht module $\mathcal{S}(\lambda)$ is simple in kS_l. Then for $n = l + 2m$, for any $m \geq 0$, we have:
(I) $\Delta_n(\lambda)$ has simple head. (Accordingly write $L(\lambda)$ for the simple head of $\Delta_n(\lambda)$ for each $\lambda \in \Lambda^n$ (N.B. we have not shown here, yet, that these are distinct over $\mathbb{C}$).)

If in addition $l > 0$ or $\delta \neq 0$:
(II) The maximal proper submodule $Q(\lambda)$ of $\Delta_n(\lambda)$ obeys $F'Q(\lambda) = 0$. That is, every simple factor $L(\mu)$ of $Q(\lambda)$ obeys $F'L(\mu) = 0$.

(III) Thus, if $k = \mathbb{C}$, every simple factor $L(\mu)$ of $Q(\lambda)$ obeys $|\mu| > l$. (Caveat: This last statement assumes the restricted label set $\Lambda^{n,0}$ in case $\delta = 0$.)

Proof. The case of $n = l$ is trivial, since $\Delta_l(\lambda) = \mathcal{S}(\lambda)$. Otherwise $n = l + 2m$ for some $m > 0$, and $\Delta_n(\lambda) = G^n_l \Delta_l(\lambda)$ by Lemma 12.3.6. The basic idea here is to check that G^n_l is a G-functor between $B_l \cong E_m B_{l+2m} E_m$ and B_n, and hence that we can use Lemma 9.7.4 (which says roughly that if N is simple and $M \subset G(N)$ then $F(M) = 0$) and Proposition 9.3 from §9.7 (themselves derived from Green [22, §6.2]).

In the cases in which $l = |\lambda| > 0$ then G^n_l is an idempotent globalisation by Lemma 11.5.5. Thus (I) follows immediately; and (II) follows once we check that F is the right adjoint. For (III) note from (**??**) that F' is also a left inverse to G^n_l, that is $F'(G^n_l(M)) \cong M$, and that every $\Delta_n(\mu)$ with $\mu \vdash l$ can be expressed as $G(M)$ for non-vanishing M. That is $F'(\Delta_n(\mu))$ is non-vanishing, and hence — by exactness of F' and (II) for μ (which holds since $k = \mathbb{C}$):

$$
\begin{tikzcd}
{} \arrow[d, "F'"] & 0 \arrow[r] & Q(\mu) \arrow[r] & \Delta(\mu) \arrow[r] & L(\mu) \arrow[r] & 0 \\
{} & 0 \arrow[r] & F'Q(\mu) = 0 \arrow[r] & F'\Delta(\mu) \arrow[r, "\sim"] & F'L(\mu) \arrow[r] & 0
\end{tikzcd}
$$

— the image of its simple head (simple by (I) for μ) is also non-vanishing.

In the case $l = 0$ and δ a unit the result follows similarly using a different idempotent to construct the G-functor.
In the other cases (δ a non-unit, i.e. $\delta = 0$, and $l = 0$) we have a surjection

$$\Delta_n((2)) \to \Delta_n(\emptyset) \to 0$$

□

(**12.3.14**) EXAMPLE. Consider $n = 4$. We have $\Delta_4(\lambda)$ for $\lambda = (4), (3,1), (2,2), (3,1,1), (1^4), (2), (1^2), \emptyset$ and $L_4(\lambda)$ for the same partitions. The lemma states that $\Delta_4(4)$ (say) is simple; and any composition factor of $\Delta_4(2)$ (say) below the head must be one of $L(4), L(3,1), ..., L(1^4)$.

(**12.3.15**) Note that Lemma(12.3.13) holds for $k = \mathbb{Q}[\delta]$. Using the Lemma, we can strengthen the result in this case to show:

PROPOSITION. For $k = \mathbb{Q}[\delta]$ the Δ-modules are simple.
Outline proof. Each algebra $B_n(\delta)$ is isomorphic to its opposite, and there is a form on each $\Delta_n(\lambda)$ providing a map to the contravariant dual. One finds that the corresponding Gram matrix determinant is non-zero in $\mathbb{Z}[\delta]$, so this map is an isomorphism over $\mathbb{Q}[\delta]$. But from (12.3.13) this map must take the simple head to the unique copy of that simple in the contravariant dual (we claim that simples are contravariant self-dual here, by character considerations), which is the simple socle. That is, everything below the head is in the kernel of the map. Since there *is no kernel*, the Δ-modules are simple. Indeed they are a complete set of simples over $\mathbb{Q}[\delta]$. (Note that this means that each Brauer algebra has the basic set-up of a 'splitting modular system'.) □

(**12.3.16**) Let A be a f.d. algebra over a field and $\mathcal{M}$ a set of modules over A. For each order of this set, and order of the simple modules of A, we have a *simple decomposition matrix*. This records the composition multiplicities of the set in the obvious way.

Note that if a collection of modules over A has an upper unitriangularisable simple decomposition matrix, then it is a basis for the Grothendieck group.

(**12.3.17**) In summary we have that the set of Δ-modules (excluding $\Delta_n(\emptyset)$ if $\delta = 0$ and $n > 1$) is a basis for the Grothendieck group for B_n over $\mathbb{C}$.

12.3.5 Simple modules revisited using G-functors

We saw in Lemma(12.2.11) a set of simple B_n-modules over $\mathbb{C}$. Once again, G-functors provide a different approach.

(**12.3.18**) PROPOSITION. *CLAIM: The set $\{head\ (\Delta_n(\lambda)) \mid \lambda \vdash n, n-2, ...\}$ contains a complete set of simple modules for $B_n(\delta)$ over any field k, for any δ.*

Proof. TO DO!

(**12.3.19**) PROPOSITION. *The set $\{head\ (\Delta_n(\lambda)) \mid \lambda \vdash n, n-2, ...\}$ is a complete set of simple modules for $B_n(\delta)$ over $k = \mathbb{C}$ for any δ.*

Proof. See (12.3.13) above. Here is a quick review: To show that head $(\Delta_n(\lambda))$ is simple, we apply Prop. 12.3.9 to Prop. 12.3.8 (recall: every $\Delta_n(\lambda)$ is $G^{\circ m}\mathcal{S}(\lambda)$ for some m). If M is any proper submodule of $\Delta_n(\lambda)$ then $F^{\circ m}M = 0$ (F and hence $F^{\circ m}$ is exact, and the Specht module is simple over $\mathbb{C}$); thus there is a unique maximal submodule Q_λ (say) of $\Delta_n(\lambda)$.

The only case not covered by this argument is $\Delta_{2m}^{\delta=0}(\emptyset)$ $(m > 1)$. Here apply right exact functor G^{m-1} to

$$0 \to \Delta_2(2) \xrightarrow{\sim} \Delta_2(\emptyset) \to 0 \tag{12.12}$$

and use that $G^{m-1}\Delta_2(2)$ has simple head.

Completeness follows from Prop. 12.2.5. □

(**12.3.20**) However regarded as a list this construction may give rise to multiple entries, depending on k and δ. Over the complex field there is no overcount with $\delta \neq 0$, and with $\delta = 0$ just the element $\lambda = \emptyset$ should be excluded (as shown by the case treated above).
This completes task (1).

12.3.6 Induction and restriction

For given n, let Ind and Res denote the induction and restriction functors associated to the injection $B_n \hookrightarrow B_{n+1}$.

(**12.3.21**) Proposition. *For given n:*
(i) We may identify the functors $Res\, G- = Ind-$.
(ii) Over the complex field we have short exact sequence

$$0 \to \bigoplus_{\mu \lhd \lambda} \Delta_{n+1}(\mu) \to Ind\, \Delta_n(\lambda) \to \bigoplus_{\mu \rhd \lambda} \Delta_{n+1}(\mu) \to 0$$

(recall $\mu \lhd \lambda$ if μ is obtained from λ by removing one box).

Proof. (i) Unpack the definitions.

(ii) Note from (i) and Prop. 12.3.8 that it is enough to prove the equivalent result for restriction. Use the diagram notation above. Consider the restriction acting on the first n strings. We may separate the diagrams out into those for which the $n+1$-th string is propagating (which span a submodule, since action on the first n strings cannot change this property), and those for which it is not. The result follows by comparing with diagrams from the indicated terms in the sequence, using the induction and restriction rules for Specht modules.

(**12.3.22**) Remark. CAUTIONARY TAIL: Induction does not always work like this for $\Delta_n^{\delta=0}(\emptyset)$. (However this is not an issue. (TO BE SHOWN))

12.4 Characters and Δ-filtration factors

(**12.4.1**) Proposition. Over the complex field the modules $\{\Delta_n(\lambda)\}_{\lambda \in \Lambda^n}$ are pairwise non-isomorphic, except precisely in the case $n = 2, \delta = 0$ in (12.1.31).

Proof. Let us compare $\Delta_n(\lambda)$, $\Delta_n(\mu)$. If $|\lambda| \neq |\mu|$, consider first $|\lambda| > |\mu| > 0$. Then we can choose m so that $2m = n - |\mu|$, so that, by Lemma 12.1.16, the trace of the E_m action distinguishes the two modules.
In case $\mu = \emptyset$, $n > 2$, E_m is not defined but $E_{m-1}\Delta_n(\emptyset) \neq 0$, so the only issue is with $|\lambda| = 2$. But it is easy to see that the module ranks do not agree in this case.

If $|\lambda| = |\mu|$ then use right-exactness of Φ' and the inequivalence of $\mathcal{S}(\lambda)$ and $\mathcal{S}(\mu)$ over $\mathbb{C}$. □

(**12.4.2**) Can we strengthen this to pairwise distinct images in the Grothendieck group? All of the distinctions are actually distinctions at the level of characters, except for the last one. However, since we are working over $\mathbb{C}$ there are idempotents in $\mathbb{C}S_{|\lambda|}$ that distinguish the various Specht modules (i.e. for each λ an idempotent that is nonvanishing only acting on the corresponding module). Can we elevate this to work in B_n with $n > |\lambda|$?

Probably yes. Try the product of commuting idempotents along the lines used for the partition algebra in [33].

(**12.4.3**) Proposition. Suppose $\delta \neq 0$ (that is, $k = \mathbb{C}$ with $\delta \in \mathbb{C}^*$). Then (I) the heads $\{L_n^k(\lambda)\}_{\lambda \in \Lambda^n}$ are pairwise non-isomorphic (and a complete set of simples, as already noted).
(II) There is a unique expression for any character in terms of Δ-characters.

Proof. (I) Under these assumptions (i) Green's Lemmas about G (§9.7.2) hold, and (ii) the Δs are all constructible as 'inflations' $G(S)$ of distinct simple Specht modules S. Thus they have simple heads.

Now suppose some simple head $L_n^k(\lambda) = L_n^k(\lambda')$ — then $F(\Delta_n(\lambda)) = F(\Delta_n(\lambda'))$ since, by Lemma 9.7.4, the maximal submodule is killed by F. But $\mathcal{S}_\lambda = F(\Delta_n(\lambda))$ so we have $\mathcal{S}_\lambda = \mathcal{S}_{\lambda'}$ and hence $\lambda = \lambda'$. Thus these heads are distinct.
They are a complete set of simples by (**??**).
(II) By (**??**) the maximal submodule of $\Delta_n(\mu)$ only has composition factors from $\{L_n^k(\lambda)\}_{|\lambda|>|\mu|}$. This means that the unique expression for a character in terms of simple characters determines the expression in terms of Δ-characters, since the coefficients are related via a matrix that is (suitably ordered) lower unitriangular.

COULD SAY MORE EXPLICITLY?... □

(**12.4.4**) Prop.12.4.3 implies that for $\delta \neq 0$ the Δ-filtration multiplicities for projectives, denoted $(P_i : \Delta_n(\lambda))$, are also uniquely defined.

12.4.1 Aside on case $\delta = 0$

This is an interesting case as an example with a 'non-square' Δ-decomposition matrix.

(**12.4.5**) For the case $\delta = 0$, when $n = 2$ (there is no idempotent inflation from $n = 0$ ensuring pairwise non-isomorphism of Δ-modules, and indeed) the isomorphism (12.12) means that Δ-filtration multiplicities are not uniquely defined. (Specifically for $n = 2$, we could simply discard one of the isomorphic modules to make them so.)

(**12.4.6**) CLAIM: For $\delta = 0$ and $n \neq 2$, we can proceed as follows. We remove the overcount for simple modules by using the index set $\Lambda^{n,0}$. This determines a labelling scheme for projectives (according to their simple heads, for which we have not yet given a construction, but which are the heads of the correspondingly labelled Δ-modules). We claim $\Delta_n(\lambda)$ can be chosen as the top section of a Δ-filtration of $P_n(\lambda)$ (for each suitable λ), and then the non-isomorphism of Δs removes the abovementioned ambiguity and Δ-filtration multiplicities are well-defined.

Proof. ...

We shall use the notation $M = A//B//C//...$ to indicate that a module M has a chain of submodules with sections $A, B, C, ...$ Then in particular, while the $n = 2$ ambiguity gives

$$P_2(2) = \Delta_2(2)//\Delta_2(\emptyset) = \Delta_2(2)//\Delta_2(2) = ... \qquad (\delta = 0)$$

(as already noted), the sectioning of projectives in the block of $\Delta_n(\emptyset)$ up to $\lambda \vdash 4$ is indicated for $n \geq 4$ by

$$P_4(2) = \Delta_4(2)//\Delta_4(\emptyset) \qquad\qquad P_4(31) = \Delta_4(31)//\Delta_4(2) \qquad\qquad (\delta = 0)$$

The $n = 4$ case is an easy direct calculation: We have

$$P_3^0(1) = \Delta_3^0(1)$$

(the module on the right is evidently projective) so by Prop. 12.3.21

$$\mathrm{Ind}\, P_3^0(1) \;=\; \mathrm{Ind}\, \Delta_3^0(1) \;\cong \Delta_4^0(2) + \Delta_4^0(1^2) + \Delta_4^0(\emptyset)$$

Since there are Δ-filtrations of this starting with either $\Delta_4^0(2)$ or $\Delta_4^0(1^2)$ (again by Prop. 12.3.21), we deduce that

$$\mathrm{Ind} P_3^0(1) = P_4^0(2) \oplus P_4^0(1^2)$$

(note that the other Δ-module is not projective). We shall see shortly that $\Delta_4^0(1^2)$ and $\Delta_4^0(\emptyset)$ are not in the same block, so we are done.

(**12.4.7**) In this sense
CLAIM: we may treat $\delta = 0$ as a degeneration of the more general case, and treat the multiplicities $(P_i : \Delta_n(\lambda))$ as uniquely defined throughout. We do this hereafter.

(**12.4.8**) Exercise: Check: What happens for $\Delta_6(2)$? Here the G functor is well-behaved and we may proceed as in the general case treated in the next section.

12.4.2 The main case

(**12.4.9**) From Prop.12.3.8 recall (for $(\delta, \lambda, n) \neq (0, \emptyset, 2)$)

$$G\Delta_n(\lambda) = \Delta_{n+2}(\lambda) \tag{12.13}$$

Further (using Lemma 12.1.16 and the construction of F)

$$F\Delta_n(\lambda) = \begin{cases} \Delta_{n-2}(\lambda) & |\lambda| < n \\ 0 & |\lambda| = n \end{cases}$$

(**12.4.10**) By Prop. 12.3.19 every B_n-module character can be expressed as a not necessarily non-negative combination of Δ-characters:

$$\chi(M) = \sum_\lambda \alpha_\lambda(M)\; \chi(\Delta(\lambda)) \qquad (\alpha_\lambda(M) \in \mathbb{Z}) \tag{12.14}$$

This expression is unique if $\delta \neq 0$. It is also unique if $\delta = 0$ and we replace Λ^n by $\Lambda^{n,0}$ (however, as already noted, in this case modules with Δ-filtrations may be given well-defined *filtration multiplicities* over Λ^n).

(**12.4.11**) *Notation:* For A a k-algebra and $N = N_0 \supset N_1 \supset ... \supset N_l = 0$ a chain of A-modules we may write $(N_i)_\supset$ for the chain; and (given this)

$$(S_i)_{//} \;=\; //_i S_i \;=\; S_0//S_1//S_2//...$$

for the corresponding list of sections $S_i = N_i/N_{i+1}$. We may write this as $N = +_i S_i$ if the datum of chain *order* is being forgotten. Now suppose $\{Q_\mu\}$ are a collection of inequivalent modules such that each $S_i \cong Q_{\mu(i)}$ for some $\mu(i)$; and define $c_\mu = \#\{j |\ S_j \cong Q_\mu\}$. Then we may write

$$N \;=\; +_\mu\, c_\mu\, Q_\mu$$

Note that this does not preclude other Q-filtrations with different multiplicities, unless for example the Qs are a basis for the Grothendieck group.

(**12.4.12**) PROPOSITION. *Let M be a $B_n(\delta)$-module over $\mathbb{C}$ with given δ ($\delta \neq 0$ if $n = 2$), such that GM has a Δ-filtration. Then M has a Δ-filtration and*

$$(GM : \Delta_{n+2}(\lambda)) = \begin{cases} (M : \Delta_n(\lambda)) & |\lambda| \leq n \\ 0 & |\lambda| = n+2 \end{cases}$$

Further, suppose that $GM = (N_i)_\supset$ is a Δ-filtration with section sequence $GM = //_i\Delta_{n+2}(\mu(i))$ (so that $(GM : \Delta_{n+2}(\lambda)) \;=\; c_\lambda = \#\{j \mid \mu(j) = \lambda\}$) then there is a section sequence $M = //_i\Delta_n(\mu(i))$.

Proof. Let $GM = N_0 \supset N_1 \supset ... \supset N_l = 0$ be a chain giving a Δ-filtration. We have, by the exactness of F, that $M = FGM = FN_0 \supseteq FN_1 \supseteq ... \supseteq FN_l = 0$ is a sequence of B_n-modules, and

$$0 \to FN_{i+1} \to FN_i \to F(N_i/N_{i+1}) = F\Delta_i \to 0$$

has $F\Delta_i = \Delta_i^n$ or $= 0$. Thus removing any zero terms gives a Δ-filtration of M with the same multiplicities for all Δ_i such that $F\Delta_i \neq 0$.

It remains to show that there *are* no zero terms. For Δ-filtration of M of length $l = 1$ we may use (12.13). Before doing the general case let us do one more low rank case.

EXAMPLE: For length 2 we may write $M = \Delta_a//\Delta_b$, that is

$$0 \to \Delta_b \to M \to \Delta_a \to 0$$

is exact. Applying right-exact functor G we have

$$X \to G\Delta_b \to GM \to G\Delta_a \to 0$$

and hence

$$X \to \Delta_b^+ \xrightarrow{f} GM \xrightarrow{g} \Delta_a^+ \to 0$$

where Δ_a^+ denotes the Δ-module in B_{n+2}-mod (we omit the + hereafter). We don't know X ab initio, but the sequence is exact at GM so $im\, f = \ker\, g$. Exactness at Δ_a tells us that GM has a filtration with top section Δ_a so we may write it as $GM = \Delta_a//...$, so the ellipsis is ker g. But the domain of f is Δ_b, so this is the biggest (as k-space) that $im\ f$ can be. On the other hand we know that one of the factors in the ellipsis is Δ_b, so $im\ f$ can be no smaller. That is, $im\ f = \Delta_b$ (so $X = 0$) and $GM = \Delta_a//\Delta_b$. That is, there are no factors Δ_i such that $F\Delta_i = 0$.

Now suppose WLOG that $M = M_0 \supset M_1 \supset ... \supset M_m = 0$ is the chain derived from the chain for GM by deleting improper inclusions. We can write $M_i/M_{i+1} = \Delta_i$ and hence

$$M = \Delta_0//\Delta_1//\Delta_2//...//\Delta_{m-1}$$

That is, a collection of short exact sequences:

$$0 \to M_{i+1} \to M_i \to \Delta_i \to 0$$

Applying G to this we get exactness of

$$X_i \to GM_{i+1} \to GM_i \to \Delta_i \to 0$$

for suitable X_i. Note that it is not assumed that GM_i has a filtration. However as a k-space we have $\dim_k GM_i \;\leq\; \dim_k G\Delta_i + \dim_k GM_{i+1}$ (with equality only if we can take $X_i = 0$).

CLAIM: Let M be *any* B_n-module with Δ-filtration of length m as above. Then for $0 \leq i < m$,

$$\dim_k GM_i \;\leq\; \Sigma_{j=i}^{m-1} \dim_k G\Delta_j \tag{12.15}$$

(with equality only if we can take every $X_i = 0$).
PROOF: By induction on i. The base case is $i = m-1$. It is true since $GM_{m-1} \;=\; \Delta_{m-1}$. Now assuming case $i+1$ we get

$$\dim_k GM_i \;\leq\; \dim_k G\Delta_i + \dim_k GM_{i+1} \;\leq\; \dim_k G\Delta_i + \Sigma_{j=i+1}^{m-1} \dim_k G\Delta_j$$

Done.

Now the last ($i = 0$) case of (12.15) bounds $\dim_k GM_0 = \dim_k GM$. On the other hand we know from above that GM contains *at least* the corresponding Δ-filtration factors. Thus the bound is saturated, and it contains no other factors. □

(**12.4.13**) Remark. Note that the saturation of the bound above forces all the $X_i = 0$. This says that every GM_i constructed in this way has a Δ-filtration.

In fact one may use the machinery of quasi-heredity to provide an alternative to the above proof; and also to show that if M has a Δ-filtration then so does GM (i.e. M has a filtration if and only if GM has).
[10]

[10]

— TAKE TWO: — (THIS SHOULD PROBABLY JUST BE DELETED NOW.)

(**12.4.14**) Proposition. *If in addition (WHAT REALLY ARE THE CONDITIONS HERE?) a module M has a Δ-filtration then the combination (12.14) is a non-negative combination and (with the caveat mentioned in (12.4.1)), then if GM also has a Δ-filtration*[11]

$$(GM : \Delta_{n+2}(\lambda)) = \begin{cases} (M : \Delta_n(\lambda)) & |\lambda| < n+2 \\ 0 & |\lambda| = n+2 \end{cases}$$

Proof. Let $M = M_0 \supset M_1 \supset ... \supset M_l = 0$ be a Δ-filtration series of M, so $0 \to M_{j-1} \to M_j \to \Delta \to 0$ exact for some Δ. We get $X \to GM_{j-1} \to GM_j \to G\Delta \to 0$ Now use [**?**] ?
... ???

Alternatively, let $GM = N_0 \supset N_1 \supset ... \supset N_l = 0$ be a Δ-filtration series of GM, so $0 \to N_{j-1} \to N_j \to \Delta \to 0$ exact for some Δ. Applying the exact functor F we have $FGM = M = FN_0 \supseteq FN_1 \supseteq ... \supseteq FN_l = 0$ a filtration series of M; and $0 \to FN_{j-1} \to FN_j \to F\Delta \to 0$ exact. But $F\Delta$ is either a Δ-module or zero, so (possibly) omiting certain terms, $M = FN_0 \supset FN_1 \supset ... \supset FN_l = 0$ is a Δ-filtration series of M. This shows that the multiplicities agree for Δ such that $F\Delta \neq 0$. Now suppose ...

12.4.3 The n-independence of $(P(\lambda) : \Delta(\mu))$

(**12.4.15**) We continue working over $\mathbb{C}$, with fixed δ. By Prop.12.3.9 the functor G takes projectives to projectives. It also preserves indecomposability, so

$$GP_n(\lambda) = P_{n+2}(\lambda) \tag{12.16}$$

Combining (12.16) with Lemma 12.4.12, using the Δ-filtration property of projectives (Prop.12.2.5) we see the following.

(**12.4.16**) Theorem. Let $k = \mathbb{C}$ and fix δ. For any suitable pair $\lambda, \mu \in \Lambda$ the multiplicity $(P_n(\lambda) : \Delta_n(\mu))$, once defined (i.e. for sufficiently large n), is independent of n. That is, there is a semiinfinite matrix D independent of n, with rows and columns indexed by Λ, such that

$$(P_n(\lambda) : \Delta_n(\mu)) \;=\; D_{\lambda,\mu}$$

for any n. Further

$$(P(\lambda) : \Delta(\lambda)) = 1$$

and otherwise

$$(P(\lambda) : \Delta(\mu)) = 0 \text{ if } |\mu| \geq |\lambda|$$

That is, the matrix D is lower unitriangularisable. □

From this we have

(**12.4.17**) Proposition. *If P is a projective module containing $\Delta(\lambda)$ with multiplicity m and no $\Delta(\mu)$ with $|\mu| > |\lambda|$, then P contains $P(\lambda)$ as a direct summand with multiplicity m.* □

The induction functor takes projective modules to projective modules, and has a behaviour with regard to standard characters determined by Prop. (12.3.21). From this we see that

(**12.4.18**) Proposition. *For e_i a removable box of λ,*

$$Ind\, P_{\lambda - e_i} \cong P_\lambda \bigoplus ...$$

Proof: By Prop.12.4.17 a projective module is a sum of indecomposable projectives including all those with labels maximal in the dominance order of its standard factors. □

(**12.4.19**) Hereafter we may write simply $//_i \Delta_i$ in place of a module that has $//_i \Delta_i$ as a Δ-section sequence (note that the module does not define a unique sequence in general, and neither does the sequence define a unique module). Indeed we may write $//_i N_i$ for any section sequence.

We have

$$\mathrm{Ind} //_i \Delta_i \cong\; //_i \mathrm{Ind} \Delta_i$$

Proof. Using $\mathrm{Ind} - = \mathrm{Res}\, G$, then prop.12.4.12, then the exactness of Res, we have

$$\mathrm{Ind} //_i \Delta_i = \mathrm{Res}\, G //_i \Delta_i = \mathrm{Res} //_i G \Delta_i = //_i \mathrm{Res}\, G \Delta_i = //_i \mathrm{Ind}\, \Delta_i$$

□

(**12.4.20**) We shall further treat the notation $//_i N_i$ as a kind of non-commutative semigroup, so that $(//_i N_i)(//_j M_j)$ makes sense, and so on.

Chapter 13

Complex representation theory of the Brauer algebra

13.1 Blocks

In light of **??** we use the labels $\lambda \in \Lambda$ for simple modules of $B_n(\delta)$ over a given field k with $\delta \in k$. Thus we say μ, λ in the same block if $L(\mu)$ and $L(\lambda)$ in the same block, which we also write as $L(\mu) \sim^{\delta} L(\lambda)$ (if we are working over $k = \mathbb{C}$, or over a given field), or indeed $\mu \sim^{\delta} \lambda$.

13.1.1 Blocks I: preliminaries

Jucys-Murphy-Nazarov (JMN) elements for $B_n(\delta)$ may be defined as follows [35].

(**13.1.1**) For $k = 1, 2, ..., n$

$$j_k = \sum_{l=1}^{k-1} (U_{lk} - (l, k)) \quad \in B_n(\delta)$$

(cf. Equation(11.1)). It is easy to check that

$$[j_k, B_{k-1}(\delta)] = 0$$

for all k, so for all k, k'

$$[j_k, j_{k'}] = 0$$

Defining Jucys-Murphy-Nazarov (JMN) elements

$$x_k = \frac{\delta - 1}{2} - j_k$$

it follows that

$$\left[\sum_{k=1}^{n} (x_k)^l, \ B_n(\delta) \right] = 0$$

for all $l = 1, 2, 3, ...$ In particular

$$T'_n = \sum_{k=1}^{n} j_k$$

is central in $B_n(\delta)$. So by Schur's Lemma T'_n acts like a scalar on each $\Delta_n(\lambda)$.

(**13.1.2**) For $\lambda \in \Lambda^n$ with $\lambda \vdash l$ let $2t = n - l \in 2\mathbb{N}_0$. Then define

$$\chi_n(\lambda) \;=\; t(\delta - 1) - \sum_{d \in \lambda} c(d)$$

where the sum is over boxes in Young diagram λ and $c(d)$ is the ordinary *content* (column position - row position) of box d.

A calculation [20] shows that

(**13.1.3**) PROPOSITION. *For all* $y \in \Delta_n(\lambda)$

$$T'_n\, y \;=\; \chi_n(\lambda)\, y$$

(**13.1.4**) EXAMPLE. With $n = 2$ we have $\lambda \in \{(2), (1^2), \emptyset\}$. Let I denote the ideal generated by U_{12}. Consider $\lambda = (2)$ (so $t = 0$). We have $e_{(2)} + I \in \Delta_2((2))$ and

$$T'_2(e_{(2)}+I) = (U_{12}-(12))(e_{(2)}+I) = (U_{12}-(12))((1+(12))+I) = -(1+(12))+I = -c((2))(e_{(2)}+I)$$

as required. Similarly we have $e_{(1^2)} + I \in \Delta_2((1^2))$ and

$$T'_2(e_{(1^2)}+I) = (U_{12}-(12))(e_{(1^2)}+I) = (U_{12}-(12))((1-(12))+I) = (1-(12))+I = -c((1^2))(e_{(1^2)}+I)$$

Meanwhile for $\lambda = \emptyset$ ($t = 1$) we have $U_{12} \in \Delta_2(\emptyset)$ and

$$T'_2 U_{12} = (U_{12} - (12))U_{12} = (\delta - 1)U_{12} = (t(\delta - 1) - c(\emptyset))U_{12}$$

(**13.1.5**) Again by Schur's Lemma we have that λ, μ are in different blocks unless this eigenvalue agrees:

$$\lambda \sim \mu \qquad \Rightarrow \qquad \chi_n(\lambda) = \chi_n(\mu)$$

(**13.1.6**) EXAMPLE. With reference to the example above we see, therefore, that (2), $\emptyset$ are in different blocks over $\mathbb{C}$ unless, possibly, $\delta = 2$; and that (1^2), $\emptyset$ are in different blocks unless, possibly, $\delta = 0$. We shall see shortly that (2), $\emptyset$ are indeed in the same block when $\delta = 2$, for all n. But comparing the actions of another central element in case $n = 2$, $(U_{12} + (12))$:

$$(U_{12} + (12))(e_{(1^2)} + I) = (U_{12} + (12))((1 - (12)) + I) = -(1 - (12)) + I = -(e_{(1^2)} + I)$$

and

$$(U_{12} + (12))U_{12} = (\delta + 1)U_{12}$$

we see that the scalars no longer agree, thus (1^2), $\emptyset$ are in different blocks, when $\delta = 0$.

(**13.1.7**) A proof of Prop.13.1.3 is as follows.

First we recall Murphy's elements of S_n:

$$R_i = (1i) + (2i) + ... + (i-1\ i) \qquad (1 < i \leq n)$$

Murphy [] (see also, e.g. Green–Diaconis []) computed the representation matrices for these elements in Young's seminormal form of each Specht module $\mathcal{S}_\lambda$:

$$\rho_\lambda(R_i) = \ diag(c_1(i), c_2(i), ...)$$

where $c_l(i)$ is the content of the box containing i in the l-th standard Young tableau of shape λ (in some chosen total order of tableau, which will not be important to us). Since $\sum_i R_i$ is central it acts like a scalar, so we only need the first diagonal entry to determine this scalar. This is, then, the sum of the contents of all the boxes in the first (or indeed any) standard tableau of shape λ. That is

$$\rho_\lambda(\sum_i R_i) = \ (\sum_b c(b))I$$

where the sum is simply over the contents of boxes of λ; and I is the identity matrix.

Now consider the action of T'_n on an element of $\Delta_n(\lambda)$. For example consider the element of form $X = \cup^{\otimes t} \otimes w$ where w is a basis element of $\mathcal{S}_\lambda$. We are only interested in the summands of T'_n that take X to itself (since we know it acts like a scalar). These are U_{12}, U_{34} and so on (t summands); together with the corresponding $-(12)$, $-(34)$ and so on, altogether making a contribution of $t(\delta - 1)$; and then various $-(ij)$s that act on the factor w. The $-(ij)$s that act diagonally here correspond to (minus) a Murphy $\sum_i R_i$ for the kS_l-subalgebra that acts non-trivially only on the w. Thus they act like $-\sum_{b\in\lambda} c(b)$. Altogether we have

$$T'_n X \ = \ \left(t(\delta-1) \ - \ \sum_b c(b) \right) X$$

as required. $\square$

13.2 Blocks II: δ-balanced pairs of Young diagrams

Section 13.1.1 gives a general indication of how it is that the block structure of B_n comes to depend on the relative content of the labelling Young diagrams. See [13] for details.

It will be convenient now to cast the appropriate content condition for blocks in various forms.

13.2.1 The original δ-balanced pairs

(**13.2.1**) From [13] Def.4.7 we have a definition of *'δ-balanced'* pairs of Young diagrams effectively as follows (we use inverted commas here since we shall employ, at least implicitly, the version on conjugate partitions, possibly without further comment).

(**13.2.2**) Firstly, from [13], the pair $\lambda \supset \mu$ are 'δ-balanced' if
(i) there exists a pairing of boxes in λ/μ such that the content of each pair sums to $1-\delta$; and
(ii) if δ even and the boxes with content $-\delta/2$ and $(2-\delta)/2$ can be paired up in columns, then the number of such columns is even.

For example, the pair $\boxed{0}\boxed{1} \supset \emptyset$ is '0-balanced' in [13] (note that the *transposed* skew is a column and so does not have a fixed row in π-rotation, and so is a MiBS).

(**13.2.3**) Finally, *a general pair λ, μ are 'δ-balanced'* if each is 'δ-balanced' with $\lambda \cap \mu$.

(**13.2.4**) In [13] Def.6.1 we then have, for $\lambda \supset \mu$ a δ-balanced pair, a definition (in effect) of a *maximal δ-balanced subpartition* μ^i (say) of λ between μ and λ, as a subpartition of λ that is maximal (with respect to the subpartition order) among those δ-balanced with λ and containing μ. Note that the role of μ in this formulation is simply to ensure that λ *has* a δ-balanced subpartition. (In [13] the formulation is slightly more involved, leading to an algorithmic construction for μ^i which makes use of μ.)

It will be useful next to develop a more constructive approach to these objects.

13.2.2 Towards a constructive treatment: δ-charge and BS

(**13.2.5**) The *δ-charge* of a box in a Young diagram is

$$chg(b) \;:=\; \delta - 1 - 2c(b)$$

(in the literature this is called *δ-conjugate-charge*); and for λ a Young diagram or skew

$$chg(\lambda) \;:=\; \sum_{b\in\lambda} chg(b)$$

(**13.2.6**) Just as for content, the lines of constant δ-charge run parallel to the main diagonal. The key difference from content is that the line of δ-charge 0 for given δ is no longer (unless $\delta = 1$) the main diagonal itself. That is, the δ-charge-0 main diagonal is shifted from the ordinary main diagonal of the Young diagram. (Indeed for δ even there are no boxes with charge 0, so the charge 0 line lies 'between' diagonal runs of charge +1 and charge -1 boxes.)

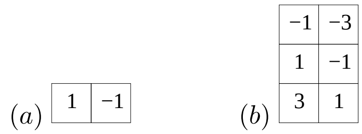


Figure 13.1: Examples of skews with charges.

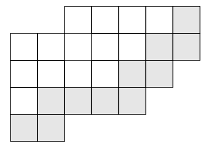


Figure 13.2: Skew outer rim construction.

(**13.2.7**) In the present setting, the point of generalising content to δ-charge is that $\mu \subset \lambda$ is in the same block only if $chg(\lambda^T/\mu^T) = 0$ (and in fact only if λ^T/μ^T contains $\pm$charge pairs).

For example, with $\delta = 2$ the skew $(2^2)/(1^2)$ contains ± 1, so potentially (and in fact, as we already found in (12.6)) we have $(2^2) \sim^{\delta=2} (2)$.

(**13.2.8**) A pair $\lambda \supset \mu \in \Lambda$ is *δ-flat* if
(i) the boxes of λ/μ can be put into pairs such that the sum of δ-charges in each pair is zero;
(ii) if there is such a pairing of boxes in which each +1,-1 pair has the two boxes side-by-side, then the number of these pairs is even.

If, for given δ, a pair $\lambda \supset \mu$ is δ-flat we shall say that skew λ/μ is a BS.

(**13.2.9**) Examples: Consider the skews with charges in Figure 13.1. Case (a) is not a BS. Case (b) is a BS(there is no pairing in which every +1,-1 pair is side-by-side).

13.2.3 π-rotation and MiBS

(**13.2.10**) Let $\lambda \supset \mu^1 \supset ... \supset \mu^l$ be any chain of Young diagrams. Then the skew μ^i/μ^{i+1} is a *skew-section* of this chain.
The *width at content (or charge) c* of a skew is the number of boxes of content c. The *maximum width* of a skew is the maximum number of boxes of any given content (or charge).

(**13.2.11**) A *rim* is a skew that is a chain. (Note that a rim has width 1.)
Define the *outer rim* of a connected skew as the collection of the bottom-right-most box of each content present. (NB this is also the bottom-right-most box of each charge present, for any δ.)

An example of a skew with outer rim shown shaded is given in figure 13.2.

(**13.2.12**) An outer rim may have removable subrims. Consider traversing the outer rim starting from the bottom-left box, facing to the right. Then the subset from any box before the first left-turn to any box between a left and a right-turn, is removable.

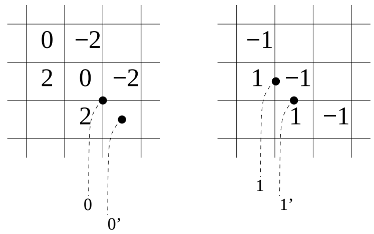


Figure 13.3: Possible π-rotation points.

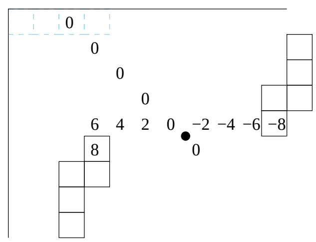


Figure 13.4: A π-rotation of rims in case $\delta = 5$.

NB, If the width of a skew goes from 2 to 1 in traversing from left to right then the outer rim has a removable subrim at that point (up to and including the last width 2 charge).

(**13.2.13**) Two rims are *δ-opposite* if there is a π-rotation of the plane about a point on the δ-charge-0 main diagonal that takes one into the other.
(Evidently this rotation is the same as reflection in the vertical line defined by the point of rotation; followed by reflection in the horizontal line defined by this point.)

(**13.2.14**) Note that any such π-rotation is necessarily about a point positioned as shown in one of the cases in Figure 13.3.

Note further that such a rotation has the effect of exchanging boxes in specific pairs, that are $\pm$charge pairs. Example (rotation of rims about the black dot shown): Figure 13.4. In this case the position of the charge-0 diagonal corresponds to $\delta = 5$.

(**13.2.15**) A *MiBS* is a skew that is a δ-opposite pair of rims such that no row of the skew is fixed by the associated π-rotation.

13.2.4 Connections and properties

(**13.2.16**) Lemma. Every MiBS is a BS. $\Box$

(**13.2.17**) We say a BS λ/μ has a section μ^i/μ^{i+1} if there is a chain $\lambda \supset \mu^1 \supset ... \supset \mu^l = \mu$ with each skew-section BS.

(**13.2.18**) Lemma. If a BS is not connected then it has a section of width 1.

Proof. It is enough to consider a two component skew. Call the components $\gamma_\pm$. Consider the outer rim of γ_+. This visits each positive charge in the skew once, and is removable. The corresponding rim of γ_- has a balancing set of charges, and is also removable, hence together they are a section. □

(**13.2.19**) Lemma. A BS has a section of width at most 2.

Proof. We construct such a section. (It is enough to consider the connected case.) Consider the part of the outer rim from the positive end to the last box in the set of consecutive diagonals, moving in the -ve direction from the charge-0 diagonal, with width ≥ 2 (NB there may be none such, in which case stop at +1). This visits all charges in this range and is removable. Remove this and consider, for what remains, the outer rim from the -ve end to the corresponding stopping point on the +ve side (NB this point is reachable since the original width to here, as it were, was ≥ 2). This is removable, so the combined skew is removable from the original; and they are balanced. □

(**13.2.20**) Corollary. Every BS has a MiBS section.

Proof. It is straightforward to see that all the sections described above contain MiBS. □

Note that this does not say that "no MiBS has a proper section". However we shall see shortly that this is also true.

Remarks on notation in CDM-I [13].

(**13.2.21**) Remark. As we shall demonstrate shortly, for $\lambda \supset \mu$, μ is a maximal δ-balanced subpartition of λ (as defined in [13]) if and only if λ^T/μ^T is a MiBS.

(**13.2.22**) The following will be clear from the definition of δ-charge.
If λ^T/μ^T is a MiBS then λ, μ are 'δ-balanced' in the sense of [13].

For this μ to be a maximal δ-balanced subpartition of λ we still need to show that there is no 'δ-balanced' partition between the two. This is our next task.

Back to MiBS

(**13.2.23**) Define a relation $(\Lambda, \leftarrow^\delta)$ by $\mu \leftarrow^\delta \lambda$ if λ/μ is a minimal δ-balanced skew. Define $(\Lambda, <^\delta)$ as the partial order that is the transitive closure of this relation.

(**13.2.24**) Lemma. Possible π-rotation points for a MiBS are of the forms shown in Figure 13.3. In case-0′ there can be no intersection of the skew with the row or column containing the point. In case-1 there can be no intersection of the skew with the row containing the point. Hence in either of these cases the skew is disconnected. □

(**13.2.25**) Lemma. (Pinning Lemma) Let π_x be a rotation as above, and b, b' two boxes comparable in the light-cone order (8.2.3), then

$$b' > b \qquad \Rightarrow \qquad \pi_x(b) > \pi_x(b')$$

□

(**13.2.26**) Proposition. *(I) If $\mu \subset \lambda$ and λ/μ a MiBS, then there is no $\mu \subset \mu' \subset \lambda$ such that μ'/μ is a MiBS.*
(II) The relation $(\Lambda, \leftarrow^{\delta})$ is the cover of the partial order $(\Lambda, <^{\delta})$.

Proof. (I): Let π_0 be the rotation fixing λ/μ and suppose (for a contradiction) that a rotation π_γ fixes $\gamma = \mu'/\mu \subset \lambda/\mu$.

The positive charge part of λ/μ is connected, so there exists a $b' \in \lambda/\mu'$ (i.e. in λ/μ and not in γ) adjacent to any $b \in \gamma$. Thus $\pi_0(b')$ lies in λ/μ adjacent to $\pi_0(b)$. Since λ/μ' is a skew over μ', we have $b' \not< b$ and hence (since adjacent) $b' > b$. Thus $\pi_0(b) > \pi_0(b')$ by Lemma 13.2.25.

Suppose for a moment that $\pi_0 = \pi_\gamma$ (i.e. they are rotations about the same point). Then $\pi_0(b') < \pi_\gamma(b)$, contradicting that γ is a skew over μ. Thus $\pi_0 \neq \pi_\gamma$.

Now, since $\pi_0 \neq \pi_\gamma$, π_0 fixes no pair $b, \pi_\gamma(b)$ in γ. That is, for each $b \in \gamma$ the boxes $b, \pi_\gamma(b), \pi_0(b), \pi_0(\pi_\gamma(b))$ are four distinct boxes in λ/μ. Observe also that b and $\pi_0(\pi_\gamma(b))$ have the same charge (and so lie in the same diagonal). Note that no charge can occur more than once in a rim or (therefore) more than twice in any MiBS. Thus for example no charge appears more than once in γ, while all the charges appearing in γ appear twice in λ/μ. Thus in particular λ/μ is connected. Note that the rotation point of π_0 is necessarily half a box down and to the right of π_γ. It then follows from Lemma 13.2.24 that γ_+ and γ_- are disconnected from each other.

Let c be the lowest charge box in γ_+. The box $\pi_0(\pi_\gamma(c))$ is below and to the right of it. Thus there is a box of λ/μ to its immediate right — call it d. There cannot be a box of λ/μ above c (since γ is a skew over μ), but there must be another box of λ/μ in the same diagonal as d (since λ/μ is connected), so the second box of λ/μ in the same diagonal as d would have to be to the right of $\pi_0(\pi_\gamma(c))$. But the π_0 image of *this* is to the left of $\pi_\gamma(c) \in \gamma$, contradicting the γ skew over μ property.

Claim (II) follows from (I) since $\mu \subset \lambda$ is a necessary condition for $\mu <^{\delta} \lambda$ so any failure of the MiBS relation to be a transitive reduction implies the existence of a μ' contradicting (I). □

(**13.2.27**) Remark. Continue with λ/μ a MiBS as above. We want also to rule out the possibility of an intermediate μ' such that λ, μ' is a balanced pair (hence showing that every MiBS corresponds to a MaBS). If λ/μ is disconnected then each charge in it occurs exactly one, and a single π-rotation is forced, once and for all, for each charge pair. Thus the same rotation would match the charge pairs in any balanced subset — i.e. it would have the properties of a MiBS, and a contradiction arises as above. What about the case of λ/μ connected? ...

...anyway, eventually we get there.

13.2.5 Δ-module maps and block relations

(**13.2.28**) Theorem. [13, Theorem 6.5] If λ/μ is a minimal δ-balanced skew then for $B_n(\delta)$ over $\mathbb{C}$

$$\mathrm{Hom}_{B_n(\delta)}(\Delta_n^{\delta}(\lambda^T), \Delta_n^{\delta}(\mu^T)) \neq 0$$

Proof. Noting the formulation in [13, Theorem 6.5], it is ETS that the definitions of MiBS and MaBS are equivalent up to transposition. It is straightforward to show that every balanced skew in the sense of [13] contains a (transposed) MiBS (see (13.2.20)). Equivalence then follows from (13.2.26). □

Write $\Lambda^{\sim\delta}$ for the RST closure of the partial order $(\Lambda, <^\delta)$. Write $[\lambda]_\delta$ for the $\Lambda^{\sim\delta}$-class of $\lambda \in \Lambda$.

(**13.2.29**) PROPOSITION. *The relation $\Lambda^{\sim\delta}$ is the (transposed) block relation for $B_n(\delta)$ over $\mathbb{C}$.*

Outline proof: We see from Prop. 13.1.3 (after some further work — see [13]) that the blocks are no bigger; and from Theorem 13.2.28 that they are no smaller. □

(**13.2.30**) Let $G_\delta(\lambda)$ be the λ-connected component of $(\Lambda, \leftarrow^\delta)$. This may thus be thought of as a directed acyclic graph. We call this the *block graph*.

13.3 The block graph

It is useful to give an alternative statement which emphasises geometrical aspects of the block condition.

Suppose λ/μ a minimal δ-balanced skew. Note that if we suspend, for intermediate steps, the dominance requirement (the requirement to work with partitions rather than arbitrary compositions) then we can build λ from μ by a sequence of transformations on pairs of rows. Each transformation extends two rows: adding part of one row, and the corresponding opposite charges in the other row. The no-row-fixed condition of (13.2.15) ensures that it is always pairs of rows (as opposed to a single row) that are involved. For each row in question one takes the leading edge of the row in μ and performs the two reflections mentioned in (13.2.13). The vertical reflection (i.e., in the horizontal) simply swaps the two rows. The other reflection takes this leading edge as far beyond the charge-0 diagonal as it was short of it beforehand. From these remarks it will be evident that the transformation $\mu \to \lambda$ can be reformulated geometrically. We shall describe this formulation in detail in (13.3.4) et seq..

(**13.3.1**) REMARK: Alternatively λ can be built by a sequence of transformations manipulating columns in pairs. The difference is firstly that, unless we transpose, the intermediate stages are neither partitions nor compositions (they are 'transpose compositions'); and secondly that it is possible in some cases to require a manipulation on a single column, rather than a pair; and thirdly that the no-row-fixed condition must still be imposed. In light of this we use here the rows-in-pairs version.
Exercise: How can we impose that row-fixed transformations do not arise in this formulation?

(**13.3.2**) Define a directed graph, G_e, with:
Vertices: subsets of $\mathbb{N}$ of even order (we call these *valley sets*);
Edges:

$$a \xrightarrow{\alpha} b \qquad \text{if} \qquad a \setminus b = \{\alpha\}, \;\; b \setminus a = \{\alpha+1\} \qquad (\alpha \in \mathbb{N})$$

$$a \xrightarrow{12} b \qquad \text{if} \qquad a \setminus b = \emptyset, \;\; b \setminus a = \{1,2\}$$

See Figure 13.6.

There is a corresponding graph G_o with vertices given by subsets of $\mathbb{N}$ of odd order. The *toggle map* between the vertex sets given by toggling the presence of 1 so as to make an odd set even is readily seen to pass to a graph isomorphism (the edge labels 1 and 12 are interchanged).

(**13.3.3**) Define a partial order $(\mathbb{R}^{\mathbb{N}}, \geq)$ by $v \geq w$ if $v_i \geq w_i$ for all i. (Write $v > w$ if $v \geq w$ and $v \neq w$.)

(**13.3.4**) For $\delta \in \mathbb{R}$ define

$$\rho_\delta = -\frac{\delta}{2}(1,1,...) - (0,1,2,...) \in \mathbb{R}^{\mathbb{N}}$$

Define $\mathbb{Z}^f$ as the subset of finitary elements of $\mathbb{Z}^{\mathbb{N}}$. Define

$$\mathsf{e}_\delta : \mathbb{Z}^f \hookrightarrow \mathbb{R}^{\mathbb{N}} \tag{13.1}$$
$$\lambda \mapsto \lambda + \rho_\delta \tag{13.2}$$

In other words, since $\Lambda \hookrightarrow \mathbb{Z}^f$, we have embedded our index set Λ into a Euclidean space. Thus our blocks now correspond to collections of points in this space.

Examples:

$$\mathsf{e}_2(\emptyset) = (0,0,0,0,...) - (1,1,1,1,...) - (0,1,2,3,...) \ = \ (-1,-2,-3,-4,...)$$

$$\mathsf{e}_{-1}(2) = (2,0,0,0,...) - \frac{-1}{2}(1,1,1,1,...) - (0,1,2,3,...)$$
$$= \ (2,0,0,0,...) + (\frac{1}{2},-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},-\frac{7}{2},...) \ = \ (\frac{5}{2},-\frac{1}{2},-\frac{3}{2},-\frac{5}{2},-\frac{7}{2},...)$$

(**13.3.5**) Note that all the image points $\mathsf{e}_\delta(\Lambda)$ are strictly descending sequences. We call such sequences *dominant*. Indeed all the image points $\mathsf{e}_\delta(\Lambda)$ are *strongly* descending sequences, meaning that $v_i - v_{i+1} \geq 1$ for all i. We write A^+ for the set of strongly decreasing sequences.

Considering for a moment the magnitudes of terms in a sequence in A^+, we see that each magnitude occurs at most twice, i.e. in a sequence of form $(...,x,...,-x,...)$. We call such a $\pm x$ pairing a *doubleton*. Define a map

$$Reg : A^+ \to A^+$$

such that $Reg(v)$ is obtained from v by removing the doubletons.

For example

$$Reg(1,-1,-3,-4,-5,-6,...) = (-3,-4,-5,-6,...)$$

(note in this case that the input is $\mathsf{e}_2((2,1))$ while the output is $\mathsf{e}_6(\emptyset)$, that is, the Reg map can increase δ);

$$Reg(4,3,1,0,-1,-5,-6,...) = (4,3,0,-5,-6,...)$$

(**13.3.6**) For $\lambda \in \Lambda$ write $p_\delta(\lambda)$ for the set of pairs of rows $\{i,j\}$ such that $(\lambda+\rho_\delta)_j = -(\lambda+\rho_\delta)_i$ (i.e. $\mathsf{e}_\delta(\lambda)_j = -\mathsf{e}_\delta(\lambda)_i$). Write $s_\delta(\lambda)$ for the *singularity* of $\mathsf{e}_\delta(\lambda)$:

$$s_\delta(\lambda) = |p_\delta(\lambda)|$$

(**13.3.7**) We say a sequence $v \in \mathbb{R}^{\mathbb{N}}$ is *regular* if no two terms have the same magnitude. Let $\mathbb{R}^{Reg}$ denote the set of regular sequences. Note that the terms in any $v \in \mathbb{R}^{Reg} \cap A^+$ have a well-defined *magnitude order*. That is, each term may be asigned a number giving its position in the list of terms ordered by increasing magnitude. For example $\frac{11}{2}$ is the 5-th term in the magnitude order of terms in $(\frac{11}{2},\frac{9}{2},\frac{3}{2},\frac{1}{2},\frac{-7}{2},\frac{-13}{2},\frac{-15}{2},...)$.

Define a map

$$o : \mathbb{R}^{Reg} \cap A^+ \ \to \mathbb{Z}^{\mathbb{N}}$$

as follows. In the i-th term, $o(v)_i$ of $o(v)$, the magnitude $|o(v)_i|$ is the position of v_i in the magnitude ordering of the set of numbers appearing in v. The sign of $o(v)_i$ is the sign of v_i, unless $v_i = 0$ in which case the sign is chosen so as to make an even number of positive terms.
(Remark: this sign choice is simply for definiteness. The definition of the function we eventually use (constructed next) will make it independent of this convention.)

Example:

$$o(\frac{11}{2}, \frac{9}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-7}{2}, \frac{-13}{2}, \frac{-15}{2}, ...) \;=\; (5, 4, 2, 1, -3, -6, -7, ...)$$

(**13.3.8**) Define

$$\begin{aligned} o_\delta : \Lambda \;&\to\; P(\mathbb{N}) \\ \lambda \;&\mapsto\; o(Reg(\mathtt{e}_\delta(\lambda)))|_+ \end{aligned} \tag{13.3}$$

where $|_+$ signifies to restrict to the initial subsequence of positive terms (which may simply be recorded as a set); and finally to apply the toggle map if this subsequence contains an odd number of terms.

Examples: $\emptyset \;\mapsto\; \mathtt{e}_2(\emptyset) = (-1, -2, -3, ...) \;\mapsto\; \emptyset$

$(1) \;\mapsto\; \mathtt{e}_2(1) = (0, -2, -3, ...) \;\mapsto\; \emptyset$

$(3, 3) \mapsto (2, 1, -3, -4, ...) \mapsto \{1, 2\}$

Note that the image sets $\mathtt{e}_\delta(\Lambda)$ do not intersect (as δ varies), so given $\mathtt{e}_\delta(\lambda)$ we can *determine* δ and λ. In the case above we have $\delta = 2$. In the next case we have $\delta = 0$:

$(3, 3, 3, 1) \mapsto \mathtt{e}_\delta(3^3 1) = (3, 2, 1, -2, -4, -5, ...) \mapsto \{1, 2\}$

$(4, 3, 3, 1) \mapsto (4, 2, 1, -2, -4, -5, ...) \mapsto \{1\} \overset{toggle}{\mapsto} \emptyset$

(**13.3.9**) Given δ and λ we define

$$o^\lambda_\delta : P_{even}(\mathbb{N}) \to \Lambda$$

as follows (indeed we could extend the domain to $P(\mathbb{N})$ by applying the toggle map to $P_{odd}(\mathbb{N})$). First construct $\mathtt{e}_\delta(\lambda)$. Note that this fixes the doubletons and (magnitudes of) singletons for its whole orbit, i.e. for every element of $o_\delta([\lambda]_\delta)$. We ignore the doubletons for a moment, and work out the magnitude order for the singletons. Note that the order in which the singletons can appear in a descending sequence is uniquely determined by their sign. Now for $v \in P_{even}(\mathbb{N})$ we give the positive sign to the corresponding singletons (in the magnitude order). Thus we have determined the singletons and their order in the sequence. The position of the doubletons is now forced, so the sequence $o_\delta(o^\lambda_\delta(v))$ is determined. But o_δ is invertible as already noted, so finally apply this inverse.

Example: $o^{(2)}_{-1}(\{1, 2, 4, 5\})$:

From an example above we see that the doubletons of $\mathtt{e}_{-1}(2)$ are just $\{5/2, -5/2\}$. The singletons have magnitudes $\{1/2, 3/2, 7/2, 9/2, 11/2, ...\}$ — where we have written them out in the magnitude order.

For $v = \{1, 2, 4, 5\}$ we give $+$ signs to $1/2$, $3/2$, $9/2$ and $11/2$ and the remaining singletons are negative. Thus

$$o^{(2)}_{-1}(\{1, 2, 4, 5\}) = (\frac{11}{2}, \frac{9}{2}, \frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{-5}{2}, \frac{-7}{2}, \frac{-13}{2}, \frac{-15}{2}, ...)$$

(**13.3.10**) Lemma. Fixing δ and λ, and hence $[\lambda]_\delta$, then o_δ and o^λ_δ are mutual inverses on $[\lambda]_\delta \leftrightarrow P_{even}(\mathbb{N})$. $\square$

(**13.3.11**) THEOREM. For each δ, λ, the map o_δ passes to an isomorphism

$$G_\delta(\lambda) \cong G_e$$

(via G_o and the toggle map in case $o_\delta(\lambda)$ of odd order).

By Lemma 13.3.10 we have that o_δ restricts to a bijection on vertex sets, with o_δ^λ the inverse map. The next few paragraphs build up to a proof (in (13.4.13)) of the graph isomorphism.

(**13.3.12**) PROPOSITION. *Fix a block-pair (δ, λ). If (v, w) is an edge in G_e with label α then the corresponding pair $(\mu, \lambda) = (o_\delta^\lambda(v), o_\delta^\lambda(w))$ gives λ/μ a minimal δ-balanced skew.*

Proof: YES – THAT WOULD BE GOOD!! BUT ACTUALLY THIS IS JUST A RESTATEMENT, AN UNPACKING, OF PART OF THE THEOREM.

13.4 The graph isomorphism, via geometrical considerations

13.4.1 Reflection group action

(**13.4.1**) A Euclidean space together with a collection of hyperplanes defines a reflection group — the group generated by reflection in these hyperplanes. Let

$$(ij) : (\lambda_1, \lambda_2, ..., \lambda_i, ..., \lambda_j, ...) \mapsto (\lambda_1, \lambda_2, ..., \lambda_j, ..., \lambda_i, ...)$$

$$(ij)_- : (\lambda_1, \lambda_2, ..., \lambda_i, ..., \lambda_j, ...) \mapsto (\lambda_1, \lambda_2, ..., -\lambda_j, ..., -\lambda_i, ...)$$

be reflection group actions on $\mathbb{R}^{\mathbb{N}}$. Write $\mathcal{D}$ for the group generated by these (all $i < j$). Write $\mathcal{D}\lambda$ for the orbit of a point under the action of $\mathcal{D}$.

(**13.4.2**) Consider how some $w \in \mathcal{D}$ acts on Λ, via its action on $\mathsf{e}_\delta(\Lambda)$ for given δ. We write $w.\lambda$ for this. The action does not lie inside Λ in general. It modifies two rows: in case (ij) this breaks the descending property, so cannot lie in Λ; in case $(ij)_-$ it extends or contracts both rows, so sometimes lies in Λ.

(**13.4.3**) EXAMPLE. MORE DETAILS!!!...

(**13.4.4**) Note that $\mathcal{D}$ does not preserve the image $\mathsf{e}_\delta(\Lambda)$, for any δ, but (it is routine to show that in a suitable sense)

$$\text{orbit} \cap \text{‘dominant’} = \text{block}$$

(**13.4.5**) LEMMA. If λ/μ is a minimal δ-balanced skew then $\mathsf{e}_\delta(\lambda)$ can be obtained from $\mathsf{e}_\delta(\mu)$ by a sequence of one or more transformations by $(ij)_-$s, extending rows in pairs of δ-balanced part-rows. Specifically

$$\mathsf{e}_\delta(\lambda) \;=\; \left(\prod_{ij} (ij)_-\right) \; \mathsf{e}_\delta(\mu)$$

where the product is over pairs of rows in the skew, from the outer pair to the inner pair.

Note also that no subset of this product, applied to $\mathsf{e}_\delta(\mu)$, results in a dominant weight.

Proof. Compare the definitions of minimal δ-balanced skew (13.2.15), e_δ and $(ij)_-$. □

(**13.4.6**) Lemma. (I) The $\mathcal{D}$ action on λ includes a traversal of the block $[\lambda]_\delta$ for each λ.
(II) This action intersects no other block.

QUESTION: WHAT HAPPENS TO THE VALLEY SET UNDER THIS TRANSF.??

Proof. (I) Follows from Lemma 13.4.5. (II) Is proved in [14]. □

(**13.4.7**) More precisely, define

$$V(v) = Dv \cap A^+$$

Define a directed graph $\mathbf{G}(v)$ with vertex set $V(v)$ by assigning an edge (t, u) if $(u-t)_i \geq 0$ for all i and this is a cover (i.e. (t, u) is not in the transitive closure of any other such pairs).

13.4.2 The graph isomorphism

(**13.4.8**) Proposition. *For $\lambda \in \Lambda$ the map e_δ restricts to a bijection $[\lambda]_\delta \to V(\lambda + \rho_\delta)$; and this bijection extends to a graph isomorphism*

$$G_\delta(\lambda) \cong \mathbf{G}(\lambda + \rho_\delta).$$

Proof: By [14] we have that e_δ defines a bijection between $[\lambda]_\delta$ and $V(\lambda+\rho_\delta)$. Note that $\mu \subset \nu \in \Lambda$ if and only if $e_\delta(\mu) < e_\delta(\nu)$. Thus, restricting this to $[\lambda]_\delta$, the graphs are covers of isomorphic partial orders. These covers thus agree on arbitrarily large finite sub-orders, and hence agree. □

Note that v is regular if and only if every sequence in Dv is regular.

(**13.4.9**) Proposition. *For $v \in A^+$ the map Reg restricts to a bijection $V(v) \to V(Reg(v))$; and this bijection extends to a graph isomorphism*

$$\mathbf{G}(v) \cong \mathbf{G}(Reg(v))$$

Proof: The set of doubletons is an invariant of the elements of $V(v)$, and there is a unique way of adding these into an element of $V(Reg(v))$ that keeps the sequence decreasing. Thus the restriction of Reg here has an inverse, i.e. the set map is a bijection. Now suppose $t, u \in A^+$ and $a \in \mathbb{R}$ such that

$$s = (t_1, t_2, ..., t_i, a, t_{i+1}, ...) \qquad s' = (u_1, u_2, ..., u_j, a, u_{j+1}, ...)$$

are in A^+. Then $t < u$ if and only if $s < s'$. The Reg map can be built from pairs of such moves, so $t < u$ if and only if $Reg(t) < Reg(u)$, which establishes the graph isomorphism. □

(**13.4.10**) To any Euclidean space and set of hyperplanes we may associate a *dual graph*. This has a vertex for each connected component of the space with the hyperplanes removed ('alcove'); and an edge whenever the closures of two alcoves intersect in a defining subset of a hyperplane.

Given a pair of a closed set of hyperplanes and a closed subset, the dominant dual graph is the intersection of the dual graph with a fundamental chamber for the subset. For example Figure 13.5 shows the dominant dual graph for affine-A_2 (generated by the hyperplanes 1,2 and 3' shown) over the subset corresponding to A_2 (generated by the hyperplanes 1 and 2).

We write G_a for the dominant part of the dual graph of our reflection group action D above. (A graph isomorphic to this, using a notation we shall explain shortly, is shown in Figure 13.6.)

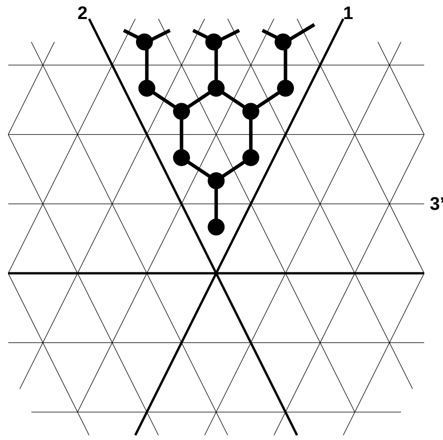


Figure 13.5: Example of a dominant dual graph: case $\hat{A}_2/A_2$

(**13.4.11**) Lemma. For any regular v, i.e. lying within an alcove, the dominant part of its D-orbit contains a point in each dominant alcove. Thus $\mathbf{G}(v) \cong G_a$.

A convenient example of a regular v is $e_2(\emptyset)$. In light of the lemma we may use the orbit of $e_2(\emptyset)$ to label dominant alcoves.

(**13.4.12**) Lemma. The map from the orbit $\mathcal{D}e_2(\emptyset)$ to subsets of $\mathbb{N}$ of even order which discards all negative entries extends to a graph isomorphism $G_a \cong G_e$.

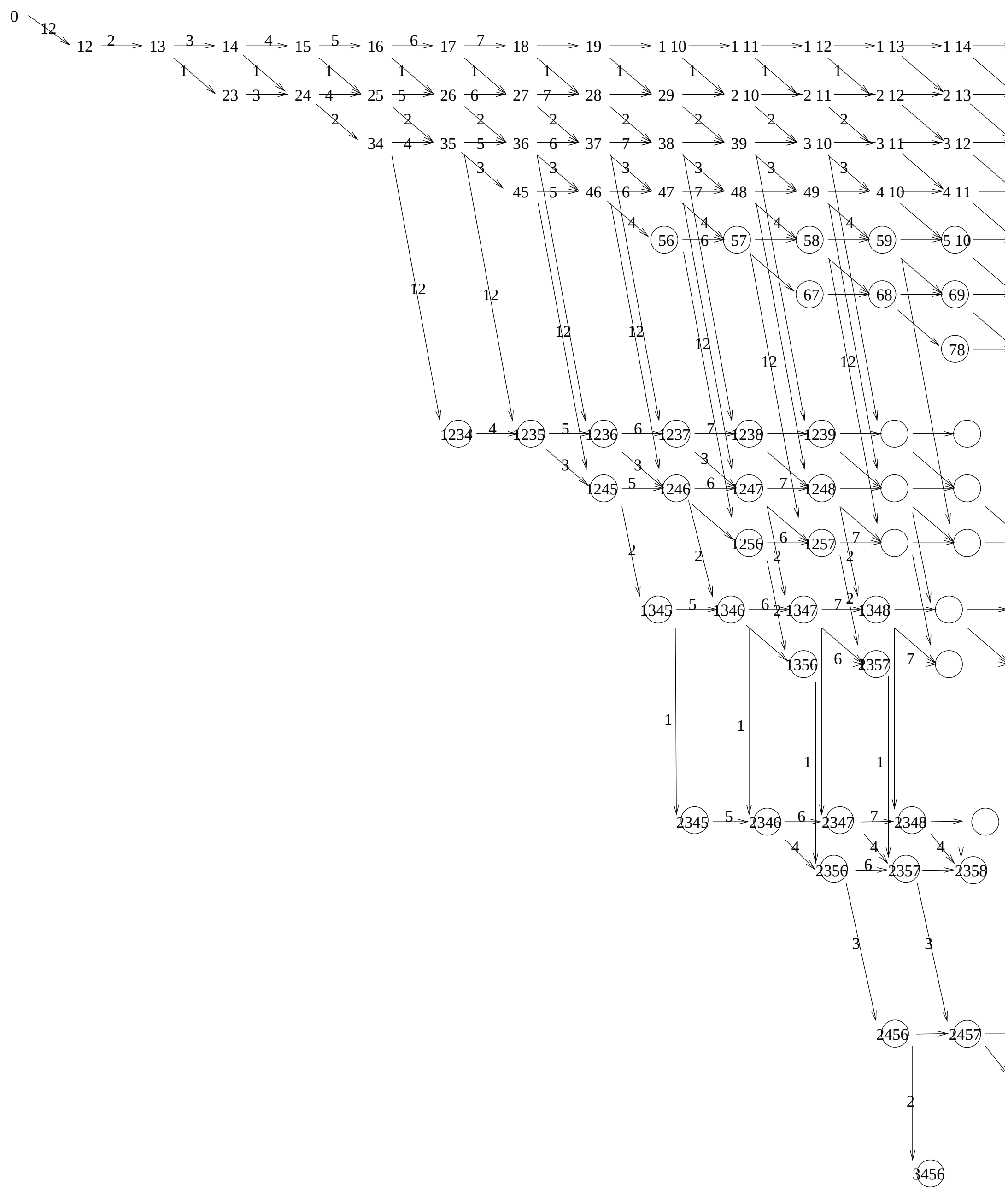
(**13.4.13**) Theorem. For all δ, λ

$$G_\delta(\lambda) \cong G_a$$

Proof. By (13.4.8), (13.4.9) and (13.4.11). □

This is a remarkable result, since the right hand side does not depend on λ or even δ.

Figure 13.6: The beginning of the graph G_e, with edge labels.



13.5 Decomposition data

13.5.1 Hypercubical decomposition graphs

(**13.5.1**) Each $a \in P(\mathbb{N})$ has a natural binary sequence representation $b(a)$. For example:

$$b : \{1, 3, 5\} \mapsto 10101$$

Define $b_\delta : \Lambda \to \{0, 1\}^{\mathbb{N}}$ by $b_\delta(\lambda) = b(o_\delta(\lambda))$.

(**13.5.2**) A *TL diagram* is here a collection of non-crossing arcs drawn in the positive quadrant of the plane, each terminating either in two vertices drawn on the horizontal part of the boundary; or in such a vertex at one end, and a crossing of the vertical part of the boundary at the other end.

(**13.5.3**) Each binary sequence has a TL diagram constructed as follows.
1. Draw a vertex for each entry (up to the last non-zero entry).
2. For each binary subsequence 01 draw an arc connecting these vertices.
3. Consider the sequence obtained by ignoring the vertices paired in 2. For each subsequence 01 draw an arc connecting these vertices (it will be evident that this can be done without crossing).
4. Iterate this process until termination (it will be evident that it terminates, since the sequence is getting shorter).
5. Note that this process terminates either in the empty sequence or in a sequence of 1s then 0s (either run possibly empty). Finally connect the run of vertices binary-labelled 1 in adjacent pairs (if any) from the left. Leave the remaining vertices as singletons.

(**13.5.4**) A number of examples are shown in Figure 13.7.

(**13.5.5**) For $a \in P(\mathbb{N})$ we write Γ_a for the list of arcs (i.e. pairs) corresponding to 01 and 11 subsequences; and Γ^a for the list of all arcs.

In particular, for example,

$$\Gamma_{1356} = \{\{2, 3\}, \{4, 5\}\} \qquad\qquad \Gamma^{1356} = \{\{2, 3\}, \{4, 5\}, \{1, 6\}\}$$

We write $\Gamma_{\delta,\lambda}$ for $\Gamma_{o_\delta(\lambda)}$.

(**13.5.6**) Each such TL diagram $d = d_\delta(\lambda)$ defines a hypercubical directed graph $h_\delta(\lambda)$, whose vertices are sequences. The top sequence is the defining sequence. There is an edge out of this for each completed arc in the TL diagram. The sequence at the other end of this edge is obtained from the original by replacing $01 \to 10$ (or $11 \to 00$) at the ends of this arc. Indeed all the edges in the hypercube follow these rules.

Example: Figure 13.8. There is a concrete example in Section 13.5.2.

(**13.5.7**) We label each edge of the hypercube (i.e. each direction) by $\{\alpha, \alpha'\}$, where α, α' are the positions of the ends of the arc associated to this edge. (If $\alpha' = \alpha + 1$ for an 01-arc we may just label the edge by α. If $\{\alpha, \alpha'\} = \{1, 2\}$ for a 11-arc we may just label the edge by 12. Note that these α-edges and 12-edges in particular then coincide with edges of G_e, although other edges do not.)

(**13.5.8**) Note that it follows from the construction that if a vertex of this hypercube is $b_\delta(\lambda)$ for some λ, then a vertex beneath it down an α or 12-edge is $b_\delta(\mu)$ for some μ a maximal δ-balanced subpartition of λ.

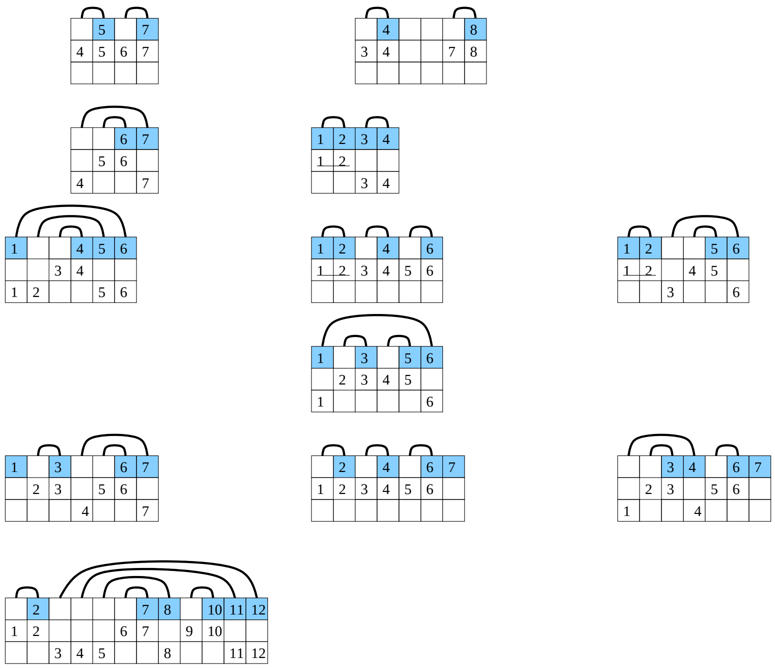


Figure 13.7: Examples for the map from sequences to TL diagrams. In each case the sequence is indicated in the first (shaded) row of boxes. The subsequent rows show sets of pairs of numbers extracted from the TL construction.

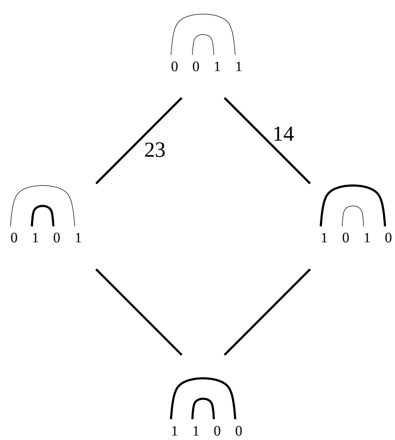


Figure 13.8: Hypercube construction.

(**13.5.9**) Let $b = (b_1, b_2, ...)$ be a binary sequence, and α a natural number. Then $\overset{\alpha}{\bigvee} b$ is the sequence obtained from b by inserting 01 in the $\alpha, \alpha+1$ positions (i.e. so that this pair become the elements in the α and $\alpha+1$ positions in the sequence, with any terms at or above these positions in b bumped two places further up in $\overset{\alpha}{\bigvee} b$).
Similarly $\overset{\alpha}{\bigvee}_{\_}\, b$ is the sequence obtained from b by inserting 10 in the $\alpha, \alpha+1$ positions.
Examples: $\overset{2}{\bigvee} 01 = 0011$, $\overset{2}{\bigvee}_{\_}\, 01 = 0101$.

(**13.5.10**) Let h be a hypercube (i.e. the $\{0,1\}$-span of any linearly independent collection of vectors, as before), and α a vector outside the span of h (or an operator that can otherwise be considered to shift all the vertices of h by the same amount in a new direction). Then by αh we mean the translate of h determined by α, and by $(1, \alpha)h$ we mean the new hypercube which contains h and a translate of h by α together with the edges in the α direction.

More specifically, if h is a hypercube whose vertices are binary sequences, all of which have 01 (or all 11) in the $\alpha, \alpha+1$ positions, then αh is the hypercube defined from h by modifying this $01 \to 10$ (respectively $11 \to 00$). In this case $(1, \alpha)h$ is the hypercubical union of h and αh.

If the bumped sequence $\overset{\alpha}{\bigvee} b_\delta(\lambda)$ makes sense, then by $\overset{\alpha}{\bigvee} h_\delta(\lambda)$ we understand the corresponding vertex-modified hypercube (insert 01 at the same position in every vertex binary sequence, and modify any edge labels affected by this accordingly). Note that this is not a hypercube of form $h_\delta(\mu)$, but a subgraph of somesuch. Similarly define $\overset{\alpha}{\bigvee}_{\_}\, h_\delta(\lambda)$ (and note that $\overset{\alpha}{\bigvee}_{\_}\, h_\delta(\lambda) = \alpha \overset{\alpha}{\bigvee} h_\delta(\lambda)$). Note that $\overset{\alpha}{\bigvee}_{\_}\, h_\delta(\lambda)$ is another hypercube not of form $h_\delta(\mu)$. However

$$(1, \alpha) \overset{\alpha}{\bigvee} h_\delta(\lambda) = h_\delta(\mu) \qquad \text{where } b_\delta(\mu) = \overset{\alpha}{\bigvee} b_\delta(\lambda) \tag{13.4}$$

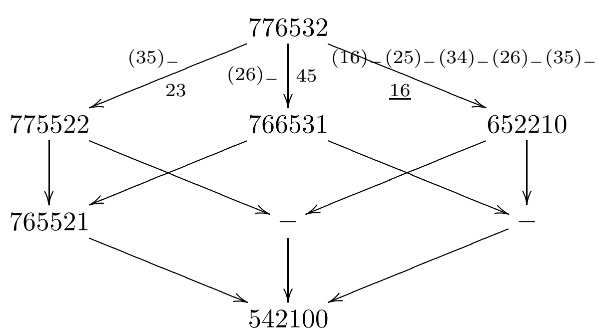
This is simply a restatement of part of the definition (13.5.6), that will be useful later.

13.5.2 Hypercubical decomposition graphs: Examples

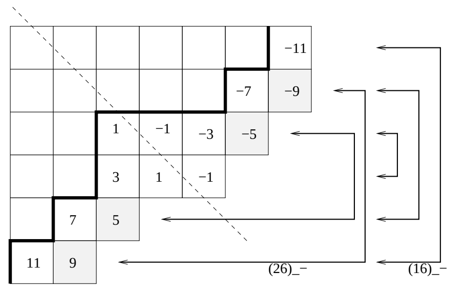
(**13.5.11**) Here is a concrete example with $\delta = 2$. We take $\lambda = (7, 7, 6, 5, 3, 2)$ so

$$\lambda + \rho_2 = (6, 5, 3, 1, -2, -4, -7, -8, ...)$$

giving $o_2(\lambda) = \{1, 3, 5, 6\}$ and hence $\Gamma_\delta^\lambda = \{\{2,3\}, \{4,5\}, \{1,6\}\}$. The specific hypercube (with weights at the vertices) is thus



Here we have recorded both the α-action and the specific reflection group action required to achieve it on each edge (for the shoulder layer). The following figure shows the explicit reflections and composite reflection in the shoulder:



Note that the composite can be built as five dominance preserving but not all commuting reflections.

(**13.5.12**) Keeping the same λ, now consider $\lambda - e_i$ in case $i = 4$.

This gives $(7,7,6,4,3,2) \rightsquigarrow (6,5,3,0,-2,-4,-7,...) \stackrel{o_2}{\rightsquigarrow} \{1,3,5,6\}$ (by the toggle rule). This means that the hypercube $h_\delta(\lambda - e_i)$ is isomorphic to that for λ above, so in particular the α-actions (the formal edge labels) are the same. Note also that the specific reflections (realising these α-actions) in the shoulder of $h_\delta(\lambda - e_i)$ are the same as for λ.

We CLAIM that so long as e_i does not separate a MiBS (in the sense of (13.6.5)) this holds true in general. That is the hypercubes are isomorphic and the reflections needed to move through the hypercube are the same.

13.6 Embedding properties of δ-blocks in Λ

In this section we consider how the block graphs embed in $\mathbb{R}^{\mathbb{N}}$ and hence how the embeddings of the different block graphs relate to each other. The result (12.3.21) means, loosely speaking, that the usual *metrical* structure on $\mathbb{R}^{\mathbb{N}}$ has relevance in our representation theory. This, together with the embedding results we develop here, will allow us to pass information between blocks.

(**13.6.1**) When δ is fixed, for $w \in \mathcal{D}$ and $\lambda \in \Lambda$, we write $w.\lambda$ for the μ such that $w\mathsf{e}_\delta(\lambda) = \mathsf{e}_\delta(\mu)$.

If λ is a vertex of G_e or (in case δ is fixed) of some $G_\delta(\mu)$, and α is the label on an edge out of λ, then we write $\alpha\lambda$ for the vertex at the other end. (Note that there is at most one edge with label α out of each vertex.)

Note that, for given δ, Lemma (13.4.5) associates a specific involutive $w \in \mathcal{D}$ to each pair $(\lambda, \alpha\lambda)$, such that $w.\lambda = \alpha\lambda$.

(**13.6.2**) The isomorphism implicit in Theorem 13.4.13 between any pair of block graphs $G_\delta(\lambda)$ and $G_\delta(\lambda')$ defines a pairing of each vertex in $G_\delta(\lambda)$ with the corresponding vertex in $G_\delta(\lambda')$. A pair of block graphs is *adjacent* if they have the same singularity, and every such pair of vertices is adjacent as a pair of partitions.

(**13.6.3**) REMARK. Fix δ. If λ, λ' are adjacent partitions such that their images $\mathsf{e}_\delta(\lambda), \mathsf{e}_\delta(\lambda')$ lie in the same $\mathcal{D}$-facet (in the alcove geometric sense) then the corresponding pair of graphs are adjacent, since the same reflection group elements serve to traverse these graphs (i.e. a sequence of reflections taking λ to μ, say, will take λ' to the isomorphic image of μ), and reflection group elements preserve adjacency of partitions. We shall need to show adjacency of a more general pairing of graphs.

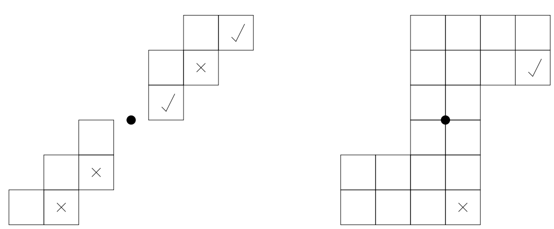
(**13.6.4**) For given λ, if $\lambda' = \lambda - e_i$ above we write

$$f_i : [\lambda]_\delta \;\to\; [\lambda - e_i]_\delta$$

for the restriction of the graph isomorphism to vertices.

(**13.6.5**) Fix δ and suppose $\lambda \in \Lambda$ has a removable box e_i. Suppose that $\lambda/\alpha\lambda$ is a MiBS containing e_i. Write π_α for the π-reflection fixing this MiBS. Then note that $\pi_\alpha(e_i)$ is an addable box of $\alpha\lambda$. If $\lambda/\alpha\lambda \setminus \{e_i, \pi_\alpha(e_i)\}$ is not a MiBS (of $\lambda - e_i$) we say that e_i *separates* $\lambda/\alpha\lambda$.

(**13.6.6**) Examples: crosses show boxes that separate; ticks show boxes that do not:



(**13.6.7**) LEMMA. (Charge-row lemma) Fix any δ. If a row i of partition λ ends in a box with charge c we have

$$(\lambda + \rho_\delta)_i = -\frac{c}{2} + \frac{1}{2}$$

13.6.1 The Relatively–regular–step Lemma

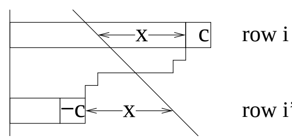
(**13.6.8**) Remark. Fix δ. For each λ, $o_\delta(\lambda)$ determines its position in the block. Thus (I) if $\lambda, \lambda - e_i$ are regular in the same alcove then $o_\delta(\lambda) = o_\delta(\lambda - e_i)$. On the other hand it is not possible to make such a small step and jump entirely from one alcove to another. Thus (II) if they are both regular, they are in the same alcove.

The argument for (I) applies if $\lambda, \lambda - e_i$ are in the same facet. However it is possible to change to a co-regular (i.e. equal singularity) but distinct facet in one step. It is not so clear that this new facet corresonds to the same block position. It turns out, though, that it does. Here we show this.

Short version:

(**13.6.9**) Lemma. Fix δ. If $\lambda/\lambda - e_i - e_{i'}$ a MiBS then $s_\delta(\lambda - e_i) > s_\delta(\lambda)$.

Proof. The picture for λ is as here:



Removing $\boxed{\text{c}}$ means rows i, i' become a singular pair, where they were not before, so $p_\delta(-)$ changes. Thus $s_\delta(\lambda - e_i) > s_\delta(\lambda)$ unless we also *lost* a singular pair i, i''. This would have to be with $i'' = i' + 1$ directly under $\boxed{\text{-c}}$, but this cannot happen since $\boxed{\text{-c}}$ is removable. $\square$

(**13.6.10**) Lemma. Fix δ. If $\lambda, \lambda - e_i \in \Lambda$ and $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then $o_\delta(\lambda) = o_\delta(\lambda - e_i)$.

Proof. Write x for $(\lambda + \rho_\delta)_i$. Thus, for some $y < x - 1$:

$$\lambda + \rho_\delta \sim (..., \underbrace{x}_{i}, y, ...), \qquad \lambda + \rho_\delta - e_i \sim (..., \underbrace{x - 1}_{i}, y, ...) \tag{13.5}$$

If $p_\delta(\lambda) = p_\delta(\lambda - e_i)$ then one can readily check that the changed row i appears in the magnitude order in both cases, and in the same position. In case $x = 1/2$ there is a sign change, but by the toggle rule $o_\delta(-)$ remains unchanged. If $p_\delta(\lambda) \neq p_\delta(\lambda - e_i)$ then from (13.5) we see firstly that $-x$ occurs in $\lambda + \rho_\delta$ and $1 - x$ occurs in $\lambda + \rho_\delta - e_i$ (for if neither occurs then p_δ does not change between them; while if only one occurs then s_δ changes);
it follows immediately that $1 - x, -x$ occur (and are adjacent) in both;
secondly, $y < x - 1$ so $x - 1$ does not occur in $\lambda + \rho_\delta$.

In computing o_δ we discount the $\pm x$ pair in $\lambda + \rho_\delta$ and the $\pm(x - 1)$ pair in $\lambda + \rho_\delta - e_i$. The discrepancy is thus now a $1 - x$ in $\lambda + \rho_\delta$ compared to a $-x$ in $\lambda + \rho_\delta - e_i$. But if $1 - x$ is the l-th largest magnitude entry in $\lambda + \rho_\delta$ then $-x$ is the l-th largest magnitude entry in $\lambda + \rho_\delta - e_i$, with all else equal, so o_δ is unchanged. $\square$

Long version:

(**13.6.11**) Lemma. Fix δ and suppose $\lambda \in \Lambda$ has a removable box e_i such that singularity $s_\delta(\lambda) = s_\delta(\lambda - e_i)$. Then
(I) $o_\delta(\lambda) = o_\delta(\lambda - e_i)$;
(II) There does not exist a weight $\lambda - e_i - e_{i'}$ δ-balanced with λ.
(III) There does not exist a weight $(\lambda - e_i) + e_i + e_{i'}$ δ-balanced with $\lambda - e_i$.

Proof. Write x for $(\lambda+\rho_\delta)_i$. That is

$$\lambda+\rho_\delta \sim (...,w,\underbrace{x}_{i},y,...), \qquad \lambda+\rho_\delta-e_i \sim (...,w,\underbrace{x-1}_{i},y,...) \tag{13.6}$$

From this we see firstly that $x-1$ cannot occur in $\lambda+\rho_\delta$ (else it would occur twice in $\lambda+\rho_\delta-e_i$, contradicting the descending property of the latter); and similarly x cannot appear in $\lambda+\rho_\delta-e_i$.

We now split into two cases, depending on whether $p_\delta(\lambda)=p_\delta(\lambda-e_i)$.

(A) If $p_\delta(\lambda)=p_\delta(\lambda-e_i)$:

(I) The argument depends on the value of x. We split into subcases (i-v).

(i) If $x-1>0$: then $-(x-1)<0$ cannot appear in either sequence (suppose it appears in the j-th position, then $\{i,j\}\in p_\delta(\lambda-e_i)$ contradicting (A));
and similarly $-x$ cannot appear in either (else again p_δ changes between them).
It follows that x appears in $Reg(\lambda+\rho_\delta)$ and $x-1$ in the corresponding position in $Reg(\lambda+\rho_\delta-e_i)$; and that these sequences otherwise agree.

Suppose then that x is, say, the l-th smallest magnitude entry in $Reg(\lambda+\rho_\delta)$. If there is a smaller magnitude entry it's magnitude is smaller than $x-1$, by the argument following Equations(13.6) and the argument above. Since all these other entries are the same for the other sequence, $x-1$ is the l-th smallest magnitude entry in $Reg(\lambda+\rho_\delta-e_i)$. Thus o_δ is unchanged.

(ii) If $x=1$: then we have

$$\lambda+\rho_\delta \sim (...,w>1,\underbrace{x=1}_{i},y<0,...), \qquad \lambda+\rho_\delta-e_i \sim (...,w>1,\underbrace{x-1=0}_{i},y<0,...)$$

We note that $-x=-1$ still cannot appear in either (else p_δ changes). Thus 1 in $Reg(\lambda+\rho_\delta)$, respectively 0 in $Reg(\lambda+\rho_\delta-e_i)$, is the smallest magnitude entry. If there are an even number of other positive entries then this entry does not contribute to o_δ in either case (in the former by the toggle rule, and in the latter by the definition of the o-map). If there are an odd number of other positive entries then this entry contributes to o_δ in both cases (similarly). Thus o_δ is unchanged.

(iii) If $x=0$: then we have

$$\lambda+\rho_\delta \sim (...,w>0,\underbrace{x=0}_{i},y<-1,...), \qquad \lambda+\rho_\delta-e_i \sim (...,w>0,\underbrace{x-1=-1}_{i},y<-1,...)$$

and this time that $-(x-1)=1$ cannot appear in either. Thus 0 in $Reg(\lambda+\rho_\delta)$, respectively -1 in $Reg(\lambda+\rho_\delta-e_i)$, is the smallest magnitude entry. If there are an even number of strictly positive entries then this entry does not contribute to o_δ in either case. If there are an odd number of positive entries then this entry contributes an element 1 to o_δ in former cases (by the definition of the o-map); the entry -1 does not contribute in the latter case, but there is an element 1 by the toggle rule. Thus o_δ is unchanged.

(iv) If $x=1/2$: then we have

$$\lambda+\rho_\delta \sim (...,w>0,\underbrace{x=1/2}_{i},y<-1,...), \qquad \lambda+\rho_\delta-e_i \sim (...,w>0,\underbrace{x-1=-1/2}_{i},y<-1,...)$$

Evidently there is no -1/2 in the former or 1/2 in the latter, so the terms in the i-th position are the smallest magnitude terms in their respective sequence, with all else equal. Again by the toggle rule o_δ is unchanged.

(v) If $x < 0$: then we have

$$\lambda + \rho_\delta \sim (..., w, \underbrace{x}_{i}, y, ...), \qquad \lambda + \rho_\delta - e_i \sim (..., w, \underbrace{x-1}_{i}, y, ...)$$

Neither $-x$ nor $-(x-1)$ can appear in either sequence (else (A) is violated much as before). The argument is then much as in (i).

(II) For $\lambda - e_i - e_{i'}$ δ-balanced with λ we would have (for $x \geq 1$)

$$\lambda + \rho_\delta - e_i - e_{i'} \sim (..., \underbrace{x-1}_{i}, ..., \underbrace{-x}_{i'}, ...)$$

But this requires $-x+1$ in the i'-position in $\lambda + \rho_\delta$, and this is already disallowed under hypothesis (A).
(The case $x = 1/2$ does not arise; and the cases $x \leq 0$ are similar to the above, with the order of i, i' reversed.)

(III) By the rules of balance $e_{i'}$ cannot be in the same row as e_i, so $(\lambda + \rho_\delta + e_{i'})_i = (\lambda + \rho_\delta)_i = x$. This would require that in the balance partner $(\lambda + \rho_\delta - e_i)_{i'} = -x$, but this is already disallowed under hypothesis (A).

(B) If $p_\delta(\lambda) \neq p_\delta(\lambda - e_i)$:

(I) Write x for $(\lambda + \rho_\delta)_i$. Then

$$\lambda + \rho_\delta \sim (..., \underbrace{x}_{i}, ...), \qquad \lambda + \rho_\delta - e_i \sim (..., \underbrace{x-1}_{i}, ...)$$

from which we see firstly that $-x$ occurs in $\lambda + \rho_\delta$ and $1 - x$ occurs in $\lambda + \rho_\delta - e_i$ (if neither occurs then p_δ does not change between them; if only one occurs then s_δ changes);
of course it follows immediately that $1 - x, -x$ occur (and are adjacent) in both;
secondly, by the same argument as above $x - 1$ does not occur in $\lambda + \rho_\delta$.

In computing o_δ we discount the $\pm x$ pair in $\lambda + \rho_\delta$ and the $\pm(x-1)$ pair in $\lambda + \rho_\delta - e_i$. The discrepancy is thus now a $1 - x$ in $\lambda + \rho_\delta$ compared to a $-x$ in $\lambda + \rho_\delta - e_i$. But if $1 - x$ is the l-th largest magnitude entry in $\lambda + \rho_\delta$ then $-x$ is the l-th largest magnitude entry in $\lambda + \rho_\delta - e_i$, with all else equal, so o_δ is unchanged.

(II) For $\lambda + \rho_\delta - e_i - e_{i'}$ δ-balanced with λ we would have

$$\lambda + \rho_\delta - e_i - e_{i'} \sim (..., \underbrace{x-1}_{i}, ..., \underbrace{-x}_{i'}, ...)$$

This requires $-x + 1$ in the i'-position in $\lambda + \rho_\delta$ as before. Although this is not disallowed here, it forces the $-x$ to lie in the next (that is, the $i' + 1$) position. This would force a second $-x$ in the same position in $\lambda + \rho_\delta - e_i - e_{i'}$, which would thus not be descending — a contradiction.

(III) Since $(\lambda + \rho_\delta - e_i)_i = x - 1$ we would require $(\lambda + \rho_\delta + e_{i'})_{i'} = 1 - x$ for balance. Thus $(\lambda + \rho_\delta)_{i'} = -x$. But we have already seen that $\lambda + \rho_\delta$ contains both $1 - x, -x$, so this would require $\lambda + \rho_\delta + e_{i'}$ containing $1 - x$ in two positions — a contradiction.

□

13.6.2 The Reflection Lemmas

(**13.6.12**) Lemma. Fix δ and suppose $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ as before. Suppose λ has an edge down labelled α; and let w be the product of commuting reflections such that $we_\delta(\lambda) = e_\delta(\alpha\lambda)$, as in Lemma (13.4.5). Then
(I) $we_\delta(\lambda - e_i)$ is dominant;
(II) $we_\delta(\lambda - e_i) = e_\delta(\alpha(\lambda - e_i))$;
(III) $\alpha(\lambda - e_i) \overset{\lhd}{\rhd} \alpha\lambda$.

Proof. (I) We split into two cases:
If e_i does not intersect $\lambda/\alpha\lambda$ then $we_\delta(\lambda - e_i)$ is the same as $we_\delta(\lambda)$ everywhere except in row i: $we_\delta(\lambda - e_i) = we_\delta(\lambda) - e_i$. Since $\lambda - e_i$ is dominant, $\lambda_i > \lambda_{i+1}$, but $(\alpha\lambda)_i = \lambda_i$ in this case, and $(\alpha\lambda)_{i+1} \leq \lambda_{i+1}$, so $(\alpha\lambda)_i > (\alpha\lambda)_{i+1}$, so $\alpha\lambda - e_i$ is dominant, so $e_\delta(\alpha\lambda - e_i) = we_\delta(\lambda - e_i)$ is dominant.
If e_i intersects $\lambda/\alpha\lambda$ then $\pi_\alpha(e_i)$ is addable to $\alpha\lambda$ as noted in (13.6.5). That is $e_\delta(\alpha\lambda + \pi_\alpha(e_i)) = we_\delta(\lambda - e_i)$ is dominant.

(II) Firstly note that $o_\delta(\lambda - e_i) = o_\delta(\lambda)$ by Lemma 13.6.11(I), so $\alpha(\lambda - e_i)$ makes sense. Similarly we have

$$o_\delta(\alpha(\lambda - e_i)) = o_\delta(\alpha\lambda)$$

(since both are equal to the formal set $\alpha o_\delta(\lambda)$ as defined via (13.3.2), which obeys

$$\alpha o_\delta(\mu) = o_\delta(\alpha\mu)$$

by construction whenever $\alpha\mu$ makes sense).

...WHAT/WHY IS THIS???

Since $we_\delta(\lambda - e_i)$ is dominant (by (I)) in the $\mathcal{D}$-orbit of $\lambda - e_i$ there is some $\mu \in [\lambda - e_i]_\delta$ such that $we_\delta(\lambda - e_i) = e_\delta(\mu)$. Since it is adjacent to $e_\delta(\alpha\lambda)$ and has the same singularity, then by Lemma (13.6.11) (applied appropriately) $o_\delta(\mu) = o_\delta(\alpha\lambda)$. That is, $\mu = \alpha(\lambda - e_i)$.

(III) Follows immediately from (II). □

(**13.6.13**) Lemma. Fix δ. Suppose $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ as before, and $\alpha\lambda/\lambda$ is MiBS (i.e. α is an edge up from λ). Then there is a reflection group element w such that $w.\lambda = \alpha\lambda$ (so $w.\alpha\lambda = \lambda$) and $w.(\lambda - e_i)$ is dominant; whereupon $w.(\lambda - e_i) = \alpha(\lambda - e_i)$.

Proof. Suppose $w.(\lambda - e_i)$ is dominant. Then it is $\mu \in [\lambda - e_i]_\delta$ adjacent to $w.\lambda = \alpha\lambda$ with the same singularity, hence the same o_δ by Lemma (13.6.11). Thus it is enough to show that $w.(\lambda - e_i)$ is dominant.

Given that $w.\lambda$ is dominant, any *failure* of dominance of $w.(\lambda - e_i)$ must involve the i-th row itself being shorter than row-$i+1$ in $w.(\lambda - e_i)$ (i.e. row-$i+1$ intersects the MiBS); or a row with which row-i is paired in w (j, say) being longer than row-$j-1$ in $w.(\lambda - e_i)$. We must consider the cases: (A) e_i lies 'behind' the skew (i.e. it's image under the π-rotation π_α that fixes $\alpha\lambda/\lambda$ extends some row of the skew); or (B) not.

(A) In this case the failure would have to be that the image of e_i under the π-rotation broke dominance, i.e. extended beyond the row above it.

Suppose e_i is behind other than the last row of the skew. Then there is a box of the skew immediately to its right and one immediately below it. The π-rotation images of these are behind and above the image of e_i, so $w.(\lambda - e_i)$ is dominant.

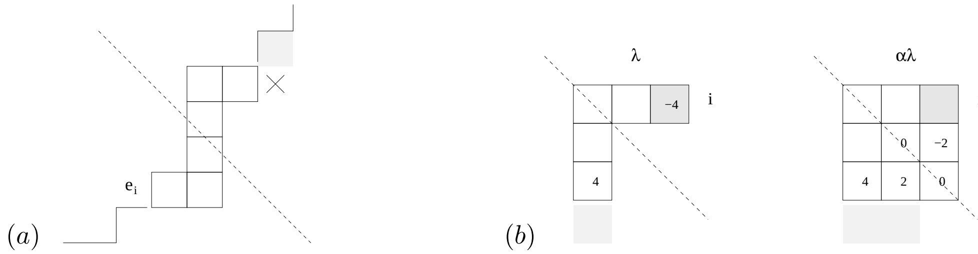


Figure 13.9: (a), (b)

On the other hand, suppose e_i is behind the last row of the skew. For example see Figure 13.9(a) (the box $\pi_\alpha(e_i)$ is marked $\times$). Here $w.(\lambda - e_i)$ is dominant unless the box above $\pi_\alpha(e_i)$ is missing from λ. But if this is missing then this row and the i-row are a singular pair in $\lambda - e_i$. Neither row can be in a singular pair in λ so this contradicts the hypothesis.

(B) If the i-th row is not moved by w then the failure would have to be that the skew $\alpha\lambda/\lambda$ includes a box directly under e_i. But in that case a δ-balanced box to e_i given by $\pi_\alpha(e_i)$ is directly to the left of the skew, and we have a setup something like Figure 13.9(b) (the δ-balanced box is the box marked 4). If there is no box below the $\pi_\alpha(e_i)$ in λ then row-i is not in a singular pair in λ, and row-i and the row containing the $\pi_\alpha(e_i)$ are a singular pair in $\lambda - e_i$, thus $s_\delta(\lambda) \neq s_\delta(\lambda - e_i)$ so we can exclude this. If there is a box below the $\pi_\alpha(e_i)$ in λ then this row and row-i are a singular pair in λ, and row-i and the row containing the $\pi_\alpha(e_i)$ are a singular pair in $\lambda - e_i$. In this case, a w which also has a factor acting on the i-th and undrawn row has the same effect on λ as one which does not. Its effect on $\lambda - e_i$ is to restore the box e_i and to add a box in the undrawn row. This $w.(\lambda - e_i)$ is dominant since the added box is under a box added in the original skew.

□

13.6.3 The Embedding Theorem

Since the block graph is connected we may use Lemmas 13.6.12 and 13.6.13 to show:

(**13.6.14**) THEOREM. (Embedding Theorem) If $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then $G_\delta(\lambda)$ is adjacent to $G_\delta(\lambda - e_i)$. □

13.7 The Decomposition Matrix Theorem

When δ is fixed, we may use the abbreviation: $P_\lambda := P_n^\delta(\lambda^T)$, and similarly for Δ_λ.

(**13.7.1**) THEOREM. For each $\delta \in \mathbb{Z}$ and $\lambda \in \Lambda$, the hypercube $h_\delta(\lambda)$ (as defined in (13.5.6)) gives the λ-th row of the (δ, λ)-block of the matrix D of Δ-filtration multiplicities of indecomposable projective modules for $B_n(\delta)$ over $\mathbb{C}$ (any n). That is

$$(P_n^\delta(\lambda) : \Delta_n^\delta(\mu)) = h_\delta(\lambda)_\mu$$

for all $n \geq |\lambda|$; or equivalently

$$P_n^\delta(\lambda) = \;+_\mu \, h_\delta(\lambda)_\mu \; \Delta_n^\delta(\mu)$$

(Recall we omit $\lambda = \emptyset$ in case $\delta = 0$.)

(**13.7.2**) REMARK. (With the $\delta = 0$ caveat Specht and standard modules coincide and we may interpret the above either as Specht characters, as required for the Cartan decomposition matrix; or as multiplicities in standard filtrations.)

Proof. We prove for a fixed but arbitrary δ, working by induction on n. The base cases are $n = 0, 1$, which are trivial (and $n = 2$ for $\delta = 0$, which is straightforward). We assume the theorem holds up to level $n - 1$, and consider $\lambda \vdash n$ (for $|\lambda| < n$ the result holds by (12.4.16) and the inductive assumption).

The λ-th row of D encodes the standard content of projective module P_λ. We apply the induction functor to a suitable $P_{\lambda - e_i}$ in level $n - 1$ (known by the inductive assumption), and use Prop.(12.4.18):

$$\text{Ind } P_{\lambda - e_i} \cong P_\lambda \bigoplus ...$$

Thus the main challenge is to determine when the projection of the new projective onto a given block splits. In general this can be complicated, but we will show that there is always a choice of $\lambda - e_i$ which makes it tractable.

Note that if λ is at the bottom of its block then the claim is trivially true. If λ is not at the bottom of its block then the binary sequence $b_\delta(\lambda)$ has at least one 01 (or initial 11) subsequence. Thus we can choose e_i to be a removable box from the skew associated to the corresponding edge α of $h_\delta(\lambda)$. (We sometimes write $\mu = \alpha\lambda$ for the partition at the other end of this edge, so the skew is $\lambda/\mu = \lambda/\alpha\lambda$.)

Note that this skew is a minimal δ-balanced skew, by (13.5.8).

The next step depends on the form of the skew: whether the skew is of form (1)+(1), or (2^2), or (2^4), or otherwise.

13.7.1 Combinatorial Lemmas

(**13.7.3**) We will say that a skew of form $\lambda/\alpha\lambda$ is *boxy* if every box in it lies within a (2^2)-shape that also lies within the skew. In our case, these are the skews in which the pair of rims fully overlap (i.e. run side-by-side). Thus in our case boxy skews have a terminal (2^2)-shape at each end, in which the largest magnitude charges reside. Note that since no (2^2)-shape has a removable box of largest magnitude charge, neither does a boxy skew (on the other hand every such shape has a removable box of next-largest magnitude, and one can see that the largest of these is removable at one end of the boxy skew or the other).

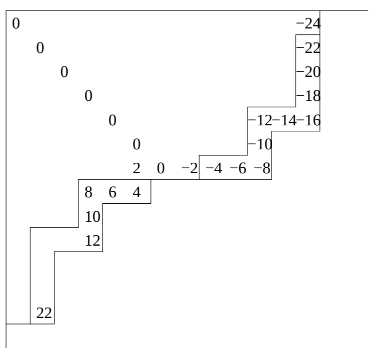
If a minimal skew is neither of form (1)+(1) nor boxy we shall say that it is generic.

(**13.7.4**) Lemma. Let $\lambda/\mu = \lambda \setminus \alpha\lambda$ be a minimal δ-balanced skew. Then there are a pair of boxes in the skew of greatest magnitude charge. In case the skew is of shape (1)+(1) both of these are removable; in the boxy cases (such as (2^2)) neither are removable (but precisely one of the next-largest is removable); and otherwise precisely one of them is removable.

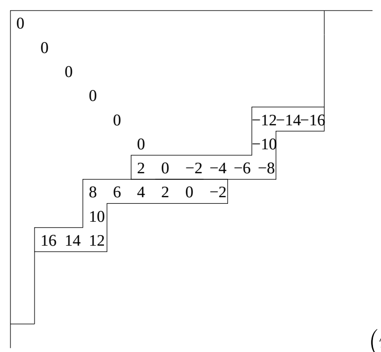
Proof. All statements are (by now) clear except the last. For this note that if both were removable this would contradict that $\alpha\lambda$ is MBS, since removing just this pair from λ would give a larger BS; while if neither were removable then again this would contradict the MBS property, since removing the complement (i.e. the boxes in $\lambda \setminus \alpha\lambda$ *not* in this pair) would give a larger BS. □

(**13.7.5**) In either (1)+(1) or general case we call a removable box of largest magnitude charge (in the given skew) a *rim-end removable box*. (Since the skew is a (possibly touching) pair of rims, and this box lies at one of the outer ends.)
In the boxy cases we shall call the removable box of next-largest magnitude charge the *rim-end removable box*.

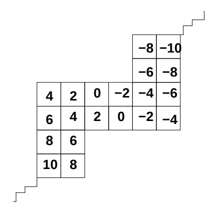
(**13.7.6**) Examples of minimal skews:



(*i*)



(*ii*)



(*iii*)

The rim-end removable boxes (as labelled by charge) are (i) 22; (ii) -16; (iii) 8.
(For $\delta = 1$ example (i) is, in greater detail,

$$\lambda + \rho_1 = (25/2, 23/2, 21/2, 19/2, 17/2, 11/2, 9/2, -3/2, -9/2, -11/2, -17/2, -19/2, -21/2, ...)$$

which is five-fold singular (in the sense of (13.3.6)), giving $o_1(\lambda) = \{2, 3\}$ for its valley set.)
[1]

[1] OLD:

(**13.7.7**) Lemma. Let e_i be a rim-end removable box of λ. If partitions λ and $\lambda - e_i$ have the same singularity then $o_\delta(\lambda) = o_\delta(\lambda - e_i)$.

Proof. It is clear that for weights differing by one box, if the singularity is unchanged then either p_δ is unchanged, or one pair is swapped for another.
(I) Suppose $p_\delta(\lambda) = p_\delta(\lambda - e_i)$. To verify this case condider moving between λ and $\lambda - e_i$ via $\lambda + \kappa e_i$, $0 \le \kappa \le 1$. This journey is too short for any hyperplane to be crossed (by the strongly descending property), and the same one cannot be left and joined by changing a single term in the sequence.
(II) Suppose $p_\delta(\lambda) \ne p_\delta(\lambda - e_i)$. To see this case note that the conditions imply $\lambda \sim (..., x+1, x, ..., -x)$ and $\lambda - e_i \sim (..., x+1, x, ..., -x-1)$, or similar. A different pair is discarded in each case by the *Reg* part of the o_δ map, but since the singletons retained are adjacent they pass to the same term in the magnitude ordering o. □

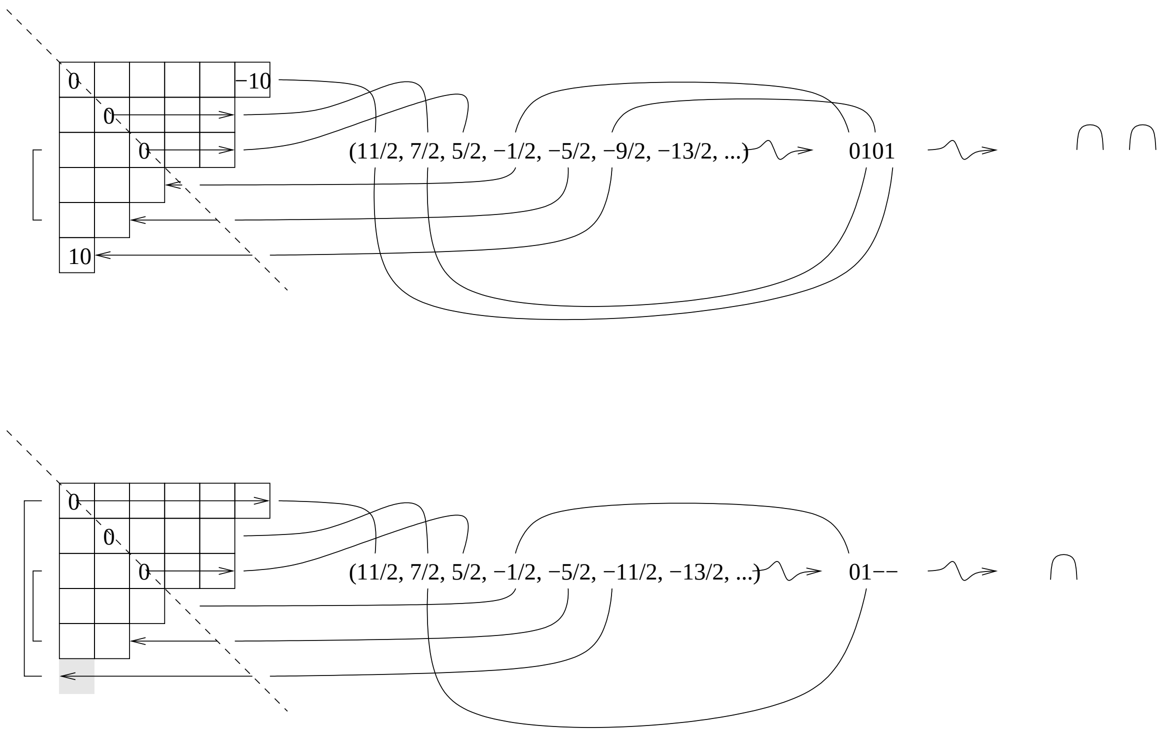


Figure 13.10: Young diagram, (case $\delta = 1$) descending sequence, binary sequence and TL diagram for two cases, λ, $\lambda - e_i$, illustrating a step up in singularity.

(**13.7.8**) Lemma 13.7.7 says that if the partitions $\lambda, \lambda - e_i$ have the same singularity then they pass to the same point on the block graph G_e, and hence are associated to the same hypercube; that is

$$h_\delta(\lambda) \cong h_\delta(\lambda - e_i)$$

with the only difference being the weight associated to each vertex in the hypercube when invoked as $h_\delta(\lambda)$ or $h_\delta(\lambda - e_i)$.

Altogether, then, this isomorphism allows us to match up each partition in the λ hypercube with the corresponding partition in the $\lambda - e_i$ hypercube. (Evidently

— OR RATHER BY ONE OF THE NEW LEMMAS??? —

these will always be adjacent partitions, although which is the larger will depend on the position in the hypercube.) Fixing λ, let us write $(\mu, f_i(\mu))$ for the μ-th such pairing. That is

$$f_i(\lambda) = \lambda - e_i$$

and so on. Thus

$$h_\delta(\lambda)_\mu = h_\delta(\lambda - e_i)_{f_i(\mu)}$$

13.7.2 The singularity lemma

(**13.7.9**) PROPOSITION. *Fix δ, and hence an identification between valley sequences and partitions. Pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda \setminus \alpha\lambda$. Then*
(I) the singularity of $\lambda - e_i$ is given by

$$s_\delta(\lambda - e_i) = \begin{cases} s_\delta(\lambda) + 1 & \text{if } |\lambda \setminus \alpha\lambda| = 2 \\ s_\delta(\lambda) & o/w \end{cases}$$

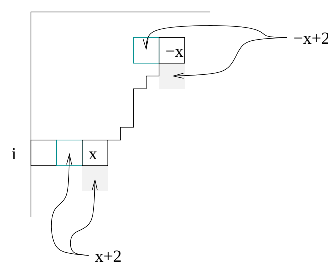
(II) In the cases in which the skew is not of form $(1)+(1)$ the valley set

$$o_\delta(\lambda - e_i) = o_\delta(\lambda)$$

thus

$${\Gamma_\delta}^{\lambda - e_i} = {\Gamma_\delta}^{\lambda}$$

Proof:
Case $|\lambda \setminus \alpha\lambda| = 2$: This is the $(1)+(1)$ (or (1^2)) case, and the charges in the boxes are (say) x and $-x$. Removing x (from row i) we get a row ending in charge $x+2$, giving $(\lambda - e_i + \rho_\delta)_i = -\frac{x+2}{2} + \frac{1}{2} = -\frac{x+1}{2}$ (by Lemma 13.6.7). The row ending in $-x$ has $(\lambda - e_i + \rho_\delta)_j = -\frac{-x}{2} + \frac{1}{2} = \frac{x+1}{2}$ thus these two rows are now a singular pair. Neither row was in a singular pair with some other row before (for any such partner would lie immediately below the other box, i.e. in one of the shaded positions as shown here:

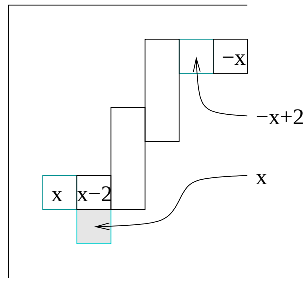


and so would prevent removability), so $s \rightsquigarrow s+1$ and we are done.

Figure 13.10 gives an example.

Other cases: Every other case is one of the following:
(i) the upper end of a rim in the skew ends in a row of length greater than 1 (as in Example 13.7.6(ii));
(ii) the lower end ends in a column of length greater than 1 (as in Example 13.7.6(i));
(iii) the skew is boxy (as in Example 13.7.6(iii)).

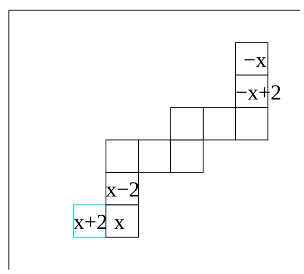
(i) If the upper end of a rim ends in a row (of length greater than 1), such as the upper rim in Example 13.7.6(ii), which ends in -16, then the end box of this row (with charge $-x$ say, in row i) is removable, but its balance partner in the skew is not (else the skew is not minimal). It follows that singularity is unchanged on removing the end-box $-x$ in row i, since this row becoming part of a singular pair while ending in $-x+2$ would imply a pair partner row ending in x, already present in λ. Such an x cannot be the balance partner of the original $-x$ since it must lie at the end of a row, but then it would lie SE of the x in the skew, as here:



— a contradiction. (One easily checks, for example using the picture above again, that λ cannot have a row ending in $x+2$, so no singular pair is *lost* in removing $-x$.)

Note here that $o_\delta(\lambda - e_i) = o_\delta(\lambda)$ as required for (II), indeed we remain in the same facet.

(ii) If the lower end of a rim ends in a column (of length greater than 1), such as the lower rim in Example 13.7.6(i), which ends in 22, then the end-box of this column (charge x, say) is removable. After removal, the new end-box charge is $x+2$. In λ the pair of rows ending in x and $-x+2$ (the box below the balance partner $-x$ to x) as here:

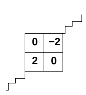


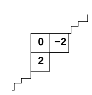
are a singular pair. In $\lambda - e_i$ the x is lost and hence so is this singular pair, but the row now ends in $x+2$, which forms a new singular pair with the $-x$. So this time $\lambda - e_i$ lies on a different set of hyperplanes to λ, but *overall* singularity is unchanged.

Since singular pairs of rows are ignored in computing $o_\delta(\lambda)$, the valley sequence is also still unchanged, as required for (II).
(In the particular example the change from $(\lambda + \rho_\delta)_i$ is $-21/2 \to -23/2$.)

(iii) For the remaining cases — the boxy cases — there are a couple of analogous variations. The largest magnitude charge removable box is either at the end of a row (the lower row at the upper end of the skew); or at the end of a column at the lower end (as in Example 13.7.6(iii)). The proofs are also analogous, and we omit them (we do, however, give a couple of representative examples below).

In the case (2^2) itself we have

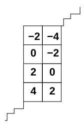


$\mapsto (..., 3/2, 1/2, \leq -5/2, ...) \quad \rightsquigarrow$


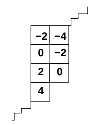
$\mapsto (..., 3/2, -1/2, \leq -5/2, ...)$

which shows that the singularity does not change. By the toggle rule these both (as far as shown) have valley set $\{1, 2\}$.

Examples: In the case (2^4) we have

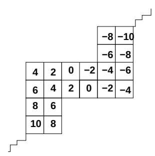


$\mapsto (..., 5/2, 3/2, 1/2, -1/2, \leq -7/2, ...) \quad \rightsquigarrow$



$\mapsto (..., 5/2, 3/2, 1/2, -3/2, \leq -7/2, ...)$

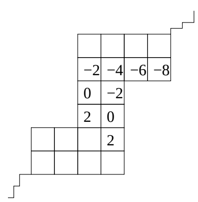
which shows that the singularity does not change, and the valley set does not change (as far as shown it is $\{1, 2\}$ again), although the wall does. In the case (3^2) we have (similarly embedded, in general) $(3^2) \mapsto (..., 2, 1, \leq -3, ...) \rightsquigarrow (32) \mapsto (..., 2, 0, \leq -3, ...)$ which has the same singularity and valley set (and wall set). A more typical boxy skew is



$\mapsto (..., 11/2, 9/2, 7/2, 5/2, -5/2, -7/2, \leq -13/2, ...)$

Removing the removable 8 here changes $-7/2 \to -9/2$, giving the same singularity (different wall) and once again the same valley set.

A final example with no change in singularity (on removing the -8 box):



$\mapsto (..., 11/2, \underbrace{9/2}_{\rightsquigarrow 7/2}, 3/2, 1/2, -1/2, -3/2, \leq -13/2, ...)$

As noted, the general argument is much the same as for the generic cases. $\square$

(**13.7.10**) REMARKS: In the simplest case this proposition looks like regular (or relatively regular) alcove geometry. If the 'distance' (more properly the skew) between a balanced pair of weights is minimal (a rank 2 skew), then the step off one of them (λ say), towards the other, must lie on the reflection wall. While if they are further apart, a single step away from λ towards the other will lie in the same alcove as λ (or at least it will be possible to take a step in the same facet as λ).

However we see that in general the proposition deals with more complex 'singularity non-changing' cases, in which the single step *does not* stay in the same facet.

13.7.3 Back to the representation theory

(**13.7.11**) LEMMA. Fix δ. No distinct pair $\lambda, \lambda - e_i + e_j \in \Lambda$ are in the same block. That is, no distinct pair $\lambda + e_i, \lambda + e_j \in \Lambda$ are in the same block.

Proof. Such a pair cannot meet the charge-pair form of the balance condition (see Defn.4.7/Cor.4.8 in [13]), since each of the skews involved has rank 1. □

(**13.7.12**) LEMMA. If $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ then for all pairs $(\mu, f_i(\mu)) \in [\lambda]_\delta \times [\lambda - e_i]_\delta$

$$\mathrm{Proj}_\lambda \mathrm{Ind}\ \Delta_n(f_i(\mu))' = \Delta_{n+1}(\mu)'$$

$$\mathrm{Proj}_{f_i(\lambda)} \mathrm{Ind}\ \Delta_n(\mu)' = \Delta_{n+1}(f_i(\mu))' \tag{13.7}$$

Proof. Note that the pair $(\mu, f_i(\mu))$ are adjacent by Theorem 13.6.14. For any ν, Prop.12.3.21 gives

$$\mathrm{Ind}\ \Delta(\nu)' = \left(+_j \Delta(\nu + e_j)'\right) + \left(+_k \Delta(\nu - e_k)'\right)$$

in the notation of §12.4.2. For $\nu = f_i(\mu)$ adjacent to μ, one of these summands is $\Delta(\mu)'$. Specifically either (i) $\mu = \nu + e_l$ (some l); or (ii) $\mu = \nu - e_l$ (some l).

In case (i) other summands are of form $\mu - e_l + e_j$, $\mu - e_l - e_k$. By Lemma (13.7.11) the former are not in $[\mu]_\delta$, and since $s_\delta(\lambda) = s_\delta(\lambda - e_i)$ we may use Lemma (13.6.11)(II) (or Lemma 13.6.9) to exclude the latter. The other case is similar. □

13.7.4 The first inductive-step lemma

(**13.7.13**) PROPOSITION. *Fix δ. For $\lambda \in \Lambda$ not at the bottom of its block, pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in skew $\lambda/\alpha\lambda$. In the cases in which the skew is neither* $(1)+(1)$ *nor* (1^2)*, this data fixes a block-graph isomorphism $f_i : [\lambda]_\delta \to [\lambda - e_i]_\delta$ as defined in (13.6.4) and we have the following.*
(i) The Δ-decomposition pattern for P_λ is the translate *of that for $P_{\lambda - e_i}$:*

$$(P_\lambda : \Delta_\mu) = (P_{\lambda-e_i} : \Delta_{f_i(\mu)}) \qquad \forall \mu \in [\lambda]_\delta$$

(ii) This verifies the inductive step for the main theorem in such cases. That is, $h_\delta(\lambda) \cong h_\delta(\lambda - e_i)$.

FOR WHICH BITS OF THIS ARE THE INDUCTIVE ASSUMPTION NEEDED?

Proof: Formally f_i can be considered to be defined for any two blocks that pass close to each other at some point. Here though (using the Singularity Lemma 13.7.9), the particular nice behaviour of the function f_i is given by the Embedding Theorem 13.6.14.

Consider the 'translation'

$$\mathrm{Proj}_\lambda \mathrm{Ind}\, P_{\lambda-e_i} \;=\; P_\lambda \oplus Q$$

($Q = \mathrm{Proj}_\lambda Q$ some projective, possibly zero), which identity follows from Prop. 12.4.18. In the cases under consideration (skew neither $(1)+(1)$ nor (1^2)) the singularities of λ and $\lambda - e_i$ are the same by (13.7.9). Thus each Δ-module occuring in $P_{\lambda-e_i}$ induces precisely one Δ-module after projection onto the block of λ, by Lemma **??**. Thus the standard content of $P_\lambda \oplus Q$ is equinumerate to that of $P_{\lambda-e_i}$. More specifically if

$$P_{\lambda-e_i} \;=\; +_\mu\, c_\mu\, \Delta_{f_i(\mu)}$$

(for some multiplicities c_μ); then

$$P_\lambda \oplus Q = \; +_\mu\, c_\mu\, \Delta_\mu$$

The same is true on inducing again and projecting back to the block of $\lambda - e_i$, by 13.7: In

$$\mathrm{Proj}_{\lambda-e_i} \mathrm{Ind}\, (P_\lambda \oplus Q) \;=\; P_{\lambda-e_i} \oplus ...$$

each standard module occuring in $(P_\lambda + Q)$ induces precisely one standard module after projection onto the block of $\lambda - e_i$. It follows that this second 'translation' results in a copy of $P_{\lambda-e_i}$. Thus the first translation cannot split, and hence is precisely P_λ — with the same decomposition pattern.

For the last part use (13.7.8).

($\square$)

13.7.5 The second inductive-step lemma

The remaining cases needed to move between level $n-1$ and n are those manifesting skews of rank 2, i.e. of form (1)+(1).

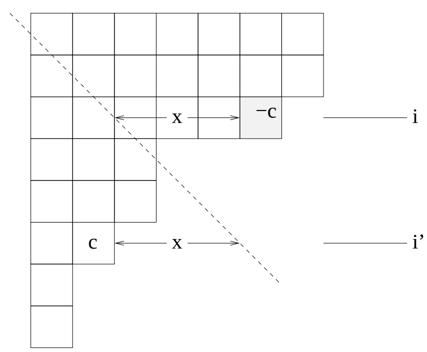
(**13.7.14**) PROPOSITION. *Fix δ. Pick $\alpha \in \Gamma_{\delta,\lambda}$ and let e_i be a rim-end removable box in $\lambda/\alpha\lambda$. Then in the cases in which the skew is of form $(1)+(1)$ we have the following.*
(Ia) The sequence $b_\delta(\lambda) = \overset{\alpha}{\vee} b_\delta(\lambda - e_i)$ (as defined in (13.5.9)).
(Ib) The hypercube

$$h_\delta(\lambda) \;=\; (1,\alpha)\bigvee^{\alpha} h_\delta(\lambda - e_i)$$

(i.e. $h_\delta(\lambda)$ has increased 'dimension' by +1 compared to $h_\delta(\lambda - e_i)$).
(Ic) The sequence $b_\delta(\alpha\lambda) = \overset{\alpha}{\vee}_{-}\, b_\delta(\lambda - e_i)$ (i.e. $b_\delta(\alpha\lambda)$ differs from $b_\delta(\lambda - e_i)$ by insertion of subsequence 10 in the α position).
(II) Given the inductive hypothesis $(P_{\lambda-e_i} : \Delta_\nu) = h_\delta(\lambda - e_i)_\nu$ for all ν, then the standard decomposition pattern for P_λ is in agreement with (I), in the sense of the equality in the main theorem: we have $(P_\lambda : \Delta_\mu) = h_\delta(\lambda)_\mu$ (for all μ).

Proof: (Ia) As shown in the proof of Prop. 13.7.9 (or see below), removing e_i makes that row part of a singular pair with the row containing the box with opposite charge. Thus $b_\delta(\lambda - e_i)$ differs from $b_\delta(\lambda)$ in that a pair which contributed an 01 sequence in the latter does not contribute to the valley sequence in the former — i.e. $b_\delta(\lambda - e_i)$ differs by the removal of this 01 sequence. (Figure 13.10 serves as an example here.) It remains to confirm the *position* of the removal.

The relevant properties of λ as appearing in this Proposition (i.e. in case $\lambda/\alpha\lambda$ of rank 2) are well illustrated by the following example:



In general we have (for some $x \geq 0$):

$$\lambda + \rho_\delta \sim (\ldots, \underbrace{\underbrace{x+1}_{i}, \ldots, -x}_{01}, \ldots)$$

Altogether the bracketed pair contribute an 01 in the binary sequence $b_\delta(\lambda)$ as indicated. The $x+1$ lies at some position l, say, in the magnitude order (as defined in (13.3.7)) of singleton terms in $\lambda + \rho_\delta$. The precise value of l, which determines the position of the 1 in the 01 in $b_\delta(\lambda)$, depends on the rest of λ. However, by definition of the α action this is $l = \alpha + 1$, as we will now show. Confer

$$\lambda - e_i + \rho_\delta \sim (..., \underbrace{x}_{i}, ..., -x, ...)$$

Here the $x, -x$ are a singular pair, so do not appear in the magnitude order — to obtain its binary representation from that of λ one deletes the binary pair in the $l, l-1$ position. That is

$$b_\delta(\lambda) = \bigvee^{l-1} b_\delta(\lambda - e_i)$$

Finally

$$\alpha\lambda + \rho_\delta \;=\; \lambda - e_i - e_{i'} + \rho_\delta \;\sim (..., \underbrace{\underbrace{x}_{i}, ..., -x-1}_{10}, ...)$$

The α action on λ here manifests (by definition (13.5.9)) as $10 \leftrightarrow 01$ in the $\alpha, \alpha+1$ position of $b_\delta(\lambda)$. Comparing with our formulation above we see that $\alpha = l - 1$ as claimed.

Noting (13.4), the assertions (Ib,c) now follow directly. **??**!

FOR LATER:
What happens for other $\mu \in h_\delta(\lambda - e_i)$ and the pair $\mu+, \mu-$ of vertices in $h_\delta(\lambda)$ that can be considered to be engendered by μ? The binary representation of μ is related to $\lambda - e_i$ by some subsequences 01 being replaced by 10 (or 11 to 00), and, by our earlier remarks **??**, μ comes from $\mu+$ by deleting a pair 01. The question is: which pair? We CLAIM it is the pair in the same $\alpha+1, \alpha$ position. HOW SEE THIS? Most likely this is also dealt with by (13.4). RIGHT?!?!

(II) Recall from the Δ-module branching rule Prop.(12.3.21)(ii) that under the present assumptions

$$\mathrm{Proj}_\lambda \mathrm{Ind} \Delta_{\lambda - e_i} = \Delta_\lambda + \Delta_{\lambda - e_i - e_{i'}} = \Delta_\lambda + \Delta_{\alpha\lambda} \tag{13.8}$$

This is a non-split sum, by []. In this case translating $P_{\lambda - e_i}$ away from and then back to $\lambda - e_i$ produces a projective whose dominant Δ-content is two copies of $\Delta_{\lambda - e_i}$ (one from each of the summands on the right) — hence, by (12.4.17), it contains two copies of the projective $P_{\lambda - e_i}$. On the other hand every Δ-factor of $P_{\lambda - e_i}$ engenders at most two factors in $\mathrm{Proj}_\lambda \mathrm{Ind} P_{\lambda - e_i}$, and hence at most two factors in $\mathrm{Proj}_{\lambda - e_i} \mathrm{Ind}(\mathrm{Proj}_\lambda \mathrm{Ind} P_{\lambda - e_i})$. Thus $\mathrm{Proj}_{\lambda - e_i} \mathrm{Ind}(\mathrm{Proj}_\lambda \mathrm{Ind} P_{\lambda - e_i}) = P_{\lambda - e_i} \oplus P_{\lambda - e_i}$. Since the section (13.8) in $\mathrm{Proj}_\lambda \mathrm{Ind} P_{\lambda - e_i}$ contributes to both summands and is non-split, it follows that $\mathrm{Proj}_\lambda \mathrm{Ind} P_{\lambda - e_i}$ is non-split, and hence

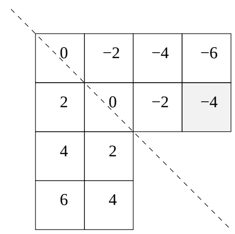
$$\mathrm{Proj}_\lambda \mathrm{Ind} P_{\lambda - e_i} = P_\lambda$$

It remains to show that $(\mathrm{Proj}_\lambda \mathrm{Ind} P_{\lambda - e_i} : \Delta_-) = h_\delta(\lambda)$. The *number* of factors is correct (double the number for the $\lambda - e_i$ hypercube). We have $(P_{\lambda - e_i} : \Delta_-) = h_\delta(\lambda - e_i)$ by the inductive assumption, and for each Δ_μ occuring in this decomposition the translation is $\Delta_\mu \rightsquigarrow \Delta_{\mu+} + \Delta_{\mu-}$ for some pair $\mu+, \mu-$ in the λ-orbit. Since $[\lambda - e_i]_\delta$ is a strictly more singular orbit than $[\lambda]_\delta$ the reflection group elements moving through $h_\delta(\lambda - e_i)$ will also serve to move the pair $\lambda, \alpha\lambda$ through these pairs $\mu+, \mu-$, thus they remain adjacent above and below μ.

For $\lambda - e_i$ itself we have seen above that $b_\delta(\lambda - e_i)$ gives $b_\delta(\lambda)$ and $b_\delta(\alpha\lambda)$ by inserting 01 (respectively 10) in the α position. For other $\mu \in h_\delta(\lambda - e_i)$, note that the relevant singular pair of rows in $\lambda - e_i$, while not contributing to the magnitude order (since they are singular) are formally permuted (in the $\mathcal{D}$-action sense) along with the rest of the rows, in the collection of reflection group actions that traverse $h_\delta(\lambda - e_i)$. Thus they (jointly) maintain a formal position in the magnitude order, between two terms that are properly consecutive in this order. The difference with μ_+, μ_- is that in these one of the pair is extended by 1, or contracted by one. Thus the singularity is broken, and the pair appear properly in the order, between the given two terms, and hence bumping up the larger of the two. Since μ is just a permutation of $\lambda - e_i$ (as far as the magnitudes are concerned), the position of the pair in the magnitude order, and hence the position of the bump in the binary representation, is the same as for $\lambda - e_i$.

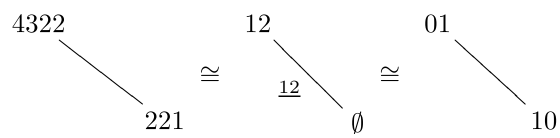
That the collection thus engendered is $h_\delta(\lambda)$ follows directly from Equation(13.4).
□

(**13.7.15**) Example: $\delta = 1$, computing for 4422 via 4322. We have



In particular $e_1(4322) = (7/2, 3/2, -1/2, -3/2, ...)$ so $o_1(4322) = \text{toggle}(\{2\}) = \{1,2\}$. By the inductive hypothesis we have

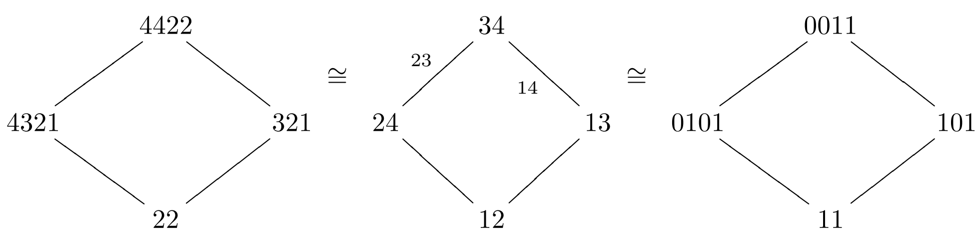
$(P_{4322} : \Delta_-) \;=\; h_1(4322) =$



where the last is the untoggled binary representation. Translating off the wall we get $4322 + 221 \to (4422 + 4321) + (321 + 22)$. In binary this corresponds to $01 \to 0**1 \to 0101 + 0011$ and $10 \to 1**0 \to 1100 + 1010$. These four sequences therefore encode the content of P_{4422}.

Meanwhile

$h_1(4422) =$



confirming the assertion of the Theorem in this case.
Note that the insertion of a binary pair in the α position, and action of α on it, modifies the labels of $h_1(4322)$ and extends it by a new generating direction (labelled by α), precisely as required to produce $h_1(4422)$.

Up to some minor variations to deal with skews containing (2^2), this completes the main inductive step for the Theorem. □

13.8 Some remarks on the block graph

The block graph G_e (or isomorphically G_o []) is an interesting thing. Here are some features.

Lemma 13.1. *Inserting sequence 01 at any point in a valley sequence $b(\lambda)$ increases the number of arcs in the TL diagram $\mathcal{T}(\lambda)$ by one.*

Proof. Note that the algorithm will remove this subsequence, and add an arc, at the first iteration; returning us to the original sequence for the next iteration.

(**13.8.1**) By Lemma **??** every vertex in G_e is the top of an ascending hypercubical subgraph, which includes all its incoming edges. We define the *dimension* of a vertex λ as the maximum number of incoming edges of any vertex in any ascending path from the root to λ.

Lemma 13.2. *The number of arcs in the TL diagram $\mathcal{T}(\lambda)$ is equal to the dimension of λ (and to $|\Gamma^\lambda|$).*

Proof. Moving up an edge we either make $00 \to 11$ at the beginning of the sequence $b(\lambda)$, or change some $10 \to 01$. It is routine to check that neither move reduces the number of arcs. If we pass to a vertex of dimension one-greater than before, each incident edge corresponds to a different pair transforming in one of these two ways. Each of these pairs separately gives rise to an arc.

Chapter 14

Example: the Temperley–Lieb algebra again

14.1 More on categories of modules

14.1.1 More fun with F and G functors

We continue here with the assumptions of §9.7.

Proposition 14.1. *Suppose that A possesses an involutive antiautomorphism that fixes e. If $N \in eAe - mod$ simple then $G(N)$ has at most one contravariant form on it. If such a form exists then its rank is the rank of the head of $G(N)$ (and this head is contravariant self-dual).*

Proof: Write a^t for the image of $a \in A$ under the antiautomorphism (so $e^t = e$). Write L for the head of $G(N)$, so that $eL \neq 0$. Thus L does not appear below the head of $G(N)$. L^o is isomorphic to L as a vector space, and the action of e on it is given by the action of e^t on L^*. Thus $eL^o \neq 0$. Thus neither L nor L^o appears anywhere except possibly in the socle of $G(N)^o$. If $L \ncong L^o$ it follows that there is no map from $G(N)$ to $G(N)^o$. If $L \cong L^o$ then $G(N)$ satisfies the assumptions of proposition 7.5.5, so we may invoke proposition 7.5.12. □

Theorem 9.6 is a powerful result. Its power is ameliorated somewhat by the failure of left-exactness in G. This motivates us to learn more about G.

Lemma 14.2. *Suppose $S_1 \overset{\psi}{\hookrightarrow} S_2$ an inclusion of left eAe-ideals, and the multiplication map associated to $G(S_1)$ is an isomorphism. Then $G(S_1) \overset{G(\psi)}{\hookrightarrow} G(S_2)$ is an injection, i.e. G behaves as if left exact.*

Proof: We have

$$\begin{array}{ccc} S_1 & \overset{\psi}{\hookrightarrow} & S_2 \\ \downarrow G & & \downarrow G \\ Ae \otimes_{eAe} S_1 & \overset{G(\psi)}{\rightarrow} & Ae \otimes_{eAe} S_2 \\ \downarrow \mu_1 & & \downarrow \mu_2 \\ AS_1 & \overset{\psi_A}{\hookrightarrow} & AS_2 \end{array}$$

Recall that $G(\psi)(ae \otimes s) = ae \otimes \psi(s) = ae \otimes s$ (keeping in mind that such a map may have a kernel, in principle — i.e. the expression $ae \otimes s$ on the right may not be identified with the $ae \otimes s$ on the left, since the one lies in $G(S_2)$ while the other lies in $G(S_1)$, and G is not left exact); and that $\mu_i(ae \otimes s) = aes$. The map ψ_A is simply the inclusion of a left A-subideal. We thus have $\psi_A(\mu_1(ae \otimes s)) = \psi_A(aes) = aes$ and $\mu_2(G(\psi)(ae \otimes s)) = \mu_2(ae \otimes s) = aes$, so that $\psi_A \circ \mu_1 = \mu_2 \circ G(\psi)$. That is, the bottom square in our diagram commutes. But if μ_1 is an isomorphism then both factors in $\psi_A \circ \mu_1$ have trivial kernel, so it is an injection, and hence so is $\mu_2 \circ G(\psi)$. Thus $G(\psi)$ is an injection. □

Suppose that S is a left sub-eAe-module of eAe (i.e. a left ideal), then there is a multiplication map

$$\begin{array}{rcl} \mu : Ae \otimes_{eAe} S & \rightarrow & AeS \\ ae \otimes s & \mapsto & aes \end{array}$$

(in the rest of this section, μ applied to a tensor product of this form will always be the appropriate multiplication map).

The surjection μ need not be an injection in general. However, suppose that there are $f, g \in eAe$ such that $S = eAef$ and $fgf = f$. (Such an f is said to satisfy the *return condition*. We call a left eAe-ideal S of form $eAef$ with f satisfying the return condition a *return ideal*.) Then there is a map $\nu : AeS \rightarrow Ae \otimes_{eAe} S$ given by

$$\nu(x) = x \otimes gf$$

so that $\mu(\nu(x)) = xgf = x$ and $\nu(\mu(a \otimes s)) = \nu(as) = as \otimes gf = a \otimes sgf = a \otimes s$. Therefore

Proposition 14.3. *If $S = eAef$ is a left ideal of eAe generated by $f \in eAe$ such that $fgf = f$ for some $g \in eAe$, then the multiplication map μ is an isomorphism*

$$G(S) \cong AeS = AS = Af$$

(NB, μ and its inverse are given explicitly). In particular the set inclusion of S in AS passes to an injection ν of S into $G(S)$. This is not an algebra-module map, but if D is a linearly independent set in S then it is linearly independent in AS and $\nu(D)$ is in $G(S)$.

Note that fg is idempotent, so

$$S = eAef \twoheadrightarrow eAefg$$

is a surjective map onto a projective eAe-module.

14.1.2 Saturated towers

stuff HERE SUPRESSED!...

14.2 On Quasi-heredity

Here A is an algebra over a field.
Lemma 14.4. *If $e \in A$ idempotent and $L \in A - mod$ simple then EITHER*

$$[A/AeA : L] \neq 0$$

OR

$$[head\ (Ae) : L] \neq 0.$$

Proof. The first inequality implies that L is not killed by the quotient, so that $eL = 0$. Every L obeying $eL = 0$ will appear in A/AeA, so the condition is also sufficient for this multiplicity to be nonzero. On the other hand, the second inequality implies $eL \neq 0$. Let e_L be an idempotent such that Ae_L is the indecomposable projective with head L. If $eL \neq 0$ then $eAe_L \neq 0$ and e_L is a primitive component of e, and Ae_L is isomorphic to a summand of Ae, so L occurs in its head. □

14.2.1 Definitions

Let A be an algebra over a field and J its radical.
Definition 14.5. An ideal S of A is a *heredity ideal* if $S^2 = S$, $SJS = 0$, and S is a projective left A-module.

By semisimplicity, every ideal of A/J is generated by an idempotent. This is of the form $f = e + J$, where $e \in A$ idempotent. Thus the image of any A-ideal S is $S + J = AeA + J$ for some e. Now $(S + J)^r = S + J^r$, and $(AeA + J)^r = AeA + J^r$, for $r = 1, 2, \ldots$. For some such r we have $J^r = 0$, so
Lemma 14.6. *Every idempotent ideal can be written in the form $S = AeA$ for some idempotent e.*

Now suppose that $S^2 = S$ is an ideal, and let e be such that $S = AeA$. Clearly $eJe = 0$ if and only if $SJS = AeAJAeA = AeJeA = 0$.
Lemma 14.7. *Let $e \in A$ be idempotent, and m the multiplication map $Ae \otimes_{eAe} eA \xrightarrow{m} AeA$.*
(I) If $(AeA)_A$ is projective then m is bijective;
(II) if $eJe = 0$ and m is bijective then $(AeA)_A$ is projective.

Proof. (I) Exercise. (II) Note that $eJe = 0$ implies *every* eAe module is projective. Since $(Ae)_{eAe}$ and $(eA)_A$ are projective, $(Ae \otimes_{eAe} eA)_A$ is projective. □

Unsurprisingly then,

Definition 14.8. An idempotent $e \in A$ is a *heredity idempotent* if
(I) eAe is a semisimple algebra
(II) the multiplication map

$$Ae \otimes_{eAe} eA \to AeA$$

is a bijection.
Definition 14.9. A sequence of idempotents in A, $(e^1, \ldots, e^l)$, is a *heredity chain* if
(I) $A = Ae^1A \supset Ae^2A \supset \ldots \supset Ae^lA$ (all proper inclusions);
(II) each e^i is a heredity idempotent modulo $Ae^{i+1}A$.
Definition 14.10. Algebra A is *quasi-hereditary* if there exists a chain of ideals

$$A = S_1 \supset S_2 \supset \ldots \supset S_l \supset S_{l+1} = 0$$

such that S_r/S_{r+1} is an heredity ideal of A/S_{r+1}.

In particular S_l is a heredity ideal of A.

An algebra is quasi-hereditary if and only if it has a heredity chain.

14.2.2 Consequences for $A - \mathbf{mod}$

Let A have heredity chain $(e^1, \ldots, e^l)$ and put $A^i = A/Ae^{i+1}A$. For each simple L_j there is a unique i such that $[\mathrm{top}(A^ie^i) : L] \neq 0$. For this i write Δ_j for the projective cover of L_j as an A^i-module (an indecomposable summand of A^ie^i.

14.3 Notes and References

Quasi-hereditary algebras were introduced by Cline, Parshall and Scott, to deal with highest weight categories (see [11], and later). The notion has turned out to be quite deep, and their definition has since been cast in a variety of ways, depending on the intended use. Here we merely review the cast(s) most useful in the context of diagram algebras. Useful sources include Dlab and Ringel [19, **?**].

14.3.1 Quasi-heredity of planar diagram algebras

stuff HERE SUPRESSED!

Chapter 15

Lie groups

15.1 Intro

King, Serre, Fulton–Harris, Murnaghan

15.2 Preliminaries

The Murnaghan convention for the Kronecker product of matrices is

$$\otimes : M_m(R) \times M_n(R) \to M_{mn}(R)$$

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots \\ a_{21}B & a_{22}B & \dots \\ \vdots & & \end{pmatrix}$$

For fixed m, n there is a matrix P such that, for all A, B,

$$A \otimes B = P(B \otimes A)P^{-1}$$

We have

$$(A_1 \otimes B_1)(A_2 \otimes B_2) \;=\; A_1A_2 \otimes B_1B_2$$

$$\det(A \otimes B) = \det(A)^n \det(B)^m$$

Example:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} aa & ab \\ ac & ad \end{pmatrix} & \begin{pmatrix} ba & bb \\ bc & bd \end{pmatrix} \\ \begin{pmatrix} ca & cb \\ cc & cd \end{pmatrix} & \begin{pmatrix} da & db \\ dc & dd \end{pmatrix} \end{pmatrix}$$

Let us write e_1, e_2 as a basis for the first factor space and similarly for the second. Thus $e_1 \otimes e_1$, $e_1 \otimes e_2 + e_2 \otimes e_1$ and $e_2 \otimes e_2$ are vectors in the product space invariant under swapping the order of the factors. Since $A \otimes A$ is invariant under swapping the order, it will fix the subspace spanned by the invariant vectors. That is

$$\begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 0 & 1 \\ & 1 & -1 & 0 \end{pmatrix} A \otimes A \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 0 & 1 \\ & 1 & -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} aa & ab & bb & 0 \\ 2ac & ad+bc & 2bd & 0 \\ cc & cd & dd & 0 \\ 0 & 0 & 0 & ad-bc \end{pmatrix} \tag{15.1}$$

where

$$\begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 0 & 1 \\ & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} e_{11} \\ e_{12} \\ e_{21} \\ e_{22} \end{pmatrix} = \begin{pmatrix} e_{11} \\ e_{12}+e_{21} \\ e_{22} \\ e_{12}-e_{21} \end{pmatrix}$$

If A is $N \times N$ we can write $e_1, e_2, ..., e_N$ for the basis of the space V (say) that it acts on. Symmetrisation of $V \otimes V$ gives $e_{11}, e_{22}, ..., e_{NN}$, $e_{12}+e_{21}, e_{13}+e_{31}, ..., e_{N-1,N}+e_{N,N-1}$ — that is, $N + \frac{N(N-1)}{2}$ vectors altogether. (And a complementary collection of vectors antisymmetric under permutation.)

Let us define $S^{(n)}(A)$ as the 'symmetrised' part of $A^{\otimes n}$, i.e. the part acting on the permutation invariant vectors in the sense of the $n=2$ case defined above (i.e. the 3x3 block in (15.1)). (Caveat: The details depend on the normalisation of the symmetrised vectors.)

15.3 Lie group

15.3.1 Example: $SU(2)$

For any N the special unitary group is the subgroup of $GL(N,\mathbb{C})$ of special (det=1) unitary ($A^\dagger A = 1$) matrices. Thus $SU(2)$ is the set of complex matrices of form

$$A = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}, \qquad \text{so} \qquad A^\dagger = \begin{pmatrix} a^* & -b \\ b^* & a \end{pmatrix},$$

with $aa^* + bb^* = 1$. This A acts on an arbitrary vector in $\mathbb{C}^2$ by $z \mapsto Az$. Consider

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = A^\dagger \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

This induces an action on functions $f(z_1, z_2) \mapsto f(z_1', z_2')$, and in particular on the space of polynomials in z_1, z_2 of fixed total degree $2n$, say (n integer or half-integer):

$$A.(\sum_{i=0}^{2n} \beta_i z_1^{2n-i} z_2^i) \;:=\; \sum_{i=0}^{2n} \beta_i (z_1')^{2n-i} (z_2')^i$$

We can use this to construct a representation of $SU(2)$ of dimension $2n+1$. We define

$$v_{(\beta_0,\beta_1,\ldots,\beta_{2n})}(z_1, z_2) = v_\beta(z) = \sum_{i=0}^{2n} \beta_i z_1^{2n-i} z_2^i$$

so that

$$A.v_\beta(z) \;=\; v_\beta(Az) = v_\beta(z')$$

and define β' by

$$v_{\beta'}(z) = v_\beta(z')$$

For example case $n = 1$ is

$$A.v_{(\beta_0,\beta_1,\beta_2)}(z_1, z_2) = v_\beta(a^* z_1 - bz_2,\; b^* z_1 + az_2,)$$

$$= \beta_0(a^* z_1 - bz_2)^2 + \beta_1(a^* z_1 - bz_2)(b^* z_1 + az_2) + \beta_2(b^* z_1 + az_2)^2 \tag{15.2}$$

$$= (\beta_0(a^*)^2 + \beta_1 a^* b^* + \beta_2 (b^*)^2) z_1^2 \;\; + (-2\beta_0 a^* b + \beta_1(aa^* - bb^*) + \beta_2 b^* a) z_1 z_2 \;\; + (\beta_0 b^2 - \beta_1 ba + \beta_2 a^2) z_2^2 \tag{15.3}$$

We then define a map from $SU(2) \to GL(2n+1, \mathbb{C})$ by

$$\beta' = R_n(A)\beta$$

We claim that this is a group representation.

For $n = 1/2$ this is just the defining representation. For $n = 1$ we have, from (15.3):

$$R_1(A) = \begin{pmatrix} (a^*)^2 & a^* b^* & (b^*)^2 \\ -2a^* b & aa^* - bb^* & 2ab^* \\ b^2 & -ab & a^2 \end{pmatrix}$$

For example, setting

$$g_t = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$$

in $SU(2)$ we have, for $t \in \mathbb{R}$,

$$R_1(g_t) = \begin{pmatrix} e^{-2it} & & \\ & 1 & \\ & & e^{2it} \end{pmatrix}$$

Indeed we claim

$$R_n(g_t) = \begin{pmatrix} e^{-2int} & & & \\ & e^{-2i(n-1)t} & & \\ & & \ddots & \\ & & & e^{2int} \end{pmatrix}$$

Note that the form of $R_n(g_t) \otimes R_1(g_t)$ is clear.

15.3.2 Lie algebra

Let $A(t)$ be any smooth path in $SU(2)$ such that $A(0) = 1$. For example, $A(t) = g_t$ above gives such a path. Then $\frac{dA(t)}{dt}$ makes sense (matrix entry by matrix entry) and can be evaluated at $t = 0$. The set of all these evaluations

$$T_1(SU(2)) = \{\frac{dA(t)}{dt}|_{t=0} \mid A(t) \text{ a smooth path; } A(0) = 1\}$$

is the *tangent space* at 1. This can be given the structure of a Lie algebra.

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Index