More notes on blob algebras

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These notes loosely correspond to some of my informal talks in Japan - particularly Kyoto - in 1995 (thanks to Jimbo, Miwa and the other organisers) and before, together with a couple of fixes and updates added since. However, N.B., plenty more fixes would be needed to smooth the many 'rough edges' in the exposition here, so the notes are not intended for publication (or, really, even wide circulation). For example, references are notably incomplete, and acknowledgements absent.

1 Introduction

Blob algebras [8] are generalisations of the Temperley–Lieb (TL) algebra [11] regarded as a diagram algebra [6]. They share with TL and with each other a number of combinatorial structures. We collect and summarize some of these here.

The core content of this paper has long been available on the web, see e.g. [5, 7]. This version is intended for those without the combined expertise in telepathy and obscure mathematical physics needed in order to read the original.

This document is just intended to be a useful collection of formulae. However it serves to unify a number of quite markedly different problems and techniques. For example, it directly relates the problem of computing determinants of Gram matrices (in representation theory) to the problem of diagonalising Hamiltonians (in quantum spin chain spectroscopy); and in particular relates the method of Smith forms, recursion equations and polynomial rings to the method of Bethe ansatz and fourier transform. In particular compare [3] with [9].

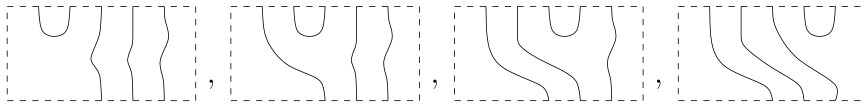
2 Temperley–Lieb Gram matrix formalities

2.1 TL diagrams

We assume that the reader is broadly familiar with the TL diagram calculus (and hence the associated k-linear monoidal categories).[1] We draw TL diagrams 'vertically', i.e. with vertices on the north and south edges. We write $H_T(n,m)$ for the set of TL diagrams with n northern and m southern vertices. We write $H_T^{=l}(n,m)$

[1] See e.g. [6, §9.5.2] (although note that the index convention used there is T_{n-1} for T_n).

for the subset of $H_T(n,m)$ with l propagating lines; and $H_T^{<l}(n,m)$ for the subset with $< l$ propagating lines (and so on). For example:

$$H_T^{=3}(5,3) = \{$$  $$\} \tag{1}$$

Let k be a given commutative ring, and $\delta \in k$. Thus the TL diagram calculus gives, for each triple n, l, m congruent modulo 2, a map:

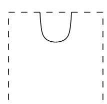
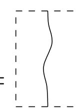
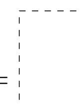
$$\circ : kH_T(n,l) \times kH_T(l,m) \to kH_T(n,m)$$

In this notation the basic diagram category is

$$\mathcal{C}_{TL} = (\mathbb{N}_0,\ kH_T(n,m)\ (n,m \in \mathbb{N}_0),\ \circ)$$

The TL algebra is $T_n = kH_T(n,n)$, with the category composition. We shall write $-^\star : H_T(n,m) \to H_T(m,n)$ for the map which turns diagrams upside-down. Thus in T_n we have $(d_1 d_2)^\star = d_2^\star d_1^\star$, so that $\star$ defines an isomorphism of T_n to its opposite.

A couple more basic examples of sets of diagrams are:

$H_T^{=0}(2,0) = \{u :=$  $\}$, $H_T^{=1}(1,1) = \{1 :=$  $\}$, $H_T(0,0) = \{1_0 :=$  $\}$

Note that in this notation $\{u^\star\} = H_T(0,2)$, and the usual TL generator $U_1 = u \circ u^\star$, and the TL loop relation is

$$u^\star \circ u = \delta 1_0$$

Then writing $\otimes$ for the *lateral* juxtaposition of diagrams, so that $1_0 \otimes d = d \otimes 1_0 = d$, we have

$$H_T^{=3}(5,3) = \{u \otimes 1 \otimes 1 \otimes 1,\ 1 \otimes u \otimes 1 \otimes 1, ...,\ 1 \otimes 1 \otimes 1 \otimes u\}$$

Note that we have

$$kH_T^{<l+2}(n,m) \cong \circ(kH_T(n,l) \times kH_T(l,m))$$

as k-modules.

2.2 Standard module bases and an inner product

By the category composition and the obvious filtration, there is an action from the left of $T_n = kH_T(n,n)$ on $kH_T^{<l}(n,m)$ for any l, and $kH_T^{<l}(n,m) \subset kH_T^{<l+2}(n,m)$ $(l < m, n)$. The (top) m-th section in this nest has basis $H_T^{=m}(n,m)$. This is a useful TL module, sometimes called a TL *standard* (or cell) module, and denoted $C^{T_n}(m)$. Thus a basis for a TL cell module looks like the example in (1). Given an ordering of this basis (such as the one in the example), we shall call the representation induced on this ordered basis the 'standard representation'.

By our construction, the action of T_n on $C^{T_n}(m)$ is by diagram concatenation from above, then using the usual diagram straightening rules, then quotienting by the subspace of diagrams with fewer propagating lines.

There is a corresponding action from the right of T_n on $H_T^{=m}(m,n)$ that we may call the right-standard module. This is a left-module for the opposite algebra, so by applying the $\star$ map (the opposite isomorphism) this becomes a left-module for T_n. This is called the costandard module. Note that the generators U_i are fixed by the $\star$ map, so the costandard representation is just given by mapping each U_i to the transpose of its image in the standard representation.

In the example in (1), which is $n=5$, $m=3$, each element of $H_T^{=m}(n,m)$ may be specified by the number of the vertex from which the non-propagating arc starts. Specifically, let us write a_i for the diagram with this arc starting in the i-th position. This works for $m=n-2$ in general, so the basis $H_T^{=n-2}(n,n-2)$ for $C^{T_n}(n-2)$ may be written

$$H_T^{=n-2}(n,n-2)=\{a_1,a_2,...,a_{n-1}\}.$$

Note that the basis has a natural order. For $C^{T_n}(n-4)$ the basis elements have two such arcs. We have

$$H_T^{=n-4}(n,n-4)=\{a_{12},a_{13},...,a_{1\,n-1},\ a_{23},a_{24},...,a_{2\,n-1},...,$$
$$a_{n-4\,n-3},a_{n-4\,n-2},a_{n-4\,n-1},a_{n-3\,n-2},a_{n-3\,n-1}\}$$

and so on. (Observe that the diagrams $a_{i\,i+1}$ are of a slightly different standing to the others — being nested.)

Note that the restriction of the category composition

$$\circ : H_T^{=m}(m,n)\times H_T^{=m}(n,m)\to kH_T^{=m}(m,m)\oplus kH_T^{<m}(m,m)$$

has either $a\circ b$ a scalar multiple of 1_m, or an element of $H_T^{<m}(m,m)$. Write $\mu_{a,b}$ for the coefficient of 1_m in $a\circ b$.

Let matrix

$$M_n(m)\ :=\ (\mu_{a,b})_{a,b}$$

In our example we can read this off from the table shown in Figure 1. That is (using δ for the loop parameter)

$$M_5(3)=\begin{pmatrix}\delta&1&0&0\\1&\delta&1&0\\0&1&\delta&1\\0&0&1&\delta\end{pmatrix}$$

It will be evident from this that, in general for $M_n(n-2)$,

$$\mu_{a_i^\star,a_j}\ =\ \delta\ \delta_{i,j}+\delta_{i,j+1}+\delta_{i,j-1} \tag{2}$$

(applying common sense appropriately at the 'boundaries', i.e. making the omission of nonsensical indices implicit). For general $M_n(m)$ the corresponding characterisation is more complicated, but see later.

Note that $\mu_{a,b}$ defines an inner product $\langle -,-\rangle : C^{T_n}(m)\times C^{T_n}(m)\to k$ (via the $\star$ map: $\langle a,b\rangle=\mu_{a^\star,b}$). That is, $M_n(m)$ is the Gram matrix.

There are a number of well-known techniques for finding the determinant of such a matrix. We will review some of them below, preparatory to certain analogous but

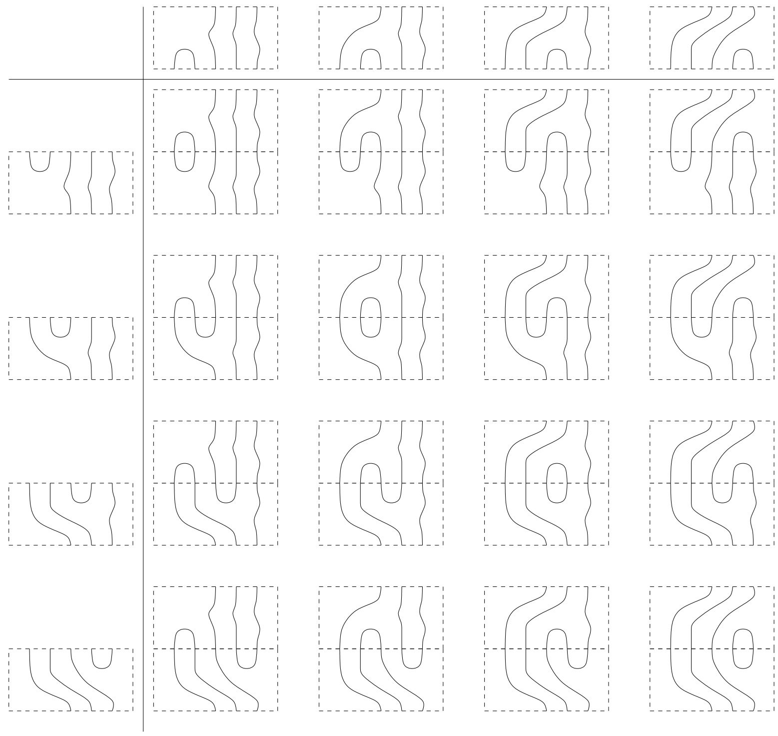


Figure 1: A step in computing the $C^{T_5}(3)$ Gram matrix $M_5(3)$.

harder calculations for blob algebras. What we will see is that if we parameterise by

$$\delta = [2] = q + q^{-1}$$

then $\det(M_n(n-2)) = [n]$. In the TL case this single result, together with some abstract representation theory (see e.g. Section 3), is enough to completely determine the simple and projective structure of TL for all n. Thus there is no real need to compute any other Gram determinants. (Another way of looking at this is to note that we can compute all other Gram determinants by reverse engineering from know representation theoretic facts.) A similar statement holds for the blob algebra. However, for other such algebras, abstract representation theory is less effective. For this reason we will also look at other Gram determinants. (The ones for $C^{T_n}(n-4)$ are discussed in section 4.2.) And for this reason it will be useful to compare and contrast some of the methods available for computing them.

3 Representation theory

For each distinct pair l, m of real numbers there is an action of the affine reflection group of type $\hat{A}_1$ on the real line [2]. One places a reflection hyperplane (a point) at l, and one at $l+m$, and forms the set $H = H(l,m)$ of reflection hyperplanes that is the closure of the action of these on each other. (The connected components of $\mathbb{R} \setminus H$ are called alcoves.)

To describe TL representation theory one sets $l = -1$, and considers this point as generating the action of the ordinary A_1 parabolic in $\hat{A}_1$. The standard/cell TL modules are indexed by dominant weights, i.e. A_1-orbits of integral points. Obviously this set of orbits may be represented by the set $\mathbb{N}_0$. (Strictly speaking, for finite n one has only the weights $\leq n$ and congruent to n modulo 2.)

The following is a mangled extract from the corresponding Theorem in [6], trimmed for our immediate purposes.

Theorem 1 *(i) If weight $\lambda = w.\mu$ for some reflection $w \in H(-1,m)$, and $\lambda > \mu$ lie in adjacent alcoves, then there is a standard/cell module homomorphism $C^{T_n}(\lambda) \to C^{T_n}(\mu)$ in case q an m-th root of 1.*
(ii) Over a field of characteristic zero the composition of any two such morphisms is zero.

One sees almost immediately that the result

$$\det(M_n(n-2)) = [n] \tag{3}$$

from Section 2 is implied by the above. This is because $C^{T_n}(n)$ has dimension 1, and maps into $C^{T_n}(n-2)$ when $[n]=0$. However this result is convenient to use in a *proof* of the Theorem. It is the only algebra specific input needed.

Considering $\mu = n-4$ one sees that maps in are possible (for suitable m) from $\lambda = n$ and $\lambda = n-2$. We read off

$$\det(M_n(n-4)) = \frac{[n-1]}{[2]}[n-2]^{n-1}$$

The denominator comes from Theorem 1(ii); and the exponent from the dimension of $C^{T_n}(n-2)$.

This result generalises immediately to all labels μ. (It is also a well-known classical result, but it will be convenient for us to have it ready to hand here.)

It will also be convenient for us to have this in a factored form, in a sense we explain in Section 4.1 (cf. (3), (5)). (We shall return to representation theory in Section 5.)

4 Determinant computation

4.1 Spectrum method (overkill): 1-particle

For any matrix M with Jordan form with diagonal entries $(\lambda_1, \lambda_2, ...)$ we have

$$det(M) = \prod_i \lambda_i$$

That is, we can compute the determinant of M by computing the eigenvalues. Indeed setting

$$M_n^- = M_{n+1}(n-1) - \delta.1_n$$

then $det(M_{n+1}(n-1)) = \prod_i (\delta + \lambda_i)$ where $\{\lambda_i\}_i$ are the eigenvalues of this M_n^-.

Our matrix M_n^- almost commutes with the row translation operator T_n:

$$T_6 = \begin{pmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(notation caveat[2]) (we say M, T almost commute if $MT - TM$ is very sparse) so we could try for eigenvectors of form $A_n(a) = (1, a, a^2, a^3, a^4, a^5, ..., a^{n-1})^t$ where $a^n = 1$.

More generally, T_n can be seen as a discrete finite-interval version of the operator on the space of real or complex-valued function[3] given by

$$Tf(x) = f(x+1)$$

In this setting our operator M_n^- becomes M, defined by

$$Mf(x) = (T + T^{-1})f(x) = f(x-1) + f(x+1)$$

We may solve the eigenvalue problem for T by fourier transform. Set

$$v_\alpha(x) = \exp(i\alpha x), \tag{4}$$

then

$$Tv_\alpha = \exp(i\alpha)v_\alpha$$

(any α). Thus

$$Mv_\alpha = 2\cos(\alpha)v_\alpha$$

and the $2\cos(\alpha)$ eigenspace of M is spanned by $v_{\pm\alpha}$.

To recover the spectrum of the discrete finite interval operator $M_n(n-2)$ we can restrict to $x \in \mathbb{Z} \subset \mathbb{R}$. We also need to fix the boundary conditions on the real x line so that we can decouple an appropriate interval. Firstly, then, we need $f(0) = 0$. Then we can extract the action of M on $f(\mathbb{N})$ as a direct summand of the overall action. Then we will need $f(L) = 0$, to decouple at the other end of the interval. If $D = D(M_n(n-2))$ is the dimension of the space that $M_n(n-2)$ acts on then $L = D + 1$, so here $L = n$.

The first condition is satisfied by selecting the subspace $\mathbb{R}(v_\alpha - v_{-\alpha})$ from each $2\cos(\alpha)$ eigenspace. That is, we consider functions of form $A_\alpha \sin(\alpha x)$. We will adopt the normalisation $f(1) = 1$ (without loss of generality), so $A_\alpha = 1/\sin(\alpha)$. We have

$$f_\alpha(x) = \frac{\sin(\alpha x)}{\sin(\alpha)}$$

The condition $f(L) = 0$ is solved by $\alpha = k\pi/L$ for $k \in \mathbb{Z}$, $0 < k < L$ (the bounds are strict to avoid triviality). Thus the spectrum of M_n^- is

$$2\cos(\pi/L),\ 2\cos(2\pi/L),\ ...,\ 2\cos((L-1)\pi/L)$$

[2]notation clash with TL algebra - hopefully distinguished by context

[3]Write $L^1(I)$ for Lebesgue integrable complex valued functions on $I = \mathbb{R}/2\pi\mathbb{Z}$.

We have

$$det(M_n(n-2)) = \prod_{i=1}^{L-1}(\delta + 2\cos(k\pi/L)) \tag{5}$$

It is convenient to write $\delta = q + q^{-1}$. Then our det (is a symmetric Laurent polynomial and) has roots at

$$q + q^{-1} = -(\exp(ik\pi/L) + \exp(-ik\pi/L))$$

That is

$$det(M_n(n-2)) = \frac{q^L - q^{-L}}{q - q^{-1}} = [L]$$

(confirming (3)).

4.2 2-particle

The next set of cell modules is those of form $C^{T_n}(n-4)$ (let us call them the 2-index or 2-particle cases). This is a little more complex. First we describe the matrix generalising (2). Suppose that $j > i+3$, then

$$\langle a_{ij}, a_{kl}\rangle = \delta^2\delta_{i,k}\delta_{j,l}+\delta(\delta_{i,k}(\delta_{j,l-1}+\delta_{j,l+1})+(\delta_{i,k-1}+\delta_{i,k+1})\delta_{j,l})+(\delta_{i,k+1}+\delta_{i,k-1})(\delta_{j,l-1}+\delta_{j,l+1})$$

$$= (\delta\delta_{i,k} + (\delta_{i,k+1} + \delta_{i,k-1}))(\delta\delta_{j,l} + (\delta_{j,l-1} + \delta_{j,l+1}))$$

The special cases are $j = i+1$, $j = i+2$ and $j = i+3$:

$$\langle a_{i\ i+3}, a_{kl}\rangle =$$

$$(\delta\delta_{i,k} + (\delta_{i,k+1} + \delta_{i,k-1}))(\delta\delta_{i+3,l} + (\delta_{i+3,l-1} + \delta_{i+3,l+1})) + \delta_{i,k}\delta_{i+3,l+2}$$

(note here that the additional term corresponds to $\langle a_{i\ i+3}, a_{i\ i+1}\rangle = 1$, and that one of the other terms is also in this case a coupling to a 'special' element: $a_{i+1\ i+2}$);

$$\langle a_{i\ i+2}, a_{kl}\rangle =$$

$$\delta^2\delta_{i,k}\delta_{i+2,l}+\delta(\delta_{i,k}\delta_{i+1,l}+\delta_{i-1,k}\delta_{i+2,l}+\delta_{i,k}\delta_{i+3,l})+\delta_{i-1,k}\delta_{i+3,l}+\delta_{i-1,k}\delta_{i+1,l}+\delta_{i+1,k}\delta_{i+3,l}\pm stuff!!!$$

$$\langle a_{i\ i+1}, a_{kl}\rangle =$$

$$\delta^2\delta_{i,k}\delta_{i+1,l} + \delta(\delta_{i,k}\delta_{i+2,l} + \delta_{i-1,k}\delta_{i+1,l} + \delta_{i+1,k}\delta_{i+3,l}) + \delta_{i-1,k}\delta_{i+2,l} + \delta_{i,k}\delta_{i+3,l} \pm stuff!!!$$

We continue this exercise with a new ansatz in §A.1.

4.3 Generic structure theorem method

Our next method for computing the gram determinant $\det(M_n(\lambda))$ uses the generic structure theorem [6, Th.1]. (Let us reiterate that there is no need, in representation theory, to compute any but the elementary $\lambda = n-2$ case. We continue merely as an exercise in service of later generalisations.)

The theorem gives a basis of the TL algebra for generic parameter over a suitable field in the form of a complete orthonormal set for the multimatrix structure. In this case we can extract a basis for each cell module such that, by orthonormality, the gram matrix is the unit matrix. Unlike the gram matrix over the defining basis, this is of no intrinsic use: manifestly since it contains no information; but implicitly since we cannot specialise the parameter as we can if we work over the 'integral' (polynomial) ring. The trick is to keep track of the renormalisations of elements in passing from the integral basis, so that we can reconstruct the determinant over the integral basis.

The basis elements we need are defined as follows (we extract directly from [6, §6.4]). Fix a cell module $\Delta_n(\lambda)$ and consider the walk enumeration of the diagram basis. The lowest walk (pair) is mapped to an element by, for example, with $m = \lambda + 1$:

$$(e_m, e_m) = (121212123...m, 121212123...m) = U_1 U_3 U_5 E_m^{(6)}$$

(see [6] for notation) (note normalisation as idempotent needed). Then if $s_i = g-1$ is a minimum of sequence s, and s^i denotes s with s_i replaced by $g+1$:

$$(s^i, t) = \sqrt{k_g k_{g+1}} \left(1 - \frac{U_i}{k_g}\right)(s, t)$$

The integral/diagram basis can be considered to start with (e_m, e_m) (up to quotients and id normalisation). The next diagram basis element, corresponding to the walk $123212123...m$, is $U_2(e_m, e_m)$. Thus we see that the new orthonormal basis element (s^i, t) in this case is got by adding a scalar multiple of the first (irrelevant for the determinant); and rescaling the second by

$$\sqrt{\frac{h-1}{h+1}\frac{h}{h-1}} = \sqrt{\frac{[h]^2}{[h+1][h-1]}}$$

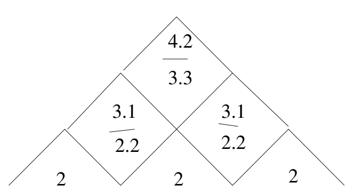
$(h = g)$. This applies both to the bra and the ket, and both rescale the determinant. For a general walk there is a factor like this for each diamond added to get to it from the $121212123...m$ walk. Thus altogether we have the following.

Lemma 1 *Schematically,*

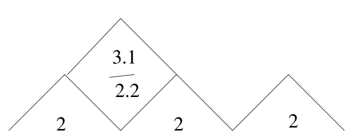
$$\det(M_n(\lambda)) = \prod_s \prod_d \frac{[h]^2}{[h+1][h-1]}$$

where the products are over the set of walks and the diamonds in each walk. □

Example: The factors for each diamond are (in abridged notation):



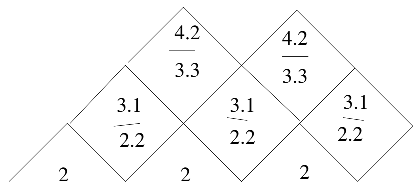
Thus in particular



gives a factor $\frac{[3][2]}{1}$. Altogether we have

$$\det(M_6(\lambda=0)) = \frac{[2]^3}{1}\frac{[3][2]}{1}\frac{[3][2]}{1}\frac{[3]^2}{[2]}\frac{[4]}{1} = [2]^4[3]^4[4] = \delta^5(\delta^2-1)^4(\delta^2-2)$$

Example II:



gives a factor $\frac{[4]^2}{[2][3]}$. Altogether this det comes out as

$$\det(M_7(\lambda=1)) = \frac{[2]^{3\times 14}}{1}\left(\frac{[3]}{[2]^2}\right)^{9+10+9}\left(\frac{[4][2]}{[3]^2}\right)^{4+4}\left(\frac{[5][3]}{[4]^2}\right)^{1} = [2]^{-6}[3]^{13}[4]^6[5]$$

JOB: show how these check against representation theory.

4.4 What about the Smith form?

More than the determinant evaluated over the ground field of interest, K say, one cares about the rank of the gram matrix (evaluated over the ground field of interest). This is because the rank determines the rank of the contravariant form, and hence the dimension of the simple head of the cell module over K. Elementary row and column operations expressible as multiplication by invertible matrices do not change the rank. Thus, if we are working over a PID (as we are in the indeterminate-δ calculations if the coefficient ring is a field) we are interested in the Smith normal form.

Reduced to the ground field of interest the SNF will give the rank directly. (Even if the ground ring is not a PID there may be a form which directly reveals the rank.) (The SNF may reveal even more than the rank in general. The power of vanishing of invariant factors vanishing at the δ-value of interest may reveal details of factor modules deeper in the Jordan–Holder series of the cell module.)

Example: Consider the gram matrix

$$\begin{pmatrix} \delta & 1 & 0 \\ 1 & \delta & 1 \\ 0 & 1 & \delta \end{pmatrix} \overset{R1:R1+R3}{\rightsquigarrow} \begin{pmatrix} \delta & 2 & \delta \\ 1 & \delta & 1 \\ 0 & 1 & \delta \end{pmatrix} \overset{R1:R1-\delta R2}{\rightsquigarrow} \begin{pmatrix} 0 & 2-\delta^2 & 0 \\ 1 & \delta & 1 \\ 0 & 1 & \delta \end{pmatrix} \overset{C2:C2-\delta C1;...}{\rightsquigarrow}$$

$$\begin{pmatrix} 0 & 2-\delta^2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & \delta \end{pmatrix} \overset{R1:R1+(\delta^2-2)R3}{\rightsquigarrow} \begin{pmatrix} 0 & 0 & -\delta(2-\delta^2) \\ 1 & 0 & 0 \\ 0 & 1 & \delta \end{pmatrix} \ldots$$

We can compute the SNF in various ways. It is

$$\begin{pmatrix} \delta(\delta^2-2) & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Method 1: elementary row and column operations.
Method 2: compute minors and take the HCF of all minors at each rank. For example the HCF of 1×1 minors in our example is clearly 1; as is that of 2×2 minors. The final invariant factor is then forced.

Of course the product of invariant factors (one for each basis element) in the SNF is the determinant, which we just computed in §?? above. Note that there is a factor for each basis element in our computation. However this is certainly not any invariant factor in general. For example the factor we computed in (??) is not 'integral'.

...

5 Structure

Exclude most of this for now, again assuming familiarity on the part of the reader. Briefly:

5.1 Generalities

Suppose k a field, and we have a bilinear form on a k-space M:

$$\langle , \rangle : M \times M \to k$$

(cf. an inner product in case $k = \mathbb{C}$, which is a sesquilinear form). This corresponds to a map

$$\begin{aligned} \psi : M &\to \mathrm{Hom}(M, k) & (6) \\ m &\mapsto f_m : M \to k & (7) \end{aligned}$$

where $f_m(m') = \langle m, m' \rangle$.

Now suppose that the form is a *contravariant form* on M as a module for a k-algebra A with an involutive antiautomorphism t. That is to say: $\langle m, am' \rangle = \langle a^t m, m' \rangle$ (all $a \in A$, $m, m' \in M$). It follows that regarding the dual right-module $\mathrm{Hom}(M, k)$ as a left-module via the antiautomorphism (let us call it the contravariant dual of M, regarded this way, and write it M^0) makes ψ a left A-module morphism.

Altogether there is a 1-1 correspondence between forms and A-module morphisms. Now suppose we know that
(i) M has simple head L, and
(ii) that this composition factor occurs with multiplicity one. Suppose further
(iii) that L is contravariant self-dual.
Then M^0 has simple socle L, and no other factor L. It follows that the only possible map (up to scalars) is one with image L. Clearly this is invertible only if $L = M$. The rank of the Gram matrix of $\langle , \rangle$ is the dimension of the image of a basis for M in M^0, and so is the dimension of L. In particular if this rank is not maximal then M is not simple.

5.2 TL case

5.3 Blob case

The blob category is a generalisation of the TL category in which lines exposed to the left may be decorated with a blob, with corresponding straightening rules. (This category is not monoidal in the same way. But this need not concern us for now.) That is, the object set is again $\mathbb{N}_0$. We write $H_b(n,m)$ for the set of (n,m)-blob diagrams, and $H_b^{<l}(n,m)$ for the subset that factor through object l. The blob algebra is $b_n = kH_b(n,n)$. For example b_1 is commutative, and has a basis consisting of two diagrams: one containing an undecorated propagating line (the identity), and one containing a blobbed propagating line (denoted e). Another basis would be two linear combinations of these, on which e acts like 1 and 0 respectively. Each of these two basis elements is the basis for a cell module, denoted $C^{b_1}(1)$ and $C^{b_1}(-1)$ respectively.

Again $kH_b^{<l}(n,m) \subset kH_b^{<l+2}(n,m)$ $(l < m,n)$. The (top) m-th section in this nest has basis $H_b^m(n,m)$. (We abuse notation and call this section $kH_b^m(n,m)$. This gives its structure as a vector space, but the actions on it as a module take account of the quotient.) This time there is an action of b_n on the left and an action of b_m on the right. If $m = n > 0$ it will be evident that $kH_b^n(n,n)$ has dimension 2, decomposing as two cell modules denoted $C^{b_n}(n)$ and $C^{b_n}(-n)$ (depending on how e acts, as before) respectively. If $m > 0$ we have left cell modules:

$$C^{b_n}(\pm m) = kH_b^m(n,m) \otimes_{b_m} C^{b_m}(\pm m)$$

5.4 Symplectic blob case

The symplectic blob category is a generalisation of the TL category in which lines exposed to the left may be decorated with a (left) blob, and lines exposed to the right may be decorated with a different (right) blob, with corresponding straightening rules. The same construction as before gives us $H_s^{<l}(n,m)$ and a section bimodule $kH_s^m(n,m)$. Looking at $kH_s^n(n,n)$ $(n > 2)$ we have a spanning set of variants of the identity diagram with decorations. We label cell modules as follows: $C^{s_n}(-n)$ (case e, f, U_i act as 0), $C^{s_n}(-(n-1))$(case e, U_i act as 0), $C^{s_n}(n-1)$ (case f, U_i act as 0), $C^{s_n}(n-2)$ (case U_i act as 0), (others depend on the relations – we will no go into details). Note that both signs and *parities* of labels appear here. Then

$$C^{s_n}(x) = kH_s^m(n,m) \otimes_{s_m} C^{s_m}(x)$$

where $x \in \{-m, -(m-1), m-1, m-2\} \setminus \{0\}$.

In this case a Gram matrix is obtained by computing with an array like

where we write $\times$ on a line to record that account must be taken of the choice of x in the tensor construction. (Note that in our specific example the choice $x = m-2 = 0$ is not included.) In our example we may take the case, say, that e acts like 1 rather than like 0 (thus the convention is that e and f act like 0 on undecorated propagating lines). For simplicity we adopt the parameterisation that e and f are both idempotent, and write L for the scalar associated to a loop decorated by a left-blob, and R for the scalar associated to a loop decorated by a right-blob. The Gram matrix itself, in examples of this form, is thus as given in equation (9).

6 Matrices

For each $d > 1$ define $d \times d$ matrix

$$M_{a,b,c}(d,L,R) = \begin{pmatrix} a & b & 0 & & & & & \\ L & [2] & 1 & 0 & & & & \\ 0 & 1 & [2] & 1 & 0 & & & \\ 0 & 0 & 1 & [2] & 1 & 0 & & \\ & & & & \ddots & & & \\ 0 & \dots & 0 & 0 & 1 & [2] & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & [2] & R \\ 0 & \dots & 0 & 0 & 0 & 0 & c & c \end{pmatrix}$$

Thus

$$\det(M_{a,a,c}(d,L,R)) = ac\det(M_{1,1,1}(d,L,R))$$

and

$$\det(M_{a,1,1}(d,L,R)) = a\det(M_{[2],1,1}(d-1,1,R)) - L\det(M_{[2],1,1}(d-2,1,R))$$

In particular

$$\det(M_{[2],1,1}(d,1,R)) = [2]\det(M_{[2],1,1}(d-1,1,R)) - \det(M_{[2],1,1}(d-2,1,R))$$

As is well known, the recurrence

$$M(d) = [2]M(d-1) - M(d-2)$$

is solved by $M(d) = \alpha[r{+}d]$ for any constants r, α. Noting that $\det(M_{[2],1,1}(2,1,R)) = [2] - R$, if we parametrise by $R = \frac{[r]}{[r+1]}$ we get

$$\det(M_{[2],1,1}(d,1,R)) = \frac{[r+d]}{[r+1]}$$

Altogether then

$$\det(M_{1,1,1}(d,L,R)) = \frac{[r+d-1]}{[r+1]} - L\frac{[r+d-2]}{[r+1]}$$

and parameterising by $L = \frac{[l]}{[l+1]}$ we get

$$\det(M_{1,1,1}(d,L,R)) = \frac{[l+1][r+d-1] - [l][r+d-2]}{[r+1][l+1]} = \frac{[l+r+d-1]}{[r+1][l+1]}$$

and

$$\det(M_{L,L,R}(d,L,R)) = \frac{[r][l][l+r+d-1]}{[r+1]^2[l+1]^2}$$

We also note

$$\det(M_{L,1,1}(d,1,R)) = \frac{[l][r+d-1] - [l+1][r+d-2]}{[r+1][l+1]} = \frac{[l-(r+d-2)]}{[r+1][l+1]} \qquad (8)$$

Define

$$M'(d,L,R) = \begin{pmatrix} L & L & 1 \\ L & [2] & 1 & 0 \\ 1 & 1 & [2] & 1 & 0 \\ 0 & 0 & 1 & [2] & 1 & 0 \\ & & & & \ddots \\ 0 & \dots & 0 & 0 & 1 & [2] & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & [2] & R \\ 0 & \dots & 0 & 0 & 0 & 0 & R & R \end{pmatrix} \tag{9}$$

Thus

$$\det(M'(d,L,R)) = R([2]-L)\det(M_{L,1,1}(d-1,1,R)) = \frac{[r][l+2]}{[r+1][l+1]}\frac{[l-(r+d-3)]}{[r+1][l+1]}$$

7 Combinatorics and generating functions

7.1 Bracket sequences and trees

Let B_n denote the set of properly nested bracket sequences of n brackets. This begins

$$\emptyset,\ \{()\},\ \{()(),(())\},\ \dots$$

Write

$$G(x) = \sum_{n=0} x^n |B_n| \ = \ x^0 + x^2 + 2x^4 + 5x^6 + \dots$$

for the generating function for the degrees of this sequence of sets — the Catalan numbers [1].

The set of rooted plane trees with n edges is in bijection with B_n. Noting that every tree with at least one edge may be 'factored' as a tree growing from the root together with a tree growing from the vertex at the end of this edge we have

$$G(x) = 1 + x^2(G(x))^2$$

so

$$G(x) = \frac{1-\sqrt{1-4x^2}}{2x^2}$$

Let B_n^l denote the set of composites of nested sequences with l propagating lines, with a total of n objects. This array begins

$$\begin{array}{ccccc} \emptyset \\ & \{|\} \\ \{()\} & & \{||\} \\ & \{()|,\ |()\} & & \{|||\} \end{array}$$

The tree version of this is a forrest of $l+1$ trees with walls between. Accordingly we have

$$\sum_{n=0} x^n |B_n^l| \ = \ G(x)(xG(x))^l \ = \ (G(x))^{l+1}x^l$$

7.2 Exclude the rest for now!

Exercise: explain the relevance of all this to blob representation theory.

8 James-Murphy Gram determinants

The following recursion was introduced by James and Murphy [4] in case $q = 1$. Let μ be an integer partition (or equivalently a Young diagram), I_μ the set of row positions of μ from which a box may be removed, and for $i \in I_\mu$, let μ^i be the corresponding subdiagram (we follow [10, Appendix B]). Define a function dim – from integer partitions to integers by

$$\dim(1) = 1$$

and

$$\dim \mu = \sum_{i \in I_\mu} \dim \mu^i$$

For $i \in I_\mu$ let J_i be the set of hook lengths of μ in the column above the removable box. Define a function from integer partitions to functions of q recursively by

$$D_{(1)} = 1$$

and

$$D_\mu = \prod_{i \in I_\mu} D_{\mu^i} \left(q^{x(\mu^i)} \prod_{j \in J_i} \frac{[j]}{[j-1]} \right)^{\dim \mu^i}$$

(here x is a function whose details need not concern us for now — see [10, Appendix B] for this, and also for a number of examples).

The point of James-Murphy's contruction is that D_μ is the Gram determinant for the S_n Specht module with label μ or (as noted by James and Mathas) the corresponding Hecke algebra module for general q.

Theorem 2 *This recursion is solved by the following explicit form in case $\mu = (\mu_1, \mu_2)$ (and $x = 0$):*

$$D'_\mu = \prod_{l=0}^{\mu_2 - 1} \left(\frac{[\mu_1 - l + 1]}{[\mu_2 - l]} \right)^{\binom{\mu_1+\mu_2}{l} - \binom{\mu_1+\mu_2}{l-1}}$$

Proof: (We will just do the cases in which $|I_\mu| = 2$.) In this case

$$\dim(\mu_1, \mu_2) = \binom{\mu_1 + \mu_2}{\mu_1} - \binom{\mu_1 + \mu_2}{\mu_1 + 1} = \binom{\mu_1 + \mu_2}{\mu_2} - \binom{\mu_1 + \mu_2}{\mu_2 - 1}$$

Sustituting D' for D in the recursion, we require to compute

$$\mathcal{K} = D'_{(\mu_1 - 1, \mu_2)} D'_{(\mu_1, \mu_2 - 1)} \left(\frac{[\mu_1 - \mu_2 + 2]}{[\mu_1 - \mu_2 + 1]} \right)^{\binom{\mu_1+\mu_2-1}{\mu_2-1} - \binom{\mu_1+\mu_2-1}{\mu_2-2}}$$

$$= \prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1-l]}{[\mu_2-l]}\right)^{\binom{\mu_1+\mu_2-1}{l}-\binom{\mu_1+\mu_2-1}{l-1}} \prod_{l=0}^{\mu_2-2} \left(\frac{[\mu_1-l+1]}{[\mu_2-l-1]}\right)^{\binom{\mu_1+\mu_2-1}{l}-\binom{\mu_1+\mu_2-1}{l-1}} \left(\frac{[\mu_1-\mu_2+2]}{[\mu_1-\mu_2+1]}\right).$$

We need to show that this can be equated with D'_μ. The first factor has numerator

$$\prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1-l]}{1}\right)^{\binom{\mu_1+\mu_2-1}{l}-\binom{\mu_1+\mu_2-1}{l-1}} = \prod_{l=1}^{\mu_2} \left(\frac{[\mu_1-l+1]}{1}\right)^{\binom{\mu_1+\mu_2-1}{l-1}-\binom{\mu_1+\mu_2-1}{l-2}}$$

$$= \chi \prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1-l+1]}{1}\right)^{\binom{\mu_1+\mu_2-1}{l-1}-\binom{\mu_1+\mu_2-1}{l-2}} \left(\frac{[\mu_1-\mu_2+1]}{1}\right)^{\binom{\mu_1+\mu_2-1}{\mu_2-1}-\binom{\mu_1+\mu_2-1}{\mu_2-2}}$$

Here we have shifted the dummy l to get the argument as in D'_μ, then applied appropriate correcting factors to get the range of the product right. In particular we have a correcting factor for the lower limit of the product

$$\chi = \left(\frac{1}{[\mu_1+1]}\right)^{\binom{\mu_1+\mu_2-1}{-1}-\binom{\mu_1+\mu_2-1}{-2}} = 1$$

The second factor has numerator

$$\prod_{l=0}^{\mu_2-2} \left(\frac{[\mu_1-l+1]}{1}\right)^{\binom{\mu_1+\mu_2-1}{l}-\binom{\mu_1+\mu_2-1}{l-1}} = \gamma \prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1-l+1]}{1}\right)^{\binom{\mu_1+\mu_2-1}{l}-\binom{\mu_1+\mu_2-1}{l-1}}$$

where

$$\gamma = \left(\frac{1}{[\mu_1-\mu_2+2]}\right)^{\binom{\mu_1+\mu_2-1}{\mu_2-1}-\binom{\mu_1+\mu_2-1}{\mu_2-2}}$$

Thus collecting these numerators we have

$$\prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1-l+1]}{1}\right)^{\binom{\mu_1+\mu_2-1}{l-1}-\binom{\mu_1+\mu_2-1}{l-2}+\binom{\mu_1+\mu_2-1}{l}-\binom{\mu_1+\mu_2-1}{l-1}} \left(\frac{[\mu_1-\mu_2+1]}{[\mu_1-\mu_2+2]}\right)^{\binom{\mu_1+\mu_2-1}{\mu_2-1}-\binom{\mu_1+\mu_2-1}{\mu_2-2}}$$

The second factor has denominator

$$\prod_{l=0}^{\mu_2-2} \left(\frac{1}{[\mu_2-l-1]}\right)^{\binom{\mu_1+\mu_2-1}{l}-\binom{\mu_1+\mu_2-1}{l-1}} = \prod_{l=1}^{\mu_2-1} \left(\frac{1}{[\mu_2-l]}\right)^{\binom{\mu_1+\mu_2-1}{l-1}-\binom{\mu_1+\mu_2-1}{l-2}}$$

Noting that

$$-\binom{\mu_1+\mu_2-1}{l-2} + \binom{\mu_1+\mu_2-1}{l} = \binom{\mu_1+\mu_2}{l} - \binom{\mu_1+\mu_2}{l-1}$$

altogether we have

$$\mathcal{K} = \prod_{l=0}^{\mu_2-1} \left(\frac{[\mu_1-l+1]}{[\mu_2-l]}\right)^{\binom{\mu_1+\mu_2}{l}-\binom{\mu_1+\mu_2}{l-1}}$$

as required. $\square$

A Appendix

A.1 Eigenvalue approach: 2-particle continued

To compute the 2-particle determinant via eigenvalues we try to generalise the ansatz, by taking (4) to:

$$v_{\alpha\beta}(x,y) = e^{\alpha x}e^{\beta y} \qquad (y > x+2)$$

by direct analogy with the previous case, but try

$$v_{\alpha\beta}(x,x+1) = e^{(\alpha+\beta)(x+1)}\phi(\alpha,\beta)$$

(where ϕ is some function, cf. [9]) in consideration of the distinct standing of these cases. For convenience we also parameterise as $v'_{a,b}(x,y) = a^x b^y$ and $v'_{a,b}(x,x+1) = (ab)^{x+1}\phi'(a,b)$.

Generically $(y > x+3)$ we have

$$Mf(x,y) = \delta^2 f(x,y) + \delta(f(x-1,y) + f(x+1,y) + f(x,y-1) + f(x,y+1))$$

$$+f(x-1,y-1) + f(x-1,y+1) + f(x+1,y-1) + f(x+1,y+1)$$

so setting $\Lambda(a) = \delta + a + a^{-1}$ and $\Lambda(a,b) = \Lambda(a)\Lambda(b)$ we have

$$Mv' = (\delta^2 + \delta(a^{-1} + a + b^{-1} + b) + ((ab)^{-1} + b/a + a/b + ab))v'$$

$$= (\delta + a + a^{-1})(\delta + b + b^{-1})v' = \Lambda(a,b)v'$$

for $v' = v'_{a,b}$. Note that

$$v' = Av'_{a,b} + Bv'_{b,a} + Cv'_{a,b^{-1}} + Dv'_{a^{-1},b}$$

(any A, B, C, D) lies in the same $\Lambda(a,b)$ crypto-eigenspace. This is 'crypto' in the sense that we have not checked the non-generic part of the action yet. We have the freedom to chose ϕ and the amplitudes A, B, C, D; but we must use this freedom to ensure that M acts with the same 'eigenvalue' in the various parts of the non-generic sector, and that the finite–discrete boundary conditions are satisfied.

If $y = x+3$ we have

$$Mf(x,x+3) = \delta^2 f(x,x+3) + \delta(f(x-1,x+3) + f(x+1,x+3) + f(x,x+2) + f(x,x+4))$$

$$+f(x-1,x+2) + f(x-1,x+4) + f(x+1,x+2) + f(x+1,x+4) + f(x,x+1)$$

giving

$$(M - \Lambda(a,b))v'_{ab} = -(a/b)(ab)^x b^3 + (ab)^{x+2}\phi'(a,b) + (ab)^{x+1}\phi'(a,b)$$

$$= -(ab)^{x+1}b + (ab)^{x+1}(ab+1)\phi'(a,b)$$

Note that setting

$$\phi'(a,b) = \frac{1}{b^{-1}+a} = \frac{b}{1+ab}$$

would make the anomalies cancel. (Alternatively one might tinker with the amplitudes.)

If $y = x + 2$ we have

$$Mf(x,x+2) = \delta^2 f(x,x+2)+\delta(f(x-1,x+2)+f(x+1,x+2)+f(x,x+1)+f(x,x+3)+f(x-1,x))$$
$$+f(x-1,x+1)+f(x-1,x+3)+f(x+1,x+3)$$

giving (with $v'_{ab} = v'_{ab}(x,x+2) = (ab)^x b^2$):

$$Mv'_{ab} = (\delta^2 + \delta(a^{-1}+b) + (a^{-1}b + ab + (ab)^{-1}))v'_{ab} + \delta\phi'(a,b)(ab)^x((ab)^2+ab+1)$$

There are two issues here: Firstly the coefficient of δ in the anomaly $(M-\Lambda(a,b))v'_{ab}$; second, the coefficient of constant. The second cannot be fixed by tuning ϕ', since it does not involve ϕ'. Instead we try to fix the amplitude between v'_{ab} and v'_{ba}. We have

$$(M-\Lambda(a,b))v'_{ab} = -\delta(a+b^{-1})v'_{ab} + \delta\phi'(a,b)(ab)^x((ab)^2+ab+1) - (a/b)v'_{ab}$$

Vanishing of the coeff of δ is:

$$(a+b^{-1})v'_{ab} = \phi'(a,b)(ab)^x((ab)^2+ab+1)$$

which is

$$\phi'(a,b) = \frac{(a+b^{-1})v'_{ab}}{(ab)^x((ab)^2+ab+1)} = \frac{(a+b^{-1})(ab)^x b^2}{(ab)^x((ab)^2+ab+1)} = \frac{(ab+1)b}{(ab)^2+ab+1}$$

which does not quite work![4]

If $y = x + 1$ we have

$$Mf(x,x+1) = \delta^2 f(x,x+1) + \delta(f(x-1,x+1)+f(x+1,x+3)+f(x,x+2))$$
$$+f(x-2,x-1)+f(x-1,x)+f(x-1,x+2)+f(x,x+3)+f(x+1,x+2)+f(x+2,x+3)$$

Thus, using $v'_{a,b}(x,x+1) = \phi'(a,b)(a/b)v'_{a,b}(x,x+2)$,

$$Mv'_{a,b} = \delta^2 v'_{a,b} + \delta\frac{1}{\phi'(a,b)}(a^{-2}+b^2+(b/a))v'_{a,b}$$

[4]WE ARE STUCK HERE FOR THE MOMENT, SINCE, NOTING THE POWERS OF δ, WE ARE GOING TO NEED A NEW TRICK TO FIX THIS. IS THERE SOME COMBINATION IN THE EIGENSPACE THAT FIXES THINGS?... ...BELOW HERE (TO THE END OF THE SECTION) NEEDS FIXING.

So in case $y = x + 2$ Thus

$$M(Av'_{ab} - v'_{ba}) = (\delta + a + a^{-1})(\delta + b + b^{-1})(Av'_{ab} - v'_{ba}) - A(\delta a^{-1} + b/a)v'_{ab} + (\delta b^{-1} + a/b)v'_{ba}$$

But $v'_{ab}(x,x+2) = a^x b^x b^2$ so $v'_{ba}(x,x+2) = (a/b)^2 v'_{ab}(x,x+2)$, so

$$-A(\delta a^{-1} + b/a)v'_{ab} + (\delta b^{-1} + a/b)v'_{ba} = (-A(\delta a^{-1} + b/a) + (\delta b^{-1} + a/b)(a/b)^2)v'_{ab}$$

This vanishes if

$$A = \frac{(\delta b^{-1} + a/b)(a/b)^2}{\delta a^{-1} + b/a}$$

$$+((ab)^{-2}+(ab)^{-1}+\frac{1}{\phi'(a,b)}(a^{-2}b+a^{-1}b^2)+ab+(ab)^2)v'_{a,b}$$

OLD VERSION:
If $y=x+1$ we have

$$Mf(x,y)=\delta^2 f(x,y)+\delta(f(x-1,y)+f(x-1,y-2)+f(x,y+1))$$
$$+f(x-1,y+1)+f(x,y+2)$$

In case $y=x+1$

$$Mv'_{ab}=(\delta^2+\delta(a+a^{-1}b^{-2}+b^{-1})+(ab^{-1}+b^{-2}))v'_{ab}$$

so

$$M(Av'_{ab}-v'_{ba})=A((\delta^2+\delta(a+a^{-1}b^{-2}+b^{-1})+(ab^{-1}+b^{-2}))v'_{ab})$$
$$-((\delta^2+\delta(a^{-1}+a^{-2}b^{-1}+b)+(ba^{-1}+a^{-2}))v'_{ba})$$

But $v'_{ab}(x,x+1)=a^x b^x b$ so $v'_{ba}=(a/b)v'_{ab}$, so

$$M(Av'_{ab}-v'_{ba})=(A(\delta^2+\delta(a+a^{-1}b^{-2}+b^{-1})+(ab^{-1}+b^{-2}))$$
$$-(\delta^2+\delta(a^{-1}+a^{-2}b^{-1}+b)+(ba^{-1}+a^{-2}))(a/b))v'_{ab}$$

Now we need to impose $f(0,y)=0$ and $f(x,n)=0$. The former will fix the relative coefficients, and the latter will quantise a,b (or equivalently α,β).

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