Chapter 10

Forms, module morphisms and Gram matrices

ss:Forms

10.1 Forms, module morphisms and Gram matrices (Draft)

Here the main objective is to consider some representation-theoretically useful forms on modules for algebras that are isomorphic to their opposites. (Many interesting algebras are isomorphic to their opposites, such as partition algebras - see e.g. (1.3.9) - and finite group algebras. For interesting examples that are *not*, consider for example the full transformation semigroups from §4.1.4.)

Aside: A k-algebra A isomorphic to its opposite via an involutive set map has an involutive antiautomorphism. (For any algebra A let $op: A \to A^{op}$ be the identity map on A as a set; and hence an algebra antihomomorphism. Since the image of op is A as a set, we can apply op again - i.e. to A^{op}: $op \circ op = 1$. Let $i: A \to A^{op}$ be an algebra isomorphism. Certainly i is a bijective set map from A to itself (not the identity map, unless A is commutative, but possibly an involution). Then $(op \circ i): A \to A$ is an antiautomorphism. Note that as a set map this is just the map i. For N in A-mod we have N^i in A-mod given by $N^i = \hom_k(N, k)$ as a set. [- ...but where, as a module, N - perhaps now denoted $\overline{N}$ - is understood to be acted on by $\overline{N} \times A \ni (m, a) \to i(a)m$ ... NO!! fix/finish.])

Overview:
The basic idea is this... Let A be a k-algebra. Firstly, *any* A-module morphism $\psi: M \to N$ gives us information about M and N. The kernel is a submodule of M for example.

If there is an algebra antiautomorphism for A then module morphisms $\psi: M \to N^i$ are in bijection with contravariant forms $\langle\rangle: M \times N \to k$ by $\langle m, n\rangle = \psi(m)(n)$ — thus contravariant forms become a useful source of morphisms.

Next, for certain special classes of modules, only morphisms from them (or to them) of special kinds are possible in principle; explicit morphisms are then particularly revealing. We give a small indicative example in (10.1.2).

(**10.1.1**) ... the ideas discussed in this section have been used historically to study the symmetric group in particular (see e.g. [80]). See also [59]. Outside the classical group representation theory

setting, it has proved very useful for 'diagram algebras' such as the partition algebra (see [117, 119], [118] for some early examples).

An outline of the section is this... We start with a brief review of basics designed to make the section as self-contained as possible. In particular it will be useful to recall properties of ordinary left-right duality. Then we discuss bilinear forms; and then contravariant forms; then examples from both the group and the diagram algebra settings; ...

Indicative (meta) Example

exa:eAeeke2 (**10.1.2**) Suppose A is a k-algebra, $e^2 = e \in A$ and $A^/$ is a quotient of A. Let $\bar{a} \in A^/$ denote the natural image of $a \in A$ (when unambiguous, we simply write a for the image of a in $A^/$). Suppose now that

$$\bar{e}A^/\bar{e} = k\bar{e}. \tag{10.1}$$ eq:eAe=ke2

Then by (1.6.7): (the image of) e is primitive; $A^/\bar{e}$ is an indecomposable projective left-$A^/$-module (and an indecomposable A-module with simple head); and $\bar{e}A^/$ is an indecomposable projective right-module. Furthermore for any $a \in A$ we have a scalar $\langle a\rangle_e \in k$ given by $\bar{e}\bar{a}\bar{e} = \langle a\rangle_e \bar{e}$.

(**10.1.3**) Given (10.1) we have a bilinear map

$$\langle\rangle : A^/\bar{e} \times \bar{e}A^/ \to k$$

given by $(a\bar{e}, \bar{e}b) \mapsto \langle a\bar{e}, \bar{e}b\rangle$ where $\langle a\bar{e}, \bar{e}b\rangle$ is given by $\bar{e}ba\bar{e} = \langle a\bar{e}, \bar{e}b\rangle\bar{e}$. That is $\langle a\bar{e}, \bar{e}b\rangle = \langle \bar{e}ba\bar{e}\rangle_e$.

If $A^/$ has an opposite isomorphism $a \mapsto a^t$ say, with $\bar{e} = \bar{e}^t$, this map amounts to the same thing as a map $A^/\bar{e} \times A^/\bar{e} \to k$ where $(a\bar{e}, b\bar{e}) \mapsto \langle a\bar{e}, (b\bar{e})^t\rangle$ and then as above.

This map $\langle\rangle$ induces a linear map

$$\psi : A^/\bar{e} \to (\bar{e}A^/)^*$$

where $(\bar{e}A^/)^* = \hom_k(\bar{e}A^/, k)$, by $ae \mapsto (eb \mapsto \langle ae, eb\rangle)$. Both the domain and the codomain now have a left A-module structure (the dual to the right projective is left injective, with simple socle). One checks if this linear map is a left A-module morphism (comparing $\psi(c(ae))$ with $c\psi(ae)$ for $c \in A$)

$$\psi(c(ae)) \;=\; (eb \mapsto \langle cae, eb\rangle) \;=\; (eb \mapsto \langle eb(cae)\rangle_e)$$
$$c\psi(ae) \;=\; c(eb \mapsto \langle ebae\rangle_e) \;=\; (eb \mapsto \psi(ae)((eb)c) = \langle (ebc)ae\rangle_e)$$

Recall (see e.g. Prop.8.6.8) that for any $A^/$-module M there is an isomorphism of abelian groups $\bar{e}M \cong \mathrm{Hom}_{A^/}(A^/\bar{e}, M)$. So in particular $\bar{e}A^/\bar{e} \cong \mathrm{Hom}_{A^/}(A^/\bar{e}, A^/\bar{e})$ (this can even be made into a ring anti-isomorphism). Also recall that (if k is algebraically closed) then composition multiplicities in any module M are given by

$$\dim_k \mathrm{Hom}_{A^/}(P_i, M) \;=\; [M : S_i] \tag{10.2}$$ eq:PiMSi

(see e.g. (9.3.12)) for any indecomposable projective P_i with simple head S_i. Thus in case $\bar{e}A^/\bar{e} = k\bar{e}$ we see that the dimension is 1 and so the head simple in indecomposable left projective $A^/\bar{e}$ occurs *only* in the head. (Caveat: expressed this way it is essentially a tautology. The hom space in (10.2) can be thought of as spanned by morphisms taking the head simple S_i in P_i to each of the various 'copies' in M.) Here the 1-d space of morphisms is spanned by the identity morphism taking head to head.

Hereafter, we are more interested in the 1-d space of morphisms from the projective to the 'corresponding' injective (necessarily taking the head to the socle, as we will see). Here the rank of the morphism is not full in general - it corresponds to the dimension of the head/socle.

In general (e.g. for an algebra not isomorphic to its opposite) the structure (the dimension; the character and so on) of the right projective $\bar{e}A^{/}$ can be quite different to the left projective. See for example (4.5). However the quotient by the radical reduces both modules to simples, and yields a semisimple (and hence opposite-isomorphic) quotient algebra, so that the head simples are paired.

It follows - see also e.g. **??** - that the simple module S' in the socle of the indecomposable left injective $(\bar{e}A^{/})^*$ above occurs only in the socle.

...We'd like to consider when the map from projective to injective is head S to socle S', i.e. when $S \cong S'$. To investigate this we consider $\bar{e}(\bar{e}A^{/})^*$. The element $f : \bar{e}m \mapsto f(\bar{e}m)$ is taken to $(\bar{e}f) : \bar{e}m \mapsto f(\bar{e}m\bar{e})$. Since ... One should compare with the quasi-heredity framework as in §9.5. And see also §2.3 for extensive examples.

Note that the dual of a projective right-module is not necessarily a projective left-module.

10.1.1 Basic preliminaries recalled: ordinary left-right duality

ss:vs dual2

(**10.1.4**) Recall the convention (e.g. from Ch.8) that we write the action of ring or algebra A on left A-module M 'on the left':

$$A \times M \;\to\; M \qquad (10.3)$$
$$(a, m) \;\mapsto\; am \qquad (10.4)$$

so that $b(am) = (ba)m$; and the action of $a \in A$ on a right A-module on the right: ma.
(If we write ma for a left action we get $(ma)b = m(ba)$ which just looks odd.)

de:dualM

(**10.1.5**) Recall that if A is an R-algebra and $M = {}_AM$ a left A-module then the *dual right module* is

$$M^* \;=\; M^*_A \;:=\; \mathrm{Hom}_R({}_AM, R)$$

Thus elements of M^* are maps

$$\mu : M \;\to\; R \qquad (10.5)$$
$$m \;\mapsto\; \mu(m) \qquad (10.6)$$

It is a *right* A-module by the action of $a \in A$ on any μ as above, to give μa as follows:

$$\mu a : m \mapsto \mu(am)$$

We check the right action property by comparing $(\mu a)b$ with $\mu(ab)$:

$$(\mu a)b : m \mapsto \mu a(bm) = \mu(a(bm))$$

$$\mu(ab) : m \mapsto \mu((ab)m)$$

(**10.1.6**) Does this $*$ lift to a functor from $A-$mod to mod$-A$? Consider the possible image of a map in $A-$mod (or indeed a sequence in $A-$mod):

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[d, "\mathrm{hom}_R(-,R)"'] & M \arrow[r, "g"] \arrow[d, "\mathrm{hom}_R(-,R)"'] & M'' \arrow[d, "\mathrm{hom}_R(-,R)"'] \\ (M')^* & M^* \arrow[l, "?"] & (M'')^* \arrow[l, "?"] \end{tikzcd}$$

Given a map $f \in \mathrm{hom}_R(M', M)$, then for each $a \in \mathrm{hom}_R(M, R)$ we can form $f^*(a) \in \mathrm{hom}_R(M', R)$ by $f^*(a) = a \circ f$. That is

$$\begin{tikzcd} M' \arrow[rr, "f^*(a)"] \arrow[dr, "f"] & & R \\ & M \arrow[ur, "a"] & \end{tikzcd}$$

Thus $f^* \in \mathrm{hom}_R(M^*, (M')^*)$ and we have

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[d, "\mathrm{hom}_R(-,R)"'] & M \arrow[r, "g"] \arrow[d, "\mathrm{hom}_R(-,R)"'] & M'' \arrow[d, "\mathrm{hom}_R(-,R)"'] \\ (M')^* & M^* \arrow[l, "f^*"] & (M'')^* \arrow[l, "g^*"] \end{tikzcd}$$

In other words $\mathrm{hom}_R(-, R) : A-\mathrm{mod} \;\to\; \mathrm{mod}-A$ defined in this way is a *contravariant* functor. Indeed this $H- = \mathrm{hom}_R(-, R)$ is a left-exact contravariant functor, meaning that an exact sequence $M' \to M \to M'' \to 0$ passes to an exact sequence $0 \to HM'' \to HM \to HM'$.

(**10.1.7**) Exercise. (I) Suppose an A-module M is simple. What can we say about M^*?
(II) Suppose we have a Jordan–Holder series (see e.g. §8.3.2) for an A-module M. What can we say about M^*?

(**10.1.8**) If in particular A is a finite dimensional algebra over a *field* R then every finitely-generated A-module M is a finite-dimensional R-vector space, and M^* has the same dimension, and we have the following (see e.g. [3, §23]).
(I) M and M^{**} are isomorphic A-modules.
(II) M is (semi)simple if and only if M^* is (semi)simple.
(III) A sequence $0 \to M' \to M \to M'' \to 0$ is (split) exact if and only if the dual sequence is (split) exact.
(IV) $\mathrm{Soc}\, M^* \cong\; (M/\mathrm{Rad}\; M)^*$.

10.1.2 Contravariant duality

ss:contra dual2

de:opM0

(**10.1.9**) Let R be a commutative ring and let A be any R-algebra. We may regard $M = {}_AM$ as a right A^{op}-module $M = M_{A^{op}}$ by defining a right action as follows:

$$ma \;:=\; am$$

(Check: recall that for $a, b \in A^{op}$, with multiplication denoted $*$ say, then $a * b = ba$ as computed in A; thus $(ma)b = b(ma) = b(am) = (ba)m = m(a * b)$). Similarly each right module gives a left module.

A left A-module homomorphism $M' \xrightarrow{f} M$ becomes a right A^{op}-module homomorphism, so this construction $\Phi : A-\mathrm{mod} \to \mathrm{mod}-A^{op}$ (say) is a (covariant) functor. We also use Φ for the right-to-left version.

`avariant duality` (**10.1.10**) A group algebra over a commutative ring R is isomorphic to its opposite (defined as for opposite ring) since $g \mapsto g^{-1}$ defines a group antiautomorphism (an isomorphism $G \cong G^{op}$); and this extends to RG via: $\sum_i r_i g_i \mapsto \sum_i r_i g_i^{-1}$.

There may be other isomorphisms. For example, a suitable group of matrices may be mapped to its opposite by $g \mapsto g^{tr}$ (transpose matrix). Here, when considering any algebra isomorphic to its opposite, we will generally fix a given involutive antiautomorphism.

`de:opM` (**10.1.11**) Let R be a commutative ring and let A be any R-algebra with an involutive antiautomorphism (generally denoted $g \mapsto g^t$, or $g \mapsto g^\tau$). We may regard $M = {}_AM$ as a right A-module by

$$ma \;:=\; a^t m$$

(Check: $(ma)b = b^t(ma) = b^t a^t m = (ab)^t m = m(ab)$); and similarly each right module gives a left module.

(**10.1.12**) It follows from (10.1.5) and (10.1.11) that for each $M \in A-\mathrm{mod}$ there is another left module M^o obtained from the dual right module M^* by applying Φ:

$$M^o \;:=\; \Phi(M^*)$$

(i.e. via the opposite isomorphism, regarding M^* as a left module for the opposite). This construction has the property that $R-\mathrm{mod}$ is invariant under taking to its dual combined with taking all $M \mapsto M^o$. (I.e. if defined, $(\mathrm{Head}\ M)^o \cong \mathrm{Soc}\ M^o$, and so on.)

We will call the map $M \mapsto M^o$ *contravariant duality* (see e.g. [59]). We have $(M^o)^o = M$.

(**10.1.13**) Exercise. Let G be a finite group and R a field. The 'contragredient' of a projective RG-module is projective (claim (10.29) in Curtis–Reiner [35]). Prove this. Give a counter-example for general finite-dimensional R-algebra A with t.
(For the symmetric group, and indeed any finite G, we will see in (10.2.1) that the regular module is contravariant self-dual for any R. Thus the collection of indecomposable summands over R a field must be fixed under duality, which verifies the claim in this case. However the regular module is not always self-dual for an algebra A with t (we shall have an example from the Temperley–Lieb algebras shortly).)

(**10.1.14**) Exercise. (Optional) Why are duals of lattices done differently in Curtis–Reiner [35] p.89 cf. p.245?

(**10.1.15**) Exercise. (Optional) Claim: Suppose A is in fact a finite group algebra over R. Let $x \in A$ be mapped to x^o by the opposite isomorphism (and regard x^o as an element of A). Then $M = Ax$ implies $M^o = Ax^o$.
Prove this, or provide a counter-example.

10.1.3 On 'Λ-standard' modules and duals of simples

`pa:dualsimple` (**10.1.16**) Given an algebra A with an involutive antiautomorphism t as above, when is a simple A-module $L = L^o$?

(**10.1.17**) Let $\Lambda = (\Lambda, \leq)$ be a poset whose underlying set indexes the simple modules of algebra A up to isomorphism. A *set* Δ of A-modules M_λ is a Λ-*standard set* if it is indexed by the poset $(\Lambda, \leq)$, with the property that (i) the set of heads is a complete set of simple modules of A; and (ii) the decomposition matrix of this set is lower unitriangular with respect to the partial order.

(**10.1.18**) Let L be the head of $M \in \Delta$ as above. The unitriangular property means that if simple L' occurs below the head in M then L will not occur below the head in the standard M' with head L'.

Note that the set of simples of A is a Λ-standard set for any order.

(**10.1.19**) Lemma. *Let A be an algebra with involutive antiautomorphism. Suppose Δ is a Λ-standard set for A; and that there is a nonzero contravariant form on each $M_\lambda \in \Delta$. Then $L \cong L^o$ for all simples L.*

Proof. Since there is a nonzero contravariant form on each $M = M_\lambda$ there is a nonzero module map from M to M^o (Prop.10.1.34). This means that there must be a copy of head L in M^o. But we know the socle of M^o is L^o, so both L and L^o are in the image of the module map from M. Thus either $L \cong L^o$ and the image is just L, or L^o lies in M below the head. But now consider the standard module M' for which L^o is the head. If $L \not\cong L^o$ then M, M' are nonisomorphic and M' must contain $L = L^{oo}$ and L^o by an analogous argument, contradicting the unitriangular property. Thus here $L \cong L^o$. □

10.1.4 On a Schur Lemma for 'standard' modules

`prepre` (**10.1.20**) Proposition. *Suppose that R is a field, and A is an R-algebra with a given involutive antiautomorphism. If left A-module M has a unique maximal proper submodule (call it M_o) and hence simple head $L = M/M_o$, and this composition factor L has multiplicity one in M, and $L \cong L^o$, then* $\dim \mathrm{Hom}_A(M, M^o) = 1$, *and $\psi \in \mathrm{Hom}_A(M, M^o)$ has rank* $\dim(L)$.

Proof: NB, every simple factor in M^o is extended by L below it. There *is* a map $\psi \in \mathrm{Hom}_A(M, M^o)$ — that which kills the unique maximal proper submodule M_o and so makes the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_o & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\cong} & & \\ 0 & \longleftarrow & (M_o)^o & \longleftarrow & M^o & \longleftarrow & L^o & \longleftarrow & 0 \end{array}$$

(or any scalar multiple thereof). No reduction is possible in the kernel, since this would require factors appearing in the image below L, which M^o does not have, as already noted. No enlargement of the image is possible since this, correspondingly, requires factors above L in M. □
(NB, the converse does not hold in general.)

ss:bilinear

10.1.5 Bilinear forms

See §1.4.9 and (1.4.56) for examples, connections and applications.

de:bilinearform

(**10.1.21**) BILINEAR FORM: For R a field (or commutative ring), a bilinear form on $M, N \in R-$mod is an R-bilinear function

$$<,>: M \times N \to R$$

(cf. Perlis [135], Maclane-Birkoff [99, §X.1]).

(**10.1.22**) Suppose M, N free R-modules. Let $B_M = \{b_1, b_2, \ldots, b_k\}$ be an ordered basis of M and $B_N = \{c_1, c_2, \ldots, c_l\}$ of N. Then a matrix $B(M,N)$ of form

$$(B(M,N))_{ij} \; = < b_i, c_j >$$

determines $<,>$.

(**10.1.23**) In an obvious notation,

$$B(M,N) = \begin{pmatrix} \langle b_1| \\ \langle b_2| \\ \vdots \\ \langle b_k| \end{pmatrix} \left(\; |c_1\rangle, \; |c_2\rangle, \cdots, \; |c_l\rangle \; \right)$$

By linearity, the effect of basis changes $B'_M = B_M U$, $B'_N = B_N W$ (B_M arranged as a row vector $(b_1, b_2, ...)$; U, W unimodular matrices, as in §8.2.4) is

$$B'(M,N) = U^t \begin{pmatrix} \langle b_1| \\ \langle b_2| \\ \vdots \\ \langle b_k| \end{pmatrix} \left(\; |c_1\rangle, \; |c_2\rangle, \cdots, \; |c_l\rangle \; \right) W \; = \; U^t B(M,N) W$$

Recall (e.g. from §8.2.4) that if R is a PID then among the possible such basis changes would be a pair that bring $B'(M,N)$ into Smith normal form.

REMARK. The reason for keeping the ground ring R at the level of generality of commutative ring (despite the fact that our eventual objects of study are typically algebras over fields) will become apparent shortly.

pa:form-map

(**10.1.24**) Keep $M, N \in R-$ mod as before. Recall that the dual $N^* = \mathrm{Hom}_R(N,R)$ has the structure of R-module (since R is commutative, 'left' and 'right' modules are the same here). Note that form $<,>$ defines an R-module homomorphism

$$\psi : M \to N^*$$

by

$$\psi(m)(n) \; = < m, n > .$$

From this perspective we can think of $B(M,N)$ (or $B'(M,N)$) as characterising the image of M in N^*.

If R is a field, then the rank of matrix $B(M,N)$ is independent of the specific choice of bases (to see this note that it is unchanged on replacing b_1 by a linear combination with nonzero component of b_1). Accordingly

$$\text{rank} \;\; <,> \;\; := \;\; \text{rank } (B(M,N)).$$

(**10.1.25**) For I an ideal of R define $L^I_{<,>}$ as the subset of M such that $< m, n > \in I$ for every $m \in L^I_{<,>}$ and $n \in N$. By linearity $L^I_{<,>}$ is a submodule.

The LEFT RADICAL $L_{<,>} = L^0_{<,>}$ of $<,>$ is the submodule M' of M such that $< m, n > = 0$ for every $m \in M'$ and $n \in N$.

If R is a field then

$$\text{rank} \;\; <,> \;\; = \;\; \dim M - \dim L_{<,>}.$$

(**10.1.26**) Note that $\psi(b_j)$ is the map such that $(\psi(b_j))(c_i) = < b_j, c_i >$. Our basis of N^* is $\{f_i\}$ such that $f_i c_j = \delta_{i,j}$ therefore

$$\psi(b_j) = \sum_i < b_j, c_i > f_i$$

(check: we have $(\sum_i \alpha_i f_i) c_j = \alpha_j$ so $(\sum_i < b_j, c_i > f_i) c_k \; = \; < b_j, c_k >$ as required).

In an example, this says that, expressing the basis of M as $(1,0,0), (0,1,0), (0,0,1)$ (say); and of N^* as $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we can express ψ acting on M by $B(M,N)$ acting on the right:

$$(1,0,0) \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{pmatrix} = (B_{11}, B_{12}) = B_{11} \begin{pmatrix} 1 \\ 0 \end{pmatrix}^T + B_{12} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T$$

or by $B(M,N)^T$ acting on the left. In this formulation the inner product can be written, say,

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^T \begin{pmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} B_{11} & B_{21} & B_{31} \\ B_{12} & B_{22} & B_{32} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = (B_{11})$$

ss:contravf

10.1.6 Contravariant forms on A-modules

In (10.1.24) we gave a correspondence between bilinear forms and maps from modules to dual-modules over a commutative ring R. Next we lift this to modules over an R-algebra A (NB, suitably changing dual right modules to left modules for a *non*-commutative algebra A requires an antiautomorphism).

This will not work for an arbitrary form $<,>: M \to N$, or corresponding $\psi : M \to N^*$, since we have the A-module morphism condition $\psi(am) = a\psi(m)$ for all $a \in A, m \in M$ to satisfy.

de:intertwine

(**10.1.27**) We can think of this condition directly or over specific bases for M and N, where a acts as a matrix $\rho_{B_M}(a)$ and $\rho_{B_N}(a)$ respectively and we must check $B\,\rho_{B_M}(a) \; = \; \rho_{B_N}(a)\,B$ for all $a \in A$ — i.e. that $B = B(M,N)$ 'intertwines' representations $\rho_{B_M}(a)$ and $\rho_{B_N}(a)$. It turns out that this condition can be expressed quite neatly.

(**10.1.28**) CONTRAVARIANT FORM.

Let A be an R–algebra with involutive antiautomorphism t as above. For $M, N \in A-$ mod, an

R–bilinear form $<,>$: $M \times N \to R$ is *contravariant* if $< am, n > = < m, a^t n >$ for all $a \in A$, $m \in M$, $n \in N$.

(**10.1.29**) Note that there is a bilinear form $<,>$ for every choice of matrix $B(N,M)$. The requirement of satisfying the constraints of *contravariant form* on A-modules, however, will in general be very restrictive on possible choices of $B(M,N)$.

de:permf (**10.1.30**) EXAMPLE. Let R be a commutative ring and G a finite group. Let $M = R\{m_1, m_2, ..., m_l\}$ be an RG-module that is free as an R-module, with the given basis. Define a bilinear form on M (i.e. on the pair $(M,N) = (M,M)$) by setting $B(M,M)$ to the identity matrix, that is, by

$$< m_i, m_j > = \delta_{m_i, m_j}.$$

We have $gm_i = m_j \iff m_i = g^{-1}m_j$. Thus *if G acts on M by permuting basis elements* we have $< gm_i, m_j > = < m_i, g^{-1}m_j >$, and hence $< am_i, m_j > = < m_i, a^t m_j >$ for $a \in RG$. Thus this $<,>$ is a contravariant form.

In particular if $M = RG$ is the left regular module then G indeed acts by permuting basis elements, so this $<,>$ is contravariant in this case. (We will give a more explicit example in (10.2.1).)

(**10.1.31**) EXERCISE. (Optional) Construct an example as above but where G does not act on M by permutation, and where the $<,>$ above is indeed not contravariant.

contraformperp (**10.1.32**) PROPOSITION. *Let A be an R-algebra with involutive antiautomorphism t, and $M, N \in A-mod$. If $<,>$: $M \times N \to R$ is a contravariant form and I an ideal in R then the subset $S = L^I_{<,>} \subseteq M$ (such that $< s, N > \in I$ for all $s \in S$) is an A-submodule of M.*

Proof. For $a \in A$, $s \in S$, $n \in N$ we have $< as, n > = < s, a^t n > \in I$ since $a^t n \in N$, so $as \in S$. □

(**10.1.33**) REMARK. Note that if $I' \subset I$ then $L^{I'}_{<,>} \subseteq L^I_{<,>}$. Of course if R is a field then the only possibility is $I = 0$.
Note that if we start with R a commutative ring and compute $S = L^0_{<,>}$; then base change to some $A^k = k \otimes_R A$ and write S^k for the corresponding submodule computed here, then S^k may be bigger than $k \otimes_R S$.

contraformx (**10.1.34**) PROPOSITION. *Let R be a commutative ring and $A = RG$ for some group G (or else an R-algebra with involutive antiautomorphism).*
(I) To each contravariant form $<,>$: $M \times N \to R$ we may associate an element

$$\psi \in \hom_A(M, N^o)$$

given by $\psi(m)(n) = <m, n>$.
(II) This association defines a bijective correspondence between such forms and morphisms.
(III) If R is a field and $M = N$ satisfies the assumptions in Proposition (10.1.20) then there is a unique form up to scalars, and the form is non–singular iff the associated ψ is an isomorphism (in particular it is non–singular if $M = N$ is simple). (cf. [59, §2.7].)

Proof. (I) We first need to show that ψ defined in this way is a homomorphism of left A–modules, i.e. that $a\psi(m) = \psi(am)$. Putting aside the way A acts on it for a moment we have $N^o = N^* = \mathrm{Hom}_R(N,R)$, so $\psi(m) \in \mathrm{Hom}_R(N,R)$. By construction we have that $\psi(am) \in \mathrm{Hom}_R(N,R)$ is given by:

$$\psi(am)(n) \; = < am, n > = < m, a^t n > = \; \psi(m)(a^t n)$$

Meanwhile for $a\psi(m)(n)$ the action of a on the left is achieved by the action of a^t on the right of N^*, which we recall is given by $(\phi a^t)(n) = \phi(a^t n)$ for any $\phi \in N^*$. Thus $(a \circ \psi(m))(n)) = (\psi(m)a^t)(n) = \psi(m)(a^t n)$ as required.

(II) Note that for given $\psi \in \hom_A(M, N^o)$ we can define a form by $<m, n>_\psi \; = \; \psi(m)(n)$.

(III) Finally observe (cf. proposition 8.5.12, noting that the difference between N^o and N^* is not relevant, since the algebra action will not be used) that the rank of the image under ψ is rank $<,>$. □

10.2 Examples

10.2.1 Examples: contravariant forms on S_n modules

ss:S_n form

See also §11.4 for generalisations of the following example.

We continue to use string notation for elements of $\mathbf{Set}(\underline{n}, \underline{m})$ as in (6.1.6) et seq, and in particular for elements in $C_{\mathbb{N}}(n,m)$, the morphism set in the $\mathbf{Set}^f$ skeleton category $C_{\mathbb{N}}$ (see (6.3.4)). For example $123111 \in C_{\mathbb{N}}(6,3)$. Thus there is an action of $C_{\mathbb{N}}(n,m)$ on $C_{\mathbb{N}}(l,n)$: more naturally seen as the category composition $\circ : C_{\mathbb{N}}(n,m) \times C_{\mathbb{N}}(l,n) \to C_{\mathbb{N}}(l,m)$. Indeed this could be seen as an 'action' from the right to the left. In particular there is an action of the monoid $C_{\mathbb{N}}(n,n)$ on $C_{\mathbb{N}}(l,n)$; and a right-action of $C_{\mathbb{N}}(l,l)$ also on $C_{\mathbb{N}}(l,n)$. Restricting these monoids to the subgroups of isomorphisms we thus have actions of S_n and S_l. To settle which is 'left' and 'right', we have to review conventions.

ex:S_3 form (**10.2.1**) EXAMPLE. The symmetric group S_3 acts on the set of sequences $T_{2,1} = \{211, 121, 112\}$ by place permutation:

$$g_1 211 = 121, \qquad g_1 112 = 112$$

and so on (g_i denotes elementary transposition $(i\, i{+}1) \in S_n$). This is a left-action:

$$g_2(g_1 211) = g_2 121 = 112 = (g_2 g_1) 211$$

The set of sequences thus forms a basis for a left $\mathbb{Z}S_3$-module, $M = \mathbb{Z}T_{2,1}$. (Or similarly over any ground ring.) The bilinear form on M given on $T_{2,1}$ by

$$< t, t' > \; = \; \delta_{t,t'}$$

is contravariant. We can see this by (10.1.30), or as follows. If $gt = t'$ for some $g \in S_3$, then $g^{-1}t' = t$ so $< gt, t' > = < t, g^{-1}t' >$.

Note that the rank of a form depends, in general, on the ground field. However in our case there is clearly no such dependence. Since this form is of full rank it defines an isomorphism between M and M^o. (Indeed since the intertwiner, as in (10.1.27), is the identity matrix, it is a rather 'uninteresting looking' isomorphism. Of course M does not satisfy the conditions of proposition 10.1.20. Next we construct a submodule which does.)

(**10.2.2**) We can restrict our form above to a form on a submodule S. For example, consider the element of $\mathbb{Z}S_3$ given by $V_{13} = 1 - (13)$ and the submodule of M generated by

$$e_{112} \; := \; V_{13} 112 \; = 112 - 211$$

This is spanned by e_{112} and $e_{121} = 121 - 211$. (Remark: This submodule will turn out to satisfy the conditions of proposition 10.1.20.)

Just for completeness we note that the antirepresentation corresponding to this basis is given as follows:

$$g_1 \begin{pmatrix} e_{112} \\ e_{121} \end{pmatrix} = g_1 \begin{pmatrix} 112-211 \\ 121-211 \end{pmatrix} = \begin{pmatrix} 112-121 \\ 211-121 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}}_{\rho'(g_1)} \begin{pmatrix} e_{112} \\ e_{121} \end{pmatrix}$$

$$g_2 \begin{pmatrix} e_{112} \\ e_{121} \end{pmatrix} = g_2 \begin{pmatrix} 112-211 \\ 121-211 \end{pmatrix} = \begin{pmatrix} 121-211 \\ 112-211 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\rho'(g_2)} \begin{pmatrix} e_{112} \\ e_{121} \end{pmatrix}$$

A couple more identities:

$$g_1(e_{112}+e_{121}) = e_{112} - 2e_{121}, \qquad g_2(e_{112}+e_{121}) = e_{112}+e_{121} \tag{10.7}$$ linalgxxx

(**10.2.3**) Writing $E = \{e_{112}, e_{121}\}$ for the ordered basis of S here, the restricted form has Gram matrix

$$B(E,E) = \begin{pmatrix} \langle e_{112}| \\ \langle e_{121}| \end{pmatrix} (\ |e_{112}\rangle, |e_{121}\rangle\ ) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

(in the obvious notation).

Via *different* changes to the left and right basis, realised here by unimodular transformations, we get that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}}_{B(E,E)} \left(\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \right) = \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}_{B(E',E'')} \tag{10.8}$$ eq:unimexa

is an equivalent (and in Smith form). This clearly gives Gram determinant 3 ('up to units').

We see from this that the Gram matrix has full rank over $\mathbb{Q}$, but not so over certain other fields. Specifically over a field of char.3 then, the rank is not full.

(**10.2.4**) What can we deduce from this? The form tells us that there is a module map out of S with a kernel $K \subset S$ of dimension equal to the corank. The kernel is a submodule. Thus S is not simple in the char.3 case.

The dimension of the quotient module S/K is equal to the gram matrix rank. But we cannot yet deduce directly from this that the quotient by the kernel is simple (of course in char.3 here the rank is 1 and we are done, but this would not be the case in general). Indeed, if 3 is a unit in the field then the rank is full. But this is *not* enough, of itself, to deduce that S is then simple (as the case of module M itself illustrates). For this we need to investigate the specifics of S a little further.

As we shall see, over a field F this submodule S of M has a simple head (which may or may not be the whole thing, depending on F). Note that if it is not the whole thing then the submodule is not a direct summand of the original module M (since this is contravariant self-dual).

We note that the unimodular transformations applied in (10.8) give

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \langle e_{112}| \\ \langle e_{121}| \end{pmatrix} (\ |e_{112}\rangle, |e_{121}\rangle\ ) \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \langle e_{112}+e_{121}| \\ \langle e_{121}| \end{pmatrix} (\ |2e_{112}-e_{121}\rangle, |-e_{112}+e_{121}\rangle\ ) = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

From (10.7), or directly, we see that the first new basis vector on the left

$$a_1 := e_{112}+e_{121}$$

(and separately $2e_{112}-e_{121}$ on the right) spans a submodule in char.3. Indeed this is a copy of the trivial module (from which we may deduce that the quotient is a copy of the alternating module).

Exercise: check that this vector lies in the kernel of the induced module map in char.3.

(**10.2.5**) Examples of this specific type — Specht modules S in permutation modules M — are very important in S_n representation theory. We finish here with some manipulations specific to this setting (see §11.4 for more details). We investigate S by returning to look at other submodules of M .

Fixing a field F, let S' be the set of vectors in M orthogonal to the above submodule S with respect to the original form ($112+121+211 \in S'$ for example, over any field). Note that S' is another submodule of M. Note that, depending on F, this new submodule is either non-intersecting of S, in which case the restricted form is of full rank, or intersects S in a submodule of dimension given by the discrepancy between the full rank and the actual rank of the form on S. (In our case $S \ni 2e_{112}-e_{121} = 2.112 - 121 - 211 \equiv -(112+121+211)$ over a field of characteristic 3, so the new module, over such a field, intersects S.) We will see that the quotient of S by $S \cap S'$ is simple (indeed absolutely irreducible). This is because of the following key result.

Evidently $V_{13}121 = 0$, so for any $m \in M$ we have $V_{13}m \in Fe_{112}$.

Suppose m is in some submodule T of M, so either $V_{13}m = 0$ or $V_{13}m \neq 0$. In the latter case $e_{112} \in T$ so $S \hookrightarrow T$. In the former case

$$0 = < V_{13}m, 112 > = < m, V_{13}^t 112 > = < m, V_{13}112 > = < m, e_{112} >$$

so $T \hookrightarrow S'$. Now suppose in particular that T is a submodule of S in M. Then either it is the whole of S, or it is also in S', and hence in $S \cap S'$. This shows that $S/S \cap S'$ is irreducible.

This is the same as to say that its Gram matrix is non-singular over F.

(**10.2.6**) There are several further related exampled later. See (11.4.5), (**??**) ...

10.2.2 Examples: T_n modules (Temperley–Lieb)

ss:Tncov01

(**10.2.7**) Recall the T_n-modules $D_n^{\text{m}}(l)$ constructed in §2.2, and the c-v form from (2.3.14). In particular

$$\mathrm{Gram}_n(n-2) = \begin{pmatrix} [2] & 1 & 0 & & \\ 1 & [2] & 1 & 0 & \\ 0 & 1 & [2] & 1 & \\ & & & \ddots & \\ 0 & \dots & 0 & 1 & [2] \end{pmatrix} \tag{10.9}$$ eq:TLgram0002

$$|\mathrm{Gram}_n(n-2)| = [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{10.10}$$ xTLgramx

(**10.2.8**) Remark. We can view this gram matrix as an intertwiner for the map $\psi : D_n^{\text{TL}}(n-2) \to D_n^{\text{TL}}(n-2)^o$ corresponding to the c-v form. To see this let us first construct the representations $\rho_{n,n-2}$ and $\rho^o_{n,n-2}$ from the diagram bases. We have for example

$$\rho_{5,3}(U_1) = \begin{pmatrix} [2] & 1 & & \\ 0 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \qquad \text{and} \qquad \rho^o_{5,3}(U_1) = \begin{pmatrix} [2] & 0 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$\rho_{5,3}(U_2) = \begin{pmatrix} 0 & & & \\ 1 & [2] & 1 & \\ & 0 & 0 & \\ & & & 0 \end{pmatrix} \qquad \text{and} \qquad \rho^o_{5,3}(U_2) = \begin{pmatrix} 0 & 1 & & \\ 0 & [2] & 0 & \\ 0 & 1 & 0 & \\ & & & 0 \end{pmatrix}$$

and so on, whereupon we check

$$\begin{pmatrix} [2] & 0 & & \\ 1 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} [2] & 1 & 0 & \\ 1 & [2] & 1 & 0 \\ 0 & 1 & [2] & 1 \\ 0 & & 1 & [2] \end{pmatrix} = \begin{pmatrix} [2] & 1 & 0 & \\ 1 & [2] & 1 & 0 \\ 0 & 1 & [2] & 1 \\ 0 & & 1 & [2] \end{pmatrix} \begin{pmatrix} [2] & 1 & & \\ 0 & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & & \\ 0 & [2] & 0 & \\ 0 & 1 & 0 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} [2] & 1 & 0 & \\ 1 & [2] & 1 & 0 \\ 0 & 1 & [2] & 1 \\ 0 & & 1 & [2] \end{pmatrix} = \begin{pmatrix} 1 & [2] & 1 & 0 \\ [2] & [2]^2 & [2] & 0 \\ 1 & [2] & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(which is symmetric, so equal to transpose, so checks) and so on (the remainder checking out by 'left-right' symmetry).

Observe that this makes sense over the 'integral' ground ring, where the gram matrix is not invertible; and over any specialisation, including cases where the gram matrix has maximal rank or various different ranks.

pa:Smithx (**10.2.9**) One easily checks that there is a Smith normal form for $\text{Gram}_n(n-2)$:

$$\mathfrak{SN}_n(n-2) \;=\; \begin{pmatrix} [n] & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ & & & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

(Indeed this comes via unimodular matrices over $\mathbb{Z}[q, q^{-1}]$, even though this is not a PID. See e.g. §8.2.4.)

(**10.2.10**) For the algebras T_n one does not need to compute any more gram matrices to determine the structure of the algebra. However, as an exercise, we can consider $\text{Gram}_n(n-4)$. We have for example

$$\text{Gram}_n(n-4) \;=$$

and in general

$$|\text{Gram}_n(n-4)| \;=\; \frac{[n-1]}{[2]}[n-2]^{n-1}$$

...

(**10.2.11**) JOBS: FIX AND FINISH THE ABOVE!
WE want to drop in the gramdetbyGST.tex notes somewhere. maybe here??

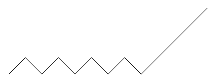
10.3 Determinant computation: Generic structure theorem method

Our next method for computing the Temperley–Lieb standard gram determinant $\det(M_n(\lambda))$ uses the generic structure theorem [106, Th.1]. (Let us reiterate that there is no need, in representation theory, to compute any but the elementary $\lambda = n-2$ case. We continue merely as an exercise in service of later generalisations.)

The theorem gives a basis of the TL algebra for generic parameter over a suitable field in the form of a complete orthonormal set for the multimatrix structure. In this case we can extract a basis for each cell module such that, by orthonormality, the gram matrix is the unit matrix. Unlike the gram matrix over the defining basis, this is of no intrinsic use: manifestly since it contains no information; but implicitly since we cannot specialise the parameter as we can if we work over the 'integral' (polynomial) ring. The trick is to keep track of the renormalisations of elements in passing from the integral basis, so that we can reconstruct the determinant over the integral basis.

The basis elements we need are defined as follows (we extract directly from [106, §6.4]). Fix a cell module $\Delta_n(\lambda)$ and consider the walk enumeration of the diagram basis (walks of length n on the positive part of the integral line that start at 1 and end at $\lambda+1$). The 'lowest' walk is

$$e_m \;=\; 121212123...m \;=$$



where $m = \lambda + 1$. Algebra basis elements correspond to pairs of walks. The lowest walk (pair) is mapped to an element by, for example, with $m = \lambda + 1$:

$$(e_m, e_m) = (121212123...m, 121212123...m) = U_1 U_3 U_5 E_m^{(6)}$$

(see [106, §6.4] for notation) (note normalisation as idempotent needed). Then if $s_i = g-1$ is a minimum of sequence s, and s^i denotes s with s_i replaced by $g+1$:

$$(s^i, t) = \sqrt{k_g k_{g+1}} \left(1 - \frac{U_i}{k_g}\right)(s, t)$$

where $k_g = \frac{[g-1]}{[g]}$.

The integral/diagram basis can be considered to start with (e_m, e_m) (up to quotients and id normalisation). The next diagram basis element, corresponding to the walk 123212123...m, is

$U_2(e_m, e_m)$. Thus we see that the new orthonormal basis element (s^i, t) in this case is got by adding a scalar multiple of the first (irrelevant for the determinant); and rescaling the second by

$$\sqrt{\frac{h-1}{h+1}\frac{h}{h-1}} = \sqrt{\frac{[h]^2}{[h+1][h-1]}}$$

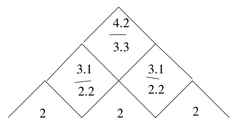
$(h = g)$. This applies both to the bra and the ket, and both rescale the determinant. For a general walk there is a factor like this for each diamond added to get to it from the 121212123...m walk. Thus altogether we have the following.

Lemma 10.1. *Schematically,*

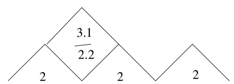
$$\det(M_n(\lambda)) = \prod_s \prod_d \frac{[h]^2}{[h+1][h-1]}$$

where the products are over the set of walks and the diamonds in each walk. □

Example: The factors for each diamond are (in abridged notation):



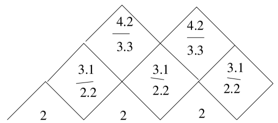
Thus in particular



gives a factor $\frac{[3][2]}{1}$. Altogether we have

$$\det(M_6(\lambda = 0)) = \frac{[2]^3}{1}\frac{[3][2]}{1}\frac{[3][2]}{1}\frac{[3]^2}{[2]}\frac{[4]}{1} = [2]^4[3]^4[4] = \delta^5(\delta^2-1)^4(\delta^2-2)$$

Example II:

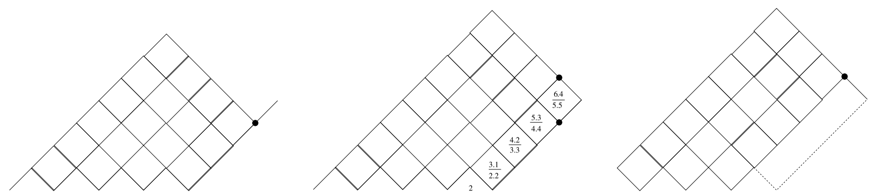


gives a factor $\frac{[4]^2}{[2][3]}$. Altogether this det comes out as

$$\det(M_7(\lambda = 1)) = \frac{[2]^{3\times 14}}{1}\left(\frac{[3]}{[2]^2}\right)^{9+10+9}\left(\frac{[4][2]}{[3]^2}\right)^{4+4}\left(\frac{[5][3]}{[4]^2}\right)^{1} = [2]^{-6}[3]^{13}[4]^6[5]$$

Observe that the determinants can be computed recursively. The walks that pass through $(\lambda - 1, n-1)$ do not pick up another diamond in the final step, so contribute a factor $det(M_{n-1}(\lambda-1))$;

and those that pass $(\lambda+1, n-1)$ all pick up the same string of diamonds, compared to their contribution to $det(M_{n-1}(\lambda+1))$. For example:



We have

$$det(M_n(\lambda)) = det(M_{n-1}(\lambda-1)) \;.\; F_n(\lambda+1)^{dim(M_{n-1}(\lambda+1))}.det(M_{n-1}(\lambda+1)) \qquad (10.11)$$ eq:det-recurs

where the diamond-string factor is given by

$$F_n(\lambda) = [2]\frac{3.1}{2.2}\frac{4.2}{3.3}\frac{5.3}{4.4}...\frac{[\lambda+1].[\lambda-1]}{[\lambda].[\lambda]} = \frac{[\lambda+1]}{[\lambda]}$$

(using the figure notation in the middle) - note that it does not actually depend on n. This gives a very quick proof (modulo knowing the structure Theorem) of the TL part of the result in [?]. We reproduce the relevant figure from [?] in Figure 10.1; and with globalisation fibres highlighted in Figure 10.2.

Example:

$$det(M_5(1)) = det(M_4(0))\,.F_5(2)^{dim(M_4(2))}.\,det(M_4(2)) = [2]^2[3].\left(\frac{[3]}{[2]}\right)^3.[4] = \frac{[4][3]^4}{[2]}$$

Now let us arrange the TL part of the result so that the fibres are in vertical array under their sl_2 label (and hence under the appropriate label in the projected Bratelli diagram):

$\lambda = 0$	1	2	3	4	5	
1						
	1					
[2]		1				
	[3]		1			
$[2]^2[3]$		[4]		1		
	$[2]^{-1}[3]^4[4]$		[5]		1	
$[2]^4[3]^4[4]$		$[2]^{-1}[4]^5[5]$		[6]		1

It is convenient to augment to also show the module dimensions (in red): Fig.10.3. And in Fig.10.4 we include a schematic indication of the recursive calculations from (10.11).

At its most succinct this scheme can be reduced to:

$$0 \underset{[2]/1}{\overset{1}{\rightleftarrows}} 1 \underset{[3]/[2]}{\overset{1}{\rightleftarrows}} 2 \underset{[4]/[3]}{\overset{1}{\rightleftarrows}} 3 \underset{[5]/[4]}{\overset{1}{\rightleftarrows}} 4 \qquad \cdots$$

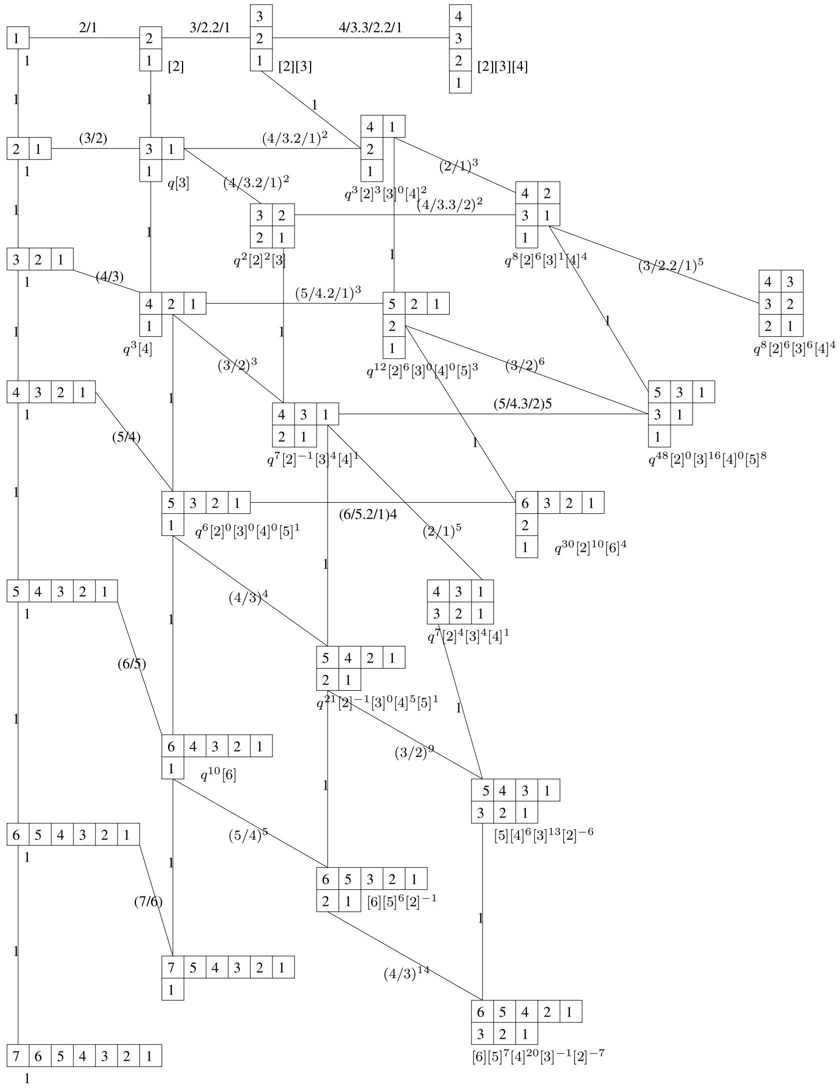


Figure 10.1: Specht Gram determinant data on Young graph (unfinished - reproduced from paper; some later q^i factors omitted, but q is a unit here so these are not relevant; the hook lengths shown are also not particularly relevant here). fig:stray1

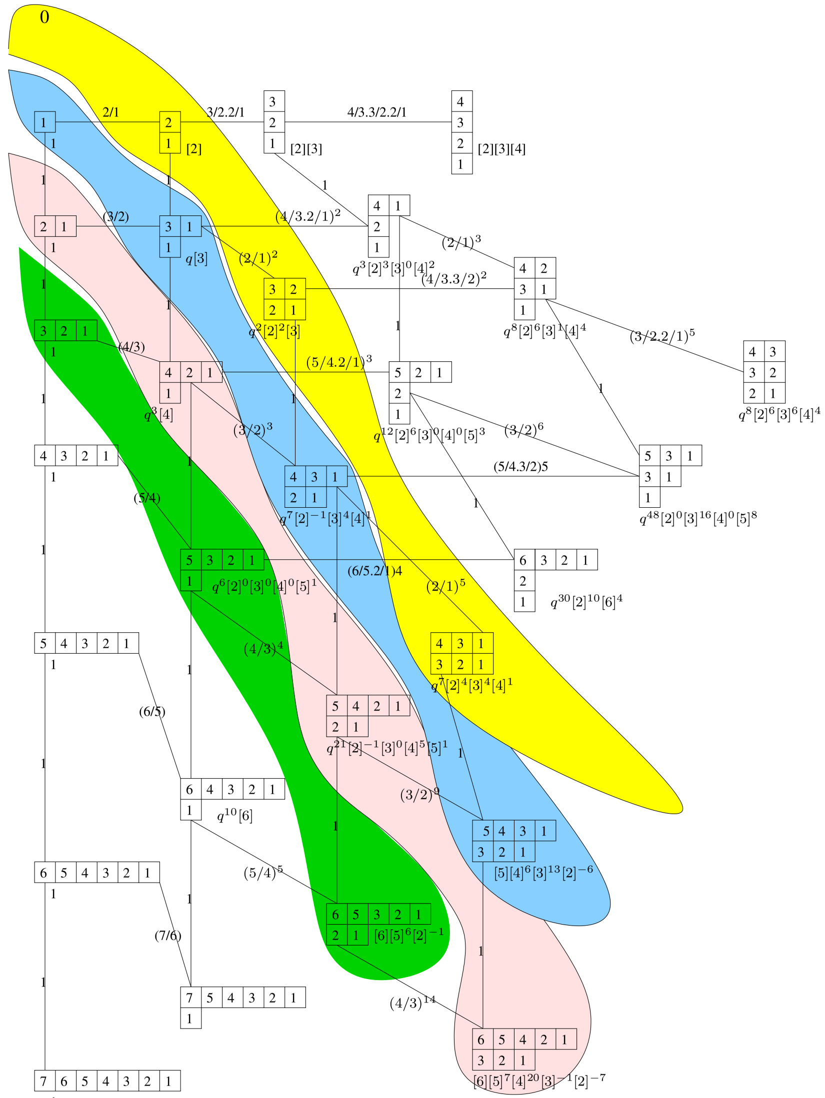


Figure 10.2: Coloured streaks indicate fibres projected to single point in TL global limit. fig:streaky1

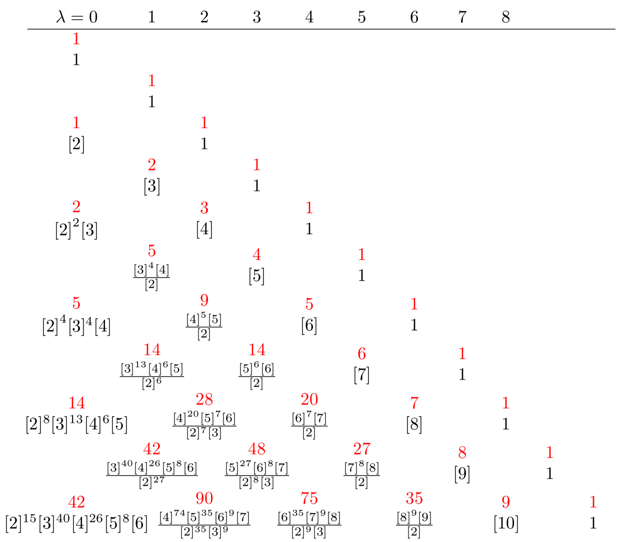


fig:gramdets01

Figure 10.3: Standard Gram determinants restricting to the TL weights and then organising by sl_2 labels.

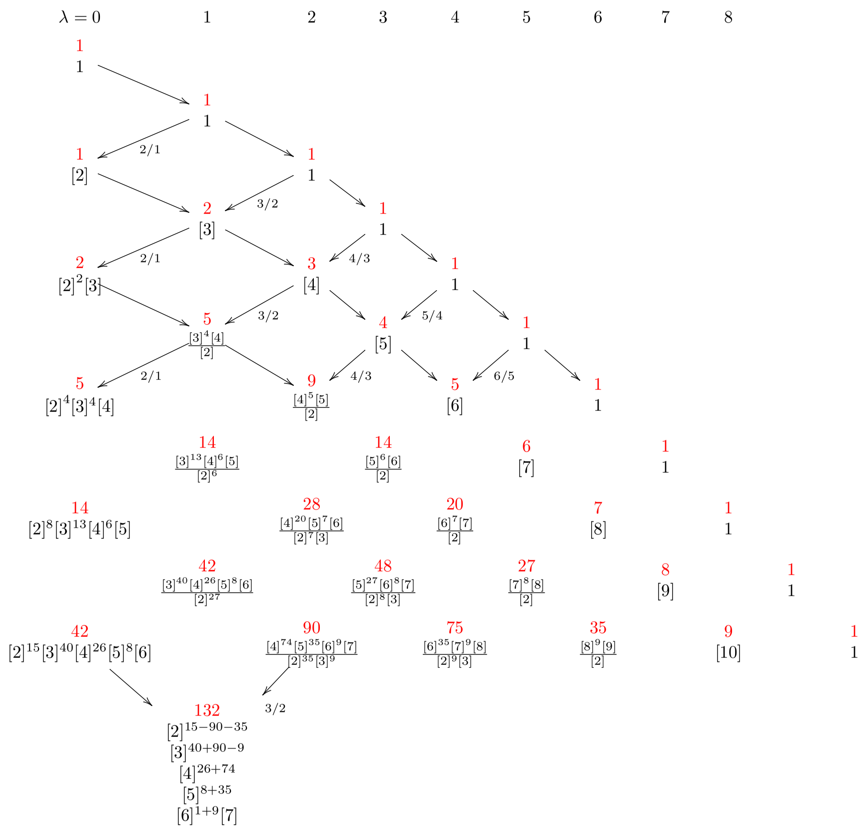


fig:gramdets031

Figure 10.4: The recursive construction for gram determinants, as in (10.11). The weightings $([\lambda+1]/[\lambda])^{dim(M_{n-1}(\lambda+1))}$ are drawn simply as $\lambda+1/\lambda$. We omit drawing edges after the various patterns are established, apart from the explicit calculation shown at the bottom.

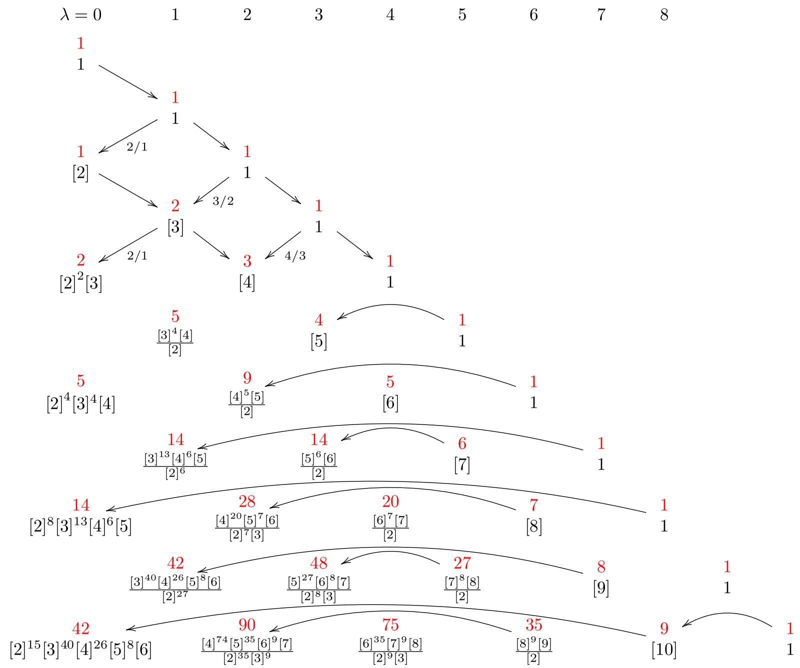


fig:gramdets03 Figure 10.5: Standard to standard module maps when $[5]=0$. This is perhaps best viewed alcove-geometrically - the ambient space is $\mathbb{R}$; the origin is at $\lambda=-1$ (it is 'ρ-shifted'); and the first affine wall at $\lambda=-1+5=4$.

meaning that $det(M_n(\lambda))$ is the product of the $det(M_{n-1}(\mu))$'s corresponding to the incoming arrows (the λ, μ labels are on the vertices); together with the factors *on* the incoming arrows raised to the power of $dim(M_{n-1}(\lambda+1)$.

To see how these determinants check against representation theory, consider for example Fig.10.5. Here we focus on the parameter value such that $[5]=0$. The first determinant that vanishes is then indicated in the figure. This vanishing indicates that there is a non-isomorphism map from the corresponding costandard module to the standard module; and hence that the standard module has a socle. In this case it is one-dimensional and it is straightforward to deduce that this corresponds to a map in from the trivial module. — And this is what the curved arrow indicates.

In the next layer we see a similar vanishing; and deduce a similar map.

In the next layer we have another. But we also have a globalisation image of the very first map - where the module mapping in is the globalisation image of the trivial, and so not the trivial. Note that this has a direct affect on the degree of vanishing of the determinant (recall we have connected this to the Smith form of the gram matrix).

Hopefully the subsequent pattern will be clear. Perhaps the next interesting observation is in the last rank shown. Here we have a map from the trivial module (of course $[5]=0$ implies $[10]=0$); and we already have an on-map, due to the globalisation property. By inspecting the degrees of vanishing of the various gram matrices we can deduce that the composite map vanishes here.

10.3.1 What about the Smith form?

More than the determinant evaluated over the ground field of interest, K say, one cares about the rank of the gram matrix (evaluated over the ground field of interest). This is because the rank determines the rank of the contravariant form, and hence the dimension of the simple head of the cell module over K. Elementary row and column operations expressible as multiplication by invertible matrices do not change the rank. Thus, if we are working over a PID (as we are in the indeterminate-δ calculations if the coefficient ring is a field) we are interested in the Smith normal form.

Reduced to the ground field of interest the SNF will give the rank directly. (Even if the ground ring is not a PID there may be a form which directly reveals the rank.) (The SNF may reveal even more than the rank in general. The power of vanishing of invariant factors vanishing at the δ-value of interest may reveal details of factor modules deeper in the Jordan–Holder series of the cell module.)

Example: Consider the gram matrix

$$\begin{pmatrix} \delta & 1 & 0 \\ 1 & \delta & 1 \\ 0 & 1 & \delta \end{pmatrix} \overset{R1:R1+R3}{\rightsquigarrow} \begin{pmatrix} \delta & 2 & \delta \\ 1 & \delta & 1 \\ 0 & 1 & \delta \end{pmatrix} \overset{R1:R1-\delta R2}{\rightsquigarrow} \begin{pmatrix} 0 & 2-\delta^2 & 0 \\ 1 & \delta & 1 \\ 0 & 1 & \delta \end{pmatrix} \overset{C2:C2-\delta C1;...}{\rightsquigarrow}$$

$$\begin{pmatrix} 0 & 2-\delta^2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & \delta \end{pmatrix} \overset{R1:R1+(\delta^2-2)R3}{\rightsquigarrow} \begin{pmatrix} 0 & 0 & -\delta(2-\delta^2) \\ 1 & 0 & 0 \\ 0 & 1 & \delta \end{pmatrix} \ldots$$

We can compute the SNF in various ways. It is

$$\begin{pmatrix} \delta(\delta^2-2) & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Method 1: elementary row and column operations.
Method 2: compute minors and take the HCF of all minors at each rank. For example the HCF of 1×1 minors in our example is clearly 1; as is that of 2×2 minors. The final invariant factor is then forced.

Of course the product of invariant factors (one for each basis element) in the SNF is the determinant, which we just computed in §**??** above. Note that there is a factor for each basis element in our computation. However this is certainly not any invariant factor in general. For example the factor we computed in (**??**) is not 'integral'.

...

Note that there is more on this topic in §2.3.1, including a table of Smith forms in Fig.2.6. - We can just reproduce the figure here - see Fig.10.6.

A notable feature here is that every factor vanishes at most linearly at any given parameter value. This corresponds broadly to the Loewy length of standard modules being at most 2 at any given complex parameter value.

Given the linearity property, then note that many of these forms are forced directly from the overall determinant. For example the case shown in blue in the figure has det $[7][6]^7/[2]$. In order to have a division chain the first factor must be divisible by [7] and [6] and so must be [7][6]; and the next must be divisible by [6] and so must be [6], and so on.

10.3.2 Examples: To Do! Full transformation semigroup

...

$$1$$

$$[2] \qquad 1$$

$$\begin{pmatrix} [3] \\ 1 \end{pmatrix} \qquad 1$$

$$\begin{pmatrix} [3][2] \\ [2] \end{pmatrix} \qquad \begin{pmatrix} [4] \\ 1 \\ 1 \end{pmatrix} \qquad 1$$

$$\begin{pmatrix} \frac{[4][3]}{[2]} \\ [3] \\ [3] \\ [3] \\ 1 \end{pmatrix} \qquad \begin{pmatrix} [5] \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad 1$$

$$\begin{pmatrix} [4][3] \\ [3][2] \\ [3][2] \\ [3][2] \\ [2] \end{pmatrix} \qquad \begin{pmatrix} [5][4] \\ ([4],3) \\ \frac{[4]}{[2]} \\ (1,4) \end{pmatrix} \qquad \begin{pmatrix} [6] \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad 1$$

$$\begin{pmatrix} \frac{[5][4][3]}{[2]} \\ (\frac{[4][3]}{[2]},5) \\ ([3],7) \\ 1 \end{pmatrix} \qquad \begin{pmatrix} \frac{[6][5]}{[2]} \\ ([5],5) \\ (1,8) \end{pmatrix} \qquad \begin{pmatrix} [7] \\ (1,5) \end{pmatrix} \qquad 1$$

$$\begin{pmatrix} [5][4][3] \\ ([4][3],5) \\ ([3][2],7) \\ [2] \end{pmatrix} \qquad \begin{pmatrix} \frac{[6][5][4]}{[3][2]} \\ ([5][4],6) \\ ([4],7) \\ (\frac{[4]}{[2]},6) \\ (1,8) \end{pmatrix} \qquad \begin{pmatrix} [7][6] \\ ([6],5) \\ \frac{[6]}{[2]} \\ (1,13) \end{pmatrix} \qquad \begin{pmatrix} [8] \\ (1,6) \end{pmatrix} \qquad 1$$

$$\begin{pmatrix} \frac{[6][5][4]}{[2]^2} \\ (\frac{[5][4][3]}{[2]},7) \\ (\frac{[4][3]}{[2]},18) \\ ([3],15) \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{[6][5][4]}{[2]} \\ ([5][4][3],7) \\ ([4][3],18) \\ ([3][2],15) \\ [2] \end{pmatrix} \qquad \ldots$$

Figure 10.6: Table of Smith normal forms - each given as the list of invariant factors - for gram matrices of cv-forms on Temperley–Lieb D-modules, working over (say) $\mathbb{Q}[\delta]$. The notation $([4],3)$ means entry [4] occurs 3 times. (This figure is repeated from 2.6; and is from unpublished notes at stripped3.tex, correcting one typo and adding part of a new line.)

;:SmithFormTLcopy