Paul Martin - Basics   ③

Physicists — sometimes need to model interacting many-body systems.
       — following energy is helpful.

$$E_{kin} = \sum_{i=1}^{N} \tfrac{1}{2} m_i v_i^2 .$$

But $\frac{dv}{dt} = F/m$, so need to understand the forces acting.

eg.
$$F_1 = \frac{q_1 q_2}{4\pi \varepsilon_0} \frac{\underline{r_{12}}}{r_{12}^3} = -F_2 .$$ — displacement vector.   Coulomb force. $1 \leftrightarrow 2$.

For a moment think of this as a force field created by q_2 acting on q_1.
Conservative $\Rightarrow \vec{F} = -\nabla \varphi$, some φ.
$\varphi(\underline{r})$ is part of potential energy of m.
$$E_1 = \tfrac{1}{2} m v_1^2 + \varphi_1 .$$
So E depends on velocities & positions of all atoms in system.
We'll never keep track, prefer to know [bulk] averaged, behaviour.

Call inaccessible complete microscopic specification "microstate" $= \sigma$. — encodes all
                                                                                        v's & posns.
For each σ, we know, in principle, $E(\sigma)$
Ask: what is probability $P(\sigma)$ of finding system in state σ at a given instant?
Expected value of bulk obs.
$$\langle O \rangle = \sum_{\sigma} O(\sigma) P(\sigma) .$$ But what is $O(\sigma)$?
↑↗
exp. value   Intensive: doesn't
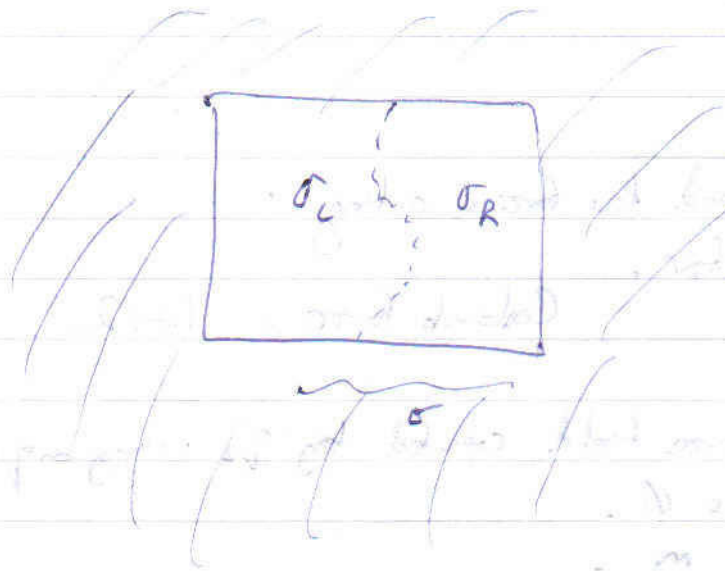   grow.

Choose σ

If all we know is the total energy E, $P(\sigma) = \begin{cases} 0 & E(\sigma) \neq E \\ 1/N_E & E(\sigma) = E \end{cases}$

We can't isolate a realistic system, so can't fix $\mathcal{E}$, but will just assume $P(\sigma) = P(E(\sigma))$.

So what is this function?

σ_L, σ_R indpt. $P(\sigma) = P(\sigma_L) P(\sigma_R)$. Can get energy from either side.

Meanwhile $E = E_L + E_R + E_{interaction}$

$\leadsto E \simeq E_L + E_R$ (geometry). — negligible, "only a line"

The L, R - subsystems will each have their own energy probability function.

$$P(E_L + E_R) = P_L(E_L) P_R(E_R).$$

E_L, E_R indpt., so $\dfrac{\partial P(E_L + E_R)}{\partial E_L} = \dfrac{\partial P(E_L + E_R)}{\partial E_R}.$

$\dfrac{P_L'(E_L)}{P_L(E_L)} = \dfrac{P_R'(E_R)}{P_R(E_R)} = -\beta$ cf wave equation.

write as $-\beta$, as must be constant.

"const of separation"

$$P'_L(E_L) = -\beta P_L(E_L)$$
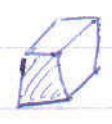
$$\therefore P(E) = C \exp(-\beta E).$$

C fixed by $\sum_\sigma P(E(\sigma)) = 1$.

β is only thing connecting subsystems

$$Z(\beta) = \sum_\sigma \exp(-\beta E(\sigma)) \left(= \frac{1}{C}\right)$$

— (inverse) temperature.

"Zustatensummer" (Boltzmann) is partition function.

$$P(E) = \frac{\exp(-\beta E)}{Z}$$

realistic systems & even with no kinetics — absorb into β.

Z is "just" a normalising factor.

$$\frac{d \ln Z}{d\beta} = -\frac{1}{Z} \sum_\sigma E(\sigma) \exp(-\beta E(\sigma)) = \text{expected energy}.$$

"internal energy"

$\frac{1}{N}$ der energy density U.

So its analysis contains physics.

Suppose $E : S \to \pm N$ (quantised).

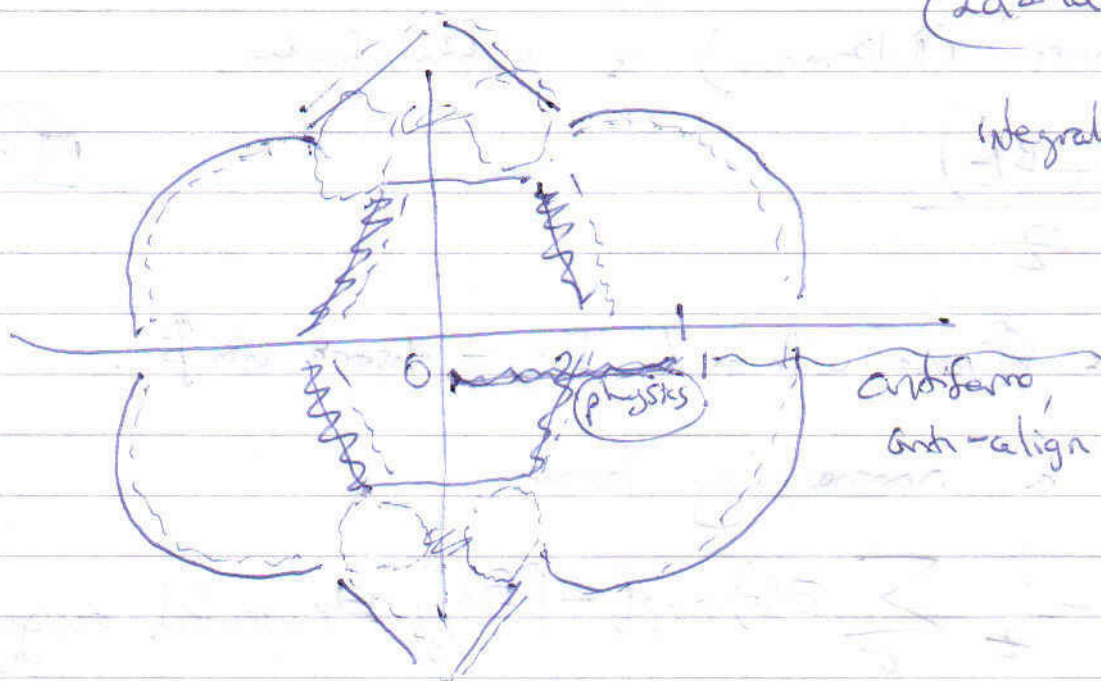Then z poly. in $\exp(\mp\beta)$.

Only analytic structure is zeros.

Descartes rule of signs.

2-state Potts model for any ~~arbitrary~~ cubic graph —

3-d Ising model
not integrable
(2d & 1d are integrable.)

enough symms →
Integrable — closed
form of p.fn.
even large N.

O $\cdots$ physics $\cdots$

antiferro,
Anti-align.

Qn What does this tell us?

Specific heat tells us the r.o.c. of internal energy with Temp.
(or in practice, other way round) :

$$S = \frac{\gamma^2 \frac{1}{N} \ln z}{\partial \beta^2}.$$

$x = e^{\beta}$.

$z = \prod_j (x - x_j)$.

Integ. means
knowing x_j's.

$$\frac{\partial(\frac{1}{N} \ln z)}{\partial \beta} = \frac{x}{N} \sum_j \frac{1}{x - x_j}.$$

— get a (large) contribution from each zero, $x - x_j \to 0$

$\frac{1}{x - x_j}$ gets large.

eg water, 2 phase transitions, use TL algebra.