CODING THEORY: SOLUTIONS 4

(1)

(a)

$$C_1: \begin{array}{cccc} 000 & 101 & 011 & 110 \\ 100 & 001 & 111 & 010 \end{array} \quad , \quad C_2: \begin{array}{cccc} 0000 & 1010 & 0101 & 1111 \\ 1000 & 0010 & 1101 & 0111 \\ 0100 & 1110 & 0001 & 1011 \\ 1100 & 0110 & 1001 & 0011 \end{array}$$

$\{3\ marks\}$

(b) (i) 1101 can be found in the third column of the standard array and therefore decodes to the top entry in the third column, namely 0101. Similaly 0011 lies in the fourth column of the standard array and so decodes to 1111. $\{1\ marks\}$

(ii) The only double errors which will be corrected are those whose error vectors are the weight 2 coset leaders. With the array constructed above, say 1111 was transmitted with errors in the first two digits and so was received as 0011. This is correctly decoded in the array as 1111. $\{1\ marks\}$

(iii) Any weight two error vector other than 1100 will give an uncorrected error using this array. Say 1111 was transmitted with errors in the last two digits and so was received as 1100. This is incorrectly decoded as 0000. $\{1\ marks\}$

(c) The code will be decoded correctly if the error vector is one of the coset leaders in the standard array. Hence,

$$\begin{aligned} P_{corr}(C_2) &= P(e=0000) + P(e=1000) + P(e=0100) + p(e=1100) \\ &= (1-p)^4 + p(1-p)^3 + p(1-p)^3 + p^2(1-p)^2 \\ &= (1-p)^4 + 2p(1-p)^3 + p^2(1-p)^2 \\ &= (0.99)^4 + 0.02(0.99)^3 + (0.01)^2(0.99)^2 = 0.9801 \end{aligned}$$

Then $P_{error}(C_2) = 1 - P_{corr}(C_2) =$ $\{3\ marks\}$

(d) An undetected transmission error occurs if and only if the error vector is a nonzero codeword. We have,

$$P_{undetec}(C) = \sum_{w=1}^{n} \delta_w p^w (1-p)^{n-w}$$

where δ_w is the number of codewords of weight w. C_2 contains 2 vectors of weight 2 and and 1 vector of weight 4 so we get that,

$$P_{undetec} = 2p^2(1-p)^2 + p^4 = 0.00019603$$

$\{2\ marks\}$

(2) We first need to turn G into standard form. This can be done by swapping the three rows around to get,

$$G = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The PCM is therefore,

$$H = \begin{bmatrix} -2 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix}$$

$\{2\ marks\}$

Now H generates $C^{\perp}$, so $C^{\perp} = \{0000, 1021, 2012\}$. $\{1\ marks\}$

(3)

(a) G is already in standard form therefore we can just write down the PCM as,

$$H = \begin{pmatrix} -2 & -3 & -5 & 1 & 0 \\ -2 & -4 & -6 & 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 5 & 4 & 2 & 1 & 0 \\ 5 & 3 & 1 & 0 & 1 \end{pmatrix}$$

$\{2\ marks\}$

(b)

$$G.H^t = \begin{pmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 5 & 6 \end{pmatrix} \begin{pmatrix} 5 & 5 \\ 4 & 3 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & 7 \\ 7 & 7 \\ 7 & 7 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\{1\ marks\}$

The columns of H^t are rows of H which generate $C^{\perp}$. $G.H^t$ assembles the various inner product calculations for C and $C^{\perp}$ which by definition must all be zero. This tells us that we have constructed the correct parity check matrix. $\{1\ marks\}$

(c) We have $00156 \in C$ as it is a row in G and $w(00156)3$ so $d(C) \leq 3$. However, H has no parallel columns, that is one columns H is not a linear combination of another, hence $d(C) \geq 3$ by special case 2 of Theorem 3.89. Hence $d(C) = 3$ as required. $\{2\ marks\}$

(d) The standard array must have all elements of $\mathbb{Z}_7^5$ and therefore will have $|\mathbb{Z}_7^5| = 7^5$ entries. As $|C| = 7^3$ we know the standard array will have 7^3 columns and therefore it must have $7^5/7^3 = 7^2$ rows. As each row starts with a coset leader we deduce there must be 7^2 coset leaders overall. Now $d(C) = 3 = 2(1)+1$ so by theorem 3.69, every vector of weight ≤ 1 will be a coset leader, so all weight 1 vectors are coset leaders. There are $5(7-1) = 30$ of these. $\{2\ marks\}$

(e) Suppose we transmit x and receive $y = 11254$. We are told a single error has occurred.

$$y.H^t = (11254)\begin{pmatrix} 5 & 5 \\ 4 & 3 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = (18, 14) \cong (4, 0)$$

Therefore we look for a coset leader of e of weight 1 such that,

$$e.H^t = (4, 0)$$

$e = (00040)$ works (check this). Hence we decode as $x = y - 00040 = 11214$.

$\{3\ marks\}$