Chapter 8

Ring–modules

ch:ringmod

8.1 Ring–modules

Here R will be a ring.

(**8.1.1**) (Left) R–Module M: M an abelian group with map $R \times M \to M$ (written $(r,m) \mapsto rm$) such that $r(x+y) = rx + ry$, $(r+s)x = rx + sx$,

$$(rs)x = r(sx) \qquad \text{('left-action condition')},$$

$1x = x$ ($r \in R$, $x, y \in M$).

Right modules defined similarly (but instead $(rs)x = s(rx)$; or equivalently we can change the notation to $(r,m) \mapsto mr$ giving $x(rs) = (xr)s$).

8.1.1 Examples

(**8.1.2**) Example. Ring R is both a left and a right R–module by the ring multiplication (on the left and on the right respectively).
Every left R–module is a right R^{op} module, where R^{op} is the opposite ring.

(**8.1.3**) Example. Let M be a left module and $m \in M$, then

$$Rm = \{rm \mid r \in R\}$$

ex:CRmod1 is a submodule.

In particular consider $\mathbb{C}$ as an $\mathbb{R}$-module. Then $\mathbb{R}1 = \mathbb{R}$ is a submodule — the real line; and $\mathbb{R}i$ is a submodule — the imaginary line; and so on. Note here that there are infinitely many such submodules, but the sum of any two of them is $\mathbb{C}$.

(**8.1.4**) Example. The set $R \times R$ is an abelian group by $(a,b) + (c,d) = (a+c, b+d)$ and an R-module by the action $r(a,b) = (ra, rb)$.

(**8.1.5**) Example. Let k be a field and V a k-space. Then the set $\mathrm{End}(V)$ of k-linear maps on V is a ring (and a k-algebra). Thus V is a module over this ring.

In the k-algebra setting, given a subset $S \in \mathrm{End}(V)$ we may write $\langle S\rangle_k$ for the smallest subalgebra containing S (the subalgebra *generated* by S). Thus V is also a $\langle S\rangle_k$-module.

(**8.1.6**) In general a k-space V as above comes with various possible choices of bilinear form defined upon it (Cf. §10.1.5). (If $k = \mathbb{R}$ or $\mathbb{C}$ we even have an *inner product* — a (conjugate) symmetric (sesqui/)bilinear form $(-,-)$ with associated positive definite quadratic form $q(v) = (v,v)$ (*positive definite*: $v \neq 0$ implies $q(v)$ real, positive) — in the sense of (**??**).)

Fixing such a product $(-,-)$, then each subspace $M \subset V$ has an *orthogonal complement* — the subset $M' \subseteq V$ such that $(m', m) = 0$ for $m' \in M'$ and all $m \in M$.

Now suppose that M is a submodule of V as a module over one of the above subalgebras of $\mathrm{End}(V)$. We can investigate the conditions under which M' is also a submodule. (We consider this further in §10.)

In case $k = \mathbb{C}$, a striking examples of an inner product is the *Hermitian product*

$$(a,b) = \sum_i (a_i)^* b_i$$

where $*$ denotes complex conjugate, and a_i is the coefficient in the standard basis. In this case evidently $M \cap M' = \{0\}$. Now consider a set S of matrices realising a linear action on V (in the standard basis) that fix M. That is, consider S such that M is an $\langle S\rangle_{\mathbb{C}}$-submodule of V. It follows (exercise) that the set of matrices obtained from S by hermitian conjugate fixes M'. And hence if S is fixed setwise by hermtitian conjugation, then it follows that $V = M \oplus M'$ (see (8.2.9)) as a module for the corresponding algebra.

8.1.2 The lattice of submodules of a module

(**8.1.7**) For any module M the set of submodules partially ordered by inclusion forms a lattice. (Meet and join are intersection and sum respectively.)

Example: Consider the R-module $R \times R$ defined above, and suppose for simplicity that R is a field. Then the submodule lattice looks, in part, like

$$\begin{tikzcd}[column sep=small]
 & & & R \times R \arrow[dlll, dash] \arrow[dll, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{r(1,0) \mid r \in R\} & \{r(0,1) \mid r \in R\} & & \{r(1,1) \mid r \in R\} & \ldots \\
 & & & \{(0,0)\} \arrow[ulll, dash] \arrow[ull, dash] \arrow[u, dash] \arrow[ur, dash] &
\end{tikzcd}$$

Setting $R_{ab} = \{r(a,b) \mid r \in R\}$ note that

$$(R_{10} + R_{01}) \cap R_{11} = R_{11}$$

$$(R_{10} \cap R_{11}) + (R_{01} \cap R_{11}) = R_{00}$$

so the distributive law does *not* hold in this lattice.

(Later we will consider finite dimensional modules for algebras over fields. There we will see that the distributive law holds iff the module has no section isomorphic to a direct sum of isomorphic simple modules. Note that our example illustrates the only-if part of this.)

8.2 R-homomorphisms and the category R-mod

ss:R-hom

(**8.2.1**) For M, N left R-modules, an R-module homomorphism, or R-homomorphism, is an element

$$f : M \to N$$

of $\hom_{\mathbf{Ab}}(M, N)$ such that $f(rm) = rf(m)$ ($r \in R$, $m \in M$).

Note that such an f is, in particular, an abelian group homomorphism, so that $\ker f$ is defined.

We write $\mathrm{Hom}_R(M, N)$ for the set of these homomorphisms. We write $\mathrm{End}_R(M)$ for $\mathrm{Hom}_R(M, M)$.

(**8.2.2**) The class $R-$mod of left R–modules is a category, with morphisms the R–module homomorphisms. ($\mathrm{mod} - R$ defined similarly for right modules.)

8.2.1 quotients

(**8.2.3**) If M' is a submodule of $M \in R - \mathrm{mod}$ then there is an action of R on the cosets of M' in M given by

$$r(m + M') = rm + M'$$

making the collection of cosets M/M' a left R-module.

For example, in (8.3.2) we have $I_d/I_{d-1} \cong I_2$ for all $d > 1$.

(**8.2.4**) If $f \in \mathrm{Hom}_R(M, N)$ then $\ker f$ is an R-submodule of M; $f(M) = \mathrm{im}\, f$ is an R-submodule of N; and $f(M)$ and $M/(\ker f)$ are isomorphic R-modules.

(**8.2.5**) Let $f \in \mathrm{Hom}_R(M, N)$. For $n \in N$ define $f^{-1}n$ (as for any map $f : M \to N$) as the set of elements m of M such that $f(m) = n$. For $S \subset N$ then $f^{-1}S$ is the subset of M whose images lie in S.

Note that f^{-1} is not an R-homomorphism (it is not even a set map). However:

pr:isRmod (**8.2.6**) Proposition. *If N' is a submodule of N then $f^{-1}N'$ is a submodule of M. If M' is a submodule of M then fM' is a submodule of N.*

Proof. If $a, b \in f^{-1}N'$ then $f(a), f(b) \in N'$ so $f(a) + f(b) \in N'$ so $f(a) + f(b) = f(a + b) \in N'$, so $a + b \in f^{-1}N'$; and if $r \in R$ then $f(ra) = rf(a) \in N'$ so $ra \in f^{-1}N'$. The proof of the other claim is similar. □

(**8.2.7**) Remark. For more on quotients see for example Zariski–Samuel [125, §III.3].

8.2.2 Direct sums and simple modules

ss:DSandSimple

(**8.2.8**) If M, N are R-modules then the external direct sum $M \dot{+} N$ is $M \times N$ with componentwise addition and

$$r(m, n) = (rm, rn)$$

This is an R-module. Further

$$M' = \{(m, 0) \mid m \in M\}$$

is a submodule (as is N' defined similarly).

de:sum mod (**8.2.9**) If M_1, M_2 submodules of R-module M then we write $M_1 + M_2$ for the obvious subset of M. This is another submodule.

(**8.2.10**) If M_1, M_2 submodules of R-module M we write

$$M_1 + M_2 = M_1 \oplus M_2$$

if $m_1 + m_2 = 0$ ($m_i \in M_i$) implies that each $m_i = 0$.

The module $M_1 \oplus M_2$ is the *(internal) direct sum* of M_1 and M_2. This extends to $\oplus_i M_i$.

Refering back to the external direct sum we have:

$$M \dotplus N = M' \oplus N'$$

p:irred (**8.2.11**) A left R-module M is *irreducible* (or *simple*) if $M' \subset M$ implies $M' = \{0\}$.

pr:schur (**8.2.12**) Proposition. [SCHUR'S LEMMA] *Let S be a simple R–module. Then $End_R(S)$ is a division ring.*

Proof. Let $f \in \mathrm{End}_R(S)$ be non–zero. The kernel of f is a submodule of S, so it is empty. Thus f is an injection. Similarly the image of f is S, so f is a surjection and hence a bijection, and so has an inverse. □

de:semisimple mod (**8.2.13**) Semisimple module M, M is a module which is a direct sum of simple modules.

(**8.2.14**) A non-zero left R-module M is *indecomposable* if it cannot be expressed as a direct sum of two non-zero submodules.

Example: The ring $T_2'(\mathbb{C})$ (from (7.1.3)) is indecomposable as a left-module for itself. It is not irreducible, since $U_2(\mathbb{C})$ is a submodule, but the only other nonzero submodule is $T_2'(\mathbb{C})$ itself, so there is evidently no direct sum decomposition.

(**8.2.15**) A diagram $L \xrightarrow{f} N \xrightarrow{g} M$ in R-mod is *exact at N* if $im(f) = ker(g)$.
A finite sequence of maps in R-mod is an *exact sequence* if it is exact at every step.
An exact sequence of form

$$0 \to L \xrightarrow{f} N \xrightarrow{g} M \to 0$$

is called a *short exact sequence.*
If such a sequence has a reverse (there is an $f' : N \to L$ with $f'f = 1_L$), it is *split.*

For example, the natural sequence

$$0 \to L \to L \oplus M \to M \to 0$$

is split.

pa:splito (**8.2.16**) Note that R-mod is an additive category (as in 6.2.9), with the category direct sum given by module direct sum. In particular, for every split short exact sequence

$$\begin{tikzcd} 0 \arrow[r, shift left] & L \arrow[l, shift left] \arrow[r, shift left, "a"] & N \arrow[l, shift left, "a'"] \arrow[r, shift left, "b'"] & M \arrow[l, shift left, "b"] \arrow[r, shift left] & 0 \arrow[l, shift left] \end{tikzcd}$$

there is an idempotent decomposition of $1_N \in \mathrm{End}_R(N)$:

$$aa' + bb' = 1_N$$

ss:free modules

8.2.3 Free modules

(**8.2.17**) A set $\{m_i\}$ of elements of an R-module M is called *R-free* if the only solution to

$$\sum_i r_i m_i = 0$$

is $r_i = 0$ for all i.

(**8.2.18**) A set $\{m_i\}$ of elements of R-module M is called a *set of generators* of M if every $m \in M$ can be expressed in the form

$$m = \sum_i r_i(m) m_i \qquad r_i(m) \in R$$

an R-linear combination of a finite number of the $\{m_i\}$.

de:basis

(**8.2.19**) A set of generators of M that is R-free is called a basis.
A (left) R-module with a basis is called a free (left) R-module.

(**8.2.20**) Free module M, M is a module with a basis.
For S any set we may write RS for the free R–module with basis S.

free module

(**8.2.21**) Proposition. *For any R–module M there is a short exact sequence*

$$0 \longrightarrow G \longrightarrow F \longrightarrow M \longrightarrow 0$$

where F is free.

(**8.2.22**) Suppose (R commutative and) that $\rho : A \to M_n(R)$ is a representation of an R-algebra A. Let $\{b_1, .., b_n\}$ be a set of symbols, and let M be the free R-module with this set as basis. Then the action of A on M given by $ab_i = \sum_j \rho(a)_{ij} b_j$ makes M an A-module.
Note however that this M is not a free A-module in general.

ss:fmbc

8.2.4 Matrices over R and free module basis change

Here we shall take R to be commutative.

(**8.2.23**) A matrix $Y \in M_n(R)$ is *unimodular* if there exists Y' such that $YY' = Y'Y = 1_n$.
Equivalently Y is unimodular if $\det(Y)$ is a unit in R.

(**8.2.24**) Matrices $S, T \in M_{m,n}(R)$ are *equivalent* if $S = Y_1 T Y_2$ with Y_i unimodular. We write $S \sim T$.

(**8.2.25**) Let M be a free R-module with ordered basis $m = (m_1, m_2, .., m_n)$. Let $Y \in M_n(R)$. Then $m' = mY$ is a basis iff Y unimodular.

(**8.2.26**) Example. From (4.1.1)

$$(-11+12+21-22, 11, 11-22, 12-21) = (11, 12, 21, 22) \begin{pmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

and $\det\, Y = -(-1 + (-1)) = 2$.

(**8.2.27**) Theorem. *Suppose R a PID. Let $S \in M_{m,n}(R)$. Then*

$$S \sim diag(d_1, d_2, .., d_l, 0, 0, .., 0)$$

where $d_i | d_{i+1} \in R$. The elements $\{d_i\}$ are the *invariant factors* of S (defined up to units in R). See e.g. [32] for a proof.

th:RPIDsubfree (**8.2.28**) Theorem. *Let R be a PID. If M is a free R-module of rank n and N is a submodule, then N is free of rank $n' \leq n$. (See e.g. [32, §16.1].) Further, there is an ordered basis $(b_1, b_2, ..., b_n)$ of M, and elements $r_1 | r_2 | ... | r_{n'} \in R^*$ such that $(r_1 b_1, r_2 b_2, ..., r_{n'} b_{n'})$ is an ordered basis for N. The r_i are uniquely determined up to units. (See e.g. [32, §16.8].)*

8.3 Finiteness issues

ss:finite1

(**8.3.1**) A module M is *finitely generated* (fg) if there is a finite set of generators.

Example: $R = R1$ is finitely generated. (Next we construct a non-fg module.)

de:egDCC (**8.3.2**) Fix a prime p. Consider the abelian additive group I of rational numbers of form $\frac{q}{p^l}$, modulo the integers. Thus writing simply x for $x + \mathbb{Z}$:

$$\frac{80}{3^4} + \frac{79}{3^4} \equiv \frac{26}{3^3}$$

Regard I as a $\mathbb{Z}$-module. The subset $I_d = \{0, \frac{q}{p^l} \mid d > l > 0\}$ is a finite subgroup and a submodule for any $d \in \mathbb{N}$. For example, $I_3 = \{\frac{q}{p^2} \mid q = 0, 1, 2, ..., p^2 - 1\}$.

We have

$$\{0\} = I_1 \subset I_2 \subset I_3 ...$$

I_2 is generated by $\frac{1}{p}$, and I_d by $\frac{1}{p^{d-1}}$, but I is not finitely generated as a $\mathbb{Z}$-module.

(**8.3.3**) The submodules of an R-module M satisfy the *ascending chain condition* (ACC) if every chain

$$M_1 \subset M_2 \subset M_3 \subset ...$$

terminates (i.e. there is an index i such that the chain cannot be extended on the right beyond M_i, except by $M_{i+1} = M_i$).

DCC means, analogously, that every descending chain terminates.

th:accfg (**8.3.4**) Theorem. *TFAE:*
(i) The submodules of R-module M satisfy ACC
(ii) Every submodule of M is finitely generated
(iii) Every collection of submodules of M contains a maximal element.

(**8.3.5**) Our example (8.3.2) above satisfies DCC but not ACC.

(**8.3.6**) Theorem. *Submodules of left R-module M satisfy DCC iff every set $\{M_i \mid M_i \subset M\}$ has a minimal element.*

(**8.3.7**) If R is a commutative integral domain in which every ideal is principle (PID), then the left ideals of R satisfy ACC.

(**8.3.8**) A ring whose left ideals satisfy DCC (i.e. a left-Artinian ring) is sometimes called a *ring with (left) minimum condition* (MC).

Examples: All finite dimensional algebras over fields are rings with MC. On the other hand the ring $\mathbb{Z}/p_1p_2\mathbb{Z}$ is finite, with p_1p_2 elements, and so has MC, but is not an algebra over a field (since this field would be finite, and any vector space over it would be finite and have prime power elements).

Ring $\mathbb{Z}$ itself does not satisfy MC.

(**8.3.9**) The *socle* of a left R-module M is the sum of its irreducible submodules.

Of course R may not have any irreducible submodules, if it does not have MC.

8.3.1 Radicals and semisimple rings

(**8.3.10**) The *radical* of a module is the intersection of its maximal submodules. (Set $\mathbf{rad}M = M$ if there are no maximal submodules.)

By Theorem (8.3.4) a nonzero fg R-module has

$$\mathbf{rad}\ M \subset M$$

(**8.3.11**) If module M satisfies DCC then $\mathbf{rad}(M) = 0$ iff M is a finite direct sum of irreducible modules.

(**8.3.12**) The *Jacobson radical* J of a ring R is the radical of R as a left-module for itself.

That is, J is the intersection of the maximal left ideals of R. (One can check that this J is a two-sided ideal; and the same as the intersection of maximal right ideals.)
Example: $\mathbf{rad}\ \mathbb{Z} = \{0\}$.
If R is a ring with Jacobson radical J then

$$\mathbf{rad}\ M_n(R) = M_n(J)$$

Let $\mathrm{Tri}_n(R)$ denote the ring of upper-triangular $n \times n$ matrices over R, and $U_n(R)$ the ideal of strictly upper-triangular matrices (0s on the diagonal). Then $\mathbf{rad}\ \mathrm{Tri}_n(K) = U_n(K)$ for any field K.

(**8.3.13**) Nil ideal I of ring R is an ideal such that for each $x \in I$ there is a natural number n such that $x^n = 0$.

(**8.3.14**) Note that every nil ideal of R is contained in the radical J.

(**8.3.15**) Nilpotent ideal I of ring R is an ideal such that there is a natural number n such that $\prod_{i=1}^n x_i = 0$ for every n-tuple $x \in I^{\times n}$.

(**8.3.16**) Let R be a ring with radical J. Note that every nilpotent ideal is a nil ideal — specifically $r \in N$ such that $N^n = 0$ implies $r^n = 0$. (In a (left) artinian ring every nil ideal is nilpotent. [1]) Thus every nilpotent ideal is contained in the radical. If R is (left) artinian then J is nilpotent.

[1] On the other hand consider the ring $\mathbb{Z}[x_1, x_2, ...]/\{x_i^i = 0\}$. The set of polynomials with vanishing constant term form an ideal I (non fg, indeed generated by $\{x_2, x_3, ...\}$ say, so the ring is not artinian) and each individual polynomial is evidently nilpotent, so the ideal is nil. However there is no n such that $I^n = 0$, since in particular $x_2x_3...x_{n+2} \in I^n$ is not zero for any n.

(**8.3.17**) A ring is said to be *semisimple* if the Jacobson radical $J = 0$.

(**8.3.18**) Cf. the definition 8.2.13 of semisimple module. Some workers define a '(left) semisimple ring' to be a ring whose left regular module is semisimple. The two definitions do not coincide in general, but they do for artinian rings:

(**8.3.19**) THEOREM. *A ring whose left regular module is semisimple is left (and right) artinian and has radical $J = 0$.*
A left artinian ring with radical $J = 0$ has semisimple left regular module.

de:mc rad (**8.3.20**) THEOREM. *For a ring R with MC the Jacobson radical coincides with the (two-sided) ideal which is the sum of all nilpotent left ideals.*

(**8.3.21**) LEMMA. *If R has MC and is semisimple then every R-module is a direct sum of irreducible modules.*
Conversely, if R has MC and is a direct sum of irreducible left modules (as a left module for itself) then it is a semisimple ring.

th:wedd (**8.3.22**) THEOREM. [Wedderburn-Artin] *If ring R is semisimple and has MC then it is isomorphic to a direct sum of a uniquely determined set of matrix rings over division rings $\{M_{d_i}(D_i) \mid i = 1, ..., r\}$. This index set $i = 1, ..., r$ also indexes the isomorphism classes of simple modules.*

8.3.2 Composition series

ss:compos

de:filtration2 (**8.3.23**) Let Γ be a set of R-modules. An R-module M has a Γ-*filtration* if there is a chain of modules $M = M_1 \supset M_2 \supset ... \supset M_l \supset M_{l+1} = \{0\}$ such that every factor M_k/M_{k+1} is isomorphic to some element of Γ.

(**8.3.24**) A chain of modules

$$M = M_1 \supset M_2 \supset ... \supset M_l \supset M_{l+1} = \{0\}$$

is a *composition series* for M if the factors M_k/M_{k+1} are irreducible.

(**8.3.25**) A left R-module M has a composition series iff it satisfies ACC and DCC.

(**8.3.26**) THEOREM. *[Jordan-Holder] Any two composition series for a left R-module are equivalent (i.e. of the same length, and the sequence of factors is the same up to order and R-isomorphism).*

Proof. Outline: Let $M \supset A_1$ and $M \supset B_1 \supset B_2$ be chains of submodules. Then

$$B_2 + (A_1 \cap B_1) = (B_2 + A_1) \cap B_1$$

(just show an inclusion both ways). Now suppose $A_1 \supset A_2$. Then one can show

$$\frac{A_1 \cap B_1}{(B_1 \cap A_2) + (A_1 \cap B_2)} \cong \frac{(A_1 + B_2) \cap B_1}{(A_2 + B_2) \cap B_1}$$

Let $M \supset ... \supset M_i \supset M_{i-1}$ and $M \supset ... \supset M'_j \supset M'_{j-1}$ be series of submodules. We can refine the first by

$$M_i \supseteq ... \supseteq (M'_j + M_{i-1}) \cap M_i \supseteq (M'_{j-1} + M_{i-1}) \cap M_i \supseteq ... \supseteq M_{i-1}$$

and the second analogously. This gives us series of equal length. There may be many equalities, in different positions, in each. But the isomorphism above allows us to match the sections up in pairs. □

(**8.3.27**) If S is an irreducible R-module we write $[M : S]$ for the multiplicity of S as a composition factor in M up to isomorphism.

(**8.3.28**) THEOREM. *If submodules of a left R-module M satisfy DCC then M can be expressed as a direct sum of finitely many indecomposable modules.*

(**8.3.29**) THEOREM. [Krull-Schmidt] *If M is a left R-module satisfying ACC and DCC then any two decompositions into a direct sum of indecomposables have the same length, and an ordering bringing the summands into pairwise isomorphism.*

(**8.3.30**) In summary, we have seen that modules satisfying ACC and DCC are characterised in large part by the list of their simple factors, together with the possible orderings of these factors. We are interested in fd algebras over fields, for which modules satisfying ACC and DCC are available. We will see that the possible orderings are determined by the radical. So interest turns naturally to the construction and 'detection' of simple modules.

8.3.3 More on chains of modules and composition series

Suppose we have a chain of R-modules $M = M_0 \supset M_1 \supset ... \supset M_l = 0$ and not every section is simple. In particular, suppose $M_i/M_{i+1} = N = N_0 \supset N_1 \supset N_2 = 0$. Can we refine the first chain using the second? I.e. can we insert M_i' in $M_i \supset M_{i+1}$ so that $M_i/M_i' \cong N_0/N_1$ and $M_i'/M_{i+1} \cong N_1/N_2 = N_1$?

pr:chain refine (**8.3.31**) PROPOSITION. *(Cf. [125, §III.4 Th.4]) Let $M_0 \supset M_1$ be R-modules, and $f : M_0 \to M_0/M_1$ be the quotient map; i.e. there is a short exact sequence*

$$0 \to M_1 \to M_0 \xrightarrow{f} M_0/M_1 \to 0$$

Every submodule $L' \subset M_0/M_1$ can be expressed as M_0'/M_1 where R-module $M_0' = f^{-1}L'$ obeys $M_0 \supset M_0' \supset M_1$.

Proof. Note that $f^{-1}L' \subset M_0$ (it is the set of elements of M_0 taken to elements of L' by f). We need to show (i) that it is an R-module; and (ii) that $M_0'/M_1 \cong L'$. For (i) we note Proposition 8.2.6. For (ii) we note that $f(f^{-1}L') = fM_0' = L'$. □

(**8.3.32**) COROLLARY: Any R-module chain that is not a composition series can be refined.

PROOF: Consider $M_i \supset M_{i+1}$ with M_i/M_{i+1} not simple. Then there is a proper submodule $L' \subset M_i/M_{i+1}$. Then for $f : M_i \to M_i/M_{i+1}$ we have $f^{-1}L' \subset M_i$, refining the chain between M_i and M_{i+1}.

8.4 Tensor product of ring-modules

ss:tensor1

de:rest funct (**8.4.1**) Given a pair of rings $R \subset S$ we have a functor

$$\mathrm{Res}_R^S : S - \mathrm{mod} \ \rightarrow \ R - \mathrm{mod}$$

where $\mathrm{Res}_R^S M = M$, an R-module via inclusion in the S-action ($r \in R \subset S$).

Indeed, given any ring homomorphism $\psi : R \to S$ each $M \in S - \mathrm{mod}$ becomes an R-module via ψ; and this extends to a functor Res_ψ. (This functor is exact — see e.g. [3, §16 Ex.1].)

(**8.4.2**) We may write ${}_S M$ to indicate that M is an S-module above. Then we may write simply ${}_R M$ for $\mathrm{Res}_R^S M$.

(**8.4.3**) The *restriction functor* $\mathrm{Res}_R^S -$ introduced in (8.4.1) will be very useful (see e.g. §8.9.1). It would also be useful to have a 'paired' functor going the other way. One of the ways to do this requires a new technical device — the *tensor product*.

The idea, mechanistically speaking, is that if we have an R, S-bimodule ${}_R M_S$ and a left S-module ${}_S N$ then we should be able to 'glue' these together at their 'dual' S-module structures to make a new left R-module in a natural way. To think about what this means in practice, the first thing is to forget about the R-module structure for a moment, and just think about how to make a suitably 'balanced gluing' of the S-module structures.

de:balanced (**8.4.4**) Let R be a ring; M a right R-module; N a left R-module; and Q an additive abelian group. A *balanced map*

$$g : M \times N \to Q$$

is (B1) a biadditive map, i.e. left additive: $g(m+m', n) = g(m, n) + g(m' + n)$, and right additive; such that (B2) $g(m, rn) = g(mr, n)$.

rem:tp universal (**8.4.5**) Remark. Given R, M, N as above, a balanced map $\psi : M \times N \to T$ is 'universal' if for every balanced map $\phi : M \times N \to Q$ there is a morphism of abelian groups $f : T \to Q$ such that $f \circ \psi = \phi$ (that is, if every ϕ factors through T).

(**8.4.6**) Let M, N as above. Let S_{MN} be the subgroup of $\mathbb{Z}(M \times N)$ generated by all formal sums of form $(m + m', n) - (m, n) - (m', n)$, $(m, n + n') - (m, n) - (m, n')$, $(m, rn) - (mr, n)$. Let $z : M \times N \to \mathbb{Z}(M \times N)/S_{MN}$ via

$$z(m, n) = (m, n) + S_{MN}$$

Then z is a balanced map ((B1) is ensured by the first two types of sums appearing in S_{MN}; and (B2) by the third type).

(**8.4.7**) We define the tensor product

$$M \otimes_R N = \mathbb{Z}(M \times N)/S_{MN}$$

We write $m \otimes n$ for the image under z of (m, n).

(**8.4.8**) Remark. A general element of $M \otimes_R N$ is of form $\sum_i m_i \otimes n_i$. It is not uncommon to find this abbreviated to $a \otimes b$ (i.e. a sum, and a suitable unpacking of a and b, is understood).

(**8.4.9**) Remark. In §8.5.2 we shall see that the object map $M_R \otimes_R - : R - \mathrm{mod} \to \mathbf{Ab}$ is a functor.

rem:tp univ (**8.4.10**) Remark. The tensor product has the universal property. (*Outline Proof*: Any balanced map $g : M \times N \to Q$ lifts to a homomorphism g' of $\mathbb{Z}(M \times N)$ to Q in the obvious way. In particular $g'(S_{MN}) = 0$. Therefore the map $g'' : M \otimes_R N \to P$ given by $g''((m,n) + S_{MN}) = g(m,n)$ is well-defined. That is, $g''(z(m,n)) = g(m,n)$. $\square$)

8.4.1 Notes and Examples of tensor products

(**8.4.11**) For any ring R and R-module N we have the isomorphism of abelian groups

$$R_R \otimes_R {}_R N \;\cong N, \tag{8.1}$$ eq:RxNN

since $r \otimes n \equiv 1 \otimes rn$.

ex:BRC1 (**8.4.12**) Example. Consider the ring $R = \mathbf{Mat}_2(\mathbb{C})$ and the right R-module B of 2-component row matrices (with the natural right action of R); and the left R-module C of 2-component column matrices. We claim that $B \otimes_R C \;\cong \mathbb{C}$ as an abelian group.

Proof: First note that $R\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C$ (C is simple). Thus every class in $B \otimes_R C$ has an element of form $v \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (the generic element $\xi \;= \sum_i m_i \otimes n_i = \;\sum_i m_i \otimes r_i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for some set of r_is in R, and this $\xi \;\equiv\; \sum_i m_i r_i \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \;\equiv\; (\sum_i m_i r_i) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by construction). Indeed every class has an element of form $(x,0) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for some $x \in \mathbb{C}$, since $v \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = v \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \equiv v\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (v_1, 0) \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Clearly every such representative (every $v_1 \in \mathbb{C}$) occurs, so it remains to show that no two distinct such representatives are in the same class. (In Re.8.4.13 we give a quick way to do this here. For now we continue with a general strategy.) So far we have shown a surjection $\mathbb{C} \to B \otimes_R C \to 0$ as an abelian group.

But the map $B \times C \to \mathbb{C}$ given by $(b,v) \mapsto b.c$ is a *balanced map* in the sense of (8.4.4). Now use a universal property as in (8.4.5,8.4.10) [**?**]. $\square$

rem:BRC2 (**8.4.13**) Remark. In the example (8.4.12) above it is perverse to consider $B \otimes_R C$ as merely an abelian group. The right module B is also a 'left' $\mathbb{C}$-module (and C is a 'right' $\mathbb{C}$-module) and one can see (exercise) that this property survives the construction, to make $B \otimes_R C$ a 'left' $\mathbb{C}$-module.

(**8.4.14**) Example. With definitions as in (8.4.12) above, noting the subring $\mathbb{C} \hookrightarrow R$, what can we say about the abelian group $B \otimes_{\mathbb{C}} C$?

In this case we claim that the Kronecker product is a balanced map. This is a 4-dimensional $\mathbb{C}$-space. Noting that the tensor product is at most 4-dimensional as a $\mathbb{C}$-space, we conclude that the Kronecker product is the tensor product.

exa:odd2 (**8.4.15**) Example. Consider $\mathbb{Q} \supset \mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$ as $\mathbb{Z}$-modules. Then $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} = 0$, since $x \otimes y = xpp^{-1} \otimes y = xp^{-1} \otimes py = 0$. Meanwhile $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$.

exa:S3S2 (**8.4.16**) Exercise. Regard $\mathbb{Z}S_3$ as a right $\mathbb{Z}S_2$-module by restriction. Compute the abelian group $(\mathbb{Z}S_3)_{\mathbb{Z}S_2} \otimes_{\mathbb{Z}S_2} M_0$ where M_0 is the trivial $\mathbb{Z}S_2$-module.
Hints: Consider

$$S_3 = \;\langle \sigma_1 = (12),\; \sigma_2 = (23) \rangle \;\;\supset\;\; S_2 = \;\langle \sigma_1 \rangle.$$

We have $\mathbb{Z}S_3 = \mathbb{Z}\{1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$. Now $\sigma_1 \otimes m = 1 * \sigma_1 \otimes m = 1 \otimes \sigma_1 m = 1 \otimes m$ and so on, so $(\mathbb{Z}S_3)_{\mathbb{Z}S_2} \otimes_{\mathbb{Z}S_2} M_0 = \mathbb{Z}\{1 \otimes m, \sigma_2 \otimes m, \sigma_1\sigma_2 \otimes m\}$.

(**8.4.17**) Suppose M' a submodule of M as above. Then another balanced map $z' : M' \times N \to M' \otimes_R N$ arises immediately. In this way there are two possible meanings for $m' \otimes n$. Unfortunately it is not generally possible to embed $M' \otimes_R N$ in $M \otimes N$ (as comparison of the two examples in (8.4.15) shows), so care is needed with this notation.

(**8.4.18**) THEOREM. *Let $f \in Hom(M, M')_R$ (right modules) and $f' \in Hom_R(N, N')$. Then the map $(m, n) \mapsto f(m) \otimes f'(n)$ is balanced; and there is a unique map*

$$f \otimes f' : M \otimes_R N \to M' \otimes_R N'$$

such that

$$(f \otimes f')(m \otimes n) = f(m) \otimes f'(n)$$

(**8.4.19**) EXERCISE. Consider the case in which the ring is a field K, and M, N finite dimensional vector spaces, with bases B_M and B_N respectively. Construct a basis for $M \otimes_K N$, regarded as a vector space via the obvious left action of K on M.
Elevate this whole picture to describe homomorphisms of based tensor spaces (matrices) constructed from homomorphisms of their tensor factor spaces (again realised as matrices).

(**8.4.20**) MORE TO GO HERE! See e.g. §13.4.1 for more examples.

8.4.2 R-lattices etc

(**8.4.21**) For R a Dedekind domain (e.g. a PID) the torsion submodule of a module M is $\tau(M) = \{m \mid rm = 0, \ \text{ some } r \in R^*\}$.

(**8.4.22**) An R-lattice for a Dedekind domain R is a f.g. torsion-free R-module.
Let R^0 be the field of fractions of R. Then $V = R^0 \otimes_R M$ is a vector space. The dimension of this space is the R-rank of M. (See [34, §4D].)

8.5 Functors on categories of modules

(**8.5.1**) A, B–BIMODULE $_A M_B$ is a left A–module and right B–module such that $a(mb) = (am)b$ for all $a \in A$, $b \in B$, $m \in M$.

(**8.5.2**) REMARK. Jacobson [62, §9], for example, uses the term *left A, right B-module* for such a bimodule.

(**8.5.3**) A *left A– left B–bimodule* $M = {}_{A,B}M$ is a left A–module and a right B–module such that $a(bm) = b(am)$ for all $a \in A$, $b \in B$, $m \in M$.
(Again Jacobson omits the 'bi'.)

`map by r mult` (**8.5.4**) Let $_A M_B$ be a left A– right B–bimodule. Then for each $b \in B$ we may define an element $b' \in \mathrm{Hom}_A({}_A M_B, {}_A M_B)$ by $b'(m) = mb$. Note that

$$(bc)'(m) = m(bc) = (mb)c = c'(b'(m)). \tag{8.2}$$ `maps by r mult`

8.5.1 Hom functors

ss:Hom

(**8.5.5**) For each $M \in R-\text{mod}$ there is a (covariant additive) functor

$$\hom_R(M,-):\ R\text{-mod} \ \to\ \mathbf{Ab}$$

with object map given by $X \mapsto \hom_R(M,X)$. The action on maps is $f \in \hom_R(X,Y)$ goes to $f_* : \hom_R(M,X) \to \hom_R(M,Y)$ given by $f_* a = fa$, $a \in \hom_R(M,X)$.

a:hom left exact (**8.5.6**) Fix M as above and consider any $A' \xrightarrow{f} A \ \in\ \hom_R(A',A)$. What can we say about $\ker f_*$? Unpacking we have

$$\hom_R(M,A') \overset{\hom_R(M,f)}{\to} \hom_R(M,A)$$

$$\begin{tikzcd} M \xrightarrow{g} A' & \mapsto & M \arrow[rr, "f\circ g"] \arrow[dr, "g"'] & & A \\ & & & A' \arrow[ur, "f"'] & \end{tikzcd}$$

We have $\ker(\hom_R(M,f)) = \{M \xrightarrow{g} A' \mid f\circ g(M)=0\} = \{M \xrightarrow{g} A' \mid g(M) \in \ker f\}$ so

$$\ker(\hom_R(M,f)) = \Psi_{A'}(\hom_R(M,\ker f))$$

— the isomorphic image of $\hom_R(M,\ker f\} \hookrightarrow \hom_R(M,A')$ got by simply enlarging the codomain to A'. Thus for $0 \to A' \xrightarrow{f} A \xrightarrow{h} A''$ exact ($\ker f = 0$; $\mathrm{im}\ f = \ker h$) we have $\ker f_* = 0$ and $\ker h_* = \Psi_A(\hom_R(M,\ker h)) = \Psi_A(\hom_R(M,\mathrm{im}\ f)) = \Psi_A(\hom_R(M,f(A'))) = \Psi_A(\hom_R(M,A'))$ by the injectivity of f. On the other hand $\mathrm{im}\ f_* = \mathrm{im}\ (\hom_R(M,f)) \cong \hom_R(M,A')$. Thus

$$\ker h_* = \mathrm{im}\ f_*$$

(Note that this is not true in general for the image of a sequence exact at A — we have used the short-exactness on the left.)

de:cvfhom (**8.5.7**) There is similarly a contravariant functor $\hom_R(-,M)$. It is contravariant because the construction takes $g \in \hom_R(Y,X)$ and builds an element g^* in $\hom_{\mathbf{Ab}}(\ \hom_R(X,M),\ \hom_R(Y,M))$ mapping $a \in \hom_R(X,M)$ to $g^*(a) = a \circ g \in \hom_R(Y,M)$.

(**8.5.8**) We may go further. Taking $M=R$, the right action of R on R (by the ring multiplication), commutes with the left action we are using, and hence survives to equip

$$X^* \ :=\ \hom_R(X,R)$$

with the property of right R–module. The functor is then from $R-\text{mod}$ to $\text{mod}-R$ (and is called *duality*). In particular the image of a sequence of modules under duality is a sequence in the other direction:

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "\hom_R(-,R)"'] & M \arrow[r] \arrow[d, "\hom_R(-,R)"'] & M'' \arrow[r] \arrow[d, "\hom_R(-,R)"'] & 0 \\ 0 & (M')^* \arrow[l] & M^* \arrow[l] & (M'')^* \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

pa:hom funct (**8.5.9**) Next we elevate the target of the $\hom_R(M,-)$ functor from $\mathbf{Ab}$ to a module category, in case M a bimodule.

We can make $\mathrm{Hom}_A({}_AM_B, {}_AL)$ an element of $B - \mathrm{mod}$ (not $\mathrm{mod} - B$, note), as follows. Applying $\mathrm{Hom}_A(-, L)$ to $b' \in \mathrm{End}_A(M)$ (from (8.5.4)) we obtain

$$\begin{array}{rcl} b'^* : \mathrm{Hom}_A(M, L) & \to & \mathrm{Hom}_A(M, L) \\ \beta & \mapsto & b'^*\beta \end{array} \tag{8.3}$$ bmapp

given by

$$b'^*\beta(m) = \beta(b'(m)).$$

Then we define the (left) action of B on $\mathrm{Hom}_A({}_AM_B, {}_AL)$ by

$$b.\beta(m) = b'^*\beta(m) \tag{8.4}$$ eq:lact

Finally we verify for $b, c \in B$, $\beta \in \mathrm{Hom}_A(M, L)$ that $(bc).\beta = b.(c.\beta)$ (i.e. that it *is* a left action):

$$((bc).\beta)(m) = ((bc)'^*\beta)(m) = (\{c'b'\}^*\beta)(m) = \beta(c'b'(m)) = (b'^*(c'^*\beta))(m) = (b.\{c.\beta\})(m).$$

(We leave it as an exercise to check that image maps are also maps in $B - \mathrm{mod}$.)

ex:isoshock (**8.5.10**) EXAMPLE. By the action in (8.4) $\mathrm{Hom}_R(R, M)$ is, in particular, a left R-module, for any ring R and R-module M. In fact, the map:

$$\psi : M \to \mathrm{Hom}_R(R, M) \tag{8.5}$$ eq:nasi

$$m \mapsto r \overset{\psi(m)}{\mapsto} rm \tag{8.6}$$

is an isomorphism of left R-modules.

$$\mathrm{Hom}_R({}_RR_R, {}_RM) \cong {}_RM$$

Proof: First note that each $\psi(m)$ is indeed in $\mathrm{Hom}_R(R, M)$ (and not just some arbitrary set map). Secondly note that ψ is R-linear. Since $\psi(m) = 0$ implies $m = \psi(m)(1) = 0$, ψ is an injection. Finally, for any $\gamma \in \mathrm{Hom}_R(R, M)$ choosing $m = \gamma(1)$ gives $\psi(m)(r) = \psi(\gamma(1))(r) = r\gamma(1)$. We have $\gamma(r) = r\gamma(1)$, so ψ is surjective. □

(**8.5.11**) Of course this says that $\mathrm{Hom}_R(R, M)$ is nonzero for any nonzero M. Thus there is a nonzero map from R to M for any such M, and in particular a surjection from R to M for any simple M. For a ring with MC, this says that every simple module appears as a composition factor[2] of R (regarded as a left module for itself); and indeed that any simple module may be chosen as the last composition factor.

dual basis (**8.5.12**) PROPOSITION. *Let $\{m_i\}$ a basis for $N \in R - \mathrm{mod}$. A set $\{\phi_j\} \subset N^*$ is a basis iff matrix $(\phi_j(m_i))_{ij}$ is full rank.*

Proof: Define $\theta_j \in N^* = \mathrm{Hom}_R(N, R)$ by $\theta_j(m_i) = \delta_{ij}$. Note that $\{\theta_j\}$ is a basis of N^*. Now use linearity. □

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[2]Hopkins Theorem says that a ring with MC has a composition series, as a left module. (See for example [33, (54.1)].)

8.5.2 Tensor functors and tensor-hom adjointness

ss:tf

(**8.5.13**) For ${}_RM \in R-\text{mod}$ define (covariant) functor

$$- \otimes_R {}_RM : \text{mod} - R \to \mathbf{Ab}$$

by $- \otimes_R {}_RM : X_R \mapsto X_R \otimes_R {}_RM$ and for $a \in \hom_R(X,Y)$, $- \otimes_R {}_RM(a) = a \otimes 1$.

(**8.5.14**) For ${}_SM_R$ a bimodule as indicated, define (covariant) functor

$${}_SM_R \otimes_R - : R-\text{mod} \;\to\; S-\text{mod}$$

by ${}_SM_R \otimes_R - : {}_RX \mapsto {}_SM_R \otimes_R {}_RX$ and for $a \in \hom_R(X,Y)$, $({}_SM_R \otimes_R -)(a) = 1 \otimes a$.

(**8.5.15**) Adjointness.
Let R, S be rings with modules ${}_RL, {}_SM_R, {}_SN$. There is an isomorphism of additive groups

$$\gamma : \hom_R(L, \hom_S(M,N)) \cong \hom_S(M \otimes_R L, N) \tag{8.7}$$

eq:adj isom

given by

$$(\gamma f)(m \otimes l) = f_l(m)$$

where $f_l \in \hom_S(M,N)$ is the image of l under f.
That is to say, the pair of functors $(M \otimes_R -,\ \hom_S(M,-))$ form an adjunction, in the sense of (6.3.7), between categories $R-\text{mod}$ and $S-\text{mod}$.

$$R-\text{mod} \underset{\hom_S(M,-)}{\overset{M\otimes_R-}{\rightleftarrows}} S-\text{mod}$$

Outline Proof: $\hom_S(M,N)$ is a left R–module, and $M \otimes_R L \in\ S-\text{mod}$, and γf is well defined. It may be shown that γ has an inverse μ defined as follows. For each $g \in \hom_S(M \otimes_R L, N)$ let $\mu g \in \hom_R(L, \hom_S(M,N))$ be given by $\{(\mu g)_l\}m = g(m \otimes l)$, where $(\mu g)_l$ is the image of l under μg. Done.

We have that $M \otimes_R -$ is *left adjoint* to $\hom_S(M,-)$.

pa:frob rep

(**8.5.16**) The canonical example of an adjunction of functors on module categories is Frobenius reciprocity. This is where we take $M = S$ and R a subring of S:

$$\hom_R({}_RL, {}_RN) \cong \hom_S({}_SS_R \otimes_R {}_RL, {}_SN)$$

Here ${}_RN = \text{Res}^S_R N$ is the obvious *restriction* to R, as is S_R, and we have used $\text{Hom}_S(S,N) \cong N$ from (8.5.10).

The functor ${}_SS_R \otimes_R -$ in this case is called *induction* from R to S.

(**8.5.17**) In particular suppose that S is an algebra over a field k, take $R = k$, and consider that $L = k$ and N is a simple S-module. Then this Frobenius reciprocity is

$$\hom_k(k, {}_kN) \cong \hom_S({}_SS, {}_SN)$$

an isomorphism of k-vector spaces of dimension $\dim N$. This tells us that there are $\dim N$ copies of ${}_SN$ in the head of the regular module ${}_SS$. We will see in (1) that this tells us that the number of copies of the indecomposable projective cover of ${}_SN$ in ${}_SS$ is $\dim N$.

8.5.3 Exact functors

(**8.5.18**) A functor F between module categories is Exact if it takes an exact sequence

$$L \xrightarrow{\lambda} M \xrightarrow{\mu} N$$

to an exact sequence

$$F(L) \xrightarrow{F(\lambda)} F(M) \xrightarrow{F(\mu)} F(N).$$

A functor F between module categories is Left Exact (respectively right exact) if it takes a short exact sequence

$$0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} N \longrightarrow 0 \tag{8.8}$$ sesy

to a sequence

$$0 \longrightarrow F(L) \xrightarrow{F(\lambda)} F(M) \xrightarrow{F(\mu)} F(N) \longrightarrow 0$$

that is exact at $F(L)$ and at $F(M)$ (respectively at $F(M)$ and at $F(N)$).

A functor which is left and right exact is exact.

th:adju (**8.5.19**) Theorem. *If functors F, G form an adjunction (F, G) between module categories*

$$C_R \underset{G}{\overset{F}{\rightleftarrows}} C_S$$

then the left adjoint F is right exact and the right adjoint G is left exact.

Proof. Applying the adjunction isomorphism to a short exact sequence

$$0 \to A' \xrightarrow{f} A \xrightarrow{g} A'' \to 0$$

in the 'N position', and any L, we get

$$\begin{tikzcd} 0 \arrow[r] & \hom_R(L, GA') \arrow[r] \arrow[d, "\cong"'] & \hom_R(L, GA) \arrow[r] \arrow[d, "\cong"'] & \hom_R(L, GA'') \arrow[d, "\cong"'] \\ 0 \arrow[r] & \hom_S(FL, A') \arrow[r] & \hom_S(FL, A) \arrow[r] & \hom_S(FL, A'') \end{tikzcd}$$

[3] The bottom row is exact since the functor $\hom_S(FL, -)$ is left exact (see (8.5.6)). Therefore the top row is exact. Note that the top row is the image under $\hom(L, -)$ of another sequence — the image of the original sequence under G. Since L can be chosen freely, if the preimage (the image

[3] In the hom/tensor case (as in (8.7)) this is:

$$\begin{tikzcd} 0 \arrow[r] & \hom_R(L, \hom_S(M, A')) \arrow[r] \arrow[d, "\cong"'] & \hom_R(L, \hom_S(M, A)) \arrow[r] \arrow[d, "\cong"'] & \hom_R(L, \hom_S(M, A'')) \arrow[d, "\cong"'] \\ 0 \arrow[r] & \hom_S(M \otimes_R L, A') \arrow[r] & \hom_S(M \otimes_R L, A) \arrow[r] & \hom_S(M \otimes_R L, A'') \end{tikzcd}$$

under G) were not exact it would pass to an inexact sequence for some choice (exercise), so it is exact. Thus G is left exact. A similar argument with all the arrows reversed shows F right exact. $\square$

In particular:

t left exact etc (**8.5.20**) PROPOSITION. *Let ${}_AV_B$ be a left $A-$ right $B-$ bimodule. The hom functor $Hom_A({}_AV_B, -)$ from $A - mod$ to $B - mod$ is left exact.*
The tensor functor F_V given by ${}_AV_B \otimes_B -$ from $B - mod$ to $A - mod$ is right exact. That is, if (8.8) is a short exact sequence in $B - mod$ then

$$ {}_AV_B \otimes_B L \stackrel{F_V(\lambda)}{\longrightarrow} {}_AV_B \otimes_B M \stackrel{F_V(\mu)}{\longrightarrow} {}_AV_B \otimes_B N \longrightarrow 0 $$

is exact in $A - mod$. $\square$

To address the question of when such functors are properly exact, it is useful to consider *projective modules*, which we do in §8.6.

(**8.5.21**) Note that we have shown that restriction is left exact (in fact it is exact); and that induction is right exact.

8.6 Simple modules, idempotents and projective modules

ss:proj21

8.6.1 Idempotents

ss:id21

Note that the only unit idempotent in a ring is 1, since if $a^2 = a$ and a has an inverse then $a = aaa^{-1} = aa^{-1} = 1$. Thus if $e^2 = e \in R$ is not 1 then $1 = e + (1 - e)$ is a sum of nonunits and R is not a local ring (as defined in §3.1.1). That is, the only idempotents in a local ring are 1 and 0.

(**8.6.1**) Two idempotents e_1, e_2 in a ring R are *orthogonal* if $e_1e_2 = e_2e_1 = 0$.
For example the idempotent elementary matrices ϵ_{ii} in $M_n(\mathbb{C})$ (see (7.1.10)) obey $\epsilon_{11}\epsilon_{22} = 0$; and

$$ \begin{pmatrix} 1 & r \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -r \\ 0 & 1 \end{pmatrix} = 0. $$

Indeed, if idempotent $e \in R$ is not 1 then $1 = e + (1 - e)$ is a decomposition of 1 into orthogonal idempotents.

(**8.6.2**) An idempotent $e \in R$ is *primitive* if it has no proper decomposition $e = e_1 + e_2$ into orthogonal idempotents.
(Note that if e, e_1, e_2 are idempotent and $e = e_1 + e_2$ so that $e_2 = e - e_1$ then $e_1e_2 = e_1(e - e_1) = e_1e - e_1$, but this is not necessarily an orthogonal decomposition.)
(Note from our example above that while it may be possible to decompose an idempotent into primitive idempotents, such a decomposition is not unique in general. However given a decomposition $1 = \sum_i e_i$ in R, every decomposition is of form $\sum_i ae_ia^{-1}$ for some unit $a \in R$.)

(**8.6.3**) PROPOSITION. *If R is a ring then an idempotent $e \in R$ is primitive if and only if Re is an indecomposable left module.*

Proof. (Only if:) Suppose for a contradiction that $Re = M \oplus M'$. Then $e = x + x'$ with $x = xe \in M$ and $x' = x'e \in M'$ nonzero. We have $M \ni (1-x)x = xe - x^2 = x(x+x') - x^2 = xx' \in M'$, so $xx' = 0$ and $x^2 = x$, an idempotent, contradicting primitivity of e.
(If:) Exercise. □

(**8.6.4**) If $1 = \sum_i e_i$ is a decomposition into orthogonal idempotents in R then

$$R = \bigoplus_i Re_i$$

as a left module. Here Re_i is indecomposable iff e_i is primitive.

`e exact` (**8.6.5**) Suppose $M \in R-$mod and $e^2 = e \in R$. Then $eM \subset M$ is the abelian group $\{em \mid m \in M\}$. (Note that eM is also an eRe-module.) Note that an R-module morphism $\psi : M' \to M$ defines an abelian group/eRe-module morphism $eM' \to eM$, by restriction ($em \mapsto \psi(em) = e\psi(em) \in eM$). Thus $M \mapsto eM$ defines a functor from R-mod to **Ab** (or to eRe-mod).

Now consider a short exact sequence $0 \to M' \to M \to M'' \to 0$. We *claim* that $0 \to eM' \to eM \to eM'' \to 0$ is short exact (i.e. that the functor is exact).

Proof. Certainly $eM' \hookrightarrow eM$. We have $eM/eM' = \{em + eM' \mid m \in M\}$ and $e(M/M') = \{e(n+M') \mid n \in M\}$. But $e(n+M') = en + eM'$. □

`de:id lifting` (**8.6.6**) Let R be a ring and I an ideal. We say idempotents can be *lifted* mod.I if for every $a \in R$ such that $a^2 - a \in I$ (i.e. every a that passes to an idempotent in the quotient ring R/I) there is an idempotent $e \in R$ such that the images of a and e in R/I coincide. (See also §7.3.4, (9.3.3) et seq..)

For example, if I is a nil ideal (such as a nilpotent ideal) of R then idempotents can be lifted. (Exercise, and see (8.6.22).)

`de:semiperfect` (**8.6.7**) A ring R is *semiperfect* if (i) idempotents can be lifted mod. the radical J; (ii) R/J is completely reducible.

For example, any left or right artinian ring is semiperfect, since in this case the radical is nilpotent. (If R is not artinian then the radical is not necessarily nilpotent and the requirement of idempotents lifting mod the radical is not automatically satisfied.)

`pr:Re M` (**8.6.8**) Proposition. *If M is an R-module and e an idempotent then*

$$\begin{aligned} f : eM &\to Hom_R(Re, M) &(8.9)\\ em &\mapsto f(em) : ae \mapsto aem &(8.10) \end{aligned}$$

is an isomorphism of abelian groups, with inverse $g(\gamma) = \gamma(e)$.

Proof. $f(em)(e) = em$ and $f(\gamma(e))(ae) = f(e\gamma(e))(ae) = ae\gamma(e) = \gamma(ae)$. □

`pr:HomRe-` (**8.6.9**) Proposition. *If $e^2 = e \in R$ then the functor $Hom_R(Re, -)$ is exact.*

Proof. Consider a short exact sequence in R-mod: $0 \to M' \to M \to M'' \to 0$. Applying the functor $\mathrm{Hom}_R(Re, -)$ we obtain a sequence of abelian groups. We *claim* that this sequence is also exact (i.e. that the functor is exact).
(Since hom-functors are only left-exact in general (see (8.5.20)), there is something to show here.)

Now the sequence $0 \to eM' \to eM \to eM'' \to 0$ is exact (by (8.6.5)). Applying (8.6.8) to this sequence we see that the $\mathrm{Hom}_R(Re, -)$ image of the original sequence is exact. □

p:feef (**8.6.10**) Let $e \in R$ be an idempotent and consider $f \in \mathrm{End}_R(Re)$. We have $f(x) = f(xe) = xf(e)$, so f may be realised by right multiplication by $f(e)$.

eRe (**8.6.11**) Proposition. *$End_R(Re) \cong (eRe)^{op}$*

Proof. Right multiplication by $ef(e)e$ would also realise $f \in \mathrm{End}_R(Re)$, as in (8.6.10), so there is a map, and $f(e) = 0$ only if $f = 0$, so it is an injection. Indeed the map $x \to xeae$ is in $\mathrm{End}_R(Re)$ for any a, so the map is surjective.

(**8.6.12**) Proposition. *A left R–module M is indecomposable if and only if $End_R(M)$ has no idempotents except 1.*

For example, if $eRe \cong \mathbb{Z}$ then Re is indecomposable and e is primitive. N.B. the converse does not hold in general.

eRe=divr (**8.6.13**) Proposition. *Let R be a ring and $e \in R$ an idempotent.*
(I) Ring eRe a local ring (e.g. a division ring or a field) implies Re indecomposable and e primitive.
(II) Ring R semiperfect (as in (8.6.7); e.g. artinian) and $e^2 = e \in R$ primitive implies eRe local.

Proof. (I) For a contradiction suppose Re has two non–zero direct summands, and that p_1 is the projection map onto the first. Then $p_1 \in \mathrm{End}_R(Re)$ and not invertible, indeed idempotent and not the identity map. Thus $\mathrm{End}_R(Re)$ is not a local ring, and by proposition 8.6.11 neither is eRe.

(II) Exercise. □

(**8.6.14**) Proposition. *A left ideal J of R is a direct summand of R (as a left-module for itself) iff*

$$J = Re$$

for some idempotent $e \in R$; whereupon

$$R = Re \oplus R(1 - e)$$

as a left-module.

8.6.2 Projective modules

There are a number of equivalent conditions for an R-module to be projective. Which one is the 'definition' depends on ones perspective. For us (for now):

de:proj1 (**8.6.15**) Projective module P, P is an R-module and the functor $\mathrm{hom}_R({}_RP_S, -) : R - \mathrm{mod} \to S - \mathrm{mod}$ (as in (8.5.9)) is exact for each bi-module structure ${}_RP_S$ on P.

Example: If $e \in R$ is idempotent then Re is projective, by Prop.8.6.9.

projective equiv (**8.6.16**) Proposition. *The following are equivalent:*

i *1. (P1: Exactness) Module P is projective (as defined in (8.6.15));*

2. *(P2: Lifting) For every R-module surjection $M \xrightarrow{f} M'' \to 0$ and homomorphism $P \xrightarrow{a} M''$ there is a homomorphism $P \xrightarrow{a'} M$ such that $fa' = a$.*

iii 3. *(P3) Module P is a direct summand of a free module;*

iiii 4. *(P4: Splitting) Every short exact sequence of the form*

$$0 \longrightarrow L \xrightarrow{\lambda} M \xrightarrow{\mu} P \longrightarrow 0 \tag{8.11}$$ short exact

   splits.

Proof. (1) implies (4) since given equation (8.11) (1) says its image under $\hom_R(P, -)$ is exact, but $1_P \in \hom_R(P, P)$ and so in particular there is a ν in $\hom_R(P, M)$ such that $\nu\mu = 1_P$, splitting equation (8.11).

Now, (4) implies (3) since by proposition 8.2.21 there is an F free such that

$$0 \longrightarrow \ker\mu \longrightarrow F \xrightarrow{\mu} P \longrightarrow 0$$

and (4) says this splits.

Next we set off to prove that (3) implies (1).

(**8.6.17**) Proposition. *Let $\{M_i\}$ be a set of right R-modules. Then for any left R-module N*

$$(\oplus_i M_i) \otimes_R N \xrightarrow{\sim} \oplus_i (M_i \otimes_R N)$$

(See for example Jacobson[64, p.154] for a proof.)

(**8.6.18**) A right R-module M is *flat* if $M \otimes_R -$ is exact.

lem:flat1 (**8.6.19**) Lemma. *A right R-module $M = \oplus_i M_i$ is flat iff each M_i is flat.*

pr:te propflat1 (**8.6.20**) Proposition. *If M_R is projective in sense (3) of (8.6.16) then the functor $M_R \otimes_R -$ is exact.*

Proof. $R \otimes_R -$ takes any sequence to an isomorphic sequence, so is flat. Thus by Lemma (8.6.19) any free R-module F is flat. For any projective P, for some such F we have $F = P \oplus P'$, by Prop. (8.6.16)(3). Thus $P \otimes_R -$ is exact by Lemma (8.6.19) again. □

(See also Hilton–Stambach[57, p.111], Anderson–Fuller[3, p.227].)

(**8.6.21**) Exercise. Show that (3) implies (1) in Prop. (8.6.16).

We omit (2) from the loop for now. □

8.6.3 Idempotent refinement

We see that idempotents are important structural tools in ring theory. We also see that if I is a nilpotent ideal in ring R then it contains no idempotent. Thus the idempotents of R and R/I are related.

Before we start, note that ideal $I^2 \subseteq I$ in R (indeed $I^2 \subset I$ if I is nilpotent). Suppose $e \in R/I$. Then $e = r_e + I$ for some $r_e \in R$. Similarly $f \in R/I^2$ is $f = r_f + I^2$ for some $r_f \in R$. Thus $f + I = r_f + I^2 + I = r_f + I \in R/I$. In other words $f + I$ makes sense, because I is, roughly speaking, a 'cruder', bigger thing than I^2.

lem:id ref (**8.6.22**) LEMMA. *If I is a nilpotent ideal in a ring R and $ee = e \in R/I$ then there is an $ff = f \in R/I^2$ such that $e = f + I$.*

Proof. Let $r \in R/I^2$ such that $e = r + I$. Then

$$0 = e(e-1) = (r+I)(r-1+I) = r(r-1) + rI + I(r-1+I)$$

so $r(r-1) \in I$ and so $r^2(r-1)^2 = 0$ in R/I^2. Note that $e_2 := (1+2(1-r))r^2$ obeys $e_2 = r + I$ and

$$e_2(e_2-1) = (1+2(1-r))r^2((1+2(1-r))r^2-1) = (1+2(1-r))r^2(-(1+2r))(r-1)^2 = 0$$

Thus we can take $f = e_2$. □

th:id ref (**8.6.23**) THEOREM. [Idempotent refinement] *(i) If I is a nilpotent ideal in a ring R and $ee = e \in R/I$ then there is an $ff = f \in R$ such that $e = f + I$.*
(ii) If $1 = \sum_i e_i$ is a primitive orthogonal idempotent decomposition in R/I then there is a corresponding decomposition $1 = \sum_i f_i$ in R with $f_i + I = e_i$. Further if $(R/I)e_i \cong (R/I)e_j$ then $Rf_i \cong Rf_j$.

Proof. (i) Since I is nilpotent, $I \supset I^2 \supset ... \supset I^i \supset I^{i+1} \supset ...$ until some $I^n = 0$. Let $f' \in R/I^2$ be idempotent passing to e as in Lemma 8.6.22. Of course $I^4 = (I^2)^2$ so there is an idempotent $f'' \in R/I^4$ that passes to f' by the same Lemma. Iterating we shall eventually reach an idempotent f in R/I^m with $m \geq n$, so that $R/I^m = R$.
(ii) Exercise.

8.7 Structure of an Artinian ring

ss:artin2

Hereafter let us suppose that R is a ring with MC.

(**8.7.1**) For an Artinian ring the set of nil ideals coincides with the set of nilpotent ideals.

(**8.7.2**) RADICAL J of Artinian ring R: J is the maximal nilpotent ideal of R.
(Recall from (8.3.20) that the Jacobson radical of an Artinian ring is the sum of all nilpotent left ideals, and so coincides with J.)

(**8.7.3**) RADICAL FILTRATION of an R–module M.
The HEAD (or TOP) of a module is M/JM, a semisimple module. We have $M \supset JM \supset J^2M \supset \cdots \supset 0$, and each section is semisimple. (The term *head* is used, for example, by Benson [7].)

(**8.7.4**) SOCLE FILTRATION of an R–module M.
Tail The SOCLE (TAIL) of a module M is the maximal semisimple submodule $\mathbf{Soc}(M)$, i.e. the sum of all simple submodules. There is a sequence of submodules $M \supset M_1 \supset M_2 \supset \cdots \supset \mathbf{Soc}(M)$ unique up to isomorphism such that each section M_i/M_{i+1} is a maximal semisimple submodule of M/M_{i+1}.

(**8.7.5**) If a module M is a direct sum of d copies of a module N we may write simply

$$M = dN$$

Since ring R has MC here, and $R/J(R)$ is semisimple and has MC, then by the Artin-Wedderburn Theorem (8.3.22) we have

$$R/J(R) \cong \bigoplus_i M_{d_i}(D_i) \cong \bigoplus_i d_i L_i$$

(where the first isomorphism is as a ring and the second is) as a left module, where L_i is the simple module. (That is, the multiplicity of a given simple module in the left regular module for a semisimple ring is given by the dimension of that simple (over the opposite of the associated division ring).) Let $1 = \sum_i e_i$ be the corresponding orthogonal idempotent decomposition in $R/J(R)$; and $1 = \sum_i f_i$ the associated decomposition in R (as in Theorem (8.6.23)). Then

$$R \cong \bigoplus_i d_i P_i$$

as a left module. That is, the multiplicity of an indecomposable projective module in R (as a left module for itself) is given by the dimension (in the same sense as before) of the corresponding simple module $L_i = P_i/(J(R)P_i)$.

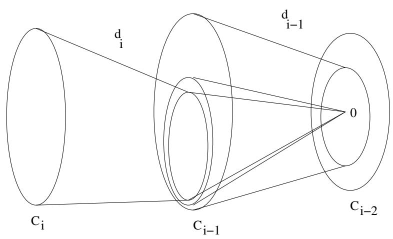
8.8 Homology, complexes and derived functors

See for example Jacobson II [64].

(**8.8.1**) A chain of R-module (or indeed abelian group) homomorphisms $d_i : C_i \to C_{i-1}$ ($i \in \mathbb{Z}$) is a *complex* (C, d) if $d_i d_{i+1} = 0$ for all i. That is, in a complex (C, d) we have

$$C_i \stackrel{d_i}{\to} C_{i-1} \stackrel{d_{i-1}}{\to} C_{i-2} \qquad (i \in \mathbb{Z})$$

and $d_{i-1}(d_i(c)) = 0$ for all $c \in C_i$ (for all i):



(**8.8.2**) A *chain homomorphism* $a : (C, d) \to (C', d')$ is a set of homomorphisms $a_i : C_i \to C'_i$ such that $a_{i-1} d_i = d'_i a_i$ for all i.

(**8.8.3**) If (C, d) is a complex and $C_i = 0$ for all $i < 0$ then (C, d) is a *positive complex.*

(**8.8.4**) Example. (I) For M an R module then $C_i = M$ and $d_i = 0 : M \to 0$ is a complex.
(II) A short exact sequence

$$0 \to M' \stackrel{\alpha}{\to} M \stackrel{\alpha'}{\to} M'' \to 0$$

defines a complex with $C_3, C_2, C_1 = M', M, M''$ and all other $C_i = 0$ and $d_2, d_3 = \alpha', \alpha$ and all other $d_i = 0$.

(**8.8.5**) Let (C, d) be a complex and note that $\ker d_i$ is a submodule of C_i; and that the image $d_{i+1}C_{i+1}$ is a submodule of $\ker d_i$. Define the i-the *homology module* of (C, d) as

$$H_i(C) = \ker d_i / d_{i+1}C_{i+1}$$

Note that the complex is exact at C_i iff $H_i(C) = 0$.

(**8.8.6**) For M an R-module, a *chain over* M is a positive complex (C, d) together with a homomorphism $e : C_0 \to M$ such that $ed_1 = 0$.

A complex over M is a *resolution* of M if the extended chain including

$$... \overset{d_2}{\to} C_1 \to C_0 \to M \to 0$$

is exact. It is a *projective* complex if every C_i is projective.

(**8.8.7**) Theorem. *If (C, e) is a projective complex over M and (C', e') is a resolution of M', and $u : M \to M'$ a homomorphism, then there is a chain homomorphism $a : C \to C'$ such that $ue = e'a_0$.*

(**8.8.8**) Let M be an R-module and

$$... \overset{d_2}{\to} C_1 \overset{d_1}{\to} C_0 \overset{e}{\to} M \to 0$$

a projective resolution of M. Now suppose we apply some functor G (from R-mod to **Ab**):

$$... \overset{Gd_2}{\to} GC_1 \overset{Gd_1}{\to} GC_0 \overset{Ge}{\to} GM \to G0$$

If G is multiplicative then $(GC, \{Gd, Ge\})$ is a positive complex over GM. If G is exact then the image sequence is exact, but this is not true in general, so $H_i(GC)$ may not vanish. Define

$$L_iGM \;:=\; H_i(GC)$$

This object map may be extended (for each i) to another functor from R-mod to **Ab**, called the i-th *left derived functor* of G. Note that it depends on the choice of resolution, but that the notation omits explicit reference to this (and in fact the dependence is, in a suitable sense, almost negligible — see e.g. [64, §6.6]).

(**8.8.9**) What is handy about L_iG is that if G is, say, right exact but not exact we can use it to develop exact sequences from the G-image of short exact sequences:

$$0 \to A \to B \to C \to 0$$

$$... \to L_1GC \to L_0GA \to L_0GB \to L_0GC \to 0$$

(**8.8.10**) Exercise. Prove it.

8.9 More on tensor products

Recall from the construction of the tensor product in (8.4) (or see for example Curtis–Reiner I [31, §12]):

bal fac **(8.9.1)** THEOREM. *For left R-module M and right R-module L, and abelian group A, every balanced map $\mu_0 : L \times M \to A$ factors through $L_R \otimes_R M$.* That is, every balanced map μ_0 induces a map $\mu : l \otimes m \mapsto \mu_0(l, m)$.

mult map2 **(8.9.2)** Suppose that left R-module M and right R-module L are in fact a principle left and right ideal respectively: $M = Rm_0$, $L = l_0R$. Note that the restriction of the ring multiplication to $L \times M$, i.e. $(l, m) \mapsto lm$, is balanced. Then by Theorem (8.9.1), associated to the tensor product $L_R \otimes_R M$ is a *multiplication map* given by

$$\mu : L_R \otimes_R M \to l_0Rm_0$$

$$l \otimes m \mapsto lm$$

(Actually we don't need the restriction to principle ideals — the main point is that multiplication makes sense.) Every element of l_0Rm_0 is clearly hit; but is this also an injection? Suppose that one of the generators is idempotent (say l_0, WLOG), then define

$$\nu : l_0Rm_0 \to L_R \otimes_R M$$

$$r \mapsto l_0 \otimes r$$

Note that in this case ν is inverse to μ.

(8.9.3) EXAMPLE. We return to Example (8.4.16). The module M_0 is isomorphic to an ideal in $\mathbb{Z}S_2$: $M_0 \cong \mathbb{Z}(1 + \sigma_1)$. This allows us to include $M_0 \subset \mathbb{Z}S_2 \subset \mathbb{Z}S_3$, and thus for M_0 to act on $\mathbb{Z}S_3$ directly by the mutliplication in this algebra. Thus we have a balanced map $\mu_0 : \mathbb{Z}S_3 \times M_0 \to \mathbb{Z}S_3$ (given by $(a, b) \mapsto ab$); and a multiplication map μ taking $\mathbb{Z}S_3 \otimes M_0 \to \mathbb{Z}S_3(1 + \sigma_1)$. The image is $\mathbb{Z}\{e', \sigma_2e', \sigma_1\sigma_2e'\}$ where $e' = 1 + \sigma_1$ (it is easy to see that these elements span; and they are linearly independent in $\mathbb{Z}S_3$). That is, the image is free of rank 3.

Preimages of the basis elements are, for example $\{1 \otimes e', \sigma_2 \otimes e', \sigma_1\sigma_2 \otimes e'\}$. We have already noted that these elements span the tensor product. We now see that they must also be linearly independent in the tensor product, since if some combination $\sum_i c_ia_i \otimes e' = 0$ then $\mu(\sum_i c_ia_i \otimes e') = \sum_i \mu(c_ia_i \otimes e') = \sum_i c_ia_ie' = 0$ so their images would be linearly dependent.

th:assoc **(8.9.4)** THEOREM. *Tensor product is associative, i.e. for L_R, ${}_RM_S$, ${}_SN$ modules as indicated:*

$$L_R \otimes_R ({}_RM_S \otimes_S {}_SN) \cong (L_R \otimes_R {}_RM_S) \otimes_S {}_SN$$

th:tp dist **(8.9.5)** THEOREM. *Distributivity:*

$$(L \oplus M)_R \otimes_R N \cong (L_R \otimes_R N) \dot{+} (M_R \otimes_R N)$$

ss:indres

8.9.1 Induction and restriction functors

We now return to one of our original motivations for introducing tensor products — the construction of an (left) adjoint to the restriction functor.

Recall that for a pair of rings with a homomorphism $\phi : R \to S$ we have a functor Res_ϕ : $S - \mathrm{mod} \to R - \mathrm{mod}$ given on objects by $\mathrm{Res}_\phi M = M$ and $rm = \phi(r)m$.

We already considered the case where ϕ is injective. See (8.5.16).

8.9.2 Globalisation and localisation functors

(**8.9.6**) Let ${}_SM_R$ and ${}_RN_S$ be bimodules as indicated. Suppose

$$ {}_SM_R \otimes_R {}_RN_S \cong S $$

as S-bimodule. Then the functor $G = {}_RN_S \otimes_S -$ is called a *globalisation*; and the functor $F = {}_SM_R \otimes_R -$ is called a localisation. We have

$$ F(G(A)) = {}_SM_R \otimes_R {}_RN_S \otimes_S A \ \cong S \otimes_S A \cong A $$

so that F is a kind of left inverse to G.

(**8.9.7**)[4] We return to consider such functors for rings that are k-algebras in §9.4.

ss:morita

8.10 Morita equivalence

See also Chapter 6 of Anderson–Fuller [3]; Section 2.2 of Benson [7]; and Chapter 3 of Jacobson II [64].

Categories A, B (not necessarily module categories) are *category equivalent* if there are a pair of functors $G : A \to B$ and $F : B \to A$ such that there are natural isomorphisms $GF \cong 1_A$ and $FG \cong 1_B$.

(**8.10.1**) Suppose S, T are module categories. Recall from (6.2.6) that a functor $F : S \to T$ is additive if, for $f, f' : L \to M$ in S, we have $F(f + f') = F(f) + F(f')$.

Rings A, B are *Morita equivalent*, denoted $A \approx B$, if there are a pair of additive functors $G : A-\mathrm{mod} \to B-\mathrm{mod}$ and $F : B-\mathrm{mod} \to A-\mathrm{mod}$ such that there are natural isomorphisms as above.

(**8.10.2**) Proposition. *Let A be a ring and $e = e^2$ in A such that $AeA = A$. Then*

$$ A \approx eAe $$

[4] This DOES NOT WORK!!! Need something more like $S = eRe$...

(**8.9.8**) With this setup, supposing also that F is exact, if L a simple S-module and B a proper submodule of $G(L)$, then $F(B) = 0$.

Proof. $F(B) \subseteq L$ by construction, but L is simple, so either $F(B) = 0$ or $F(B) = L$.

Proof. ...

First note that for each $M \in A-$mod there is a morphism $X_M : Ae \otimes_{eAe} eA \otimes_A M \to M$ given by $ae \otimes eb \otimes m \mapsto aebm$. Consider

$$\begin{tikzcd} M \arrow[r, "f"] & N \\ Ae \otimes_{eAe} eA \otimes_A M \arrow[u, "X_M"] \arrow[r] & Ae \otimes_{eAe} eA \otimes_A N \arrow[u] \end{tikzcd}$$

We note that the two composite morphisms coincide: Through M we have $ae \otimes eb \otimes m \mapsto f(aebm)$. Through $Ae \otimes_{eAe} eA \otimes_A N$ we have $ae \otimes eb \otimes m \mapsto ae \otimes eb \otimes f(m) \mapsto aebf(m) = f(aebm)$. That is, $X_- : (ae \otimes eb \otimes -) \to 1_{A\text{–mod}}$ is a natural homomorphism.

If $AeA = A$ then we claim X_- is even a natural isomorphism.

...

pr:MEconseq1 (**8.10.3**) Proposition. *If S, T are rings and $F : S - mod \to T - mod$ is a Morita equivalence functor then the lattice of submodules of a module ${}_SM$ is isomorphic to the lattice of submodules of $F({}_SM)$.*

Proof. See e.g. Anderson-Fuller §21.

(**8.10.4**) It follows from Prop.8.10.3 for example that $F(M)$ is indecomposable iff M is.

Chapter 9

Algebras

ch:alg

9.1 Algebras and A-modules

ss:alg1

Here R is a commutative ring. We start by recalling the (second) definition from (1.2.17).

de:algebra (**9.1.1**) R–ALGEBRA A: A a ring and an R–module such that $ax.y = x.ay = a(xy) \quad \forall x, y \in A$, and $a \in R$.
Examples:
(i) Any ring K is a $\mathbb{Z}$–algebra with $na = a + a + \cdots + a$ (n summands).
(ii) Let G be a finite group or monoid, R a commutative ring and RG the free R–module with basis G. Then R–linear extension of the group multiplication equips RG with the property of R–algebra.
(iii) Let A' be the free abelian monoid generated by 1 and a symbol x, and let RA' be the monoid algebra as above. Let RA be the quotient of this algebra by the ideal generated by $x^2 - 2$. If $R = \mathbb{Q}$ then $\{1, x\}$ is a basis for RA.

(**9.1.2**) REMARK. Let A be an R-algebra. It is interesting to recast the definition in terms of commutative diagrams. The ring operations of A itself obey $(x + y)z = xz + yz$ and $z(x + y) - zx - zy = 0$ of course, and the commutativity of R means that the interaction with the R-module structure can be written $(ax)y = (xa)y = x(ay)$ $(a \in R)$. Thus the multiplication factors through a balanced map (compare (8.4.4)), and may be considered as an R-linear map $\nabla : A \otimes_R A \to A$. Similarly the multiplicative identity induces an R-linear map $\eta : R \to A$. In these terms associativity becomes commutativity of

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\nabla \otimes id"] \arrow[d, "id \otimes \nabla"] & A \otimes A \arrow[d, "\nabla"] \\
A \otimes A \arrow[r, "\nabla"'] & A
\end{tikzcd}$$

where we use the already asserted associativity of tensor product (Theorem 8.9.4) to give the two