Topology

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Based partly on some lovely notes by Ben Sharp,
Josh Cork, Derek Harland and others

2021

Contents

Chapter 1

Introduction

"Continuity" is a very useful notion in human thought. But what does it mean, exactly? A topology is the minimal extra structure with which we must equip a set (such as physical space) so that the idea of "continuity" makes sense.

The following two sections are introductory descriptions of topology, from different perspectives. (The two sections are similar — almost the same. What does it mean to say that they are almost the same? That itself is a question in topology.)

1.1 Introduction: a generalists viewpoint

Are you like me? If you are, then you wake up on Wednesday morning feeling that you are pretty much the same person you were when you went to bed on Tuesday night. This notion of same-ness is part of our sense of self. While sleep may have interupted the continuity of consciousness, there is a deeper feeling of continuity of the self that survives sleep. The subject of 'Topology' is about studying and using the general notion of continuity. It underpins our sense of self, but it is also very useful in countless other ways, as we will see.

A set X becomes a topological space if it has 'a topology'. A topology is a collection of special subsets of X that must satisfy three fairly simple requirements (see Definition 5.1). In order to have a reasonable notion of continuity for functions f on a space X to a space Y, we only require the existence of a topology on X and Y.

(This remarkable eventuality seems just to be a piece of good luck. It is worth admitting that. Just so that we can grasp the intellectual starting point for this branch of study. But because of this good luck, we can study many profound and useful organisational tools for life at quite a good level of mathematical rigour.)

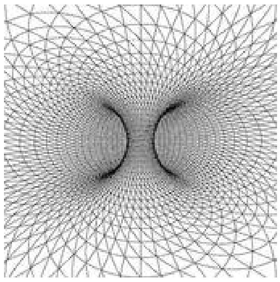
We will determine rigorous and abstract properties of topological spaces which will help us distinguish between topological spaces. Specifically, we will introduce notions of connectedness, compactness and 'shape'.

When it comes down to detail, different people may have different notions of sameness (think of the different possible notions associated to “she bought the same newspaper every day”). A Topologist’s notion of sameness equates any two spaces if (and only if) one can be continuously mapped to the other, bijectively, and such that the inverse map is also continuous. We will explain all these terms.

Why is this useful? (Apart from in understanding one’s sense of self.)
Well, its a big scary world. Some tiger-sized things will eat you. Some won’t. We need a reliable and practical way to detect the dangerous ones. (Choosing the tiger example in a country with no wild tigers is to choose a lighthearted example, but hopefully the transferability of the point will be clear.) If a given tiger eats you (or someone) then we can mark that tiger down as dangerous. But do you want to give other tigers the benefit of the doubt, or lump them together, just to be safe? Topology is the maths behind the how and what of this ‘lumping together’.

Topology sees equivalence between the boundary of a cube, and the boundary of a solid ball (a sphere) — there is a continuous bijection from one to the other whose inverse is also continuous. (What does this map look like? Put the cube inside the sphere and think of light rays emanating from a point inside!) Similarly, the surface of a ring donut is equivalent to the surface of a coffee mug, from the perspective of 'holdability' topology (both have a 'handle').

Can one continuously, and bijectively map the surface of a solid ball to that of a donut with a map whose inverse is continuous? We can rigorously prove that the answer is "NO!"



One can sometimes think of topological manifolds (special kinds of topological spaces) as being made out of rubber, so that a lot of bending, stretching and shrinking is allowed without changing the underlying topological structure.

The tools and ideas of topology are used wherever organisation and classification are useful, and hence across all realms of thought. In the course we will prove some powerful results in different fields. E.g. the fundamental theorem of algebra, and a 'ham sandwich' theorem (which proves that any sandwich made from bread, butter, and ham can always be sliced (with a single cut) into two parts, so that each part consists of equal quantities of the three separate ingredients). We can also prove that at each moment in time, there exist antipodal points on the surface of the earth which have the same temperature *and* pressure (assuming that temperature and pressure vary continuously in space).

In the realm of risk assessment we mentioned before — the risk posed by various tiger-sized things — what we need is a classification scheme for such things: a way of quickly identifying them into a grouping of established risk level. For tiger-like things in particular we tend to do this by looking at the surface (assuming the appropriate size of course). The exact shape is not helpful because tigers articulate their bodies when they run, but can we classify things like tigers, say, according to some common properties of their surfaces? In fact we can classify all surfaces up to the kind of articulations and movements that tigers can do. (In practice most hunted animals do their risk assessment classifications essentially subconsciously, rather than with maths research. But it is empowering, and transferable, to know how this works.)

1.2 Introduction: a pure maths viewpoint

(I have borrowed this beautiful short essay directly from Ben's notes for comparison.)

At its heart, topology is concerned with spaces upon which it is possible to discuss/define continuous functions (a topological space), and is geared towards rigorously classifying all such spaces. A space X is a topological space if it has a topology. A topology is a collection of special subsets of X that must satisfy three requirements (see Definition 5.1) - these special subsets are usually called **open subsets**. Remarkably, in order to have a reasonable notion of continuity for functions f on a space X to a space Y, we only require the existence of a topology on X and Y.

We will determine rigorous and abstract properties of topological spaces - topological invariants - which will help us distinguish between topological spaces. Specifically, we will introduce notions of connectedness (what does it mean for a topological space to be connected?), compactness (perhaps the most important concept to pure mathematicians) and 'shape' (or more precisely, homotopy). Topologists equate any two spaces if one can be continuously mapped to the other, bijectively, and whose inverse is also continuous[1]. If no such map exists then the spaces are topologically different.

A topologist sees no difference between the boundary of a cube, and the boundary of a football (a sphere); there is a continuous bijection from one to the other whose inverse is also continuous; what does this map look like?[2] Similarly, the surface of a donut is no different to the surface of a coffee mug, from the perspective of topology (can you imagine why?).

Question: can one continuously, and bijectively map the surface of a football to that of a donut with a map whose inverse is continuous? We'll be able to rigorously prove that the answer is "NO!" by the end of the course. Your intuition should tell you that this would be impossible without tearing one or the other surface.

You can sometimes think of topological spaces (more precisely topological manifolds) as being made out of rubber, so that any bending, stretching or shrinking is allowed without changing the underlying topological structure. However the rubber is so strong that an 'infinite amount' of stretching may survive this process[3], but the following are **not** allowed: tearing of the rubber; folding so hard that two regions become merged; or squeezing so hard that you 'lose dimensions'.

The abstract tools/ideas of topology are used heavily across all subfields of mathematics. We will not have time to go into the more algebraic side of things (via homology and cohomology), however we will introduce homotopy groups and use these to distinguish between different topological spaces. By the end of the course we will also be able to prove some powerful results in different fields: e.g. the fundamental theorem of algebra, and the ham sandwich theorem[4]. One more thing we'll be able to prove by the end of the course: at any moment in time, there exist antipodal points on the surface of the earth which have the same temperature *and* pressure[5]. To give you an idea of the power of topology, see if you can prove this before reading the notes...

[1] such a map is called a *homeomorphism*

[2] Put the cube inside the sphere and think of light rays emanating from a point inside!

[3] e.g. the continuous function $f : (0, 1) \to (1, \infty)$, $f(x) = \frac{1}{x}$ continuously stretches out a bounded interval to an unbounded one: you can check that the inverse exists and is also continuous

[4] which proves that any sandwich made from bread, butter, and ham can always be sliced (with a single cut) into two parts, so that each part consists of equal quantities of the three separate ingredients

[5] We are making the assumption that temperature and pressure vary continuously in space here

Chapter 2

Preliminaries

We assume familiarity with set theory ideas from earlier, but we will review some of them here in Chapter 2 (and see also Section A).

2.1 Some reminders on sets

We assume here that you are reasonably happy with the idea of a collection of "objects". This is a bit vague and potentially troublesome. But it *is* very useful, and we have to start somewhere. We will use the term 'set' for a collection of objects.

Suppose that we (you and I) both have in mind a set. Let's call it S. To say that we both have it is to say that we agree on what the "elements" are — the objects that are collected in S. Thus if we both have in mind an object x (say), we can agree if the statement 'x is in S' (written $x \in S$) is true or false (if false then we write $x \notin S$).

What might constitute a good "object"? In practice this is anything that we can agree is a good object. Just to get things started with a minimum of trouble, we can say that a set itself can be an object. Let us also say that there is one formal set, call it $\emptyset$, that does not contain any objects — thus postponing the general issue of what an object is by avoiding it. Thus the statement '$x \in \emptyset$' is false for every object x.

Putting these two ideas together, we have another set: the set containing only the set $\emptyset$.

If we have given a name to an object, like $\emptyset$, or X perhaps, then we can write the set containing only that object as $\{X\}$. The only concrete example of this that we have so far is $\{\emptyset\}$. For this at least we can say $\emptyset \in \{\emptyset\}$ and $x \notin \{\emptyset\}$ for all other objects x.

We say that two sets are equal if they contain the same elements; and otherwise they are unequal. Thus $\emptyset \neq \{\emptyset\}$.

Suppose that x and y and z represent objects, somehow agreed between us. One way of writing that x and y and z are in S (that $x, y, z \in S$) is $S = \{x, y, z, ...\}$. Another way is $S = \{y, x, z, ...\}$. If x, y, z are the only elements in S then we can write $S = \{x, y, z\}$. The extension of this notation to more (or fewer) elements can be guessed. (For the moment the question of precisely what the objects x, y, z here are remains mysterious.)

And then, using this notation, another set with un-mysterious objects is $\{\emptyset, \{\emptyset\}\}$. Notice that this is not equal as a set either to $\emptyset$ or to $\{\emptyset\}$. And notice that we can 'iterate' this construction: the set containing all the sets we have so far as elements is a new set; and now we can make another new set by adding this new set as a new element.

With such unappealing constructions of new sets, and hence new objects, we can at least delay the discussion of more interesting (but maybe not clearly defined) objects. We do now have many objects available — just by iterating the construction of adding a new set to a set of sets.

One more device before we really get started. Suppose that a and b represent objects (not even necessarily distinct). An *ordered pair*, denoted (a,b), is a set $\{\{a\},\{a,b\}\}$.
(Caveat: this notation (a,b) can be used in other contexts as well, to represent other things. So, to be safe, if we do mean it to denote an ordered pair then we will say so explicitly.)

Because of the way we write (and talk; and think) it sometimes looks like there is order in expressions like $\{a,b\}$ already. But note from above that $\{a,b\} = \{b,a\}$ so there is not. However note that $(a,b) \neq (b,a)$ (unless $a = b$) — this is a good exercise to prove.

Some further reading:
Beginning Finite Mathematics (Schaum's Outline Series), S Lipschutz et al.
Discrete Mathematics, J K Truss.
Sets, Logic and Categories, P J Cameron (Springer).
Algebra Volume 1, P M Cohn.

2.2 Elementary set theory notations and constructions

As in Green [**?**], let

$$\underline{n} \; := \; \{1,2,..,n\}$$

Similarly here $\underline{n}' \; := \; \{1',2',..,n'\}$, $\underline{n}'' \; := \; \{1'',2'',..,n''\}$ and so on.

(**2.2.1**) For S a set, let $\mathtt{P}(S)$ denote the *power set*, the set of subsets of S. (We may consider $\mathtt{P}(S)$ to be partially ordered by inclusion. As such it is a lattice (see 2.4.8).) Let $\mathtt{P}_n(S) \subset \mathtt{P}(S)$ be the subset of elements of order n.

(**2.2.2**) For S,T sets, let $U_{S,T}$ denote the set of relations on S to T. That is,

$$U_{S,T} = \mathtt{P}(S \times T).$$

Let $U_S = U_{S,S}$. Even in this case we may consider the left-hand 'input' set to be distinct from the right-hand 'output' set — elements are distinguished by their position in the ordered pair.

A relation on S to T is *'simple'* if no element of the left-hand set S appears twice.

2.2.1 Functions

(**2.2.3**) For $(a,b) \in S \times T$, set $\pi_1(a,b) = a$. For $\rho \in U_{S,T}$ let $\mathrm{dom}(\rho) \;\; := \;\; \pi_1(\rho)$. Let $T^S \; = \; \hom(S,T) \; \subset \; U_{S,T}$ be the subset of simple relations with $\mathrm{dom}(\rho) = S$, or *functions*.

For example

$$\underline{2}^{\underline{2}} \; = \{\{(1,1),(2,1)\},\{(1,1),(2,2)\},\{(1,2),(2,1)\},\{(1,2),(2,2)\}\} \tag{2.1}$$

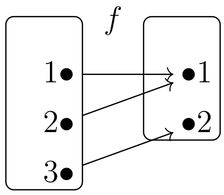
(**2.2.4**) For I a set and S_i a non-empty set for each $i \in I$, a choice function is a function that takes an element $i \in I$ as input and returns an element from S_i. (The axiom of choice says that such a function exists for any such family of sets.) Then $\prod_{i \in I} S_i$ is the set of all such functions.

2.2.2 Composition of functions

(**2.2.5**) It will be useful to have in mind the *mapping diagram* realisation of finite functions such as in (2.1). For example

$$f = \{(1,1),(2,1),(3,2)\} \in \underline{2}^{\underline{3}}$$

is



(**2.2.6**) If T, S finite it will be clear that any total order on each of T and S puts T^S in bijection with $\underline{|T|}^{\underline{|S|}}$. We may represent the elements of T^S as S-ordered lists of elements from T. Thus

$$\underline{2}^{\underline{2}} = \{11, 12, 21, 22\}, \qquad \underline{2}^{\underline{3}} = \{111, 112, 121, 122, 211, 212, 221, 222\}$$

(for example $22(1) = 2$, since the first entry in 22 is the image of 1).

(**2.2.7**) A *composition* of n is a finite sequence λ in $\mathbb{N}_0$ that sums to n. We write $\lambda \vDash n$.

We define the *shape* of an element f of $\underline{m}^{\underline{n}}$ as the composition of n given by

$$\lambda(f)_i = |f^{-1}(i)|$$

Example: for $111432525 \in \underline{6}^{\underline{9}}$ we have $\lambda(111432525) = (3, 2, 1, 1, 2, 0)$.

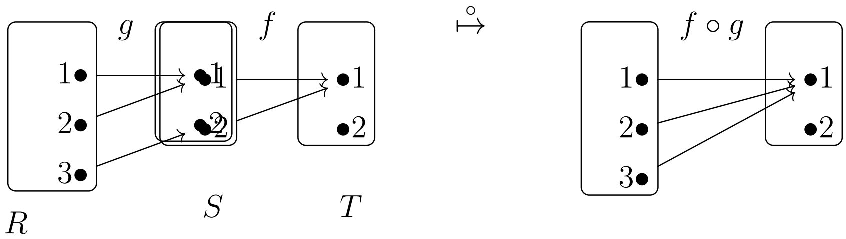
If $\lambda \vDash n$ we write $|\lambda| = n$.

(**2.2.8**) Composition of functions defines a map

$$\hom(S, T) \times \hom(R, S) \;\; \to \;\; \hom(R, T) \tag{2.2}$$
$$(f, g) \mapsto f \circ g \tag{2.3}$$

where as usual $(f \circ g)(x) = f(g(x))$. For example $11 \circ 22 = 11$ (since $11(22(1)) = 11(2) = 1$; and so on).

The *mapping diagram realisation* of composition is to first juxtapose the two functions so that the two instances of the set S coincide, then define a direct path from R to T for each path of length 2 so formed:



(2.4)

(**2.2.9**) If the image $f(S)$ of a map $f : S \to T$ is of finite order we shall say that f has order $|f(S)|$ (otherwise it has infinite order).

For $R \xrightarrow{f} S \xrightarrow{g} T$ we have the *bottleneck principle*

$$|(g \circ f)(R)| \;\leq\; \min(|g(S)|, |f(R)|)$$

To see this note that evidently $g(S) \supseteq g \circ f(R)$, from which the first inequality follows; meanwhile clearly $|f^{-1}(R)| = |f(R)|$ for any $f \in \hom(R, -)$, leading to the second inequality.

(**2.2.10**) Proposition. *(i) For S a set, $S^S = \hom(S, S)$ is a monoid under composition of functions.*
(ii) For each $d \in \mathbb{N}$ then set $\hom^d(S, S) := \{f \in S^S \mid |f(S)| < d\}$ is an ideal (hence a sub-semigroup) of S^S.

Proof. (i) Hint: $(f \circ (g \circ h))(x) = f(g(h(x))) = ((f \circ g) \circ h)(x)$
Exercise: explain this argument in terms of mapping diagrams.
(ii) Consider $g \circ f$, say. Evidently $g(S) \supseteq g \circ f(S)$. Thus $\hom^d(S,S) \circ f \subset \hom^d(S,S)$ for all f. Meanwhile $f(s) = f(t)$ implies $g \circ f(s) = g \circ f(t)$ so the partition $p = f^{-1}(S)$ of S implied by f cannot be refined in passing to the partition implied by $g \circ f$. Of course $|f^{-1}(S)| = |f(S)|$ for any f. Thus $g \circ \hom^d(S,S) \subset \hom^d(S,S)$ for all g. □

2.2.3 Set partitions

(**2.2.11**) Let $\mathsf{E}_S \subset U_S$ denote the set of equivalence relations (reflexive, symmetric, transitive/RST relations) on set S. Let P_S denote the set of partitions of S. Note the natural bijection

$$\mathsf{E}_S \underset{\kappa}{\overset{\epsilon}{\leftrightarrows}} \mathsf{P}_S.$$

For $\rho \in U_S$ let $\bar{\rho} \in U_S$ be the smallest transitive relation containing ρ. The relation $\bar{\rho}$ is called the *transitive closure* of ρ.

(**2.2.12**) Let a, b be RS relations on any two finite sets. Then $a \cup b$ is an RS relation on the union. Let $ab := \overline{a \cup b}$ be the transitive closure of $a \cup b$.

Note that $\overline{a \cup b}$ is an equivalence relation on the union of the two finite sets. Note that

$$\overline{\bar{a} \cup b} = \overline{a \cup b} \tag{2.5}$$

If a, b partitions then $\epsilon(a), \epsilon(b)$ are RS (indeed RST), and we will understand by ab the partition given by $ab = \kappa(\epsilon(a)\epsilon(b))$.

(**2.2.13**) Proposition. *For a, b, c RS relations*

$$a(bc) = (ab)c$$

Proof.

$$(ab)c = \overline{(\overline{a \cup b}) \cup c} \overset{(2.5)}{=} \overline{(a \cup b) \cup c} = \overline{a \cup b \cup c} = \overline{a \cup \overline{b \cup c}} = a(bc)$$

□

(**2.2.14**) Let $\mathsf{P}_{n,m} = \mathsf{P}_{\underline{n} \cup \underline{m'}}$; and $\mathsf{P}_n = \mathsf{P}_{n,n}$. Let $\mathsf{E}_{n,m} = \mathsf{E}_{\underline{n} \cup \underline{m'}}$ similarly. For $a \in \mathsf{P}_{n,m}$ let a' be the partition of $\underline{n'} \cup \underline{m''}$ obtained by adding a prime to each object in every part.

For $a \in \mathsf{P}_{l,m}$, $b \in \mathsf{P}_{m,n}$ partitions (and hence $\epsilon(a)$, $\epsilon(b)$ equivalence relations) note that $\epsilon(a)\epsilon(b')$ is an equivalence relation on $\underline{l} \cup \underline{m'} \cup \underline{n''}$. Restricting to $\underline{l} \cup \underline{n''}$ this equivalence relation gives again a partition, call it $r(ab')$ (indeed if a, b are pair-partitions then so is $r(ab')$).

For $x \in \underline{l} \cup \underline{n''}$ let $u(x) \in \underline{l} \cup \underline{n'}$ be the image under the action of replacing double primes with single.

We may define a map

$$\circ : \mathsf{P}_{l,m} \times \mathsf{P}_{m,n} \to \mathsf{P}_{l,n}$$

by

$$a \circ b = u(r(ab')) \in \mathsf{P}_{l,n}$$

— the image under the obvious application of the u map.

(**2.2.15**) Proposition. *For each $n \in \mathbb{N}$ the map $\circ : (a,b) \mapsto u(r(ab'))$ defines an associative unital product on P_n, making it a monoid, with identity $1_n = \{\{1,1'\},\{2,2'\},...,\{n,n'\}\}$.*

Proof. To show associativity note that ab' encodes $a \circ b$ directly, except that it is encoded via the unprimed and double-primed 'vertices'. Thus $(ab')c''$ encodes $(a \circ b) \circ c$ via the unprimed and triple-primed vertices. Meanwhile $b'c''$ encodes $b \circ c$ via the primed and triple-primed vertices; thus $a(b'c'')$ encodes $a \circ (b \circ c)$ via the unprimed and triple-primed vertices. But by Prop.2.2.13 we have $a(b'c'') = (ab')c''$.

To show unital with identity 1_n: Exercise. ■

(**2.2.16**) A convenient pictorial realisation of such a set partition p, i.e. a realisation as a picture in the plane, is as follows. (See also **??**.)

Firstly, a digraph G (as in 2.4.12) on vertex set V determines a relation on V in the obvious way. In particular a graph determines a symmetric relation. Hence a graph G determines an equivalence relation on V (take the RT closure); and hence also an equivalence relation on (or partition of) any subset of V. Thus it is enough to realise a suitable graph G of p as a picture.

To depict such a G one draws a set of points for the vertices V, and specifies an injective map from the underlying set of p to V; and then draws a 'regular' collection of 'edges'. Here a picture edge is a piecewise smooth line between two vertices. A collection is *regular* if two lines never meet at points where they do not have distinct tangents. The collection consists of one picture edge for each vertex pair that are associated by an edge in G. ('Incidental' vertices in the picture are those *not* associated to the underlying set.)

Note that two elements from the underlying set are in the same part in p if there is a path between their vertices.

(**2.2.17**) For a partition in P_n one may arrange the underlying-set vertices naturally as two parallel rows of vertices (if there are incidental vertices these are drawn between the two rows). In this realisation the product $\circ$ may be computed, schematically, by concatenating the two pictures so as to identify certain vertices in pairs between two rows — one row from each picture (thus forming a 'middle' row).

(**2.2.18**) Let $J_S \subset \mathsf{P}_S$ be the set of pair-partitions of S. Let

$$J_{n,m} = J_{\underline{n} \cup \underline{m}'} \subset \mathsf{P}_{n,m}$$

Set $J_n = J_{n,n}$.

(**2.2.19**) PROPOSITION. *The compostion $\circ$ restricts to make J_n a monoid.*

Proof. Exercise. ■

(**2.2.20**) A partition is *non-crossing* if there is such a pictorial realisation having the property that all lines are drawn in the interior of the interval defined by the two rows, and no two lines cross. Let T_n denote the subset of non-crossing pair partitions.

One easily checks that the product above restricts to make T_n a monoid. This is sometimes called the n-th *Temperley–Lieb monoid.*

(**2.2.21**) One could similarly imagine drawing a realisation of a partition on a cylinder. This leads us to a notion of cylinder-non-crossing pair partitions. There are several further subsets of the set of partitions that are characterised in terms of geometrical embeddings.
Exercise: Find some more submonoids of P_n.

2.2.4 Exercises on closed binary operations

(**2.2.22**) A closed binary operation on a (finite) set S (of degree n) may be given by a multiplication table — an element of $S^{S \times S}$. There are $|S^{S \times S}| = n^{(n^2)}$ of these. Note that an ordering of S induces an ordering on the set of closed binary operations (read order the entries in the multiplication table and dictionary order the ordered lists).

Define a natural notion of isomorphism of closed binary operations on S, and determine the number of isomorphism classes for $n = 2$. Is commutativity a class property? If so, how many of these classes are commutative?

Which of the following are semigroups/ monoids / groups?:

$$
\begin{array}{c|cc} & a & b \\ \hline a & aa & ab \\ b & ba & bb \end{array}
\quad : \quad
\begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & a & a \end{array}
\quad
\begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & a & b \end{array}
\quad
\begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & b & a \end{array}
\quad
\begin{array}{c|cc} & a & b \\ \hline a & a & a \\ b & b & b \end{array}
\quad
\begin{array}{c|cc} & a & b \\ \hline a & a & b \\ b & b & a \end{array}
\quad
\begin{array}{c|cc} & a & b \\ \hline a & b & a \\ b & a & a \end{array}
$$

(Hint: S,M,X ($b(ab) = b$),S,G,X ($(aa)b = a$).)

Explain the following statement: "For $n = 3$, most binary operations are not associative." (Hint: 113 of them are associative.)

2.3 Basic tools: topology

See e.g. Mendelson [**?**], Hartshorne [**?**].

(**2.3.1**) A *sigma-algebra* over a set S is a subset Σ of the power set $\mathtt{P}(S)$ which includes S and $\emptyset$ and is closed under countable unions, and complementation in S.

Any subset S' of $\mathtt{P}(S)$ defines a sigma-algebra — the smallest sigma-algebra generated by S'. For example $\{\{1\}\} \subset \mathtt{P}(\{1, 2, 3\})$ generates $\Sigma = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$.

(**2.3.2**) A *topological space* is a set S together with a subset T of the power set $\mathtt{P}(S)$ which includes S and $\emptyset$ and is closed under unions and finite intersections.

The set T is called a *topology* on S. The elements of T are called the *open sets* of this topology. A set is *closed* if it is the complement in S of an open set.

(**2.3.3**) A function between topological spaces is *continuous* if the inverse image of every open set is open. Two spaces are *homeomorphic* if there is a bijection between them, continuous in both directions.

(**2.3.4**) Example. Consider the set $\mathbb{R}^n$ together with the set of subsets 'generated' by (unions and finite intersections of) the set of open balls. This is a topological space.

In particular this makes $M_n(\mathbb{R})$ a topological space — as a topological space it is $\mathbb{R}^{n^2}$. The subgroup $GL_n(\mathbb{R})$ of invertible matrices may be considered as a topological space by restriction. Note that $GL_n(\mathbb{R})$ is open and not closed (its complement is not open) in $M_n(\mathbb{R})$, but it is open and closed in the restricted topology.

(**2.3.5**) Given a topological space (S, T), the restriction of T to $S' \subset S$ is a topology on S', called the *subspace topology*.

A subset S' of a topological space (S, T) is *irreducible* if $S' = S_1 \cup S_2$ with S_1 closed implies S_2 not closed.

(**2.3.6**) Let k be a field. A polynomial $p \in k[x_1, ..., x_r]$ determines a map from k^r to k by evaluation. For $P = \{p_i\}_i \subset k[x_1, ..., x_r]$ define

$$Z(P) = Z(\{p_i\}_i) = \{x \in k^r \ : \ p_i(x) = 0 \ \forall \ i\}$$

An *affine algebraic set* is any such set, in case k algebraically closed. An *affine variety* is any such set, that cannot be written as the union of two proper such subsets. (See for example, Hartshorne [**?**, I.1].)

(**2.3.7**) Example. $Z(x_1x_2 - 1) = Z(\{p(x_1, x_2) = x_1x_2 - 1\})$ is a variety in k^2. Its points $(x_1, x_2) = (\alpha, \beta)$ may be given by a free choice of α (say) from $k^\times$, with β then determined.

(Note that this latter characterisation looks like an open subset of k (specifically the complement of $Z(x)$), but the original formulation makes it clear that it is closed in k^2.)

(**2.3.8**) The set of affine varieties in k^r satisfy the axioms for closed sets in a topology. This is called the *Zariski topology.* The Zariski topology on an affine variety is simply the corresponding subspace topology.

The set $I(P) \in k[x_1, ..., x_r]$ of all functions vanishing on $Z(P)$ is the ideal in $k[x_1, ..., x_r]$ generated by P. We call

$$k_P = k[x_1, ..., x_r]/I(P)$$

the *coordinate ring* of $Z(P)$.

(**2.3.9**) Let Z be an affine variety in k^r and $f : Z \to k$. We say f is *regular* at $z \in Z$ if there is an open set containing z, and $p_1, p_2 \in k[x_1, ..., x_r]$, such that f agrees with p_1/p_2 on this set.

(**2.3.10**) A morphism of varieties is a Zariski continuous map $f : Z \to Z'$ such that if V is open in Z' and $g : V \to k$ is regular then $g \circ f : f^{-1}(V) \to k$ is regular.

(**2.3.11**) Given affine varieties X, Y then $X \times Y$ may be made in to an algebraic variety in the obvious way.

(**2.3.12**) An *algebraic group* G is a group that is an affine variety such that inversion is a morphism of algebraic varieties; and multiplication is a morphism of algebraic varieties from $G \times G$ to G.

2.4 Partial orders, lattices and graphs

2.4.1 Posets and lattices

General references on posets and lattices include Birkhoff [**?**], and Burris and Sankappanavar [**?**, §1].

(**2.4.1**) A *relation* on a set S is a subset of $S \times S$ as in (2.2.1). Thus the intersection of any set of relations on S is a relation. Indeed the intersection of any set of transitive relations is transitive.

The *transitive closure* of a relation ρ on S is the intersection of all transitive relations containing ρ. (This transitive relation is non-empty since $S \times S$ is a transitive relation.)

(**2.4.2**) A *poset* is a reflexive, antisymmetric, transitive relation.

An acylic (no cyclic chains) relation ρ on S defines a partial order, by taking the transitive reflexive closure $TR(\rho)$. Note that every relation in the interval $[\rho, TR(\rho)]$ (with respect to the inclusion partial order) is acyclic.

We may consider the set of all relations having the same transitive reflexive closure. If the closure is a poset then all the relations 'above' it are acyclic. A minimal such relation is a *transitive reduction* (of any element of this set).

If S is finite then there is a unique transitive reduction of an acylic relation. Otherwise there may be no (or one, or multiple) transitive reductions.

If there is a unique transitive reduction of an acyclic relation we call this the *covering relation.*

(**2.4.3**) If we use the notation $(S, \geq)$ for a poset then we may write $a > b$ for $a \geq b$ and $a \neq b$. In this case the relation $(S, >)$ induces the same poset. Further we may write $(S, \leq)$ for the opposite relation, which is another poset.

(**2.4.4**) Let $(S, \geq)$ be a poset, and $s, t \in S$. We say *s covers t* if $s > t$ and there does not exist $s > u > t$.

(**2.4.5**) The notion of cover/covering relation leads to the notion of *Hasse diagram*, as for example in [**?**, **?**].

(**2.4.6**) A poset satisfies ACC (is *Noetherian*) if every ascending chain terminates.

For example, the poset of ideals, ordered by inclusion, of the ring $\mathbb{Z}$ satisfies ACC.

(**2.4.7**) By convention if we declare a poset $(S, \leq)$ then $a \leq b$ can be read as *a is less than or equal to b* (although the opposite relation is a perfectly good poset, and we could in principle have associated the relation symbol $\leq$ to that).

(**2.4.8**) With the above convention, a poset S is a *join semilattice* if every pair $s, t \in S$ has a least upper bound (join) in S.
A poset is a *lattice* if both it and its opposite are join semilattices.

(**2.4.9**) Example. The power set $\mathsf{P}(S)$ of a finite set with the inclusion order is a lattice. An upper bound of $s, t \in \mathsf{P}(S)$ is any set containing sets s, t; and the least upper bound is the union. That is

$$s \vee t \;=\; s \cup t.$$

(**2.4.10**) A lattice is *modular* if

$$S \wedge (T \vee U) \;=\; (S \wedge T) \vee U \qquad \Rightarrow \qquad S \geq U.$$

(**2.4.11**) For P a lattice, the interval

$$[a, b] := \;\{c \in P \mid a \leq c \leq b\}$$

is sometimes called a *quotient* (see e.g. Faith [**?**]).

2.4.2 Digraphs and graphs

(**2.4.12**) A *digraph* is a triple (V, E, f) where V, E are (finite) sets and f a function $f : E \to V \times V$.

(**2.4.13**) Given a digraph $G = (V, E, f)$, we say V is the set of 'vertices' and E the set of 'edges'. An edge $e \in E$ with $f(e) = (a, a)$ is a 'loop'. If $f(e) = (a, b)$ then e is an edge 'on' (a, b) or from a to b.

An *edge colouring* of a digraph is a map from E to a set of 'colours', and hence a partition of E into same-coloured subsets.

(**2.4.14**) A digraph can be represented by a picture with a labeled node for each vertex and a directed labeled arc for each edge. Examples:

$$G_1 = \begin{tikzcd} a \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & b \\ c \arrow[u, "z"] \arrow[ur, "t"'] & \end{tikzcd} \qquad\qquad G_2 = \begin{tikzcd} a \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] & b \arrow[loop above, "u"] \arrow[dl, bend left, "s"] \\ c \arrow[u, "z"] \arrow[ur, "t"] & \end{tikzcd} \tag{2.6}$$

(**2.4.15**) A *simple digraph* is a digraph (V, E, f) in which f is an inclusion.

This amounts to saying that we can use a subset of $V \times V$ as the edge set. Thus we do not need labels on edges in a picture. That is, a simple digraph is just a relation on V.

(**2.4.16**) Given a digraph, if there is a proper path (along directed edges) from a to b then the *'distance'* from a to b is the minimum number of edges in such a path. (Note that this is not a true distance function. The distance from b to a may be different, for example.)

A digraph is *acyclic* if there is no proper path (along directed edges) from a to a for any $a \in V$.

(**2.4.17**) A digraph is *rooted* with root $r \in V$ if there is a vertex $r \in V$ such that every vertex is reachable by a directed path from r.

Note that if a digraph is acyclic then it has at most one root.

(**2.4.18**) Two simple digraphs (V, E, f) and (V', E', f') are *isomorphic* if there is a bijection $\psi : V \to V'$ such that (a, b) is in $f(E)$ iff $(\psi(a), \psi(b))$ is in $f'(E')$.

(**2.4.19**) We will say that two (not necessarily simple) digraphs are isomorphic if there is a bijection $\psi : V \to V'$ such that $f^{-1}(a, b)$ has the same order as $f'^{-1}(\psi(a), \psi(b))$ for all a, b (thus each of these pairs of sets could be placed in explicit bijection, but such a set of bijections is not necessarily given).

That is, two digraphs are isomorphic if their pictures can be 'morphed' into each other, using ψ, but ignoring the edge labels.

(**2.4.20**) Remark. We do not require the *finite* set condition for digraphs here. In practice our digraphs are either finite or inverse limits of sequences of finite graphs. This means in particular that there are only finitely many edges associated to any given pair of vertices, i.e. $f^{-1}(v, w)$ is always finite.

(**2.4.21**) Let G be a digraph with a countable vertex set. The *adjacency matrix* $M^G = A(G)$ is a vertex indexed square array such that entry M^G_{ij} is the number of edges from i to j in G.

Example from (2.6) above:

$$A(G_1) = \left(\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 2 & 0 \\ b & 0 & 0 & 0 \\ c & 1 & 1 & 0 \end{array} \right)$$

(**2.4.22**) The opposite graph of a digraph has the same V and E but $f^{op}(e) = f(e)^{op}$ (i.e. if $f(e) = (a, b)$ then $f^{op}(e) = (b, a)$).

(**2.4.23**) A *graph* is a digraph that is isomorphic to its opposite.

(**2.4.24**) We say a graph is *connected* if for any pair of vertices there is a finite chain of edges connecting them.

(**2.4.25**) Example. Let G be a group and S a set of elements. The *Cayley graph* $\Gamma(G, S)$ is the digraph with vertex set G and an edge s_a on (a, b) whenever $b = as$ for some $s \in S$.

(**2.4.26**) Notes:

1. $s = a^{-1}b$ so there is at most one edge on (a, b), i.e. $\Gamma(G, S)$ is simple.
2. If $s \in S$ is an involution then edges involving s are effectively undirected. Some workers define S to include inverses (write this as $S = S^{-1}$), so again $\Gamma(G, S)$ is undirected.
3. We consider that S excludes the identity, so $\Gamma(G, S)$ is loop-free.
4. Some workers require that S generates G. Then $\Gamma(G, S)$ is connected. If $S = G$ then $\Gamma(G, S)$ is the complete graph.
5. If all generators are involutions (or $S = S^{-1}$) then the graph is effectively undirected by construction. However one can sometimes 'direct' such a $\Gamma(G, S)$, by using a *length function*...

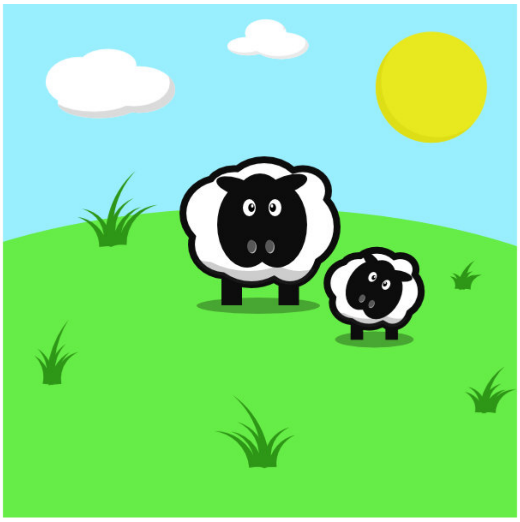
The root vertex is the identity 1, and there is a well-defined distance (really minimum distance, since there are undirected adjacent pairs) from 1 to any vertex g, denoted $l(g)$. If there is no edge between vertices of equal distance then we can 'direct' edges away from the root.

(**2.4.27**) Example. For $\Gamma(S_n, S)$ where S is the set of adjacent pair permutations, one can show that $l(gs) \neq l(g)$. See §**??**.

Chapter 3

The real field and geometry

For examples and motivation topology makes heavy use of properties of the real number field. These are familiar to us — the familiar properties of real arithmetic. But they are amazing and important. So let's review them a little. It is safe to skip this Chapter on first reading.



Our starting point here is the observation that the real line is a magical thing — meaning that it is amazing and familiar but hard to fully understand.

Here is a picture (of just a bit of it):

It is already quite amazing that I didn't need to tell you *which* bit that was. This is because, from one point of view, one chunk of the real line is much like another. If I mark a point anywhere on it, then that partitions the line into three parts: left of the point; the point; right of the point.

Notice that is not a property held by just any old set. If I have a random set and a pick an element, no further structual implications arise.

We often do mark a point on the real line. We call the first such marked point 0. We then have an additive structure on the line, with respect to which 0 is the identity. We can do 'slide-rule' addition, meaning we can compute $a + b$ by taking another copy of the line and sliding until the second copy of 0 is at a then looking at the position of the second copy of b. (Caveat: To do the slide we had to place the line in a bigger universe in which the second copy could also live! This was a physical-world operation rather than a maths one. To do it in maths-world we have to be a bit more patient.)

Notice that there was not actually anything special about the point 0 as far as the line was concerned. We chose it and made it special.

We often mark another special point on the line: we call it 1. Again this is a choice (except on the real line it should not be the same point as 0). We can use it as the identity of the $\times$ operation.

(**3.0.1**) Recall that the real numbers form a field, with $+$ and $\times$ as operations and 0 and 1 as their identities.

As usual we write $-a$ for the additive inverse of a.

(**3.0.2**) An ordered field F is one with a non-empty subset P_F (P for positive) that is closed under $+, \times$, does not contain 0, and for all $a \in F$ either a or $-a$ is in P_F.

In an ordered field we write $a > b$ if $a + (-b) \in P_F$.

Is this a partial order on F? Is it true that $a > b$ and $b > c$ implies $a > c$?

(**3.0.3**) Anyway, an example is our real field $\mathbb{R}$, taking the positive numbers for $P_{\mathbb{R}}$. Indeed here $>$ has its usual meaning corresponding to ordering on the real line.

Another example is $\mathbb{Q}$.

3.1 Arithmetic

(**3.1.1**) For $r \in \mathbb{R}$ we define $|r|$ to be the element of $\{r, -r\}$ that lies in $P_{\mathbb{R}}$ (as above) if $r \neq 0$, and define $|0| = 0$.

Note that $|r| = 0$ implies $r = 0$. Otherwise $|r| > 0$.

(**3.1.2**) For $a, b \in \mathbb{R}^n$ recall the notation

$$a.b = \sum_{i=1}^{n} a_i b_i$$

Note that $a.a > 0$. An interesting notation is to set $|a|^2 = a.a$.

An interesting property of $\mathbb{R}$ is that for $r \in \mathbb{R}$ if $r > 0$ then the equation $x^2 - r = 0$ has a positive solution in $\mathbb{R}$ (and also a negative solution). We write $\sqrt{r}$ for the positive solution. Then for $a \in \mathbb{R}^n$ the notation $|a|$ means the positive square root of $|a|^2$.

Chapter 4

Metric Spaces

4.1 Metrics and metric spaces

In this section we assume you know quite a lot about the real numbers — the set $\mathbb{R}$ together with its properties as an ordered field. (Do not worry. These are properties with which you *are* familiar, like $3\pi > 0$; and that positive numbers have real square roots, as we'll see.)

Definition 4.1. Let X be a set. A function $d : X \times X \to \mathbb{R}$ is a **metric** on X if:

(M1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;

(M2) $d(x, y) = d(y, x)$ for all $x, y \in X$ (symmetry);

(M3) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$ (triangle inequality).

The pair (X, d) is called a **metric space**.

Example 4.2. On any set X, the function

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

is a metric. Axioms (M1) and (M2) are satisfied directly. (M3) holds if $x = z$, for then $d(x, y) + d(y, z) \geq 0 = d(x, z)$. If $x \neq z$ then y cannot coincide with both x and z, so $d(x, y) + d(y, z) \geq 1 = d(x, z)$.

Example 4.3. Let $X = \mathbb{R}$ and let

$$d(x, y) = |x - y|.$$

(M1) and (M2) are satisfied here by (3.1.1). (M3) can be checked case-by-case. We can assume that $x \leq z$ without loss of generality. If $x \leq y \leq z$ then $d(x, z) = d(x, y) + d(y, z)$. If $y < x \leq z$ or $x \leq z < y$ then $d(x, z) < d(x, y) + d(y, z)$.

This d is thus a metric. It is sometimes called the **standard metric** on $\mathbb{R}$.

Example 4.4. Fix $n \in \mathbb{N}$. Let $X = \mathbb{R}^n$ and let

$$d(x, y) = \sqrt{\sum_{i-1}^{n} (x_i - y_i)^2} = \sqrt{|x - y|^2} = |x - y|$$

Here (M1) and (M2) follow from (3.1.2). To prove (M3), we can use the inequality:

$$a \cdot b \leq \sqrt{|a|^2}\sqrt{|b|^2} \quad \forall a, b \in \mathbb{R}^n. \tag{CS}$$

This is sometimes called the *Cauchy-Schwartz inequality*. Here is a short proof. Note that

$$\begin{aligned} 0 &\le \sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_j - a_jb_i)^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i^2b_j^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} a_j^2b_i^2 - 2\sum_{i=1}^{n}\sum_{j=1}^{n} a_ib_ia_jb_j \\ &= 2\sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2 - 2\left(\sum_{i=1}^{n} a_ib_i\right)^2. \end{aligned}$$

So $|a|^2|b|^2 - (a \cdot b)^2 \ge 0$, and rearranging yields the inequality (CS).

Returning to our proof of the triangle inequality, we have that

$$\begin{aligned} (d(x,y) + d(y,z))^2 &= |x-y|^2 + |y-z|^2 + 2\sqrt{|x-y|^2}\sqrt{|y-z|^2} \\ &\ge |x-y|^2 + |y-z|^2 + 2(x-y)\cdot(y-z) \\ &= (x-y+y-z)\cdot(x-y+y-z) \\ &= d(x,z)^2. \end{aligned}$$

Example 4.5. Let $X = \mathbb{R}^n$ and let

$$d(x,y) = \max_{i=1,\ldots,n} |x_i - y_i|.$$

Then d is a metric. For (M3), suppose j is such that $d(x,z) = |x_j - z_j|$. We know from Example 4.3 that $|x_j - z_j| \le |x_j - y_j| + |y_j - z_j|$. So

$$\begin{aligned} d(x,z) &\le |x_j - y_j| + |y_j - z_j| \\ &\le \max_{i=1,\ldots,n} |x_i - y_i| + \max_{i=1,\ldots,n} |y_i - z_i| \\ &= d(x,y) + d(y,z). \end{aligned}$$

4.2 Open balls

Definition 4.6. Let (X,d) be a metric space. An **open ball** is a subset $B_\varepsilon(x)$ of X of the form

$$B_\varepsilon(x) = \{y \in X \;:\; d(x,y) < \varepsilon\}$$

where x is any element of X and $\varepsilon > 0$ is any positive real number.

Lemma 4.7. *Let (X, d) be a metric space, let $x \in X$ and let $\delta > 0$. If $y \in B_\delta(x)$ then there exists an $\varepsilon > 0$ such that $B_\varepsilon(y) \subseteq B_\delta(x)$.*

Proof. Let $\delta > 0$. We need to choose $\varepsilon > 0$ such that $z \in B_\varepsilon(y) \Rightarrow z \in B_\delta(x)$. By the triangle inequality, any $z \in B_\varepsilon(y)$ satisfies

$$d(z, x) \leq d(z, y) + d(y, x) < \varepsilon + d(y, x)$$

So if $\varepsilon = \delta - d(y, x)$ then every $z \in B_\varepsilon(y)$ satisfies $d(x, z) < \delta$ as required. This choice of ε is positive, because $d(x, y) < \delta$. □

Definition 4.8. Let (X, d) be a metric space. We say that $U \subseteq X$ is ***d*-open** if and only if: for all $x \in U$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$.

Lemma 4.9. *In a metric space, open balls are d-open.*

Proof. Follows from Lemma 4.7. □

Lemma 4.10. *Let (X, d) be a metric space and denote*

$$\tau_d = \{U \subseteq X : U \text{ is } d\text{-open}\} \subseteq \mathcal{P}(X)$$

Then we have

(T1) *$\emptyset \in \tau_d$ and $X \in \tau_d$.*

(T2) *If $U \in \tau_d$ and $V \in \tau_d$ then $U \cap V \in \tau_d$.*

(T3) *If $\{U_\lambda\}_{\lambda \in \Lambda}$ is a family of subsets of X such that $U_\lambda \in \tau_d$ for all $\lambda \in \Lambda$, then $\bigcup_{\lambda \in \Lambda} U_\lambda \in \tau_d$.*

Proof. (T1): for the first claim there is nothing to show; and for the second it is true by construction.
(T2): if $x \in U \cap V$ there exits $\varepsilon_1, \varepsilon_2 > 0$ such that $B_{\varepsilon_1}(x) \subset U$ and $B_{\varepsilon_2}(x) \subset V$. Letting $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ gives $B_\varepsilon(x) \subset U \cap V$.
(T3): exercise. □

Definition 4.11. For reasons that will be explained shortly, we call the set of subsets τ_d the **metric topology** on the metric space (X, d).

Definition 4.12. Let X be a metric space. A subset A of X is called ***d*-closed** if $X \setminus A$ is open.

The conditions (T2) and (T3) have a different flavour. What would happen if we consider arbitrary intersections in (T2) instead of only pairwise? Consider for example the Euclidean metric above. We claim:

$$\cap_{e>0} B_e(x) \;=\; \{x\}$$

By definition this intersection is the set of elements that are in every such ball around x. The element x itself is in every such ball. But for any $y \neq x$ we have $d(y, x) > 0$ so one of the smaller balls does not contain y.

The above argument says that if we modified (T2) to allow arbitrary intersections then each $\{x\}$ would be an open set. Since we allow arbitrary unions, then every subset would be open.

4.3 Equivalent metrics

Definition 4.13. Two metrics d and d' on a set X are called **equivalent** if there exist real numbers $D \geq C > 0$ such that

$$Cd'(x,y) \leq d(x,y) \leq Dd'(x,y) \quad \forall x,y \in X.$$

Lemma 4.14. *Let d and d' be equivalent metrics on a set X. Then $U \subset X$ is d-open if and only if $U \subset X$ is d'-open. Thus $\tau_d = \tau_{d'}$.*

Proof. We need to show that any set which is open with respect to one metric is also open with respect to the other.

Suppose that $U \subset X$ is d'-open. This means that for each $x \in U$ there exists an $\varepsilon > 0$ such that $B'_\varepsilon(x) \subset U$, where $B'_\varepsilon(x)$ denotes the open ball in the metric d'. We must construct an open ball in the metric d that contains x and is contained in U.

By definition $B_{C\varepsilon}(x) \subset B'_\varepsilon(x)$; if $z \in B_{C\varepsilon}(x)$ then $d(x,z) \leq C\varepsilon \implies Cd'(x,z) \leq d(x,z) \leq C\varepsilon$ so $z \in B'(x)$. Thus we have shown that U is open with respect to d.

Now, if U is d-open a similar argument (exercise) shows that it is also d'-open.

□

Example 4.15. The metrics on $\mathbb{R}^n$ defined in examples 4.5 and 4.4 are equivalent and therefore define the same set of d-open sets. This is because

$$\left(\max_{i=1,\ldots,n} |x_i - y_i|\right)^2 = \max_{i=1,\ldots,n} |x_i - y_i|^2 \leq \sum_{i=1}^{n} (x_i - y_i)^2$$

and

$$\sum_{i=1}^{n} (x_i - y_i)^2 \leq n \max_{i=1,\ldots,n} |x_i - y_i|^2 = n \left(\max_{i=1,\ldots,n} |x_i - y_i|\right)^2.$$

4.4 Metric continuous functions

Recall that $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at x if and only if: for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

We're now going to write this in a different way using d - the standard metric on $\mathbb{R}$, $d(x,y) = |x - y|$.

We have: $f : \mathbb{R} \to \mathbb{R}$ is continuous at x if and only if: for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $d(f(x), f(y)) < \varepsilon$.

Using yet more notation from the world of metric spaces I could have equivalently written (check this!), $f : \mathbb{R} \to \mathbb{R}$ is continuous at x if and only if: for all $\varepsilon > 0$ there exits $\delta > 0$ such that $f(B_\delta(x)) \subset B_\varepsilon(f(x))$.

Once more, since $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x))) \iff f(B_\delta(x)) \subset B_\varepsilon(f(x))$ we will write another equivalent statement, $f : \mathbb{R} \to \mathbb{R}$ is continuous at x if and only if: for all $\varepsilon > 0$ there exits $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$.

This motivates the following

Definition 4.16. A function $f : (X,d) \to (Y,d')$ between metric spaces is **metric continuous** at $x \in X$ if and only if: for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$B_\delta(x) \subset f^{-1}(B'_\varepsilon(f(x))).$$

Furthermore we say that $f : X \to Y$ is metric continuous if and only if it is continuous for all $x \in X$.

Remark 4.17. NB: This definition is the usual definition of continuity when X, Y are the vector spaces $\mathbb{R}^n, \mathbb{R}^m$ equipped with the standard metrics, only now it is suitable for general metric spaces.

I use the notation B' to denote an open ball with respect to the metric on Y, d'. But from here on out I'll forget this notation since it's clear from the context in which space the balls lie!

Theorem 4.18. *$f : (X, d) \to (Y, d')$ is metric continuous if and only if for all d'-open $U \subset Y$, $f^{-1}(U) \subset X$ is d-open.*

Proof. Suppose that for all open $U \subset Y$, $f^{-1}(U) \subset X$ is open. Then for all $\varepsilon > 0$ we know that $f^{-1}(B_\varepsilon(f(x))) \subset X$ is open, since $B_\varepsilon(f(x))$ is open in Y. By definition (since $x \in f^{-1}(B_\varepsilon(f(x)))$) this implies there exists $\delta > 0$ so that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$, thus f is metric continuous.

For the converse suppose that f is metric continuous. Let $U \subset Y$ be open so we want to prove that $f^{-1}(U)$ is open. Pick $x \in f^{-1}(U)$ giving $f(x) \in U$. Since U is open, there exists $\varepsilon > 0$ so that $B_\varepsilon(f(x)) \subset U$ and since f is metric continuous, there exists $\delta > 0$ so that $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x))) \subset f^{-1}(U)$ so we are done (remember that, since X is a metric space, $f^{-1}(U)$ is open if and only if, for all $x \in f^{-1}(U)$ there exists $\delta > 0$ so that $B_\delta(x) \subset f^{-1}(U)$). $\square$

Chapter 5

Topological spaces

5.1 Abstract topological spaces

Notice that in order to define continuity on metric spaces we only need the notion of open sets by Theorem 4.18.

In essence, a topology on a set X is a collection of subsets τ which we **declare to be open**. We will need this collection to satisfy some axioms; you will notice (after reading the definition!) that the metric topology is indeed a topology (cf Lemma 4.10).

Definition 5.1. Let X be a set. A **topology** on X is a set τ of subsets of X having the properties that

(T1) $\emptyset \in \tau$ and $X \in \tau$.

(T2) If $U \in \tau$ and $V \in \tau$ then $U \cap V \in \tau$.

(T3) If $\{U_\lambda\}_{\lambda\in\Lambda}$ is a family of subsets of X such that $U_\lambda \in \tau$ for all $\lambda \in \Lambda$, then $\bigcup_{\lambda\in\Lambda} U_\lambda \in \tau$.

A **topological space** is a pair (X, τ) where X is a set and τ is a topology on X. The elements of X are called the **points** of the space. The elements of τ are called the **open sets** of the space, and are said to be **open** in (X, τ).

Remark 5.2. It follows by induction from (T2) that the intersection of any *finite* collection of open sets is open. Note however that the intersection of an *infinite* collection of open sets need not be open. By contrast, in axiom (T3) the indexing set Λ could be finite, countably infinite, or uncountably infinite.

5.1.1 Examples

Example 5.3. If as in examples 4.3 and 4.5 $X = \mathbb{R}^n$ and d is the standard (or Euclidean) metric, the associated metric topology is called the **standard topology**.

Example 5.4. Let X be any set and let $\tau = \{\emptyset, X\}$. Then certainly τ satisfies (T1). (T2) holds because $\emptyset \cap X = \emptyset \in \tau$, and (T3) holds because $\emptyset \cup X = X \in \tau$. So τ is a topology on X; it is known as the *trivial topology* or **indiscrete topology**.

Example 5.5. Let X be any set and let τ be the power set of X (i.e. the set of all subsets of X). You can check for yourself that τ satisfies the axioms (T1), (T2) and (T3). So τ defines a topology on X; it is known as the **discrete topology** on X.

Example 5.6. Let (X, d) be as in Example 4.2. Then the associated metric topology is the discrete topology.

To prove this, we need to first consider what open balls look like in this metric. If $\varepsilon > 1$ then $B_\varepsilon(x)$ is equal to the whole of X, whereas if $\varepsilon \leq 1$ $B_\varepsilon(x) = \{x\}$ is a singleton (a set with one element). Thus the collection of all open balls consists of all singletons in X, together with X itself. In particular *any* subset of X is open (since it is either empty, or a union of singletons).

Example 5.7. Let $X = \{1, 2\}$ and let $\tau = \{\emptyset, \{1\}, \{1, 2\}\}$. Then τ is a topology on X. Checking this is a matter of going through the axioms once more. (T1) is immediate. (T2) holds because

$$\emptyset \cap \{1\} = \emptyset \in \tau, \quad \emptyset \cap \{1, 2\} = \emptyset \in \tau, \quad \{1\} \cap \{1, 2\} = \{1\} \in \tau.$$

(T3) holds, because

$$\emptyset \cup \{1\} = \{1\} \in \tau, \quad \emptyset \cup \{1, 2\} = \{1, 2\} \in \tau, \quad \{1\} \cup \{1, 2\} = \{1, 2\} \in \tau, \quad \emptyset \cup \{1\} \cup \{1, 2\} = \{1, 2\} \in \tau.$$

Example 5.8. Let X be any set and let

$$\tau = \{A \subset X \ : \ X \setminus A \text{ is finite}\} \cup \{\emptyset\}.$$

Then τ defines a topology on X – checking this is an Exercise. This topology is known as the **finite complement topology** or *general Zariski topology.*

Definition 5.9. Let X be a topological space. A subset A of X is called **closed** if $X \setminus A$ is open.

Example 5.10. In the discrete topology on a set X, any subset is both open and closed.

Example 5.11. In the indiscrete topology on set X, the only closed subsets are X and $\emptyset$. All other sets are neither open nor closed.

The above examples show that sets can be open, closed, both, or neither!

Proposition 5.12. *Let X be a topological space. Then*

1. *X and $\emptyset$ are closed;*

2. *The union $A \cup B$ of two closed sets A and B is closed;*

3. *If $\{A_\lambda\}_{\lambda \in \Lambda}$ is any indexed family of closed subsets of X, the intersection $\bigcap_{\lambda \in \Lambda} A_\lambda$ is closed.*

Proof. These all follow from the axioms for a topological space (using de Morgan's rules for the second two). $\square$

If we know two different topologies on the same set, then it is useful to be able to compare them:

Definition 5.13. Let τ and σ be two topologies on a set X. If τ is a subset of σ (i.e. if A is any subset of X then $A \in \tau \Rightarrow A \in \sigma$) then τ is said to be **coarser** than σ, and σ is said to be **finer** than τ.

It follows from this definition that the discrete topology is finer than any other topology. Similarly, axiom (T1) implies that the indiscrete topology is coarser than any other topology.

5.2 Neighbourhoods and separated spaces

Definition 5.14. Let X be a topological space and let $x \in X$. A **neighbourhood** U of x is an open set $U \subset X$ such that $x \in U$.

Definition 5.15. A topological space X is called a **separated (or Hausdorff) space** if

(H) for each pair x, y of distinct points in X there exist neighbourhoods U of x and V of y which are disjoint.

The Hausdorff condition (H) will allow us to prove strong theorems for separated spaces that do not apply to non-separated spaces.

Theorem 5.16. *Metric topologies are separated.*

Proof. Let (X, d) be a metric space equipped with the metric topology. Suppose that x_1, x_2 are two distinct points in X. Then $d(x_1, x_2) > 0$. Let $\varepsilon = d(x_1, x_2)/2$, $B_1 = B_\varepsilon(x_1)$ and $B_2 = B_\varepsilon(x_2)$. Then certainly $x_1 \in B_1$ and $x_2 \in B_2$. It remains to check that $B_1 \cap B_2 = \emptyset$. Suppose for contradiction that $B_1 \cap B_2$ is non-empty, and let $y \in B_1 \cap B_2$. Then, by the triangle inequality,

$$d(x_1, x_2) \leq d(x_1, y) + d(y, x_2) < \varepsilon + \varepsilon = d(x_1, x_2).$$

It cannot happen that $d(x_1, x_2) < d(x_1, x_2)$, so $B_1 \cap B_2$ must be empty. □

It is not true that every separated topology is induced by a metric.

5.3 New topologies from old

In this section we will describe two ways to generate more examples of topological spaces.

Definition 5.17. Let (X, σ) and (Y, τ) be two topological spaces. The **product topology**, $(X \times Y, \sigma \times \tau)$ is defined by:

$$\{W \subset X \times Y \ : \forall\, (x, y) \in W\ \exists U \in \sigma \text{ and } V \in \tau \text{ such that } (x, y) \in U \times V \subset W\}. \tag{5.1}$$

Notice that it is equivalent to say that $W \subset X \times Y$ is open if there exist $\{U_\lambda\}_{\lambda \in \Lambda}$ open in X and $\{V_\lambda\}_{\lambda \in \Lambda}$ open in Y such that

$$W = \bigcup_{\Lambda} (U_\lambda \times V_\lambda)\,.$$

For this definition to make sense we must explain why the collection (5.1) of subsets of $X \times Y$ is a topology. (T1) and (T3) are straightforward to check using the definition (exercise). For (T2) if W_1, W_2 satisfy (5.1) then given $(x, y) \in W_1 \cap W_2$, let $U_1, U_2 \subset X$ open and $V_1, V_2 \subset Y$ open be such that $(x, y) \in U_1 \times V_1 \subset W_1$ and $(x, y) \in U_2 \times V_2 \subset W_2$. Now setting $U = U_1 \cap U_2 \ni x$ (which is open in X) and $V = V_1 \cap V_2 \ni y$ (open in Y) we have

$$(x, y) \in U \times V \subset W_1 \cap W_2.$$

Example 5.18. Consider $\mathbb{R}^p$ and $\mathbb{R}^q$ for $p, q \in \mathbb{Z}_+$ and $p + q = n \geq 2$. Let each space be equipped with the standard topology. Then the product topology on $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^q$ coincides with the standard topology on $\mathbb{R}^n$.

Let τ_n denoted the product topology and τ the standard topology. We will show that $W \subset \mathbb{R}^n$ is τ_n-open if and only if it is τ-open.

First suppose that W is τ_n-open and pick $x \in W$. We need to show that there exists $\varepsilon > 0$ so that $B^n_\varepsilon(x) \subset W$, where the open ball is defined with respect to the standard metric on $\mathbb{R}^n$. Since W is τ_n-open there exist open $U \subset \mathbb{R}^p$ and open $V \subset \mathbb{R}^q$ such that $x = (x_1, x_2) \in U \times V \subset W$. Since they are both open there exits $\varepsilon_1, \varepsilon_2 > 0$ such that $B^p_{\varepsilon_1}(x_1) \subset U$ and $B^q_{\varepsilon_2}(x_2) \subset V$. Set $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$ and notice that $B^n_\varepsilon(x) \subset B^p_{\varepsilon_1}(x_1) \times B^q_{\varepsilon_2}(x_2) \subset U \times V \subset W$ and we are done. The first inclusion follows since if $y = (y_1, y_2) \in B^n_\varepsilon(x) = B^n_\varepsilon((x_1, x_2))$ we have

$$\min\{\varepsilon_1, \varepsilon_2\}^2 = \varepsilon^2 > |y - x|^2 = |y_1 - x_1|^2 + |y_2 - x_2|^2,$$

giving $y_1 \in B^p_{\varepsilon_1}(x_1)$ and $y_2 \in B^q_{\varepsilon_2}(x_2)$.

The converse is similar so I won't spell it out. If W is τ-open. Pick $x = (x_1, x_2) \in W$ and notice that we can choose $\varepsilon > 0$ so that $B^n_\varepsilon(x) \subset W$. Now set $\varepsilon_1 = \frac{\varepsilon}{\sqrt{2}}$ and $\varepsilon_2 = \frac{\varepsilon}{\sqrt{2}}$ to give $B^p_{\varepsilon_1}(x_1) \times B^q_{\varepsilon_2}(x_2) \subset B^n_\varepsilon(x) \subset W$ since if $y = (y_1, y_2) \in B^p_{\varepsilon_1}(x_1) \times B^q_{\varepsilon_2}(x_2)$ then

$$\varepsilon^2 = \varepsilon_1^2 + \varepsilon_2^2 > |y_1 - x_1|^2 + |y_2 - x_2|^2 = |y - x|^2.$$

Thus setting $U = B^p_{\varepsilon_1}(x_1)$, $V = B^q_{\varepsilon_2}(x_2)$ gives $x \in U \times V \subset W$ and W is τ_n-open.

Definition 5.19. Let (X, τ) be a topological space and let $A \subset X$ be a subset of X. The **subspace topology** on A is

$$\tau_A = \{A \cap U \ : \ U \in \tau\}, \tag{5.2}$$

and the topological space (A, τ_A) is called a **subspace** of (X, τ).

To justify this definition we must explain why the collection τ_A of sets is a topology. Axiom (T1) is satisfied because $A = A \cap X$ and $\emptyset = A \cap \emptyset$ both belong to τ_A. τ_A satisfies axiom (T2) because τ does: if $A \cap U$ and $A \cap V$ are members of τ_A then their intersection $A \cap U \cap V$ is again in τ_A, because $U \cap V \in \tau$. Similarly, τ_A satisfies axiom (T3) because τ does: if $A \cap U_\lambda \in \tau_A$ is a collection of sets in τ_A indexed by $\lambda \in \Lambda$ then

$$\bigcup_{\lambda \in \Lambda} (A \cap U_\lambda) = A \cap \left(\bigcup_{\lambda \in \Lambda} U_\lambda \right) \in \tau_A.$$

Proposition 5.20. *Let A be a subspace of a topological space X, and let $B \subset A$. Then B is closed in A if and only if there exists a closed subset C of X such that $B = C \cap A$.*

Proof. See exercise sheet 2. □

The following lemma gives a useful way to identify open sets in some special circumstances:

Lemma 5.21. *Let X be a topological space and let Y be an open subset of X. If $U \subset Y$ is open in the subspace topology then U is an open subset of X.*

Proof. Let $U \subset Y$ be any open subset of Y. Then by definition there exists an open subset $V \subset X$ such that $U = Y \cap V$. Then U must be open, because both Y and V are. □

Remark 5.22. The analogous result to Lemma 5.21 holds for closed subsets of closed subspaces as an easy corollary to Proposition 5.20: i.e. if $B \subset X$ is closed, and $C \subset B$ is closed in B (with the subspace topology), then $C \subset X$ is closed.

Recall that separated spaces are special kinds of topological spaces that satisfy the additional axiom (H). Happily, the operations of product and subspace preserve the Hausdorff property:

Proposition 5.23. *Any subspace of a separated space is Hausdorff. The product of any two separated spaces is Hausdorff.*

Proof. First suppose that X is a separated space and that A is a subspace. Let $x, y \in A$ be two points such that $x \neq y$. Since X is separated there exist open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Then the open subsets $A \cap U$ and $A \cap V$ satisfy the condition that $x \in A \cap U$, $y \in A \cap V$, and $(A \cap U) \cap (A \cap V) = \emptyset$. So A is Hausdorff.

The case of a product is left as an exercise! □

5.4 Continuous functions on topological spaces

As suggested previously, the notion of continuity can now be defined using open sets:

Definition 5.24. Let X and Y be two topological spaces and let $f : X \to Y$ be a function. f is called **continuous** if for every open subset $A \subset Y$, $f^{-1}(A)$ is an open subset of X.

Remark 5.25. Theorem 4.18 tells us that if X and Y are metric spaces equipped with the metric topologies then $f : X \to Y$ is continuous if and only if it is metric continuous. So there is only one notion of continuity, and it only relies on open sets.

Example 5.26. Constant functions are continuous.

Let X, Y be two topological spaces, pick a point $c \in Y$ and let $f(x) = c \ \forall x \in X$. Now suppose that $U \subset Y$ is an open subset of Y. Then either $c \in U$ and $f^{-1}(c) = X$, or $c \notin U$ and $f^{-1}(U) = \emptyset$. In either case, $f^{-1}(U)$ is open, so f must be continuous.

Example 5.27. The identity map is continuous.

Let X be a topological space and let $f(x) = x \ \forall x \in X$. Suppose that $U \subset X$ is an open subset of X. Then $f^{-1}(U) = U$ so $f^{-1}(U)$ is open. Thus f is continuous

Example 5.28. Let Y be any topological space, let X be a set equipped with the discrete topology, and let $f : X \to Y$. Then for any open set $U \subset Y$, $f^{-1}(U)$ is open, since all subsets of X are open by definition. Therefore f is continuous.

Example 5.29. Let X be any topological space, let Y be a set equipped with the indiscrete topology, and let $f : X \to Y$. The only open sets in Y are Y and $\emptyset$: $f^{-1}(Y) = X$ and $f^{-1}(\emptyset)$ are both open, so f is continuous.

Proposition 5.30. *Let X, Y be topological spaces. Then $f : X \to Y$ is continuous if and only if for every closed subset C of Y, $f^{-1}(C)$ is closed in X.*

Proof. See exercise sheet 2. □

Now we will prove some basic properties of continuous functions.

Proposition 5.31. *Let X, Y and Z be topological spaces and let $f : X \to Y$ and $g : Y \to Z$ be continuous functions. Then $g \circ f : X \to Z$ is continuous.*

Proof. Suppose that $U \subset Z$ is open. Since g is continuous, $g^{-1}(U) \subset Y$ is open, and since f is continuous $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open. Thus $g \circ f$ is continuous. □

Proposition 5.32. *Suppose that A is a subspace of a topological space X. Then*

(a) the inclusion map $\iota_A : A \to X$, $x \mapsto x$ is continuous.

(b) if Y is any topological space and $f : X \to Y$ is continuous, the restriction $f_A : A \to Y$ of f to A is continuous.

(c) if Z is any topological space and $f : Z \to X$ satisfies $f(Z) \subset A$, then $f : Z \to X$ is continuous $\iff$ $f : Z \to A$ is continuous.

Proof. For part (a), suppose that U is an open subset of X. Then $\iota_A^{-1}(U) = A \cap U$. This set is by definition an open subset of A, so ι_A is continuous.

For part (b), note that $f_A = f \circ \iota_A$. So f_A is continuous by Proposition 5.31.

For part (c), ($\implies$), if $U \subset A$ is open then $U = A \cap \tilde{U}$ for $\tilde{U} \subset X$ open. Thus $f^{-1}(U) = f^{-1}(A \cap \tilde{U}) = f^{-1}(\tilde{U})$ since $f(Z) \subset A$. So $f^{-1}(U) \subset Z$ is open.

For ($\impliedby$), if $\tilde{U} \subset X$ is open, then again by the assumption on f we have $f^{-1}(\tilde{U}) = f^{-1}(\tilde{U} \cap A)$. But $\tilde{U} \cap A \subset A$ is open by definition, to $f^{-1}(\tilde{U}) \subset Z$ is open. □

Proposition 5.33. *Let X and Y be two topological spaces. Then*

(a) the projection maps $\pi_X : X \times Y \to X$, $(x, y) \mapsto x$ and $\pi_Y : X \times Y \to Y$, $(x, y) \mapsto y$ are continuous.

(b) if Z is any topological space then $f : Z \to X \times Y$ is continuous if and only if $\pi_X \circ f$ and $\pi_Y \circ f$ are both continuous.

Proof. For part (a), suppose that U is an open subset of X. Then $\pi_X^{-1}(U) = U \times Y$. This is clearly open in $X \times Y$.[1] Thus π_X is continuous; a similar argument shows that π_Y is continuous.

For the "only if" part of (b), suppose that $f : Z \to X \times Y$ is continuous. Then Proposition 5.31 implies that $\pi_X \circ f$ and $\pi_Y \circ f$ are both continuous, because π_X and π_Y are both continuous.

For the "if" part of (b), assume that $\pi_X \circ f$ and $\pi_Y \circ f$ are both continuous, and let W be an open subset of $X \times Y$. We must show that $f^{-1}(W)$ is an open subset of Z. By Definition 5.17 W can be written as:

$$W = \bigcup_{\lambda \in \Lambda} U_\lambda \times V_\lambda, \quad \text{with } U_\lambda \subset X,\ V_\lambda \subset Y \text{ open } \forall \lambda \in \Lambda.$$

By Proposition A.8, we have that

$$f^{-1}\left(\bigcup_{\lambda \in \Lambda} U_\lambda \times V_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(U_\lambda \times V_\lambda).$$

For each $\lambda \in \Lambda$,

$$\begin{aligned} f^{-1}(U_\lambda \times V_\lambda) &= \{z \in Z \ :\ f(z) \in U_\lambda \times V_\lambda\} \\ &= \{z \in Z \ :\ \pi_X \circ f(z) \in U_\lambda \text{ and } \pi_Y \circ f(z) \in V_\lambda\} \\ &= (\pi_X \circ f)^{-1}(U_\lambda) \cap (\pi_Y \circ f)^{-1}(V_\lambda) \end{aligned}$$

Since we have assumed that $\pi_X \circ f$ and $\pi_Y \circ f$ are continuous, this set is the intersection of two open sets and hence open. Therefore $f^{-1}(W)$ is a union of open sets, hence open. □

Lemma 5.34 (glue lemma)**.** *Let X and Y be two topological spaces and let A and B be subsets of X such that $X = A \cup B$. Let $g : A \to Y$ and $h : B \to Y$ be two continuous functions such that $g(x) = h(x)$ for all $x \in A \cap B$. Define $f : X \to Y$ by*

$$f(x) = \begin{cases} g(x) & \text{if } x \in A \\ h(x) & \text{if } x \in B. \end{cases}$$

Then

(a) if A and B are both open subsets of X then f is continuous; and

(b) if A and B are both closed subsets of X then f is continuous.

Proof.

(a) Suppose that A and B are open. Let U be an open subset of Y. Then

$$f^{-1}(U) = g^{-1}(U) \cup h^{-1}(U).$$

Since g and h are both continuous the subsets $g^{-1}(U) \subset A$ and $h^{-1}(U) \subset B$ are open in the subspace topology. The subsets $A \subset X$ and $B \subset X$ are open, so by Lemma 5.21, $g^{-1}(U)$ and $h^{-1}(U)$ are open subsets of X. Therefore $f^{-1}(U)$ is a union of open sets, hence open.

[1] Clearly $U \subset X$ and $Y \subset Y$ are open. Thus for all $(x, y) \in U \times Y$ we have $(x, y) \in U \times Y \subset U \times Y$

(b) Suppose now that A and B are closed. Let C be a closed subset of Y. By Proposition 5.30, it suffices to show that $f^{-1}(C)$ is closed. We have that

$$f^{-1}(C) = g^{-1}(C) \cup h^{-1}(C).$$

Since g and h are both continuous the subsets $g^{-1}(C) \subset A$ and $h^{-1}(C) \subset B$ are closed. Then by Remark 5.22, since $A \subset X$, $B \subset X$ are closed, we know that $g^{-1}(C) \subset X$, $h^{-1}(C) \subset X$ are closed. Since the union of two closed sets is closed we have $f^{-1}(C) \subset X$ is closed.

□

5.5 Sequences and limit points

A point of terminology: a **neighbourhood of a point** x is an open set U containing x.

Definition 5.35. Let X be a topological space and let A be a subset of X. The **closure** $\overline{A}$ of A is the intersection of all closed sets containing A: $\overline{A} = \bigcap_{\lambda \in \Lambda} C_\lambda$, where $\{C_\lambda\}_{\lambda \in \Lambda}$ is the collection of all closed sets containing A. A is called **dense** if $\overline{A} = X$.

Note that $\overline{A}$ is closed, by Theorem 5.12. Note also that $A \subset \overline{A}$. In fact, $\overline{A}$ is the smallest closed set containing A.

Exercise 5.1. Prove that A is closed if and only if $A = \overline{A}$.

Definition 5.36. Let X be a topological space and let A be a subset of X. A point $x \in X$ is called a **limit point** of A if for every neighbourhood U of x, $U \cap A \neq \emptyset$.

Proposition 5.37. *Let A be any subset of a topological space X. Then*

$$\overline{A} = \{\textit{limit points of } A\}.$$

Proof. We prove each set is contained in the other.

Let $x \in \overline{A}$. Then for any closed set $C \supset A$, $x \in C$. Suppose, for contradiction, that x is not a limit point of A. Then there exists a neighbourhood U of x, such that $U \cap A = \emptyset$. Then $K = X \setminus U_x$ is closed, and $A \subset K$. But $x \notin K$, contradicting $x \in \overline{A}$. Hence x is a limit point of A.

Conversely, let x be a limit point of A and assume for contradiction that $x \notin \overline{A}$. Then there is a closed set C such that $C \supset A$ and $x \notin C$. Therefore $U = X \setminus C$ is an open set containing x, and $U \cap A = \emptyset$, contradicting the fact that x is a limit point of A. Hence $x \in \overline{A}$. □

Proposition 5.38. *Let X be a separated space and let $x \in X$. Then $\overline{\{x\}} = \{x\}$. In particular, $\{x\}$ is closed.*

Proof. Note that $x \in \overline{\{x\}}$ because $\{x\} \subset \overline{\{x\}}$. Let $y \in X$ be any point with $y \neq x$. Then there exist neighbourhoods U of x and V of y such that $U \cap V = \emptyset$. Then $V \cap \{x\} = \emptyset$, so y is not a limit point of $\{x\}$. So by Proposition 5.37, $y \notin \overline{\{x\}}$. □

Now we turn our attention to sequences.

Definition 5.39. Let X be a topological space, let $x \in X$, and let $\{x_n\}_{n \in \mathbb{Z}^+}$ be a sequence of points in X. The sequence x_n is said to **converge to** x, and x is called a **limit** of the sequence x_n, if for every neighbourhood U of x there exists $N \in \mathbb{Z}^+$ such that $x_n \in U$ for all $n \geq N$. We write $x_n \to x$.

Example 5.40. In general, limits may not be unique!

In the indiscrete topology, all sequences converge to everything. Let X be set with the indiscrete topology, let x_n be any sequence in X and let x be any point in X. The only neighbourhood of x is X, and $x_n \in X$ whenever $n \geq 1$. So $x_n \to x$.

There is an equivalent way to check convergence in metric spaces:

Theorem 5.41. *Let (X, d) be a metric space and let x_n be a sequence in X. Then x_n converges to a point $x \in X$ if and only if for every $\varepsilon > 0$ there exists an $N \in \mathbb{Z}^+$ such that $x_n \in B_\varepsilon(x)$ whenever $n \geq N$.*

Exercise 5.2. Prove that, in a metric space (X, d), $x_n \to x$ if and only if $d(x_n, x) \to 0$. This follows by writing this out in the usual ε-N language.

This theorem implies that in $\mathbb{R}$ with the standard (metric-induced) topology, our definition of convergence agrees with one that you have seen before.

Proof. The "only if" part is straightforward, because any open ball $B_\varepsilon(x)$ is a neighbourhood of x.

For the "if" part, suppose that x_n is a sequence with the property that $\forall \varepsilon > 0\ \exists N \in \mathbb{Z}^+$ such that $x_n \in B_\varepsilon(x)$ whenever $n \geq N$. Let U be a neighbourhood of x. Since U is open and $x \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. Let N be such that $x_n \in B_\varepsilon(x)$ whenever $n \geq N$. Then $x_n \in U$ whenever $n \geq N$. So $x_n \to x$. □

Now we return to the question of how many limits a sequence can have. Everything works nicely in separated spaces:

Theorem 5.42. *In separated spaces, limits of convergent sequences are unique.*

Proof. Let X be a separated space and suppose for contradiction that x_n is a sequence in X with two distinct limits x and y. Since X is separated there exist disjoint neighbourhoods U and V of x and y. Since $x_n \to x$ and $x_n \to y$ there exist $M, N \in \mathbb{Z}^+$ such that $x_n \in U$ whenever $n \geq M$ and $x_n \in V$ whenever $n \geq N$. If $n \geq \max\{M, N\}$ then $x_n \in U \cap V$, contradicting the fact that U and V are disjoint. So a sequence x_n can have only one limit. □

Is every limit point of a set the limit of some sequence within that set? In general no, but the answer is yes in metric spaces!

Lemma 5.43. *In a metric space (X, d), let $A \subset X$, and x be a limit point of A. Then there exists a sequence $\{x_n\}_{n \in \mathbb{Z}^+} \subset A$ such that $x_n \to x$.*

Proof. Consider the ball $B_{\frac{1}{n}}(x)$, for $n \in \mathbb{Z}^+$. This is certainly a neighbourhood of x for all $n \in \mathbb{Z}^+$. Hence, as x is a limit point of A, we must have

$$B_{\frac{1}{n}}(x) \cap A \neq \emptyset.$$

Thus, there are points $x_n \in B_{\frac{1}{n}}(x) \cap A$ for each $n \in \mathbb{Z}^+$ which define a sequence in A. Let $\varepsilon > 0$, and choose $N \in \mathbb{Z}^+$ such that $N > \frac{1}{\varepsilon}$. Then for all $n \geq N$,

$$d(x_n, x) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon,$$

so $x_n \in B_\varepsilon(x)$ for all $n \geq N$, which means $x_n \to x$ by Theorem 5.41. □

Lemma 5.44. *Let X be a topological space and $C \subset X$. If C is closed, then for all convergent sequences $\{x_n\} \subset C$ such that $x_n \to x \in X$, we have $x \in C$.*

If (X, d) is a metric space and $C \subset X$ then the converse is true. If $C \subset X$ satisfies: for all convergent sequences $\{x_n\} \subset C$ such that $x_n \to x \in X$, we have $x \in C$. Then C is closed.

Proof. For the first part, let $\{x_n\} \subset C$ such that $x_n \to x \in X$, then x is a limit point of C since for all neighbourhoods U of x there exists some N s.t. $n \geq N$ implies $x_n \in U$. i.e. $U \cap C \neq \emptyset$ for all such U.

Assuming (X, d) is a metric space let $x \in \overline{C}$. Then by Lemma 5.43 there exists a sequence $\{x_n\} \subset C$ such that $x_n \to x$. By assumption we have $x \in C$ and thus $\overline{C} \subset C$, in which case $C = \overline{C}$ is closed. □

5.6 The quotient topology

This will allow us to make rigorous sense of gluing and/or deforming topological spaces to make new ones via equivalence relations. Recall that an equivalence relation $\sim$ on a set X partitions X into equivalence classes, denoted $[x]$. It satisfies

- $x \sim x$ (reflexivity)
- $x \sim y \implies y \sim x$ (symmetry)
- $x \sim y, y \sim z \implies x \sim z$ (transitivity),

and

$$[x] = \{y \in X : x \sim y\}.$$

The **quotient set** is simply the collection of all equivalence classes

$$X/_{\sim} = \{[x] : x \in X\}.$$

There is a natural map $p : X \to X/_{\sim}$, given by $p : x \mapsto [x]$ called the **quotient map**. This allows us to equip $X/_{\sim}$ with a topology:

Definition 5.45. Let (X, τ) be a topological space, let $\sim$ be an equivalence relation on X, and let $p : X \to X/_{\sim}$ be the map $x \to [x]$. The **quotient topology** on $X/_{\sim}$ is

$$\tau_{\sim} = \tau_p = \{U \subset X/_{\sim} \ : \ p^{-1}(U) \in \tau\}.$$

The quotient set $X/_{\sim}$ equipped with the topology $\tau_{\sim}$ is called the **quotient space**.

Remark 5.46. The quotient map p is continuous by definition.

This forms a topology because of the following

Lemma 5.47. *Let (X, τ) be a topological space, and Y be any* set *(not yet a topological space), and let $f : X \to Y$ be any function. Then*

$$\tau_f := \{U \subset Y \ : \ f^{-1}(U) \in \tau\}$$

is a topology on Y.

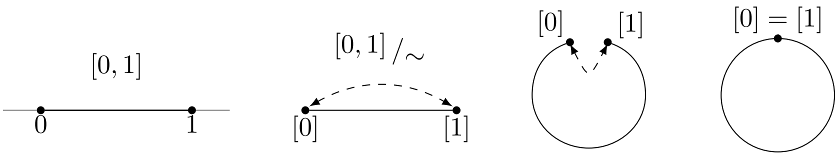
Remark 5.48. τ_f is the largest/finest topology on Y which makes f continuous.

Proof. To prove this lemma we just need to check the axioms. (T1) holds because $f^{-1}(Y) = X \in \tau$ and $f^{-1}(\emptyset) = \emptyset \in \tau$. (T2) holds because if $U, V \in \tau_f$ then $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V) \in \tau$. (T3) holds because if $\{U_\lambda\}_{\lambda \in \Lambda}$ is a collection of open sets in τ_f then $f^{-1}(\bigcup_{\lambda \in \Lambda} U_\lambda) = \bigcup_{\lambda \in \Lambda} f^{-1}(U_\lambda) \in \tau$. □

Example 5.49. Let $X = [0,1] \subset \mathbb{R}$ and $\sim$ be defined by $0 \sim 1$: it goes without saying that $x \sim x$ for all $x \in [0,1]$. The equivalence classes are thus

$$[x] = \{x\} \text{ when } x \in (0,1) \quad \text{and} \quad [0] = [1] = \{0,1\}.$$

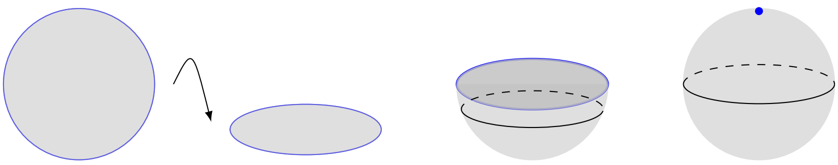
Defining $S^1 = \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ with the subspace topology, we will see later that there is no difference between the topological spaces ${[0,1]}/{\sim}$ and S^1 as the following picture indicates:



Example 5.50. Let $X = D^2 \subset \mathbb{R}^2$ where $D^2 := \{x = (x_1,x_2) : |x|^2 = x_1^2 + x_2^2 \leq 1\}$ and $\sim$ be defined by $x \sim y$ if and only if $x = y$ or $|x| = |y| = 1$. The equivalence classes are thus

$$[x] = \{x\} \text{ when } |x| < 1 \quad \text{and} \quad [x] = \{y \in S^1\} \text{ when } |x| = 1.$$

Defining $S^2 = \{z = (z_1,z_2,z_3) \in \mathbb{R}^3 : |z|^2 = 1\}$, we will see later that there is no difference between the topological spaces ${D^2}/{\sim}$ and S^2:



The examples above fall into an important category of quotient spaces In general, let X be a topological space and $A \subset X$. Define an equivalence relation on X via

$$x \sim y \quad \Longleftrightarrow x = y, \quad \text{or} \quad x, y \in A.$$

You should check this is an equivalence relation. Then the equivalence classes are A and the sets $\{x\}$ such that $x \in X \setminus A$.

Notation 5.51. In the case of the above equivalence relation, the resulting quotient space is often denoted by ${X}/{A}$.

Example 5.52. Let $X = [0,1] \times [0,1] \subset \mathbb{R}^2$. Let $\sim$ be the equivalence relation on X such that

$$(0,y) \sim (1,y) \quad \forall y \in [0,1].$$

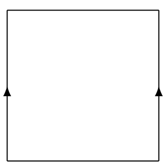
The equivalence classes for this equivalence relation consist of 1-point sets,

$$\{(x,y)\} = [(x,y)] \quad \text{when } x \in (0,1) \text{ and } y \in [0,1],$$

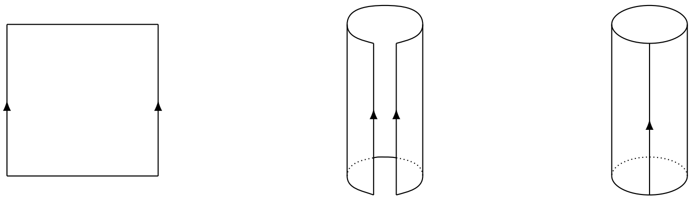
and 2-point sets

$$\{(0,y),(1,y)\} = [(0,y)] = [(1,y)] \quad \text{when } y \in [0,1].$$

This equivalence relation identifies the left- and right-hand edges of the square. The quotient ${X}/{\sim}$ is a topological space known as the **cylinder**. We represent the cylinder with the following picture:



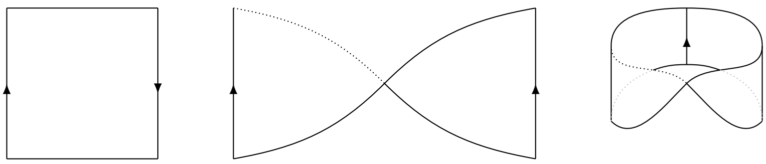
The arrows indicate the direction in which the edges have been identified and you should be able to see why this space is called a cylinder:



Example 5.53. Another example of an equivalence relation on $X = [0,1] \times [0,1]$ is

$$(0, y) \sim (1, 1-y) \quad \forall y \in [0,1].$$

The quotient of ${}^{X}/_{\sim}$ in this case is known as the **Möbius strip**. The picture looks like this (you should see a band with a twist in it):

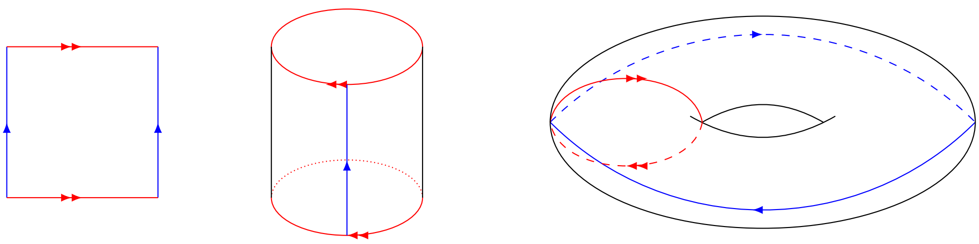


Once again, the arrows indicate the directions in which the edges have been identified.

Example 5.54. The equivalence relation (again on $X = [0,1] \times [0,1]$)

$$\begin{aligned} (0, y) &\sim (1, y) \quad \forall y \in [0,1], \\ (x, 0) &\sim (x, 1) \quad \forall x \in [0,1], \end{aligned}$$

gives a topological space called the **torus**. In the following picture, the single arrows indicate the identification of the left and right edges (blue), and the double arrows indicate the identification of the upper and lower edges (red). Once you have made this identification you end up with something that looks like a bagel.

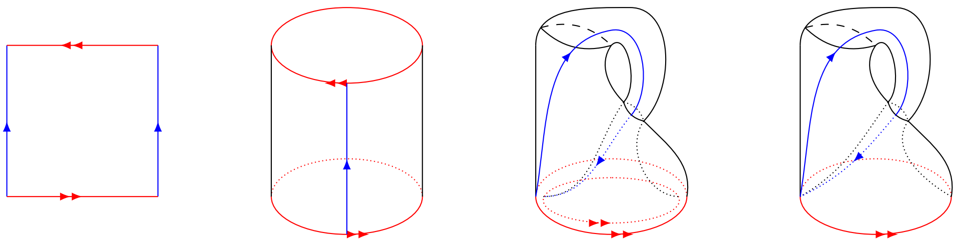


Example 5.55. The equivalence relation (again on $X = [0,1] \times [0,1]$)

$$\begin{aligned} (0, y) &\sim (1, y) \quad \forall y \in [0,1], \\ (x, 0) &\sim (1-x, 1) \quad \forall x \in [0,1], \end{aligned}$$

gives a topological space called the **Klein bottle**.[2] Unlike the previous examples, this one can't be realised/imagined (in three dimensions) without self-intersections:

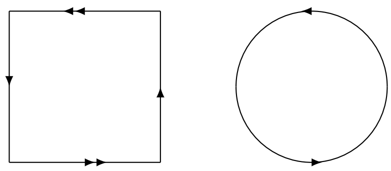
[2]Why is the Klein bottle called a bottle? The German word for bottle is *Flasche*, and the German word for surface is *Fläsche*. Probably this topological space was originally known in German as the Klein Fläsche, but someone made a mistake translating it to English. Would this surface be any use as a bottle? Notice that it doesn't have a well-defined inside or outside: see a youtube video of the construction here: https://www.youtube.com/watch?v=yaeyNjUPVqs



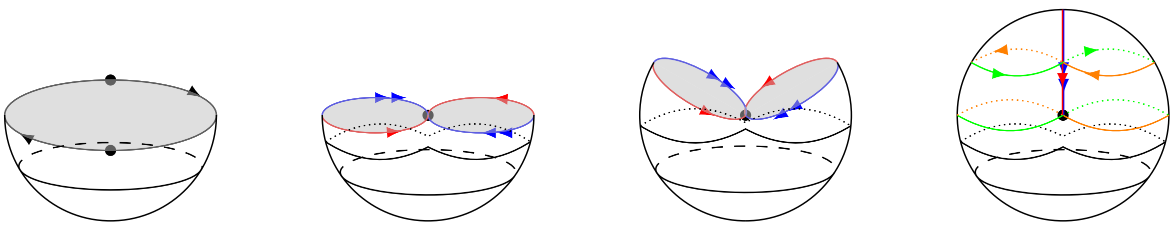
Example 5.56. The equivalence relation (again on $X = [0,1] \times [0,1]$)

$$
\begin{aligned}
(0, y) &\sim (1, 1-y) \quad \forall y \in [0,1], \\
(x, 0) &\sim (1-x, 1) \quad \forall x \in [0,1],
\end{aligned}
$$

gives a topological space called the **projective plane**, denoted $\mathbb{RP}^2$. By smoothing out the edges of the square, you can imagine that the pictures below are both fair representations of this topological space:

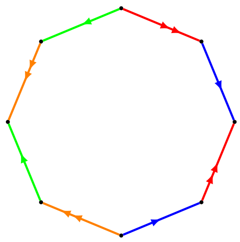


Here, the right hand picture is ${D^2}/{\sim}$ where $x \sim y$ if $|x| = 1$ and $x = -y$. In other words we identify opposite points on the boundary of D^2. Once again the projective plane cannot be realised/imagined in three-dimensions without self intersections. Here's one way to do this, called the cross-cap:[3]



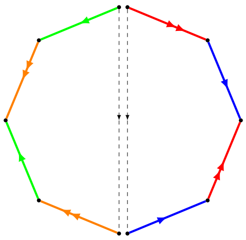
You should be able to convince yourselves now that in fact $\mathbb{RP}^2$ can be equivalently obtained/defined via ${S^2}/{\sim}$ where $\sim$ is defined via $x \sim -x$.

Example 5.57. We won't write the next one down precisely, but consider an octagon with its edges identified as suggested in the picture below

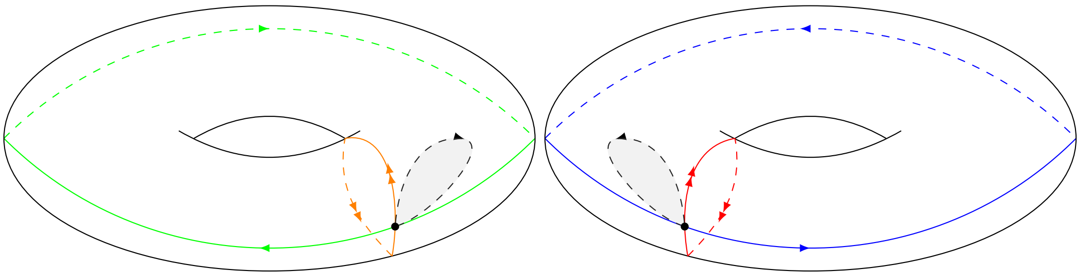


To see what we end up with it is best to first split the picture down the middle and cut it into two:

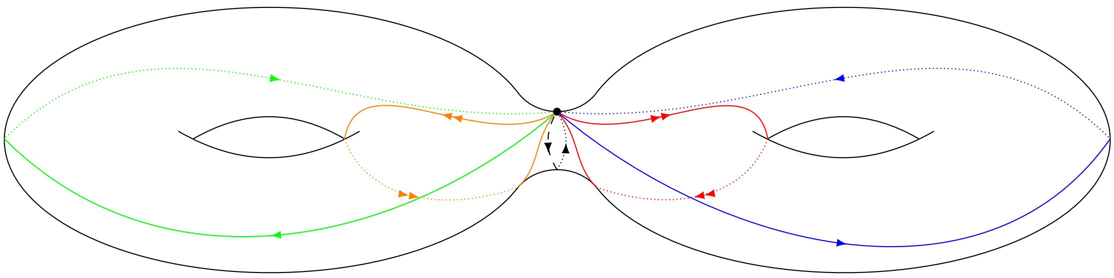
[3]find a youtube video of this construction here: https://www.youtube.com/watch?v=W-sKLN0VBkk



We end up with two pieces that each look like a torus, but with an extra curve and a piece missing. Try to imagine why we end up with the following (the grey areas indicate parts that are missing):



Now we must stitch the surface together again along the black dashed line - notice that the vertices of the hexagon have now all been identified to the same point. We end up with:



Recall that the product, subspace and metric constructions of topologies respect the Hausdorff property. The quotient topology does not, as the following counterexample shows:

Counterexample 5.58. Let $X = \mathbb{R}$ with its Euclidean topology and let $\sim$ be the equivalence relation

$$x \sim y \;\Leftrightarrow\; \exists\, \lambda \in \mathbb{R} \setminus \{0\} \text{ such that } x = \lambda y,$$

You may check for yourself that this is symmetric, reflexive, and transitive. Then the quotient space has just two elements, namely

$$X/\sim \;= \{\{0\}, \mathbb{R} \setminus \{0\}\}.$$

This is true since any non-zero $x, y \in \mathbb{R}$ are related via $\lambda = x/y$. In order to calculate its topology, let $p : X \to X/\sim$ denote the projection. The subset $\{\{0\}\} \subset X/\sim$ is not open since $p^{-1}\{\{0\}\} = \{0\}$ and $\{0\}$ is not an open subset of $\mathbb{R}$ in the standard topology. All other subsets are open, so the topology is given by

$$\tau = \left\{\emptyset, \{\mathbb{R} \setminus \{0\}\}, X/\sim\right\}.$$

Then $X/\sim$ is not Hausdorff: for if $x = \{\{0\}\}$ and $y = \{\mathbb{R} \setminus \{0\}\}$ then the only open subset of $X/\sim$ which contains x is $U = X/\sim$. But then any open set V containing $\mathbb{R} \setminus \{0\}$ has to intersect U, because $\mathbb{R} \setminus \{0\} \in U$.

Despite this, the five quotient spaces defined above (cylinder, Möbius strip, torus, Klein bottle, projective plane) *are* separated spaces.

Thinking about continuous maps between quotient spaces can sometimes be tricky. The next result helps us to do this.

Proposition 5.59. *Let X and Y be two topological spaces, let $\sim_X$ and $\sim_Y$ be equivalence relations on X and Y, and let $f : X \to Y$ be a continuous function. Suppose that f has the property that $\forall x, x' \in X$,*

$$x \sim_X x' \quad \Rightarrow \quad f(x) \sim_Y f(x').$$

Then

$$\widetilde{f} : X /_{\sim_X} \to Y /_{\sim_Y}, \quad \widetilde{f} : [x] \to [f(x)]$$

is a well-defined continuous function with respect to the quotient topologies.

Proof. First we check that $\widetilde{f}$ is well-defined, i.e. that if $[x] = [x']$ then $\widetilde{f}([x]) = \widetilde{f}([x'])$. Note that $[x] = [x']$ if and only if $x \sim_X x'$. Thus by assumption $[x] = [x']$ implies $f(x) \sim_Y f(x')$, which implies that $[f(x)] = [f(x')]$, or in other words, $\widetilde{f}([x]) = \widetilde{f}([x'])$.

Now we check that $\widetilde{f}$ is continuous. Let $p_X : X \to X /_{\sim_X}$ and $p_Y : Y \to Y /_{\sim_Y}$ be the two projections.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p_X"'] & Y \arrow[d, "p_Y"] \\
X /_{\sim_X} \arrow[r, "\widetilde{f}"] & Y /_{\sim_Y}
\end{tikzcd}$$

Observe that

$$\widetilde{f} \circ p_X = p_Y \circ f$$

(because $p_Y(f(x)) = [f(x)] = \widetilde{f}([x]) = \widetilde{f}(p_X(x))$).

Let $U \subset Y /_{\sim}$ be open. We must show that $\widetilde{f}^{-1}(U)$ is open, which by definition means that $p_X^{-1}(\widetilde{f}^{-1}(U)) \subset X$ is open. By the identity above, $p_X^{-1}(\widetilde{f}^{-1}(U)) = (\widetilde{f} \circ p_X)^{-1}(U) = (p_Y \circ f)^{-1}(U) = f^{-1}(p_Y^{-1}(U))$. Now $p_Y^{-1}(U)$ is open because U is open, and $f^{-1}(p_Y^{-1}(U))$ is open because f is continuous. □

The proof of the above proposition also allows us to understand continuous functions from $X /_{\sim}$ to Y via continuous functions $f : X \to Y$.

Corollary 5.60. *Let X and Y be two topological spaces, let $\sim$ be an equivalence relation on X, and let $f : X \to Y$ be a continuous function. Suppose that f has the property that $\forall x, x' \in X$,*

$$x \sim x' \quad \Rightarrow \quad f(x) = f(x').$$

Then

$$\widetilde{f} : X /_{\sim} \to Y, \quad \widetilde{f} : [x] \to f(x)$$

is a well-defined continuous function with respect to the quotient topology on X.

Proof. This time we have $\widetilde{f} \circ p = f$ and so exactly the same reasoning as above gives the result. □

Remark 5.61. If $f : X \to Y$ is continuous and $\sim$ is an equivalence relation on Y then $p : Y \to Y /_{\sim}$ is continuous so $\widetilde{f} = p \circ f : X \to Y /_{\sim}$ is continuous by Proposition 5.31 - i.e. $\widetilde{f}(x) = [f(x)]$, is always a continuous function when f is.

Chapter 6

Topological Invariants

These are properties of topological spaces which allow us to determine whether two spaces are different.

6.1 Homeomorphisms

In this section we will start comparing topological spaces. The most important concept is that of "homeomorphism": two spaces which are homeomorphic are topologically the same, up to a relabelling of the points (as will become apparent below).

Definition 6.1. Let X and Y be two topological spaces and let $f : X \to Y$. Then f is called a **homeomorphism** if it is bijective and both f and f^{-1} are continuous. In this case the spaces X and Y are called **homeomorphic**, denoted by $X \cong Y$.

Remark 6.2. Notice that a homeomorphism induces a bijection between the topologies on X and Y: for all open $V \subset Y$ there exists a unique open $U \subset X$ such that $f(U) = V$. Thus they are considered to be the 'same' topological space.

Example 6.3. $\mathbb{R}$ is homeomorphic to $(0, \infty)$. The function $f : \mathbb{R} \to (0, \infty)$, $f : x \mapsto e^x$ is a bijection; both f and its inverse $f^{-1} : (0, \infty) \to \mathbb{R}$, $f^{-1} : y \mapsto \ln y$ are continuous.

Example 6.4. Let $X = \{1, 2\}$, $\sigma = \{\emptyset, \{1\}, \{1, 2\}\}$, $Y = \{3, 4\}$, and $\tau = \{\emptyset, \{3\}, \{3, 4\}\}$. Then $f : X \to Y$, $f(1) = 3$, $f(2) = 4$ is a homeomorphism but $g : X \to Y$, $g(1) = 4$, $g(2) = 3$ is not ($g^{-1}(\{3\}) = \{2\}$ is not open).

Example 6.5. Let $X = [0, 1)$ and $Y = S^1 = \{x \in \mathbb{R}^2 \ : \ x_1^2 + x_2^2 = 1\}$, and equip X and Y with their subspace topologies inherited from $\mathbb{R}$ and $\mathbb{R}^2$. Let $f : X \to Y$ be the map $f : t \to (\cos(2\pi t), \sin(2\pi t))$. Then f is a continuous bijection, but f is *not* a homeomorphism. This is because $f^{-1} : Y \to X$ is not continuous.

To prove that f^{-1} is not continuous, we must find one open subset $U \subset X$ such that $(f^{-1})^{-1}(U) = f(U)$ is not open in Y. Consider $U = [0, 1/2)$. Then U is an open subset of X (because $U = (-1/2, 1/2) \cap X$), but $f(U)$ is not open. The reason for this is that $f(0) = (1, 0) \in f(U)$ but any open set containing $f(0)$ is not a subset of $f(U)$ (I'll leave you to check the details).

Example 6.6. You will prove in the exercises that for $a, b \in \mathbb{R}$, $a < b$ then the open interval $(a, b) \cong \mathbb{R}$.

Reminder: Continuity of multivariable functions Consider the spaces $\mathbb{R}^n$, $\mathbb{R}^m$ equipped with their standard topologies. By Proposition 5.33

$$f(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), f_2(x_1, \ldots x_n), \ldots f_m(x_1, \ldots, x_n))$$

is continuous if and only if $f_i(x_1, \ldots, x_n)$ is continuous for all $i = 1, \ldots, m$.

So we really only need to be concerned with $f_i(x_1, \ldots, x_n)$ in order to check continuity of f.

Example 6.7. Let $B_r^n(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$, i.e. the open ball or radius r centred at x_0 in $\mathbb{R}^n$. Then for all $x_0, y_0 \in \mathbb{R}^n$ and $r, R > 0$ we have $B_r^n(x_0) \cong B_R^n(y_0)$ since $f : B_r^n(x_0) \to B_R^n(y_0)$ defined by

$$f(x) = \frac{R}{r}(x - x_0) + y_0$$

is a continuous bijection with inverse $f^{-1}(y) = \frac{r}{R}(y - y_0) + x_0$, which is also continuous.

Example 6.8. Define $f : B_1^n(0) \to \mathbb{R}^n$ by

$$f(x) = \frac{x}{1 - |x|}.$$

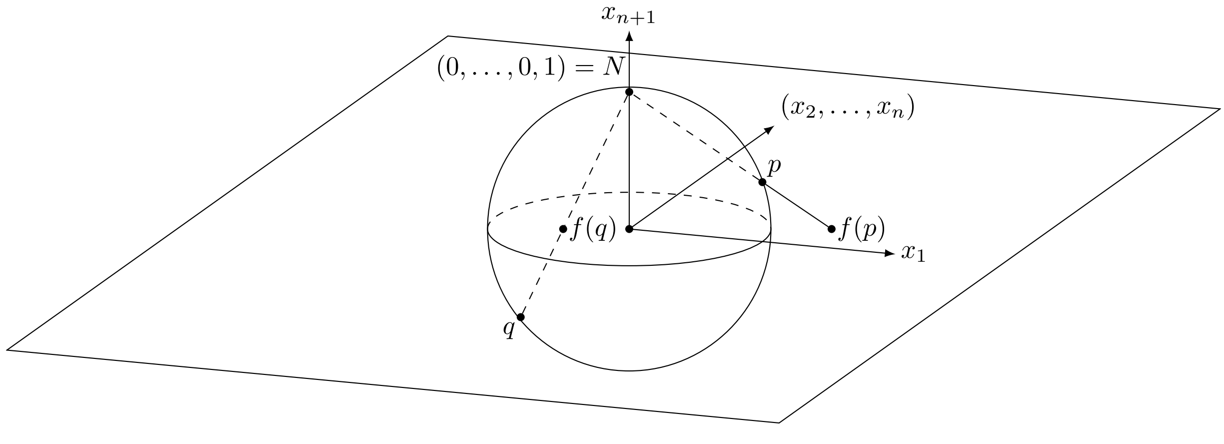
You can check that f is continuous, and directly compute its inverse to be $f^{-1} : \mathbb{R}^n \to B_1^n(0)$

$$f^{-1}(y) = \frac{y}{1 + |y|},$$

which is also continuous. Thus $B_1^n(0) \cong \mathbb{R}^n$.

Example 6.9. Let $S^n := \{x \in \mathbb{R}^{n+1} : |x|^2 = 1\}$.. Defining the North Pole $N = (0, \ldots, 0, 1)$, we now define the *stereographic projection* $f : S^n \backslash \{N\} \to \mathbb{R}^n$ in the following way. First off, we make the identification $\mathbb{R}^n \cong \{x \in \mathbb{R}^{n+1} : x^{n+1} = 0\}$.

Pick $p \in S^n \backslash \{N\}$ and draw the unique straight line in $\mathbb{R}^{n+1}$ joining p and N. Extending this line *ad infinitum*, it will strike $\mathbb{R}^n$ once, and only once at a point, which we define to be $f(p)$.



As an exercise, you should be able to check that

$$f(x_1, \ldots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \ldots, x_n),$$

which is continuous on $S^n \backslash \{N\}$; since $f : \mathbb{R}^{n+1} \backslash \{N\} \to \mathbb{R}^n$ is continuous and $S^n \backslash \{N\} \subset \mathbb{R}^{n+1} \backslash \{N\}$ has inherited the subspace topology (see Proposition 5.32). Furthermore, you can check that

$$f^{-1}(y_1, \ldots, y_n) = \frac{2}{1 + |y|^2}\left(y_1, \ldots, y_n, \frac{|y|^2 - 1}{2}\right),$$

is also continuous (also by Proposition 5.32), proving that f is a homeomorphism, i.e. $S^n \backslash \{N\} \cong \mathbb{R}^n$.

A similar construction can be carried out if we single out any point of S^n (i.e. not just for the North Pole N). So in fact we have $S^n \backslash \{p\} \cong \mathbb{R}^n$ for any $p \in S^n$. The easiest way to do this is to let $R : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a rotation or reflection (i.e. an orthogonal matrix[1]) satisfying $R(p) = N$. Then $f \circ R : S^n \backslash \{p\} \to \mathbb{R}^n$ is our desired homeomorphism. In particular, if $S = (0, \dots, 0, -1)$ is the South Pole, then we set $R_S(x_1, x_2, \dots, x_{n+1}) = (x_1, \dots, x_n, -x_{n+1})$ and $f \circ R_S : S^n \backslash \{S\} \to \mathbb{R}^n$ is a homeomorphism.

Example 6.10. Let $D^n = \overline{B_1^n(0)} \subset \mathbb{R}^n$, so that $D^n = \{y \in \mathbb{R}^n : |y| \leq 1\}$. Let $S_+^2 = \{x \in S^n : x_{n+1} \geq 0\}$ be the upper hemisphere. Now we'll show that $D^n \cong S_+^n$.

Let $f : D^n \to \mathbb{R}^{n+1}$ be given by

$$f(y_1, y_2, \dots, y_n) = (y_1, \dots, y_n, \sqrt{1 - |y|^2}).$$

By now I hope it is obvious that f is continuous, and furthermore $|f|^2 = 1$ for all $y \in D^n$ and $f_{n+1}(y_1, \dots, y_n) \geq 0$. So that $f : D^n \to S_+^n$ is in fact continuous.

f^{-1} is given simply by

$$f^{-1}(x_1, x_2, \dots, x_{n+1}) = (x_1, x_2, \dots, x_n)$$

and again, is clearly continuous. Thus f is a homeomorphism.

Proposition 6.11. *Homeomorphism is an equivalence relation.*

Proof. First check symmetry: suppose that X is homeomorphic to Y. Then there exists a homeomorphism $f : X \to Y$. The inverse $f^{-1} : Y \to X$ is also a homeomorphism, so Y is homeomorphic to X.

Next check reflexivity: the identity map $\mathrm{id}_X : X \to X$, $\mathrm{id}_X : x \mapsto x$ is a homeomorphism, so X is homemorphic to X.

Finally, check transitivity: if $f : X \to Y$ is a homemorphism and $g : Y \to Z$ is a homemorphism then $g \circ f : X \to Z$ is also a homeomorphism, hence X is homeomorphic to Z. □

Definition 6.12. A **topological invariant** is a property of a topological space which is preserved by homeomorphism. Specifically, it is an assignment $\varphi : \{\text{Top. spaces}\} \to \mathcal{S}$, where $\mathcal{S}$ is a set, such that if $X \cong Y$, then $\varphi(X) = \varphi(Y)$.

Remark 6.13. There are several important topological invariants that we will cover in this course:

- the Hausdorff property
- connectedness
- compactness
- the fundamental group.

Anyone interested in taking the study of topology further would do well to learn about homology and cohomology (and for surfaces, the Euler characteristic).

Topological invariants are useful because they let us prove that two spaces are not homeomorphic. Indeed, if $\varphi(X) \neq \varphi(Y)$, then $X \ncong Y$. Note however, we cannot use topological invariants to prove that two spaces are homeomorphic (there are many examples of invariants and spaces such that $\varphi(X) = \varphi(Y)$, but $X \ncong Y$).

[1] R satisfies $R^{-1} = R^T$

Proposition 6.14. *The Hausdorff property is a topological invariant.*

Proof. Suppose that $f : X \to Y$ is a homeomorphism between two topological spaces X and Y and suppose in addition that Y is a separated space. We must show that X is a separated space.

Let x_1, x_2 be two distinct points in X. Then $f(x_1), f(x_2)$ are two distinct points in Y, and since Y is separated there exist disjoint open subsets $V_1, V_2 \subset Y$ such that $f(x_i) \in V_i$ for each $i = 1, 2$. Then $U_i = f^{-1}(V_i)$ satisfy $x_i \in U_i$ for each $i = 1, 2$ and $U_1 \cap U_2 = \emptyset$. So X is Hausdorff. $\square$

Example 6.15. If X is a set with the indiscrete topology, Y is a set with the discrete topology and X has at least two elements, then X is not homeomorphic to Y. This is because Y is separated and X is not, and the previous proposition says that if X is homeomorphic to a separated space it must be Hausdorff.

It is **not** true that two separated spaces must be homeomorphic (for example, if X and Y are both separated but X has two elements and Y three then X and Y cannot be homeomorphic).

Proposition 6.16. *If $f : X \to Y$ is a homeomorphism and Z is a subspace of X then $f_Z : Z \to f(Z)$ is a homeomorphism.*

Proof. f_Z is surjective by definition, and it is injective because f is. Therefore $f_Z : Z \to f(Z)$ is a bijection. f_Z is continuous by Proposition 5.32. The only tricky part is proving that f_Z^{-1} is continuous. To do this we use the following lemma:

Lemma 6.17. *Let $g : X \to Y$ be an injective map between two sets. Then for any subsets $A, B \subset X$, $g(A \cap B) = g(A) \cap g(B)$.*

Proof. You should have already proved in problem set 1 that $g(A \cap B) \subset g(A) \cap g(B)$, so we only need to show that $g(A \cap B) \supset g(A) \cap g(B)$. Let $y \in g(A) \cap g(B)$. Then there exist $x \in A$ and $x' \in B$ such that $g(x) = y$ and $g(x') = y$. Since g is injective, $x = x'$, so $x \in A \cap B$. Therefore $y \in g(A \cap B)$. $\square$

Returning to the proof of Proposition 6.16, suppose that $U \subset Z$ is open. By definition there exists a $V \subset X$ such that $U = V \cap Z$. By the lemma,

$$(f_Z^{-1})^{-1}(U) = f_Z(U) = f(U) = f(V \cap Z) = f(V) \cap f(Z).$$

The set $f(V) = (f^{-1})^{-1}(V)$ is open in Y because f^{-1} is continuous, so $f(V) \cap f(Z)$ is open in $f(Z) \subset Y$. Therefore f_Z^{-1} is continuous. $\square$

Corollary 6.18. *If $f : X \to Y$ is a homeomorphism and $\{x_1, x_2, \ldots, x_n\} \subset X$ then $f : X \backslash \{x_1, x_2, \ldots, x_n\} \to Y \backslash \{f(x_1), f(x_2), \ldots, f(x_n)\}$ is a homeomorphism.*

Proof. Exercise. $\square$

6.2 Connectedness

Definition 6.19. Let X be a topological space. A pair of **nonempty open** subsets U and V of X such that $U \cap V = \emptyset$ and $X = U \cup V$ is called a **partition** of X. X is called **disconnected** if it admits a partition, and **connected** if it does not.

Remark 6.20. Notice that this definition forces the empty topological space to be connected (it certainly does not admit a non-empty subset, let alone a pair of disjoint ones!).

X is connected if and only if the only subsets of X which are both open and closed are X and $\emptyset$ (you will prove this in the exercise sheet).

Example 6.21. Any set X with the indiscrete topology is connected. This is because there is only one non-empty open set (the set X itself).

Example 6.22. If X has at least 2 elements then X with the discrete topology is disconnected. This is because we can write $X = \{x\} \cup (X \setminus \{x\})$ for some $x \in X$; the sets $\{x\}$ and $X \setminus \{x\}$ are both open and do not intersect one another.

Example 6.23. The subspace $Y = [0, 1] \cup [2, 3]$ of $\mathbb{R}$ is disconnected. This is because $U = [0, 1] = (-1, 2) \cap Y$ and $V = [2, 3] = (1, 4) \cap Y$ are open subsets of Y that form a partition.

Example 6.24. The subspace $\mathbb{Q}$ of $\mathbb{R}$ is disconnected. Let a be any irrational number (e.g. $a = \sqrt{2}$); then $U = (-\infty, a) \cap \mathbb{Q}$ and $V = (a, \infty) \cap \mathbb{Q}$ are open subsets of $\mathbb{Q}$ that form a partition.

We have now seen some examples of disconnected subspaces of $\mathbb{R}$ – one might then hesitate to ask if there are any examples of connected subspaces of $\mathbb{R}$? Proving that a topological space is connected turns out to be quite a bit harder than proving that a space is disconnected. To help answer the question, we will introduce some terminology:

Definition 6.25. A nonempty subset J of $\mathbb{R}$ is called an **interval** if for every $x, y \in J$, $[x, y] \subset J$.

According to this definition, all of the following subsets of $\mathbb{R}$ are intervals:

$$(a, b),\ (a, b],\ [a, b),\ [a, b],\ (-\infty, a),\ (-\infty, a],\ (a, \infty),\ [a, \infty),\ (-\infty, \infty) = \mathbb{R}.$$

In addition, a singleton set of the form $\{a\}$ for some $a \in \mathbb{R}$ is an interval according to the definition.

Theorem 6.26. *A subset of $\mathbb{R}$ is connected if and only if it is an interval.*

Recall that $\mathbb{Q}$ is disconnected, so $\mathbb{R}$ is very different from $\mathbb{Q}$ as a topological space. Unsurprisingly, the proof of this theorem relies on the property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$, namely that every bounded subset of $\mathbb{R}$ has a least upper bound (called the supremum, or sup).

Proof. First we prove the "only if" part. Suppose that $J \subset \mathbb{R}$ is connected and suppose for contradiction that J is not an interval. Then there exist $x, y \in J$ and $c \notin J$ with $x < c < y$. Let $U = J \cap (-\infty, c)$ and $V = J \cap (c, \infty)$. Then U and V are open $U \cap V = \emptyset$ and $U \cup V = J \setminus \{c\} = J$. So U and V form a partition of J, contradicting the supposition that J is connected.

Now we prove the "if" part: suppose that J is an interval, and let U and V be two open non-empty subsets of J such that $U \cap V = \emptyset$. By definition, these take the form $U = U_0 \cap J$ and $V = V_0 \cap J$ for some open subsets U_0, V_0 of $\mathbb{R}$. We will prove that there exists a number $c \in J$ with $c \notin U \cup V$; this means that $J \neq U \cup V$, and hence that J is connected.

Since U and V are non-empty and non-intersecting we may pick points $a \in U, b \in V$ with $a \neq b$. By changing notation if necessary, we may arrange things so that $a < b$. Since J is an interval, $[a, b]$ is a subset of J.

Now let $U_1 = U \cap [a, b] = U_0 \cap [a, b]$ and $V_1 = V \cap [a, b] = V_0 \cap [a, b]$, and let

$$c = \sup U_1$$

be the least upper bound of U_1. We know that $c \in [a, b]$, because $a \in U_1$ and b is an upper bound for U_1. Therefore $c \in J$, and it remains to show that c does not belong to U_1 or V_1.

If $c \in U_1 \subset U_0$ then $c \neq b$, so $c \in [a, b)$. Then there exists an interval (d_1, d_2) with $c \in (d_1, d_2) \subset U_0$, and we may assume $d_2 < b$ by shrinking this interval if necessary. Then

$(c + d_2)/2$ is an element of U_1 and is greater than c, contradicting the statement that c is an upper bound for U_1.

If $c \in V_1 \subset V_0$ there exists an interval (e_1, e_2) with $c \in (e_1, e_2) \subset V_0$. So no real numbers $x \in (e_1, e_2)$ belong to U_1. In fact, since c is an upper bound for U_1 and $c > e_1$, no real numbers $x > e_1$ belong to U_1. So $(c + e_1)/2$ is an upper bound for U_1 which is less than c, contradicting the statement that c is a least upper bound. □

Now we will begin to explore the properties of connected topological spaces

Proposition 6.27. *The continuous image of a connected topological space is connected.*

Proof. Let X and Y be two topological spaces and let $f : X \to Y$ be a continuous function. Suppose that the subspace $f(X) \subset Y$ is disconnected; we must show that X is disconnected.

Since $f(X)$ is disconnected there exist open subset U and V of Y such that $f(X) \cap U$ and $f(X) \cap V$ form a partition of Y. This means that $f(X) \cap U \neq \emptyset$, $f(X) \cap V \neq \emptyset$, $(f(X) \cap U) \cap (f(X) \cap V) = \emptyset$ and $(f(X) \cap U) \cup (f(X) \cap V) = f(X)$. We claim that $f^{-1}(U)$ and $f^{-1}(V)$ form a partition for X. These sets are open because f is continuous. They are non-empty because $f(X) \cap U$ and $f(X) \cap V$ are. Their intersections and unions are

$$f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V) = f^{-1}(f(X) \cap U \cap V) = f^{-1}(\emptyset) = \emptyset$$

and

$$f^{-1}(U) \cup f^{-1}(V) = f^{-1}(U \cup V) = f^{-1}(f(X) \cap (U \cup V)) = f^{-1}(f(X)) = X.$$

□

This proposition is useful for proving that spaces are connected.

Example 6.28. The circle $S^1 = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ (with its subspace topology) is connected. This is because it is the image of the continuous function $f : [0, 1] \to \mathbb{R}^2$, $f : t \mapsto (\cos 2\pi t, \sin 2\pi t)$, and $[0, 1]$ is connected.

Proposition 6.27 has two important theoretical consequences. The first is that it generalises the intermediate value theorem that you learnt in your real analysis course.

Corollary 6.29. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then for any number y between $f(a)$ and $f(b)$ there exists an $x \in [a, b]$ such that $f(x) = y$.*

Proof. According to Theorem 6.26 $[a, b]$ is a connected topological space. By Proposition 6.27 its image $f([a, b])$ is connected. By Theorem 6.26 its image is an interval. Since $f(a), f(b) \in f([a, b])$ any y such that $f(a) < y < f(b)$ or $f(b) < y < f(a)$ is an element of $f([a, b])$. Then by definition there exists an $x \in [a, b]$ such that $f(x) = y$. □

The second important consequence is that

Corollary 6.30. *Connectedness is a topological invariant.*

Proof. Let X, Y be two topological spaces, let $f : X \to Y$ be a homeomorphism, and suppose that X is connected. Since f is surjective Y is equal to the image of $f(X)$ of X. By proposition 6.27 $f(X)$ is connected. Therefore Y is connected. □

We can use the property of connectedness to prove that topological spaces are not homeomorphic

Example 6.31. $[a, b]$ is not homeomorphic to (c, d) for any a, b, c, d such that $a \le b$, $c < d$.

Suppose that $f : [a, b] \to (c, d)$ is a homeomorphism. Then $f_{(a,b]} : (a, b] \to (c, d) \setminus \{f(a)\}$ is also a homeomorphism, by Proposition 6.16. The topological space $(a, b]$ is connected by Theorem 6.26. The topological space $(c, d) \setminus \{f(a)\}$ is not, because it can be written as union of two disjoint non-empty open subsets: $(c, d) \setminus \{f(a)\} = (c, f(a)) \cup (f(a), d)$. These statements contradict corollary 6.30.

Using the same ideas you can prove that (for a, b, c, d as above) $[a, b]$ is not homeomorphic to $(c, d]$. Similarly $(a, b]$ is not homeomorphic to (c, d).

Another useful consequence of 6.27 is the following:

Corollary 6.32. *Let X and Y be two topological spaces. $X \times Y$ is connected if and only if X and Y are connected.*

Proof. ($\Longrightarrow$): Recall from Proposition 5.33 that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous maps, and that the image of $\pi_X(X \times Y) = X$ and $\pi_Y(X \times Y) = Y$. Therefore X and Y are the images of connected maps, hence connected by Proposition 6.27.

($\Longleftarrow$): Assuming that X and Y are connected, we first note that for fixed $y \in Y$

$$f_y : X \to X \times Y \quad x \mapsto (x, y)$$

is continuous (exercise) and therefore $X \times \{y\} \subset X \times Y$ is connected for each fixed y.

Similarly, $\{x\} \times Y \subset X \times Y$ is connected for each fixed x.

For a contradiction, suppose that $X \times Y$ is not connected, so there exits a non-empty open partition U, V of $X \times Y$.

Claim: For each fixed $x \in X$ we have

$$U \cap (\{x\} \times Y) = \begin{cases} \{x\} \times Y & \text{or} \\ \emptyset. \end{cases}$$

To prove the claim, suppose it is false. In particular $U_x = U \cap \{x\} \times Y$ and $V_x = V \cap \{x\} \times Y$ is a non-empty open partition of $\{x\} \times Y$. This contradicts the connectedness of $\{x\} \times Y$ so the claim is true.

The claim also proves that

$$V \cap (X \times \{y\}) = \begin{cases} X \times \{y\} & \text{or} \\ \emptyset \end{cases}$$

for any fixed $y \in Y$. Now, letting $(x, y) \in U$ and $(x', y') \in V$ we see that $\{x\} \times Y \subset U$ and $X \times \{y'\} \subset V$. Therefore $(x, y') \in U \cap V \neq \emptyset$ and we have our contradiction.

□

6.3 Path-connectedness

In this section we study another topological invariant, called path connectedness. Path connectedness is related to connectedness, and is usually easier to work with. There is a difference, however, and it turns out that path-connectedness is a stronger condition.

Definition 6.33. Let X be a topological space. Given two points $x, y \in X$, a **path from x to y** is a continuous function $\alpha : [0, 1] \to X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. The point x is called the **initial point** of the path and the point y is called the **final point** of the path.

Definition 6.34. X is called **path-connected** if for any two points $x, y \in X$ there exists a path from x to y. Notice that the empty set automatically satisfies this, so it is itself path connected.

Proposition 6.35. *Any path-connected topological space is connected.*

Proof. Let X be a path-connected topological space. Suppose for contradiction that X is not connected. Let U, V be a partition of X, and let $a \in U$ and $b \in V$. Since X is path-connected there exists a continuous function $\alpha : [0,1] \to X$ such that $\alpha(0) = a$ and $\alpha(1) = b$.

Consider the subspace $\alpha([0,1]) \subset X$. It is straightforward to show that the open subsets $\alpha([0,1]) \cap U$ and $\alpha([0,1]) \cap V$ form a partition for $\alpha([0,1])$: they are nonempty because $a \in \alpha([0,1]) \cap U$ and $b \in \alpha([0,1]) \cap V$; their union is $\alpha([0,1])$ because $U \cup V = X$, and they are disjoint because U and V are. So $\alpha([0,1])$ is disconnected. However, Theorem 6.27 and Proposition 6.26 imply that $\alpha([0,1])$ is connected, because it is the continuous image of a connected set $[0,1]$. So we have a contradiction, and it must be the case that X is connected. □

Using this proposition makes it much easier to check that a space is connected. Be warned however that the converse to Proposition 6.35 is false: connected spaces are not necessarily path-connected. We will give a counter-example below.

Example 6.36. $\mathbb{R}^n$ is path connected, hence connected.

Let x and y be two points in $\mathbb{R}^n$. Then

$$\alpha : [0,1] \to \mathbb{R}^n, \quad \alpha : t \mapsto (1-t)x + ty$$

is a continuous function such that $\alpha(0) = x$ and $\alpha(1) = y$.

Example 6.37. When $n \geq 2$, $\mathbb{R}^n \setminus \{(0,\dots,0)\}$ is path connected, hence connected.

Let x and y be two points in $\mathbb{R}^n \setminus \{(0,\dots,0)\}$. If x and y do not lie on opposite sides of the origin we may use the function α written down in the previous example to define a continuous $\alpha : ([0,1]) \to \mathbb{R}^n \setminus \{(0,\dots,0)\}$ such that $\alpha(0) = x$ and $\alpha(1) = y$. If x and y do lie on opposite sides of the origin (i.e. $x = -\lambda y$ for some $\lambda > 0$) we must use a different path. Thinking geometrically, there is always some non-zero vector $m \in \mathbb{R}^{n+1}$ so that $x \cdot m = 0$[2]. In particular $\tilde{x} = x + m$ is non-zero and does not lie on opposite sides of the origin to y or x. Thus we first run a straight line (as above) from x to $\tilde{x}$ and then from $\tilde{x}$ to y. But we do each one at double speed! The following one is suitable:

$$\alpha : t \mapsto \begin{cases} (1-2t)x + 2t\tilde{x} & \text{if } t \in [0,1/2] \\ (2-2t)\tilde{x} + (2t-1)y & \text{if } t \in [1/2,1]. \end{cases}$$

You should check for yourself that this function is continuous (you could use, for example, Lemma 5.34).

Example 6.38. $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}$.

To prove this, suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is a homeomorphism. Then the restriction $f_{\mathbb{R}^2\setminus\{(0,0)\}} : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R} \setminus \{f(0,0)\}$ is also a homeomorphism, by Proposition 6.16. However, $\mathbb{R}^2 \setminus \{(0,0)\}$ is connected and $\mathbb{R} \setminus \{f(0,0)\}$ is not, so they cannot be homeomorphic. Therefore there is no homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}$.

Now we will prove an analog of Proposition 6.27

Proposition 6.39. *The continuous image of a path-connected topological space is path-connected.*

[2] actually there is an $n-1$-dimensional space of choices for such an m, please prove why this is

Proof. Let X and Y be topological spaces such that X is path-connected, and let $f : X \to Y$ be a continuous function. We must show that $f(X)$ is path-connected. Let $y_1, y_2 \in f(X)$. Then there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is path-connected there exists a path α from x_1 to x_2. Then $f \circ \alpha$ is a path from y_1 to y_2. It is continuous because α is continuous by definition, and the composition of continuous functions is continuous, and it satisfies $f \circ \alpha(0) = y_1$, $f \circ \alpha(1) = y_2$ because $\alpha(0) = x_1$ and $\alpha(1) = x_2$. So $f(X)$ is path-connected. $\square$

Corollary 6.40. *Path-connectedness is a topological invariant.*

Proof. Exercise. $\square$

Proposition 6.41. *Let X and Y be two topological spaces. Then $X \times Y$ is path-connected if and only if X and Y are path-connected.*

Proof. Suppose first that X, Y are path connected. Let (x_1, y_1) and (x_2, y_2) be two points in $X \times Y$. Then there exist paths α fom x_1 to x_2 and β from y_1 to y_2. The function $F : [0,1] \to X \times Y$, $F(t) = (\alpha(t), \beta(t))$ is continuous by Proposition 5.33 and satisfies $F(0) = (x_1, y_1)$ and $F(1) = (x_2, y_2)$, so is a path from (x_1, y_1) to (x_2, y_2). Therefore $X \times Y$ is path-connected.

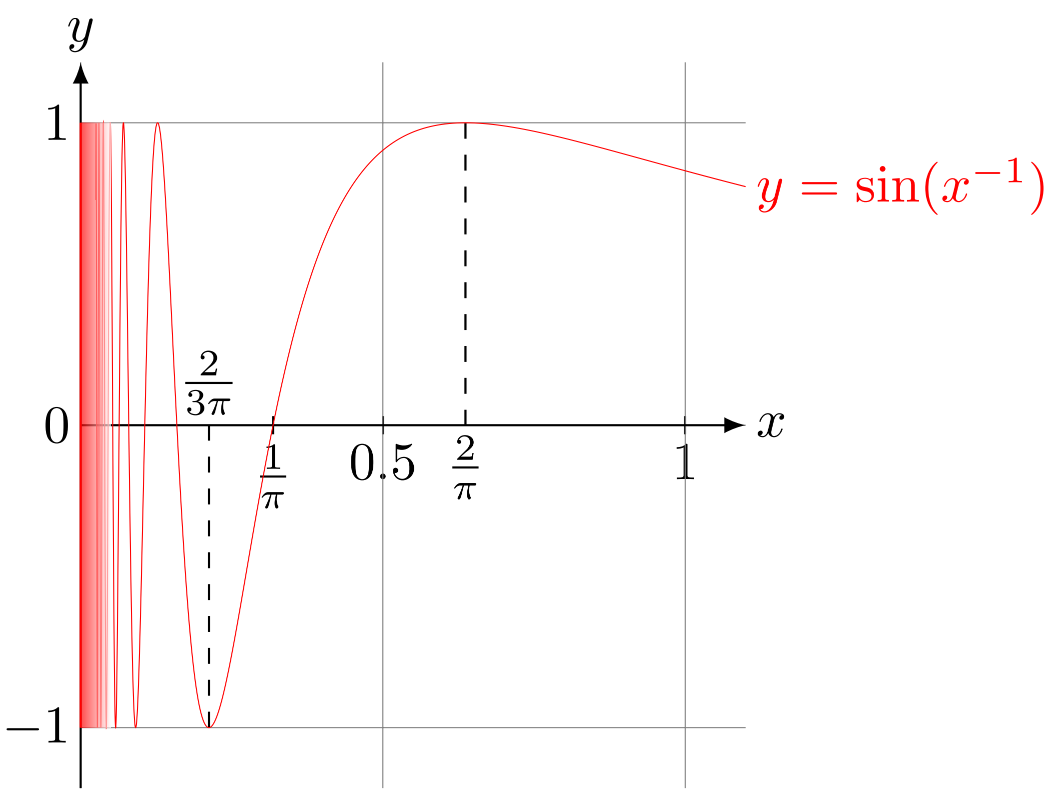
The proof of the converse (i.e. that $X \times Y$ path connected implies X and Y path connected) is similar to the proof of corollary 6.32 and left as an exercise. $\square$

Finally, here is the promised counter-example.

Counterexample 6.42. The "topologist's sine curve". Let $X \subset \mathbb{R}^2$ be the space

$$X = S \cup Y, \quad Y = \{x = 0, \, y \in [-1, 1]\}, \quad S = \{(x, \sin(1/x)) \; : \; x \in (0, \infty)\}$$

i.e. X is the union of the graph of the function $\sin(1/x)$ with the interval $[-1, 1]$ in the y-axis. As $x \to 0$, $\sin(1/x)$ will rise and fall, hitting all points in the interval $[-1, 1]$ infinitely many times (see the image below). Thus all the limit points of S are given exactly by Y so that $X = \overline{S}$. It turns out that X is connected but not path-connected.



To see that X is connected, suppose that U and V form a partition for X. Neither of these sets can equal Y because it is not an open subset of X. Therefore $U \cap S$, $V \cap S$ form a partition for S. But S is the image of the connected set $(0, \infty)$ under the continuous function $x \mapsto (x, \sin(1/x))$, hence connected, so we have a contradiction.

To see that X is not path-connected. Pick a continuous path $\alpha : [0,1] \to X$ so that $\alpha(0) = (0,0)$ and $\alpha(1) = (2/\pi, 1)$. Recall that the projections $\pi_y : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto y$ and $\pi_x : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x$ are continuous.

Let $t_0 = \sup_{t \in [0,1]}\{\alpha(t) \in Y\} = \sup_{t\in[0,1]}\{(\pi_x \circ \alpha)(t) = 0\}$. By continuity of α we must have $\alpha(t_0) \in Y$ and $t_0 < 1$. Let $\tilde{\alpha} = \pi_x \circ \alpha : (t_0, 1] \to \mathbb{R}$. By assumption $\tilde{\alpha}(t) > 0$ (equivalently $\alpha(t) \in S$) for all $t \in (t_0, 1]$ and thus $\alpha(t) = (\tilde{\alpha}(t), \sin(1/\tilde{\alpha}(t)))$ for all $t \in (t_0, 1]$.Now notice that $\hat{\alpha} = \pi_y \circ \alpha : [0,1] \to [-1,1]$ is continuous. But, for any $w \in [-1,1]$ we can choose a sequence $\{x_n\} \subset (t_0, 1]$ with $x_n \to t_0$ so that $\hat{\alpha}(x_n) = \sin(1/\tilde{\alpha}(x_n)) \to w$, contradicting the continuity of $\hat{\alpha}$ (in particular our conditions imply that $\hat{\alpha}(t_0) = w$ for any $w \in [-1,1]$!). This contradiction shows that no such continuous α can exist and X is not path connected.

More generally, one can show that (exercise) if $S \subset Y$ is connected, then $\overline{S}$ is connected. Again, the above example shows that if $S \subset Y$ is path connected, then we do not necessarily have $\overline{S}$ is path connected.

6.4 Compactness

This is perhaps the most important mathematical concept that you will meet in this course.

Definition 6.43. Let X be a topological space and let A be a subset of X. An **open cover** for A is an indexed family $\{U_\lambda\}_{\lambda\in\Lambda}$ of **open** subsets of X such that

$$A \subset \bigcup_{\lambda\in\Lambda} U_\lambda.$$

An open cover is called **finite** if the set Λ contains finitely-many elements. If $\{U_\lambda\}_{\lambda\in\Lambda}$ is an open cover for A and $\Lambda' \subset \Lambda$ is such that

$$A \subset \bigcup_{\lambda\in\Lambda'} U_\lambda$$

then $\{U_\lambda\}_{\lambda\in\Lambda'}$ is called a **subcover** of $\{U_\lambda\}_{\lambda\in\Lambda}$.

Definition 6.44. Let X be a topological space and let A be a subset of X. A is called a **compact subset** of X if every open cover for A has a finite subcover. X is called a **compact topological space** if the set X is a compact subset of X.

Example 6.45. If X is any set with the indiscrete topology, then every subset of X is compact. This is because the only possible open covers for any subset of X are $\{X\}$ and $\{X, \emptyset\}$; both of these are already finite, so they both admit finite subcovers.

Example 6.46. Let X be any topological space and let A be any subset of X with finitely many elements. Then A is compact.

To prove this, suppose that $\{U_\lambda\}_{\lambda\in\Lambda}$ is any open cover for A. Then for each $x \in A$ we may choose a λ_x such that $x \in U_{\lambda_x}$. Set

$$\Lambda' = \{\lambda_x \,:\, x \in A\}.$$

We claim that $\{U_\lambda\}_{\lambda\in\Lambda'}$ is a finite subcover. We need to check that $A \subset \bigcup_{\lambda\in\Lambda'} U_\lambda$: let $x \in A$; then $x \in U_{\lambda_x}$ and $U_{\lambda_x} \subset \bigcup_{\lambda\in\Lambda'} U_\lambda$, so $x \in \bigcup_{\lambda\in\Lambda'} U_\lambda$ as required. We also need to check that Λ' is finite: this is true because A is finite and $|\Lambda'| \leq |A|$.

Example 6.47. Let X be any set with the discrete topology and let $A \subset X$. Then A is compact if and only if A is finite.

We have already proved the "if" part of this statement. For the "only if" part, suppose that A is compact. Let $U_x = \{x\}$ for any x in A. Then $\{U_x\}_{x \in A}$ is an open cover for A. If we remove one set U_y from the collection $\{U_x\}_{x \in A}$ it will no longer be an open cover for A. So the only subcover of $\{U_x\}_{x \in A}$ is $\{U_x\}_{x \in A}$ itself.

Since A is compact we know that $\{U_x\}_{x \in A}$ admits a finite subcover. Therefore $\{U_x\}_{x \in A}$ must be a finite cover for A. Therefore A has finitely many elements.

Example 6.48. $\mathbb{R}$ with its standard topology is not a compact topological space.

To prove this we need to give an example of an open cover that does not admit a finite subcover. $\{(-n, n)\}_{n \in \mathbb{Z}^+}$ is such an example. It should be clear that this is an open cover. If $M \subset \mathbb{Z}^+$ is a finite subset then $\{(-n, n)\}_{n \in M}$ is not an open cover for $\mathbb{R}$: for example, the real number $x = \max M$ does not belong to $\bigcup_{n \in M}(-n, n)$.

Example 6.49. $\mathbb{R}^n$ with its standard topology is not a compact topological space, for any $n \geq 1$. Set $B^n_M(0) = \{|x| < M\}$. Then $\{B_M\}_{M \in \mathbb{Z}^+}$ is an open cover for $\mathbb{R}^m$ that does not admit a finite subcover.

Example 6.50. The subset $(0, 1) \subset \mathbb{R}$ is not compact. $\{(1/3n, 1 - 1/3n)\}_{n \in \mathbb{Z}^+}$ is an example of an open cover that does not admit a finite subcover.

Proposition 6.51. *Let (X, τ) be a topological space, let $A \subset X$ and let τ_A be the subspace topology on A. Then A is a compact subset of X if and only if (A, τ_A) is a compact topological space.*

Proof. "if" part: suppose that (A, τ_A) is a compact topological space. Let $\{V_\lambda\}_{\lambda \in \Lambda}$ be an open cover for A, where $V_\lambda \in \tau$. Set $U_\lambda = V_\lambda \cap A$; then $\{U_\lambda\}_{\lambda \in \Lambda}$ is a cover for A by open subsets of A. Since A is a compact topological space this admits a finite subcover $\{U_\lambda\}_{\lambda \in \Lambda'}$. Then $\{V_\lambda\}_{\lambda \in \Lambda'}$ is an open subcover of $\{V_\lambda\}_{\lambda \in \Lambda}$.

"only if" part: suppose that A is a compact subset of X. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover for A, where $U_\lambda \in \tau_A$. By definition there exist open sets $V_\lambda \in \tau$ such that $U_\lambda = A \cap V_\lambda$. Then $\{V_\lambda\}_{\lambda \in \Lambda}$ is a cover for A by open subsets of X. Since A is a compact subset of X this admits a finite subcover $\{V_\lambda\}_{\lambda \in \Lambda'}$. Then $\{U_\lambda\}_{\lambda \in \Lambda'}$ is an open subcover of $\{U_\lambda\}_{\lambda \in \Lambda}$. □

Are there any compact subsets of $\mathbb{R}$? The following theorem, (which is a special case of the Heine-Borel theorem) shows that there are:

Theorem 6.52. *Every interval $[a, b]$ such that $a < b$ is a compact subset of $\mathbb{R}$.*

Proof. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover for $[a, b]$. Clearly, for any $x \in [a, b]$, $\{U_\lambda\}_{\lambda \in \Lambda}$ is an open cover for $[a, x]$. The main idea of this proof is to consider the question: for which $x \in [a, b]$ does this open cover have a finite subcover? We will define

$$C = \big\{x \in [a, b] \ : \ \text{the open cover } \{U_\lambda\}_{\lambda \in \Lambda} \text{ of } [a, x] \text{ admits a finite subcover}\big\}.$$

We claim that there exists an $\varepsilon > 0$ such that

$$[a, a + \varepsilon) \subset C.$$

To prove this, let $\lambda \in \Lambda$ be such that $a \in U_\lambda$. Since U_λ is open there exists an ε such that $(a - \varepsilon, a + \varepsilon) \subset U_\lambda$. Suppose that $x \in [a, a + \varepsilon)$; then $\{U_\lambda\}$ is a finite subcover for $[a, x]$ and $x \in C$. Therefore $[a, a + \varepsilon) \subset C$.

Since the set C is non-empty, and we may define

$$s = \sup C.$$

Since $[a, a+\varepsilon) \subset C$, $s \in (a, b]$. We will show below that (i) $s \in C$ and (ii) $s = b$. It follows that the open cover $\{U_\lambda\}_{\lambda \in \Lambda}$ for $[a, b]$ has a finite subcover, and hence that $[a, b]$ is compact.

(i) Let us show that $s \in C$. Let $\mu \in \Lambda$ be such that $s \in U_\mu$. Since U_μ is open we may choose a $\delta > 0$ such that $(s-\delta, s+\delta) \subset U_\mu$. There must exist an $x \in (s-\delta, s]$ such that $x \in C$, for otherwise $s - \delta$ would be an upper bound for C that is strictly less than s. Since this x belongs to C there exists a finite subcover $\{U_\lambda\}_{\lambda \in \Lambda'}$ for $[a, x]$. Then $\{U_\lambda\}_{\lambda \in \Lambda' \cup \{\mu\}}$ is a finite subcover for $[a, s]$, so $s \in C$.

(ii) Now we show that $s = b$. Suppose for contradiction that $s < b$. Again, let $\mu \in \Lambda$ be such that $s \in U_\mu$ and let $\delta > 0$ be such that $(s-\delta, s+\delta) \subset U_\mu$. Now let y be such that $y \in (s, s+\delta)$ and $y < b$. We claim that $y \in C$, in contradiction with the fact that s is an upper bound for C. To prove this claim, let $\{U_\lambda\}_{\lambda \in \Lambda''}$ be a finite subcover for $[a, s]$; then $\{U_\lambda\}_{\lambda \in \Lambda'' \cup \{\mu\}}$ is a finite subcover for $[a, y]$ so $y \in C$. Since the assumption $s < b$ lead to a contradiction it must be that $s = b$. □

Now we explore some properties of compact topological spaces. The following is similar to Propositions 6.27 and 6.39 above

Proposition 6.53. *The continuous image of a compact set is compact.*

Proof. Let X, Y be topological spaces, let $f : X \to Y$ be a continuous function and let $A \subset X$ be a compact subset of X. We must show that $f(A)$ is a compact subset of Y.

Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover for $f(A)$. Then $\{f^{-1}(U_\lambda)\}_{\lambda \in \Lambda}$ is an open cover for A, because

$$\bigcup_{\lambda \in \Lambda} f^{-1}(U_\lambda) = f^{-1}\left(\bigcup_{\lambda \in \Lambda} U_\lambda\right) \supset f^{-1}(f(A)) \supset A.$$

Since A is compact this open cover admits a finite subcover $\{f^{-1}(U_\lambda)\}_{\lambda \in \Lambda'}$. Then $\{U_\lambda\}_{\lambda \in \Lambda'}$ is a finite subcover for $f(A)$, because

$$f(A) \subset f\left(\bigcup_{\lambda \in \Lambda'} f^{-1}(U_\lambda)\right) = f\left(f^{-1}\left(\bigcup_{\lambda \in \Lambda'} U_\lambda\right)\right) \subset \bigcup_{\lambda \in \Lambda'} U_\lambda.$$

□

Proposition 6.53 may be used to prove that spaces are compact:

Example 6.54. The circle $S^1 = \{x \in \mathbb{R}^2 \,:\, |x|^2 = 1\}$ is compact. To prove this, note that S^1 is equal to the image of the following continuous function f from the compact topological space $[0, 1]$ to $\mathbb{R}^2$:

$$f : t \mapsto (\cos 2\pi t, \sin 2\pi t).$$

Corollary 6.55. *Compactness is a topological invariant.*

Proof. Exercise. □

Next we investigate compactness of the product topology.

Theorem 6.56 (Tychonoff)**.** *Let X and Y be two topological spaces. Then $X \times Y$ is compact if and only if X and Y are compact.*

Proof. For the "only if" part, suppose that $X \times Y$ is compact. Recall from Proposition 5.33 that the projections $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are continuous. Then X and Y are compact by Proposition 6.53 because they are equal to the images of the projection maps.

For the "if" part, suppose that X and Y are compact and let $\{W_\lambda\}_{\lambda\in\Lambda}$ be an open cover for $X\times Y$. For each $(x,y) \in X\times Y$ there exists a $\lambda_{(x,y)} \in \Lambda$ such that $(x,y) \in W_{\lambda_{(x,y)}}$. By definition there exist open sets $U_{(x,y)} \subset X$ and $V_{(x,y)} \subset Y$ such that $(x,y) \in U_{(x,y)} \times V_{(x,y)} \subset W_{\lambda_{(x,y)}}$.

Consider the open cover

$$\{U_{(x,y)} \times V_{(x,y)}\}_{(x,y)\in X\times Y}.$$

We will show that this open cover has a finite subcover $\{U_{(x,y)} \times V_{(x,y)}\}_{(x,y)\in\Omega}$, where $\Omega \subset X\times Y$ is a finite subset. It will follow that the original open cover $\{W_\lambda\}_{\lambda\in\Lambda}$ has a finite subcover $\{W_{\lambda_{(x,y)}}\}_{(x,y)\in\Omega}$.

For each $b \in Y$ the family $\{U_{(x,b)}\}_{x\in X}$ is an open cover for X. Therefore it admits a finite subcover $\{U_{(x,b)}\}_{x\in\Xi_b}$, where Ξ_b is a finite subset of X. We will define

$$V_b = \bigcap_{x\in\Xi_b} V_{(x,b)}.$$

Now $\{V_y\}_{y\in Y}$ is an open cover for Y. Therefore it admits a finite subcover $\{V_y\}_{y\in\Upsilon}$, where Υ is a finite subset of Y.

We claim that

$$\{U_{(x,y)} \times V_{(x,y)} \;:\; y \in \Upsilon \text{ and } x \in \Xi_y\}.$$

is a finite open cover of $X \times Y$. To prove this, note first that

$$X \times V_y \subset \bigcup_{x\in\Xi_y} U_{(x,y)} \times V_y \subset \bigcup_{x\in\Xi_y} U_{(x,y)} \times V_{(x,y)}.$$

It follows that

$$X \times Y \subset \bigcup_{y\in\Upsilon} X \times V_y \subset \bigcup_{y\in\Upsilon}\bigcup_{x\in\Xi_y} U_{(x,y)} \times V_{(x,y)}.$$

□

Remark 6.57. It follows by induction that any finite product $X_1 \times X_2 \times \ldots \times X_n$ of compact topological spaces is compact.

Example 6.58. The rectangle $[a,b] \times [c,d]$ is compact for any $a < b$ and $c < d$. Equally the n-dimensional cubes

$$[a_1,b_1] \times [a_2,b_2] \times \cdots \times [a_n,b_n] \subset \mathbb{R}^n$$

are compact.

6.4.1 Closed sets and compact sets

There are close relations between closed subsets of topological spaces and compact subsets of topological spaces. We will investigate some of these below.

Proposition 6.59. *A closed subset of a compact space is compact.*

Proof. Let X be a compact topological space and let C be a closed subset of X. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover for C. We must find a finite subcover.

The collection $\{U_\lambda \;:\; \lambda \in \Lambda\} \cup \{X \setminus C\}$ is an open cover for X. Therefore it admits a finite subcover of the form $\{U_\lambda \;:\; \lambda \in \Lambda'\} \cup \{X \setminus C\}$, where Λ' is a finite subset of Λ. Then $\{U_\lambda \;:\; \lambda \in \Lambda'\}$ is a finite open cover for C, because $C \cap (X \setminus C) = \emptyset$. □

Example 6.60. $S^n = \{x \in \mathbb{R}^{n+1} : |x|^2 = 1\}$ is a closed subset of the compact space $[-1,1] \times \cdots \times [-1,1]$, since $S^n = f^{-1}(\{1\})$, where

$$f(x) = |x|^2$$

is continuous. Therefore S^n is compact.

Remark 6.61. Both hypotheses in this theorem are necessary:

- $(0,1)$ is a subset of the compact space $[0,1]$ but is not compact.
- $[0,\infty)$ is a closed subset of $\mathbb{R}$ but is not compact.

Note that in a compact space, compact does not imply closed:

Counterexample 6.62. Let $X = \{1,2\}$ and let $\tau = \{\emptyset, \{1\}, \{1,2\}\}$. Then X is compact because it contains finitely many elements. The set $\{1\}$ is compact (because it is finite), but is not closed (because $\{2\} \notin \tau$).

However, compact *does* imply closed in a separated space:

Proposition 6.63. *A compact subset of a separated space is closed.*

Proof. Let X be a separated space and let A be a compact subset of X. We must show that $X \setminus A$ is open.

Let $x \in X \setminus A$. For any $a \in A$ there exist disjoint open sets U_a, V_a such that $a \in U_a$ and $x \in V_a$. The family $\{U_a\}_{a\in A}$ forms an open cover for A. Since A is compact, it admits a finite subcover $\{U_{a_1}, U_{a_2}, \ldots, U_{a_n}\}$ (where $a_1, \ldots, a_n$ are finitely many points in A). Define

$$V_x = \bigcap_{i=1}^{n} V_{a_i}.$$

This set is open, because it is a finite intersection of open sets. Moreover, $x \in V_x$ and $V_x \cap A = \emptyset$ (the latter because V_x is disjoint from all the sets U_{a_i} in the finite open cover of A).

We claim that

$$X \setminus A = \bigcup_{x\in X\setminus A} V_x.$$

$X \setminus A \subset \bigcup_{x\in X\setminus A} V_x$ because every $x \in X \setminus A$ belongs to at least one set V_x in the union, and $X \setminus A \supset \bigcup_{x\in X\setminus A} V_x$ because $V_x \cap A = \emptyset$ for all x. So $X \setminus A$ is a union of open sets, hence open. □

Corollary 6.64. *In a separated space, arbitrary intersections of compact subsets are compact.*

Proof. Let X be a separated space and let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of compact subsets of X. The sets A_λ are closed, by Proposition 6.63. The intersection $\bigcap_{\lambda\in\Lambda} A_\lambda$ is closed, by Theorem 5.12. Pick some $\mu \in \Lambda$; then $\bigcap_{\lambda\in\Lambda} A_\lambda$ is a closed subset of the compact space A_μ, hence compact by Proposition 6.59. □

Without the Hausdorff condition, it is not even true that finite intersections of compact subsets are compact. Here's a counterexample that shows why:

Counterexample 6.65. Let X be any infinite set and let a and b be any two distinct points in X. Let $\tau = \{U \subset X \setminus \{a,b\}\} \cup \{X\}$. Then τ defines a topology on X (check this!).

Let $A = X \setminus \{a\}$ and $B = X \setminus \{b\}$. Then A is compact: any open cover for A must contain the set X because X is the only open set containing b; therefore any open cover for A has the finite subcover $\{X\}$. Similar reasoning shows that B is compact.

However, $A \cap B$ is not a compact topological space (and hence not a compact subset of X). This is because the subspace topology on $A \cap B$ is equal to the discrete topology on $A \cap B$. We showed earlier that the only compact subsets in the discrete topology are the finite sets, so $A \cap B$, being an infinite set, is not compact.

Corollary 6.66. *A continuous bijection from a compact topological space to a separated space is a homeomorphism.*

Proof. Let X be a compact space, let Y be a Hausdorff space and let $f : X \to Y$ be a continuous bijection. We must show that f^{-1} is continuous.

f^{-1} is continuous if and only if for every closed subset $C \subset X$, $(f^{-1})^{-1}(C) = f(C)$ is closed. Let $C \subset X$ be any closed subset. Then C is compact by Theorem 6.59, $f(C)$ is compact by Proposition 6.53, and $f(C)$ is closed by Proposition 6.63. $\square$

The above result is **very** useful.

Example 6.67. Going back to Example 5.49, setting $F : [0,1] \to S^1$ to be

$$F(x) = (\cos(2\pi x), \sin(2\pi x))$$

then by Proposition 5.59 F induces a continuous bijection $\tilde{F} : [0,1]/_\sim \to S^1$ which must be a homeomorphism since the domain is compact, and the target is Hausdorff. In particular $[0,1]/_\sim \cong S^1$.

Corollary 6.68. *The only separated topology on a finite set is the discrete topology.*

Proof. Let X be a finite set and let τ be a separated topology on X. Let τ_0 denote the discrete topology on X. Let $f : (X, \tau_0) \to (X, \tau)$ be the function $f : x \mapsto x$. Then f is a bijection and is continuous (because any function from a set with the discrete topology is continuous). The topological space (X, τ_0) is compact because X is finite and the topological space (X, τ) is separated by assumption, so by corollary 6.66 f is a homeomorphism. It follows that $\tau = \tau_0$. $\square$

One may wonder how these results come in handy in practice. Corollary 6.66 is very powerful as it enables us to show homeomorphisms without having to explicitly construct them or their inverse.

6.4.2 Compactness in metric spaces

The first result shows that continuous functions defined on a compact metric space satisfy a stronger notion of continuity:

Lemma 6.69. *Let (X, d_1), (Y, d_2) be metric spaces with X compact. Let $f : X \to Y$ be continuous, then f is uniformly continuous. i.e. for all $\varepsilon > 0$ there exists $\delta > 0$ s.t. for any $x \in X$, $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$.*

Remark 6.70. In the simple case that $f : [a,b] \to \mathbb{R}$ is continuous, the Lemma above shows that f is uniformly continuous. In the usual language: for all $\varepsilon > 0$ there exists $\delta > 0$ so that for any $x, y \in [a,b]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

The point here is that $\delta > 0$ is uniform in x. For instance the function $f : (0,1) \to \mathbb{R}$, $x \mapsto 1/x$ is *not* uniformly continuous (of course $(0,1)$ is not compact).

Proof. Let $\varepsilon > 0$. f is continuous so for all $x \in X$ there exist $\delta_x > 0$ (that depends on x) s.t. $B_{2\delta_x}(x) \subset f^{-1}(B_{\varepsilon/2}(f(x)))$. The family $\{B_{\delta_x}(x)\}_{x \in X}$ covers X and thus admits a finite subcover, given by $x_1, \ldots, x_n$ say (with their associated $\delta_1 \ldots \delta_n$). Set $\delta = \min_i \delta_i > 0$.

Now pick $x \in X$ and notice that $x \in B_{\delta_i}(x_i)$ for some i.

For arbitrary $y \in B_\delta(x)$ then

$$d_1(y, x_i) \leq d_1(y, x) + d_1(x, x_i) < \delta + \delta_i \leq 2\delta_i.$$

This implies that $y \in B_{2\delta_i}(x_i)$ so that $f(y) \in B_{\varepsilon/2}(f(x_i))$. But since $x \in B_{\delta_i}(x_i)$ we also have $f(x) \in B_{\varepsilon/2}(f(x_i))$ giving

$$d_2(f(x), f(y)) \leq d_2(f(x), f(x_i)) + d_2(f(x_i), f(y)) < \varepsilon.$$

In particular $f(y) \in B_\varepsilon(f(x))$ and we are done: for all $\varepsilon > 0$ there exists $\delta > 0$ s.t. for any $x \in X$, $B_\delta(x) \subset f^{-1}(B_\varepsilon(f(x)))$. □

Definition 6.71. Let (X, d) be a metric space and let A be a subset of X. Then A is called **bounded** if there exists an $x \in X$ and $M > 0$ such that $A \subset B_M(x)$.

Proposition 6.72. *Every nonempty compact subset of a metric space is closed and bounded.*

Proof. Let (X, d) be a metric space and let A be a compact subset of X. The topology on X is separated by Theorem 5.16, so A is closed by Proposition 6.63. It remains to show that A is bounded.

Let x be any point in A. We note that $A \subset \bigcup_{n\in\mathbb{Z}^+} B_n(x)$, because if $y \in A$ then $y \in B_n(x)$ for any n large enough that $n > d(x, y)$. So $\{B_n(x)\}_{n\in\mathbb{Z}^+}$ form an open cover for A, and since A is compact there exists a finite subset $\Gamma \subset \mathbb{Z}^+$ such that $\{B_n(x)\}_{n\in\Gamma}$ is a finite subcover. It follows that $A \subset B_M(x)$, where M is equal to the largest element in Γ. □

In general it is not true that closed and bounded subsets of metric spaces are compact, however, there is one important example of a metric space whose closed and bounded subsets are compact.

Theorem 6.73 (Heine-Borel)**.** *A nonempty subset of $\mathbb{R}^n$ (with the standard topology) is compact if and only if it is closed and bounded.*

Proof. Proposition 6.63 shows that if $A \subset \mathbb{R}^n$ is compact it must be closed and bounded. Suppose then that A is a closed and bounded subset of $\mathbb{R}^n$; we must show that A is compact. Since A is bounded there exists an $a \in A$ and $M > 0$ such that $A \subset B_M(a)$. But

$$B_M(a) \subset [a_1 - M, a_1 + M] \times [a_2 - M, a_2 + M] \times \ldots \times [a_n - M, a_n + M].$$

Each set $[a_i - M, a_i + M]$ is compact by Theorem 6.52, so the product is compact by Tychonoff's Theorem 6.56. So A is a closed subset of a compact set, and therefore compact by Proposition 6.59. □

Warning 6.74. A closed and bounded set is **not** compact in a general metric space. For example a set X with the discrete topology is the topological space induced by the discrete metric

$$d(x, y) = \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

X is bounded in this metric since $X = B_2(x)$ for any $x \in X$. However, X is compact if and only if X is finite, hence for infinite discrete spaces, we have a counter example.

A better example is the space of bounded infinite sequences $l^\infty = \{\{x_n\}_{n\in\mathbb{Z}^+} \subset \mathbb{R} : \sup_n |x_n| < \infty\}$ with the metric

$$d(\{x_n\}, \{y_n\}) = \sup_n |x_n - y_n|.$$

You can check that this is a metric if you like (this space is an example of a Banach space and is infinite-dimensional).

Consider the set

$$D_1(0) = \{\{x_n\} : d(\{x_n\}, \{0\}) \leq 1\}$$

which turns out to be closed and bounded. However, this is by no means compact: let $\{x_n\}_{\lambda_\pm} = \{0, \ldots, 0, \pm 1, 0 \ldots 0, \ldots\}$ be the sequence with ± 1 in the $\lambda_\pm^{th}$ position. Then

$$\{B_1(\{x_n\}_{\lambda_\pm}), B_1(\{0\})\}_{\lambda_\pm \in \mathbb{Z}^+}$$

is an open cover of D_1 with no finite subcover.

You may recall that connectedness provided a means to generalise a fundamental theorem from real analysis, the intermediate value theorem. The notion of compactness allows us to generalise another important theorem of real analysis: the maximum value theorem.

Corollary 6.75 (Maximum value theorem)**.** *Let X be a compact topological space and let $f : X \to \mathbb{R}$ be a continuous function. Then there exist points $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.*

In the particular case $X = [a, b]$ for some $a < b$ this is just the usual maximum value theorem.

Proof. Let $A = f(X)$. By Proposition 6.53, A is a compact subset of $\mathbb{R}$. Thus A is closed and bounded by Heine-Borel. Let $m = \inf A$ and $M = \sup A$ so that $A \subset [m, M]$. By Lemma 5.44 $m, M \in A$, thus there exist $c, d \in X$ s.t. $f(c) = m$, $f(d) = M$. □

6.4.3 Lebesgue numbers and sequential compactness

Again we will focus on metric spaces (X, d). Our goal is to prove that **sequential compactness** is equivalent to compactness in this setting:

Definition 6.76. A subspace $C \subset X$ of a metric space (X, d) is **sequentially compact** if for any sequence $\{x_n\} \subset C$, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x \in C$.

You should be able to see using Theorem 5.42 and Lemma 5.44 that this implies C is closed.

Theorem 6.77. *A subspace $C \subset X$ of a metric space (X, d) is compact if and only if it is sequentially compact.*

We will start with the first half of the above theorem:

Proposition 6.78. *If a subspace $C \subset X$ of a metric space (X, d) is compact then it is sequentially compact.*

Proof. First of all let's re-phrase what it means to find a convergent subsequence. There is a convergent subsequence $\{x_{n_k}\}$ if and only if: there exists $x_0 \in C$ so that for all $\varepsilon > 0$, there is a $K \in \mathbb{Z}^+$ so that $k \geq K$ implies $x_{n_k} \in B_\varepsilon(x_0)$. This is true if and only if: there exists $x_0 \in C$ so that for all $\varepsilon > 0$, $\sup\{n : x_n \in B_\varepsilon(x_0)\} = \infty$.

By this reasoning, if there is no convergent subsequence: for any $y \in C$ there exists $\varepsilon(y) > 0$ such that $\sup\{n : x_n \in B_\varepsilon(y)\} < \infty$.

The family $\{B_{\varepsilon(y)}(y)\}_{y \in C}$ is surely an open cover of C to which there exists a finite subcover $\{B_{\varepsilon(y_i)}(y_i)\}_{i=1}^L$. But notice that for each i, there are only finitely many elements of the sequence in each $B_{\varepsilon(y_i)}(y_i)$, and thus the sequence only has finitely many elements in C. This contradiction finishes the proof. □

Definition 6.79. Let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open cover of a subset $A \subset X$ of a metric space (X, d). We call $\varepsilon > 0$ a **Lebesgue number** of the open cover if: for all $x \in A$ there exits $\lambda' \in \Lambda$ so that $B_\varepsilon(x) \subset U_{\lambda'}$.

Proposition 6.80. *Suppose that $C \subset X$ is a sequentially compact subset of a metric space (X, d). Then for any open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of C, there exits a Lebesgue number $\varepsilon > 0$ for $\{U_\lambda\}_{\lambda\in\Lambda}$.*

Proof. Suppose the statement is false. In other words, there exits an open cover $\{U_\lambda\}_{\lambda\in\Lambda}$ of C, such that that for all $\varepsilon > 0$, there exits $x \in C$ with $B_\varepsilon(x) \not\subset U_\lambda$ for all λ.

For this open cover, let $\varepsilon(n) = 1/n$ for $n \in \mathbb{Z}^+$ and pick x_n so that

$$B_{\varepsilon(n)}(x_n) = B_{1/n}(x_n) \not\subset U_\lambda \tag{6.1}$$

for all λ. Since C is sequentially compact, this sequence $\{x_n\}$ admits a convergent subsequence $\{x_{n_k}\}$ converging to some $x \in C$. Pick λ' so that $x \in U_{\lambda'}$ and notice that for all $\eta > 0$ there exists K so that $k \geq K$ implies $x_{n_k} \in B_\eta(x)$.

Since η is arbitrary we pick it so that $B_{2\eta}(x) \subset U_{\lambda'}$ (which is possible since $U_{\lambda'}$ is open). Now we can pick $K' \geq K$ so that $1/n_k < \eta$ whenever $k \geq K'$.

We have, for $k \geq K'$, $B_{1/n_k}(x_{n_k}) \subset B_{2\eta}(x) \subset U_{\lambda'}$ which contradicts (6.1). □

We now finish the proof of Theorem 6.77 by proving the converse to Proposition 6.78.

Proof. Suppose that C is sequentially compact and let $\{U_\lambda\}_{\lambda\in\Lambda}$ be an arbitrary open cover. By Proposition 6.80 there exits a Lebesgue number $\varepsilon > 0$ for this open cover.

We now **claim** that there exits a finite number of points $x_1, \ldots, x_n$ so that $C \subset \cup_{i=1}^n B_\varepsilon(x_i)$ and thus, for all i there exits U_{λ_i} such that $B_\varepsilon(x_i) \subset U_{\lambda_i}$. In particular $C \subset \cup_{i=1}^n U_{\lambda_i}$ and we have found our finite subcover.

To prove the claim, suppose that it is false. Now pick x_1 arbitrary: we must have $C \not\subset B_\varepsilon(x_1)$. Thus there is $x_2 \in C \backslash B_\varepsilon(x_1)$ and again we must have $C \not\subset B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$. Since the claim is false, for all $n \in \mathbb{Z}^+$, we can inductively find $x_n \in C \backslash \cup_{i=1}^{n-1} B_\varepsilon(x_i)$ such that $C \not\subset \cup_{i=1}^n B_\varepsilon(x_i)$.

This sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ with $x_{n_k} \to x \in C$, by assumption. Since we have convergence of this subsequence, for any $\eta > 0$ there exits $K \in \mathbb{Z}^+$ so that $k \geq K$ implies $x_{n_k} \in B_\eta(x)$. So pick $\eta = \varepsilon/2$ and notice that $x_{n_k} \in B_{\varepsilon/2}(x) \subset B_\varepsilon(x_{n_k})$ for all $k \geq K$. But this contradicts that $x_{n_{k+1}} \notin B_\varepsilon(x_{n_k})$. □

Chapter 7

Homotopy theory

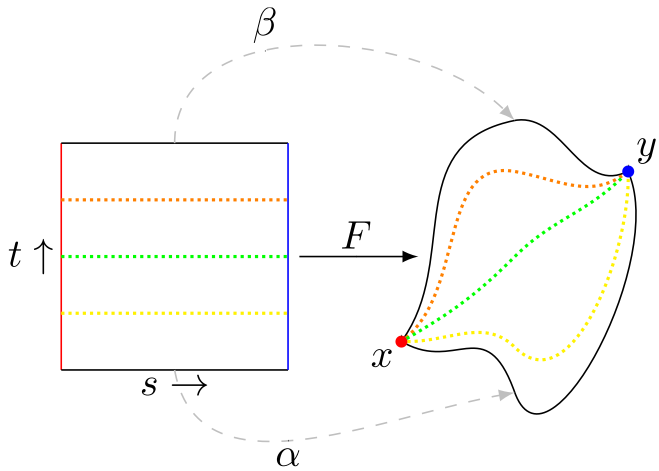
7.1 Homotopies

This section makes frequent use of the concept of a *path*. Remember that a path is a continuous function α from the interval $[0,1]$ to a topological space X.

Definition 7.1. Let X be a topological space, let $x, y \in X$ and let α, β be two paths from x to y. Then α is said to be **path homotopic** to β (written $\alpha \simeq \beta$) if there exists a continuous function $F : [0,1] \times [0,1] \to X$ such that

$$\begin{aligned} F(s,0) &= \alpha(s) \quad \forall s \in [0,1] \\ F(s,1) &= \beta(s) \quad \forall s \in [0,1] \\ F(0,t) &= x \quad \forall t \in [0,1] \\ F(1,t) &= y \quad \forall t \in [0,1]. \end{aligned}$$

Such a function F is called a **path homotopy** from α to β.



Think of F as being a path of paths from α to β: for **fixed** t, the path $f_t(s) = F(s,t)$ is another path from x to y. If we let t vary from 0 to 1 the resulting paths f_t will go from $\alpha = f_0$ to $\beta = f_1$. The above diagram is showing $f_{1/4}$ (yellow), $f_{1/2}$ (green) and $f_{3/4}$ (orange).

Of course, in general there is no reason that the intermediate paths f_t will not intersect one another - I've drawn an example of quite a nice homotopy above.

It's also possible that F doesn't change the "image" of α as t moves, but just the parametrisation. For example, letting

$$F(s,t) = \begin{cases} \alpha\left(\frac{2s}{2-t}\right) & \text{if } s \leq 1 - t/2 \\ \alpha(1) & \text{if } s \geq 1 - t/2 \end{cases}$$

gives a homotopy from $\alpha(s) = F(s, 0)$ to

$$\beta(s) = \begin{cases} \alpha(2s) & \text{if } s \leq 1/2 \\ \\ \alpha(1) & \text{if } s \geq 1/2 \end{cases}$$

which is just a re-parametrisation of α (β runs along α twice as fast and then waits at the end point!).

Example 7.2. Let $x, y \in \mathbb{R}^n$ and let $\alpha, \beta : [0,1] \to \mathbb{R}^n$ be any two paths from x to y. Then the following function is a homotopy from α to β:

$$F(s,t) = (1-t)\alpha(s) + t\beta(s).$$

Clearly $F(s, 0) = \alpha(s)$ and $F(s, 1) = \beta(s)$. The functions $(s,t) \mapsto t$, $(s,t) \mapsto \alpha(s)$ and $(s,t) \mapsto \beta(s)$ are continuous by Proposition 5.33, and so F is continuous by the calculus of limits. This homotopy is called the **straight line homotopy**.

Example 7.3. The following paths in $\mathbb{R}^2 \setminus \{(0,0)\}$ are *not* path homotopic:

$$\begin{aligned} \alpha(s) &= (\cos(\pi s), \sin(\pi s)) \\ \beta(s) &= (\cos(\pi s), -\sin(\pi s)) \end{aligned}$$

We will next show that homotopy is an equivalence relation; to do this we need to recall the glue lemma (Lemma 5.34).

Lemma 7.4. *The relation of path homotopy is an equivalence relation.*

Proof. We need to check that the relation is reflexive, symmetric, and transitive. Let X be a topological space and let α, β, γ be paths in X from x to y.

Reflexive: we must show $\alpha \simeq \alpha$. $F(s,t) = \alpha(s)$ is a homotopy from α to α so $\alpha \simeq \alpha$.

Symmetric: we must show $\alpha \simeq \beta \Rightarrow \beta \simeq \alpha$. If $\alpha \simeq \beta$ there is a path homotopy from α to β. Then $G : [0,1] \times [0,1] \to X$, $G(s,t) = F(s, 1-t)$ is a path homotopy from β to α, so $\beta \simeq \alpha$.

Transitive: suppose $\alpha \simeq \beta$ and $\beta \simeq \gamma$. Then there are path homotopies F from α to β and G from β to γ. We claim the following function is a path homotopy from α to γ.

$$H(s,t) = \begin{cases} F(s, 2t) & t \in [0, \frac{1}{2}] \\ G(s, 2t-1) & t \in [\frac{1}{2}, 1]. \end{cases}$$

The restrictions of H to the closed subsets $[0,1] \times [0, \frac{1}{2}]$ and $[0,1] \times [\frac{1}{2}, 1]$ of $[0,1] \times [0,1]$ are both continuous. On the overlap region $[0,1] \times \{\frac{1}{2}\}$ we have that $F(s, 2t) = \beta(s) = G(s, 2t - 1)$. Therefore H is continuous, by the glue lemma. Clearly $H(s, 0) = \alpha(s)$, $H(s, 1) = \gamma(s)$, $H(0,t) = x$ and $H(1,t) = y$ so H is a path homotopy from α to γ and $\alpha \simeq \gamma$. $\square$

Definition 7.5. Let X be a topological space, let x, y, z be three points in X, let α be a path from x to y and let β be a path from y to z. The **join** of α and β is the path $\alpha * \beta : [0,1] \to X$ from x to z defined by

$$\alpha * \beta(s) = \begin{cases} \alpha(2s) & s \in [0, \frac{1}{2}] \\ \beta(2s-1) & s \in [\frac{1}{2}, 1]. \end{cases}$$

The **reverse** of α is the path $\overline{\alpha} : [0,1] \to X$ from y to x defined by

$$\overline{\alpha}(s) = \alpha(1-s).$$

For any point $x \in X$, the **constant path at** x is the path $e_x : [0,1] \to X$ from x to x defined by

$$e_x(s) = x \quad \forall s \in [0,1].$$

I leave it as an exercise to check that $\alpha * \beta$, $\overline{\alpha}$ and e_x are paths (you will need the glue lemma for $\alpha * \beta$).

Theorem 7.6. *The operations $*$ and $\bar{}$ are well-defined on path-homotopy equivalence classes, moreover they have the following properties:*

1. *Associativity: $[\alpha] * ([\beta] * [\gamma]) = ([\alpha] * [\beta]) * [\gamma]$ for all paths α, β, γ such that $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$.*
2. *Identity: if α is a path from x to y then $[e_x] * [\alpha] = [\alpha]$ and $[\alpha] * [e_y] = [\alpha]$.*
3. *Inverse: if α is a path from x to y then $[\alpha] * [\overline{\alpha}] = [e_x]$ and $[\overline{\alpha}] * [\alpha] = [e_y]$.*

Proof. First we show that the operations are well-defined. Suppose that $\alpha, \alpha', \beta, \beta'$ are paths such that $\alpha(1) = \beta(0)$, $\alpha \simeq \alpha'$ and $\beta \simeq \beta'$. We need to check that $[\alpha * \beta] = [\alpha' * \beta']$, i.e. that $\alpha * \beta \simeq \alpha' * \beta'$. Let F be a path homotopy from α to α' and let G be a path homotopy from β to β'. We claim that the following function H is a path homotopy from $\alpha * \beta$ to $\alpha' * \beta'$:

$$H(s,t) = \begin{cases} F(2s,t) & (s,t) \in [0, \frac{1}{2}] \times [0,1] \\ G(2s-1,t) & (s,t) \in [\frac{1}{2}, 1] \times [0,1]. \end{cases}$$

This function is continuous by the glue lemma, because the sets $[0, \frac{1}{2}] \times [0,1]$ and $[\frac{1}{2}, 1] \times [0,1]$ are closed, and $F(2s,t) = G(2s-1,t)$ on their intersection. It is straightforward to check that $H(0,t) = \alpha(0)$ and $H(1,t) = \beta(1)$ $\forall t \in [0,1]$, and that $H(s,0) = \alpha*\beta(s)$ and $H(s,1) = \alpha'*\beta'(s)$, $\forall s \in [0,1]$.

We also need to show that $\overline{\alpha} \simeq \overline{\alpha}'$. It is straightforward to check that the function $(s,t) \mapsto F(1-s,t)$ is a path homotopy from $\overline{\alpha}$ to $\overline{\alpha}'$.

Next we prove that constant paths are identity elements. Let α be a path from x to y. We need to show that

$$e_x * \alpha(s) = \begin{cases} x & s \in [0, \frac{1}{2}] \\ \alpha(2s-1) & x \in [\frac{1}{2}, 1] \end{cases}$$

is path homotopic to α. We make an ansatz for a path homotopy of the form:

$$J(s,t) = \begin{cases} x & s \in [0, \frac{t}{2}] \\ \alpha\left(As + B\right) & s \in [\frac{t}{2}, 1] \end{cases}.$$

Notice that the ranges of the intervals depend on t! In order to satisfy the condition $J(1,t) = y$ we require that $\alpha(A+B) = y = \alpha(1)$ and hence that

$$A + B = 1.$$

In order to make this function J well-defined and continuous we require that $\alpha(At/2 + B) = x$ and hence that

$$A\frac{t}{2} + B = 0.$$

These simultaneous equations are solved by $A = 2/(2-t)$, $B = -t/(2-t)$. We leave it as an exercise to check using the glue lemma J is continuous and moreover that J is a path homotopy from $e_x * \alpha$ to α. We also leave it as an exercise to find a path homotopy form $\alpha * e_y$ to α.

Next we prove that reverses are inverses. Let α be a path from x to y. We need to show that

$$\overline{\alpha} * \alpha(s) = \begin{cases} \alpha(1-2s) & s \in [0, \frac{1}{2}] \\ \alpha(2s-1) & s \in [\frac{1}{2}, 1] \end{cases}$$

is homotopic to the constant path e_y. We make an ansatz for a homotopy of the form

$$K(s,t) = \begin{cases} \alpha(As+B) & s \in [0, \frac{1}{2}] \\ \alpha(Cs+D) & s \in [\frac{1}{2}, 1] \end{cases}.$$

In order to satisfy the conditions $K(0,t) = y = K(1,t)$ we impose that

$$B = 1, \quad C + D = 1.$$

In order to make this a homotopy from $\overline{\alpha} * \alpha$ to e_y we also require that $K(1/2, t) = \alpha(t)$ (so that $K(1/2, 0) = \overline{\alpha} * \alpha(1/2)$, $K(1/2, 1) = e_y(1/2)$). This condition is met if

$$A/2 + B = t, \quad C/2 + D = t.$$

The above simultaneous equations are solved by

$$A = 2t - 2,\ B = 1,\ C = 2 - 2t,\ D = 2t - 1.$$

It is left as an exercise to check using the glue lemma that K is continuous and show that it satisfies the definition of a path homotopy. It is also left as an exercise to check that $\alpha * \overline{\alpha} \simeq e_x$.

Finally we prove associativity. We need to find a path homotopy from

$$\alpha * (\beta * \gamma)(s) = \begin{cases} \alpha(2s) & s \in [0, \frac{1}{2}] \\ \beta(4s-2) & s \in [\frac{1}{2}, \frac{3}{4}] \\ \gamma(4s-3) & s \in [\frac{3}{4}, 1] \end{cases}$$

to

$$(\alpha * \beta) * \gamma(s) = \begin{cases} \alpha(4s) & s \in [0, \frac{1}{4}] \\ \beta(4s-3) & s \in [\frac{1}{4}, \frac{1}{2}] \\ \gamma(2s-1) & s \in [\frac{1}{2}, 1]. \end{cases}$$

We seek a path homotopy of the form,

$$I(s,t) = \begin{cases} \alpha(As+B) & 0 \leq s \leq \frac{2-t}{4} \\ \beta(Cs+D) & \frac{2-t}{4} \leq s \leq \frac{3-t}{4} \\ \gamma(Es+F) & \frac{3-t}{4} \leq s \leq 1. \end{cases}$$

Notice how the endpoints of the intervals depend on t, such that when $t = 0, 1$ they agree with those for $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$. We choose the constants A, B, C, D, E, F so that I satisfies the definition of a path homotopy. The conditions $I(0,t) = \alpha(0)$, $I(1,t) = \gamma(1)$ are satisfied if

$$B = 0, \quad E + F = 1.$$

Looking at the endpoints of the intervals, the function I is well-defined and continuous if

$$A\frac{2-t}{4} + B = 1, \quad C\frac{2-t}{4} + D = 0, \quad C\frac{3-t}{4} + D = 1, \quad E\frac{3-t}{4} + F = 0.$$

These simultaneous equations are solved by

$$A = \frac{4}{2-t},\ B = 0,\ C = 4,\ D = t - 2,\ E = \frac{4}{t+1},\ F = \frac{t-3}{t+1}.$$

It is left as an exercise to check using the glue lemma that I is continuous and show that it satisfies the definition of a path homotopy. $\square$

7.2 The fundamental group

The fundamental group of a topological space is a group defined using equivalence classes of paths in the space. Before saying precisely how, let's remind ourselves what a group is:

Definition 7.7. A **group** $(G, \cdot)$ consists of a set G and a map $\cdot : G \times G \to G$, $(g,h) \mapsto g \cdot h$ satisfying the following conditions:

(G1) (Associativity) $(g \cdot h) \cdot k = g \cdot (h \cdot k)$ for all $g, h, k \in G$.

(G2) (Identity) There exists an element $e \in G$ such that $g \cdot e = g = e \cdot g$ for all $g \in G$.

(G3) (Inverses) For every $g \in G$ there exists an element $g^{-1} \in G$ such that $g^{-1} \cdot g = e = g \cdot g^{-1}$.

If in addition $g \cdot h = h \cdot g$ for every $g, h \in G$ the group is called **abelian** (or *commutative*).

Definition 7.8. Let X be a topological space and let $x \in X$. A path from x to x is called a **loop based at** x. The set of path homotopy equivalence classes of loops based at x together the operation $*$ is called the **fundamental group of X relative to the base point** x, and is denoted $\pi_1(X, x)$.

Remark 7.9. The fundamental group is a group! This follows from the Theorem 7.6 proved above.

Remark 7.10. The fundamental group is also known as the **first homotopy group**. There is also an n-th homotopy group $\pi_n(X, x)$ defined using maps from the n-dimensional disc to X. Fully understanding these groups remains an important open problem.

Example 7.11. For any $x \in \mathbb{R}^n$, $\pi_1(\mathbb{R}^n, x)$ is trivial (i.e. it has only one element). This is because every loop based at $x \in \mathbb{R}^n$ is path homotopic to the constant path at x (via the straight line homotopy described above).

The fundamental group $\pi_1(X, x)$ involves a choice of base point. The next definition and theorem explains precisely how the group depends on the base point:

Definition 7.12. Let G, H be two groups and let $\phi : G \to H$. Then ϕ is called a **homomorphism** if

$$\phi(g_1 \cdot g_2) = \phi(g_1) \cdot \phi(g_2) \quad \forall g_1, g_2 \in G.$$

ϕ is called an **isomorphism** if in addition it is a bijection; in this case G and H are said to be **isomorphic** and we write $G \cong H$.

Groups which are isomorphic are essentially the same, up to a relabelling of their elements!

Theorem 7.13. *Let X be a topological space, let x, y be points in X, and let α be a path from x to y. Then the map*

$$\begin{array}{rccc} \hat{\alpha} : & \pi_1(X, x) & \to & \pi_1(X, y) \\ & [\beta] & \mapsto & [\overline{\alpha}] * [\beta] * [\alpha] \end{array}$$

is an isomorphism.

Proof. First we show that $\hat{\alpha}$ is a bijection. We do so by showing that $\hat{\alpha}$ has an inverse. The inverse is the map

$$\begin{array}{rccc} \hat{\overline{\alpha}} : & \pi_1(X, y) & \to & \pi_1(X, x) \\ & [\beta] & \mapsto & [\alpha] * [\beta] * [\overline{\alpha}]. \end{array}$$

That this is an inverse follows directly from the properties proved in Theorem 7.6, indeed, for any loop β based at x,

$$\begin{aligned}\hat{\overline{\alpha}} \circ \hat{\alpha}([\beta]) &= [\alpha] * ([\overline{\alpha}] * [\beta] * [\alpha]) * [\overline{\alpha}] \\ &= ([\alpha] * [\overline{\alpha}]) * [\beta] * ([\alpha] * [\overline{\alpha}]) \\ &= [e_x] * [\beta] * [e_x] \\ &= [\beta].\end{aligned}$$

So $\hat{\overline{\alpha}} \circ \hat{\alpha} = \mathrm{id}_{\pi_1(X,x)}$. By a similar calculation, $\hat{\alpha} \circ \hat{\overline{\alpha}} = \mathrm{id}_{\pi_1(X,y)}$.

Next we show that $\hat{\alpha}$ is a homomorphism, i.e. that $\hat{\alpha}([\beta] * [\gamma]) = \hat{\alpha}([\beta]) * \hat{\alpha}([\gamma])$ for all loops β, γ based at x. By Theorem 7.6,

$$\begin{aligned}\hat{\alpha}([\beta]) * \hat{\alpha}([\gamma]) &= ([\overline{\alpha}] * [\beta] * [\alpha]) * ([\overline{\alpha}] * [\gamma] * [\alpha]) \\ &= [\overline{\alpha}] * [\beta] * ([\alpha] * [\overline{\alpha}]) * [\gamma] * [\alpha] \\ &= [\overline{\alpha}] * [\beta] * [\gamma] * [\alpha] \\ &= \hat{\alpha}([\beta] * [\gamma])\end{aligned}$$

□

Remark 7.14. Note that different paths α from x to y may induce different isomorphisms. Also if there is no path from x to y, then there is no reason to expect that $\pi_1(X, x) \cong \pi_1(X, y)$. For this reason, we tend to study fundamental groups of path connected spaces only. Of course the above theorem tells us that if X is path connected then the fundamental group is independent of the choice of base point, and in this case we may well suppress the base point and write simply $\pi_1(X)$ to denote the fundamental group.

The fundamental group provides a means to test whether two paths are path homotopic:

Lemma 7.15. *Let X be a topological space, let $x, y \in X$ and let α, β be two paths from x to y. Then $\alpha \simeq \beta$ if and only if $[\alpha] * [\overline{\beta}]$ is equal to the identity element in $\pi_1(X, x)$.*

Proof. First suppose that $[\alpha] * [\overline{\beta}] = [e_x]$. Then by Theorem 7.6,

$$\begin{aligned}[\alpha] &= [\alpha] * [e_y] \\ &= [\alpha] * [\overline{\beta}] * [\beta] \\ &= [e_x] * [\beta] \\ &= [\beta].\end{aligned}$$

Conversely, if $[\alpha] = [\beta]$ then

$$[\alpha] * [\overline{\beta}] = [\beta] * [\overline{\beta}] = [e_x].$$

□

Definition 7.16. A topological space X is called **simply connected** if it is path connected and the fundamental group of X is trivial (i.e. it is the group with only one element). The latter property is often denoted $\pi_1(X) \cong 0$.

Lemma 7.17. *In a simply connected topological space, any two paths having the same initial and final points are path homotopic.*

Proof. Direct consequence of previous lemma. □

7.3 The fundamental group of the circle

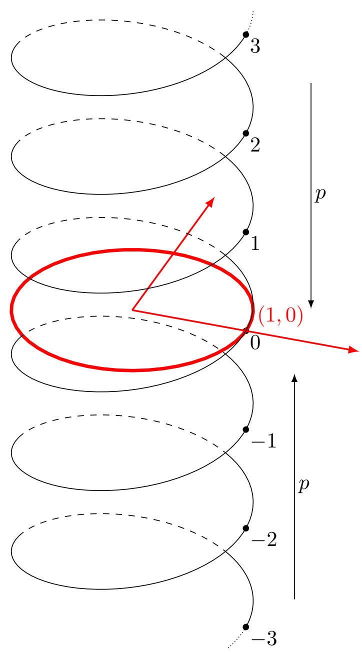
We have learnt quite a bit about the general properties of the fundamental group. Now it is time to calculate the fundamental group of a specific space: the circle.

Fix the base point $b = (1, 0)$ in the circle $S^1 = \{x \in \mathbb{R}^2 \; : \; |x|^2 = 1\}$, and let α be a loop based at b. Roughly speaking, the winding number $N(\alpha)$ of α is the number of times the path goes around the circle anticlockwise, minus the number of times it goes around clockwise.

In order to work rigorously with this number we first need to see $\mathbb{R}$ as the **universal cover** of S^1 (I will not define this formally in this course).

Let $p : \mathbb{R} \to S^1$ be the map $p : u \mapsto (\cos(2\pi u), \sin(2\pi u))$. First of all notice that $p(u) = p(u + m)$ for all $u \in \mathbb{R}, m \in \mathbb{Z}$. Also, when we restrict p to any open interval of length one, i.e. in the form (u_0, u_0+1), we have that $p|_{(u_0,u_0+1)} : (u_0, u_0+1) \to S^1 \backslash \{p(u_0)\}$ is a homeomorphism onto its image: since p is a continuous bijection here with inverse $p|^{-1}_{(u_0,u_0+1)}(x) = u_0 + \frac{\theta(x,p(u_0))}{2\pi}$ where $\theta(x, p(u_0)) \in (0, 2\pi)$ is the anticlockwise angle between x and $p(u_0)$.[1]

You should be imagining $\mathbb{R}$ (the black coil) sitting over S^1 (in red) and winding around so that, for $x \in S^1$, $p^{-1}(\{x\}) \subset \mathbb{R}$ is the collection of points lying directly above x in the picture. I have drawn on $p^{-1}(\{(1, 0)\}) = \mathbb{Z}$.



A **lift** of α is defined to be a path $\tilde{\alpha}$ in $\mathbb{R}$ such that $p \circ \tilde{\alpha} = \alpha$. The **winding number** of α is defined to be

$$N(\alpha) = \tilde{\alpha}(1) - \tilde{\alpha}(0). \qquad (*)$$

This number $N(\alpha)$ is in fact an integer, because $\tilde{\alpha}(0), \tilde{\alpha}(1) \in p^{-1}(\{b\})$ and $p^{-1}(\{b\}) = \mathbb{Z}$.

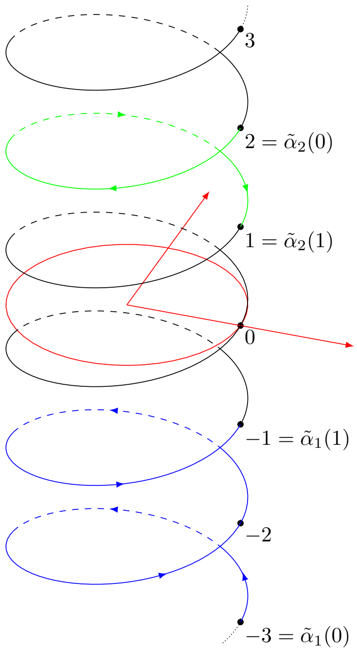
Of course there are infinitely many lifts: if $\tilde{\alpha}$ is a lift of α then $\hat{\alpha}(s) = \tilde{\alpha}(s) + m$ is another lift for any $m \in \mathbb{Z}$. But the point is that $N(\alpha)$ is independent of the lift that we use.

For example if $\alpha_1(s) = (\cos(2\pi \times 2s), \sin(2\pi \times 2s))$ then $\tilde{\alpha}_1 = -3 + 2s$ is a lift of α_1 (in blue below) and $N(\alpha_1) = 2$. Similarly if $\alpha_2(s) = (\cos(2\pi s), -\sin(2\pi s))$ then $\tilde{\alpha}_2(s) = 2 - s$ is a lift of α_2 (in green) and $N(\alpha_2) = -1$:

[1] If you want an expression for θ, first write the anticlockwise rotation of $p(u_0)$ by $\pi/2$ as $p(u_0)^{\perp}$. Then we have

$$\theta(x, p(u_0)) = \begin{cases} \arccos p(u_0) \cdot x & \text{if } p(u_0)^{\perp} \cdot x \geq 0 \\ 2\pi - \arccos p(u_0) \cdot x & \text{if } p(u_0)^{\perp} \cdot x \leq 0, \end{cases}$$

where $\arccos \in [0, \pi]$. You can check that this is continuous if you like.



Thankfully, we only need to specify the starting point of a lift in order that it is unique.

Lemma 7.18. *Let $\alpha : [0,1] \to S^1$ be any loop based at $b = (1,0)$ (so $\alpha(0) = \alpha(1) = (1,0)$). Then for each $q \in p^{-1}(\{b\}) = \mathbb{Z}$, there exits a unique lift $\tilde{\alpha} : [0,1] \to \mathbb{R}$ satisfying $\tilde{\alpha}(0) = q$ and $p \circ \tilde{\alpha} = \alpha$.*

See the end of this section for a proof (but this is non-examinable material).

Notice that this Lemma immediately tells us that N is well-defined. Since if $\tilde{\alpha}$ is a lift of α starting at $q_1 \in \mathbb{Z}$ (which is unique), and if $\hat{\alpha}$ is another lift starting at $q_2 \in \mathbb{Z}$, then setting $\tilde{\alpha}_1(s) = \hat{\alpha}(s) + q_1 - q_2$, we must have $\tilde{\alpha}_1(s) \equiv \tilde{\alpha}(s)$. Surely $\tilde{\alpha}_1$ defined in this way is another lift starting at q_1, but since lifts are unique when we fix the initial point, then $\tilde{\alpha} = \tilde{\alpha}_1$.

Thus

$$N(\alpha) = \tilde{\alpha}(1) - \tilde{\alpha}(0) = \hat{\alpha}(1) - \hat{\alpha}(0).$$

The next Lemma will guarantee that not only is N well-defined, but it only depends upon the homotopy class of α, i.e. $N(\alpha) = N([\alpha])$.

Lemma 7.19. *Suppose that $F : [0,1] \times [0,1] \to S^1$ is continuous and satisfies $F(0,t) = F(1,t) = b = (1,0)$ for all $t \in [0,1]$. Then for any $q \in p^{-1}(\{b\}) = \mathbb{Z}$, there exits a unique continuous map $\tilde{F} : [0,1] \times [0,1] \to \mathbb{R}$ satisfying $\tilde{F}(0,t) = q$ for all t and $p \circ \tilde{F} = F$.*

Once again, the proof is non-examinable but you can find it at the end of this section. A consequence of this result is crucial to us: it tells us that $N(\alpha)$ only depends on the homotopy class $[\alpha]$ of α.

Specifically, if $[\alpha] = [\beta] \in \pi_1(S^1, b)$ and F is a path homotopy between them, we can let $\tilde{F} : [0,1] \times [0,1] \to \mathbb{R}$ be the lift of F described by the previous Lemma for some fixed $q \in \mathbb{Z}$. Notice that, by definition $\tilde{F}(s,0) = \tilde{\alpha}(s)$ is a lift of α and $\tilde{F}(s,1) = \tilde{\beta}(s)$ is a lift of β. Furthermore, $p \circ \tilde{F}(1,t) = b = (1,0)$ for all t, so $\tilde{F}(1,t) \in \mathbb{Z}$ for all t. This implies that $\tilde{F}(1,t) = q_0 \in \mathbb{Z}$ is fixed for all t and in particular $\tilde{\alpha}(1) = \tilde{F}(1,0) = \tilde{F}(1,1) = \tilde{\beta}(1) = q_0$.

Of course we already know that $\tilde{F}(0,t) = q$ for all t which implies $\tilde{\alpha}(0) = \tilde{F}(0,0) = \tilde{F}(0,1) = \tilde{\beta}(0) = q$.

We have shown that $[\alpha] = [\beta] \implies N(\alpha) = N(\beta)$.

Theorem 7.20. *The map*

$$\phi : \pi_1(S^1, b) \to \mathbb{Z},$$
$$\phi : [\alpha] \mapsto N(\alpha)$$

is an isomorphism.

Proof. Lemmata 7.18 and 7.19 guarantee that ϕ is well-defined (which is an important step). It remains to check that it is a bijective group homomorphism (i.e. an isomorphism).
ϕ is surjective: Let $n \in \mathbb{Z}$; we must find a loop α such that $\phi(\alpha) = n$. A suitable loop is given by

$$\alpha : t \mapsto (\cos(2\pi nt), \sin(2\pi nt)).$$

A lift of this map is given by $\tilde{\alpha} : [0,1] \to \mathbb{R}$, $\tilde{\alpha} : t \mapsto nt$; clearly $\tilde{\alpha}(1) - \tilde{\alpha}(0) = n$, as required.

ϕ is injective: Let α, β be two loops in S^1 such that $N(\alpha) = N(\beta)$; we must show that $[\alpha] = [\beta]$. Let $\tilde{\alpha}, \tilde{\beta} : [0,1] \to \mathbb{R}$ be lifts of these paths, both starting at the same point q. Since $N(\alpha) = N(\beta)$ they also have the same final point. Therefore there exists a path homotopy $\hat{F}$ from $\tilde{\alpha}$ to $\tilde{\beta}$ (fixing end points). The map $F = p \circ \hat{F} : [0,1] \times [0,1] \to S^1$ is a path homotopy from α to β, so $[\alpha] = [\beta]$.

ϕ is a homomorphism: Let α, β be any two loops based at b; we must show that

$$N(\alpha * \beta) = N(\alpha) + N(\beta).$$

Let $\tilde{\alpha}, \tilde{\beta} : [0,1] \to \mathbb{R}$ be lifts of these paths, but this time suppose that $\tilde{\alpha}$ is a lift starting at q and $\tilde{\beta}$ a lift starting at $\tilde{\alpha}(1)$, i.e. $\tilde{\beta}(0) = \tilde{\alpha}(1)$. The join $\tilde{\alpha} * \tilde{\beta}$ is a lift of $\alpha * \beta$, because

$$p \circ \tilde{\alpha} * \tilde{\beta} : s \mapsto \begin{cases} p \circ \tilde{\alpha}(2s) & 0 \leq s \leq \frac{1}{2} \\ p \circ \tilde{\beta}(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases} = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}.$$

So

$$N(\alpha * \beta) = \tilde{\alpha} * \tilde{\beta}(1) - \tilde{\alpha} * \tilde{\beta}(0) = \tilde{\beta}(1) - \tilde{\beta}(0) + \tilde{\alpha}(1) - \tilde{\alpha}(0) = N(\beta) + N(\alpha)$$

as required. □

Proof of Lemma 7.18. Here we will need to use the notion of a Lebesgue number that we met at the end of the previous chapter. Consider $U = S^1 \backslash \{b\}$ and $V = S^1 \backslash \{-b\}$ which is an open cover of S^1. Furthermore, $p^{-1}(U) = \mathbb{R} \backslash \mathbb{Z}$ and $p^{-1}(V) = \mathbb{R} \backslash (\mathbb{Z} + 1/2)$.

Given any $m \in \mathbb{Z}$ and an interval of the form $(m, m+1)$ then $p_m = p|_{(m,m+1)} : (m, m+1) \to S^1 \backslash \{b\}$ is a homeomorphism onto its image. Similarly $p_{m/2} = p|_{(m-1/2, m+1/2)} : (m - 1/2, m + 1/2) \to S^1 \backslash \{-b\}$ is a homeomorphism onto its image.

Since α is continuous then $\alpha^{-1}(U)$ and $\alpha^{-1}(V)$ form an open cover of $[0,1]$ so let $\varepsilon > 0$ be the Lebesgue number of this cover (which exists because $[0,1]$ is compact cf. Proposition 6.80). In particular we can find $0 = t_0 < t_0 < \cdots < t_{n-1} < t_n = 1$ so that $\alpha([t_i, t_{i+1}]) \subset U$ **or** $\alpha([t_i, t_{i+1}]) \subset V$ for all $i = 0, \ldots, n-1$ (e.g. as long as $t_{i+1} - t_i < \varepsilon/2$ this is guaranteed by the definition of a Lebesgue number). Now by fixing $q \in \mathbb{Z}$ we define $\tilde{\alpha} : [t_0, t_1] \to \mathbb{R}$ by $\tilde{\alpha} = p_{q/2}^{-1} \circ \alpha$ which is well defined since on this interval $\alpha([t_0, t_1]) \subset V$ (it cannot be a subset of U since $\alpha(0) \notin U$). Notice that $\tilde{\alpha}([t_0, t_1]) \subset (q - 1/2, q + 1/2)$ by definition.

If $\alpha([t_1, t_2]) \subset V$ as well then simply extend $\tilde{\alpha}$ in exactly the same fashion to $[t_0, t_2]$. : i.e. set $\tilde{\alpha} = p_{q/2}^{-1} \circ \alpha$ on $[t_0, t_2]$ which is well defined since $\alpha([t_0, t_2]) \subset V$.

If, on the other hand, if $\alpha([t_1, t_2]) \not\subset V$ then we must have $\alpha([t_1, t_2]) \subset U$. In particular $\tilde{\alpha}(t_1) \neq q$. We first ask: is $\tilde{\alpha}(t_1) \in (q, q + 1/2)$ or $\tilde{\alpha}(t_1) \in (q - 1/2, q)$?[2] It must be one or the other, so if it's the first we define $\tilde{\alpha} : [t_1, t_2] = p_q^{-1} \circ \alpha$ and if it's the second we define $\tilde{\alpha} : [t_1, t_2] = p_{q-1}^{-1} \circ \alpha$.

Notice that in either case $\tilde{\alpha} : [t_0, t_2] \to \mathbb{R}$ is well-defined and continuous (e.g. by the glue Lemma): it is also uniquely determined in each interval $[t_0, t_1]$ and $[t_1, t_2]$[3] since we have

[2]i.e. did the curve start winding anticlockwise, or clockwise?

[3]you might we wondering what happens if $\alpha([t_i, t_{i+1}]) \subset U$ **and** $\alpha([t_i, t_{i+1}]) \subset V$: in this case we must have $\alpha([t_i, t_{i+1}])$ (since it is connected) is contained in one of the connected components of $U \cap V = S^1 \backslash \{\pm b\}$: thus for all $r \in \mathbb{Z}$ we have $p_r^{-1} = p_{r/2}^{-1}$ and $p_{r-1}^{-1} = p_{r/2}^{-1}$ in either of these regions

sufficiently restricted p, $p|$ so that $p|^{-1}$ has a specified target and is a homeomorphism onto that target.

Now, we can inductively keep extending the definition of $\tilde{\alpha}$ in this way and obtain a path $\tilde{\alpha} : [0,1] \to \mathbb{R}$ with $\tilde{\alpha}(0) = q$. Furthermore, $\rho \circ \tilde{\alpha} = \alpha$ by construction, and since it was uniquely determined on each interval $[t_i, t_{i+1}]$ it must be uniquely determined by the initial choice of $q \in \mathbb{Z}$.

□

Proof of Lemma 7.19. This is really the same proof as the previous Lemma 7.18 in spirit, so I will only sketch the details. Once again, let $\varepsilon > 0$ be a Lebesgue number for the cover $\{F^{-1}(U), F^{-1}(V)\}$ of $[0,1] \times [0,1]$ (guaranteed to exists since $[0,1] \times [0,1]$ is compact cf. Proposition 6.80).

In this case, we can find $0 = t_0 < t_1 < \cdots < t_n = 1$ so that each sub-cube $[t_i, t_{i+1}] \times [t_j, t_{j+1}] \subset F^{-1}(U)$ or $[t_i, t_{i+1}] \times [t_j, t_{j+1}] \subset F^{-1}(V)$ for all $i, j \in \{0, \ldots, n\}$.

Again, we can construct $\tilde{F}$ uniquely on each sub-cube, under the starting condition that $\tilde{F}(0,0) = q$: start with $[t_0, t_1] \times [t_0, t_1]$ and notice that $F([t_0, t_1] \times [t_0, t_1]) \subset V$. Then move onto $[t_1, t_2] \times [t_0, t_1]$ and notice that $\tilde{F}(\{t_1\} \times [t_0, t_1])$ has already been determined by the previous step. Again we can uniquely extend $\tilde{F}$ to be defined in $[t_0, t_2] \times [t_0, t_1]$.

Inductively we keep going along the bottom row of sub-cubes, exactly as before, until $\tilde{F}$ is defined on $[0,1] \times [t_0, t_1]$. Now we start at $[t_0, t_1] \times [t_1, t_2]$, notice that $\tilde{F}$ has already been uniquely determined by $\tilde{F}([t_0, t_1] \times \{t_1\})$ so we can uniquely extend it into this cube. Again work horizontally so that we end up having uniquely determined $\tilde{F}$ on $[0,1] \times [t_0, t_2]$. At this point we start on $[t_0, t_1] \times [t_2, t_3]$... etc etc.

At each stage of the extension, at least one edge of the next cube (and always a connected component of it) has been uniquely determined and we can continue extending the definition of $\tilde{F}$ in this way until we have exhausted all sub-cubes. □

7.4 Deformation retracts

Definition 7.21. Let X and Y be topological spaces, let $x \in X$ and let $h : X \to Y$ be a continuous function. The **homomorphism induced by** h is the map

$$\begin{array}{rccc} h_* : & \pi_1(X, x) & \to & \pi_1(Y, h(x)) \\ & [\alpha] & \mapsto & [h \circ \alpha]. \end{array}$$

For this definition to make sense, we need to confirm two things: that the map h_* is well-defined, and that it is a homomorphism.

Checking that the map is well-defined means confirming that if α, α' are two loops based at x such that $\alpha \simeq \alpha'$, then $h \circ \alpha \simeq h \circ \alpha'$. Suppose then that F is a path homotopy from α to α'. We claim that $h \circ F$ is a path homotopy from $h \circ \alpha$ to $h \circ \alpha'$. This function is continuous because it is the composition of two continuous functions, and it is straightforward to check that

$$h \circ F(s, 0) = h \circ \alpha(s), \quad h \circ F(s, 1) = h \circ \alpha'(s), \quad h \circ F(0, t) = h \circ F(1, t) = h(x)$$

$\forall s, t \in [0,1]$. Therefore $h \circ \alpha \simeq h \circ \alpha'$.

Checking that h_* is a homomorphism means confirming that $h_*([\alpha] * [\beta]) = h_*([\alpha]) * h_*([\beta])$ for all loops α, β based at x. This property follows from the following calculation:

$$\begin{aligned} (h \circ \alpha) * (h \circ \beta)(s) &= \begin{cases} h(\alpha(2s)) & 0 \leq s \leq \frac{1}{2} \\ h(\beta(2s - 1)) & \frac{1}{2} \leq s \leq 1 \end{cases} \\ &= h(\alpha * \beta(s)). \end{aligned}$$

Theorem 7.22. *Inducing homomorphisms is functorial, that is*

(a) If X, Y, Z are topological spaces, $x \in X$, and $f : X \to Y$, $g : Y \to Z$ are continuous functions then $(g \circ f)_* = g_* \circ f_*$.

(b) If X is a topological space, $x \in X$ and $\mathrm{id}_X : X \to X$ is the identity function on X, then $(\mathrm{id}_X)_*$ is the identity function on $\pi_1(X, x)$.

Proof. Exercise. □

Corollary 7.23. *If X, Y are topological spaces and $f : X \to Y$ is a homeomorphism then f_* is an isomorphism. In other words, the isomorphism class of the fundamental group is a topological invariant.*

Proof. As f^{-1} is continuous, we have that $(f^{-1})_* \circ f_* \cong \mathrm{id}_{\pi_1(X,x)}$ and $(f)_* \circ (f^{-1})_* \cong \mathrm{id}_{\pi_1(Y,f(x))}$, so $(f^{-1})_*$ is the inverse of f_* and f_* is a bijection. □

Definition 7.24. Let X be a topological space and let $A \subset X$ be a subspace. A is said to be a **strong deformation retract** of X if there exists a continuous map $H : X \times [0, 1] \to X$ such that

$$\begin{aligned} H(x, 0) &= x \quad \forall x \in X \\ H(x, 1) &\in A \quad \forall x \in X \\ H(a, t) &= a \quad \forall a \in A,\ t \in [0, 1]. \end{aligned}$$

The map H is called a **strong deformation retraction** in this case.

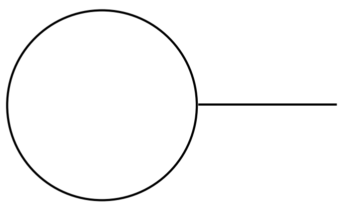
Example 7.25. $\{0\}$ is a strong deformation retract of $[0, 1]$.

Let $H : [0, 1] \times [0, 1] \to [0, 1]$ be the function $H(x, t) = (1 - t)x$. Then H is continuous (by the calculus of limits). Clearly $H(x, 0) = x$ and $H(x, 1) \in \{0\}$ $\forall x \in [0, 1]$. Also, $H(0, t) = 0$ $\forall t \in [0, 1]$. So H is a strong deformation retraction from $[0, 1]$ to $\{0\}$.

Example 7.26. S^{n-1} is a strong deformation retract of $\mathbb{R}^n \setminus \{0\}$.

Let $H : \mathbb{R}^n \setminus \{0\} \times [0, 1] \to \mathbb{R}^n \setminus \{0\}$ be the function $H(x, t) = (1 - t)x + t\frac{x}{|x|}$. Then H is continuous by the calculus of limits. Clearly $H(x, 0) = x$ and $H(x, 1) \in S^{n-1}$ for any $x \in \mathbb{R}^n \setminus \{0\}$. And if $x \in S^{n-1}$, $H(x, t) = x$ for any $t \in [0, 1]$. So H is a strong deformation retraction.

Example 7.27. There is a strong deformation retraction from the following subset of $\mathbb{R}^2$ to the circle:



Appealing to your geometric intuition, the strong deformation retraction shrinks the straight line down to a point (just as in the first example the interval $[0, 1]$ was shrunk to a point).

Proposition 7.28. *Let A be a strong deformation retract of X and let $a \in A$. Then the inclusion $i : A \to X$ induces an isomorphism $i_* : \pi_1(A, a) \to \pi_1(X, a)$ of fundamental groups.*

Proof. Let $H : X \times [0,1] \to X$ be a strong deformation retraction and let $h : X \to A$ be the map $h(x) = H(x,1)$. We claim that h_* is the inverse of i_*; it follows that i_* is an isomorphism.

To prove the claim we must show that $h_* \circ i_*$ and $i_* \circ h_*$ are the identity maps on $\pi_1(A,a)$ and $\pi_1(X,a)$ respectively. First note that $h \circ i$ is equal to the identity map on A. So by parts (a) and (b) of Theorem 7.22 $h_* \circ i_* = (h \circ i)_*$ is equal to the identity map on $\pi_1(A,a)$.

Now consider the map $i_* \circ h_*$, which by Theorem 7.22 is equal to $(i \circ h)_*$. Let α be any loop in X based at a. We claim that the function $F : [0,1] \times [0,1]$ defined by $F(s,t) = H(\alpha(s),t)$ is a path homotopy from α to $i \circ h \circ \alpha$; from this it follows that $(i \circ h)_*([\alpha]) = [\alpha]$. F is continuous by Theorem 5.33. Since $\alpha(0) = \alpha(1) = a \in A$ and H is a deformation retraction, $F(0,t) = \alpha(0)$ and $F(1,t) = \alpha(1)$ $\forall t \in [0,1]$. $F(s,0) = \alpha(s)$ $\forall s \in [0,1]$ because H is a deformation retraction, and $F(s,1) = i \circ h \circ \alpha(s)$ $\forall s \in [0,1]$ by the definition of h. □

Corollary 7.29. *Let $n \geq 1$, then $\pi_1(\mathbb{R}^{n+1} \backslash \{(0)\}) \cong \pi_1(S^n)$. In particular we have $\pi_1(\mathbb{R}^2 \backslash \{0\}) \cong \mathbb{Z}$.*

Remark 7.30. We will see later that $\pi_1(S^n) \cong 0$ when $n \geq 2$.

Remark 7.31. Using deformation retracts it is relatively straightforward to show that the fundamental groups of the cylinder and Möbius band are isomorphic to $\mathbb{Z}$ (they can both be deformation retracted to S^1).

7.4.1 The Van Kampen Theorem

The Seifert–Van Kampen Theorem is a powerful tool for calculating the fundamental group. This theorem is best stated using the language of "presentations of groups" so we begin this section by introducing the necessary group theory.

Free groups and presentations

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite set. A **word** in A is a finite sequence of symbols a_i and a_i^{-1}. For example, the following are all words in the set $\{a_1, a_2\}$:

$$a_1, \quad a_1 a_1^{-1}, \quad a_1 a_2 a_1^{-1} a_2 a_2.$$

The **empty word** is the word " " in which no symbols occur – this is denoted 1 or e to avoid confusion. The **product** $w_1 w_2$ of two words w_1 and w_2 is defined by joining them together, and the **inverse** w^{-1} of a word w is defined by reversing the order of the elements and replacing a_i with a_i^{-1} and a_i^{-1} with a_i. Thus:

$$(a_1)(a_1 a_1^{-1}) = a_1 a_1 a_1^{-1}, \quad (a_1 a_2 a_1^{-1} a_2 a_2)^{-1} = a_2^{-1} a_2^{-1} a_1 a_2^{-1} a_1^{-1}.$$

It is also common to use the shorthand $a_i^2 = a_i a_i$, $a_i^{-3} = a_i^{-1} a_i^{-1} a_i^{-1}$ etc.

Now let $R = \{r_1, r_2, \ldots, r_m\}$ be a finite set of words in A. We define an equivalence relation $\sim$ on the set of all words by saying that two words w_1 and w_2 are equivalent if w_1 can be obtained from w_2 by finitely many operations of the following type:

1. inserting $a_i a_i^{-1}$ or $a_i^{-1} a_i$, where $a_i \in A$, anywhere into the word,
2. removing $a_i a_i^{-1}$ or $a_i^{-1} a_i$, where $a_i \in A$, anywhere from the word,
3. inserting r or r^{-1}, where $r \in R$,
4. removing r or r^{-1}, where $r \in R$.

I leave it as an exercise to check this is an equivalence relation. Let G be the quotient of the set of all words by this equivalence relation. We claim that G (with the product and inverse as defined above) is a group, and write

$$G = \langle a_1, a_2, \dots, a_n;\ r_1, r_2, \dots, r_m \rangle$$

The elements of A are called **generators** for G and the elements of R are called **relations**; together the sets A and R are called a **presentation** for G.

As this is not a group theory course we will not take the time to prove that G is a group (although it is not hard to do so). Instead we will give some examples.

Example 7.32. The infinite cyclic group.

Let $A = \{a\}$ be a set with one element and let $R = \emptyset$. Then any word is equivalent to one of the following words:

$$\dots, a^{-2}, a^{-1}, e = a^0, a, a^2, \dots.$$

For example, the word $aa^{-1}aa$ is equivalent to $aa = a^2$. Moreover, it can be shown that no two words on this list are equivalent to each other. So

$$G = \langle a;\ \rangle = \{a^m\ :\ m \in \mathbb{Z}\}.$$

This group is isomorphic to $\mathbb{Z}$; the map $\phi : \mathbb{Z} \to G$, $\phi(m) = a^m$ is an isomorphism (ϕ is clearly a bijection, and $\phi(m+n) = a^{m+n} = a^m a^n$).

Example 7.33. Cyclic groups of order n.

Let $A = \{a\}$ as before and let $R = \{a^n\}$ for some $n \in \mathbb{Z}^+$. We claim that any word is equivalent to one of the following:

$$e, a, \dots, a^{n-1}.$$

For example, $a^n \sim e$, and $a^{-1} \sim a^n a^{-1} = aa\dots a(aa^{-1}) \sim a^{n-1}$. Moreover, it can be shown that no two words on this list are equivalent to each other. So the group G has only n elements. This group is called the cyclic group of order n and is denoted $(\mathbb{Z}_n, +_n)$. Of course this is just the group $\{0, 1, \dots, n-1\}$ under addition modulo n.

Example 7.34. Free group on n generators.

Let $A = \{a_1, \dots, a_n\}$ and let $R = \emptyset$. The group $G = \langle a_1 \dots, a_n; \rangle$ is once again infinite. Unlike previous examples, this group is **not** abelian.

Example 7.35. Let $A = \{a, b\}$ and let $R = \{aba^{-1}b^{-1}\}$ so $G = \langle a, b; aba^{-1}b^{-1} \rangle$. Thus, in G we have $aba^{-1}b^{-1} = e$ and so $ab = ba$. In particular G must be abelian and any element can be written in the form $a^m b^n$ for some m, n. It can now easily be shown that the group is isomorphic to $(\mathbb{Z}^2, +)$.

Example 7.36. Let $G = \langle a, b; abab^{-1} \rangle$. This group is *not* abelian, and I claim that $G \cong \langle c, d; c^2 d^2 \rangle$. Without writing a formal proof (which isn't difficult), letting $c = ab$ and $d = b^{-1}$ notice that any word in $\{a, b\}$ can be written as a word in $\{c, d\}$ (and vice versa) since $a = cd$ and $b = d^{-1}$. Furthermore the relation $abab^{-1} = e$ is equivalent to $c^2 d^2 = e$.

Note that if A is empty then the only word is the empty word e – so $\langle\ ;\ \rangle$ is the trivial group $\{e\}$.

Theorem 7.37 (Seifert-Van Kampen)**.** *Suppose that U, V are path-connected open subsets of a topological space X such that $X = U \cup V$ and $U \cap V$ is nonempty and path connected. Let $x \in U \cap V$. Suppose that the fundamental groups of $U, V, U \cap V$ have the following presentations:*

$$\pi_1(U, x) \cong \langle a_1, \dots a_l;\ r_1, \dots r_i \rangle$$
$$\pi_1(V, x) \cong \langle b_1, \dots b_m;\ s_1, \dots s_j \rangle$$
$$\pi_1(U \cap V, x) \cong \langle c_1, \dots c_n;\ t_1, \dots t_k \rangle.$$

Let $\phi : U \cap V \to U$, $\psi : U \cap V \to V$ be the inclusions. Then

$$\pi_1(X, x) \cong \langle a_1, \dots a_l, b_1, \dots b_m;\ r_1 \dots r_i, s_1 \dots s_j, \phi_*(c_1)\psi_*(c_1)^{-1}, \dots \phi_*(c_n)\psi_*(c_n)^{-1}\rangle.$$

Remark 7.38. Each generator corresponds to some homotopy class of loops based at x, $a_i = [\alpha_i]$, say.

Notice that $\phi_*(c_i) \in \pi_1(U, x)$ and $\psi_*(c_i) \in \pi_1(V, x)$, thus we respectively have that $\phi_*(c_i)$ is a word in $\{a_1, \dots, a_l\}$ and $\psi_*(c_i)$ is a word in $\{b_1, \dots, b_m\}$. The new relations are therefore perfectly well-defined and tell us how the homotopy classes of the fundamental groups $\pi_1(U, x)$, $\pi_1(V, x)$ interact with one-another.

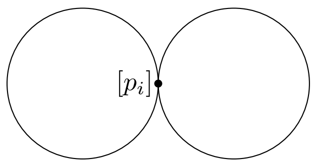
The proof of this theorem is quite long and will not be included in this course (although it can be found in many textbooks). In this course we will only use the theorem to calculate some fundamental groups.

Examples

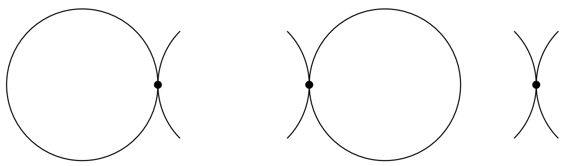
We shall start with an important example – "a bouquet of circles" – before calculating the fundamental groups of some fundamental topological spaces.

Before we start: given X and Y two topological spaces, then we can form the **disjoint union** of these spaces to get a new topological space $Z = X \coprod Y$. The topology on Z is formed by considering sets $W = U \coprod V$ for $U \subset X$ open and $V \subset Y$ open. Inductively we can do this for any finite number of topological spaces.

Example 7.39. Let $X = S^1_1 \coprod S^1_2$ be the disjoint union of two copies of $S^1 \subset \mathbb{R}^2$. Pick points $p_i \in S^1_i$ and define $\sim$ on X via $p_1 \sim p_2$ and if $x, y \notin \{p_1, p_2\}$ then $x \sim y \iff x = y$. We end up with $X/\sim$ looking like:



Pick $U \subset X/\sim$ and $V \subset X/\sim$ both open and path connected sets as in the next picture (with $U \cap V$ shown at the end)

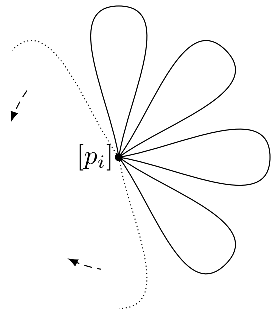


Both U and V deformation retract into a circle S^1_i, and $U \cap V$ deformation retracts to a point $[p_i]$. So we have $\pi_1(U, [p_i]) \cong \langle a; \cdot\rangle \cong \mathbb{Z}$, $\pi_1(V, [p_i]) \cong \langle b; \cdot\rangle \cong \mathbb{Z}$ and $\pi_1(U \cap V, [p_i]) \cong \langle \cdot; \cdot\rangle \cong 0$.

Van Kampen's Theorem tells us, therefore, that $\pi_1(X/\sim) \cong \langle a, b; \cdot\rangle$ which is a free group on two generators (and is *not* Abelian). Notice that a is the homotopy class of a loop winding once around the first S^1 (say anti-clockwise), and b is the homotopy class of a loop winding once around the second S^1 (clockwise): Van Kampen's Theorem is thus telling us that $\pi_1(X/\sim)$ is completely determined by a and b (and there is no relation between these two loops, as you would expect). Notice that we do not really need to be precise about which "directions" a and b are winding, as long as they go around once and once only.

Example 7.40. Let $X = \coprod_{i=1}^n S^1_i$, the disjoint union of n copies of $S^1 \subset \mathbb{R}^2$. Pick a point $p_i \in S^1_i$ for each i, and define $\sim$ on X via $p_i \sim p_j$ for all i, j, otherwise $x \sim y \iff x = y$. We end up with $X/\sim$ being n copies of S^1 joined together at a single point $[p_i] = \{p_j : j = 1, \dots, n\}$.

Pictorially:



By induction on the previous example we end up with $\pi_1(X/_\sim) \cong \langle a_1, \dots a_n; \cdot \rangle$ which is a free group on n generators. Once again, each a_i is a homotopy class of a loop winding once around S^1_i.

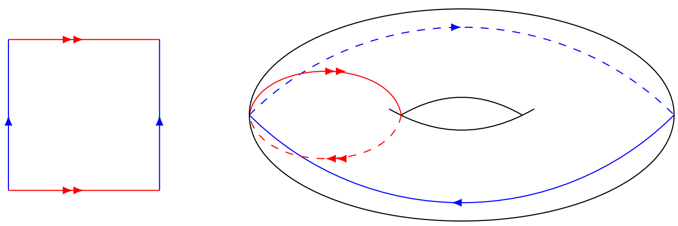
Example 7.41. The spheres $S^n \subset \mathbb{R}^{n+1}$ for $n \geq 2$. Recalling Example 6.9 (the stereographic projection) we know that $U = S^n \backslash \{N\}$ and $V = S^n \backslash \{S\}$ are both homeomorphic to $\mathbb{R}^n$. In particular they are simply connected. Furthermore $S^n = U \cup V$ and $U \cap V = S^n \backslash \{N, S\} \cong \mathbb{R}^n \backslash \{0\}$ is path connected (since $n \geq 2$).

It follows from Van Kampen's Theorem that, for $x \notin \{N, S\}$, $\pi_1(S^n) = \pi_1(S^n, x) \cong \langle\, ; \,\rangle$ is trivial. Note that in this example we did not need to calculate $\pi_1(U \cap V, x)$, because a group which is presented with no generators is always trivial. Actually, I'm telling a lie here. We should at least check that $\pi_1(U \cap V, x)$ is of the form $\langle c_1, \dots, c_n; t_1, \dots t_k \rangle$ (*a-priori* there is no guarantee of this). However there are two ways of getting around this. The first is that there is a more general version of Van Kampen's Theorem that applies in this situation. The second (and more preferable one) is the following:

Theorem 7.42. *Suppose that X is a topological space and $U, V \subset X$ are open and simply connected. Suppose that $X = U \cup V$ and $U \cap V \neq \emptyset$ is path connected. Then X is simply connected.*

The proof of this is relatively straightforward using Lebesgue numbers - but we will not give the details here.

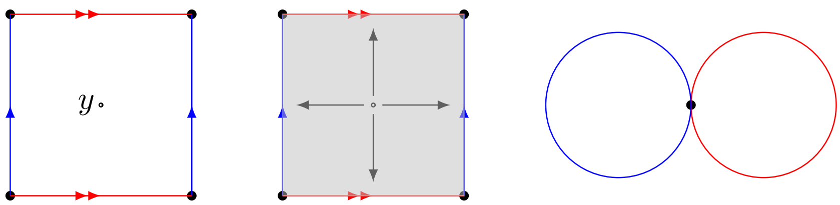
Example 7.43. The torus T^2: recall that this topological space is a quotient space obtained via the square $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and the equivalence relation $\sim$ as shown in the diagram below:



We'll use Van Kampen's Theorem to prove that $\pi_1(T^2) = \pi_1(T^2, x) = (\mathbb{Z}^2, +)$.

Letting $y = (1/2, 1/2)$, $\tilde{U} = X \backslash \{y\}$ and $\tilde{V} = X \backslash \partial X = (0, 1) \times (0, 1)$ we'll set $U = \tilde{U}/_\sim$ and $V = \tilde{V}/_\sim = \tilde{V}$, giving $U \cap V = V \backslash \{[y]\}$.

For U notice that it deformation retracts onto $\partial X/_\sim$, the boundary of the cube (under an equivalence relation), as the following picture indicates. Notice that $\partial X/_\sim$ is nothing more than two circles joined together at a single point as in Example 7.39:



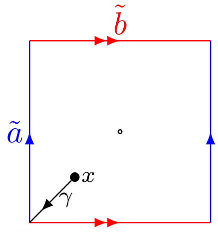
Thus if we let $\tilde{a}$ denote the homotopy class of a curve going once around the blue loop and $\tilde{b}$ denote the homotopy class of a curve going once around the red loop (in some prescribed directions) we have

$$\pi_1(U, [(0,0)]) \cong \pi_1\left(\partial X/_{\sim}, [(0,0)]\right) \cong \langle \tilde{a}, \tilde{b}; \cdot \rangle$$

by Proposition 7.28.

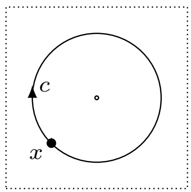
However, we want to compute $\pi_1(U, x)$, $\pi_1(V, x)$ and $\pi_1(U \cap V, x)$ for $x \in U \cap V$. To that end, let $x = [(1/4, 1/4)]$, say.

By Theorem 7.13 we have $\pi_1(U, x) \cong \pi_1(U, [(0,0)])$ via the curve γ:



Thus if we set $a = [\gamma] * \tilde{a} * [\overline{\gamma}]$ and $b = [\gamma] * \tilde{b} * [\overline{\gamma}]$ then we have $\pi_1(U, x) \cong \langle a, b; \cdot \rangle$.

V is clearly simply connected so $\pi_1(V, x) \cong \langle \cdot; \cdot \rangle$. It remains to study $\pi_1(U \cap V, x)$ which is $(0,1) \times (0,1) \backslash \{[y]\}$ as in the diagram below.



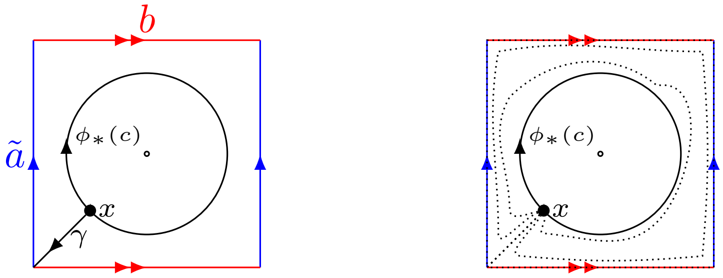
We should be able to see at this point that $U \cap V$ deformation retracts onto S^1 (or perhaps a circle of radius $1/2$), in which case $\pi_1(V, x) = \langle c; \cdot \rangle \cong \mathbb{Z}$. The homotopy class c is represented in the above diagram and corresponds to winding once around $[y]$ in a clockwise direction.

Now we have computed all the fundamental groups, we need to use Van Kampen's Theorem to knit them together and give the fundamental group $\pi_1(T^2)$.

So what does $\phi_*(c)$ look like in U? We can prove that

$$\phi_*(c) = aba^{-1}b^{-1} = [\gamma] * \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} * [\overline{\gamma}].$$

Please convince yourselves of this, I have given an indication of this below (the dashed curves indicate a homotopy from a representative of $[\gamma] * \tilde{a}\tilde{b}\tilde{a}^{-1}\tilde{b}^{-1} * [\overline{\gamma}]$ to a representative of $\phi_*(c)$).



Finally, Van Kampen's Theorem gives us

$$\pi_1(T^2) \cong \langle a, b; aba^{-1}b^{-1}\rangle \cong (\mathbb{Z}^2, +).$$

Example 7.44. The real projective spaces $\mathbb{RP}^n$. Define $\mathbb{RP}^n := \mathbb{R}^{n+1}\backslash\{0\} \big/ \sim$ where $x \sim y$ if and only if there exists $\lambda \in \mathbb{R}_*$ so that $x = \lambda y$. In other words $\mathbb{RP}^n$ is the topological space of lines passing through the origin in $\mathbb{R}^n$.

Consider the map $F : \mathbb{R}^{n+1}\backslash\{0\} \to S^n$ given by $F(x) = x/|x|$ and notice that $x \sim y$ if any only if $F(x) = \pm F(y)$ and so by Proposition 5.59 F induces a continuous map $\tilde{F} : \mathbb{RP}^n \to S^n \big/ \sim_1$ where $x \sim_1 y \iff x = \pm y$. $\tilde{F}$ is in fact a homeomorphism with inverse $\tilde{F}^{-1}([z]_1) = [z]$. Notice that this immediately tells us that $\mathbb{RP}^n$ is compact, connected and Hausdorff.

We now trivially have that

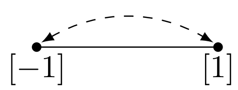
$$G : S^n \big/ \sim_1 \to S^n_+ \big/ \sim_1, \qquad G([x]_1) = [x]_1$$

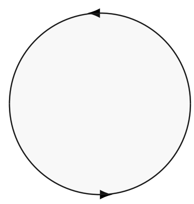
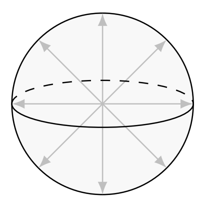
is a homeomorphism. Recall from example 6.10 that $S^n_+ \cong D^n$ via $f : D^n \to S^n_+$. Remember that $D^n = \{x \in \mathbb{R}^n : |x| \le 1\}$ and we now let $x \sim_2 y \iff x = y$, or $|x| = |y| = 1$ **and** $x = \pm y$. Notice that $x \sim_2 y$ if and only if $f(x) \sim_1 f(y)$ so in fact $\tilde{f} : D^n \big/ \sim_2 \to S^n_+ \big/ \sim_1$ is a homeomorphism. In particular

$$\tilde{f}^{-1} \circ G \circ \tilde{F} : \mathbb{RP}^n \to D^n \big/ \sim_2$$

is a homeomorphism, i.e. $\mathbb{RP}^n \cong D^n \big/ \sim_2$.

Below are some representations of $\mathbb{RP}^n \cong D^n \big/ \sim_2$ for $n = 1$, $n = 2$ and $n = 3$

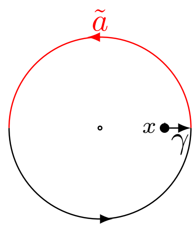


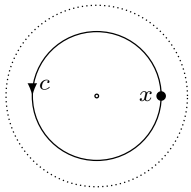
We can now see that $\mathbb{RP}^1 \cong S^1$ (using something similar to examples 5.49, 6.67), giving $\pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$.

Similarly $\mathbb{RP}^2$ is exactly as it appears in Example 5.56 and we will prove that $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$. Letting $\tilde{U} = D^2\backslash\{0\}$ and $\tilde{V} = B^2_1(0)$ we let $U = \tilde{U} \big/ \sim_2$ and $V = \tilde{V} \big/ \sim_2 = \tilde{V}$. Thus $U \cap V = B^2_1\backslash\{0\}$.

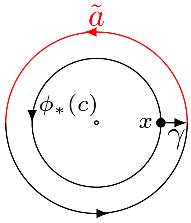
As in the torus case, U deformation-retracts onto $S^1 \big/ \sim_1 \cong S^1_+ \big/ \sim_1 \cong D^1 \big/ \sim_2 \cong \mathbb{RP}^1 \cong S^1$. Letting $\tilde{a}$ represent a generator of $\pi_1(S^1)$ we have that $\pi_1(U) \cong \langle \tilde{a}; \rangle$. Once again, we want to realise $\pi_1(U, x)$, this time for $x = [(0, 0.75)]$ (say) and we do this via γ as below, and set $a = [\gamma] * \tilde{a} * [\overline{\gamma}]$. Thus we have $\pi_1(U, x) \cong \langle a; \rangle$.



We trivially have $\pi_1(V, x) \cong \langle ; \rangle$, and almost as trivially (by now) we see that $\pi_1(U \cap V, x) \cong \langle c; \rangle$.



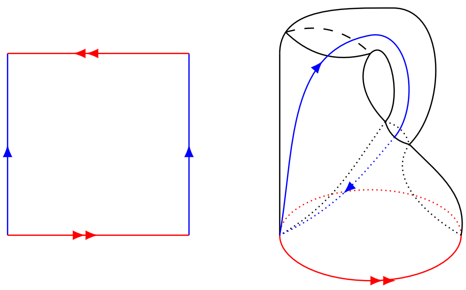
Again we must consider what $\phi_*(c) \in \pi_1(U, x)$ is in terms of a. The picture below suggests that we have $\phi_*(c) = a^2$ (which you can check directly, if you need to).



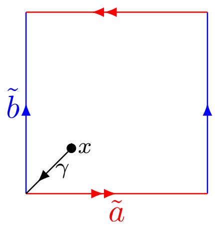
Thus Van Kampen's Theorem tells us that $\pi_1(\mathbb{RP}^2) \cong \langle a; a^2 \rangle \cong \mathbb{Z}_2$.

In fact, one can prove that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ for all $n \geq 2$ using Van Kampen's Theorem and the same method as above. By induction we may assume that we know this to be true for $\mathbb{RP}^{n-1}$ and we aim to prove it for $\mathbb{RP}^n \cong D^n \big/_{\sim_2}$ when $n \geq 3$. In exactly the same way as above we set $U = D^n \backslash \{0\} \big/_{\sim_2}$ and $V = B_1^n(0)$. The crux of the proof is that U deformation retracts onto $S^{n-1} \big/_{\sim_1} \cong \mathbb{RP}^{n-1}$ and thus $\pi_1(U) \cong \mathbb{Z}_2$. Now, since $n \geq 3$ we have $\pi_1(V) \cong \pi_1(U \cap V) \cong \{e\}$. Thus $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$.

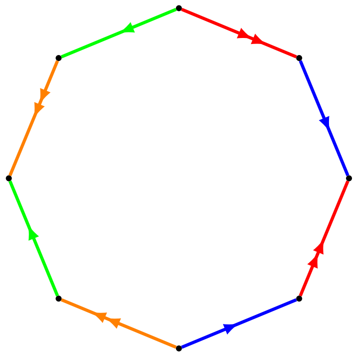
Example 7.45. The Klein bottle, K^2. Recall that this is the topological space obtained as a quotient of the square $X = [0,1] \times [0,1]$ as suggested below.

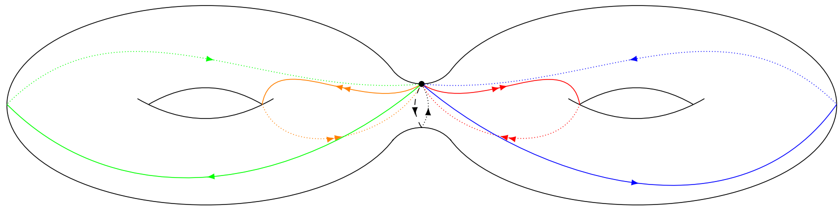


Computing the fundamental group here is very much like the torus above, so we will be very brief. Consider the curves $a = [\gamma] * \tilde{a} * [\overline{\gamma}]$ and $b = [\gamma] * \tilde{b} * [\overline{\gamma}]$ - noting from the picture below that this is subtlety different to the torus case. I will simply point out that we have $a * b * a * b^{-1} = [e_x]$ and leave it to the reader to prove (using Van Kampen and the same method for the torus) that $\pi_1(K^2, x) \cong \langle a, b; abab^{-1} \rangle \cong \langle \hat{a}, \hat{b}; \hat{a}^2 \hat{b}^2 \rangle$ (see example 4.11).



Example 7.46. A genus two surface Σ_2^2. Recall that this is obtained from an octagon via the gluing suggested by the colours and arrows of the edges.



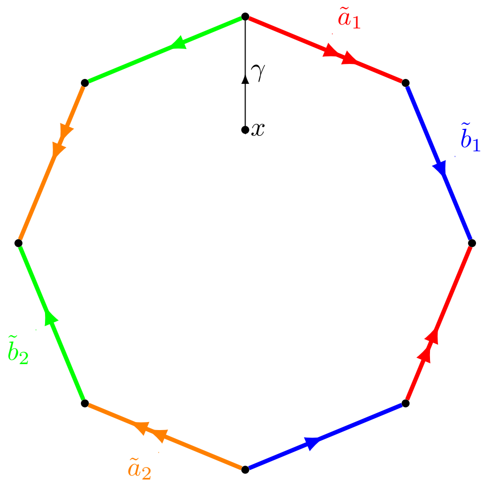


Define $\tilde{a}_i$ and $\tilde{b}_i$ as suggested below, and use γ to define $a_i = [\gamma] * \tilde{a}_i * [\overline{\gamma}]$, $b_i = [\gamma] * \tilde{b}_i * [\overline{\gamma}]$. First of all letting $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ you can check that

$$e_x \simeq [a_1, b_1][a_2, b_2].$$

Thus, using Van Kampen's Theorem (analogously as to the torus and Klein bottle) we have

$$\pi_1(\Sigma_2^2, x) \cong \langle a_1, b_1, a_2, b_2; [a_1, b_1][a_2, b_2] \rangle \cong \langle a_1, b_1, a_2, b_2; a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \rangle.$$



Let's review what we have learned. The fundamental group of $S^1 = \mathbb{RP}^1$ is $\mathbb{Z}$ under addition. The fundamental groups of the sphere S^n is trivial when $n \geq 2$. The fundamental group of the projective spaces $\mathbb{RP}^n$ is $\mathbb{Z}_2$, so is non-trivial but finite when $n \geq 2$. The fundamental group of the torus is infinite and abelian. The fundamental group of the Klein bottle is infinite and non-abelian. Finally the fundamental group of a genus two surface is infinite, non-abelian, and is generated by 4 generators. When we set $n = 2$, none of these groups are isomorphic to each other, thus S^2, $\mathbb{RP}^2$, T^2, K^2 and Σ_2^2 are topologically distinct from one-another (because the fundamental group is a topological invariant). So the fundamental group is a powerful tool!

7.5 Topological manifolds and the classification of curves and surfaces

In this section we will not provide proofs of any of the statements. I have included this in the course in order that you can see some beautiful classification results for one and two dimensional topological manifolds Before we give the definition of a topological manifold, we first need to define a basis for a topology.

Definition 7.47. Let X be any set. A **basis for a topology** on X is a collection β of subsets of X such that

(B1) For each $x \in X$, there is at least one $B \in \beta$ such that $x \in B$.

(B2) If $B_1, B_2 \in \beta$ and $x \in X$ are such that $x \in B_1$ and $x \in B_2$, there exists a third set $B_3 \in \beta$ such that $x \in B_3$ and $B_3 \subset B_1 \cap B_2$.

The sets $B \in \beta$ are called **basis elements**.

Definition 7.48. Let β be a basis for a topology on a set X. The **topology τ generated by β** is

$$\tau = \{U \subset X \ : \text{ for each } x \in U \text{ there exists a } B \in \beta \text{ such that } x \in B \text{ and } B \subset U\}\,.$$

Equivalently

$$\tau = \{U \subset X : \exists\{B_\lambda\}_{\lambda\in\Lambda} \subset \beta \ s.t. \ U = \cup_\lambda B_\lambda\}.$$

Of course, one should check that this is a topology (which it is). We have seen many examples of topologies generated by bases. For instance the metric topology on a metric space (X, d) is generated by the basis of open balls $\beta = \{B_r(x) : x \in X, \ r > 0\}$. Furthermore, the product topology on $X \times Y$ is generated by the basis $\{U \times V : U \subset X$ and $V \subset Y$ are open.$\}$.

Definition 7.49. X is said to be **second countable** if it is generated by a basis with countably many elements.

For example, $\mathbb{R}^n$ (and any subspace of it) is second countable since all balls of rational radii centred at points with rational coordinates, is a basis for the standard topology (and thus a basis for any subspace). Similarly, *any* metric space (X, d) which has a countable subset $A \subset X$ such that $\overline{A} = X$, is second countable. Furthermore if X and Y are second countable, then $X \times Y$ is second countable.

If we let R denote the real numbers equipped with the discrete topology, then it is *not* second countable.

Definition 7.50. A topological space M^n is called a **topological manifold of dimension n** if

1. M is separated

2. M is second countable

3. For all $x \in M$ there exists a neighbourhood U_x and a homeomorphism $f : U_x \to B_1^n(0) \subset \mathbb{R}^n$ with $f(x) = 0$: this last condition can be stated simply that M is locally homeomorphic to $\mathbb{R}^n$.

Of course $\mathbb{R}^n$ (and any open subset thereof) is a topological n-manifold. We also have that S^n, $\mathbb{RP}^n$, the torus, the Klein bottle and the genus two surface are topological manifolds.

If we let R be as above and $\mathbb{R}$ denote the real numbers with the standard topology, then $N = \mathbb{R} \times R$ is separated and locally homeomorphic to $\mathbb{R}$, but it is *not* second countable with the product topology.

Definition 7.51. A topological space M^n is called a **topological manifold with boundary** (again of dimension n), if M satisfies

1. M is separated

2. M is second countable

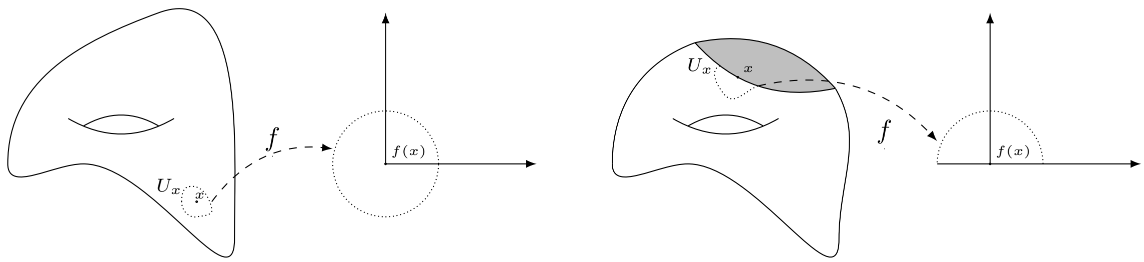
3. For all $x \in M$ one of the following holds, and we enforce that both scenarios occur[4]:
   (a) there exists a neighbourhood U_x and a homeomorphism $f : U_x \to B_1^n(0)$ with $f(x) = 0$
   (b) there exists a neighbourhood U_x and a homeomorphism $f : U_x \to B_1^n(0)^+ = \{x \in B_1^n(0) : x_n \geq 0\}$ with $f(x) = 0$.

If x satisfies $3(a)$ then it is an interior point of M, and if it satisfies $3(b)$ then it is a boundary point. One can check directly from the definition that the interior and boundary of M are disjoint from one-another (no point x can satisfy both (a) and (b)). Moreover the interior of M, denoted $\mathring{M}$, is open and not closed in M, and the boundary is closed. We then have that the boundary points of M are the set $\partial\mathring{M}$ in M, often just written simply as ∂M. Beware that this notation is far from being consistent - the boundary of a manifold is not the same definition of topological boundary we have seen in the exercise sheets!

One can again prove using the definition, that the manifold boundary ∂M is a topological $(n-1)$-manifold (without boundary).

The examples of manifolds with boundary that we have seen are the discs $D^n := \{x \in \mathbb{R}^n : |x| \leq 1\}$, the cylinders $[0,1] \times S^n$ and the Möbius band. Be aware that the open ball $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$ and the open cylinder $(0,1) \times S^n$ are topological manifolds *without* boundary! This is in direct conflict with the definition of boundary we defined earlier in the course - so be very careful when considering manifolds with boundary verses the boundary of a set $A \subset X$.

Below I have attempted to draw a topological surface without boundary (on the left) and with boundary (on the right - the grey area is *not* part of the surface).



We will deal exclusively with *connected* topological 1 and 2 manifolds.

Theorem 7.52. *Let M^1 be a connected topological 1-manifold either with or without boundary. Then M is homeomorphic to one of four options depending on whether it is compact, or has boundary. To be precise, M is homeomorphic to one of the following*

1. *If $\partial M = \emptyset$ and M is not compact then $M \cong \mathbb{R} \cong (0,1)$.*
2. *If $\partial M = \emptyset$ and M is compact then $M \cong S^1$.*
3. *If $\partial M \neq \emptyset$ and M is not compact then $M \cong [0,\infty) \cong [0,1)$. In this case $|\partial M| = 1$.*
4. *If $\partial M \neq \emptyset$ and M is compact then $M \cong [0,1]$. In this case $|\partial M| = 2$.*

[4] If (b) does not occur then this is no different to the definition of a topological manifold. Furthermore, if (b) holds somewhere then there must be points where (a) holds, by definition.

Once again be aware that the notation ∂M does not mean the same thing as the boundary you met in exercise sheet two - here we are thinking of boundary purely in the topological manifold sense defined above.

If we now restrict to topological surfaces, or topological 2-manifolds, there is a very useful operation, called the connected sum, which we will now discuss. We will, from now on, deal with connected and compact topological surfaces without boundary.

Definition 7.53. Let M_1^2, M_2^2 be two connected and compact topological surfaces **without** boundary. Pick $x_i \in M_i$ and let $f_i : U_{x_i} \to B_1^2(0)$ be the respective local homeomorphisms. Defining $\mathfrak{B}(x_i) := f_i^{-1}(B_{1/2}^2(0))$, and $\mathcal{S}_i^1(x_i) = f_i^{-1}(S_{1/2}^1(0))$ we let $\widetilde{M}_i^2 := M_i^2 \setminus \mathfrak{B}(x_i)$ and define

$$M_1 \# M_2 := \widetilde{M}_1 \coprod \widetilde{M}_2 /_\sim$$

where $x \sim y \iff x = y$ or $x \in \mathcal{S}_1^1(x_1)$, $y \in \mathcal{S}_2^1(x_2)$ and $f_1(x) = f_2(y)$.

Remarkably this operation is well-defined, in the sense that $M_1 \# M_2$ is another connected and compact topological surface without boundary; furthermore it is independent (up to homeomorphism) of the choice of $x_i \in M_i$, the neighbourhoods U_{x_i} and the homeomorphisms f_i. Finally if $M_1 \cong N_1$ and $M_2 \cong N_2$ then

$$M_1 \# M_2 \cong N_1 \# N_2.$$

We do not have time to justify these statements, but we can at least use the definition to write down the classification of compact topological surfaces without boundary:

Theorem 7.54. *Suppose that M^2 is a connected and compact topological surface* ***without boundary****. Then M is homeomorphic to either S^2; a finite connected sum of tori; or a finite connected sum of projective planes $\mathbb{RP}^2$. Moreover any two distinct members of this list are topologically distinct.*

If M^2 is homeomorphic to a connected sum of g tori, it is said to have orientable genus g and

$$\pi_1(M) \cong \langle a_1, b_1, \dots a_g, b_g; [a_1, b_1][a_2, b_2] \dots [a_g, b_g] \rangle.$$

Similarly if M^2 is homeomorphic to a connected sum of k projective planes, it is said to have non-orientable genus k and in this case

$$\pi_1(M) \cong \langle a_1, \dots a_k; a_1^2 a_2^2 \dots a_k^2 \rangle.$$

Remark 7.55. Some background and complementary facts that are worth bearing in mind when considering the above. Let M be a compact, connected topological surface without boundary. Then

1. S^2 is said to have orientable genus zero.

2. $M \# S^2 \cong M$ for all topological surfaces M.

3. The genus two surface we have seen earlier in the course should really be called the "orientable genus two surface".

4. $\mathbb{RP}^2 \# \mathbb{RP}^2$ is homeomorphic to the Klein bottle.

5. If M has orientable genus g then $M \# \mathbb{RP}^2$ has non-orientable genus $k = 2g + 1$, so is of the form $\mathbb{RP}^2 \# \mathbb{RP}^2 \# \dots \# \mathbb{RP}^2$ for $2g + 1$ copies of $\mathbb{RP}^2$ - this is why "cross" terms do not occur in our list.

7.6 Applications of the fundamental group

In this last section we will prove some interesting theorems using our knowledge of the fundamental group.

Theorem 7.56 (Brouwer)**.** *Let $D^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$. Any continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. First note that S^1 is not a deformation retract of D^2 since $\pi_1(S^1, (1, 0)) \cong \mathbb{Z}$ and $\pi_1(D^2, (1, 0)) \cong \{e\}$.

Now suppose for contradiction that there exists a continuous map $f : D^2 \to D^2$ with no fixed point, i.e. such that $f(x) \neq x$ for all $x \in D^2$. For any $x \in D^2$, let L_x be the line starting at $f(x)$ and passing through x. If we continue this line far enough beyond x it will hit the boundary. Thus we can find $t > 0$ such that $L_x = \{(1 - t)f(x) + tx : t > 0\}$ hits S^1, i.e. solve

$$(1 - t)^2|f(x)|^2 + 2(1 - t)t\langle x, f(x)\rangle + t^2|x|^2 = 0$$

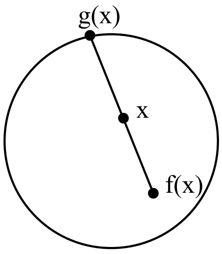
for $t > 0$. You can check that we have

$$t(x, f(x)) = \frac{-\langle f(x), x - f(x)\rangle + \sqrt{\langle f(x), x - f(x)\rangle^2 + |x - f(x)|^2(1 - |f(x)|^2)}}{|x - f(x)|^2} > 0,$$

which is a continuous function of x since $|x - f(x)| > 0$ for all x.

Let $g(x)$ be the point of intersection of L_x with the boundary S^1 of D^2 that is furthest from $f(x)$, so that

$$g(x) = (1 - t(x, f(x)))f(x) + t(x, f(x))x.$$



Then $g : D^2 \to S^1$ is a continuous map with $g(x) = x$ when $x \in S^1$. But then $H(x, t) = (1 - t)x + tg(x)$ is a deformation retract of D^2 to S^1 which is our contradiction.

□

Theorem 7.57 (Borsuk-Ulam)**.** *There does not exist a continuous map $f : S^2 \to S^1$ such that $f(-x) = -f(x)$ for all $x \in S^2$.*

This theorem can be proved using only the fundamental group of S^1. Before presenting the proof, we will investigate some of its consequences.

Corollary 7.58. *Let $f : S^2 \to \mathbb{R}^2$ be a continuous map. Then there exists a point $x \in S^2$ such that $f(-x) = f(x)$.*

This corollary implies that at any given moment in time there exist two antipodal points on the surface of the Earth that have the same temperature and pressure.

Proof. Suppose for contradiction that there exists a continuous map $f : S^2 \to \mathbb{R}^2$ such that $f(-x) \neq f(x)$ $\forall x \in S^2$. Then we may define $g : S^2 \to S^1$ by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

This definition is valid because $|f(x) - f(-x)| \neq 0$, moreover g is obviously continuous. However, g satisfies
$$g(-x) = \frac{f(-x) - f(x)}{|f(x) - f(-x)|} = -g(x).$$
But this contradicts the Borsuk-Ulam Theorem, which states that no continuous map g with this property exists. Therefore the map f cannot exist. □

Corollary 7.59. *No subset of* $\mathbb{R}^2$ *is homeomorphic to* S^2

Proof. Suppose that $f : S^2 \to U$ is a homeomorphism from S^2 to a subset U of $\mathbb{R}^2$. Let $i : U \to \mathbb{R}^2$ be the inclusion; then $i \circ f : S^2 \to \mathbb{R}^2$ is continuous and injective. By the previous corollary there exists a point $x \in S^2$ such that $i \circ f(x) = i \circ f(-x)$, but this contradicts the fact that $i \circ f$ is injective. Therefore no such homeomorphism f exists. □

Corollary 7.60. $\mathbb{R}^2$ *is not homeomorphic to* $\mathbb{R}^3$.

Proof. If $f : \mathbb{R}^3 \to \mathbb{R}^2$ is a homeomorphism, then when we restrict $f|_{S^2} : S^2 \to f(S^2) \subset \mathbb{R}^2$, it is a homeomorphism onto its image. Corollary 7.59 rules this out so no such f can exist. □

Now for the proof of the Borsuk-Ulam Theorem:

Proof of Borsuk-Ulam. We start with an important

Lemma 7.61. *Suppose that* $h : S^1 \to S^1$ *is continuous and satisfies* $h(-x) = -h(x)$ *for all* x. *Defining* $\alpha(s) = h(\cos(2\pi s), \sin(2\pi s))$ *so that* $\alpha : [0,1] \to S^1$ *is a loop (based at some point* $y = h((1,0)) \in S^1$*), we have* $N(\alpha) = 2k + 1$ *for some* $k \in \mathbb{Z}$*, thus* $[\alpha] \neq [e_y]$ *is homotopically non-trivial.*

Proof of Lemma 7.61. The condition that $h(-x) = -h(x)$ implies that for all $s \in [0, 1/2]$, $\alpha(s) = -\alpha(s + 1/2)$. Letting $\tilde{\alpha}$ be a lift of α, we have that $\tilde{\alpha}(s + 1/2)$ differs from $\tilde{\alpha}(s)$ by $k(s) + 1/2$ for some $k(s) \in \mathbb{Z}$, and for all $s \in [0, 1/2]$. In other words, for all $s \in [0, 1/2]$ there exists $k(s) \in \mathbb{Z}$ such that
$$\tilde{\alpha}(s + 1/2) - \tilde{\alpha}(s) = k(s) + 1/2.$$
Since the left hand side of the above expression is continuous for $s \in [0, 1/2]$, it implies that $k(s)$ is continuous and thus $k(s) \equiv k$ is constant (since it must always be an integer). In particular,
$$\tilde{\alpha}(1) = \tilde{\alpha}(1/2) + k + 1/2 = \tilde{\alpha}(0) + 2k + 1.$$
We have proved that $N(\alpha) = \tilde{\alpha}(1) - \tilde{\alpha}(0) = 2k + 1$. □

Now, assuming that $f : S^2 \to S^1$ is continuous and satisfies $f(-x) = -f(x)$ we will derive a contradiction. We restrict f to the upper hemisphere $f|_{S^2_+} : S^2_+ \to S^1$. By Example 6.10, $S^2_+ \cong D^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$ via $G(x_1, x_2, x_3) = (x_1, x_2)$ and we therefore have a continuous function $g = f|_{S^2_+} \circ G^{-1} : D^2 \to S^1$. Letting $h = g|_{S^1} : S^1 \to S^1$, which is continuous, we also have, for all $x \in S^1$, $h(-x) = h(-x_1, -x_2) = f(-x_1, -x_2, 0) = -f(x_1, x_2, 0) = -h(x)$.

Defining α as in Lemma 7.61, $H(s,t) = g((1-t)\cos(2\pi s) + t, (1-t)\sin(2\pi s))$ is a homotopy from α to e_y, i.e. $[e_y] = [\alpha] \neq [e_y]$ which is our contradiction, so no such f can exist.

□

Theorem 7.62 (Fundamental Theorem of algebra)**.** *Every non-constant complex polynomial has a root.*

Corollary 7.63. *Every complex polynomial of degree* $k \in \mathbb{Z}_+$ *has* k *roots, counted with multiplicities.*

Proof. Suppose for contradiction that there exists a non-constant polynomial P with no root. We may assume without loss of generality that the polynomial takes the form

$$P(z) = a_0 + a_1 z + \ldots + a_{k-1} z^{k-1} + z^k$$

for some $k \geq 1$. If $a_0 = 0$ then $z = 0$ is a root, so it must be that $a_0 \neq 0$.

Choose $R > 0$ large enough that

$$R^k > |a_0| + |a_1|R + \ldots + |a_{k-1}|R^{k-1}. \tag{$*$}$$

Let $\alpha : [0,1] \to \mathbb{C} \setminus \{0\}$ be the following loop based at 1:

$$\alpha(s) = \frac{P(Re^{2\pi i s})}{P(R)}.$$

We claim that:

- α is path homotopic to the constant loop e_1
- α is path homotopic to the loop $\beta : s \mapsto e^{2\pi i k s}$.

Together these lead to a contradiction, because e_1 has winding number 0 and β has winding number k.

To prove the first claim, let $F : [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$ be the following function:

$$F(s,t) = \frac{P(Rte^{2\pi i s})}{P(Rt)}.$$

The denominator in this expression is never zero because P has no root. The image of this function is contained in $\mathbb{C} \setminus \{0\}$ for the same reason. F is continuous with $F(0,t) = F(1,t) = 1$ $\forall t \in [0,1]$, and $F(s,0) = 1$, $F(s,1) = \alpha(s)$ $\forall s \in [0,1]$. So F is a path homotopy from e_1 to α.

To prove the second claim, let $G : [0,1] \times [0,1] \to \mathbb{C} \setminus \{0\}$ be the following function:

$$G(s,t) = \frac{(1-t)(a_0 + a_1 Re^{2\pi i s} + \ldots + a_{k-1}(Re^{2\pi i s})^{k-1}) + (Re^{2\pi i s})^k}{(1-t)(a_0 + a_1 R + \ldots + a_{k-1}R^{k-1}) + R^k}.$$

The inequality $(*)$ implies that the numerator and denominator in this expression are nonzero, because

$$\begin{aligned}
|(1-t)(a_0 + a_1 Re^{2\pi i s} + \ldots + a_{k-1}(Re^{2\pi i s})^{k-1}) + (Re^{2\pi i s})^k| \\
&\geq R^k - (1-t)\left(|a_0| + |a_1|R + \ldots + |a_{k-1}|R^{k-1}\right) \\
&> 0,
\end{aligned}$$

and thus G is continuous. We have that $G(0,t) = G(1,t) = 1$ $\forall t \in [0,1]$ and that $G(s,0) = \alpha(s)$ and $G(s,1) = \beta(s)$ $\forall s \in [0,1]$. Therefore G is a path homotopy from α to β. $\square$

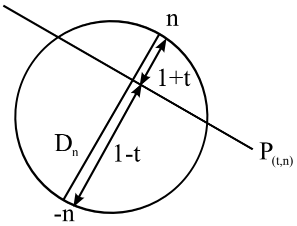
Theorem 7.64 (Ham Sandwich Theorem)**.** *Let A, B, C be bounded subsets of $\mathbb{R}^3$. Then there is a plane in $\mathbb{R}^3$ which simultaneously divides each region exactly in half by volume.*

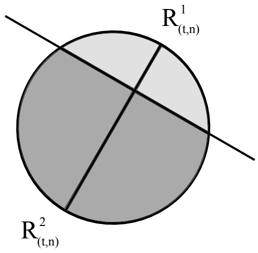
Remark 7.65. How would you adapt this theorem to prove an analogous statement in $\mathbb{R}^2$? You would consider two bounded subsets A, B of $\mathbb{R}^2$ and prove that there was a line which simultaneously divides both subsets in half by area. After you understand the proof below you should try to prove this (easier) version!

Do you think there is an analogous statement in $\mathbb{R}^n$? (You should!)

Do you think the above theorem works for **four** bounded subsets A, B, C, D of $\mathbb{R}^3$? You should not, but why?

Proof. We may re-scale our space such that A, B and C are contained in the closed unit ball. For any $n \in S^2$ let D_n denote the diameter of the ball through n, and for any $t \in [-1, 1]$ let $P_{(t,n)}$ be the plane orthogonal to D_n which passes through the point at distance $1 + t$ from n. The plane $P_{(t,n)}$ divides the unit ball into two regions $R^+_{(t,n)}$ and $R^-_{(t,n)}$ such that $n \in R^+_{(t,n)}$ and $-n \in R^-_{(t,n)}$.





For any fixed $n \in S^2$, let

$$V(t) = \mathrm{Vol}(A \cap R^+_{(t,n)}) - \mathrm{Vol}(A \cap R^-_{(t,n)}).$$

Then $V : [-1, 1] \to \mathbb{R}$ is continuous (this is intuitively clear, but requires a bit of measure theory to prove rigorously). Moreover, $V(-1) = -\mathrm{Vol}(A)$ and $V(1) = \mathrm{Vol}(A)$. We wish to choose a number α such that $V(\alpha) = 0$. Such a number exists by the intermediate value theorem. Since $V(t)$ is monotonically increasing, the set of points t such that $V(t) = 0$ is either a singleton $\{b\}$ or a closed interval $[c, d]$. In the former case we define $\alpha(n) = b$, and in the latter case we define $\alpha(n) = (c + d)/2$.

By repeating this process for every $n \in S^2$ we obtain a function $\alpha : S^2 \to [-1, 1]$. This function is continuous (this is intuitively clear, but again requires a bit of measure theory to make rigorous). It satisfies $\alpha(-n) = -\alpha(n)$, because $P_{(-t,n)} = P_{(t,-n)}$.

Now let

$$\begin{aligned} f(n) &= \mathrm{Vol}(B \cap R^+_{(\alpha(n),n)}) \\ g(n) &= \mathrm{Vol}(C \cap R^+_{(\alpha(n),n)}). \end{aligned}$$

The function $F : S^2 \to \mathbb{R}^2$ defined by $F(n) = (f(n), g(n))$ is again continuous (apply intuition and measure theory!). By corollary 7.58 there exists a point $n \in S^2$ such that $F(-n) = F(n)$. Thus,

$$\begin{aligned} \mathrm{Vol}(B \cap R^+_{(\alpha(n),n)}) &= \mathrm{Vol}(B \cap R^+_{(\alpha(-n),-n))}) \\ &= \mathrm{Vol}(B \cap R^+_{(-\alpha(n),-n)}) \\ &= \mathrm{Vol}(B \cap R^-_{(\alpha(n),n)}), \end{aligned}$$

and we have that $P_{(\alpha(n),n)}$ divides B in half by volume. By a similar calculation $P_{(\alpha(n),n)}$ divides C in half by volume, and by construction $P_{(\alpha(n),n)}$ divides A in half by volume.

□

Appendix A

Set Theory

A.1 Sets

A **set** X is a collection of 'objects'.

If an object x belongs to the set X we write $x \in X$ or $X \ni x$. If x does not belong to the set X we write $x \notin X$.

A **subset** A of a set X is a set A such that $x \in A \Rightarrow x \in X$. A shorthand for this is $A \subset X$. Note that $X \subset X$ (the notation $\subseteq$ may also be used).

You may be familiar with the following sets:

- $\mathbb{R}$ the set of real numbers
- $\mathbb{Z}$ the set of integers
- $\mathbb{Z}^+$ the set of positive integers (note $0 \notin \mathbb{Z}^+$)
- $\mathbb{N}$ the set of non-negative integers, or natural numbers (note $0 \in \mathbb{N}$)
- $\mathbb{Q}$ the set of rational numbers
- $\emptyset$ the empty set (the unique set containing no elements).

A.2 Set arithmetic

Given two sets X and Y,

- their **intersection** is the set $X \cap Y = \{z \,:\, z \in X \text{ and } z \in Y\}$;
- their **union** is the set $X \cup Y = \{z \,:\, z \in X \text{ or } z \in Y\}$;
- and the **difference** of X and Y is the set $X \setminus Y = \{z \,:\, z \in X \text{ and } z \notin Y\}$.

If A is a subset of X then $X \setminus A$ is sometimes called the **complement** of A. If $X \cap Y = \emptyset$ then X and Y are called **disjoint**.

The **Cartesian product** of two sets X and Y is the set

$$X \times Y = \{(x, y) \,:\, x \in X \text{ and } y \in Y\}.$$

The **power set** of a set X is the set of all subsets of X.

A.3 Relations and Functions

Given two sets X and Y, a **function** $f : X \to Y$ is a rule that assigns to every element $x \in X$ and element $f(x) \in Y$. In this case the set X is called the **domain** of f and the set Y is called the **codomain** of f.

If $A \subset X$, the **image** of A under f is the subset $f(A) \subset Y$ defined by $f(A) := \{f(a) : a \in A\}$. The **range** of f is the set $f(X)$. If $B \subset Y$, the **preimage** of B under f is the subset $f^{-1}(B) \subset X$ defined by $f^{-1}(B) = \{x \in X \ : \ f(x) \in B\}$. Note that the preimage $f^{-1}(B)$ is well-defined regardless of whether f is invertible. The preimage behaves well with respect to set theoretic operations, as opposed to the image.

A function $f : X \to Y$ is called **injective** if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, and is called **surjective** if $f(X) = Y$. A function which is both injective and surjective is called **bijective**. One can rephrase these definitions in terms of the preimage of sets. Indeed, let $A \subset X$ and $B \subset Y$. Then

- $A \subset f^{-1}(f(A))$, with equality if f is injective.
- $f(f^{-1}(B)) \subset B$, with equality if f is surjective.

Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be functions. The **composition** $g \circ f$ is defined by $(g \circ f)(x) = g(f(x))$.

A.4 Collections of sets

The remainder of this section will explain how to deal with collections of sets, and in particular how to take the intersection and union of these.

Definition A.1. Let X and Λ be two sets such that $\Lambda \neq \emptyset$. For each $\lambda \in \Lambda$, let A_λ be a subset of X. Then the collection of all sets A_λ with $\lambda \in \Lambda$ is called an *indexed family of subsets of* X, and is denoted $\{A_\lambda\}_{\lambda\in\Lambda}$ or $\{A_\lambda \ : \ \lambda \in \Lambda\}$. The set Λ is called the **indexing set**.

Definition A.2. Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be an indexed family of subsets of X. The **union** of this family is the set

$$\bigcup_{\lambda\in\Lambda} A_\lambda := \{x \in X \ : \ \text{there exists } \lambda \in \Lambda \text{ such that } x \in A_\lambda\}.$$

The **intersection** of this family is the set

$$\bigcap_{\lambda\in\Lambda} A_\lambda := \{x \in X \ : \ \text{for every } \lambda \in \Lambda, \ x \in A_\lambda\}.$$

Remark A.3. Actually, if Λ is uncountable, we need the axiom of choice to even make sense of $\bigcap_{\lambda\in\Lambda} A_\lambda$, but let's ignore this subtle point.

Remark A.4. The following remarks follow immediately from the definition:

- If Λ is a finite set, say $\Lambda = \{1, 2, \ldots, n\}$, then

$$\bigcup_{\lambda\in\Lambda} A_\lambda = \bigcup_{\lambda=1}^{n} A_\lambda = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcap_{\lambda\in\Lambda} A_\lambda = \bigcap_{\lambda=1}^{n} A_\lambda = A_1 \cap A_2 \cap \cdots \cap A_n.$$

- $\bigcup_{\lambda\in\Lambda} A_\lambda$ and $\bigcap_{\lambda\in\Lambda} A_\lambda$ are both subsets of X.

- For every $\mu \in \Lambda$,

$$U_\mu \subset \bigcup_{\lambda \in \Lambda} U_\lambda, \quad \text{and} \quad \bigcap_{\lambda \in \Lambda} U_\lambda \subset U_\mu.$$

Example A.5. $X = \mathbb{R}$, $\Lambda = \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, $A_n = \big(-\frac{1}{n}, \frac{1}{n}\big)$

$$\bigcup_{n \in \mathbb{Z}^+} A_n = (-1, 1), \qquad \bigcap_{n \in \mathbb{Z}^+} A_n = \{0\}.$$

Example A.6. $X = \mathbb{R}^2$, $\Lambda = (0, \infty)$, $A_\lambda = \{(x, y) \in \mathbb{R}^2 \, : \, x^2 + y^2 = \lambda^2\}$

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \mathbb{R}^2 \backslash \{(0,0)\}, \qquad \bigcap_{\lambda \in \Lambda} A_\lambda = \emptyset$$

Proposition A.7 (De Morgan's Laws)**.** *Let A_λ be any indexed family of subsets of X. Then*

$$X \setminus (\bigcup_{\lambda \in \Lambda} A_\lambda) = \bigcap_{\lambda \in \Lambda} (X \setminus A_\lambda) \tag{DM1}$$

$$X \setminus (\bigcap_{\lambda \in \Lambda} A_\lambda) = \bigcup_{\lambda \in \Lambda} (X \setminus A_\lambda). \tag{DM2}$$

Proof. The proof of (DM1) follows from elementary operations in logic:

$$\begin{aligned} x \in X \setminus (\bigcup_{\lambda \in \Lambda} A_\lambda) &\Leftrightarrow x \in X \text{ and } x \notin \bigcup_{\lambda \in \Lambda} A_\lambda \\ &\Leftrightarrow x \in X \text{ and } \forall \lambda \in \Lambda \; x \notin A_\lambda \\ &\Leftrightarrow \forall \lambda \in \Lambda \; (x \in X \text{ and } x \notin A_\lambda) \\ &\Leftrightarrow \forall \lambda \in \Lambda \; x \in X \setminus A_\lambda \\ &\Leftrightarrow x \in \bigcap_{\lambda \in \Lambda} X \setminus A_\lambda \end{aligned}$$

The proof of (DM2) is left as an exercise. □

Proposition A.8. *Let $f : X \to Y$ be a function, $\{A_\lambda \, : \, \lambda \in \Lambda\}$ be an indexed family of subsets of X and $\{B_\gamma \, : \, \gamma \in \Gamma\}$ be an indexed family of subsets of Y. Let $S \subset X$, $T \subset Y$. Then*

(a)	$f^{-1}(\bigcup_{\gamma \in \Gamma} B_\gamma) = \bigcup_{\gamma \in \Gamma} f^{-1}(B_\gamma)$,	*(b)*	$f^{-1}(\bigcap_{\gamma \in \Gamma} B_\gamma) = \bigcap_{\gamma \in \Gamma} f^{-1}(B_\gamma)$,
(c)	$f(\bigcup_{\lambda \in \Lambda} A_\lambda) = \bigcup_{\lambda \in \Lambda} f(A_\lambda)$,	*(d)*	$f(\bigcap_{\lambda \in \Lambda} A_\lambda) \subset \bigcap_{\lambda \in \Lambda} f(A_\lambda)$,
(e)	$f^{-1}(Y \setminus T) = f^{-1}(Y) \setminus f^{-1}(T)$,	*(f)*	$f(X \setminus S) \supset f(X) \setminus f(S)$.

Proof. See exercise sheet 1. □