2.1.7 Irreducible content of TL representations

Recall that quite generally the simple modules ($\{L_\lambda : \lambda \in \Lambda\}$, say) of an algebra are a basis for the Grothendieck group. This means that the character of rep ρ determines its irreducible content.

How does this work in practice?

The character of ρ is the map from the algebra to scalars given by trace. Given the basis theorem above the character is determined by the images of $|\Lambda|$ elements that are independent in this sense. Picking such a subset of elements, the character becomes a vector. We then have

$$\chi_\rho = \sum_\lambda m_\lambda \chi_\lambda$$

where m_λ is the multiplicity.

Example: The characters of TL standard modules are easy. Recall that the index set in case n (n strings) is the set $(n), (n-1,1), (n-2,2), ...$ There are roughly $n/2$ of these and it follows that a suitable independent set of elements is $1, U_1, U_1U_3, ...$ We have

$$\chi_\lambda = \begin{pmatrix} tr_\lambda(1) \\ tr_\lambda(U_1) \\ ... \end{pmatrix}$$

For example, setting $\delta = \sqrt{Q}$, the characters of the standard modules are

$$\chi_{(4)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \chi_{(3,1)} = \begin{pmatrix} 3 \\ \sqrt{Q} \\ 0 \end{pmatrix}, \qquad \chi_{(2,2)} = \begin{pmatrix} 2 \\ \sqrt{Q} \\ Q \end{pmatrix}$$

Meanwhile

$$\chi_{Potts} = \begin{pmatrix} Q^2 \\ \sqrt{Q}Q \\ Q \end{pmatrix}$$

so we have

$$m_{(4)} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + m_{(3,1)} \begin{pmatrix} 3 \\ \sqrt{Q} \\ 0 \end{pmatrix} + m_{(2,2)} \begin{pmatrix} 2 \\ \sqrt{Q} \\ Q \end{pmatrix} = \begin{pmatrix} Q^2 \\ \sqrt{Q}Q \\ Q \end{pmatrix}$$

This is easy to solve. Exercise.

We get $m_{(4)} = Q^2 - 3Q + 1$, $m_{(3,1)} = Q - 1$ and $m_{(2,2)} = 1$. It is interesting to consider some specific cases.

In case $Q = 3$ we get mutliplicities 1,2,1, which have nice physical interepretations.

In case $Q = 2$ we get -1,1,1. This requires mathematical interpretation! The point is that we used characters for standard modules, and these are not the simple modules in this case. Specifically we already know from [89] that the (2,2) standard is not simple. Indeed it contains (4) as a submodule. Thus we have (from the minus 1) the cancellation we need! We get only the simple part of the standard module.