Geometric partition categories: On short Brauer algebras and their blob subalgebras

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Abstract: In Part 1 we prove the Conjecture in Remark 6.7 of [13]. The main result here is thus an algebra(/linear category) isomorphism between a geometrically defined subcategory $\mathcal{J}_0^1$ of a low-height (short) Brauer category $\mathcal{J}_0$ and a certain one-parameter specialisation of the blob category $\mathfrak{b}$. We also define a sequence of generalisations $\mathcal{J}_{i-1}^i$ of the category $\mathcal{J}_0^1$.

Keywords: diagram algebra, topological spin chain.

1 Introduction

The motivating aim here is to study the structure of the k-linear categories $\mathcal{J}_l$ from [13], and in particular the representation theory of the corresponding k-algebras (with k a field) $\mathcal{J}_{l,n}$ in the non-semisimple cases. These structures are of intrinsic interest (cf. [11, 12, 19, 8]); and see also [13] for a discussion of some of the extrinsic motivations for this study — in short one seeks generalisations of the intriguing examples of Kazhdan–Lusztig theory [14, 25, 2] observed [19] in the representation theory of the Brauer category $\mathcal{B} = \mathcal{J}_\infty$ [6].

The study strategy in Part 1 (§2-4) is to try to relate the problem to the representation theory of the blob category $\mathfrak{b}$ and the blob algebra $\mathfrak{b}_n$ [20], which is contrastingly very well understood (see e.g. [9]), itself with deep and tantalising connections to Kazhdan–Lusztig theory [21, 5]. This also allows us to make contact with the original physical motivations for these algebras, as the algebras of physical systems with boundaries and interfaces [20].

As we shall see, in the simplest non-trivial case the algebras are (at least) related by inclusion $\mathfrak{b}_m \hookrightarrow \mathcal{J}_{0,n}$. Inclusion is not in general a directly helpful relationship in representation theory. (For example the Temperley–Lieb algebra T_n [26] is a subalgebra of $\mathfrak{b}_n$, but the representation theories of these algebras are radically different: cf. [17] and [9].) However the inclusion here is of high index, so there is hope that it will indeed shed light on the open problem. In Part 2 (§5) we include some indicative results on $\mathcal{J}_{0,n}$ representation theory. These are obtained by working directly with $\mathcal{J}_{0,n}$, but serve as a first step in this direction (analysis of these results is demoted to a separate paper).

In Section 2 we introduce concepts and notations. In §3 we define for each category $\mathcal{J}_l$ a new subcategory. In §4 we examine the relationship to the blob category. In particular in Section 4 we state and prove the main theorem. In section 5 we consider consequences for the algebras $\mathcal{B}_n^l = \mathcal{J}_{l,n}$ themselves. In Section 6 we discuss several further interesting open problems.

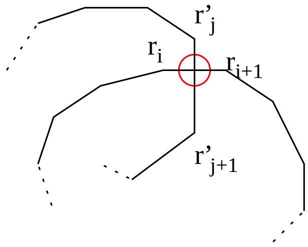


Figure 1: A regular intersection between lines (circled).

2 Preliminary definitions

Define $\underline{n} = \{1, 2, ..., n\}$, $\underline{n}' = \{1', 2', ..., n'\}$ and so on. Define $V^n_m = \underline{n} \cup \underline{m}'$. Write $J(n, m)$ for the set of set partitions of $\underline{n} \cup \underline{m}'$ into subsets of order 2. Fix a commutative ring k and $\delta \in k$. In §2.1-2.2 we recall the Brauer partition category

$$\mathcal{B} = (\mathbb{N}_0, kJ(n, m), *)$$

with loop-parameter δ. The category $\mathcal{B}$ has an infinite family of subcategories $\mathcal{J}_l$ introduced in [13]. The key new ingredient in [13] is the definition of the *left-height* of a partition. We recall it here in §2.3. In §2.4, we recall the blob category [20].

2.1 Brauer pictures

We need to recall the definition of multiplication $*$ in $\mathcal{B}$ and relate to 'pictures' of partitions.

(**2.1**) A *line* or *polygonal path* in the plane $\mathbb{R}^2$ is a subset, denoted (r), consisting of an ordered list of finitely many points $r^1, r^2, ..., r^l$ and the straight lines $[r^i, r^{i+1}]$ between each pair r^i, r^{i+1}, such that each point r^i lies in at most two straight lines.

(**2.2**) Let $R \subset \mathbb{R}^2$ be a rectangle, with frame ∂R and interior $(R) = R \setminus \partial R$. A set of lines is *regular in* R if:
(R1) each line $(r) \subset R$ touches ∂R only if $r^1 \neq r^l$ and then only at its end-points r^1, r^l;
(R2) the point-list $r^1, r^2, ..., r^l$ of each line has no intersection with any other line. See Fig. 1.

(**2.3**) Remark. Given a regular set of lines D, consider the subset $p(D)$ of R defined by D. Note that we can recover the points of D from $d = p(D)$ except for those points r^i that are colinear with their neighbours r^{i-1}, r^{i+1}. Note that colinearity is not a generic condition. The fibre $p^{-1}(d)$ of regular sets of lines over d includes a representative D' in which no line has an i with r^{i-1}, r^i, r^{i+1} colinear. The fibre consists of sets obtained by inserting such colinear points in lines.

Note that d completely determined D up to such inserted colinear points.

(**2.4**) A *Brauer picture* d of a partition p in $J(m, n)$ is a rectangle $R \subset \mathbb{R}^2$ with m points, called 'vertices', on the northern edge, and n on the southern edge (as in Figure 2); and a subset of R consisting of a regular set of lines (r) in R as follows. Each line is either a loop ($r^1 = r^l$) or else connects vertices pairwise in accordance with the pairs in p.

See Fig.2 and Fig.3 for examples. Remark: By [22, §6], piecewise linear and piecewise smooth lines are effectively indistinguishable as far as physically drawn figures are concerned.

(**2.5**) Note that the construction ensures a well-defined map π from pictures to partitions: for each $i \in V^n_m$ the line from i in d may be followed unambiguously to the other end; and one takes a line with end-points i, j to $\{i, j\}$.

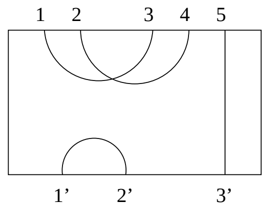


Figure 2: A picture of $\{\{3,1\},\{5,3'\},\{4,2\},\{2',1'\}\} \in J(5,3)$. N.B. Drawings of piecewise-smooth approximations to piecewise-linear embeddings are safe to use here.

(**2.6**) Write $[p]$ for the set of pictures d such that $\pi(d) = p$.

(**2.7**) Fixing commutative ring k and $\delta \in k$, we extend the π-map to $\Pi(d) = \delta^{\#(d)}\pi(d) \in kJ(m,n)$ where $\#(d)$ is the number of loops in the picture d.

For d a picture let $\hat{d}$ denote d with all loops removed. Thus $\Pi(d) = \delta^{\#(d)}\Pi(\hat{d})$.

(**2.8**) Consider pictures d_1 for $p_1 \in J(m,n)$ and d_2 for $p_2 \in J(n,q)$.

We have not specified exactly where the n vertices lie on the southern frame of R in d_1 (or indeed where the southern frame lies in $\mathbb{R}^2$), but it will be clear that there are representative pictures of p_1, p_2 for which the n vertices from d_1 match up with the n from d_2. These pictures can then be stacked (with d_1 over d_2) so that the n vertex sets in each 'factor' coincide. Note that the concatenation is again a Brauer picture. It is denoted by $d_1 \,|\, d_2$.

Note then (1) that $d_1|d_2$ can be seen as a picture of an element of $kJ(m,q)$;
(2) that if d_2 and d_3 also stack then

$$(d_1|d_2)|d_3 = d_1|(d_2|d_3) \tag{1}$$

2.2 The Brauer partition category $\mathcal{B}$

(**2.9**) Let $p_1 \in J(m,n)$ and $p_2 \in J(n,q)$. Given a set p of symbols, let p' be the set obtained by adding primes to symbols in p (note that this can work recursively). Now for p_1, p_2 above, form $p_1 \cup p_2'$. Note each single-primed element appears twice and others once. Consider a maximal chain $\{a_0,a_1\},\{a_1,a_2\},...,\{a_{k-1},a_k\}$ in $p_1 \cup p_2'$. (It is helpful to note that this corresponds to the vertices in a connected component of $d_1|d_2$ (for *any* suitable choice of pictures).) The chain is either all primed, and closed, or $\{a_0,a_k\}$ lies in $\underline{m} \cup \underline{q}''$. (This pair corresponds to the vertices at the ends of the connected component of $d_1|d_2$ (again for *any* suitable choice of pictures). Thus the image in $\pi(d_1|d_2)$ if $a_0 \neq a_k$ is $\{a_0,a_k\}$ (more precisely $\{a_0,a_k\}$ with any $i'' \rightsquigarrow i'$).) In this way $p_1 \cup p_2'$ determines an element $p_1.p_2$ of $J(m,q)$; and also a number $\#$ of closed chains. Define

$$p_1 * p_2 = \; \delta^{\#} p_1.p_2 \; \in kJ(m,q).$$

(**2.10**) From the parenthetical notes in (2.9) we have, independently of the choice of pictures,

$$p_1 * p_2 = \Pi(d_1 \,|\, d_2).$$

For example, from Figure 4

$$\{\{4',2\},\{3,5'\},\{1,3'\},\{1',2'\}\} * \{\{2,1\},\{4,5\},\{1',3\}\} \;=\; \delta\{\{3,2\},\{1,1'\}\} \;\in kJ(3,1)\;.$$

Note we use the convention that a picture d_1 for $p_1 \in J(m,n)$ has m (resp. n) points on its northern (southern) edge. Thus $d_1 \,|\, d_2$ is the concatenation of d_1 on top of d_2.

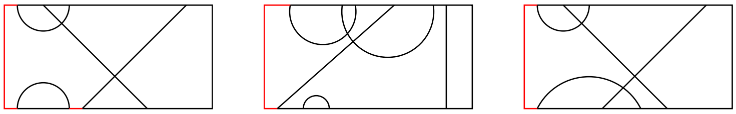


Figure 3: 'Lowest-height' pictures of elements of $J(4,4)$.

(**2.11**) Theorem. [13] *Composition $*$ is associative. This defines the category $\mathcal{B}$.*

Proof. By the parenthetical points in (2.9) above we may obtain $p_1 * p_2$ from $d_1|d_2$ (independently of the choices of these pictures). Associativity follows since $(d_1|d_2)|d_3 = d_1|(d_2|d_3)$. $\square$

(**2.12**) Define $\otimes : J(n,m) \times J(n',m') \to J(n+n', m+m')$ as the composition corresponding to side-by-side concatenation of pictures. Note the following.

(**2.13**) Lemma. *The composition $\otimes$ makes $\mathcal{B}$ into a monoidal category.* $\square$

2.3 The subcategory $\mathcal{J}_l$ of category $\mathcal{B}$

The regularity property (R2) of a picture d means that the number, and position, of crossings of lines in d is well-defined. A *path* through a picture d is a further line in R that satisfies (R2).

Fix a picture d, let x be a point in d, and consider all paths from x to the left edge. Then $h_d(x)$ is the minimum number of crossings with lines of the picture among such paths.

The *left-height* $h(d)$ of a picture d is the maximum $h_d(x)$ of all crossing points x of lines in d; or, if there are no crossings, then $h(d) = -1$.

For $p \in J(n,m)$ let $h(p)$ denote the minimum $h(d)$ among pictures in $[p] = \{d : \pi(d) = p\}$. Define

$$J_{\leq l}(n,m) = \{p \in J(n,m) \mid h(p) \leq l\}$$

Define $[p]'$ as the subset of $[p]$ of pictures of p of the minimum height. That is,

$$[p]' = \{d \in [p] \mid h(d) = h(p)\}.$$

See Fig.3 for examples of minimum height pictures.

As shown in [13], $h(p_1 * p_2) \leq \max(h(p_1), (h(p_2))$. A consequence is the following.

(**2.14**) Theorem. [13] *Category $\mathcal{B}$ has a subcategory $\mathcal{J}_l = (\mathbb{N}_0, kJ_{\leq l}(n,m), *)$.* $\square$

(**2.15**) Given a subset S of a rectangle R, an *alcove* of S is a connected component of $R \setminus S$.

(**2.16**) Remark. Let $d \in [p]$. Note that alcoves of d have well-defined height. Note that the heights of the intervals of the frame of R are determined by p, and are otherwise independent of d.

Proof: Note that there exist paths w from the left edge to points on ∂R such that w lies in a neighbourhood of ∂R. Note that there are such paths that have the lowest number of crossings. By construction two pictures $d, d' \in [p]$ are close to identical in a neighbourhood of ∂R. In particular there are paths to points on ∂R that lie in such a neighbourhood; have the lowest number of crossings; and that have the same number of crossings in d and d'.

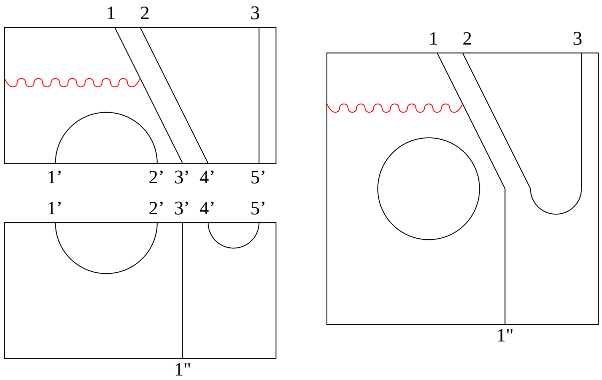


Figure 4: A left-exposed line is a part of a left-exposed line after concatenation.

2.4 The blob category $\mathfrak{b}$

Now we recall the blob category [20]. Note that for $p \in J_{-1}(m,n)$ a pictures d in $[p]'$ has no crossings. Thus for each pair v in p the corresponding line l_v in $d \in [p]'$ has the property that every point x on l_v has the same value of $h_d(x)$. Furthermore this $h_d(l_v) = h_d(x)$ depends only on v and not on d. Thus v has a well-defined $h(v) = h_d(l_v)$. A *left-exposed* pair in $p \in J_{-1}(m,n)$ is a pair v with $h(v) = 0$. (That is, there exists a path from the left edge of R to l_v, which path does not intersect any line of d. The line l_v is literally left-exposed.)

(**2.17**) Lemma. *Let $p_1 \in J_{-1}(m,n)$ and $p_2 \in J_{-1}(n,q)$, and $d_i \in [p_i]$. Let v be a left-exposed pair in p_1 or p_2 and let l_v be the corresponding line in d_i. Then the line l in $d_1 \,|\, d_2$ containing l_v has $h_d(l) = 0$.*

Proof. A path connecting the line l_v to the left edge without intersection in d_1 or d_2 also connects the line in $d_1|d_2$ which contains l_v to the left edge without intersection (cf. Fig.4). □

(**2.18**) For $p \in J_{-1}(m,n)$ let S_p^L denote the subset of left-exposed pairs. Define

$$\mathsf{J}^\bullet(m,n) \;=\; \{(p,s) \,|\, p \in J_{-1}(m,n),\; s \subset S_p^L\} \tag{2}$$

(**2.19**) A picture with blobs is a pair (d,b) where d is a picture in the sense of (2.4); and b is a set of points (called *blobs*) in the interiors of lines of d. We require that each blob lies in exactly one line (if $d \in [p]'$ with $p \in J_{-1}$ then lines here are non-crossing and this is automatic).

(**2.20**) Given a picture with blobs (d,b) we define

$$\pi' : (d,b) \mapsto (\pi(d), s(b)) \;\in \mathsf{J}^\bullet(m,n)$$

where π is as in (2.5) and $s(b)$ is the set of pairs associated to the lines decorated by b.

(**2.21**) An element $(p,s) \in \mathsf{J}^\bullet(m,n)$ can be represented by a pair (d,b), where d is a no-loop picture of p; and b consists of at least one point in the interior of each line of d corresponding to a pair in s. See Fig.5.

For an example note that Figure 5 is a picture $(d, \{P_1, P_2, P_3\})$ for an element

$$(p,s) = (\{\{2',1'\},\{9,10\},\{3,4\},\{7',5\},\{1,2\},\{8,8'\},\{3',6'\},\{5',4'\},\{6,7\}\},\{\{5,7'\},\{1,2\},\{6',3'\}\})$$

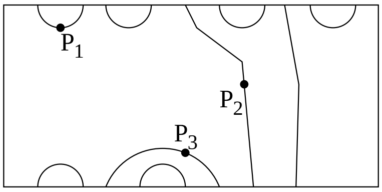


Figure 5: An illustrative blob picture.

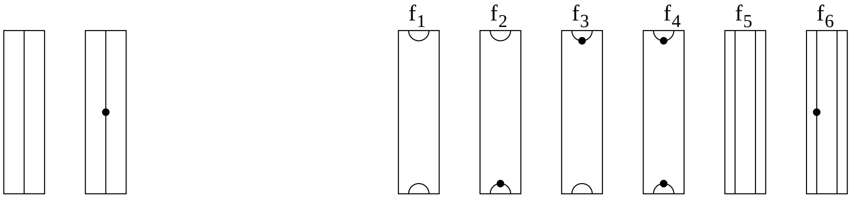


Figure 6: (a) The basis $\mathsf{J}^{\bullet}(1,1)$. (b) The basis $\mathsf{J}^{\bullet}(2,2)$.

$\in \mathsf{J}^{\bullet}(10,8)$, where the three blobs $P_i, i = 1,2,3$ are mapped to the left exposed pairs $\{2,1\}$, $\{5,7'\}$, $\{6',3'\}$, respectively.

(**2.22**) We fix $\delta, \delta' \in k$ and define

$$\Pi(d,b) \;\; := \;\; \delta^{\#}\delta'^{\#'}\;\pi'(d,b) \;\;\in k\mathsf{J}^{\bullet}(m,n),$$

where $\# = \#(d,b)$ (respectively $\#' = \#'(d,b)$) is the number of loops without (respectively with) blobs.

(**2.23**) Let $(p_1,s_1) \in \mathsf{J}^{\bullet}(m,n)$ and $(p_2,s_2) \in \mathsf{J}^{\bullet}(n,q)$. Let $d_i \in [p_i]'$ and (d_i,b_i) be no-loop pictures with appropriate blobs (hereafter we just write d_i, including blobs). By Lem.2.17 the concatenated picture $d_1 \,|\, d_2$ is a picture of some $p \in J(m,q)$ plus possible loops and blobs.

Recall $p_1.p_2$ and $p_1 * p_2$ from (2.9). Define $\overline{s_1 s_2}$ to be the set of those pairs $\{a_0,a_k\}$ in $p_1.p_2$, from chains $\{a_0,a_1\},\{a_1,a_2\},\ldots,\{a_{k-1},a_k\}$ where at least one pair comes from $s_1 \cup s_2$. Define $\#$ as the number of closed chains with no pair from $s_1 \cup s_2$, and $\#'$ as the number of remaining closed chains.

Define the composition $\circ : \mathsf{J}^{\bullet}(m,n) \times \mathsf{J}^{\bullet}(n,q) \to k\mathsf{J}^{\bullet}(m,q)$ by

$$(p_1,s_1)\circ(p_2,s_2) = \;\delta^{\#}\delta'^{\#'}(p_1.p_2, \overline{s_1 s_2}).$$

By an argument similar to (2.10) we have, independently of choices,

$$(p_1,s_1)\circ(p_2,s_2) = \;\Pi(d_1|d_2).$$

(**2.24**) Theorem. *Fix a commutative ring k and $\delta,\delta' \in k$. Then $\mathfrak{b} = (\mathbb{N}_0, k\mathsf{J}^{\bullet}(n,m), \circ)$ is a category.*

Proof. We require to prove associativity of the product, and this follows analogously to (2.10) and (2.11). □

(**2.24.1**) Corollary. *The End sets* $k\mathsf{J}^\bullet(m,m)$ *have the structure of an associative algebra.*

We may denote these algebras by $J_m^{\bullet\,\delta,\delta'}$, indicating the fixed parameters from k in the definition of the multiplication only when needed for clarity.

(**2.25**) Examples. Consider Fig.6(a,b). Applying the rules of multiplication we have e.g., $1e = e1 = ee = e$, $f_1^2 = \delta f_1$, $f_2^2 = \delta' f_2$, $f_2 f_1 = f_2 f_3 = \delta' f_1$, $f_6 f_1 = f_6 f_3 = f_3$ and so on.

2.5 Generators and relations, etc.

(**2.26**) Theorem. [20, 18] *Consider the algebra defined by generators* $U^e = \{e, U_1, U_2, ..., U_{n-1}\}$ *and relations*
$\tau' = \{\ U_i^2 = \delta U_i,\ U_i U_{i\pm 1} U_i = U_i,\ U_i U_j = U_j U_i,\ j \neq i \pm 1,\ ee = e,\ U_1 e U_1 = \delta' U_1,\ U_i e = e U_i,\ i > 1\ \}$.
(I) The map

$$U_i \mapsto \{\underbrace{\{\{1,1'\}\{2,2'\},...,\{i,i+1\},\{i',i+1'\},...,\{n,n'\}\}}_{p},\underbrace{\ \emptyset\ }_{s}\}$$

$$e \mapsto \{\underbrace{\{\{1,1'\},\{2,2'\},...,\{n,n'\}\}}_{p},\underbrace{\{\{1,1'\}\}}_{s}\}$$

extends to an algeba isomorphism $k\langle U^e\rangle/\tau' \cong \mathfrak{b}_n$.
(II) Every element of the partition basis $\mathsf{J}^\bullet(n,n)$ *can be expressed as a word in these generators. In particular* $(p,s) \in \mathsf{J}^\bullet(n,n)$ *can be expressed as a word in which* e *appears as a factor* $|s|$ *times.* □

2.6 Aside on generator algebras and spin chain Physics

The original motivation for the blob algebra [20] was to study XXZ spin chains (and other related spin chains [1, 16]) with various boundary conditions via representation theory. In the simplest formulation (see [24, 15, 20] for details) one notes that there are boundary conditions for which the n-site XXZ chain Hamiltonian may be expressed in the form

$$H = \sum_{i=1}^{n-1} U_i$$

where U_i acts on $(\mathbb{C}^2)^n$ by

$$U_i = 1_2 \otimes 1_2 \otimes ... \otimes \begin{pmatrix} 0 & & & \\ & q & 1 & \\ & 1 & q^{-1} & \\ & & & 0 \end{pmatrix} \otimes 1_2 \otimes ... \otimes 1_2$$

It is known that these matrices give a faithful representation of the Temperley–Lieb algebra. In this sense the algebra controls the spectrum of H. A centraliser algebra is $U_q sl_2$, and so this can equivalently be seen as controlling the spectrum.

There are several reasons for wanting to generalise away from this particular choice of boundary conditions. For example: (1) periodic boundary conditions may be desirable for reasons of computability or to minimise boundary effects at finite size. (2) one may be interested in critical bulk physics in the presence of a doped boundary. (3) one may be interested in physics on the

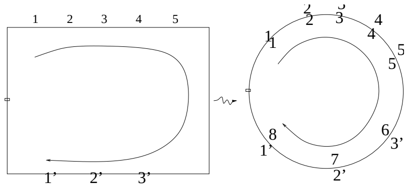


Figure 7: Disk order on vertices.

surface of a bulk system (typically perhaps a 2D surface in 3D, but *modelled* more simply by a thickened finite interval at the end of an infinite line).

The generalisation required for the periodic case requires quite delicate tuning — see [20]. But the simplest form of such generalisation is simply to modify the first or last operator in the chain:

$$H' = U'_0 + \sum_{i=1}^{n-1} U_i$$

where U'_0 acts only on the first tensor factor (and then of course to take the thermodynamic limit) [15]. The corresponding extension of the Temperley–Lieb algebra is covered by the blob algebra. Another challenge is to dope with a more complex operator at the boundary. Algebraically this generalisation can get difficult quite quickly. A version which at least lies within the Brauer algebra is to have a Temperley–Lieb chain with some permutation operators at the end. That is, one first considers the Brauer algebra B_n as the algebra generated by its sub-symmetric group and Temperley–Lieb Coxeter generator elements

$$B_n = \langle \sigma_i, U_i \; : \; i = 1, 2, ..., n-1 \rangle \tag{3}$$

Note that this is not a minimal generating set. For example, all but one U_i can be discarded.

(**2.27**) Trivially one can then define for each n a 'Coxeter subalgebra'

$$B_{l,m,n} = \langle \sigma_i, \; : \; i = 1, 2, ..., l-1, \;\; U_i, \; : \; i = m, m+1, ..., n-1 \rangle$$

It is clear that various values of l, m reduce to known cases. For example if $m > l$ then we just have a product of S_l and T_{n-m}. So the interesting cases are $m \leq l$. ...

The Hamiltonians for such systems have been considered [4], but in the present work we focus on the abstract algebraic aspects.

It is clear that $B_{l,1,n} \hookrightarrow \mathcal{J}_{l,n}$; and that $B_{2,2,n} \hookrightarrow \mathcal{J}^1_{0,n}$. It is conjectured that these inclusions are isomorphisms. ... And in this spirit we can ask about a geometrical and categorical characterisation of $B_{l,l,n}$.

(**2.28**) The *disk order* on elements of $\{1, 2, ..., m\} \cup \{1', 2', ..., n'\}$ is given by renumbering $i' \mapsto m+n+1-i$. In our convention for pictures of p this is clockwise order on the topological marked disk R — see fig.7. For $\{i, j\}$ a pair in $p \in J(m, n)$, with $i < j$ in the disk order, we understand by $[i, j]$ the interval from i to j with respect to the disk order.

3 The subcategory $\mathcal{J}^i_{i-1}$ of $\mathcal{J}_{i-1}$

(**3.1**) Let S be a totally ordered set and p a partition of S into pairs (we have in mind the disk order as in (2.28)). Via the total order, the restriction of p to any two pairs induces a partition

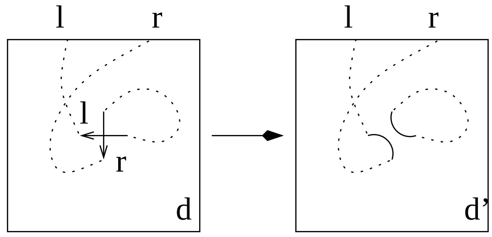


Figure 8: Schematic for removing a line self-crossing in a picture.

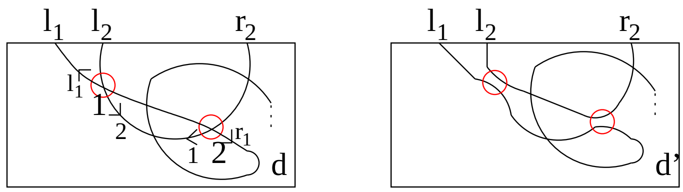


Figure 9: Cancellation: Small neighbourhoods of two crossings of two lines in a picture $d \in [p]$ modified such that the resulting picture satisfies $d' \in [p]$ with two fewer crossings.

of $\{1,2,3,4\}$. The set of such partitions is $\{\{\{1,3\},\{2,4\}\},\ \{\{1,2\},\{3,4\}\},\ \{\{1,4\},\{2,3\}\}\}$.

A pair in p is *crossing in* p if there is another pair in p such that the partition induced by the restriction of p to these two pairs is $\{\{1,3\},\{2,4\}\}$.

Remark: The point of this terminology is that if the total order is the disk order then in every picture d of p the line for the 'crossing' pair must cross another line. (This follows from the Jordan curve theorem [22].)

3.1 The crossing number χ_p of a partition

(**3.2**) We denote the number of crossings of a picture $d \in [p]$ by $\#^c(d)$

(**3.3**) Let p be a Brauer partition and let $\{\{l_1, r_1\}, \{l_2, r_2\}\}$ be two pairs in p. Note by the Jordan Curve Theorem that if precisely one of l_2, r_2 lies in the (disk)-interval from l_1 to r_1 then a picture of p must have at least one crossing of the lines corresponding to these two pairs — we say the two pairs 'cross'. We thus have a lower bound on the number of crossings in a picture d of p:

$$\# \geq \chi_p$$

where

$$\chi_p = \sum_{\{\{l_1,r_1\},\{l_2,r_2\}\}\subset p} \delta(\{\{l_1, r_1\}, \{l_2, r_2\}\})$$

where the sum is over pairs of pairs and $\delta(-)$ is 1 if they cross and zero otherwise.

(**3.4**) Lemma. *Consider $p \in J(n,m)$. There exist pictures of p achieving the minimum χ_p.*

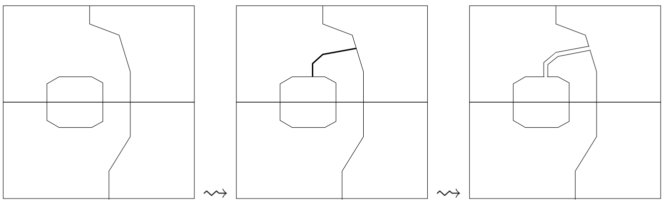
Proof. First consider a picture $d \in [p]$ with a line self-crossing. Then note from (2.1) that removing the open segment of the line corresponding to the loop starting and ending at the self-crossing produces another picture d°, such that $\pi(d^\circ) = \pi(d)$. Next, see Fig.9. It shows that whenever there are two crossings of the same two lines in $d \in [p]$ these crossing neighbourhoods may be 'cancelled' to make $d' \in [p]$ with two fewer crossings. $\square$

(**3.5**) LEMMA. *Consider $p \in J(n,m)$. There exist low-height pictures of p with χ_p crossings.*

Proof. We claim that we do not increase the height by either $d \rightsquigarrow d^\circ$ or by $d \rightsquigarrow d'$ in (3.4) above. The first is due to Lemma 2.31 of [13]. For $d \rightsquigarrow d'$ we proceed as follows. Consider a crossing x of d that remains a crossing in d'. Consider a specific path in d from x to the left edge. Note that this path is also a path in d' provided that the differing neighbourhoods are taken small enough so that the path avoids them. Therefore the number of crossings of the path with segments of the picture is identical in d'. This puts an upper bound on the height of this crossing point in d'. Since this applies for each crossing point, the height of d' is bounded above by the height of d. In particular if d is low-height then so must d' be. $\square$

(**3.6**) Let $[p]''$ denote the set of low height pictures of p with minimum number of crossings.

(**3.7**) The following procedure will be useful later. Given a picture d for a non-empty non-crossing partition p where d has a loop, as when a loop appears in composition, we can modify the picture by a 'garden path' as here:



so the new picture is non-crossing, for the same p, and has no loop.

3.2 $L1$-chain partitions

(**3.8**) Consider the underlying set of a set partition, equipped with a total order — for example recall the disk order on vertices as in (2.28).

Let p be a Brauer partition. An ordered subset $\{\{l_1, r_1\}, \{l_2, r_2\}, ...\{l_j, r_j\}\}$ of p is called a *chain* (of length j) if $l_{i-1} < l_i < r_{i-1} < r_i$ for all i.

(**3.9**) Note that a chain c has a 'canonical' (isotopy class of) picture in which the lines for each successive pair of links cross exactly once. Indeed this is the unique χ-minimal class.

(**3.10**) A chain c in a partition p divides ∂R (and the corresponding ordered vertex set) into intervals of height 0,1,2 with respect to this chain only — denoted ht_c. In particular the sequence of these heights from the top-left of ∂R is

$$seq_c \; = \; 01(21)^{j-1}0$$

In the example in fig.10(a) the hts are given for the blue (thin line) chain only.

These heights agree with the heights of point in ∂R in a canonical picture d in the usual sense, but once c in p is given then we may also consider them as invariants of p — see (2.16).

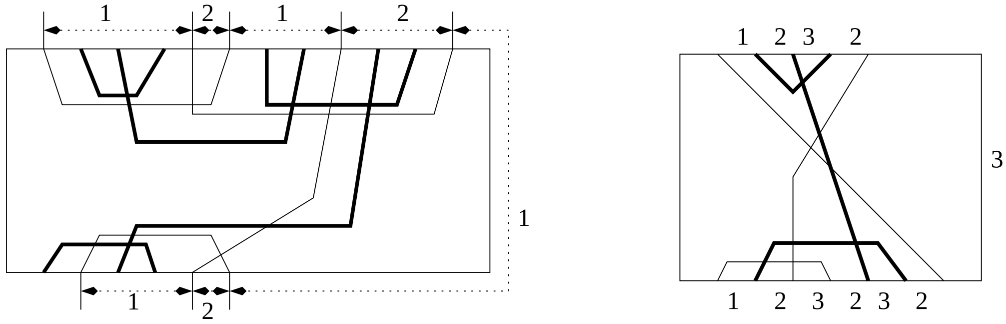


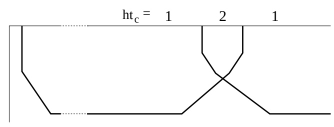
Figure 10: (a),(b) Pictures of pair-partitions decomposable as two chains.

($\mathbf{3.11}$) A partition p is called $L1$-*chain* if $ht(p) \leq 0$ and there is a chain with $l_1 = 1$ and $r_j = 1'$.

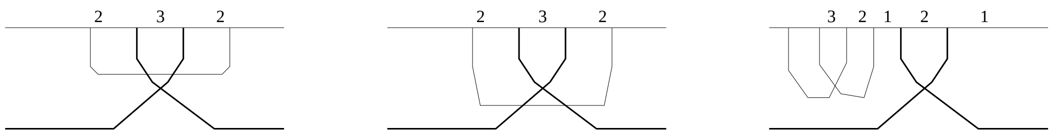
($\mathbf{3.12}$) A partition p is called $L1$-*simple* if there is a chain with $l_1 = 1$ and $r_j = 1'$.
Write $J^1(n,m) \subset J(n,m)$ for the set of $L1$-simple partitions.

($\mathbf{3.13}$) Lemma. *Every $L1$-chain partition p consists of a unique chain from 1 to $1'$ together with a collection of pairs that are non-crossing in p.*

Proof. Firstly p has at least one chain from 1 to $1'$ by definition. Pick one such, and start with a canonical picture d_c of this chain c and the alcoves it defines. Schematically an interval of the chain looks like:



Consider a pair in p that is not part of the chain. By (3.10) a path for this pair, in any picture extending d_c, starts in an alcove of ht_c value 1 or 2. Such a path itself defines at least one new alcove, so the true heights in some region over the corresponding interval are 2 or 3. One sees:



that such a pair, if it is crossing in p, must give rise to a crossing touching height 3. This contradicts the $ht(p) \leq 0$ condition. Thus a pair in $p \setminus c$ is non-crossing. Finally no non-crossing pair can be part of a 'long' chain, so c is unique. $\square$

($\mathbf{3.14}$) Note that the minimum number of crossings in a picture $d \in [p]$ here is $j - 1$.

($\mathbf{3.15}$) Remark: The definition of $L1$-chain is equivalent to left-simple (as in (**??**)) and height 0.

Proof. It is clear that L-simple implies left-simple. The converse is not true. However finish me.

3.3 Li-chain partitions

(**3.16**) A pair of chains in $p \in J(n,m)$ is *exclusive* if their individual ht 2 intervals do not intersect.
The example in fig.10(a) is not exclusive. The example in fig.10(b) is exclusive.

(**3.17**) ht with respect to a pair of chains together is $ht_{c_1} + ht_{c_2}$. Thus exclusive pair gives hts up to 3, and hence crossing hts up to 1.

Consider an exclusive pair whose initial points are adjacent and whose final points are adjacent. Their combined boundary height sequence is of the form

$$seq_{c_1 \cup c_2} \;=\; 01(23)^l 210$$

for some l. A ht 3 region is necessarily a link region; and all links arise this way. Thus if partition p is an exclusive pair of this form then there is only one way in which it is an exclusive pair.

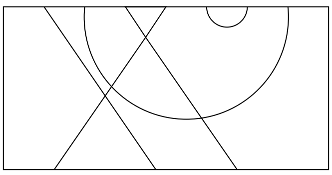
(**3.18**) A *canonical drawing* of an exclusive pair c_1, c_2 has arcs of c_2 that contain ht 2 regions of c_1 drawn *over* the corresponding crossings.

(**3.19**) Define $\#_i : J(n,m) \to \mathbb{N}_0$ so that $\#_i(p)$ is the number of pairs that start in $\{1,2,...,i\}$ and end in $\{1',2',...,i'\}$.

(**3.20**) A partition $p \in J(n,m)$ is *Li-chain* if it has $ht < i$ and i chains that are pairwise exclusive, starting in $\{1,2,...,i\}$ and ending in $\{1',2',...,i'\}$.

N.B. We consider pairs in p that meet both $\{1,2,...,i\}$ and $\{1',2',...,i'\}$ as chains of length 1, and hence trivially exclusive with any other chain.

(**3.21**) Write $J^i_{<i}(n,m)$ or $J^i_{\leq i-1}(n,m)$ for subset of Li-chain partitions.



$$p = \{\underbrace{\{1,2'\}}_{c_1}, \underbrace{\{2,7\},\{4,3'\}}_{c_2}, \underbrace{\{3,1'\}}_{c_3}, \{5,6\}\} \qquad (4)$$

(**3.22**) Example. The figure in (4) above gives a p which is $L3$-chain, in $J^3_{<3}(7,3)$.

(**3.23**) Given an Li-chain decomposition of a Li-chain partition p:

$$p = \cup_{j=1}^i c_j \cup p'$$

where the c_js are pairwise exclusive chains from $[1,i]$ to $[i',1']$, then the restriction to any two chains obeys the $L2$-chain property. Thus the boundary height sequence of p is of the form

$$seq_{\cup_i c_i} \;=\; 012...i(i+1\; i)^l...210,$$

independent of any decomposition. Hence, as in 3.17, the link positions are determined, and the decomposition of p is unique.

(**3.24**) Lemma. *Let $p \in J^i_{i-1}(m,n)$ with $m,n \geq i$, so that $p = \cup_{j=1}^i c_j \cup p'$ as above. Then p' is a set of pairs not crossing with each other or any chain.*

Proof. Noting (3.23), this is analogous to Lem.3.13. ☐

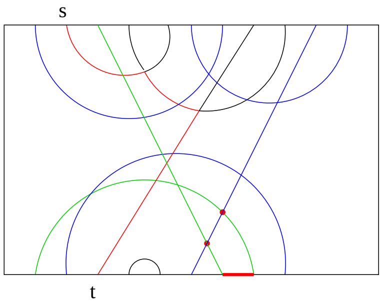


Figure 11: Picture of a partition that is *not* L3-chain (blue and green chains not exclusive).

3.4 A $B_{i+1,i+1,m}$-module of Li-chain partitions

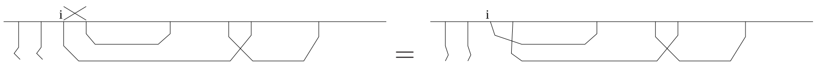
Note that (for $m > i$) the algebra $kJ_{\leq i-1}(m,m)$ has a natural subalgebra isomorphic to the symmetric group algebra kS_{i+1}.

(**3.25**) LEMMA. *The space $kJ^i_{\leq i-1}(m,n)$ is closed under the left action of $B_{i+1,i+1,m}$.*

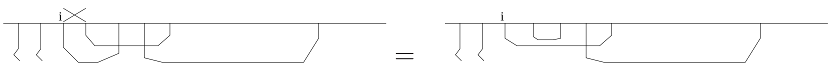
Proof. Consider $p \in J^i_{i-1}(m,n)$, and let $c_1, c_2, ..., c_i$ be the unique chain decomposition of p as in (3.23).

We can write out chains as sequences of pairs in the chain order: $\{j_1, j_2\}, \{j_3, j_4\}, ...$, or even as lists $j_1, j_2, j_3, j_4, ...$

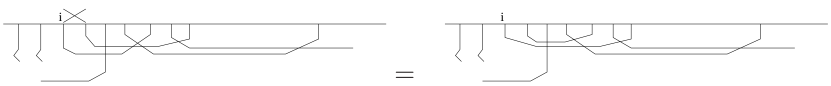
Consider the action of generators $\sigma_1, ..., \sigma_i$ and U_j $(j > i)$ on p as follows:
Case 1: σ_j with $j < i$. This changes only c_j and c_{j+1}, swapping their first terms. Since these terms are adjacent it follows that the Li-chain property is preserved.
Case 2: σ_i. We can subdivide into three possibilities here.
(i) vertex $i+1$ not in any c_j. In this case $i+1$ is in a non-crossing pair by the Li-chain property (Lemma 3.24). Schematically, drawing only chain c_i and the $i+1$ pair:



That is, the partition $\sigma_i p$ has a chain c_i with an extra link. Note that the new exclusive region for this chain comes from the non-crossing part of the original partition. Thus it cannot overlap an exclusive region for any of the undrawn and unchanged chains, and the exclusive property is preserved.
(iii) vertex $i+1$ in c_i. Here there is a chain $\{i, j\}, \{i+1, l\}, \{k, m\}, ...$ This becomes $\{i, l\}, \{k, m\}, ...$ and leaves $\{i+1, j\}$. The first of these is a chain from i. The second a non-crossing pair. Schematically:

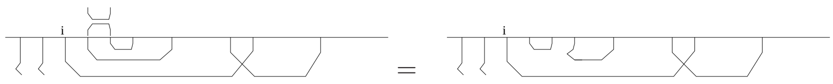


Here one link region is removed and so there is nothing to check for exclusivity of $\sigma_i p$.
(ii) vertex $i+1$ in $c_{j<i}$. Note that $i+1$ cannot be in a pair directly with j since then there is no way to on-link this chain, so it must be the first vertex in the second pair of c_j. So here there is a chain $\{j, k\}, \{i+1, l\}, ...$ and a chain $\{i, r\}, ...$ Schematically:

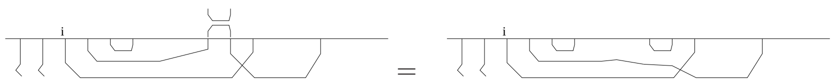


The new chains are $\{i, l\}, ...$ and $\{j, k\}, \{i+1, r\}, ...$. Note that these are chains from the same starting points. It is clear that they are pairwise exclusive; and that pairwise exclusivity with other chains is not affected.

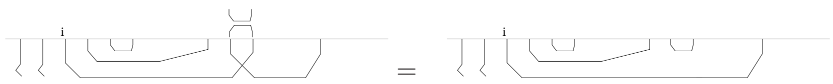
Case 3: U_j with $j > i$. The subcases here are (i) U_j touches no chain c_k. Schematically:



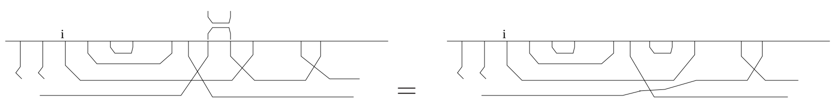
Note here that there is nothing to check for exclusivity of the new partition.
(ii) U_j touches one vertex of a pair in some chain c_k, and no other:



Here note that the size of a link region changes, but only into or out of a non-crossing region. Thus exclusivity is not affected.
(iii) U_j touches some chain c_k at two points:



Here c_k has one fewer link and one fewer link region so exclusivity is preserved.
(iv) U_j touches two chains:



Here two adjacent link regions (for the two touched chains) become combined as a single link region. Since they are adjacent there is no other link region between them in p, and the combination does not affect exclusivity. $\square$

In Appendix B we give explicit examples of σ_j actions in figures with colour-coded chains. If you view in colour they may help to reinforce the Lemma.

3.5 The Chain-basis Theorem

For given k, and $i < m$, let $J^i_{i-1,m} := kJ^i_{\leq i-1}(m,m)$. Recall the Brauer algebra $B_m = (kJ(m,m), *)$ as defined in (3).

(**3.26**) THEOREM. *Fix k and $\delta \in k$. Fix i and consider $m \geq i$. Then $B_{i+1,i+1,m} = J^i_{i-1,m}$ as a k-module and hence as a subalgebra of B_m.*

Proof. It is clear that $(J^i_{i-1,i}, *) = (kS_i, *) = B_{i+1,i+1,i}$. So consider $kJ^i_{i-1,m}$ with $i < m$. Note that the Coxeter generating set of $B_{i+1,i+1,m}$ (from (2.27)) is in $J^i_{i-1,m}$. Thus it is enough to show $J^i_{i-1,m} \subseteq B_{i+1,i+1,m}$.

We work by induction on the number of crossings χ_p. Let

$$J^{i,c}_{\leq i-1}(m,m) := \{p \in J^i_{\leq i-1}(m,m) \mid \chi_p \leq c\}$$

The base of the induction concerns $p \in J^{i,0}_{\leq i-1}(m,m)$. By the Li condition this means that $p = 1_i \otimes p'$ for some non-crossing p'. Then p is generated by $\{U_j, j \geq i\}$ so it is clear that

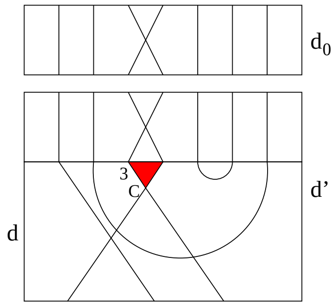


Figure 12: The decomposition of a $d \in [p]''$, $p \in J^3_{\leq 2}(7,3)$ into $d = d_0|d'$, with $j = 3$, $k = 4$. We can strip off some crossings with low i and expose ...

$p \in B_{i+1,i+1,m}$. For the inductive step we assume that $J^{i,c}_{\leq i-1}(m,m) \subseteq B_{i+1,i+1,m}$ and require to show that this implies that $J^{i,c+1}_{\leq i-1}(m,m) \subseteq B_{i+1,i+1,m}$.

Consider $p \in J^{i,c+1}_{\leq i-1}(m,m)$. Note that if the pair from vertex i is non-crossing in p (in the sense of (3.1)) then it must be $\{i, i'\}$ and it is clear that p lies in $B_{i+1,i+1,m}$. So suppose the pair from vertex i is crossing. If this pair crosses a pair from a lower vertex then the pairs from some adjacent pair $j, j+1$ with $j < i$ cross. (Note that the Li-chain condition implies that the pairs starting in $\{1, ..., i\}$ all pass out of this set — to some set $\{p(1), ..., p(i)\}$ say. This permutation may be considered as generated from the non-crossing one by the natural action of S_{i-1} on $\{1, ..., i\}$. Since this action can be expressed in a reduced form in the Coxeter generators it includes, if any crossing, a crossing of an adjacent pair.) In this case $\sigma_j p$ has crossing number c and hence lies in $B_{i+1,i+1,m}$ by assumption. But then $\sigma_j \sigma_j p = p$ also lies in $B_{i+1,i+1,m}$ and we are done.

It remains to consider the case in which the pair from i crosses a pair from some $k > i$. Consider in particular the lowest such k. Call this crossing C^{ik}. By Lem.3.24 the vertices between i and k form a set w of non-crossing pairs.

Consider the partition $\mu = 1_i \otimes w^* \otimes 1 \otimes w \otimes 1_{m-k}$ in B_m. See the middle layer in Fig.13 for an example. Note that μ lies in $B_{i+1,i+1,m}$. But then $\tau = \sigma_i \mu$ lies in $B_{i+1,i+1,m}$ *and* has a single crossing, which is a crossing of the pairs from i' and k'. Hence τ cancels C^{ik} in the partition given by the product τp. (In general there are some loops in τp, but each loop includes pairs from τ, so there is a 'garden path' modification (3.7) $\overline{\tau}$ of τ, also in $B_{i+1,i+1,m}$, so that $\overline{\tau} p$ gives the same partition as τp but without loops.)

By Lem.3.25 we have that τp is in $J^i_{i-1,m}$. By the cancellation we have $\chi_{\tau p} = \chi_p + \chi_\tau - 2 = c+1+1-2$. Thus τp lies in $J^{i,c}_{\leq i-1}(m,m)$ and hence, by the inductive assumption, in $B_{i+1,i+1,m}$. Since τ^* also lies in $B_{i+1,i+1,m}$ we have that $\tau^* \tau p$ lies in $B_{i+1,i+1,m}$. But $\tau^* \tau = 1_i \otimes w \otimes w^* \otimes 1_{m-k+1}$ so $\tau^* \tau p = \delta^{2|w|} p$ — see Fig.13 for an illustration. Unless $\delta = 0$ this directly implies $p \in B_{i+1,i+1,m}$ completing the inductive step. (Alternatively there is an analogue of $\overline{\tau}$ so that $\overline{\tau^*} \overline{\tau} p = p$, which completes the inductive step in general — see Fig.14 for an illustration.) $\square$

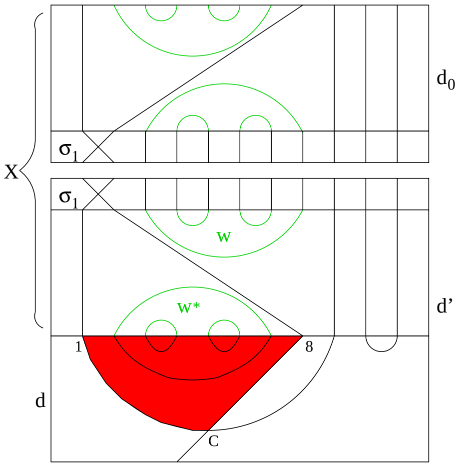


Figure 13: The decomposition $d \sim X|d$ or $d \sim d_0|d'$ with $j = 1$, $k = 8$.

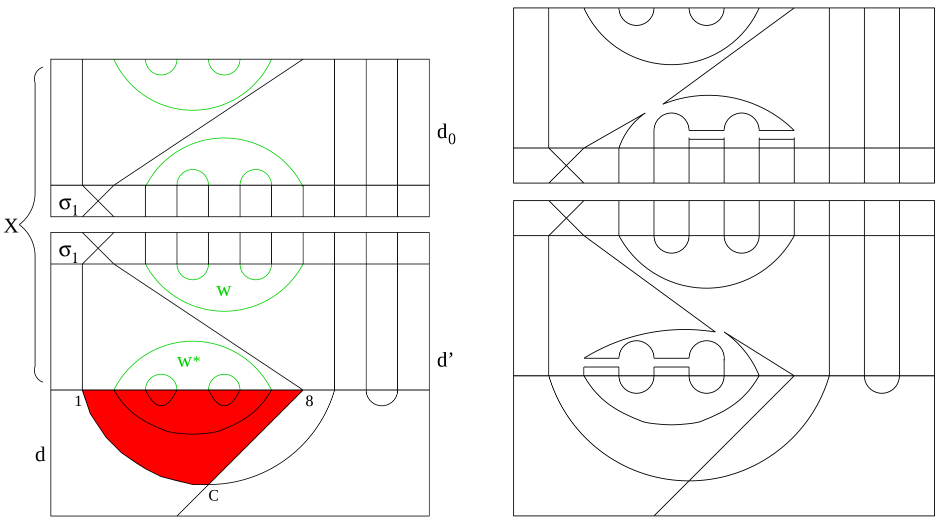


Figure 14: (Right) A picture of the same partition modified such that there are no loops in it.

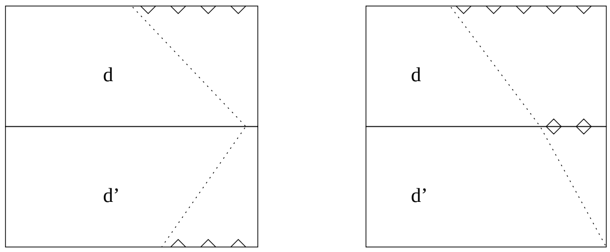


Figure 15: Pictures of $\overline{p}^r$ and $\overline{p'}^r$ from partitions $p \in J(m,n)$ and $p' \in J(n,q)$ with $n > m > o$ (left) and $m > n > o$ (right).

3.6 The category $\mathcal{J}^l_{l-1}$

(**3.27**) Theorem. *There is a subcategory of $\mathcal{J}_{l-1}$ given by*

$$\mathcal{J}^l_{l-1} = (\mathbb{N}_0, kJ^l_{\leq l-1}(m,n), *)$$

Proof. (There are different ways to approach this.) Let us consider the Hom set bases $J^i_{\leq i-1}(m,n)$ and $J^i_{\leq i-1}(n,o)$. We define the composition $J^i_{\leq i-1}(m,n) \times J^i_{\leq i-1}(n,o) \to J^i_{\leq i-1}(m,o)$ by embedding the Hom sets into $J^i_{i-1,r} = B_{i+1,i+1,r}$ with $r = \max(m,n,o)$ as follows. Let u, u^* be the unique elements in $J(0,2), J(2,0)$, respectively. The embedding uses monoidal composition with powers of u, u^*, as illustrated in Fig.15. For $p \in J(s,t)$ and $r \in \mathbb{N}$ define $\theta(x) = \max(x/2, 0)$ and

$$\overline{p}^r = p \otimes u^{\otimes \theta(r-s)} \otimes (u^*)^{\otimes \theta(r-t)} ,$$

whenever the arguments of θ are even.

Now let $p \in J^i_{\leq i-1}(m,n)$, $p' \in J^i_{\leq i-1}(n,o)$. We claim that $\overline{p}^r, \overline{p'}^r \in B_{i+1,i+1,r}$. They are clearly in $J_{\leq i-1}(r,r)$. But the Li-chain property is also preserved under monoidal composition with an element from $J_{-1}(-,-)$ on the right: it corresponds to adding a noncrossing partition in the disk intervals $[m,n']$ of p and $[n,o']$ of p'.

By the same token it is clear that there is a unique $p'' \in J^i_{\leq i-1}(m,o)$ such that

$$\overline{p}^r * \overline{p'}^r \propto \overline{p''}^r .$$

Formally we could define

$$p * p' := \delta^{-\theta(r-n)} p'' . \quad (5)$$

More generally, e.g. when $\delta \notin k^*$, we can eliminate the $\theta(r-n)$ loops from the product by a mild generalisation of $\overline{p}^r$ using suitable garden paths.

Note that any number $r+2w, w \in \mathbb{N}$ can be used above without changing the definition of the product. Using this observation one can explicitly verify associativity by a direct calculation. □

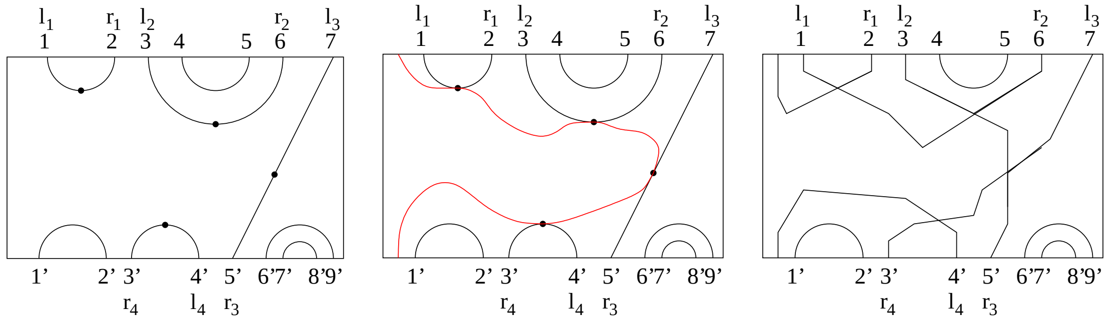


Figure 16: (a) A picture $d \in [p]'$ for $p \in J^\bullet(7, 9)$. (b) Insert a path c into d. (c) Cross at blobs.

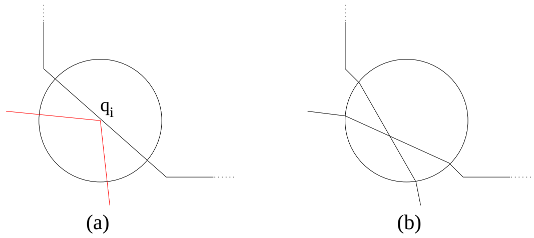


Figure 17: (a): a small neighbourhood of q_i, with the path c in red. (b) reconnection of the line segments in the small neighbourhood according to the map Ψ.

4 The blob isomorphism Theorem

4.1 The initial set map $\Psi : \mathsf{J}^\bullet(m, n) \to J_0^1(m + 1, n + 1)$

Consider $(p, s) \in \mathsf{J}^\bullet(m, n)$, as in §2.4. See Fig.16(a) for a picture d of an example, with $d \in [p]'$. (Note that here such pictures are unique up to *ambient* isotopy.) We can use such a picture d of (p, s) to describe a modification $\Psi(p, s) \in J_0^1(m + 1, n + 1)$ with the following steps:
(1) add new boundary points $0, 0'$ and draw a path between them that touches the lines of d only at each of the blobs, as in Fig.16(b).
(2) noting that each blob now has four line segments incident, replace this with two crossing lines as in Fig.17, to obtain d' (Fig.16(c)).
(3) Set $\Psi(p, s) = \pi(d')$.

(4.1) Lemma. *(I) The procedure Ψ gives a map*

$$\Psi : \mathsf{J}^\bullet(m, n) \to J_0^1(m + 1, n + 1)$$

(II) Ψ is a bijection.

Proof. (I) It will be clear that the procedure gives an element of $J_0^1(m+1, n+1)$. Thus it remains to show that this is independent of the choice of $d \in [p]'$ representing $(p, s) \in \mathsf{J}^\bullet(m, n)$. This may be shown by recasting the procedure at the original set-theoretic level. The details do not illuminate subsequent arguments, so we demote them to Appendix A.

(II) Note that the construction of picture d' in Ψ (at step (2)) is unique up to ambient isotopy and produces a *canonical* picture. Note also that steps (1) and (2) are reversible. By (3.9) we

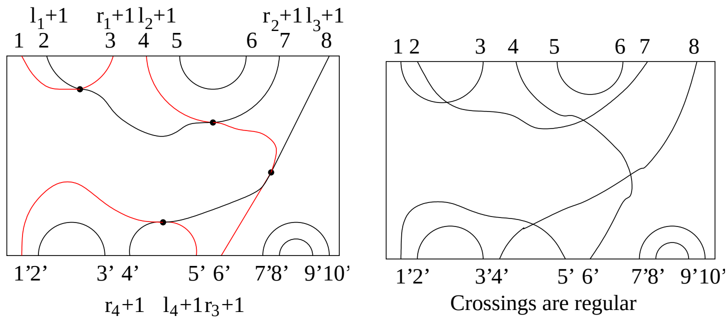


Figure 18: (a) renumbering the boundary vertices and modifications of the small neighbourhoods of the blobs q_i, (b) the resulting object is a picture of $\Psi(p,s) \in J_0^1(8,10)$.

can pass from $p \in J_0^1(m+1,n+1)$ to a canonical picture. Thus Ψ is invertible. $\square$

4.2 An algebra isomorphism

(**4.2**) There is a map Θ on the $\mathfrak{b}_n$ generator set U^e (cf. Theorem 2.26) to $kJ_0^1(n+1,n+1)$ given by:

$$U_i \mapsto U_{i+1} \ = \Psi(U_i), \qquad\qquad e \mapsto \frac{1}{2}(1+\sigma_1) \ = \frac{1}{2}(\Psi(1)+\Psi(e)) \tag{6}$$

(note the uses of notation U_i distinguished by context).

(**4.3**) Theorem. *Fix a commutative ring k and $\delta \in k$, and set $\delta' = \frac{\delta+1}{2}$. Map Θ extends to an algebra isomorphism $\Theta : \mathfrak{b}_n \to J^1_{0,n+1}$ (with $J^1_{0,n+1}$ as defined in §3.5).*

Proof. To verify that Θ extends to an algebra homomorphism it is enough, by Theorem 2.26, to check that the images obey the relations τ' from that Theorem. The relations on the $\{U_i\}$ are all clear. The image of $ee = e$ is clear. Next $U_1 e U_1 = \delta' U_1$:

$$U_2 \frac{1}{2}(1+\sigma_1)U_2 = \frac{\delta+1}{2} U_2$$

The remaining commutation relations will be clear.

This homomorphism is surjective on $B_{2,2,n+1}$ since it hits the generators. On the other hand $B_{2,2,n+1} = J^1_{0,n+1}$ by Theorem 3.26. But $\mathfrak{b}_n$ and $J^1_{0,n+1}$ are isomorphic as vector spaces by Lem.4.1. $\square$

4.3 Category version

(**4.4**) For $m,n \in \mathbb{N}_0$ define

$$\begin{aligned} \Phi : \mathsf{J}^{\bullet}(m,n) \ &\to \ kJ_0^1(m+1,n+1) \\ (p,s) \ &\mapsto \ 2^{-|s|} \sum_{z \subseteq s} \Psi(p,z) \end{aligned}$$

and extend k-linearly.

(**4.5**) Recall (Thm.2.26(II)) that every (blob-)partition (p,s) can be expressed as a product of generators, and write $\#(p,s)$ for the length (the minimum number of factors in such a word). Remark: it is not in general easy to determine length from (p,s) itself. It is easy to determine the number of factors e, since this is just $|s|$. Note also that disk order induces a total order on s. For example if $\{a,b\}$ is the last pair in s with $a<b$ in disk order then the image under Ψ includes $\{a+1,1'\}$.

(**4.6**) LEMMA. *The map Φ restricts to an algebra isomorphism $\mathfrak{b}_n(\delta,(\delta+1)/2) \to J^1_{0,n}$. That is, $\Phi = \Theta$.*

Proof: We work by induction on word length. The base case is trivial.

Consider (p,s) with $|s| = j > 0$ and consider the same partition with one fewer blobs, $(p, s\setminus a)$ say (here a is one of the pairs in s). Then

$$\Phi(p,s) = \frac{1}{2^{|s|}}\left(\sum_{z\subseteq s\setminus a}\Psi(p,z) + \sum_{z\subseteq s\setminus a}\Psi(p,z\cup\{a\})\right) \tag{7}$$

We assume $\Phi(w) = \Theta(w)$ for words w of length $\leq n$, and aim to show for words of length $n+1$. Every such word may be written in the form wx where w has length n and x is a generator. First consider $x=e$. If $(p,s) = we$ in $\mathfrak{b}_n$ then s has an element $a = \{1',j\}$ for some j, and $w = (p, s\setminus a)$. Note that in this case $\Psi(p, z\cup\{a\}) = \Psi(p,z)\Psi(e)$. So in this case (7) becomes

$$\Phi(p,s) = \frac{1}{2}\left(\Phi(p,s\setminus a) + \sum_{z\subseteq s\setminus a}\Psi(p,z)\Psi(e)\right) = \Phi(p,s\setminus a)\frac{1}{2}\left(\Psi(1)+\Psi(e)\right) = \Phi(p,s\setminus a)\Theta(e)$$

by (6). By inductive assumption, $\Phi(w) = \Theta(w)$, so we have made the inductive step in this case.

Similarly if $(p,s) = wU_i$ for some i, with wU_i longer than $w = (p_w, s_w)$, then either $w \mapsto wU_i$ does not change s, $p = p_wU_i$ and

$$\Phi(p,s) = \frac{1}{2^{|s|}}\sum_{z\subseteq s}\Psi(p,z) = \frac{1}{2^{|s|}}\sum_{z\subseteq s}\Psi(p_wU_i,z) = \frac{1}{2^{|s|}}\sum_{z\subseteq s}\Psi(p_w,z)\Psi(U_i) = \Phi(p_w,s)\Theta(U_i)$$

or $w\mapsto wU_i$ changes a single element of s by $\{i',j\}\mapsto\{k,j\}$ (where k is the element in a pair with $i+1'$ in p_w) and similarly $\Psi(p,s) = \Psi(p_w,s_w)\Theta(U_i)$. This completes the inductive step. $\square$

(**4.7**) LEMMA. *Let $(p,s)\in J^\bullet(m,n)$ and $p_0 \in J_{-1}(m_0,n_0)$. Then (i) (confer (2.18) and (2.12)) $(p,s)\mapsto(p\otimes p_0,s)$ defines an injection $J^\bullet(m,n)\hookrightarrow J^\bullet(m+m_0,n+n_0)$; and (ii)*

$$\Phi(p\otimes p_0,s) = \Phi(p,s)\otimes p_0. \tag{8}$$

Proof. (i) $(p\otimes p_0,s)\in J^\bullet(m+m_0,n+n_0)$ since the height of a picture is not increased by concatenating a non-crossing piece to the right and $s\subseteq S^L_p$ implies $s\subseteq S^L_{p\otimes p_0}$. (ii) Now

$$\Phi(p\otimes p_0,s) = 2^{-|s|}\sum_{z\subseteq s}\Psi(p\otimes p_0,z) = 2^{-|s|}\sum_{z\subseteq s}\Psi(p,z)\otimes p_0 = \Phi(p,s)\otimes p_0\ , \tag{9}$$

where the second equality holds by construction since the p_0 part has no blobs. $\square$

(**4.8**) Theorem. *Map Φ gives an equivalence of categories $\mathfrak{b}$ (with $\delta' = (1+\delta)/2$) and $\mathcal{J}_0^1$.*

Proof: We require to show:
(1) Map Φ is a k-vector space isomorphism $\Phi : k\mathsf{J}^\bullet(m,n) \to kJ_0^1(m+1,n+1)$.
(2) For any $(p,s) \in \mathsf{J}^\bullet(m,n)$ and $(p',s') \in \mathsf{J}^\bullet(n,q)$

$$\Phi(p,s) * \Phi(p',s') = \Phi((p,s) \circ (p',s')).$$

Claim (1) follows from (4.6) and (4.7). For (2) let $t = \max(m,n,q)$. For any $p_0 \in J_{-1}(t-m, t-n)$ and $p_0' \in J_{-1}(t-n, t-q)$ note that $(p \otimes p_0, s)$, $(p' \otimes p_0', s') \in J^\bullet(t,t)$. Since $\Phi = \Theta$ is an algebra isomorphism we have

$$\Phi((p \otimes p_0, s) \circ (p' \otimes p_0', s')) = \Phi(p \otimes p_0, s) * \Phi(p' \otimes p_0', s') \ . \tag{10}$$

Let us expand the left-hand side first. By construction

$$(p \otimes p_0, s) \circ (p' \otimes p_0', s') = ((p,s) \circ (p',s')) \otimes (p_0 * p_0')$$

hence, using Lemma 4.7, we have

$$\Phi((p \otimes p_0, s) \circ (p' \otimes p_0', s')) = \Phi((p,s) \circ (p',s')) \otimes (p_0 * p_0') \ . \tag{11}$$

Now, the r.h.s. of (10) using Lemma 4.7 for both terms reads

$$\Phi(p \otimes p_0, s) * \Phi(p' \otimes p_0', s') = (\Phi(p,s) \otimes p_0) * (\Phi(p',s') \otimes p_0') = (\Phi(p,s) * \Phi(p',s')) \otimes (p_0 * p_0') \ , \tag{12}$$

where the right equality holds by construction. Since p_0, p_0' were arbitrary, the statement of the lemma follows from the equality of the rhs. of (11) and (12). why? □

5 On representation theory consequences

5.1 Summary of relevant results for $\mathfrak{b}_n$

Let us restrict to the case $k = \mathbb{C}$. From a representation theory perspective the natural parameterisation of $\mathfrak{b}_n$ is $\delta = [2]$ and $\delta' = \frac{[m+1]}{[m]}$. Then if $m \notin \mathbb{Z}$ we know that $\mathfrak{b}_n$ is semisimple, with a well-known structure [20]. If $m \in \mathbb{Z}$ but q not a root of unity then the algebras are no longer semisimple (for sufficiently large n), but the structure is still relatively simple to describe. The most interesting case is $m \in \mathbb{Z}$ and q a root of unity. The structure in this case is quite complicated. See e.g. [9] for a full description. With this summary in mind, note that due to (4.3) we are interested in the cases when

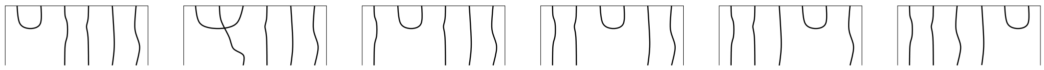
$$\frac{[m+1]}{[m]} = \frac{[2]+1}{2}$$

This is solved for example by $m = 1$ when $[2] = 1$.

5.2 Gram matrices, towers of recollement

Here we assume familiarity with the representation theory as treated in [13], including the construction of standard modules.

Here we restrict consideration to height 0. For example, the diagram basis for the $n = 6$, $l - 1 = 0$ standard module corresponding to $\lambda = (4, +)$ can be drawn as:



where we omit the (2)-symmetrizer sitting on the first two propagating lines (thus we can draw the $\lambda = (4, -)$ case similarly, provided we keep in mind the omission, which affects calculations). Note that the basis (so drawn) contains one extra diagram compared to the $l = -1$/Temperley–Lieb case.

The extra diagram has an interesting effect on the gram matrix of the natural contravariant form (see [13]). As for the TL case this can be computed in terms of Chebyshev polynomials (or equivalently fourier transforms). But here the initial conditions are different. We have

$$\Delta_1^3 = \begin{pmatrix} \delta & 1 & 1 \\ 1 & \delta & 1 \\ 1 & 1 & \delta \end{pmatrix}, \qquad \Delta_{2,\pm}^4 = \begin{pmatrix} \delta & 1 & 1 & 0 \\ 1 & \delta & 1 & \pm 1 \\ 1 & 1 & \delta & 1 \\ 0 & \pm 1 & 1 & \delta \end{pmatrix}, \qquad \Delta_{n-2,+}^n = \begin{pmatrix} \delta & 1 & 1 & 0 & 0 & 0 \\ 1 & \delta & 1 & 1 & 0 & 0 \\ 1 & 1 & \delta & 1 & 0 & 0 \\ 0 & 1 & 1 & \delta & 1 & 0 \\ 0 & 0 & 0 & 1 & \delta & 1 \\ 0 & 0 & 0 & 0 & 1 & \delta \end{pmatrix}$$

(we give the $n = 6$ example, but the general pattern will be clear). Laplace explanding $D_m^n = |\Delta_m^n|$ with respect to the bottom row we get a Chebyshev recurrence

$$D_{n-2,\pm}^n = \delta D_{n-3,\pm}^{n-1} - D_{n-4,\pm}^{n-2}$$

where the initial conditions are $D_1^3 = (x-1)^2(x+2)$ and $D_{2,+}^4 = x(x-1)(x^2+x-4)$ and $D_{2,-}^4 = (x-1)(x+1)(x-2)(x+2)$ (by direct calculation). We may write x for δ, simply for reasons of laziness.

Note from [10] (the tower-of-recollement method) and [13] that the other gram determinants and indeed the 'reductive' representation theory can be determined from this subset of gram determinants. We will address this task in a separate paper. Here we restrict to some of the (interesting) preliminary observations.

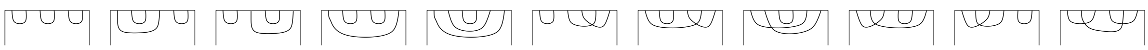
Recall (see e.g. [17]) that the Chebyshev polynomials are the polynomials d_n determined by the recurrence above, thus $d_{n+2} = xd_{n+1} - d_n$, but with initial conditions $d_0 = 1, d_1 = x$. The first few are $d_n = 1, x, x^2 - 1, x^3 - 2x, x^4 - 3x^2 + 1, \ldots$ $(n = 0, 1, 2, 3, \ldots)$. Evidently they are a basis for the space of polynomials; and the recurrence is linear, so we can express our recurrence in terms of them (and hence make use of their more 'fourier-like' formulations: $d_{n-1} = [n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, where $x = q + q^{-1}$).

Thus we express D_1^3 in the Chebyshev basis:

$$D_1^3 = (x-1)^2(x+2) = (x^2 - 2x + 1)(x+2) = x^3 - 3x + 2 = d_3 - d_1 + 2d_0$$

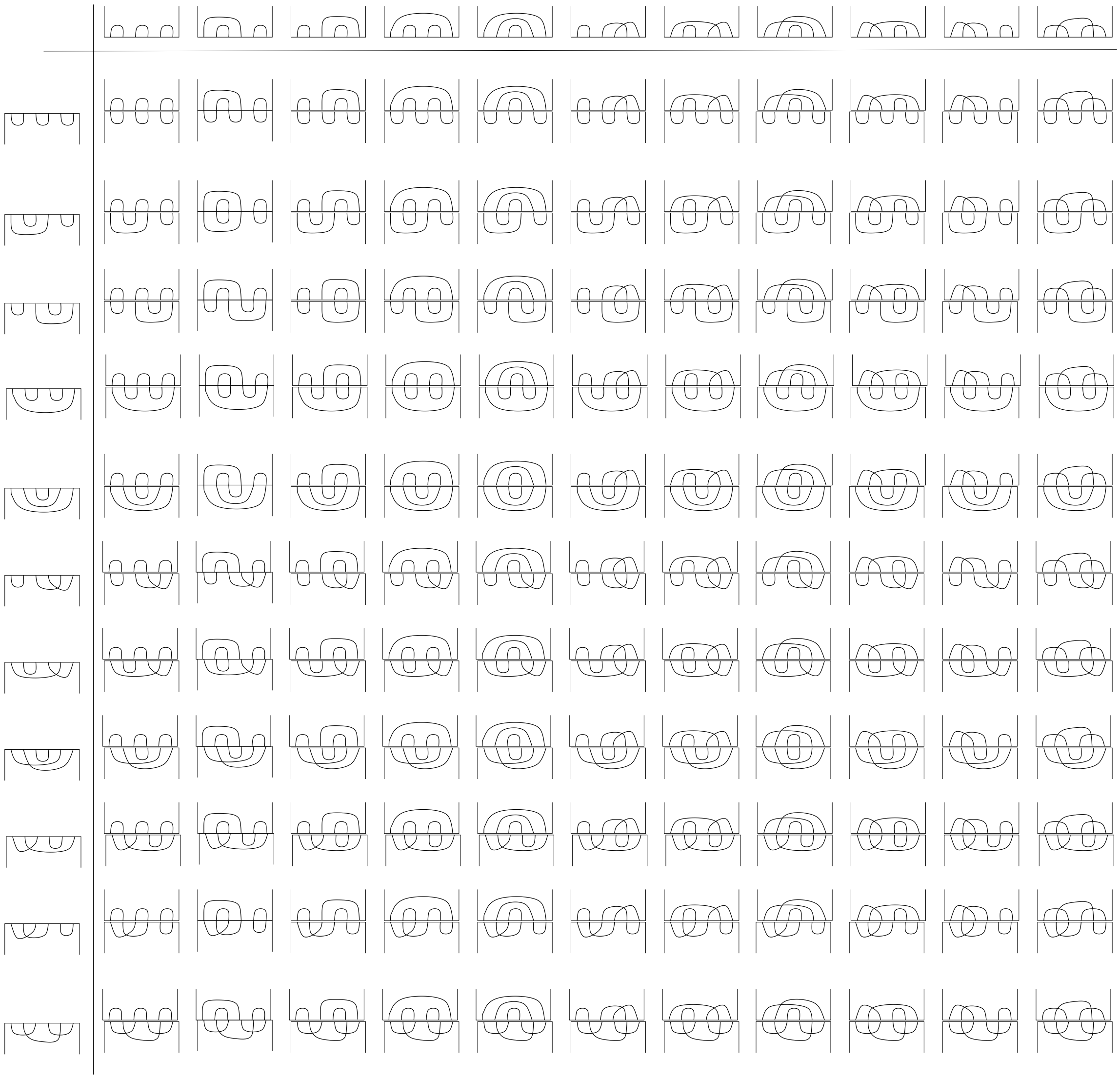
$$D_{2,+}^4 = x(x-1)(x^2 + x - 4) = x^4 - 5x^2 + 4x \quad = \quad d_4 - 2d_2 + 4d_1 + \ldots$$

As a further illustration, here is the basis for $n = 6$ and $\lambda = 0$:



(NB We do not need such cases. It is enough to use $\lambda = n - 2$. We include it for entertainment/sanity checking.)

The corresponding gram matrix then comes from the following array:



Thus, writing j for δ^j (with j the number of connected components in a diagram), the gram matrix is given by

$$\left(\begin{array}{ccccc|ccccc|c}
3 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 2 & 1 \\
2 & 3 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\
2 & 1 & 3 & 2 & 1 & 2 & 1 & 1 & 2 & 1 & 1 \\
1 & 2 & 2 & 3 & 2 & 1 & 2 & 1 & 2 & 1 & 1 \\
2 & 1 & 1 & 2 & 3 & 1 & 1 & 2 & 1 & 1 & 2 \\
\hline
2 & 1 & 2 & 1 & 1 & 3 & 2 & 1 & 1 & 1 & 2 \\
1 & 2 & 1 & 2 & 1 & 2 & 3 & 2 & 1 & 1 & 2 \\
2 & 1 & 1 & 1 & 2 & 1 & 2 & 3 & 2 & 1 & 1 \\
1 & 1 & 2 & 2 & 1 & 1 & 1 & 2 & 3 & 2 & 2 \\
2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 2 \\
\hline
1 & 1 & 1 & 1 & 2 & 2 & 2 & 1 & 2 & 2 & 3
\end{array}\right)$$

The determinants D^p_n of the Gram matrices $\Delta^{n,p}_{n-2}$ (superscript: rank and $p = \pm$ corresponding to the symmetric and antisymmetric simple module of S_2, subscript: number of propagating lines) of a subset of the standard modules of $J^{1,x}_0$ can be expressed in terms of the Chebyshev polynomials ($d_0 = 1, d_1 = x, d_{n+2} = xd_{n+1} - d_n$):

$$\begin{aligned} D^+_n &= (x-1)\left[(x+2)(x-1)d_{n-3} - 2xd_{n-4}\right] \\ D^-_n &= (x-1)(x+2)\left[(x-1)d_{n-3} - 2d_{n-4}\right] \end{aligned}$$

Explicitly, the low rank example are as follows:

$$\begin{aligned}
D^3_1 &= (x-1)^2(x+2) \\
D^4_0 &= (x-1)^2x^3(x+2) \\
D^{4+}_2 &= (x-1)x(x^2+x-4) \\
D^{4-}_2 &= (x-1)(x+1)(x-2)(x+2) \\
D^5_1 &= (x-1)^{12}(x+1)(x-2)(x+2)^6(x^2+x-4) \\
D^{5+}_3 &= (x-1)(x^4+x^3-5x^2-x+2) \\
D^{5-}_3 &= (x-1)(x+2)(x^3-x^2-3x+1) \\
D^6_0 &= (x-1)^{12}x^{11}(x+1)(x-2)(x+2)^6(x^2+x-4) \\
D^{6+}_2 &= (x-1)^8x^5(x+1)(x-2)(x+2)(x^2+x-4)^6(x^4+x^3-5x^2-x+2) \\
D^{6-}_2 &= (x-1)^8(x+1)^6(x-2)^6(x+2)^7(x^2+x-4)(x^3-x^2-3x+1) \\
D^{6+}_4 &= (x-1)^2x(x^3+2x^2-4x-6) \\
D^{6-}_4 &= (x-1)^2(x+2)(x^3-4x-2)
\end{aligned}$$

(the cases not computed by recursion may be computed by brute force, e.g. via Mathematica).

Taken in combination with tower of recollement methods these results 'seed' the reductive representation theory (the determination of decomposition matrices). We address this analysis in a separate paper.

6 Discussion

Some notable open questions follow.
Q1. How generalise the 'short Brauer' construction to BMW [3, 23]?

6.1 Two parameters

Q2. One question (related to Q1.) is how we can relate the usual two-parameter version of the blob algebra to the low-height Brauer algebras — which by the original construction have only a single parameter.

Recall that there is, essentially trivially, a two-parameter version of T_n. First recall that T_n has a basis of non-crossing Brauer diagrams [27, 7] up to ambient isotopy (see §2 for a summary of Brauer diagram concepts — ambient isotopy does not include, for example, the Reidemeister moves included in general Brauer diagram equivalence, but it is sufficient in the non-crossing case, and this is key here). The elements of the basis can be seen as partitioning the interval into alcoves. These alcoves can be shaded black or white with the property that
(A1) the colour changes across each boundary; and
(A2) the leftmost alcove is white, say.
(NB Another way of saying this is that arcs have a well-defined 'height' in the sense of this paper,

which is either odd or even.)
Thus in composition both black and white loops may form. The number of each separately is an invariant of ambient isotopy. It follows that we may associate a different parameter to each.

Thus we have an algebra $T_n(\delta_b, \delta_w)$, say. It is easy to see that $T_n(\delta_b, \delta_w) \cong T_n(\alpha\delta_b, \delta_w/\alpha)$ for any unit α, so the difference can usually be scaled away.

It is even easier to see this phenomenon from the point of view of the standard alternative definition of T_n in terms of generators and relations. Recall:

(**6.1**) THEOREM. [17] *Consider the algebra defined by generators* $U = \{U_1, U_2, ..., U_{n-1}\}$ *and relations* $\tau = \{\ U_i^2 = \delta U_i,\ U_i U_{i\pm 1} U_i = U_i,\ U_i U_j = U_j U_i,\ j \neq i \pm 1\ \}$. *The map*

$$U_i \mapsto u_i = \{\{1,1'\}\{2,2'\}, ..., \{i, i+1\}, \{i', i+1'\}, ..., \{n, n'\}\} \qquad (i = 1, 2, ..., n-1)$$

extends to an algeba isomorphism $k\langle U\rangle/\tau \cong T_n$. □

To see the isomorphic two-parameter version consider the effect on the relations of the map $U_i \mapsto \alpha U_i$ (i odd), $U_i \mapsto \alpha^{-1} U_i$ (i even).

The blob algebra $\mathfrak{b}_n$ can be seen as the subalgebra of $T_{2n}(\delta_b, \delta_w)$ generated by diagrams with a lateral-flip symmetry. In this subalgebra, however, it is *not* possible to scale away the second parameter.

The short Brauer algebras are, from one perspective, generalisations of T_n. It is interesting to consider if there are analogous generalisations of the two-parameter version that (like the blob) have the property that the second parameter becomes material.

This is far from obvious. The generalisation immediately destroys the two-tone alcove construction...

How does the two-tone construction look in the categorical setting? Here we write $T(n,m)$ for the subset $J_{-1}(n,m)$ of $J(n,m)$ of non-crossing pair-partitions. We fix $\delta \in k$ and note that $\mathcal{T} = (\mathbb{N}_0, kT(n,m), *)$ is a subcategory of $\mathcal{B}$. Indeed $\mathcal{T} = \mathcal{B}^{-1}$. The inclusion is of k-linear categories, and also of monoidal k-linear categories.

As in the algebra case we note that in the non-crossing setting we can count the number of black and white loops separately (i.e. these numbers are separately well-defined). Note however that the monoidal structure on $\mathcal{T}$ does not preserve this property. It is the axiom (A2) that is the problem.

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Appendix

A Proof of Lem.4.1 concluded

Given $(p, s) \in \mathsf{J}^\bullet(m, n)$ we write the pairs in s as $\{l_k, r_k\}, k = 1, ..., |s|$, where $l_k < r_k$ and this is the k-th pair, both orderings with respect to the disk-order.

See Fig.18(a) for an example: the set of lines with blobs project to the set of pairs, which correspond to $\{l_1 = 1, r_1 = 2\}$, $l_2 = 3$, $r_2 = 6$, $l_3 = 7$, $r_3 = 5'$, $l_4 = 4'$, $r_4 = 3'$.

Note that the sequence $l_1, r_1, l_2, r_2, \ldots, l_{|s|}, r_{|s|}$ is a subsequence of $1, 2, \ldots, m, n', (n-1)', \ldots, 1'$.

Given a pair partition $p \in J(m, n)$ denote by p^+ the relabeling obtained by adding one to each label. That is, p^+ is a pair partition of the set $\{2, 3, \ldots m+1, 2', 3', \ldots (n+1)'\}$.

(**A.1**) Lemma. *There is a well-defined map*

$$\Psi : \mathsf{J}^{\bullet}(m,n) \to J_0^1(m+1,n+1)$$

given by

$$\Psi(p,s) = x \cup (p \backslash s)^+ \tag{13}$$

where, with s as above,

$$x = \{\{1, r_1+1\}, \{l_1+1, r_2+1\}, \dots \{l_{|s|-1}+1, r_s+1\}, \{l_{|s|}, 1'\}\}$$

Proof. We will verify $\Psi(p,s) \in J_0^1(m+1,n+1)$ by constructing the picture of it. Let $d \in [p]'$. Choose $|s|$ points, which are not from the r-set of the lines corresponding to the left-exposed pairs in s. Denote the chosen $|s|$ points by $q_k, k = 1, 2, \dots, |s|$ according to the disk-ordering of the left points l_i of the pairs.

Denote the middle point between the top left (bottom left) corner of R and the most left northern (southern) endpoint of a line by 0 ($0'$, respectively).

We continue the proof of Lemma A.1 after the proof of the following lemma.

(**A.2**) Lemma. *Consider d as above. There exists a connected path c from 0 to $0'$ without intersecting any line of the picture, which connects $0, q_1, q_2, \dots, q_{|s|}, 0'$ in this order with the following property. At a neighbourhood of the point q_i, c consists of two consecutive straight segments, whose common point is q_i, see Fig 17(a) and also Fig. 18(a) for an example of the full path c in red.*

Proof. The points $0, q_1, q_2, \dots, q_{|s|}, 0'$ are in the boundary of the $\mathcal{A}_0$, which is a connected set, and they are oriented naturally as $0, q_1, q_2, \dots, q_{|s|}, 0'$. Therefore, the line c can be drawn along the boundary of $\mathcal{A}_0$, but slightly inwards, except at the points. □

Now we conclude the proof of Lemma A.1. Consider the union of d and a path c as in Lemma A.2. We relabel the vertices of d in ∂R as $2, 3, \dots m+1$ ($2', 3', \dots, (n+1)'$) in the northern (southern) edge and the beginning and endpoint of c by 1 and $1'$, respectively. Take a small neighbourhood of q_k ($k \in \{1, 2, \dots, |s|\}$ and change it slightly such that the connection of segments change according to the formula for the map Ψ, see Fig. 17b. The resulting object has height 0 since the path c is in $\mathcal{A}_0$ and all intersections are in c. It is also left-simple by construction. Hence, the resulting object is a picture for $\Psi(\{p,s\}) \in J_0^1(m+1,n+1)$ (see figure 18). □

B Colour pictures

Consider Lemma 3.25. When $j < i$, the initial points of chain from i and $i+1$ are interchanged (see Fig. 19a)). When $j = i$ we have three different cases: (i) The line from $i+1$ is part of a chain from $[1,i]$, distinct from that from i, (ii) both lines from i and $i+1$ belong to the same chain, (iii) the line from $(i+1)$ is non intersecting. Observe from Fig. 19b),c),d) that the resulting partitions are $Li-$simple with i exclusive chains from $[1,i]$ to $[1',i']$ and standalone pairs with no intersecting region with any other pair or chains.

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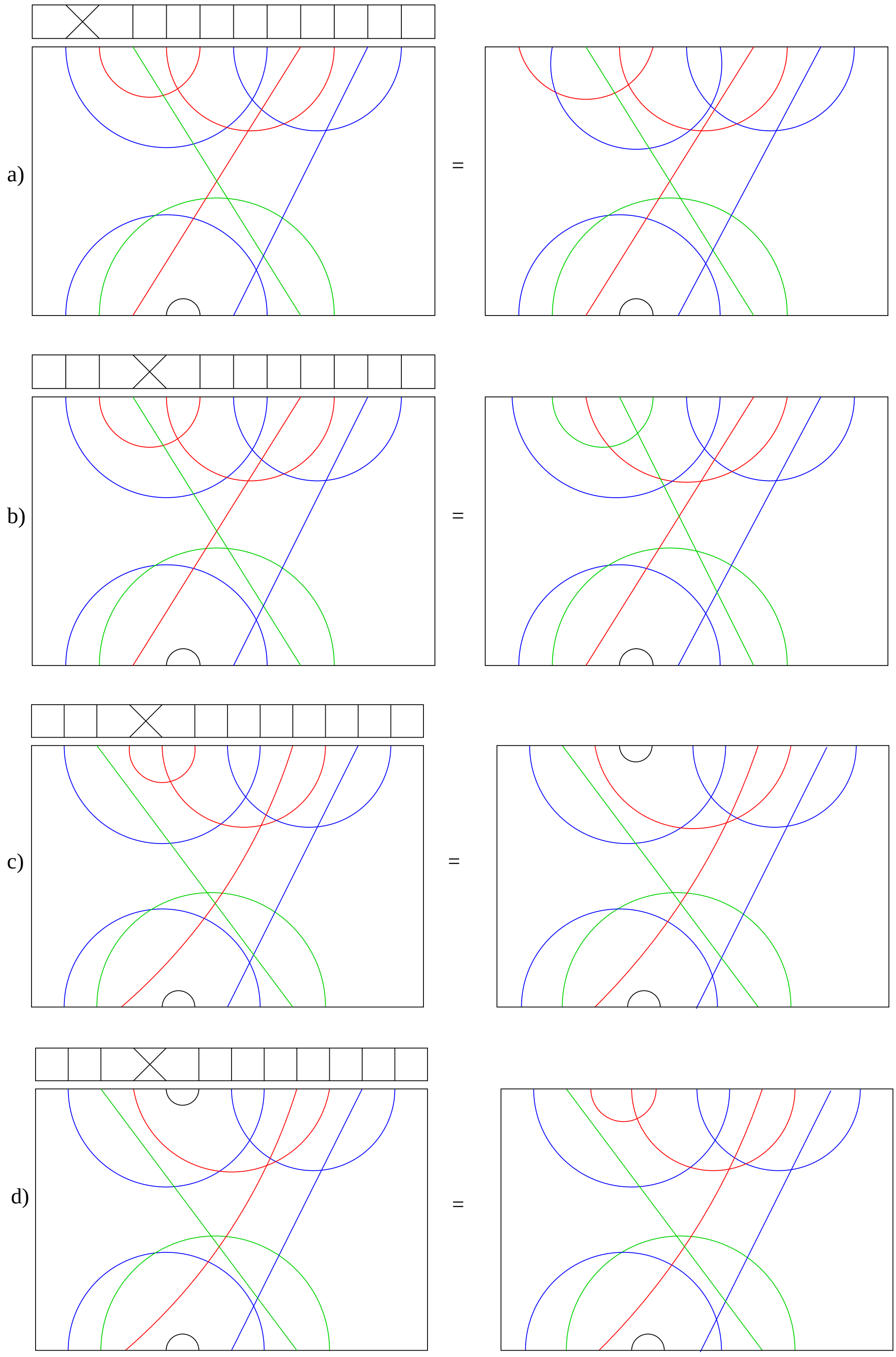

Figure 19: Examples of left actions of S_4 elements ((a): σ_1, b),c),d): σ_3) on elements of $J_2^3(11,9)$. They correspond to the four prototype cases discussed in the proof of Lemma (3.25).

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