MATH1225 Introduction to Geometry, 2017/2018

Tutorial Sheet 1 - Answers

1. Give an example of a statement and its converse (not the same as in Question 1).

Answer: Let n be an integer. If n is odd then n^3 is odd.

Converse: Let n be an integer. If n^3 is odd then n is odd.

2. Consider the statement "Let T be a triangle. If T has an acute angle then T has a right angle." Explain why this statement is **not** true.

Answer: The statement is not true because, for example, an equilateral triangle has an acute angle but no right angle.

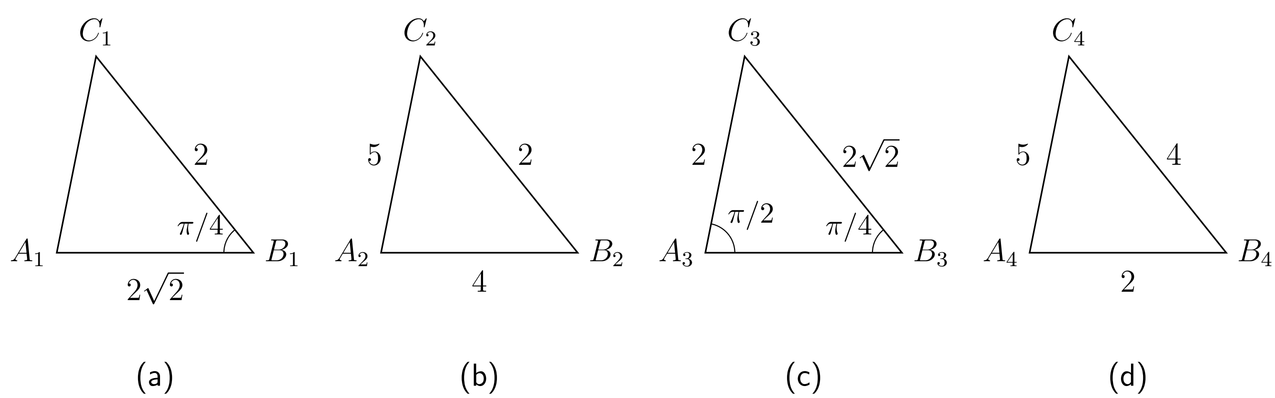
Write down the converse to the above statement, and explain why the converse **is** true.

Answer: Let T be a triangle. If T has a right angle then T has an acute angle.

Let T be a triangle with a right angle. Since the angles in a triangle add up to π, the remaining two angles of T add up to $\pi/2$. Since they are both non-zero, neither can be equal to $\pi/2$, so they must be acute.

3. Determine, with reasons, which of the following triangles are congruent, given the sides and angles shown. (Note that the triangles are not drawn to scale).

Answer: We label the vertices of the triangles as shown below.



In triangle (c), angle $\angle B_3C_3A_3$ is $\pi - \pi/2 - \pi/4 = \pi/4$, since the sum of the angles in a triangle is π. Hence, by (SAS), triangle $A_1B_1C_1$ is congruent to $B_3C_3A_3$, i.e. triangles (a) and (c) are congruent.

By (SSS), triangle $A_2B_2C_2$ is congruent to $C_4B_4A_4$, so triangles (a) and (d) are congruent.

Since triangle (a) cannot have a side of length 4 or a side of length 5, it cannot be congruent to triangle (b) or triangle (d). Similarly, triangle (c) cannot be congruent to triangle (b) or triangle (d).

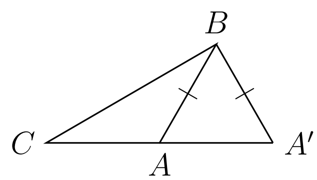
So triangles (a) and (c) are congruent, triangles (b) and (d) are congruent, and there are no more congruences.

4. Consider the following statement, which is similar (but different) to the statements (SSS) and (SAS) from the lectures.

(SSA)=Side-Side-Angle.

Let ABC and $A'B'C'$ be triangles. Assume that $AB = A'B'$, $BC = B'C'$ and $\angle BCA = \angle B'C'A'$. Then the triangles ABC and $A'B'C'$ are congruent.

By considering the triangles ABC and $A'BC$ in the following diagram, or otherwise, determine whether or not (SSA) is true.

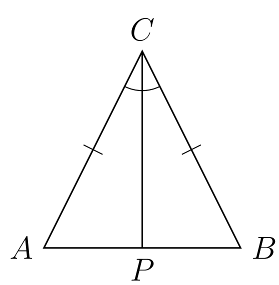


Answer: Choose an isosceles triangle $AA'B$ as shown, and let C be a point on $A'A$ extended, distinct from A. Let $B' = B$ and $C' = C$. Then $AB = A'B$, $BC = BC$ and $\angle BCA = \angle BCA'$, so the triangles ABC and $A'B'C'$ satisfy the assumptions of (SSA). But $AC \neq A'C$, so the triangles ABC and $A'B'C' = A'BC$ are not congruent, and therefore do not satisfy the conclusion. Hence ABC and $A'B'C'$ are a counter-example to (SSA) and (SSA) is false.

5. Given an angle $\angle ABC$, the *bisector* of the angle is a ray starting at B which cuts the angle $\angle ABC$ into two equal parts.

Let ABC be a triangle with $AC = BC$ (recall that a triangle with two sides the same length is called an *isosceles triangle*). Prove that the angles $\angle BAC$ and $\angle CBA$ are equal. (*Hint:* Consider the bisector of the angle $\angle ACB$).

Answer:



We take a bisector of the angle $\angle ACB$, and let P be the intersection of the bisector with AB, giving the figure shown. Then $AC = BC$, $CP = CP$ and $\angle ACP = \angle PCB$. Hence, by (SAS), the triangles APC and BPC are congruent (note that this means A corresponds to B, P corresponds to P and C corresponds to C). Hence $\angle PAC = \angle CBP$, i.e. $\angle BAC = \angle CBA$, as required. Thus we have shown that, in an isosceles triangle, the angles opposite the equal sides are also equal.

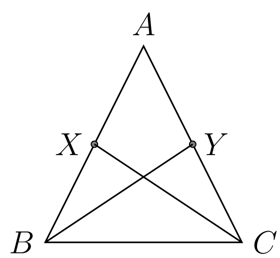
What can you say if all three sides of ABC have the same length?

Answer: Recall that a triangle with this property is known as an *equilateral triangle*. Suppose that $AB = AC = BC$. Applying the same argument to the pair AB and AC, we obtain that $\angle ACB = \angle CBA$. Hence all three of the angles in the triangle coincide. Since the sum of the angles in the triangle is π, they must all be equal to $\pi/3$.

6. In the situation in Question 5, show that $AP = BP$ and that the angles $\angle CPA$ and $\angle BPC$ are both right angles.

Answer: In Question 5, we have shown that the triangles APC and BPC are congruent. Hence $AP = BP$. Furthermore, $\angle CPA = \angle BPC$. But the sum of these two angles is the straight angle, π, so they must both be equal to $\pi/2$.

7. Consider the following figure, and suppose that $AB = AC$ and $AX = AY$. Show that $XC = YB$.



Answer: Since $AX = AY$, $CA = BA$ and $\angle XAC = \angle BAY$, it follows from (SAS) that the triangles AXC and AYB are congruent. Hence $XC = YB$ as required.

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