Notes in representation theory
(Rough Draft)

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Contents

Foreword

0.0.1 Definition summary

ss:defsum

There follows a list of definitions in the form
ALGEBRAIC SYSTEM $A = (A \text{ a set, } n-\text{ary operations})$, axioms.
(The selection of a special element $u \in A$, say, counts as a 0-ary operation.)
Extended examples are postponed to the relevant sections.

SEMIGROUP	$S = (S, \Box)$, $\Box$ a closed associative binary operation on S.
MONOID	$M = (M, \Box, u)$, $(M, \Box)$ a semigroup, $u \in M$ a *unit element* (i.e. $au = a = ua \ \ \forall a \in M$). Example: $(\mathbb{N}_0, +, 0)$.
GROUP	$G = (G, ., u)$, G a monoid, $\forall a \in G \exists a'$ such that $aa' = u = a'a$.
ABELIAN GROUP	$G = (G, +, 0)$, G a group, $a + b = b + a$.
RING	$R = (R, +, ., 1, 0)$, $(R, +, 0)$ an abelian group, $(R, ., 1)$ a monoid, $a(b + c) = ab + ac$, $(a + b)c = ac + bc$.
DIVISION RING	D, D a ring, every non–zero element has a multiplicative inverse.
LOCAL RING	A, A a ring, sum of two nonunits is a nonunit (a a nonunit means there does not exist b such that $ab = ba = 1$). [1]
DOMAIN	K, K a ring, $0 \neq 1$, $mn = 0$ implies either $m = 0$ or $n = 0$.
INTEGRAL DOMAIN	K, K a ring, . commutative, $0 \neq 1$, $mn = 0$ implies either $m = 0$ or $n = 0$. (I.e. an integral domain is a commutative domain.)
PRINCIPAL IDEAL DOMAIN	K, K an integral domain, every ideal $J \subseteq K$ is principal (i.e. $\exists$ $a \in K$ such that $J = aK$).
FIELD	F, F an integral domain, every $a \neq 0$ has a multiplicative inverse.

Our other core definitions are, for S a semigroup, R a ring as above:

S–IDEAL J: $J \subset S$ and $rj, jr \in J$ for all $r \in S$, $j \in J$.
R–IDEAL J: $J \subset R$ and $rj, jr \in J$ for all $r \in R$, $j \in J$.

(LEFT) R–MODULE M: M an abelian group with map $R \times M \to M$ (we write rx for the image of (r, x)) such that $r(x + y) = rx + ry$, $(r + s)x = rx + sx$, $(rs)x = r(sx)$, $1x = x$ ($r \in R$, $x, y \in M$). Right modules defined similarly, but with $(rs)x = s(rx)$.

(LEFT) R–MODULE HOMOMORPHISM : Ψ from left R-module M to N is a map $\Psi : M \to N$ such that $\Psi(x + y) = \Psi(x) + \Psi(y)$, $\Psi(rx) = r\Psi(x)$ for $x, y \in M$ and $r \in R$.

(**0.0.1**) EXERCISE. $\mathbb{Z}$ is a ring. Form examples of as many of the other structures as possible from this one. (And some non-examples.)

In the following table k is a field and $\mathbb{H}$ is the ring of real quaternions (see §3.1.3).

	$DivR$	LR	ID	PID
$\mathbb{Z}$	$\times$	$\times$	$\surd$	$\surd$
$\mathbb{Z}[x]$	$\times$	$\times$	$\surd$	$\times$
$k[x]$	$\times$	$\times$	$\surd$	$\surd$
$k[x, y]$	$\times$	$\times$	$\surd$	$\times$
$\mathbb{H}$	$\surd$	$\surd$	$\times$	$\times$

0.0.2 Glossary

		alternatives and references
$M_N(R)$	ring of $N \times N$ matrices over ring R	[6]
$GL(N)$	general linear group on $\mathbb{C}^N$	GL_N, $GL(N, \mathbb{C})$
$SL(N)$	special (det=1) linear group on $\mathbb{C}^N$	SL_N, $SL(N, \mathbb{C})$
$O(N)$	orthogonal ($g^T g = 1$) group on $\mathbb{R}^N$	$O(N, \mathbb{R})$
$O(N, \mathbb{C})$	orthogonal ($g^T g = 1$) group on $\mathbb{C}^N$	$O(N, \mathbb{C})$
$SO(N, \mathbb{C})$	special orthogonal group on $\mathbb{C}^N$	$SO(N, \mathbb{C})$
$U(N)$	unitary ($g^\dagger g = 1$) group on $\mathbb{C}^N$	U_N
$SU(N)$	special unitary group on $\mathbb{C}^N$	SU_N
Λ	set of integer partitions	$\mathcal{P}$ [78, I.1]
Λ_n	set of integer partitions of n	$\mathcal{P}_n$ [78, I.1]
P_S	partitions of a set S	
J_S	pair partitions of a set S	
$\mathtt{P}(S)$	power set (lattice) of a set S	
$\mathtt{P}_n(S)$	subset of $\mathtt{P}(S)$ of sets of order n	
$U_{S,T}$	the set of relations $\mathtt{P}(S \times T)$	
E_S	set of equivalence relations on set S	
$\underline{n}$	$\{1, 2, ..., n\}$	[52, §2]
$\underline{l}^{\underline{n}}$	set of functions $f : \underline{n} \to \underline{l}$	$I(l, n)$ [52, §2]
Σ_n, S_n	symmetric group $\subset (\underline{n}^{\underline{n}}, \circ)$	S_n [78, I.7], $G(n)$ [52, §2]
$\Lambda(l, n)$	S_n orbits of $\underline{l}^{\underline{n}}$ / compositions of n into l parts	[52, §3.1]
$\Lambda^+(l, n)$	S_l orbits of $\Lambda(l, n)$ / partitions of n into l parts	[52, §3.1]
Set	category of sets and set maps	
$Z(\mathbb{H})$	polyhedral complex defined by set of hyperplanes $\mathbb{H}$	
$\Gamma(G, S)$	Cayley graph of group G over subset S	
$G(W, S)$	directed Cayley graph of Coxeter system (W, S)	
$W_{\mathbb{H}}$	reflection group generated by set of hyperplanes $\mathbb{H}$	
$D(\mathbb{H})$	dual graph of complex defined by hyperplanes $\mathbb{H}$	
$\mathcal{C}_{\mathbb{H}}$	set of chambers of defined by $\mathbb{H}$	
$\mathcal{C}_W$	set of chambers of reflection group W	
$\mathcal{A}_{\mathbb{H}}$	set of alcoves of defined by $\mathbb{H}$	
$\mathbb{H}_a$	subset of hyperplanes, walls of chamber $a \in \mathcal{C}_{\mathbb{H}}$	

Chapter 1

Introduction

ch:basic

Chapters 1 and 2 give a brief introduction to representation theory, and a review of some of the basic algebra required in later Chapters. A more thorough grounding may be achieved by reading the works listed in §1.14: *Notes and References*.

Section 1.1 (upon which later chapters do not depend) attempts to provide a sketch overview of topics in the representation theory of finite dimensional algebras. In order to bootstrap this process, we use some terms without prior definition. We assume you know what a vector space is, and what a ring is (else see Section 2.1.1). For the rest, either you know them already, or you must intuit their meaning and wait for precise definitions until after the overview.

1.1 Representation theory preamble

s:ov

1.1.1 Matrices

ss:matrices1

Let $M_{m,n}(R)$ denote the additive group of $m \times n$ matrices over a ring R, with additive identity $0_{m,n}$. Let $M_n(R)$ denote the ring of $n \times n$ matrices over R.

Define a block diagonal composition (matrix direct sum)

$$\begin{aligned} \oplus : M_m(R) \times M_n(R) &\rightarrow M_{m+n}(R) \\ (A, A') &\mapsto A \oplus A' = \begin{pmatrix} A & 0_{m,n} \\ 0_{n,m} & A' \end{pmatrix} \end{aligned}$$

(sometimes we write $\oplus^{\cdot}$ for matrix/exterior $\oplus$ for disambiguation).

Define Kronecker product

$$\otimes : M_{a,b}(R) \times M_{m,n}(R) \rightarrow M_{am,bn}(R) \tag{1.1}$$

eq:kronecker12

$$(A, B) \mapsto \begin{pmatrix} a_{11}B & a_{12}B & ... \\ a_{21}B & a_{22}B & ... \\ \vdots & & \end{pmatrix} \tag{1.2}$$

In general $A \otimes B \neq B \otimes A$, but (if R is commutative then) for each pair A, B there exists a pair of permutation matrices S, T such that $S(A \otimes B) = (B \otimes A)T$ (if A, B square then $T = S$ — the *intertwiner* of $A \otimes B$ and $B \otimes A$).

1.1.2 Aside: binary operations, magmas and associativity

Most of the algebraic structures we consider here satisfy an associativity condition (or something similarly strong). Here we say a few words about the more general case, for context. See §2.2.4 for some exercises.

(**1.1.1**) A set with a closed binary operation is sometimes called a *magma*.

We may define the *free magma* M_S generated by a set S as follows. First of all the elements $S \subset M_S$ (elements of *length* 1). Given a pair of elements x, y then the free magma product is the ordered pair (x, y). Thus in particular $S \times S \subset M_S$ (elements of length 2). But then obviously we also get $((x, y), z)$ and $((x, y), (y, z))$ and so on. For $n > 0$ define sets $S^{!n}$ iteratively as follows: $S^{!1} = S$; $S^{!2} = S \times S$; then

$$S^{!n} = \bigcup_{a+b=n} S^{!a} \times S^{!b}$$

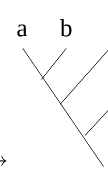
We have $M_s = \bigcup_n S^{!n}$.

(**1.1.2**) Proposition. *The product $a * b = (a, b)$ closes on M_S.*

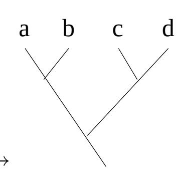
(**1.1.3**) Magma $M_S = (M_S, *)$ is free in the sense that no conditions have been imposed on the product. It is also free in the sense that if $f : S \to G$ is any map to a magma, then this extends uniquely to a magma map $f' : M_S \to G$.

(**1.1.4**) Note that an element of $S^{!n} \subset M_S$ corresponds to a word w in S of length n together with a planar binary tree with n leaves. One labels the leaves by w in the natural way, then reads off the order of composition from the tree. Examples:

$((ab)c)d \quad \mapsto$



$(ab)(cd) \quad \mapsto$



(**1.1.5**) We say a few words about imposing a congruence on M_S corresponding to associativity. Let us define a relation on M_S by $a(bc) \sim (ab)c$. (We assume this notation engenders no ambiguity.)

Consider the pentagon of congruences induced by this relation on $((ab)c)d$:

$$\begin{tikzcd}
& ((ab)c)d \arrow[ld, "ab"'] \arrow[rd, "g"] & \\
(ab)(cd) \arrow[rdd, "cd"'] & & (a(bc))d \arrow[d, "bc"] \\
& & a((bc))d) \arrow[ld, "g"] \\
& a(b(cd)) == a(b(cd)) &
\end{tikzcd}$$

Each step is a congruence implied by an element of the relation. (The edge label g means it is via a relation $\sim$ exactly as originally written. The label ab means that ab is an atom in the congruence.) The composites are congruences by transitive closure. Note that the two routes to the bottom result not only in congruent elements but in *identical* elements.

1.1.3 Aside: Some notations for monoids and groups

(See §2.2 for a more extended discussion of set theory notations. See §2.2.4 for exercises on binary operations.)

de:freemonoid

(**1.1.6**) Given a set S, then a *word* in S is a finite sequence from 'alphabet' S, i.e. a map from $\underline{n}$ to S for some $n \in \mathbb{N}_0$. E.g. for $S = \{a, b, c\}$ then write $w = abc$ for the word $abc : \underline{3} \to S$ given by $abc(1) = a$ and so on.

The *free monoid* S^* is the set of words in the alphabet S, together with the operation of juxtaposition: $a * b = ab$. (Note associativity.) That is, for $w : \underline{n} \to S$ (written, for example, as $w = w_1 w_2 ... w_n$, with $w_i = w(i)$) and $v : \underline{m} \to S$ we have $w * v : \underline{n+m} \to S$ given by

$$(w * v)(i) = \begin{cases} w(i) & i \leq n \\ v(i-n) & i > n \end{cases}$$

i.e. $w * v = w_1 w_2 ... w_n v_1 v_2 ... v_m$.

pr:f1

(**1.1.7**) If M is a monoid with generating subset S' in bijection with set S (bijection $s \leftrightarrow s'$, say) then there is a map $f : S^* \to M$ given by $f(s) = s'$.

(**1.1.8**) Let ρ be a relation on set S, a monoid. Then ρ is *compatible* with monoid S if $(s,t),(u,v) \in \rho$ implies $(su, tv) \in \rho$.

We write $\rho\#$ for the intersection of all compatible equivalence relations ('congruences') on S containing relation ρ.

(**1.1.9**) If ρ is an equivalence relation on set S then S/ρ denotes the set of classes of S under ρ.

(**1.1.10**) If ρ is a congruence on semigroup S then S/ρ has a semigroup structure by:

$$\rho(a) * \rho(b) = \rho(a * b)$$

(Exercise: check well-definedness and associativity.)

(**1.1.11**) For set S finite we can define a monoid by *presentation*. This is the monoid $S^*/\sim$, where the presentation $\sim$ is a relation on S.

...

(**1.1.12**) For more on semigroups see for example Howie [57].

(**1.1.13**) A monoid M is *regular* if $m \in mMm$ for all $m \in M$.

Fix a monoid M. The equivalence relation $\mathcal{J}$ on M is given by $a\mathcal{J}b$ if $MaM = MbM$. Note that the classes are partially ordered by inclusion.

de:solvableg (**1.1.14**) A group G is *solvable* if there is a chain of subgroups $...G_i \subset G_{i+1}...$ such that $G_i \trianglelefteq G_{i+1}$ (normal subgroup) and G_{i+1}/G_i is abelian.

(**1.1.15**) Example. $(\mathbb{Z}, +)$ and S_3 are solvable; S_5 is not.

(**1.1.16**) A group G is *simple* if it has no proper normal subgroups.

(**1.1.17**) Example. The alternating group A_n is simple for $n > 4$; S_n is not simple for $n > 2$.

1.1.4 Group representations

de:rep (**1.1.18**) A matrix representation of a group G over a commutative ring R is a map

$$\rho : G \to M_n(R) \tag{1.3}$$ try345

such that $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$. In other words it is a map from the group to a different system, which nonetheless respects the extra structure (of multiplication) in some way. The study of representations — models of the group and its structure — is a way to study the group itself.

(**1.1.19**) The map ρ above is an example of the notion of representation that generalises greatly. A mild generalisation is the representation theory of R-algebras that we shall discuss, but one could go further. Physics consists in various attempts to model or represent the observable world. In a model, Physical entities are abstracted, and their behaviour has an image in the behaviour of the model. We say we understand something when we have a model or representation of it mapping to something we understand (better), which does not wash out too much of the detailed behaviour.

de:repIII (**1.1.20**) Representation theory itself seeks to classify and construct representations (of groups, or other systems). Let us try to be more explicit about this.

(I) Suppose ρ is as above, and let S be an arbitrary invertible element of $M_n(R)$. Then one immediately verifies that

$$\begin{aligned} \rho_S : G &\to M_n(R) && (1.4) \\ g &\mapsto S\rho(g)S^{-1} && (1.5) \end{aligned}$$ aaas

is again a representation.
(II) If ρ' is another representation (by $m \times m$ matrices, say) then

$$\begin{aligned} \rho \oplus \rho' : G &\to M_{m+n}(R) && (1.6) \\ g &\mapsto \rho(g) \oplus \rho'(g) && (1.7) \end{aligned}$$ dsum

is yet another representation.
(III) For a finite group G let $\{g_i \;:\; i = 1, ..., |G|\}$ be an ordering of the group elements. Each element g acts on G, written out as this list $\{g_i\}$, by multiplication from the left (say), to permute the list. That is, there is a permutation $\sigma(g)$ such that $gg_i = g_{\sigma(g)(i)}$. This permutation can be recorded as a matrix,

$$\rho_{Reg}(g) = \sum_{i=1}^{|G|} \epsilon_{i\ \sigma(g)(i)}$$

(where $\epsilon_{ij} \in M_{|G|}(R)$ is the i, j-elementary matrix) and one can check that these matrices form a representation, called the *regular representation*.

Clearly, then, there are unboundedly many representations of any group. However, these constructions also carry the seeds for an organisational scheme...

(**1.1.21**) Firstly, in light of the ρ_S construction, we only seek to classify representations *up to isomorphism* (i.e. up to equivalences of the form $\rho \leftrightarrow \rho_S$).

Secondly, we can go further (in the same general direction), and give a cruder classification, by *character*. (While cruder, this classification is still organisationally very useful.) We can briefly explain this as follows.

Let c_G denote the set of classes of group G. A *class function* on G is a function that factors through the natural set map from G to the set c_G. Thus an R-valued class function is completely specified by a c_G-tuple of elements of R (that is, an element of the set of maps from c_G to R, denoted R^{c_G}). For each representation ρ define a *character* map from G to R

$$\begin{aligned} \chi_\rho : G &\to R && (1.8) \\ g &\mapsto \mathrm{Tr}(\rho(g)) && (1.9) \end{aligned}$$ eq:ch1

(matrix trace). Note that this map is fixed up to isomorphism. Note also that this map is a class function. Fixing G and varying ρ, therefore, we may regard the character map instead as a map χ_- from the collection of representations to the set of c_G-tuples of elements of R.

Note that pointwise addition equips R^{c_G} with the structure of abelian group. Thus, for example, the character of a sum of representations isomorphic to ρ lies in the subgroup generated by the character of ρ; and $\chi_{\rho \oplus \rho'} = \chi_\rho + \chi_{\rho'}$ and so on.

We can ask if there is a small set of representations whose characters '$\mathbb{N}_0$-span' the image of the collection of representations in R^{c_G}. (We could even ask if such a set provides an R-basis for R^{c_G} (in case R a field, or in a suitably corresponding sense — see later). Note that $|c_G|$ provides an upper bound on the size of such a set.)

(**1.1.22**) Next, conversely to the direct sum result, suppose $R_1 : G \to M_m(R)$, $R_2 : G \to M_n(R)$, and $V : G \to M_{m,n}(R)$ are set maps, and that a set map $\rho_{12} : G \to M_{m+n}(R)$ takes the form

$$\rho_{12}(g) = \begin{pmatrix} R_1(g) & V(g) \\ 0 & R_2(g) \end{pmatrix} \qquad (1.10)$$ eq:plus

(a matrix of matrices). Then ρ_{12} a representation of G implies that both R_1 and R_2 are representations. Further, $\chi_{\rho_{12}} = \chi_{R_1} + \chi_{R_2}$ (i.e. the character of ρ_{12} lies in the span of the characters of the smaller representations). Accordingly, if the isomorphism class of a representation contains an element that can be written in this way, we call the representation *reducible*.

(**1.1.23**) For a finite group over $R = \mathbb{C}$ (say) we shall see later that there are only a finite set of 'irreducible' representations needed (up to equivalences of the form $\rho \leftrightarrow \rho_S$) such that every representation can be built (again up to equivalence) as a direct sum of these; and that all of these irreducible representations appear as direct summands in the regular representation.

We have done a couple of things to simplify here. Passing to a field means that we can think of our matrices as recording linear transformations on a space with respect to some basis. To say that ρ is equivalent to a representation of the form ρ_{12} above is to say that this space has a G-subspace (R_1 is the representation associated to the subspace). A representation is irreducible if there is no such proper decomposition (up to equivalence). A representation is *completely reducible* if for every decomposition $\rho_{12}(g)$ there is an equivalent identical to it except that $V(g) = 0$ — the direct sum.
Theorem [Mashke] Let ρ be a representation of a finite group G over a field K. If the characteristic of K does not divide the order of G, then ρ is completely reducible.
Corollary Every complex irreducible representation of G is a direct summand of the regular representation.

Representation theory is more complicated in general than it is in the cases to which Mashke's Theorem applies, but the notion of irreducible representations as fundamental building blocks survives in a fair degree of generality. Thus the question arises:
Over a given R, what are the irreducible representations of G (up to $\rho \leftrightarrow \rho_S$ equivalence)?
There are other questions, but as far as physical applications (for example) are concerned, this is arguably the main interesting question.

(**1.1.24**) Examples: In this sense, of constructing irreducible representations, the representation theory of the symmetric groups S_n over $\mathbb{C}$ is completely understood! (We shall review it.) On the other hand, over other fields we do not have even so much as a conjecture as to how to organise the statement of a conjecture! So there is work to be done.

1.1.5 Unitary and normal representations

A complex representation ρ of a group G in which every $\rho(g)$ is unitary is a *unitary representation* (see e.g. Boerner [11, III§6]). A representation equivalent to a unitary representation is *normal*.

(**1.1.25**) Theorem. *Let G be a finite group. Every complex representation of G is normal. Every real representation of G is equivalent to a real orthogonal representation.*

1.1.6 Group algebras, rings and modules

`de:lset` (**1.1.26**) For a set S, a map $\psi : G \times S \to S$ (written $\psi(g, s) = gs$ where no ambiguity arises) such that

$$(gg')s = g(g's),$$

equips S with the property of *left G-set*.

(**1.1.27**) For example, for a group $(G, *)$, then G itself is a left G-set by left multiplication: $\psi(g, s) = g * s$. (Cf. (1.1.20)(III).)

On the other hand, consider the map $\psi_r : G \times G \to G$ given by $\psi_r(g, s) = s * g$. This obeys $\psi_r(g * g', s) = s * (g * g') = (s * g) * g' = \psi_r(g', \psi_r(g, s))$. This ψ_r makes G a *right G-set*: in the notation of (1.1.26) we have

$$(gg')s = g'(gs). \tag{1.11}$$ eq:rset

The map $\psi_- : G \times G \to G$ given by $\psi_-(g, s) = g^{-1} * s$ obeys $\psi_r(g * g', s) = (g * g')^{-1} * s = (g'^{-1} * g^{-1}) * s = g'^{-1} * (g^{-1} * s) = \psi_-(g', \psi_-(g, s))$. This ψ_- makes G a *right G-set.*

rem:Rn (**1.1.28**) Remark: When working with R a *field* it is natural to view the matrix ring $M_n(R)$ as the ring of linear transformations of vector space R^n expressed with respect to a given ordered basis. The equivalence $\rho \leftrightarrow \rho_S$ corresponds to a change of basis, and so working up to equivalence corresponds to demoting the matrices themselves in favour of the underlying linear transformations (on R^n). In this setting it is common to refer to the linear transformations by which G acts on R^n as the representation (and to spell out that the matrices are a *matrix* representation, regarded as arising from a choice of ordered basis).

Such an action of a group G on a set makes the set a G-set as in 1.1.26. However, given that R^n is a set with extra structure (in this case, a vector space), it is a small step to want to try to take advantage of the extra structure.

(**1.1.29**) For example, continuing for the moment with R a field, we can define RG to be the R-vector space with basis G (see Exercise 1.15.1), and define a multiplication on RG by

$$\left(\sum_i r_i g_i\right)\left(\sum_j r'_j g_j\right) = \sum_{ij} (r_i r'_j)(g_i g_j) \tag{1.12}$$ groupalgmult

which makes RG a ring (see Exercise 1.15.2).

One can quickly check that

$$\begin{aligned} \rho : RG &\to M_n(R) && (1.13)\\ \sum_i r_i g_i &\mapsto \sum_i r_i \rho(g_i) && (1.14) \end{aligned}$$

extends a representation ρ of G to a representation of RG in the obvious sense. Superficially this construction is extending the use we already made of the multiplicative structure on $M_n(R)$, to make use not only of the additive structure, but also of the particular structure of 'scalar' multiplication (multiplication by an element of the centre), which plays no role in representing the group multiplication *per se*. The construction *also* makes sense at the G-set/vector space level, since linear transformations support the same extra structure.

de:RG-module (**1.1.30**) The same formal construction of RG works when R is an arbitrary commutative ring (called the *ground ring*), except that RG is not then a vector space. Instead, in respect of the vector-space-like aspect of its structure, it is called a *free R-module with basis G* (see also §7.2.3). The idea of matrix representation goes through unchanged.

If one wants a generalisation of the notion of G-set for RG to act on, the additive structure is forced from the outset. This is called a *(left) RG-module*. A formal definition may be given as follows. (The definition of left module makes sense with RG replaced by an arbitrary ring H, so we state it as such. We keep in mind the ring $H = RG$.) A left H-module is, then, an abelian group $(M, +)$ with a suitable action of H defined on it: $r(x+y) = rx + ry$, $(r+s)x = rx + sx$,

$$(rs)x = r(sx), \tag{1.15}$$ eq:lmodule

$1x = x$ $(r, s \in H,\ x, y \in M)$. That is, M is a kind of 'H-set', just as the original vector space R^n was in (1.1.28).

Several examples of modules are given in §1.3.1. One thing that is new at this level is that such a structure may not have a basis (a *free* module has a basis), and so may not correspond to any class of matrix representations.

(**1.1.31**) Exercise. Construct an RG-module without basis.
(Possible hints: With G trivial we have, simply, an R-module. The caveat already applies here — it is enough to look for an R-module without basis for some commutative ring R. 1. Consider $R = \mathbb{Z}$, G trivial, and look at §7.3. 2. Consider the ideal $\langle 2, x\rangle$ in $\mathbb{Z}[x]$.)

(**1.1.32**) Remark. The above exercise concerns a different issue to the formal one which may arise if the module is in fact a vector space. A finite-dimensional vector space has a basis by definition; but it general it is (only) axiomatic that every vector space has a basis. (It can be seen as a consequence of Zorn's Lemma: If a partially ordered set P is such that every chain in P has an upper bound, then P has a maximal element.) Consider the case of $(\mathbb{R}, +)$ regarded as a $\mathbb{Q}$-module.

From this point the study of representation theory may be considered to include the study of both matrix representations and modules.

1.1.7 Algebras

(**1.1.33**) What other kinds of systems can we consider representation theory for?
A natural place to start studying representation theory is in Physical modeling. Unfortunately we don't have scope for this in the present work, but we will generalise from groups at least as far as rings and algebras.

The generalisation from groups to *group algebras RG* over a commutative ring R is quite natural as we have seen. The most general setting within the ring-theory context would be the study of arbitrary ring homomorphisms from a given ring. However, if one wants to study this ring by studying its modules (the obvious generalisation of the RG-modules introduced above) then the parallel of the matrix representation theory above is the study of modules that are also free modules over the centre, or some subring of the centre. (For many rings this accesses only a very small part of their structure, but for many others it captures the main features. The property that *every* module over a commutative ring is free holds if and only if the ring is a field, so this is our most accessible case. We shall motivate the restriction shortly.) This leads us to the study of *algebras*.

To introduce the general notion of an algebra, we first write $\mathrm{cen}\,(A)$ for the centre of a ring A

$$\mathrm{cen}\, A = \{a \in A \mid ab = ba\ \forall b \in A\}$$

de: alg1 (**1.1.34**) An algebra A (over a commutative ring R), or an R-algebra, is a ring A together with a homomorphism $\psi : R \to \mathrm{cen}\,(A)$, such that $\psi(1_R) = 1_A$.

de:groupalgebra Examples: Any ring is a $\mathbb{Z}$-algebra. Any ring is an algebra over its centre. The group ring RG is an R-algebra by $r \mapsto r1_G$. The ring $M_n(R)$ is an R-algebra.

Let $\psi : R \to \mathrm{cen}\,(A)$ be a homomorphism as above. We have a composition $R \times A \to A$:

$$(r, a) = ra = \psi(r)a$$

so that A is a left R-module with

$$r(ab) = (ra)b = a(rb) \tag{1.16}$$ eq: alg12

Conversely any ring which is a left R-module with this property is an R-algebra.

(**1.1.35**) An R-representation of A is a homomorphism of R-algebras

$$\rho : A \to M_n(R)$$

(**1.1.36**) The study of a group algebra RG depends heavily on R as well as G. The study of such R-algebras takes a relatively simple form when R is an algebraically closed field; and particularly so when that field is $\mathbb{C}$. We shall aim to focus on these cases. However there are significant technical advantages, even for such cases, in starting by considering the more general situation. Accordingly we shall need to know a little ring theory, even though general ring theory is not the object of our study.

Further, as we have said, neither applications nor aesthetics restrict attention to the study of representations of groups and their algebras. One is also interested in the representation theory of more general algebras.

1.2 Group and Partition algebras — some quick examples

ss:pa0001

Our study of representation theory will benefit from plentiful examples. We use algebras such as the *partition algebra* [84, 87] to generate examples.

The objective can be considered to be determining representation theory data, such as (A0-III) from (1.4.1), for various *Artinian algebras* (as in (1.3.24)). (The aim is to illustrate various tools for doing this kind of thing.) We follow directly the argument in [87].

This Section can be skipped at first reading. We start by very briefly recalling the partition algebra construction but, essentially, we assume for now that you know the definition and some notations for the partition algebras (else see §2.2.3 and §12, or [87]).

Implicit in this section are a number of exercises, requiring the proof of the various claims.

1.2.1 Defining an algebra: by basis and structure constants

Let k be a commutative ring. How might we define an algebra over k?

One way to define an algebra is to give a basis and the 'structure constants' — the associative multiplication rule on this basis. (See also §2.2.)

(**1.2.1**) Example. A group algebra for a given group, as in 1.1.34, is a very simple example of this.

de:Pn (**1.2.2**) For S a set, P_S is the set of partitions of S. Let $n, m \in \mathbb{N}$. Define $\underline{n} = \{1, 2, ..., n\}$ and $\underline{n'} = \{1', 2', ..., n'\}$ and $N(n,m) = \underline{n} \cup \underline{m'}$. We recall the *partition algebra*.

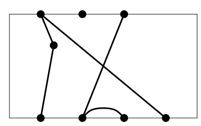
Fix a commutative ring k, and $\delta \in k$. Firstly, the partition algebra $P_n = P_n(\delta)$ over k is an algebra with a basis $\mathsf{P}_{N(n,n)}$. That is, as a k-module,

$$P_n \;=\; k\mathsf{P}_{N(n,n)} \tag{1.17}$$ de:Pn1

In order to describe a suitable multiplication rule on $\mathsf{P}_{N(n,n)}$ it is convenient to proceed as follows. (One can alternatively proceed purely set-theoretically. See e.g. [86].)

de:regu (**1.2.3**) A graph g determines a partition $\pi(g)$ of its vertex set V (into the connected components of g) — and hence determines a partition $\pi_{V'}(g)$ of any subset V' of V by restriction. We may represent a partition of $N(n,m)$ as an (n,m)-*graph*. An (n,m)-graph is a 'regular' drawing d of a graph g in a rectangular box with vertex set including $N(n,m)$ on the frame — unprimed $1, 2, ..., n$ left-to-right on the northern edge; primed $1', 2', ..., m'$ on the southern.

'Regular' means in effect that d determines g. We show in (1.2.8) that such drawings exist. Here is an example of a (3,4)-graph:

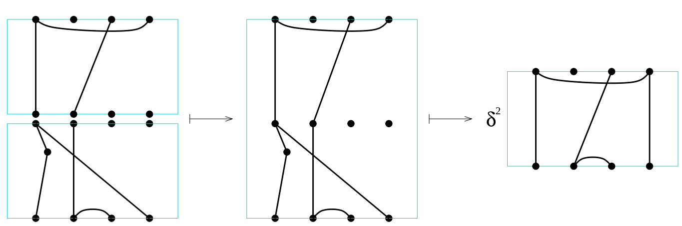


(1.18) eq:reggrapheg1

(**1.2.4**) If d is such a graph drawing, then $\pi_{n,m}(d) \in \mathsf{P}_{N(n,m)}$ is the partition with $i, j \in N(n,m)$ in the same part if they are in the same connected component in d.

For us any d such that $\pi_{n,m}(d) = p$, and such that every vertex is in a connected component with an element of $N(n,m)$, serves as a picture of p. A connected component in such a graph is *internal* if it has no vertices on either external edge. A graph d with $l_i(d)$ internal components denotes an element $\pi^{\delta}_{n,m}(d) \;=\; \delta^{l_i(d)}\pi_{n,m}(d)$ of $k\mathsf{P}_{n,m}$. (We also extend this k-linearly in the obvious way.)

(**1.2.5**) Note that a suitable (n,m)-graph d will stack over an (m,l)-graph d' to make an (n,l)-graph $d|d'$ in the manner indicated in the first step in (1.19):



(1.19) eq:Ppic1

(the second step shown tidies up to a scalar×graph with the same image). We then compute the product $p * p'$ of $p, p' \in \mathsf{P}_{N(n,n)}$ by

$$p * p' \;=\; \delta^{l_i(d|d')}\; \pi_{n,n}(d|d') \tag{1.20}$$ eq:palgx1

where d, d' are pictures for p, p' respectively.

Assuming that the general idea for diagram composition is clear from this picture (!), then in this approach to P_n we next have to check the following.

(**1.2.6**) PROPOSITION. *The composition $*$ is well-defined and associative.*

For now this is left as an exercise (see §2.2.3 or Chapter 12).

We extend $*$ k-linearly to $k\mathsf{P}_{N(n,n)}$.

(**1.2.7**) Remark: By (1.17) the rank of P_n as a free k-module is the Bell number B_{2n}. In particular if k is a field then P_n is Artinian (cf. 1.3.25).

1.2.2 Aside on pictures of partitions

In (1.2.3) we said of a drawing d of a graph d that 'Regular' means in effect that d determines g. We show in (1.2.8) that such drawings exist.

de:regdraw (**1.2.8**) Let $\mathcal{G}[S]$ denote the class of finite graphs whose vertex set contains 'external' ordered subset S. A polygonal embedding of $g \in \mathcal{G}[S]$ with full vertex set V is an embedding e in $\mathbb{R}^3$ — vertices to points; edges to polygonal arcs ending at the appropriate points. We also require that y values in $e(g)$ lie in an interval $[0, h]$ for some 'height' h, with the bounds saturated only by the points in $e(S)$; and that external vertex points lie (at WLOG integral points?) on $(x, 0, 0)$ or $(x, h, 0)$.

A *regular embedding* is one such that the projection $p(x, y, z) = (x, y)$ into $\mathbb{R}^2$ is regular in the usual knot theory sense [29]. The point is that one can recover g from the datum $d = (V, \lambda, L)$ consisting of the injective map $\lambda : V$ where $\lambda = p \circ e|_V$, which amounts to a labelling of certain points in the image $L = p(e(g))$; and the image L itself. We call d a regular drawing. (Note that h is not necessarily determined by d and that if $h > 0$ then one can rescale to any other $h > 0$. Note that an analogous finite 'width' of d can be chosen, and is similarly subsidiary to the main datum.)

Note that such an embedding exists for every g (cf. e.g. [29] or §**??**). Let $\mathcal{E}[S]$ denote the class of regular drawings over $\mathcal{G}[S]$.

A regular drawing d is a containing rectangle R in $\mathbb{R}^2$; a set V and an injective map $\lambda : V \to R$; and a subset L of R that is the projection p of a regular embedding of some $g \in \mathcal{G}[S]$ (i.e. a collection of possibly crossing lines). That is (suppressing R) $\quad d = (V, \lambda, L)$.

PROPOSITION. There is a surjective map $\Pi : \mathcal{E}[S] \to \mathcal{G}[S]$. ■

On this basis, when we confuse/identify a drawing with the graph it determines, we mean the graph.

Note that in the case of an (n, m)-graph we can even omit the vertex labels, since these are determined by the ordering on the line for external vertices, and are unimportant for other vertices.

1.2.3 Examples and useful notation for set partitions

[illegible] (**1.2.9**) See Table 1.1 for examples and notations. Given a partition p of some subset of $N(n, m)$, take $p^\star$ to be the image under toggling the prime. Define partition $p_1 \otimes p_2$ by side-by-side concatenation of diagrams (and hence renumbering the p_2 factor as appropriate). See Table 1.1 for examples.

$\mathsf{v} = \{\{1\}\} =$	$\mathsf{U} = \{\{1,2\}\} =$	$\mathsf{u} = \mathsf{v} \otimes \mathsf{v}^\star =$
$\mathsf{v}^\star = \{\{1'\}\} =$	$\Gamma = \{\{1,2,1'\}\} =$	$\mathsf{u}_1 := \mathsf{u} \otimes 1 \otimes 1 \otimes \ldots \otimes 1 =$
$1 = \{\{1,1'\}\} =$	$\sigma = \{\{1,2'\},\{2,1'\}\}$	$\mathsf{u}_2 := 1 \otimes \mathsf{u} \otimes 1 \otimes \ldots \otimes 1$
$\mathsf{u} = \{\{1\},\{1'\}\} =$	$\square = \{\{1,2,1',2'\}\}$	$\mathsf{e} := \mathsf{U} \otimes \mathsf{U}^\star$

Table 1.1: Set partitions: examples and notations tab:part1

de:pnotations (**1.2.10**) Let $\mathsf{P}_{n,m} := \mathsf{P}_{N(n,m)}$. We say a part in $p \in \mathsf{P}_{n,m}$ is *propagating* if it contains both primed and unprimed elements. Write $\mathsf{P}_{n,l,m}$ for the subset of $\mathsf{P}_{n,m}$ with l propagating parts; and $\mathsf{P}^l_{n,m}$ for the subset of $\mathsf{P}_{n,m}$ with at most l propagating parts. Thus

$$\mathsf{P}^l_{n,m} = \bigsqcup_{l=0}^{l} \mathsf{P}_{n,l,m} \qquad \text{and} \qquad \mathsf{P}_{n,m} = \bigsqcup_{l=0}^{n} \mathsf{P}_{n,l,m}.$$

E.g. $\mathsf{P}_{2,2,2} = \{1 \otimes 1, \sigma\}$, $\mathsf{P}_{2,1,1} = \{\mathsf{v} \otimes 1, 1 \otimes \mathsf{v}, \Gamma\}$, $\mathsf{P}_{2,0,0} = \{\mathsf{v} \otimes \mathsf{v}, \mathsf{U}\}$ and

$$\mathsf{P}_{2,1,2} = \mathsf{P}_{2,1,1}\mathsf{P}_{1,1,2} = \{\mathsf{u} \otimes 1, 1 \otimes \mathsf{u}, \mathsf{v} \otimes 1 \otimes \mathsf{v}^\star, \mathsf{v}^\star \otimes 1 \otimes \mathsf{v}, \Gamma\Gamma^\star, \ldots\}.$$

Note that $\mathsf{P}_{n,n,n}$ spans a multiplicative subgroup:

$$\mathsf{P}_{n,n,n} \cong S_n \tag{1.21}$$ eq:PnSnsub

Define $L : \mathsf{P}_{n,l,m} \to S_l$ by deleting all but the (top and bottom) leftmost elements in each propagating part, and renumbering consecutively. Define $\mathsf{P}^L_{n,l,m}$ as the subset with $L(p) = 1 \in S_l$.

(**1.2.11**) We have $P_0 \cong k$, $P_1 = k\{1, \mathsf{u}\}$ and

$$P_2 = k(\mathsf{P}_{2,2,2} \cup \mathsf{P}_{2,1,2} \cup \mathsf{P}_{2,0,2}) = k(\mathsf{P}_{2,2,2} \cup \mathsf{P}_{2,1,2} \cup \{\cup \otimes \cup^\star, (\mathsf{v} \otimes \mathsf{v}) \otimes \cup^\star, (\mathsf{v} \otimes \mathsf{v})^\star \otimes \cup, \mathsf{u} \otimes \mathsf{u}\}).$$

We have $\mathsf{u}^2 = \delta\mathsf{u}$ (but see Ch.12 for the definition of the algebra/category composition) and $\mathsf{v}^\star\mathsf{v} = \delta\emptyset$ and $\mathsf{v}\mathsf{v}^\star = \mathsf{u}$.

1.2.4 Defining an algebra: as a subalgebra

(**1.2.12**) Given a ring with 1 like P_n we can consider any subset S and ask what is the ring *generated* by S in P_n — the smallest subring containing this subset. For example, the ring generated by $\emptyset$ is the smallest subring, the ring $k1$.

de:TLn (**1.2.13**) Let $\mathsf{T}_{n,n} \subset \mathsf{P}_{n,n}$ be the subset of non-crossing pair partitions. (Here we follow [84, §9.5].) For example, $\mathsf{e} := \{\{1,2\},\{1',2'\}\} = \mathsf{U} \otimes \mathsf{U}^\star \in \mathsf{T}_{2,2}$; and for given n, $\mathsf{e}_1 := \mathsf{e} \otimes 1 \otimes 1 \otimes \ldots \otimes 1 \in \mathsf{T}_{n,n}$.

PROPOSITION. The $P_n = P_n(\delta)$ product $*$ from (1.20) closes on $k\mathsf{T}_{n,n}$. ■

Accordingly the subalgebra of P_n generated by $\mathsf{T}_{n,n}$ is also *spanned* k-linearly by $\mathsf{T}_{n,n}$ and we may define T_n as the subalgebra of the k-algebra P_n with basis $\mathsf{T}_{n,n}$:

$$T_n = T_n(\delta) = (k\mathsf{T}_{n,n}, *)$$

(**1.2.14**) Exercise. Show that there is also a subalgebra J_n with a basis of arbitrary pair-partitions.

(**1.2.15**) Remark. Historically the subalgebra J_n of P_n with basis of pair-partitions comes first [13] — the *Brauer algebra* B_n. We look at this in §**??** et seq.

1.2.5 Defining an algebra: by a presentation

For R a commutative ring, the free R-algebra on a set S is the R-monoid-algebra of the free monoid on S (all words in S, multiplied by concatentation, as in (1.1.6)). The elements of S are called *generators* of the algebra.

Given an algebra A, the quotient by an ideal I is another algebra, A/I. The quotient by the ideal generated (as an ideal) by an element a has the *relation* $a = 0$. Every algebra is isomorphic to the quotient of some free algebra by (an ideal defined by) some relations.

(**1.2.16**) Exercise. (I) Determine a minimal subset of $\mathsf{P}_{n,n}$ that generates P_n.
(II) Determine generators and relations for an algebra isomorphic to P_n.

de:TLiebn (**1.2.17**) For k a commutative ring, and $\delta \in k$, define the Temperley–Lieb algebra TL_n as the quotient of the free k-algebra generated by the symbols $U_1, U_2, ..., U_{n-1}$ by the relations

$$U_i^2 = \delta U_i$$

$$U_i U_{i\pm 1} U_i = U_i$$

$$U_i U_j = U_j U_i \qquad |i-j| \neq 1$$

Thus for example TL_2 has basis $\{1, U_1\}$; while $TL_3 = k\{1, U_1, U_2, U_1U_2, U_2U_1\}$ as a k-space. Note in the case TL_2 that the obvious bijection from this basis/generating set to $\{1, \mathsf{e}\}$ extends to an isomorphism $TL_2 \cong T_2$. We have the following.

(**1.2.18**) Theorem. (See e.g. [84, Co.10.1]) *Fix a commutative ring k and $\delta \in k$. For each n, $TL_n \cong T_n$.* ■

Hint: check that the map from the generators of TL_n to T_n given by $U_i \mapsto \mathsf{e}_i$ extends to an algebra homomorphism.

:TLbraidquotient (**1.2.19**) Suppose q a unit in k such that $\delta = q + q^{-1}$. The elements $g_i = 1 - qU_i$ in T_n obey the braid relations: $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$, $g_j g_i = g_i g_j$ $(|i-j| \neq 1)$. This establishes the following.

Proposition. Fix k and δ. Then T_n is a quotient of the group algebra of the braid group $\mathfrak{B}_n$ over k. □

1.2.6 More exercises

(**1.2.20**) Proposition. Assuming δ a unit,

$$P_{n-1} \cong \mathsf{u}_1 P_n \mathsf{u}_1 \tag{1.22}$$ eq:PUPU

$$P_n / P_n \mathsf{u}_1 P_n \cong kS_n. \tag{1.23}$$ eq:PPUPx

■

Remark: Our idea is to determine the representation theory of P_n (over a suitable algebraically closed field k) inductively from that of P_m for $m < n$, using (1.22). To this end we need to connect the two algebras. We will return to this problem shortly.

(**1.2.21**) Proposition. Assuming δ a unit,

$$T_{n-2} \cong \mathtt{e}_1 T_n \mathtt{e}_1 \tag{1.24}$$ eq:UTU2

$$T_n / T_n \mathtt{e}_1 T_n \cong k \tag{1.25}$$ eq:TTeT1

■

1.3 Modules and representations

The study of algebra-modules and representations for an algebra over a field has some special features, but we start with some general properties of modules over an arbitrary ring R. (NB, this topic is covered in more detail in Chapter 7, and in our reference list §1.14.)

A module over an arbitrary ring R is defined exactly as for a module over a group ring — (1.1.30) (NB our ring R here has taken over from RG not the ground ring, so there is no requirement of commutativity).

We assume familiarity with exact sequences of modules. See Chapter 7, or say [75], for details.

de:ideal0 (**1.3.1**) A *left ideal* of R is a submodule of R regarded as a left-module for itself. A subset $I \subset R$ that is both a left and right ideal is a *(two-sided) ideal* of R.

1.3.1 Preliminary examples of ring and algebra modules

:module examples

ex:ring001 (**1.3.2**) Example. Consider the ring $R = M_n(\mathbb{C})$. This acts on the space $M = M_{n,1}(\mathbb{C})$ of n-component column matrices by matrix multiplication from the left. Thus M is a left R-module.

ex:ring01 (**1.3.3**) Example. Consider the ring $R = M_2(\mathbb{C}) \oplus^{\cdot} M_3(\mathbb{C}) \subset M_5(\mathbb{C})$ as in §1.1.1. A general element in R takes the form

$$r = r_1 \oplus^{\cdot} r_2 \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus^{\cdot} \begin{pmatrix} e & f & g \\ h & i & j \\ k & l & m \end{pmatrix} \;\in\; R$$

Here, $M = \mathbb{C}\{(1,0)^T, (0,1)^T\} = \{\begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in \mathbb{C}\}$ is a left R-module with r acting by left-multiplication by $r_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; $M'' = M_2(\mathbb{C})$ is a left module with r acting in the same way; $M' = \{\begin{pmatrix} s \\ t \\ u \end{pmatrix} \mid s, t, u \in \mathbb{C}\}$ is a left module with r acting by r_2; and M'' is also a right module by right-multiplication by r_1.

Note that the subset of M'' of form $\begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix}$ is a left submodule.

(**1.3.4**) Our next example concerns a commutative ring, where the distinction between left and right modules is void. Consider the ring $\mathbb{Q}$. This acts on $(\mathbb{R},+)$ in the obvious way, making $(\mathbb{R},+)$ a left (or right) $\mathbb{Q}$-module. Here $(\mathbb{Q},+) \subset (\mathbb{R},+)$ is a submodule — indeed it is a minimal submodule, in the sense that any submodule containing 1 must contain this one. Note that this submodule (generated by 1) and the submodule generated by $\sqrt{2} \in \mathbb{R}$ do not intersect non-trivially. Note that here there is no 'maximal submodule'.

exe:funny1 (**1.3.5**) Exercise. Consider the ring R_χ of matrices of form $\begin{pmatrix} q & 0 \\ x & y \end{pmatrix} \in \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$. (Note that this is not an algebra over $\mathbb{R}$ and is not a finite-dimensional algebra over $\mathbb{Q}$.) Determine some submodules of the left-regular module.

Answer: (See also (1.3.26).) Consider the submodules of the left-regular module R_χ generated by a single element. Firstly:

$$\begin{pmatrix} q & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$$

— that is, there is a submodule of matrices of the form on the right, with $y \in \mathbb{R}$. Note that this submodule itself has no non-trivial submodules (indeed it is a 1-d $\mathbb{R}$-vector space). Then:

$$\begin{pmatrix} q & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$$

is again a 1-d $\mathbb{R}$-vector space. Finally consider

$$\begin{pmatrix} q & 0 \\ x & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ x & 0 \end{pmatrix}$$

Note that the submodule generated here, while not an $\mathbb{R}$-vector space, itself has the first case above as a submodule. The quotient has no non-trivial submodule (and indeed is a 1-d $\mathbb{Q}$-vector space).

(**1.3.6**) Our next example is a commutative finite dimensional algebra over a field k. As a k-space it is $R_A = k\{1, x, y\}$. The associative commutative ring multiplication is given on the generators by

$*$	1	x	y
1	1	x	y
x	x	0	0
y	y	0	0

Note that $R_A \cong k[x,y]/(x^2, y^2, xy)$.

As always the (left) regular module is generated by 1. Here $k\{x,y\}$ is a 2d submodule. Indeed any nonzero element of form $bx+cy$ spans a 1d submodule (indeed a nilpotent ideal); and the quotient of R_A by this submodule has a 1d submodule. We can construct the (left)-regular representation as follows. We first write the actions out in matrix form:

$$x\begin{pmatrix} 1 \\ x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ x \\ y \end{pmatrix}$$

$$y\begin{pmatrix}1\\x\\y\end{pmatrix}=\begin{pmatrix}y\\0\\0\end{pmatrix}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}1\\x\\y\end{pmatrix}$$

The matrices give, as usual, the regular antirepresentation. Since R_A is commutative this is also a representation — the 'cv-dual' representation ρ^o. Considering the action of a general element $\rho^o(a.1+b.x+c.y)$ on the corresponding 3d module we have

$$\begin{pmatrix}a&b&c\\0&a&0\\0&0&a\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}a\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}a&b&c\\0&a&0\\0&0&a\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}b\\a\\0\end{pmatrix},\qquad\begin{pmatrix}a&b&c\\0&a&0\\0&0&a\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}c\\0\\a\end{pmatrix}$$

Note that the first vector spans a simple submodule (on which x, y act like zero); and that the first and second vectors span a submodule; and the first and third (or the first and any linear combination of the second and third). The 'Loewy structure' is M^o here:

$$M^o = \begin{array}{ccc}\alpha & & \alpha\\ & \diagdown\ \diagup & \\ & \alpha & \end{array}, \qquad\qquad M = \begin{array}{ccc} & \alpha & \\ & \diagup\ \diagdown & \\ \alpha & & \alpha\end{array}$$

(but we will not explain this notation until §1.4.1). The transposes of these matrices give the regular representation, with the structure M above, as already noted.

pr:simpleinreg (**1.3.7**) Given a ring R and a left R-module M, then consider the set map $f: M \to \mathrm{Hom}_R(R, M)$ given by $f(m)(r) = rm$. Define the map $g: \mathrm{Hom}_R(R, M) \to M$ by $g(\psi) = \psi(1)$. For any $\psi \in \mathrm{Hom}_R(R, M)$ we have $f(\psi(1))(r) = r\psi(1) = \psi(r)$, so $f \circ g(\psi) = f(\psi(1)) = \psi$. Meanwhile $g \circ f(m) = g(r \mapsto rm) = 1m = m$. Thus f and g are inverse. We have shown the following.

PROPOSITION.

$$\mathrm{Hom}_R(R, M) \cong M$$

as sets.

It follows in particular that there is a nonzero module map from the regular module to each nonzero module.

1.3.2 Simple, semisimple and indecomposable modules

(**1.3.8**) A left R-module (for R an arbitrary ring) is *simple* if it has no non-trivial submodules. (See §7.2 for more details.)

In Example 1.3.3 both M and M' are simple; while R is a left-module for itself which is not simple, and M'' is also not simple.

de:semisim (**1.3.9**) A module M is *semisimple* if equal to the sum of its simple submodules.

de:dirsum01 (**1.3.10**) Suppose M', M'' submodules of R-module M. They *span* M if $M' + M'' = M$; and are *independent* if $M' \cap M'' = 0$. If they are both independent and spanning we write

$$M = M' \oplus M''$$

(*(module) direct sum*). A module is *indecomposable* if it has no proper direct sum decomposition.

(**1.3.11**) Example. Suppose $e^2 = e \in R$, then

$$Re \oplus R(1-e) = R \tag{1.26}$$ eq:projid1

as left-module.

Proof. For $r \in R$, $r = re + r(1-e)$ so $Re + R(1-e) = R$; and $re \in R(1-e)$ implies $re = re(1-e) = 0$. □

1.3.3 Jordan–Holder Theorem

(**1.3.12**) Let M be a left R-module. A *composition series* for M is a sequence of submodules $M = M_0 \supset M_1 \supset M_2 \supset ... \supset M_l = 0$ such that the section M_i/M_{i+1} is simple.

In particular if a composition series of M exists for some l then M_{l-1} is a simple submodule.

The sections of a composition series for M (if such exists) are *composition factors*. Their multiplicities up to isomorphism are called *composition multiplicities*. Given a composition series for M, write $(M : L)$ for the multiplicity of simple L.

th:JH (**1.3.13**) **Theorem.** (Jordan–Holder) Let M be a left R-module. (JHA) All composition series for M (if such exist) have the same factors up to permutation; and (JHB) the following are equivalent:
(I) M has a composition series;
(II) every ascending and descending chain of submodules of M stops (these two stopping conditions separately are known as *ACC* and *DCC*);
(III) every sequence of submodules of M can be refined to a composition series.

Proof. Obviously (III) implies (I). See §7.3.2 for the rest.

(**1.3.14**) Note that this form of the Theorem does not address the question of conditions for a module to have a composition series. For now note the following.

le:JHkA (**1.3.15**) Lemma. *Suppose A is a finite dimensional algebra over a field. Then every finite dimensional A-module M has a composition series. And, by (JHA), multiplicity $(M : L)$ is well-defined independently of the choice of series. (Exercise.)*

1.3.4 Radicals, semisimplicities, and Artinian rings

de:nilideal0 (**1.3.16**) A *nil ideal* of R is a (left/right/two-sided) ideal in which every element r is nilpotent (there is an $n \in \mathbb{N}$ such that $r^n = 0$). A *nilpotent ideal* of R is an ideal I for which there is an $n \in \mathbb{N}$ such that $I^n = 0$. (So I nilpotent implies I nil.)

de:JacRad0 (**1.3.17**) The *Jacobsen radical* of ring R is the intersection of its maximal left ideals.

th:JL0 (**1.3.18**) Theorem. *The Jacobsen radical of ring R is the subset of elements that annihilate every simple module.* ■

(**1.3.19**) Ring R itself is a *semisimple ring* if its Jacobsen radical vanishes.

Remark: This term is sometimes used for a ring that is semisimple as a left-module for itself (in the sense of (1.3.9)). The two definitions coincide under certain conditions (but not always). See later.

de:lss (**1.3.20**) For the moment we shall say that a ring R is *left-semisimple* if it is semisimple as a left-module ${}_RR$ (cf. e.g. Adamson [2, §22]). There is then a corresponding notion of *right-semisimple*, however: THEOREM. A ring is right-semisimple if and only if left-semisimple.

The next theorem is not trivial to show:
THEOREM. The following are equivalent:

(I) ring R is left-semisimple.
(II) every module is semisimple (as in (1.3.9)).
(III) every module is projective (every short exact sequence splits — see also 1.3.54).

(**1.3.21**) THEOREM. The Jacobsen radical of ring R contains every nil ideal of R. ■[1]

Remark: In general the Jacobsen radical is not necessarily a nil ideal. (But see Theorem 1.3.27.)

(**1.3.22**) An element $r \in R$ is *quasiregular* if $1_R + r$ is a unit. The element $r' = (1_R + r)^{-1} - 1$ is then the *quasiinverse* of r. (See e.g. Faith [**?**].)

(**1.3.23**) THEOREM. If J is the Jacobsen radical of ring R and $r \in J$ then r is quasiregular. ■

1.3.5 Artinian rings

de:artinian (**1.3.24**) Ring R is *Artinian* (resp. *Noetherian*) if it has the DCC (resp. ACC, as in (1.3.13)) as a left and as a right module for itself.

th:fdalgebraa (**1.3.25**) Example: THEOREM. A finite dimensional algebra over a field is Artinian.

Proof. A left- (or right-)ideal here is a finite dimensional vector space. A proper subideal necessarily has lower dimension, so any sequence of strict inclusions terminates. □

de:funny ring (**1.3.26**) Aside: We say more about chain conditions in §7.3. Here we briefly show by an example that the left/right distinction is not vacuous (although, as the contrived nature of the example perhaps suggests, it will be largely irrelevant for us in practice). Consider the ring R_χ of matrices of form $\begin{pmatrix} q & 0 \\ x & y \end{pmatrix} \in \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$ as in (1.3.5). (Note that this is not an algebra over $\mathbb{R}$ and is not a finite-dimensional algebra over $\mathbb{Q}$.) We claim that R_χ is Artinian and Noetherian as a left module for itself. However we claim that there are an infinite chain of right-submodules of R_χ as a right-module for itself between $\begin{pmatrix} 0 & 0 \\ \mathbb{Q} & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ \mathbb{R} & 0 \end{pmatrix}$. Thus R_χ is left Artinian but not right Artinian.

To prove the left-module claims one can show that all possible candidates are $\mathbb{R}$-vector spaces, and finite dimensional. To prove the infinite chain claim, recall that one can form a set of infinitely many $\mathbb{Q}$-linearly-independent elements in $\mathbb{R}$ (else $\mathbb{R}$ is countable!). Order the beginning of this set as $B_n = \{1, b_1, b_2, ..., b_n\}$ (we have taken the first element as 1 WLOG), for $n = 0, 1, 2, ...$. We have $\mathbb{Q}B_0 = \mathbb{Q}$ and $\mathbb{Q}B_n \subset \mathbb{Q}B_{n+1}$ for all n, thus an infinite ascending chain. On the other hand there is an inverse limit B of the sequence B_n contained in $\mathbb{R}$ (perhaps this requires Zorn's Lemma/the axiom of choice!), so we can define a sequence B^n by eliminating 1 then b_1 and so on from $B = B^0$, giving an infinite descending chain $\mathbb{Q}B^n \supset \mathbb{Q}B^{n+1}$.

[1]We shall use ■ to mean that the proof is left as an exercise.

(**1.3.27**) THEOREM. If ring R Artinian then the Jacobsen radical is the maximal two-sided nilpotent ideal of R (i.e. it is nilpotent and contains all other nilpotent ideals). ■

th:nilrad0

(**1.3.28**) THEOREM. If ring R Artinian then ideal I nil implies I nilpotent. ■

(**1.3.29**) THEOREM. If a ring is left-semisimple (as in 1.3.20) then it is (left and right) Artinian and left Noetherian, and is semisimple (i.e. has radical zero). ■(See e.g. [2, Th.22.2].)

th:ARLJ

(**1.3.30**) THEOREM. *If ring R is Artinian with radical J then every simple left R-module is also a well-defined simple R/J-module; and this identification gives a complete set of simple R/J-modules.* ■

1.3.6 Schur's Lemma

Schur's Lemma appears in various useful forms. We start with a general one, then discuss a couple of special cases of particular interest for the representation theory of algebras over algebraically closed fields. (See §**??** for more details.)

lem:Schur

(**1.3.31**) **Theorem.** (Schur's Lemma) Suppose M, M' are nonisomorphic simple R modules. Then the ring $hom_R(M, M)$ of R-module homomorphisms from M to itself is a division ring; and $hom_R(M, M') = 0$.

Proof. (See also 7.2.12.) Let $f \in \hom_R(M, M)$. M simple implies $\ker f = 0$ and $\operatorname{im} f = M$ or 0, so f nonzero is a bijection and hence has an inverse. Now let $g \in \hom_R(M, M')$. M simple implies $\ker g = 0$ and M' simple implies $\operatorname{im} g = M = M'$ or zero, so $g = 0$. □

ex:ring01a

(**1.3.32**) EXAMPLE. Let us return to ring R and module M from Example 1.3.3. In this case $\hom_R(M, M) \subset \hom_{\mathbb{C}}(M, M)$, and $\hom_{\mathbb{C}}(M, M)$ is all $\mathbb{C}$-linear transformations, so realised by $M_2(\mathbb{C})$ in the given basis. We see that $\hom_R(M, M)$ is the subset that commute with the action of R. This is the centre of $M_2(\mathbb{C})$, which is $\mathbb{C}1_2$, which is isomorphic to $\mathbb{C}$.

On the other hand, $\hom(M, M')$ is realised by matrices $\tau \in M_{3,2}(\mathbb{C})$:

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ \tau_{31} & \tau_{32} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \tau_{11}x + \tau_{12}y \\ . \\ . \end{pmatrix}$$

Here in $\hom_R(M, M')$ we look for matrices τ such that

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ \tau_{31} & \tau_{32} \end{pmatrix} r \begin{pmatrix} x \\ y \end{pmatrix} = r \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ \tau_{31} & \tau_{32} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

for all r, that is

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ \tau_{31} & \tau_{32} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e & f & g \\ h & i & j \\ k & l & m \end{pmatrix} \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \\ \tau_{31} & \tau_{32} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

but since $a, b, c, d, e, ..., m$ may be varied independently we must have $\tau = 0$.

(**1.3.33**) Remark. Cf. the occurence of the division ring in the general proof with the details in our example. We can consider the occurence of the division ring in Schur's Lemma as one of the main reasons for studying division rings alongside fields.

Next we talk about the specifics of the division ring $\hom_R(L,L)$ from Schur's Lemma, and the case where R is an algebra (over $\mathrm{Cen}(R)$ say), and then specifically an algebra over an algebraically closed field as in Ex.1.3.32.

We start with the case that R is the simplest kind of semisimple ring — a simple ring — which has only one possibility for L.

de:simplering (**1.3.34**) A ring R is a *simple ring* if R is semisimple and has no proper ideals. (Equivalently to the ideal condition we can say that there is only one isomorphism class of simple left modules.)

(**1.3.35**) An algebra that is simple as a ring is a *simple algebra*.

If A is a simple k-algebra and $k = \mathrm{Cen}(A)$ then we call A a *full simple algebra*. (Others call this a *central simple algebra*, see e.g. [**?**].)

If algebra A is division as a ring we call it a *division algebra*.

(**1.3.36**) Suppose A a simple algebra and L a simple A-module. Then the ring $E = \mathrm{Hom}_A(L,L)$ is division (Schur). In fact here one can show that $A \cong \mathrm{Hom}_E(L,L)$. And

$$\mathrm{Cen}(A) \;\cong\; \mathrm{Cen}(E) \tag{1.27}$$ eq:cen1

and, writing r for the number of copies of L in ${}_AA$ then

$$A \;\cong\; \mathrm{Hom}_E(L,L) \;\cong\; M_r(E^{op})$$

And

$$\mathrm{Cen}(E) \;\cong\; \mathrm{Cen}(E^{op}) \tag{1.28}$$ eq:cen2

Via (1.27) and (1.28) we have that E is a k-algebra, and finally (cf. (1.6.24))

$$A \;\cong\; M_r(k) \otimes_k E^{op}$$

(**1.3.37**) TO DO!

(**1.3.38**) Suppose R is an algebra over an algebraically closed field k (as in Example(1.3.32)). Then $\hom_R(M,M) \cong k$ in Schur's Lemma. It follows that any element of the centre of R acts like a scalar on simple M. Indeed we have the following.

Proposition. (I) A central element of R acts like a scalar on any indecomposable module (in the sense of 1.3.10 or §7.2.2). (II) A central element of R acts like the same scalar on every simple module in the same block (as defined in 1.3.41). ■

(**1.3.39**) Example. Consider the twist element of the braid group as in [84, §5.7.2]. The double-twist is clearly central. Hence its image is central in a quotient (such as T_n). We can use it to (partially) separate blocks. First we will need some indecomposable T_n-modules to work with. We will use $D_n^{\text{m}}(l)$ as in (1.35).

1.3.7 Ring direct sum, blocks, Artin–Wedderburn Theorem

de:ringdirectsum (**1.3.40**) Suppose that ring R has a decomposition of 1 into orthogonal central idempotents: $1 = \sum_i e_i$. Then each $R_i = Re_i$ is an ideal of R and a ring with identity e_i. In this case we say that R is a *ring direct sum* of the rings R_i, and write $R = \oplus_i R_i$. (Note that this is consistent with Example (1.3.3).)

de:block01 (**1.3.41**) A refinement of a central idempotent e is a decomposition $e = e' + e''$ where e', e'' are central orthogonal idempotents. A central idempotent e is *primitive central* if it cannot be written $e = e' + e''$ where e', e'' are central orthogonal idempotents.

If $1 = \sum_i e_i$ in (1.3.40) above is a primitive central idempotent (PCI) decomposition then it is unique up to reordering. (Proof: Suppose $1 = \sum_j e'_j$ is another. Since $e_i = \sum_j e_i e'_j$ this is a refinement of e_i unless $e_i e'_k = e_i$ for some k and other summands vanish. Similarly $e_i e'_k = e'_k$.)

(**1.3.42**) Example. Consider the algebra T_3 over the field of rational polynomials. This is an algebra of dimension 5. The element $\frac{1}{\delta}\mathsf{e}_1$ is idempotent, but not central. In fact the PCI decomposition is given by $1 = F + (1 - F)$ where

$$F = \frac{1}{\delta^2 - 1}(\delta(\mathsf{e}_1 + \mathsf{e}_2) - (\mathsf{e}_1\mathsf{e}_2 + \mathsf{e}_2\mathsf{e}_1)) \tag{1.29}$$ eq:T3PCI

(This result is not particularly easy to find, or even check, by brute arithmetic. It helps to know, as we shall later show in (**??**), that every PCI of T_n is fixed under the 'flip' automorphism.)

(**1.3.43**) If R is Artinian then there is a primitive central idempotent decomposition (cf. Th.1.5.7), and the rings R_i for the primitive decomposition are called the *blocks* of R.

A central idempotent acts like 1 or 0 on a simple module L. Thus if R is Artinian then precicely one primitive central idempotent acts like 1 on L. We say L is in block i if $e_i L = L$.

(**1.3.44**) Example. In our T_3 example in (1.29) above we see that there are two blocks. This computation of F also works, by evaluation, to give the PCI over any field k in which $\delta^2 - 1$ has an inverse. And in other cases ($k = \mathbb{C}$ and $\delta = 1$ say) we may deduce that there is no possible PCI except 1, and hence only one block.

Note that if a primitive central idempotent such as F lies in a subalgebra then it is also a central idempotent there. But it is not necessarily primitive (since there may be more idempotents that are central in the subalgebra — the test for centrality may require commutation with fewer elements).

In particular note that F lies in the fixed ring of T_3 under the flip automorphism. It is not primitive there. We have orthogonal idempotents

$$E_\pm = \frac{1}{2(\delta \pm 1)}(\mathsf{e}_1 + \mathsf{e}_2 \pm (\mathsf{e}_1\mathsf{e}_2 + \mathsf{e}_2\mathsf{e}_1))$$

obeying $F = E_+ + E_-$. These idempotents are not central in T_3, but they are in the fixed ring.

(**1.3.45**) On the other hand a PCI, such as F, is also idempotent in a superalgebra (such as T_4 say). However here it may not be primitive or central.

th:AWI (**1.3.46**) **Theorem.** (Artin–Wedderburn) Suppose R is semisimple and Artinian. Then R is a direct sum of rings of form $M_{n_i}(K_i)$ ($i = 1, 2, ..., l$, some l) where each K_i is a division ring.

Proof. Exercise. (See also §7.3 or e.g. Benson [7, Th.1.3.5].) ■

(**1.3.47**) Note that a central idempotent decomposition of 1_R leads to an ideal decomposition of R; while an arbitrary orthogonal idempotent decomposition of 1_R leads to a left-module decomposition of R.

Evidently a central idempotent decomposition is an orthogonal idempotent decomposition, but such a decomposition may be refinable once the central condition is relaxed. The matrix algebra $M_n(K)$ has the n elementary matrix idempotents $\{e_i^n\}_i$, which are orthogonal and such that

$$1_{M_n(K)} = \sum_{i=1}^{n} e_i^n$$

so this gives us one way to refine the central idempotent decomposition of 1_R in a semisimple Artinian ring (as in 1.3.46) to an (ordinary) orthogonal idempotent decomposition:

$$1_R = \sum_{i=1}^{l} \sum_{j=1}^{n_i} e_j^{n_i}$$

(here the first sum needs interpretation — it comes formally from the direct sum). We say more about this in §1.5.

(**1.3.48**) With A-W in mind we can consider the ring $M_n(K)$ over division ring K as a left-module for itself. We have

$$_{M_n(K)}M_n(K) \cong nL \ := \ L \oplus L \oplus ... \oplus L$$

(module direct sum as in (1.3.10)) where L is simple. Note from 1.3.7 that this L is the *only* simple module of $M_n(K)$.

th:AW2 (**1.3.49**) Thus a *general* semisimple Artinian ring as in the A-W Theorem becomes, as a left-module for itself, a direct sum of simple modules $\{L_i\}_i$ (n_i copies of L_i for each i). Again by 1.3.7 *every* simple module arises in the left-regular module in this way.

(**1.3.50**) Typically (for us) our Artinian ring R *is* a finite-dimensional algebra over a field k (k lying in the centre of R). What can we say about dimensions?

For a ring of form $M_n(k)$ with k a field, the dimension of L above is n. However if R is a finite-dimensional algebra over a field k it does not follow automatically that the division rings K_i in A-W can be indentified with k.

th:ASTIcaveat (**1.3.51**) Note therefore that the above does not say, for an k-algebra over a field, that $\dim L_i = n_i$ in 1.3.49. For example, the $\mathbb{Q}$-algebra $A = \mathbb{Q}\{1, x\}/(x^2 - 2)$ is a simple module for itself of dimension 2. That is, Artin–Wedderburn here is rather trivial: $A = M_1(A)$.

Another perspective on this is that left-module $_AA$ in our example is simple, but it is not 'absolutely irreducible'. A k-algebra module is *absolutely irreducible* if it remains simple when we extend the ground field k (see e.g. §**??**). If we extend $\mathbb{Q} \subset \mathbb{C}$ by adding $\sqrt{2}$ then

$$1 \ = \ (1 + \frac{1}{\sqrt{2}}x) + (1 - \frac{1}{\sqrt{2}}x)$$

split semisimple is an orthogonal idempotent decomposition, so $_AA$ is no longer simple.

If every simple module of semisimple k-algebra A is absolutely irreducible then we say A is *split* semisimple.

pr:sumsquares

PROPOSITION. A sufficient condition for $\dim L_i = n_i$ in A–W is that k is algebraically closed. In this case we see that the k-dimension of the algebra is the sum of squares of the simple dimensions.

1.3.8 Krull–Schmidt Theorem over Artinian rings

Krull

(**1.3.52**) **Theorem.** (Krull–Schmidt) If R is Artinian then as a left-module for itself it is a finite direct sum of indecomposable modules (as in (1.3.10) or §7.2.2); and any two such decompositions may be ordered so that the i-th summands are isomorphic.

Proof. Exercise. (See also §7.3.2.)

1.3.9 Projective modules over arbitrary rings

ss:proj0001

(**1.3.53**) If $x : M \to M'$, $x' : M' \to M$ are R-module homomorphisms such that $x \circ x' = 1_{M'}$ then x is a *split surjection* (and x' a split injection).

de:iproj

(**1.3.54**) An R-module is *projective* if it is a direct summand of a free module (an R-module with a linearly independent generating set).

(**1.3.55**) EXAMPLE. $e^2 = e \in R$ implies left-module Re projective, since it is a direct summand of free module R, by (1.26).

th:proj intro

(**1.3.56**) **Theorem.** TFAE
(I) R-module P is projective;
(II) whenever there is an R-module surjection $x : M \to M'$ and a map $y : P \to M'$ then there is a map $z : P \to M$ such that $x \circ z = y$;
(III) every R-module surjection $t : M \to P$ splits.

Proof. Exercise. (See also §7.6.)

1.3.10 Structure of Artinian rings

:structArtinian1

th:ASTI

(**1.3.57**) If R is Artinian and J_R its radical then R/J_R is semisimple so by (1.3.46):

$$R/J_R \ = \ \oplus_{i \in l(R)} M_{n_i}(R_i)$$

for some set $l(R)$, numbers n_i and division rings R_i. There is a simple R/J_R-module (L_i say) for each factor, so that *as a left module*

$$R/J_R \cong \oplus_i n_i L_i$$

(i.e. n_i copies of L_i). There is a corresponding decomposition of 1 in R/J_R:

$$1 = \sum_i e_i$$

into orthogonal idempotents. One may find corresponding idempotents in R itself (see later) so that $1 = \sum_i e'_i$ there. This gives left module decomposition

$$R = \oplus_i n_i P_i$$

where (by (1.3.52)) the P_is are a complete set of indecomposable projective modules up to isomorphism.
(See also §7.7.)

1.3.11 Finite dimensional algebras over algebraically closed fields

(**1.3.58**) Let A be a finite dimensional algebra over an algebraically closed field k. Let $\{L_i\}_{i\in\Lambda}$ be a set of isomorphism classes of simple A-modules L_i. Then $\dim A \;\geq\; \sum_{i\in\Lambda}(\dim L_i)^2$; with equality iff the set is complete and A semisimple.

Proof. Cf. Prop.1.3.51 and 1.3.49. Exercise.

a2 (**1.3.59**) THEOREM. *For A as above, and J_A the radical, suppose ${}_AA$ filtered by a set $\{S_i\}$. Then $\sum_i(\dim S_i)^2 \;\geq\; \dim(A/J_A)$ with equality iff $\{S_i\}$ a (necessarily complete) set of simples.*

1.4 Nominal aims of representation theory

So, what are the aims of representation theory? For Artinian algebras they are, broadly and roughly speaking, to describe the (finite dimensional) modules, and their homomorphisms. One might also be looking for representations (i.e. module bases) with special properties (perhaps motivated by physics). But in any case, it is worth being a bit more specific about this ‘description’.

Typically, to start with, one is looking for *invariants* — properties of modules that would be manifested by any isomorphic algebra; so that one can, say, determine from representation theory whether two algebras are isomorphic (or more easily, that two algebras are *not* isomorphic).

An example of an invariant would be the number of isomorphism classes of simple modules — this would be the same for any isomorphic algebra... See (1.2.17) for a specific example.

de:fund inv (**1.4.1**) Given an Artinian algebra R (let us say specifically a finite dimensional algebra over an algebraically closed field k, so that each $R_i = k$ in (1.3.57)), we are called on
(A0) to determine a suitable indexing set $l(R)$ as in (1.3.57),
(A0’) to determine the blocks as a partition of $l(R)$,
(AI) to compute the fundamental invariants $\{n_i \;\;:\;\; i \in l(R)\}$,
(AII) to give a construction of the simple modules L_i,
(AIII) to compute composition multiplicites for the indecomposable projective modules P_i,
(AIV) to compute Jordan-Holder series for the modules P_i.
(AV) to compute some further invariants (see e.g. (1.4.9) below).

(**1.4.2**) Note that (AI) contains (A0), and completely determines the maximal semisimple quotient algebra up to isomorphism (by the Artin–Wedderburn Theorem). Aim (AII) is not an invariant, so does not have a unique answer; but having at least one such construction is clearly desirable in studying an algebra (and any answer for (AII) contains (AI)).

Of course there are unboundedly many nonisomorphic algebras with the same maximal semisimple quotient in general, so we need more information to classify non-semisimple algebras.

The aim (AIII) is an invariant, and tells us more about a non-semisimple algebra. Aim (AIV) contains (AIII). But still, (AIV) is not enough to classify algebras in general. It is very useful partial data, however. And we will usually consider this to be 'enough' for most purposes (applications, for example). We will say a little next about futher (and possibly complete) invariants; before returning to study the above aims in detail.

(**1.4.3**) At a further level, we might also try the following. To investigate the isomorphism classes of indecomposable modules (beyond projective modules).

(**1.4.4**) Some invariants are invariants of isomorphism classes of algebras. Some are invariants of 'Morita' equivalence classes of algebras (see §1.6.2). This latter is a weaker (but very useful) notion. The number $l(R)$ is an invariance of Morita equivalence. The multiset $\{n_i\}$ is an invariance of isomorphism.

ss:Loewy1

1.4.1 Radical series and socle of a module

(**1.4.5**) Fix an algebra A. Given an A-module M, its *radical* $\mathrm{Rad}(M)$ is the intersection of maximal submodules. The *radical series* of M is

$$M \supset \mathrm{Rad}\ M \supset \mathrm{Rad}\ \mathrm{Rad}\ M \supset ...$$

The sections $\mathrm{Rad}\ ^i M/\mathrm{Rad}\ ^{i+1}M$ are the *radical layers*. In particular

$$\mathrm{Head}(M) \ = \ M/\mathrm{Rad}\ M$$

$$\mathrm{Shoulder}(M) \ = \ \mathrm{Rad}\ M/\mathrm{Rad}\ ^2M \ = \ \mathrm{Head}(\mathrm{Rad}\ M)$$

pr:mradM

(**1.4.6**) Proposition. *(I) Module M is semisimple (of finite length) iff Artinian and Rad $M = 0$. (II) If a module M is Artinian then $M/Rad\ M$ is semisimple.* ■

(**1.4.7**) The *socle* $\mathrm{Soc}(M)$ of a module is the maximal semisimple submodule. One can form socle layers: $\mathrm{Soc}(M)$, $\mathrm{Soc}(M/\mathrm{Soc}(M))$, $\mathrm{Soc}((M/\mathrm{Soc}(M))/\mathrm{Soc}(M/\mathrm{Soc}(M)))$, ... in the obvious way. These layers do not agree, in general, with the reverse of the radical layers; but the lengths of sequences agree if defined.

(**1.4.8**) Let A be a finite dimensional algebra over an algebraically closed field. (Then the radical series of any finite dimensional module terminates; and the sections are semisimple modules, by Prop.1.4.6.) Here we put indexing set $l(A) = \Lambda(A)$. For the indecomposable projective A-modules $\{P_i\}_{i\in\Lambda(A)}$ then

$$\{P_i\}_{i\in\Lambda(A)} \leftrightarrow \{S_i = \mathrm{Head}(P_i)\}_{i\in\Lambda(A)}$$

is a bijection between indecomposable projectives and simples. In general we have

$$\mathrm{Head}(M) \ \cong \ \bigoplus_{i\in\Lambda(A)} \ \underbrace{m_i^0(M)}_{multiplicity} \ S_i$$

$$\text{Shoulder}(M) \;\cong\; \bigoplus_{i\in\Lambda(A)} m_i^1(M)\; S_i$$

(and so on) for some multiplicities $m_i^l(M) \in \mathbb{N}_0$.

A *radical Loewy diagram* of an Artinian module M gives the radical layers:

$$\begin{array}{ccllllll} M & = & S_{0,1} & S_{0,2} & S_{0,3} & ... & S_{0,l_0} & \\ & & S_{1,1} & S_{1,2} & S_{1,3} & S_{1,4} & ... & S_{1,l_1} \\ & & S_{2,1} & S_{2,2} & ... & & & \\ & & ... & & & & & \end{array}$$

(the multiset of simple modules $\{S_{0,1}, S_{0,2}, ...\}$ encodes Head(M) and so on). We give some examples in §1.4.2.

1.4.2 The ordinary quiver of an algebra

ss:quiv00

de:quiv1 (**1.4.9**) The ordinary quiver of an algebra. (...See §2.5 for details.)

How *do* we classify finite dimensional algebras (over an algebraically closed field) up to isomorphism; or up to Morita equivalence?

(**1.4.10**) An algebra is *connected* if it has no proper central idempotent. Every algebra is isomorphic to a direct sum of connected algebras, so it is enough to classify connected algebras (and then, for an arbitrary algebra, give its connected components).

de:basicalg0 (**1.4.11**) An algebra is *basic* if every simple module is one-dimensional. (See also (1.5.9).) Every algebra is Morita equivalent to (i.e. has an equivalent module category to) a basic algebra. So it is enough to classify basic connected algebras.

(**1.4.12**) The *Ext-matrix* $\mathcal{M}(A)$ of algebra A is given by the 'shoulder data'

$$\mathcal{M}(A)_{ij} \;=\; m_i^1(P_j)$$

A necessary condition for algebra isomorphism $A \cong B$ is that there is an ordering of the index sets such that $\mathcal{M}(A) = \mathcal{M}(B)$.

The *Ext-quiver* or *ordinary quiver* $Q(A)$ of algebra A is the matrix $\mathcal{M}(A)$ expressed as a graph. Note that $Q(A)$ is connected as a graph if A is connected as an algebra. Isomorphism $A \cong B$ implies isomorphic Ext-quivers, but not v.v.. However one can characterise any connected basic algebra A up to isomorphism using a quotient of the *path algebra* $kQ(A)$ of $Q(A)$ (given a quiver Q, then kQ is the k-algebra with basis of walks on Q and composition on walks by concatenation where defined, and zero otherwise [2]), as we describe in §**??**. Specifically we have the following.

(**1.4.13**) Theorem. [48, §4.3] *For any connected basic algebra A there is an ideal I_A in $kQ(A)$ (contained in $I_{\geq 2}$ and containing $I_{\geq m}$ for some m) such that*

$$A \cong kQ(A)/I_A$$

[2]Note that walks of length at least l span an ideal in kQ. Write $I_{\geq l}$ for this ideal.

Proof. First note that there is a surjective algebra homomorphism $\Psi : kQ(A) \to A$. The walks of length-0 pass to a set of idempotents such that $P_i = Ae_i$. The walks of length-1 from i to j pass to a basis for $e_i J_A e_j / e_i J_A^2 e_j$.

Next we need to show that the kernel of Ψ has the required form. See e.g. [7, Prop.1.2.8]. ■

(**1.4.14**) Thus we can determine (characterise up to isomorphism) such a connected basic A by computing $Q(A)$ and then giving elements of $kQ(A)$ that generate I_A. (Note however that generators for I_A are not unique in general.)

More generally then, one can determine an arbitrary algebra A by giving the corresponding data for its connected components; together with the dimensions of the simple modules.

(**1.4.15**) Given $A \cong kQ(A)/I_A$, we can recover structural data about the indecomposable projective modules as follows. Write e_a for the path of length 0 from vertex a (sometimes we just write $a = e_a$ for this). This is an idempotent in $kQ(A)$. Then

$$P_a = Ae_a$$

(identifying A with $kQ(A)/I_A$ here without loss of generality). Thus a basis for P_a is the set of all paths from a 'up to the quotient'. This is the path of length 0 (corresponding to the head); and all the paths of length 1 (the shoulder); and some paths of length 2; and so on.

Note that (the image of) $I_{\geq 1}$ lies in the radical of kQ/I_A, since the m-th power lies in $I_{\geq m} \equiv 0$. Hence the image of $I_{\geq 1}$ *is* the radical.

(**1.4.16**) Let us give some low-dimensional examples of algebras of form Q/I_A, where $I_A \subset I_{\geq 2}$ and $I_A \supset I_{\geq m}$ for some m.

For Q a single point then kQ is one-dimensional and $I_{\geq 2} = 0$. Indeed any kQ with $I_{\geq 1} = 0$ is semisimple — the quiver is just a collection of points. Let us give some non-semisimple examples. For

$$\begin{tikzcd} a \arrow[loop above, "u"] \end{tikzcd} \qquad \text{with relation } u^2 = 0$$

we have a 2d algebra with 1 simple S_a. The corresponding projective P_a is $P_a = Aa = k\{a, ua\}$ (it terminates here since $aua = ua = u$ and $u^2 a = 0$ and so on), in which $k\{ua\}$ is a submodule (of, in a suitable sense, length-1 elements) isomorphic to S_a. That is, a radical Loewy diagram for P_a is

$$\begin{array}{ccc} P_a & = & S_a \\ & & S_a \end{array}$$

There is a 1-simple algebra in each dimension obtained by replacing $u^2 = 0$ by $u^d = 0$.

Alternatively in 3d, we can take the quiver with 1 vertex and two loops u, v, together with the relations $uu = uv = vu = vv = 0$. The quiver

$$\begin{tikzcd} a & b \arrow[l, "x"'] \end{tikzcd} \qquad \text{with no relations}$$

(again $I_{\geq 2} = 0$ here) gives another 3d algebra, this time with 2 simples.

The quiver

$$\begin{tikzcd} a \arrow[r, bend left, "x"] & b \arrow[l, bend left, "s"] \end{tikzcd} \qquad \text{with } sx = 0$$

has basis $\{a, b, xa, sb, xsb\}$. (Note that the given relation is sufficient to make kQ/I_A finite, but otherwise an arbitrary choice for an example here.) The indecomposable projective Aa is generated by walks out of a: a, xa, $sxa = 0$, that is, it terminates after one step. The projective $P_b = Ab$ has walks b, sb, xsb, $sxsb = 0$.

(**1.4.17**) What about this?:

$$\begin{tikzcd} a \arrow[r, bend left, "x_{ab}"] & b \arrow[l, bend left, "x_{ba}"] \arrow[r, bend left, "x_{bc}"] & c \arrow[l, bend left, "x_{cb}"] \end{tikzcd} \qquad \text{with } x_{bc}x_{ab},\ x_{ba}x_{cb},\ x_{ba}x_{ab} \text{ and } x_{ab}x_{ba} - x_{cb}x_{bc} \text{ in } I_A.$$

(These relations are another arbitrary finite choice here. However these particular relations will appear 'in the wild' later.) We have $P_a = Aa = k\{a, x_{ab}a\}$. Next $P_b = Ab = k\{b, x_{ba}b, x_{bc}b, x_{ab}x_{ba}b\}$. Finally $P_c = Ac$. Note the submodule structure of P_b. As ever there is a unique maximal submodule Rad $P_b = k\{x_{ba}b, x_{bc}b, x_{ab}x_{ba}b\}$. The intersection of the maximal submodules of this, in turn, is spanned by $x_{ab}x_{ba}b$. Thus the radical layers of the projectives look like this:

$$P_a = \begin{array}{c} S_a \\ S_b \end{array} \qquad P_b = \begin{array}{ccc} & S_b & \\ S_a & & S_c \\ & S_b & \end{array} \qquad P_c = \begin{array}{c} S_c \\ S_b \\ S_c \end{array}$$

Remark. This case exemplifies a very interesting point: that the presence of a simple module as a compostion factor for a module always allows for a corresponding homomorphism from the indecomposable projective cover of that simple module. Here in particular there is no homomorphism from S_a to P_b, say, but there is a homomorphism from P_a to P_b. See later.

(**1.4.18**) What about this?:

$$\begin{tikzcd} a \arrow[r, bend left, "x"] & b \arrow[l, bend left, "s"] \arrow[loop above, "u"] \end{tikzcd}$$

Determine some conforming relations to make a finite quotient of kQ. ...

1.5 Idempotents, Morita hints, primitive idempotents

ss:xxid

1.5.1 Morita hints

We started by thinking about matrix representations of groups, and this has led us naturally to consider modules over algebras. Two components of this progression have been (i) the passage to natural new algebraic structures (from groups to rings to algebras) on which to study representation theory; and (ii) the organisation of representations into equivalence classes (de-emphasising the basis). Representation theory studies algebras by studying the structure preserving maps between algebras (a map from the algebra under study to a known algebra gives us the modules for the known algebra as modules for the new algebra). We could go further and de-emphasise the modules in favour of the maps between them. This is one route into using 'category theory' (cf. §1.6).

(**1.5.1**) Let A be an algebra over k and $e^2 = e \in A$ (e not necessarily central, cf. 1.3.40). The *Peirce decomposition* (or Pierce decomposition! [30, 32, §6]) of A is

$$A = \; eAe \oplus (1-e)Ae \oplus eA(1-e) \oplus (1-e)A(1-e) \quad = \quad \bigoplus_{i,j} e_i A e_j$$

where $e_1 = e$ and $e_2 = 1-e$. (Question: What algebraic structures are being identified here? This is an identification of vector spaces; but the algebra multiplication is also respected. On the other hand not every summand on the right is unital.)

This decomposition is non-trivial if $1 = e + (1-e)$ is a non-trivial decomposition. Set $A(i,j) = e_i A e_j$. These components are not-necessarily-unital 'algebras', and non-unit-preserving subalgebras of A. The cases $A(i,i)$ are unital, with identity e_i.

Can we study A by studying the algebras $A(i,i)$?

(**1.5.2**) Example. Consider $M_3(\mathbb{C})$ and the idempotent $e_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. We have the corresponding vector space decomposition (not confusing $\oplus$ with $\oplus^{\cdot}$)

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

(which is not necessarily a particularly interesting decomposition, but see later).

de:primid1 (**1.5.3**) If we can further decompose e into orthogonal idempotents then there is a corresponding further Peirce decomposition. This decomposition process terminates when some $e = e_\pi$ has no decomposition in A (it is 'primitive'). What special properties does $e_\pi A e_\pi$ have then?

(**1.5.4**) Later we will provide detailed answers to the questions raised above. For now, our next objective will be to construct some interesting examples. We return to this discussion in (7.6.13) and §8.4.1 and §11.11.2.

1.5.2 Primitive idempotents

(**1.5.5**) An orthogonal decomposition of 1 into primitive idempotents (in the sense of 1.5.3) is called a 'complete' orthogonal decomposition.

For examples see §8.3.1.

(**1.5.6**) Aside: Let $1 = \sum_{i \in H} e_i$ be an orthogonal idempotent decomposition, and extend the definition of $A(i,j)$ to this case. Note that we have a composition $A(i,j) \times A(k,l) \to A(i,l)$ given by $a \circ b = ab$ in A. But in particular $ab = 0$ unless $j = k$. Thinking along these lines we see that the orthogonal idempotent decomposition of $1 \in A$ gives rise to a category (see §1.6,§5.1) 'hiding' in A. The category is $A_H = (H, A(i,j), \circ)$.

th:eRe-Re1 (**1.5.7**) Theorem. *If a ring R is left or right Artinian then it has a complete orthogonal idempotent decomposition of 1, $1 = \sum_{i=1}^{l} e_i$ say, with $e_i R e_i$ a local ring.*

If $e_i R e_i$ is local then e_i is primitive and $R e_i$ is indecomposable projective. ■

(**1.5.8**) EXAMPLE. Fix a field k and $\delta \in k^*$. Recall the algebra T_n and idempotents $\frac{1}{\delta}\mathsf{e}_1$, $\frac{1}{\delta}\mathsf{e}_2 = \frac{1}{\delta}1 \otimes \mathsf{e} \otimes 1 \otimes 1 \otimes ...$ and so on. Consider the quotient algebra $T_n' = T_n/T_n\mathsf{e}_1\mathsf{e}_3T_n$.
PROPOSITION. (1) For $n > 3$ the element $\frac{1}{\delta}\mathsf{e}_1$ is idempotent but not primitive in T_n.
(2) For $i = 1, 2, ..., n-1$ the (image of the) idempotent $\frac{1}{\delta}\mathsf{e}_i$ is primitive in T_n'.
(3) We have a left-T_n'-module isomorphism $T_n'\mathsf{e}_i \cong T_n'\mathsf{e}_j$ for all i, j.

Proof. (1) We can see that $\mathsf{e}_1 T_n \mathsf{e}_2 \cong T_{n-2}$ as an algebra. But T_{n-2} is not local ring for $n > 3$.
(2) Note that $\mathsf{e}_1 T_n' \mathsf{e}_1 = k\mathsf{e}_1 \cong k$.
(3) Exercise. ■

On the other hand we have the following. Consider the fixed subring $\overline{T}_n$ of T_n with respect to the left-right diagram flip involutive automorphism. What can we say about analogous quotient algebras and analogous primitive idempotents in this case.

de:basicalgebra (**1.5.9**) An Artinian ring R, with complete set $\{e_1, e_2, ..., e_l\}$ of orthogonal idempotents, is *basic* if $Re_i \cong Re_j$ as left-R-modules implies $i = j$. (Cf. also (1.4.11).)

(**1.5.10**) EXAMPLE. The k-subalgebra of $M_2(k)$ given by $A_{1,1} = \{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in k\}$ has a complete set $\{e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}$. One easily checks that $A_{1,1}e_1 \not\cong A_{1,1}e_2$ (consider the action of e_1 on each side, say), so $A_{1,1}$ is basic.

On the other hand $M_2(k)$ has the same complete set, but $M_2(k)e_1 \cong M_2(k)e_2$, so $M_2(k)$ is not basic.

(**1.5.11**) One can check that if a finite-dimensional k-algebra A is basic then every simple R-module is 1-dimensional.

(**1.5.12**) (We will see shortly that) For every finite-dimensional k-algebra there is a basic algebra having an 'equivalent module category'.

1.5.3 General idempotent localisation

If $e^2 = e \in A$ and M an A-module, then eM is an eAe-module.

pr:eMsimple (**1.5.13**) PROPOSITION. *If M is a simple A-module; and $e^2 = e \in A$. Then eM is a simple eAe-module or zero.* ■*(See e.g. §11.11.2.)*

pr:eMJH (**1.5.14**) PROPOSITION. [Jordan-Holder localisation] *Let k be a field, and A a finite dimensional k-algebra. Let M be an A-module. Let $M \supset M_1 \supset ...$ be a Jordan–Holder series for M, with simple factors $L_i = M_i/M_{i+1}$. Let $e^2 = e \in A$. Then*
(I) $eM \supseteq eM_1 \supseteq ...$ becomes a JH series for ${}_{eAe}eM$ on deleting the terms for which $eM_i/eM_{i+1} = eL_i = 0$. In particular if $eL \neq 0$ for some simple L then composition multiplicity

$$(M : L) = (eM : eL).$$

Thus in particular, (II) if ${}_{eAe}eM$ is simple then the composition factors of M include a factor $L_e = eL_e$ appearing once, and any other factors L obey $eL = 0$.

(III) If ${}_{eAe}eAe$ is simple (i.e. if eAe is a copy of the ground field k) then the composition factors of $M = Ae$ are a simple head factor $L_e = eL_e$ appearing once, and any other factors L obey $eL = 0$.

Proof. (I,II) See e.g. (11.26). (III) Note that eAe simple as a left-module implies that it is local as a ring, so Ae is indecomposable projective, so has a unique maximal submodule M_1. Noting (II), we need only show that the head M/M_1 is not killed by e. For a contradiction suppose $e(M/M_1) = 0$. For any $M \supset M'$ we have $e(M/M') = eM/eM'$ (just unpack the definitions). Thus $e(M/M_1) = 0$ implies $eM/eM_1 = 0$, which implies $eM = eM'$. But $AeM = AeAe = Ae$ while $AeM_1 \subset Ae$, giving a contradiction. ■

In particular for the proof of (I) it will be convenient to have a category theoretic context...

1.6 Small categories and categories

ss:cat0001

See §5.1 for more details. Categories are useful from at least two different perspectives in representation theory. One is in the idea of de-emphasising modules in favour of the (existence of) morphisms between them. Another is in embedding our algebraic structures (our objects of study) in yet more general settings.

A *small category* is a triple $(A, A(-,-), \circ)$ consisting of a set A (of 'objects'); and for each element $(a, b) \in A \times A$ a set $A(a, b)$ (of 'arrows'); and for each element $(a, b, c) \in A^{\times 3}$ a composition: $A(a, b) \times A(b, c) \to A(a, c)$, satisfying associativity and identity conditions (for each a there is a 1_a in $A(a, a)$ such that $1_a \circ f = f = f \circ 1_b$ whenever these make sense).

(A *category* is a similar structure allowing larger classes of objects and arrows.)

(**1.6.1**) Example: A monoid is a category with one object.

(**1.6.2**) Example: $A = \mathbb{N}$ and $A(m, n)$ is $m \times n$ matrices over a ring R.

(**1.6.3**) Example: A is a set of R-modules and $A(M, N)$ is the set of R-module homomorphisms from M to N. (The category $R-$mod is the category of all left R-modules.)

de:Pcat1 (**1.6.4**) The product in (1.19) generalises to a category P in an obvious way, with object set $\mathbb{N}_0$. There is a corresponding T subcategory.

(**1.6.5**) We may construct an 'opposite' category A^o from category A, with the same object class, by setting $A^o(a, b) = A(b, a)$ and reversing the compositions.

1.6.1 Functors

(**1.6.6**) A *functor* is a map between (small) categories that preserves composition and identities.

de:functoreg0001 (**1.6.7**) Example: (I) If R is a ring and $e^2 = e \in R$ then there is a map $F_e : R-\text{mod} \to eRe-\text{mod}$ given by $M \mapsto eM$ that extends to a functor.

de:homfunctintro (**1.6.8**) (II) If R is a ring and N a left R-module then there is a map

$$\text{Hom}(N, -) : R-\text{mod} \to \mathbb{Z}-\text{mod}$$

given by $M \mapsto \text{Hom}(N, M)$. This extends to a functor by $L \xrightarrow{f} M \mapsto (N \xrightarrow{g} L \mapsto N \xrightarrow{f \circ g} M)$.

de:homfunctproj (**1.6.9**) The functor $\text{Hom}(N, -)$ has some nice properties. Consider a not-necessarily short-exact sequence $0 \longrightarrow M' \xrightarrow{\mu} M \xrightarrow{\nu} M'' \longrightarrow 0$ and its not-necessarily exact image

$$0 \longrightarrow \text{Hom}(N, M') \overset{\mu_N = \text{Hom}(N,\mu)}{\longrightarrow} \text{Hom}(N, M) \overset{\nu_N = \text{Hom}(N,\nu)}{\longrightarrow} \text{Hom}(N, M'') \longrightarrow 0.$$

$$N \xrightarrow{f} M' \qquad \mapsto \qquad N \xrightarrow{\mu\circ f} M$$

We can ask (i) if exactness at M' implies $\ker \mu_N = 0$; (ii) if exactness at M implies $\operatorname{im}\mu_N = \ker \nu_N$; (ii') if $\nu \circ \mu = 0$ implies $\nu_N \circ \mu_N = 0$; (iii) if exactness at M'' implies $\operatorname{im}\nu_N = \operatorname{Hom}(N, M'')$?

(i) Since μ injective, $\mu\circ f = \mu\circ g$ implies $f = g$. But then $\mu\circ f = 0$ implies $f = 0$, so $\ker \mu_N = 0$.

(ii) See (7.5.6). (The answer if yes if exact at M' *and* M.)

(ii') $\operatorname{Hom}(N, \nu) \circ \operatorname{Hom}(N, \mu) = \operatorname{Hom}(N, \nu \circ \mu) = 0$.

(iii) This does not hold in general. However if N is projective then by Th.1.3.56(II), given exactness at M'', every $\gamma \in \operatorname{Hom}(N, M'')$ can be expressed $\nu \circ g$ for some $g \in \operatorname{Hom}(P, M)$, so then (iii) holds.

We will give some more examples shortly — see e.g. (1.6.10).

ex:functy (**1.6.10**) Let $\psi : A \to B$ be an map of algebras over k. We define functor

$$\operatorname{Res}_\psi : B\!-\!\operatorname{mod} \;\to\; A\!-\!\operatorname{mod}$$

by $\operatorname{Res}_\psi M = M$, with action of $a \in A$ given by $am = \psi(a)m$ for $m \in M$; and by $\operatorname{Res}_\psi f = f$ for $f : M \to N$.
We need to check that Res_ψ extends to a well-defined functor, i.e. that every B-module map $f : M \to N$ is also an A-module map. We have $bf(m) = f(bm)$ for $b \in B$ and $m \in M$. Consider $af(m) = \psi(a)f(m) = f(\psi(a)m)$, where the second identity holds since $\psi(a) \in B$. Finally $f(\psi(a)m) = f(am)$ and we are done.

See §1.10.6 for properties of Res_ψ.

(**1.6.11**) In order to develop a useful notion of equivalence of categories we need the notion of a natural transformation — a map between functors.

1.6.2 Natural transformations, Morita equivalence, adjoints

ss:MEO

For now see (5.1.26) for natural transformations. A natural isomorphism is a natural transformation whose underlying maps are isomorphisms.

Two categories A, B are equivalent if there are functors $F : A \to B$ and $G : B \to A$ such that the composites FG and GF are naturally isomorphic to the corresponding identity functors.

(**1.6.12**) Two categories are equivalent if there are functors between them whose composite is in a suitable sense isomorphic to the identity functor. We talk about making this precise later. For now we will rather aim to build some illustrative examples.

de:adjointI (**1.6.13**) Consider functors $C \leftrightarrows^F_G C'$. Then (F, G) is an *adjoint pair* if for each suitable object pair M, N there are natural bijections $\operatorname{Hom}(FM, N) \mapsto \operatorname{Hom}(M, GN)$.

1.6.3 Aside: Special objects and arrows

(**1.6.14**) An arrow f is *epi* if $gf = g'f$ implies $g = g'$ (see e.g. Mitchell [**?**]).

Given a category $\mathcal{A}$ we write $A \overset{f}{\twoheadrightarrow} B$ if f is epi.

(**1.6.15**) An arrow f is *mono* if $fg = fg'$ implies $g = g'$.

Given a category $\mathcal{A}$ we write $A \overset{f}{\hookrightarrow} B$ if f is mono.

If $A \overset{f}{\hookrightarrow} B$ then we say A is a *subobject* of B.

(**1.6.16**) Next we should define the notions of isomorphism; isomorphic subobject; and balanced category.

de:projincat1 (**1.6.17**) An object P is *projective* if for every $P \xrightarrow{h} B$ and $A \stackrel{f}{\twoheadrightarrow} B$ then $h = ff'$ for some $P \xrightarrow{f'} A$. (Cf. (1.3.56)(II).)

(**1.6.18**) A category $\mathcal{A}$ *has enough projectives* if there is an $P \stackrel{f}{\twoheadrightarrow} A$, with P projective, for each object A.

de:zeroobject (**1.6.19**) An object O in category $\mathcal{A}$ is a *zero object* if every $\mathcal{A}(M,O)$ and $\mathcal{A}(O,M)$ contains a single element.
If there is a unique zero object we denote it 0. In this case we also write $M \xrightarrow{0} 0$ and $0 \xrightarrow{0} M$ for all the 'zero-arrows' (even though they are distinct); and $M \xrightarrow{0} N$ for the arrow that factors through 0.

de:kernelI (**1.6.20**) Here we suppose that $\mathcal{A}$ has a unique zero-object.
A *prekernel* of $A \xrightarrow{f} B$ is any pair $(K, K \xrightarrow{k} A)$ such that $fk = 0$.
A *kernel* of $A \xrightarrow{f} B$ is a prekernel $(K, K \xrightarrow{k} A)$ such that if $(K', K' \xrightarrow{k'} A)$ is another prekernel then there is a unique $K' \xrightarrow{g} K$ such that $kg = k'$.

(**1.6.21**) Note that if $(K, K \xrightarrow{k} A)$ is a kernel of f then k is mono, and K is an isomorphic suboject of A to every other kernel object of f (see later).
Exercise: consider the existence and uniqueness of kernels.

(**1.6.22**) Next we should define normal categories and exact categories; define exact sequences. —FINISH THIS SECTION!!!—

(**1.6.23**) A category of modules has a lot of extra structure and special properties compared to a generic category (see Freyd [45] or §**??** for details). For example: (EI) The arrow set $A(M,N) = \mathrm{Hom}(M,N)$ is an abelian group; composition of arrows is bilinear. (An *additive* functor between such categories respects this extra structure.) (SII) There is a unique object 0 such that $\mathrm{Hom}(M,0) \cong \mathrm{Hom}(0,M) \cong \{0\}$ for all M (by $0 : M \to 0$ we mean this zero-arrow — an abuse of notation!). (SIII) Given objects M, N there is a categorical notion of an object $M \oplus N$, and these objects exist. (SIV) There is a function *ker* associating to each arrow $f \in \mathrm{Hom}(M,N)$ an object K_f and an arrow $k_f \in \mathrm{Hom}(K_f, A)$ such that $f \circ k_f = 0$ (in the sense above), and (K_f, k_f) is in a suitable sense universal (see later).

This extra structure is useful, and warrants the treatment of module categories almost separately from generic categories. This raises the question of what aspects of representation theory are 'categorical' — i.e. detectable from looking at the category alone, without probing the objects and arrows as modules and module morphisms per se.

For example, the property of projectivity is categorical. (Exercise. Hint: consider $\mathrm{Hom}(P, -)$ and short exact sequences.) The property of an object being a set is not categorical (although this concreteness is a safe working assumption for module categories, fine details of the nature of this set are certainly not categorical).

1.6.4 Aside: tensor products

e:tensorprod0001 (**1.6.24**) Let R be a ring and $M = M_R$ and $N = {}_R N$ right and left R-modules respectively. Then there is a *tensor product* — an abelian group denoted $M \otimes_R N$ constructed as follows. Consider the formal additive group $\mathbb{Z}(M \times N)$, and the subgroup S_{MN} generated by elements of form $(m+m', n) - (m, n) - (m', n)$, $(m, n+n') - (m, n) - (m, n')$ and $(mr, n) - (m, rn)$ (all $r \in R$). We set $M \otimes_R N = \mathbb{Z}(M \times N)/S_{MN}$. (In essence $M \otimes_R N$ is equivalence classes of $M \times N$ under the relation $(mr, n) = (m, rn)$. See §7.4 for details.)

This construction is useful because it gives us, for each M_R, a functor $M_R \otimes -$ from R-mod to the category $\mathbb{Z}$-mod (of abelian groups). This has many useful generalisations.

1.6.5 Functor examples for module categories: globalisation

ss:glob1

de:GF1 (**1.6.25**) Let A be an algebra over k and $e^2 = e \in A$ as in §1.5 above. We define functor $G = G_e$

$$G_e : eAe\text{−mod} \;\to\; A\text{−mod}$$

by $G_e M = Ae \otimes_{eAe} M$ (as defined in §7.4) and $F_e : A - \text{mod} \to eAe - \text{mod}$ by $F_e N = eN$. (Exercise: check that there are suitable mappings of module maps.)

ex:GF1 (**1.6.26**) Exercise. Show the following.
(I) Pair (G_e, F_e) is an adjunction (as in (5.3.7)).
(II) Functor F_e is exact.
(III) Functor G_e is right exact, takes projectives to projectives and indecomposables to indecomposables. (See Th.7.5.19 et seq.)
(IV) The composite $F_e \circ G_e : eAe\text{−mod} \;\to\; eAe\text{−mod}$ is a category isomorphism.

Note from these facts that there is an embedded image of eAe−mod in A−mod (the functorial version of an inclusion). Cf. Fig.1.1. Functor G_e does not take simples to simples in general. (One can see this either from the construction or 'categorically'.) However since simples and indecomposable projectives are in bijective correspondence, we can effectively 'count' simples in A-mod by counting those in eAe-mod and then adding those which this count does not include. It is easy to see the following.
Proposition. Functor F_e takes a simple module to a simple module or zero. ■

th:simp0001 (**1.6.27**) Theorem. Let us write $\Lambda(A)$ for some index set for simple A-modules (up to isomorphism); and $\Lambda_e(A)$ for the subset on which e acts as zero. It follows from (1.6.26) that we may take $\Lambda(A) \setminus \Lambda_e(A)$ as index set $\Lambda(eAe)$, and hence

$$\Lambda(A) = \Lambda(eAe) \sqcup \Lambda_e(A).$$

Of course simples on which e acts as zero are also the simples of the quotient algebra A/AeA, so $\Lambda_e(A) = \Lambda(A/AeA)$. ■

Let us consider some examples.

pr:lams (**1.6.28**) Proposition. *Recall the partition algebra P_n from (1.2.2); and T_n from (1.2.13).*
For $\delta \in k$ a unit, we may take $\Lambda(P_n) = \Lambda(P_{n-1}) \sqcup \Lambda(kS_n)$. Thus

$$\Lambda(P_n) = \sqcup_{i=0,1,\ldots,n} \Lambda(kS_i).$$

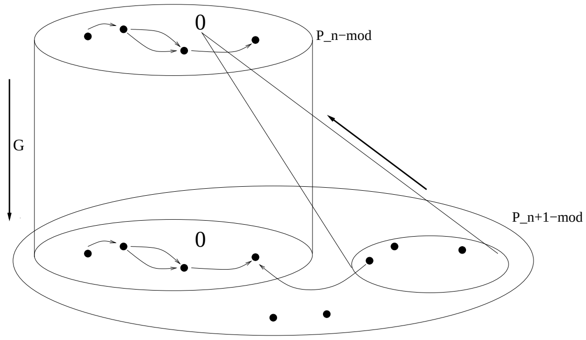


Figure 1.1: Schematic for the G-functor. fig:Pnmodembed1

Similarly $\Lambda(T_n) = \Lambda(T_{n-2}) \sqcup \Lambda(k)$. *Thus*

$$\Lambda(T_n) = \sqcup_{i=n,n-2,\ldots,1/0} \Lambda(k).$$

Proof. Consider in particular the functor $G = G_{\mathsf{u}_1}$ and use (1.22) (resp. 1.24) and (1.6.27). □

de:PDelt (**1.6.29**) Note that every simple module of P_n is associated to a symmetric group S_i irreducible for some $i \leq n$. Symmetric group irreducibles can be found in the heads of symmetric group Specht modules $\Delta^S_\lambda := kS_i v_\lambda$ (suitable $v_\lambda \in S_i$; these are classical constructions for irreducible modules over $\mathbb{C}$ that are well defined over any ground ring). Accordingly we define P_n-module $\Delta(\lambda) = \Delta_n(\lambda)$ by applying G-functors to Δ^S_λ as many times as necessary:

$$\Delta_n(\lambda) \;=\; G^{n-i}\Delta^S_\lambda \qquad (\lambda \vdash i)$$

Note that it follows that

$$F\Delta_n(\lambda) \;= \mathsf{u}_1\Delta_n(\lambda) \;\cong\; \Delta_{n-1}(\lambda)$$

and hence (by the Jordan-Holder localisation Theorem) that

$$(\Delta_n(\lambda) : L) \;=\; (\Delta_{n-1}(\lambda) : \mathsf{u}_1 L)$$

whenever the RHS makes sense (i.e. whenever $\mathsf{u}_1 L \neq 0$).

(**1.6.30**) If $k \supset \mathbb{Q}$ then v_λ can be chosen idempotent (indeed primitive). It follows that $\Delta(\lambda)$ is indecomposable projective in a suitable quotient algebra of P_n. Thus it has simple head. It follows that every module's structure can be investigated by investigating morphisms from these modules.

(**1.6.31**) Remark. The preceeding example will be very useful for analysing P_n-mod by induction on n. But first we think about some other examples, and how module categories and functors work with representation theory in general.

1.7 Modular representation theory

ss:mod0001

Sometimes an algebra is defined over an arbitrary commutative ring k. We may focus on the representation theory over the cases of k a field in particular. But the idea of considering all cases together provides us with some useful tools. (This follows ideas of Brauer [14]. See also, for example, Curtis–Reiner [**?**, Ch.2], Benson [**?**, Ch.1].)

Let R be a commutative ring with a field of fractions (R_0) and quotient field k (quotient by some given maximal ideal). (Ring R a complete rank one discrete valuation ring would be sufficient to have such endowments.) Let A be an R-algebra that is a free R-module of finite rank. Let $A_0 = R_0 \otimes_R A$ and $A_k = k \otimes_R A$ (we call these constructions 'base changes' from R to R_0 and to k respectively).

The working assumption here is that A_0 is relatively easy to analyse. (The standard example would be a group algebra over a sufficiently large field of characteristic zero; which is semisimple by Mashke's Theorem.) And that A_k is the primary object of study.

In particular, suppose that we have a complete set of simple modules for A_0. One can see (e.g. in (**??**)) that:

lem:liftem

(**1.7.1**) Lemma. For every A_0-module M there is a finitely generated A-module (that is a free R-module) that passes to M by base change. ■

Remark: Note that there can be multiple non-isomorphic A-modules all passing to M. (We will give examples shortly.)

(**1.7.2**) Let

$$\mathcal{D} \;=\; \{D^R(l) \;:\; l \in \Lambda = \{1, 2, ..., m\}\;\}$$

be an ordered set of A-modules that passes by base change to a complete ordered set $\mathcal{D}_0$ of m simple A_0-modules $D(l) = A_0 \otimes_A D^R(l)$. Let $D^k(l) = k \otimes D^R(l)$. Write

$$\mathcal{L}^k \;=\; \{L^k_\lambda \;:\; (\lambda \in \Lambda^k)\}$$

for a complete ordered set of simple A_k-modules.

(**1.7.3**) Fix k, and the ordering of Λ^k. There is then a decomposition matrix for any ordered set of modules. In particular, the choice of ordering of Λ gives us a $\mathcal{D}$-decomposition matrix D:

$$\mathsf{D}_{i\lambda} = [D^k(i) : L^k_\lambda]$$

(note that the index sets $\Lambda = \{1, 2, ..., m\}$ and Λ^k are not the same in general).
Remark: because all possible choices for $\mathcal{D}$ come from $\mathcal{D}_0$ we will see that the matrix D does not depend on $\mathcal{D}$ (althought there is potentially plenty of choice in $\mathcal{D}$).

Note that A^k is Artinian. Write P^k_λ for the projective cover of L^k_λ (the indecomposable projective with head L^k_λ), and e^k_λ for a corresponding primitive idempotent. One can show the following.

th:liftee

(**1.7.4**) Proposition. *(We assume suitable conditions on our base rings — see later.) There is a primitive idempotent in A that passes to e^k_λ, and an indecomposable projective A-module, $P^{k,R}_\lambda$ say, that passes to P^k_λ by base change.* (Caveat: A is not Artinian in general.)

For examples see §8.3.1.

(**1.7.5**) Since P^k_λ is projective, $\mathsf{D}_{i\lambda} = \dim \hom(P^k_\lambda, D^k(i))$. (Proof: For any indecomposable projective P^k_λ we have $\dim \hom(P^k_\lambda, M) = [M : L^k_\lambda]$ by the exactness property (as in (1.6.9)) of the functor

$\mathrm{Hom}(P^k_\lambda, -)$. For example one can use exactness and an induction on the length of composition series.)

On the other hand the free R-module $\hom(P^{k,R}_\lambda, D^R(i))$ has a basis which passes to a basis of $\hom(P^k_\lambda, D^k(i))$; and to a basis of $\hom(A_0 \otimes P^{k,R}_\lambda, A_0 \otimes D^R(i))$. Now suppose that A_0 is semisimple. A basis of the latter hom-set is the collection of maps, one for each simple factor of the direct sum $A_0 \otimes P^{k,R}_\lambda$ isomorphic to the simple module $A_0 \otimes D^R(i)$. That is, the dimension is the multiplicity of the A_0-simple module in $A_0 \otimes P^{k,R}_\lambda$. We have the following.

pr:mod recip (**1.7.6**) PROPOSITION. *(Modular reciprocity) Let (A, A_0, A_k) be as above, with A_0 semisimple (indeed split semisimple as in 1.3.51). Then*

$$[D^k(i) : L^k_\lambda] \;=\; [A_0 \otimes P^{k,R}_\lambda : A_0 \otimes D^R(i)].$$

■

(**1.7.7**) REMARK. (I) The Prop. does not say that P^k_λ has a filtration by $\{D^k(l)\}_l$. Indeed $\mathcal{D}$ could be a mixture of Specht and coSpecht modules, so that such a filtration would be unlikely. (While on the other hand such filtrations are certainly sometimes possible.)
(II) However D does not depend on the choice of $\mathcal{D}$.
(III) The Prop. does not determine any decomposition numbers. However, we have the following.

(**1.7.8**) For given k this says in particular that the Cartan decomposition matrix (with rows and columns indexed by Λ^k) is

$$C \;=\; \left([P^k_\lambda : L^k_\mu]\right) \;=\; \left(\sum_i \underbrace{(P^k_\lambda : D^k(i))}_{*}[D^k(i) : L^k_\mu] \right) \;=\; \mathsf{D}^T \mathsf{D} \tag{1.30}$$ eq:Cartan0001

(here $*$ is undefined, but can be understood here as in the Prop.). For an example see §1.11.1.

1.7.1 Modularity and localisation together

Now suppose there is an idempotent e in the algebra A in §1.7. With the 'localised' algebra $B = eAe$ we also have algebras $B_0 = eA_0e$ and $B_k = eA_ke$. With the quotient algebra $A^{(e)} = A/AeA$ we have $A^{(e)}_0 = A_0/A_0eA_0$ and so on.

Write Λ for the index set $\underline{m}$ here. Let the set

$$\Lambda_e \;:=\; \{l \in \Lambda \mid eD(l) \neq 0\}$$

and $\Lambda^k_e \;= \{\lambda \in \Lambda^k \mid eL^k_\lambda \neq 0\}$. By (1.6.26) we have a complete set of simple B_0-modules

$$\mathcal{D}^e_0 \;=\; \{eD(l) \mid l \in \Lambda_e\}$$

and a complete set of simple B_k-modules $\mathcal{L}^{k(e)} \;=\; \{eL^k_\lambda \mid \lambda \in \Lambda^k_e\}$.

The triple B, B_0, B_k and the sets $\mathcal{D}^e_0$ and $\mathcal{L}^{k(e)}$ obey the conditions in §1.7 so we can define

$$\mathsf{D}^e_{i\lambda} \;=\; [eD^k(i) : eL^k_\lambda]$$

whenever $i \in \Lambda_e$ and $\lambda \in \Lambda^k_e$. This gives a decomposition matrix for the B_k-modules $eD^k(i)$.

th:modlocal (**1.7.9**) THEOREM. [Modular localisation] $\mathsf{D}^e_{i\lambda} = \mathsf{D}_{i\lambda}$ *(i.e., whenever $i \in \Lambda_e$ and $\lambda \in \Lambda^k_e$).* ■

In other words

$$\mathsf{D} = \; i \left(\begin{array}{c|c} \mathsf{D}^e & \cdots \\ \hline \cdots & \cdots \end{array}\right) \; i \in \Lambda_e$$

That is, the multiplicities we do not know in terms of D^e include those of the modules $D^k(l)$ with $eD^k(l) = 0$. These are also modules for the quotient algebra $A_k^{(e)}$. Indeed any module obeying $eM = 0$ is also a module for the quotient.

See e.g. Pr.(14.6.12).

Note the following.

(**1.7.10**) LEMMA. *Suppose L a composition factor of M, a module for an Artinian algebra. Then $eL \neq 0$ implies $eM \neq 0$.* ■

Therefore $eM = 0$ implies $eL = 0$ and so the lower block (giving composition factors L obeying $eL \neq 0$ of $D^k(i)$'s obeying $eD^k(i) = 0$) is zero:

$$\mathsf{D} = \; i \left(\begin{array}{c|c} \mathsf{D}^e & \cdots \\ \hline 0 & \check{\mathsf{D}}^e \end{array}\right) \; i \in \Lambda_e$$

Meanwhile $\check{\mathsf{D}}^e$ encodes the multiplicities of simples obeying $eL = 0$ in $D^k(i)$'s obeying $eD^k(i) = 0$. Note that these are all modules of the quotient algebra $A_k^{(e)}$. So $\check{\mathsf{D}}^e$ can be considered as a decomposition matrix for certain modules of this algebra.

(**1.7.11**) Now suppose that the quotient algebra $A_k^{(e)}$ is semisimple. Then its simple modules are also projective.

CLAIM: there is an ordering so that $\check{\mathsf{D}}^e$ is the identity matrix.

Proof: 1. The Cartan decomposition matrix is the identity matrix. 2. ???

Examples

1.8 Example of (almost) everything: $TL_n(1)$

Here we look at a small Artinian ring which is non-commutative with non-zero radical. (This Artinian ring example is not entirely 'generic', however. It is isomorphic to its opposite. And it is basic. For other small examples see e.g. §3.1.1.)

Fix commutative ring k. Set $A = TL_3(1)$, i.e. TL_3 with $\delta = 1$. Recall from (1.2.17) that

$$TL_3(1) = k\langle 1, U_1, U_2, \rangle / \sim$$

where $\sim$ is the relations $U_i^2 = U_i$, $U_1U_2U_1 = U_1$ and $U_2U_1U_2 = U_2$.

1.8.1 Generalities

It will be clear from the relations that A is spanned by the five words 1, U_1, U_2, U_1U_2, U_2U_1. Thus if k is a field we have an Artinian ring/k-algebra. (Exercise: Show that these words are independent in A.)

(**1.8.1**) We have (as usual, see e.g. (9.1.4) for details) the contravariant functor $\mathrm{Hom}_k(-,k) : A-\mathrm{mod} \to \mathrm{mod}-A$: For every left-$A$-module N there is a dual right-module $N^* = \mathrm{Hom}_k(N,k)$. For a basis B_N of N the dual basis is the set of linear maps $\{f_b \mid b \in B_N\}$ given by $f_b(a) = \delta_{a,b}$ for $a \in B_N$. Alternatively N^* can be viewed as a left-module for the opposite algebra.

Consider the representation ρ_N of A afforded by B_N. The transpose matrices give a representation of the opposite algebra.

de:cv-functor1 (**1.8.2**) If A is isomorphic to its opposite then this gives an action of A on N^* again — we write N° for this contravariant dual module. Note that this construction lifts to a contravariant functor on $A-\mathrm{mod}$.

de:cv-rep1 (**1.8.3**) In our case A is isomorphic to its opposite under the map $i_A : A \to A^{op}$ that fixes the generators U_i (with $i_A(U_1U_2) = U_2U_1$ and so on). Thus the map from A to matrices given by the map on generators U_i to the transpose matrices $\rho_N(U_i)^t$ is also a representation of A. We write ρ_M° for this (the representation afforded by the contravariant dual module).

1.8.2 Regular module, basis and representation

ss:TL211

de:rep afforded (**1.8.4**) We may encode the linear action of $a \in A$ on a k-basis of a left A-module as a matrix $M(a)$, as follows. We arrange the basis as a column vector V (merely for convenience), on which a acts pointwise, then there is a unique $M(a)$ such that $aV = M(a)V$. The *representation afforded by ordered basis* V of the left module is given by the transposes of the matrices $M(a)$ (one easily sees that $a \mapsto M(a)$ is an antirepresentation, cf. also §3.1.1 and **??**).

In our case we have, for the left regular module:

$$U_1 \begin{pmatrix} 1 \\ U_1 \\ U_2 \\ U_1U_2 \\ U_2U_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ U_1 \\ U_2 \\ U_1U_2 \\ U_2U_1 \end{pmatrix}, \quad U_2 \begin{pmatrix} 1 \\ U_1 \\ U_2 \\ U_1U_2 \\ U_2U_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ U_1 \\ U_2 \\ U_1U_2 \\ U_2U_1 \end{pmatrix}$$

The representation afforded by this basis of the left regular module ${}_AA$ is given by the transposes of the above 5×5 matrices $M(U_i)$. This is the left-regular representation ρ_A (cf. (1.1.20)).

(**1.8.5**) Note that if we use row vectors V^t the corresponding matrices appear on the right: $U_iV^t = (U_iV)^t = (M(U_i)V)^t = V^t(M(U_i))^t = V^t\rho_A(U_i)$, and do not require transposition. (Note that we are still encoding the left action here. If we encode the right action on V: $VU_i = N(U_i)V$ then the matrices $N(U_i)$ again give a representation without transposition — the right-regular representation.)

(**1.8.6**) On the other hand, the algebra is isomorphic to its opposite via a map that fixes these generators (and transpose maps a matrix ring to its opposite), so these two matrices $M(U_i)$ themselves also give rise to a representation, the k-dual of the right-regular representation: $\rho_{A^*_A} : A \to M_5(k)$.

It is interesting to note that ρ_A and $\rho_{A^*_A}$ are not isomorphic in this case.

(**1.8.7**) It will be clear from the definition of A that there are two one-dimensional representations: $\rho_0(U_i) = 0$ and $\rho_1(U_i) = 1$.

lem:informa (**1.8.8**) By reciprocity the composition multiplicity of a simple module L_λ in the regular module is equal to $\dim P_\lambda$, and so at least equal to $\dim L_\lambda$. The bound is saturated for all simples if and only if A is semisimple — the dimension of the radical is $\dim(A) - \sum_\lambda \dim(L_\lambda)^2$. It follows that
(1) the 1d modules above are a complete set of simple A-modules;
(2) the dimension of the radical is 3;
(3) ${}_A A \cong P_0 \oplus P_1$, with dimensions 3 and 2 respectively.

1.8.3 Morphisms, bases and Intertwiner generalities

ss:IntertwineI

e:intertwinermat (**1.8.9**) An *intertwiner matrix* corresponding to a left-A-module map

$$\psi : M \to N$$

for the representations $\rho_M,\ \rho_N$ afforded by given bases, is a matrix X such that

$$X\ \rho_M(a) \quad = \quad \rho_N(a)\ X \qquad \forall\, a \in A \tag{1.31}$$ eq:intertwined

(N.B. I think that Curtis–Reiner [30, §29] have this the wrong way round.)

intertwinerspace (**1.8.10**) Define $\mathrm{Int}(\rho, \rho')$ as the k-space of intertwiners from representation ρ to ρ'.

Note that to verify $X \in \mathrm{Int}(\rho, \rho')$ it is sufficient to check (1.31) on generators of A.

de:cvimagemat (**1.8.11**) In cases (like $TL_3(1)$) with a generator-fixing opposite isomorphism we can effectively look simultaneously for $X \in \mathrm{Int}(\rho, \rho')$ and for $Y \in \mathrm{Int}(\rho'^{\circ}, \rho^\circ)$, since the transpose of (1.31) (on generators) gives the latter — the contravariant image of X.

(**1.8.12**) Note that there is a submodule (a left ideal) $M = \Delta_1$ of ${}_A A$ as in §1.8.2 spanned by $B_M = \{U_1, U_2 U_1\}$. We have

$$U_1 \begin{pmatrix} U_1 \\ U_2 U_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 U_1 \end{pmatrix}, \qquad U_2 \begin{pmatrix} U_1 \\ U_2 U_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 U_1 \end{pmatrix}$$

affording a corresponding representation ρ_M and cv dual ρ_M°.

(**1.8.13**) Exercise: look for intertwiners for ρ_M and ρ_M° to and from the 1d representations. (We do this shortly.)

1.8.4 Intertwiners between $M = \Delta_1$ and ${}_A A = TL_3(1)$

(**1.8.14**) Exercise: look for intertwiners corresponding to the module map

$$\psi : M \hookrightarrow {}_A A$$

and other module maps $\mu : M \rightarrow {}_A A$; and for possible maps $\phi : M^\circ \rightarrow {}_A A^\circ$.

de:IntX1 (**1.8.15**) We can look for ψ directly or (just because we can!) by looking for the cv image $\psi^\circ : {}_A A^\circ \rightarrow M^\circ$ (and then taking transpose). We have for $\rho^\circ_M X = X \rho^\circ_{{}_A A}$:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus $X \in \mathrm{Int}(\rho^\circ_{{}_A A}, \rho^\circ_M)$; and $X^t \in \mathrm{Int}(\rho_M, \rho_{{}_A A})$. (Note from the form of X^t how it specifically realises the inclusion ψ of M.)

(**1.8.16**) Are there other independent intertwiners in $\mathrm{Int}(\rho^\circ_M, \rho^\circ_{{}_A A})$? We have to simultaneously solve

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \end{pmatrix} = \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \end{pmatrix} = \begin{pmatrix} a & b & c & d & e \\ f & g & h & i & j \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

These give $a = c = e = f = g = i = 0$, $b = j$ and $d = h$, so $\mathrm{Int}(\rho_M, \rho_{{}_A A})$ is spanned by X above and

$$X' = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

(**1.8.17**) Is there an intertwiner in the other direction? Is there a splitting idempotent?

(**1.8.18**) Now for maps $\psi : A \rightarrow M$ we can look directly or at $\psi^\circ : M^\circ \rightarrow {}_A A^\circ$. For the latter we have

$$\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \\ i & j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \\ 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \\ i & j \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \\ i & j \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ e & f \\ g & h \\ i & j \end{pmatrix}$$

The space of intertwiners is thus the space of matrices of form X

$$X = \begin{pmatrix} a & b \\ a+b & 0 \\ 0 & a+b \\ a+b & 0 \\ 0 & a+b \end{pmatrix}$$

What about composite maps $M \to A \to M$? Note

$$\begin{pmatrix} 0 & x & 0 & y & 0 \\ 0 & 0 & y & 0 & x \end{pmatrix} \begin{pmatrix} a & b \\ a+b & 0 \\ 0 & a+b \\ a+b & 0 \\ 0 & a+b \end{pmatrix} = \begin{pmatrix} (x+y)(a+b) & 0 \\ 0 & (x+y)(a+b) \end{pmatrix}$$

That is, there are maps whose composite is the identity, and maps whose composite is zero.

1.8.5 Structure of $M = \Delta_1$ (maps between M and L_0 and L_1)

(**1.8.19**) Module M has a simple submodule $L_0 = k\{U_2U_1 - U_1\}$ (spanned by a single element) giving rise to the representation $\rho_0(U_i) = (0)$. We have

$$\phi : L_0 \hookrightarrow M$$

$$\rho_M(U_1)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}(0)$$

$$\rho_M(U_2)\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}(0)$$

which give $a + b = 0$. That is, the hom-space is 1d.

Evidently (for example from the trace) the composition factors of M are L_0 and L_1. However if we look for an intertwiner for $L_1 \to M$ (replace (0) by (1) above) we get $a + b = a$, $0 = b$, $0 = a$ and $a + b = b$, that is $a = b = 0$, so there is no intertwiner. It follows that M is non-split:

$$0 \to L_0 \to M \to L_1 \to 0$$

de:IntY1 (**1.8.20**) We can confirm that we also have $\mu : M^\circ \to L_0$ with intertwiner Y such that $\rho_0 Y = Y \rho_M^\circ$:

$$(0)(1, -1) = (1, -1)\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

$$(0)(1,-1) = (1,-1)\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

(**1.8.21**) Next we can look for $L_1 \to M^\circ$, or (its cv image) $M \to L_1$.

...

(**1.8.22**) We can combine Y from (1.8.20) with, say, X from (1.8.15):

$$\rho_0\, YX \;=\; Y\rho^\circ_M X \;=\; YX\rho^\circ_{AA}$$

1.8.6 Structure of A/M

Consider

$$U_1\begin{pmatrix} 1+M \\ U_2+M \\ U_1U_2+M \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1+M \\ U_2+M \\ U_1U_2+M \end{pmatrix}$$

$$U_2\begin{pmatrix} 1+M \\ U_2+M \\ U_1U_2+M \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1+M \\ U_2+M \\ U_1U_2+M \end{pmatrix}$$

Exercise: complete this analysis.

1.9 Modules and ideals for the partition algebra P_n

1.9.1 Ideals

We continue to use the notations as in (1.2.10) and so on.

(**1.9.1**) Note that the number of propagating components cannot increase in the composition of partitions in P_n (the 'bottleneck principle'). Hence $k\mathsf{P}^m_{n,n}$ is an ideal of P_n for each $m \leq n$, and we have the following ideal filtration of P_n

$$P_n = k\mathsf{P}^n_{n,n} \supset k\mathsf{P}^{n-1}_{n,n} \supset \dots \supset k\mathsf{P}^0_{n,n}. \tag{1.32}$$ eq:Pstar01

Note that the sections $\mathfrak{P}^m_{n,n} \;:=\; k\mathsf{P}^m_{n,n}/k\mathsf{P}^{m-1}_{n,n}$ of this filtration are bimodules, with bases $\mathsf{P}_{n,m,n}$.

(**1.9.2**) Write

$$P_n^{/m} \;:=\; P_n/k\mathsf{P}^m_{n,n}$$

for the quotient algebra.

(**1.9.3**) Note the natural inclusion

$$\mathsf{P}_{n,l,m} \otimes \mathsf{v}^\star \hookrightarrow \mathsf{P}_{n,l,m+1}$$

lem:natdecomp (**1.9.4**) Lemma. *For any $l \leq n$ there is a natural bijection*

$$\mathsf{P}_{n,l,n} \xrightarrow{\sim} \mathsf{P}^L_{n,l,l} \times \mathsf{P}_{l,l,l} \times \mathsf{P}^L_{l,l,n}$$

(the inverse map is essentially category composition in P as in 1.6.4).

1.9.2 Idempotents and idempotent ideals

(**1.9.5**) Lemma. If $\delta \in k^*$ then $\mathsf{u}_1 \in P_n$ is an unnormalised idempotent and
(I) The ideal $k\mathsf{P}^m_{n,n} \; = \; P_n(\mathsf{u}^{\otimes(n-m)} \otimes 1_m)P_n$
(II) $k\mathsf{P}_{n,m} = k\mathsf{P}^m_{n,m} \cong \; P_n(\mathsf{u}^{\otimes(n-m)} \otimes 1_m)$ as a left P_n-module.

(**1.9.6**) Note that $k\mathsf{P}^m_{n,l}$ is a left P_n-module (indeed a $P_n - P_l$-bimodule) for each l, m, and $k\mathsf{P}^{m-1}_{n,l} \subset k\mathsf{P}^m_{n,l}$ (assuming $n \geq l \geq m$). Hence there is a quotient bimodule

$$\mathfrak{P}^l_{n,l} = k\mathsf{P}^l_{n,l}/k\mathsf{P}^{l-1}_{n,l}$$

with basis $\mathsf{P}_{n,l,l}$.

There is a natural right action of the symmetric group S_l on this module (NB $S_l \subset P_l$), which we can use. Let $v_\lambda \in kS_l$ be such that $kS_l v_\lambda$ is a Specht S_l-module (an irreducible S_l-module over $\mathbb{C}$). Then define left P_n-module

$$D_\lambda \; = \; k\mathsf{P}_{n,l,l}\; v_\lambda.$$

(**1.9.7**) If $k \supset \mathbb{Q}$ then v_λ can be chosen idempotent, and this module D_λ is a quotient of an indecomposable projective module, and hence has simple head. It follows that if P_n is semisimple then the modules of this form are a complete set of simple modules.

(**1.9.8**) Exercise. What can we say about $\mathrm{End}_{P_n}(D_\lambda)$?

(**1.9.9**) Exercise. Construct some examples. What about contravariant duals?

(**1.9.10**) The case $n = 1$, $k = \mathbb{C}$. Fix δ. Artinian algebra P_1 has dimension 2. By (1.3.57) and (1.3.51) this tells us that either it is semisimple with two simple modules, or else it has one simple module.

Unless $\delta = 0$ then u/δ is idempotent so there are two simples. If $\delta = 0$ then u lies in the radical $J(P_1)$, and $P_1/J(P_1)$ is one-dimensional (semi)simple.

(**1.9.11**) The case $n = 2$, $k = \mathbb{C}$. Fix δ. Artinian algebra P_2 has dimension 15.
As we shall see, for most values of δ we have $P_2 \cong M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_3(\mathbb{C}) \oplus M_2(\mathbb{C})$.

(**1.9.12**) We have $P_n \subset P_{n+1}$ via the injection given, say, by $p \mapsto p \cup \{\{n+1, (n+1)'\}\}$, which it will be convenient to regard as an inclusion.

1.10 Modules and ideals for the algebra T_n

ss:ModidTn

Recall the definition (1.2.13) of T_n over k.

de:flippy (**1.10.1**) Note that the flip map $t \mapsto t^\star$ from (1.2.9) obeys $(t_1 t_2)^\star = t_2^\star t_1^\star$. It follows that the flip $\star$ defines an involutive antiautomorphism of T_n. Thus T_n is isomorphic to its opposite.

(**1.10.2**) Fix n. Set $\mathsf{e}_1^{2l-1} = \mathsf{e}_1\mathsf{e}_3 ... \mathsf{e}_{2l-1}$ (l factors). Thus $\mathsf{e}_1^{2l-1} \in \mathsf{T}_{n,n-2l,n}$ ($n - 2l$ propagating parts, cf. 1.2.10). If $\delta \in k^*$ set $\bar{\mathsf{e}}_1^{2l-1} = \delta^{-l}\mathsf{e}_1\mathsf{e}_3 ... \mathsf{e}_{2l-1}$. Then the ideal $T_n \mathsf{e}_1\mathsf{e}_3 ... \mathsf{e}_{2l-1} T_n$ has basis $\mathsf{T}^{n-2l}_{n,n}$ ($n - 2l$ or fewer propagating parts, cf. 1.2.10). Write

$$T_n^{/n-2l} \; := \; T_n/(T_n \mathsf{e}_1\mathsf{e}_3 ... \mathsf{e}_{2l-1} T_n)$$

for the quotient algebra by this ideal (with a basis of diagrams with more that $n-2l$ propagating lines). In particular, (1.25) becomes $T_n^{/n-2} \cong k$.

Note that $\mathsf{e}_1 T_n^{/n-4} \mathsf{e}_1 \cong T_{n-2}^{/n-4} \cong k$ and $\mathsf{e}_1\mathsf{e}_3 T_n^{/n-6} \mathsf{e}_1\mathsf{e}_3 \cong T_{n-4}^{/n-6} \cong k$ and so on. By 1.5.7 this says that $\frac{1}{\delta}\mathsf{e}_1$ is a primitive idempotent in $T_n^{/n-4}$ and $\bar{\mathsf{e}}_1^3$ is primitive in $T_n^{/n-6}$ and so on:

pr:idqT1 (**1.10.3**) PROPOSITION. *Suppose $\delta \in k^*$. The image of $\bar{\mathsf{e}}_1^{2l-1}$ is a primitive idempotent in the quotient algebra $T_n^{/n-2l-2}$.* □

1.10.1 Propagating ideals

Let $\mathsf{T}_{n,m}^l$ denote the subset of $\mathsf{T}_{n,m}$ of partitions with $\leq l$ propagating lines as above. Note

$$k\mathsf{T}_{n,m}^l = k\mathsf{T}_{n,l} * k\mathsf{T}_{l,m}. \qquad (1.33)$$ eq:catfilti

Analogously to the P_n case (1.32) we have an ideal filtration:

$$T_n = k\mathsf{T}_{n,n}^n \supset k\mathsf{T}_{n,n}^{n-2} \supset ... \supset k\mathsf{T}_{n,n}^{0/1}$$

Similarly $k\mathsf{T}_{n,m}^l \supseteq k\mathsf{T}_{n,m}^{l-2}$ for any l, m, n. Write

$$\mathfrak{T}_{n,m}^l = k\mathsf{T}_{n,m}^l / k\mathsf{T}_{n,m}^{l-2}$$

for the section bimodule, with basis $\mathsf{T}_{n,l,m}$. Note that for $l \leq n, m$ we have a bijection

$$* : \mathsf{T}_{n,l,l} \times \mathsf{T}_{l,l,m} \xrightarrow{\sim} \mathsf{T}_{n,l,m} \qquad (1.34)$$ eq:cartax

The inverse is called 'polar decomposition' of a TL diagram.

1.10.2 C-modules ('half-diagram modules')

As a *left*-module $\mathfrak{T}_{n,n}^l$ decomposes as a direct sum:

$$_{T_n}\mathfrak{T}_{n,n}^l \;\cong\; \bigoplus_{w \in \mathsf{T}_{l,l,n}} k\mathsf{T}_{n,l,l} w$$

where each $k\mathsf{T}_{n,l,l}w$ is a left-module by the algebra action on the quotient; and these modules are pairwise isomorphic. In other words we have a filtration of the regular module T_n by the modules

$$C_n^{\text{TL}}(l) \;=\; \mathfrak{T}_{n,l}^l \;=\; k\mathsf{T}_{n,l,l}\;,$$

$l = n, n-2, ..., 1/0$.

th:Clemma (**1.10.4**) THEOREM. *For each n we have the following. (0) The left-regular module T_n is filtered by C-modules. (I) $\sum_l (\dim C_n^{\text{TL}}(l))^2 = \dim T_n$. (II) If k a field and T_n semisimple then $\{C_n^{\text{TL}}(l)\}_l$ is a complete set of simples.*

Proof. (0) By construction. (I) Consider (1.34) and the analysis preceeding it. (II) Cf. (I) and Th.(1.3.59). □

Next we will show that these modules $\{C_n^{\text{TL}}(l)\}_l$ are indecomposable.

1.10.3 D-modules ('standard modules')

By Prop.1.10.3 the $T_n^{/n-4}$-module $D_n^{\text{TL}}(n-2) = T_n^{/n-4}\mathsf{e}_1$ is indecomposable projective (we assume $\delta \in k^*$ for now); and hence also indecomposable with simple head as a T_n-module. Generalising, define

$$D_n^{\text{TL}}(n-2l) \;:=\; T_n^{/n-2l-2}\mathsf{e}_1^{2l-1} \tag{1.35}$$ eq:DTe

We have:

pr:DTL1 (**1.10.5**) Proposition. *If $\delta \in k^*$, or $l \neq 0$, then $D_n^{\text{TL}}(l)$ is indecomposable with simple head as a T_n-module. Furthermore, by Prop.1.5.14 all the factors below the head obey $\mathsf{e}_1^{2l-1}L = 0$.* ■

co:DTL1 (**1.10.6**) Corollary. *Every projective T_n-module has a filtration by D-modules. (We will see shortly that the multiplicities are well-defined.)* ■

pr:basisDTL (**1.10.7**) Proposition. *(I) $\mathsf{T}_{n,l,l}$ is a basis for $D_n^{\text{TL}}(l)$. (II) $D_n^{\text{TL}}(l) \cong C_n^{\text{TL}}(l)$.* ■

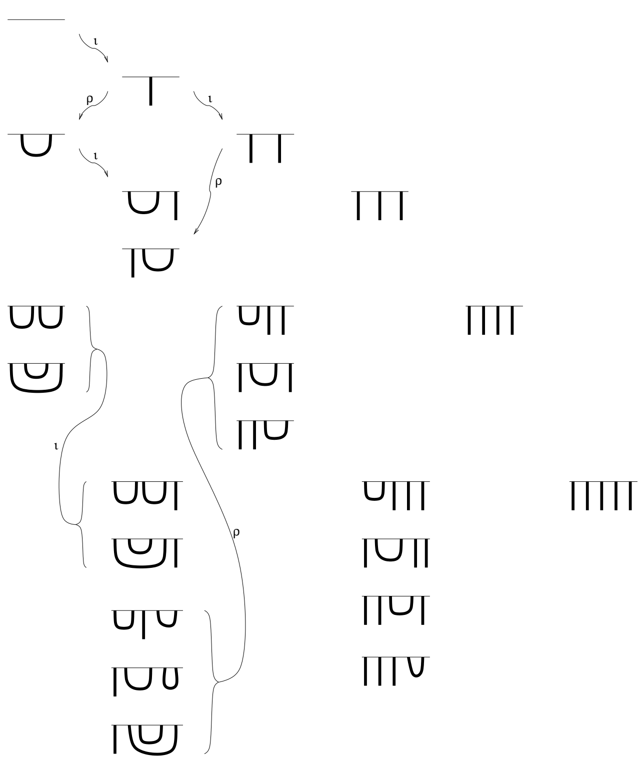


Figure 1.2: Truncated Pascal triangle enumerating sets $\mathsf{T}_{n,l,l}$. Here we have only drawn the northern edge of the frame rectangle for each diagram.
fig:bratt001

A construction for all such bases is given in Fig.1.2 (n increases top to bottom; l left to right). Map $\iota : \mathsf{T}_{n,l,l} \to \mathsf{T}_{n+1,l+1,l+1}$ adds a line on the right. Map $\rho : \mathsf{T}_{n,l,l} \to \mathsf{T}_{n+1,l-1,l-1}$ bends the bottom of the last propagating line back to the top.

(**1.10.8**) Note that there is a directly corresponding construction to (1.35) of indecomposable *right*-modules with analogous properties.

(**1.10.9**) There is also the construction of right-modules from the $D_n^{\text{TL}}(l)$ themselves by taking the ordinary duals, i.e. by applying the contravariant functor

$$()^* : T_n\text{-mod} \;\; \to \text{mod-}T_n$$

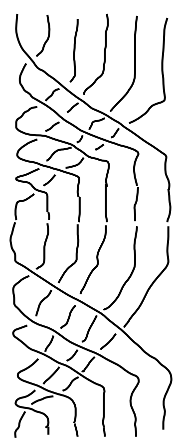
$$()^* : M \mapsto \text{Hom}_k(M, k)$$

(the duals $(D_n^{\text{TL}}(l))^*$ are also indecomposable on general grounds; but they need not have the other 'standard' properties from Prop.1.10.5 in general). We give a concrete example in (1.10.12).

One can ask how these two kinds of right modules are related. In general they are *not* isomorphic (but do have the same composition factors), as we shall see.

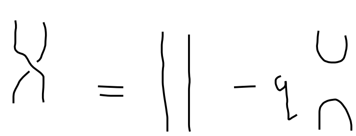
1.10.4 Aside: action of a central element in T_n on D-modules

Note from Prop.1.2.19 (the special feature) that T_n (with $\delta = q + q^{-1}$ for some $q \in k^*$) is a quotient of the braid group B_n. We consider the action of the central double-twist braid element M^2

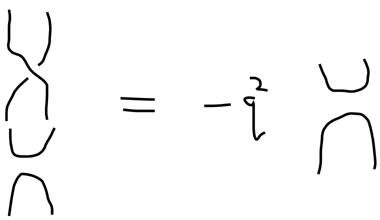


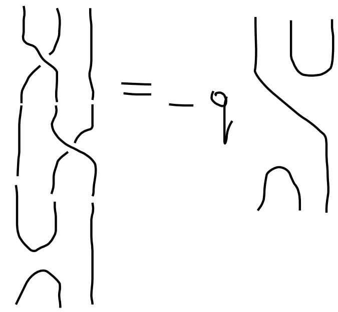
on our indecomposable D-modules.

This action can be computed using some hybrid diagrammatic rules, where crossings are understood as linear combinations of TL diagrams. First recall that the quotient map takes the braid generator g_i to $g_i \mapsto 1 - qU_i$. Informally we can generalise our diagrams for TL elements to incorporate this as:



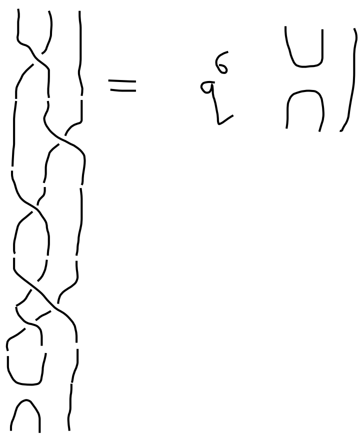
This gives us actions of braids on TL diagrams (and half-diagrams). In particular we have 'move 1' and 'move 2':

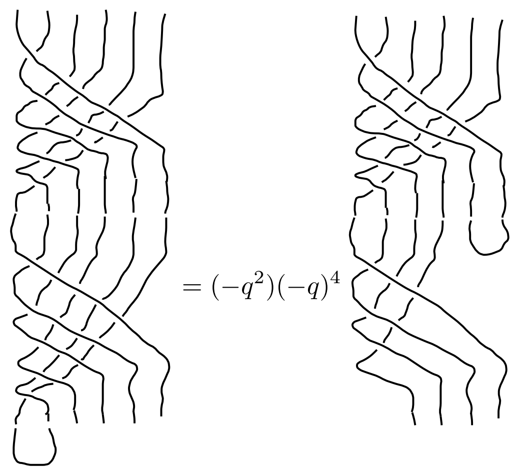




Note that the braids *look* like partition diagrams, but we *cannot* consider these as partition diagrams any more!

Applying the moves we get, for example,





We can think of the computation for the action of M^2 as passing the 'U' from the bottom-left through the various braids, first using move-1 ($-q^2$); then move-2 ($n-2$) times ($(-q)^{(n-2)}$); then a 'right-to-left over' version of move-2 ($n-2$) times ($(-q)^{(n-2)}$); then move-1 again. This gives a factor q^{2n} altogether; and what is left to act is M^2_{n-2} — the double-twist from B_{n-2} — on the remaining part of the basis element. (Thus if there is another 'U' then we will get a factor $q^{2(n-2)}$, and so on.)

In this way we can easily compute the action of M^2 on a basis element for any one of our modules from Fig.1.2. Besides the moves, the other feature is that because of the quotient by which the modules are defined, a braid acts like 1 on parallel lines in a module basis element.

The results are given in Fig.1.3. For $b \in D^{\mathrm{TL}}_n(l)$ we have:

$$M^2 b = q^{(n-l)(n+l+2)/2} b \tag{1.36}$$ eq:TLblock01

$n \setminus l$	0	1	2	3	4	5	6
0	1						
1		1					
2	q^4		1				
3		q^6		1			
4	q^{12}		q^8		1		
5		q^{16}		q^{10}		1	
6	q^{24}		q^{20}		q^{12}		1

Figure 1.3: Scalars by which M^2 acts on indecomposable T_n-modules $\Delta_n^{TL}(l)$. fig:Mact0001

Note in particular that the actions are all by powers of q, and that for given n they are all by different powers of q. By (**??**) this tells us that no two D-modules are in the same block (in the sense of 1.3.41) unless q is a root of unity.

pr:TLgensimp01 (**1.10.10**) PROPOSITION. *The algebra T_n over a field k is semisimple unless ($\delta = q + q^{-1}$ where) q is a root of unity.*

Proof. Exercise.

1.10.5 Some module morphisms

(**1.10.11**) It follows from (1.10.1) that every right T_n-module M can be made into a left T_n-module $\Pi_\star(M)$ by allowing T_n to act via the $\star$-map (the flip map). Note that a submodule of M passes to a submodule of $\Pi_\star(M)$. Indeed this map extends to a covariant functor between the categories of modules (in either direction):

$$\Pi_\star : \text{mod}-T_n \quad \leftrightarrow \quad T_n-\text{mod}$$

In particular, every exact sequence of right modules passes to an exact sequence of left modules.

Furthermore each module M has a contravariant (c-v) dual[3], here denoted $\Pi^o(M)$:

$$\Pi^o(M) \;=\; \Pi_\star(\text{Hom}_k(M,k)) \;=\; \Pi_\star(M^*)$$

exa:422 (**1.10.12**) Example: What does the c-v dual of $M = D_n^{\text{TL}}(l)$ look like? As a k-module it is $\text{Hom}_k(M,k)$. Given a basis $\{b_1, b_2, ..., b_r\}$ of M, the usual choice of basis of this ordinary dual is the set of linear maps f_i such that $f_i : b_j \mapsto \delta_{i,j}$. (The right-action of $a \in T_n$ is given by $(f_i a)(b_j) = f_i(ab_j)$. Thus $((f_i a)a')(b_j) = (f_i a)(a'b_j) = f_i(a(a'b_j))$ and $(f_i(aa'))(b_j) = f_i((aa')b_j) = f_i(a(a'b_j))$, so $((f_i a)a')(b_j) = (f_i(aa'))(b_j)$ as required.)

[3]The c-v dual of a module M over such a k-algebra is the ordinary dual right-module $\text{Hom}_k(M,k)$ made into a left-module via $\star$.

$n \setminus l$	$+0$	-0	$+1$	-1	$+2$	-2	$+3$	-3	$+4$	-4	$+5$	-5	$+6$	-6	$+7$	-7
0	1															
1			1													
2	$-q^2$	q^2			1											
3			q^3	$-q^3$			1									
4	q^6	$-q^6$			$-q^4$	q^4			1							
5			q^8	$-q^8$			q^5	$-q^5$			1					
6	$-q^{12}$	q^{12}			q^{10}	$-q^{10}$			$-q^6$	q^6			1			
7			q^{15}	$-q^{15}$			q^{12}	$-q^{12}$			q^7	$-q^7$			1	
8	q^{20}	$-q^{20}$			$-q^{18}$	q^{18}			q^{14}	$-q^{14}$			$-q^8$	q^8		

fig:tab11

Figure 1.4: Scalars by which M acts on indecomposable modules $\Delta_n(l)$ of the fixed subring of T_n under the left-right diagram flip (see **??**).

In our case let us order the basis of $M = D_n^{\text{TL}}(l)$ as in Fig.1.5. Then our basis for the dual is $\{f_1, f_2, ..., f_{n-1}\}$.
Exercise: What is the right action of T_n on this k-module? For example, what is $f_1 U_1$?

de:headshot

(**1.10.13**) It follows from (1.10.5) that the only copy of the simple head L_l (say) of $D_n^{\text{TL}}(l)$ occuring in the c-v dual lies in the simple socle (note that e_1^{2l-1} is fixed under $\star$). It then follows from Schur's Lemma 1.3.31 that there is a unique T_n-module map, up to scalars, from $D_n^{\text{TL}}(l)$ to its contravariant dual — taking the simple head to the simple socle. (In theory the socle, which is the simple dual of the simple head, might not be isomorphic to it; allowing no map. But we will show the existence of at least one map explicitly.)

As we will see, it follows from this abstract representation theoretic argument that $D_n^{\text{TL}}(l)$ has a contravariant form (a bilinear form such that $< x, ay > = < a^\star x, y >$, as in (**??**)) defined on it that is unique up to scalars.

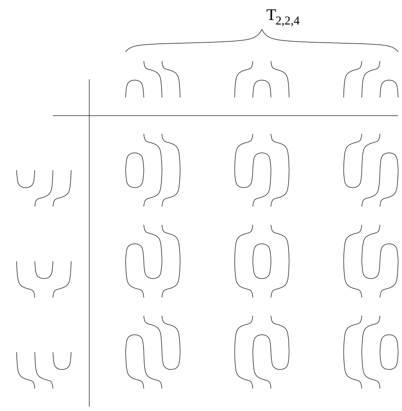
Actually *finding* the explicit c-v form could be difficult in general. But in fact we can construct such a form here for all δ simultaneously (over a ring with δ indeterminate, as it were). We can use this to determine the structure of the module.

(**1.10.14**) For a, b in the basis $\mathsf{T}_{n,l,l}$ (from (1.10.7)) then define $\alpha(a,b) \in k$ as follows. Note that $a^\star b \in \mathsf{T}_{l,l}$ (up to a scalar), thus either $a^\star b = \alpha(a,b)c$ with $c \in \mathsf{T}_{l,l,l}$ (indeed $c = 1_l$) for some $\alpha(a,b) \in k$; or $a^\star b \in k\mathsf{T}_{l,l}^{l-2}$, in which case set $\alpha(a,b) = 0$. Define an inner product on $k\mathsf{T}_{n,l,l}$ by $< a, b > = \alpha(a,b)$ and extending linearly.

ex:gramTL1

Example: Fig.1.5. The corresponding matrix of scalars is called the *gram matrix* with respect

Figure 1.5: The array of diagrams $a^\star b$ over the basis $\mathsf{T}_{4,2,2}$. fig:epud



to this basis. From our example we have (in the handy alternative parameterisation $\delta = q + q^{-1}$):

$$\mathrm{Gram}_n(n-2) \;=\; \begin{pmatrix} [2] & 1 & 0 & & \\ 1 & [2] & 1 & 0 & \\ 0 & 1 & [2] & 1 & \\ & & & \ddots & \\ 0 & \ldots & 0 & 1 & [2] \end{pmatrix} \qquad \text{so } |\mathrm{Gram}_n(n-2)| = [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{1.37}$$ eq:TLgram0001

pr:innprodcov1 (**1.10.15**) PROPOSITION. *The inner product defined by* $<-,->$ *is a contravariant form on* $D_n^{\mathrm{TL}}(l)$*.* ■

(**1.10.16**) Consider the k-space map

$$\phi_{\langle\rangle} : D_n^{\mathrm{TL}}(l) \;\;\rightarrow\;\; \Pi^o(D_n^{\mathrm{TL}}(l)) \tag{1.38}$$
$$\phi_{\langle\rangle} : m \;\;\mapsto\;\; \phi_{\langle\rangle}(m) \tag{1.39}$$

where $\phi_{\langle\rangle}(m) \in \hom(D_n^{\mathrm{TL}}(l), k)$ is given by

$$\phi_{\langle\rangle}(m)(m') \;=\; <m|m'> .$$

(**1.10.17**) PROPOSITION. *The map* $\phi_{\langle\rangle}$ *is a* T_n*-module homomorphism (unique up to scalars) from* $D_n^{\mathrm{TL}}(l)$ *to its contravariant dual.*

Proof. This map is a module morphism by Prop.1.10.15. To show uniqueness note that by (1.10.5) the contravariant dual must have the simple head of $D_n^{\mathrm{TL}}(l)$ as its simple socle (and only in the socle). Thus a head-to-socle map is the only possibility. □

(**1.10.18**) EXAMPLE. In our example we have (from the gram matrix, using (1.10.12))

$$\phi_{\langle\rangle} : \; \cup|\,| \;\;\mapsto\;\; [2]f_1 + f_2$$

$$\phi_{\langle\rangle} : \; |\cup| \;\mapsto\; f_1 + [2]f_2 + f_3$$
$$\phi_{\langle\rangle} : \; ||\cup \;\mapsto\; f_2 + [2]f_3$$

and for instance

$$\phi_{\langle\rangle} : \; \cup|| - [2]\,|\cup| - [3]\,||\cup \;\;\mapsto\;\; [4]f_3$$

The point of this case is to show that the module map $\phi_{\langle\rangle}$ has a kernel when $[4] = 0$. Obviously, in general,

PROPOSITION. If a T_n-module map has a kernel then the kernel is a submodule of the domain.

Thus in our case, when $[4] = 0$, the domain is not simple.

It will also be clear from the example that if the rank of the gram matrix is maximal then the morphism $\phi_{\langle\rangle}$ has no kernel, and so is an isomorphism. This does not, of itself, show that the domain is a simple module, but we already showed in (1.10.13) that in our case the image must be simple, so the domain *is* simple.

de:gramdetzero

(**1.10.19**) If $D^{\mathrm{TL}}_n(l)$ is in fact simple then $\phi_{\langle\rangle}$ is an isomorphism and the contravariant form is non-degenerate. Otherwise the form is degenerate.

It will be clear from our example that if the determinant of the gram matrix is non-zero then $D^{\mathrm{TL}}_n(l)$ is simple; and otherwise it is not. (Note that the case $\delta = 0$ is excluded here, for brevity. It is easy to include it if desired, via a minor modification.) In particular if the determinant is zero then $D^{\mathrm{TL}}_n(n-2)$ has composition length 2; and the other composition factor is the simple module $D^{\mathrm{TL}}_n(n)$.

(**1.10.20**) PROPOSITION. *Given a c-v form (with respect to involutive antiautomorphism $\star$) on A-module M and $Rad_{<>}M = \{x \in M \; : < y, x >= 0 \; \forall y\}$ then*
(I) $Rad_{<>}M$ is a submodule, since $x \in Rad_{<>}M$ implies $< y, ax >=< a^\star y, x >= 0$.
(II) Thus $\dim Rad_{<>}M \;=\; corank\, \mathrm{Gram}_{<>}M$. ■

(**1.10.21**) In our example rows 2 to $(n-1)$ of the $(n-1)\times(n-1)$ matrix $\mathrm{Gram}_n(n-2)$ are clearly independent, while replacing $\boxed{\cup||...|}$ (the basis element in the first row) by

$$w = \boxed{\cup||...|} - [2]\boxed{|\cup|...|} + [3]\boxed{||\cup...|} - ...$$

(a sequence of elementary row operations adding to the first row multiples of each of the subsequent rows) replaces the first row of $\mathrm{Gram}_n(n-2)$ with $(0, 0, ..., 0, [n])$. That is, $\mathrm{Rad}_{<>}D^{\mathrm{TL}}_n(n-2) = 0$ unless $[n] = 0$. If $[n] = 0$ then w spans the Rad.

Explicit check in case $n = 4$: $U_1 w = ([2] - [2] + 0)\boxed{\cup||} = 0$; $U_2 w = (1 - [2]^2 + [3])\boxed{|\cup|} = 0$; $U_3 w = (0 - [2] + [2][3])\boxed{||\cup}$.

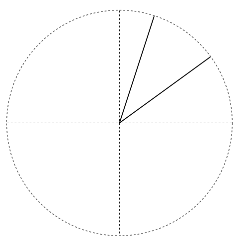
(**1.10.22**) PROPOSITION. The T_n-module $D^{\mathrm{TL}}_n(n-2)$ is simple unless $[n] = 0$, in which case $D^{\mathrm{TL}}_n(n) \hookrightarrow D^{\mathrm{TL}}_n(n-2)$ and the quotient is simple.

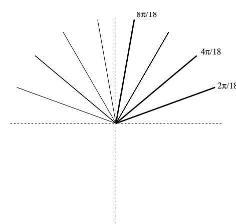
The condition $[n] = 0$ is satisfied when q is a solution to $q^{2n} = 1$ excluding $q = \pm 1$. One should compare this with the block data in Fig.1.3.

What values of q do we need to consider, to capture all possible algebra structures arising up to isomorphism in case $k = \mathbb{C}$? (1) Complex conjugation of q of magnitude 1 does not change δ, so it is enough to consider cases of q of nonnegative imaginary part. (2) It is easy to see that the algebra with $\delta \to -\delta$ is isomorphic to the original (via the invertible map $U_i \to -U_i$), and hence that $q \to -q$ also gives an isomorphism. Thus the algebras with $q^r = 1$ with r odd (satisfying $q^{2r} = 1$) can be treated with $q^r = -1$ and hence in the primitive $q^{2r} = 1$ cases. We will obfuscate this slightly by using the sign change to take representatives all in the non-negative real part region (some of which will not then be primitive $2r$-th roots), and hence give representatives in the positive (nonnegative) quadrant.

Cases:
$q^4 = 1$ yields $q = i$ and $\delta = 0$
$q^6 = 1$ yields $q = \frac{1 \pm \sqrt{-3}}{2}$ and $\delta = 1$
$q^8 = 1$ yields $q = \frac{1 \pm \sqrt{-1}}{\sqrt{2}}$ and $\delta = \sqrt{2}$
$q^{10} = 1$ yields two new positive δ values $\frac{\sqrt{5}+1}{2}$ and $\frac{\sqrt{5}-1}{2}$
$q^{12} = 1$ yields $q = \frac{\sqrt{3} \pm \sqrt{-1}}{2}$ and $\delta = \sqrt{3}$ ...

'Principle' q values for $q^{10} = 1$: and for example $q^{18} = 1$:





In the 10 case the coprime numbers are 1,3,7,9. We discard 7,9 as complex conjugates and make $3 \to 2$.

We will see in Th.1.11.5 that the *structure* of the algebra depends only on the r for which q is a primitive $2r$th root, if any, and not directly on the value of q. (This does not imply that the explicit construction of simple modules, say, only depends on r. For a possible meta-example note that there are some constructions of representations that only work when δ^2 is a natural number. See **??**.)

(**1.10.23**) It is easy to write down the cv form explicitly, particularly for $l = n - 2$, and compute the determinant. We can use this to determine the structure of the algebra. First we will need a couple of functors.

(**1.10.24**) REMARK. In case M is a matrix over a PID, the *Smith form* of M (see e.g. [5]) is a certain diagonal matrix equivalent to M under elementary operations.

One sees from the proposition and example that the rank, or indeed a Smith form, of Gram$D$ is potentially more useful than the determinant. However note that working over $\mathbb{Z}[\delta]$ as we partly are, a Smith form may not exist until we pass specifically to $\mathbb{C}$, say (or at least to a PID $k[\delta]$ with k a field); and they are harder to compute when they do exist.

See §11.10 for more on this.

1.10.6 Aside on Res-functors (exactness etc)

`ss:aside res`

(**1.10.25**) Note the limits of what functor Res_ψ (from (1.6.10)) says about A-modules in practice. For each B-module there is an A-module identical to it as a k-space. And for each B-module homomorphism there is an A-module homomorphism. It *does not* say that if $\mathrm{Hom}_B(M,N) = 0$ then so is $\mathrm{Hom}_A(M,N) = 0$.

In the particular case when ψ is surjective then M simple implies $\mathrm{Res}_\psi M$ simple — i.e. M simple as an A-module (any A-submodule M' of M would also be a B-submodule, since in this case the B action is contained in the A action).

(**1.10.26**) We can also think about what happens to exact sequences under this functor Res_ψ. Suppose $M' \hookrightarrow M \twoheadrightarrow M''$ is a short-exact sequence of B-module maps. As we have just seen, it is again a sequence of A-module maps. The sequence is of the form $M' \hookrightarrow M \twoheadrightarrow M''$ since injection and surjection are properties of the underlying k-modules; but such a sequence is short-exact if $dim(M') + dim(M'') = dim(M)$ — again a property of the underlying k-modules. In other words Res_ψ is an *exact* functor on finite dimensional modules.

We can also ask about split-ness. If the B-module sequence is split (i.e. $M = M' \oplus M''$) then there is another SES reversing the arrows, which again passes to an A-module sequence. If the B-module sequence is *non-split* what happens? Suppose that the A sequence is split. This means that there is an A-submodule of M isomorphic to M'', i.e. (up to isomorphism) $aM'' \in M''$ for all a. Note that *if ψ is surjective* [4] then every B action can be expressed as an A action (via ψ), so M'' is also a B-submodule, contradicting non-splitness. That is,

LEMMA. If algebra map ψ surjective then Res_ψ takes a non-split extension to a non-split extension. □

1.10.7 Functor examples for module categories: induction

(**1.10.27**) Functor Res_ψ makes B a left-A right-B-bimodule; and there is a similar functor making B a left-B right-A-bimodule. Hence define

$$\mathrm{Ind}_\psi : A - \mathrm{mod} \to B - \mathrm{mod}$$

by $\mathrm{Ind}_\psi N = B \otimes_A N$ (cf. 1.6.25).

REMARK. This construction is typically used in case $\psi : A \to B$ is an inclusion of a subalgebra (in which case Res is called restriction).

(**1.10.28**) EXERCISE. Investigate these functors for possible adjunctions. Hints: Consider the map

$$a : \mathrm{Hom}_B(\mathrm{Ind}_\psi M, N) \to \mathrm{Hom}_A(M, \mathrm{Res}_\psi N)$$

given as follows. For $f \in \mathrm{Hom}_B(\mathrm{Ind}_\psi M, N)$ we define $a(f) \in \mathrm{Hom}_A(M, \mathrm{Res}_\psi N)$ by $a(f)(m) = f(1 \otimes m)$. Given $g \in \mathrm{Hom}_A(M, \mathrm{Res}_\psi N)$ we define $b(g) \in \mathrm{Hom}_B(\mathrm{Ind}_\psi M, N)$ by $b(g)(c \otimes m) = cg(m)$. One checks that $b = a^{-1}$, since $b(a(f)) = b(f(1 \otimes -)) = 1f = f$.

(**1.10.29**) EXAMPLE. We have in (1.23) above a surjective algebra map $\psi : P_n \to S_n$. It follows that every S_n-module is also a P_n-module via ψ. Of course every S_n-module map is also a P_n-module map.

`pr:pr ind pr` (**1.10.30**) PROPOSITION. *The functor Ind_ψ takes projectives to projectives.* ■

[4] needed?

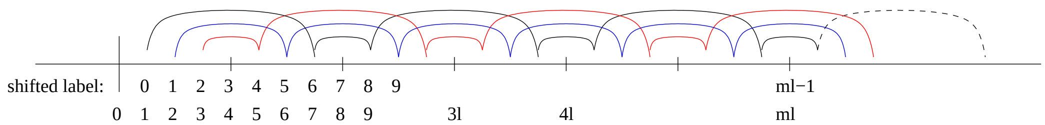


Figure 1.6: Orbits of an affine reflection group on $\mathbb{Z}$ giving blocks for T_n with $l = 4$. fig:TLalcoves1

1.10.8 Back to P_n for a moment

(**1.10.31**) Fix n. It follows from the results assembled in §1.6.5 (e.g. 1.6.29) that for each $\lambda \vdash l \in \{n, n-1, ..., 0\}$ we have a P_n-module $\Delta_\lambda = G^{n-l} S_\lambda$, where S_λ is a symmetric group Specht module. (Note that this notation omits n, so care is needed. We can write Δ^n_λ to emphasise n.)

Fix $k = \mathbb{C}$, so that every S_λ is simple. It follows from 1.6.26(III) and 1.6.28 that if P_n is semisimple for some given choice of δ (and some given n) then the set of Δ_λ modules is a complete set of simple modules for this algebra.

(**1.10.32**) More generally, if P_n is non-semisimple then at least one Δ_λ is not simple. Further, if Δ^n_λ is not simple, then Δ^{n+1}_λ is not simple. Thus, for fixed δ, we may think of the 'first' non-semisimple case (noting that P_0 is always simple), and hence a 'first' (one or more) non-simple Δ_λ — at level n say. We note (from 1.5.14, say) that this first non-simple case is manifested by a homomorphism from some Δ_ν with $\nu \vdash n$.

There are a number of ways we can 'detect' these homomorphisms. One approach starts by noting another adjunction: the (ind,res) adjunction corresponding to the inclusion $P_n \hookrightarrow P_{n+1}$. One can work out $\mathrm{Res}\Delta_\lambda$ by constructing an explicit basis for each Δ_λ. One can work out $\mathrm{Ind}\,\Delta_\lambda$ by using the formula $\mathrm{Ind} = \mathrm{Res}G$. It then follows from the (ind,res) adjoint isomorphism that any such homomorphism implies a homomorphism to Δ_λ with $|\lambda|$ 'close' to n. These modules take a relatively simple form, and it is possible to detect morphisms to them explicitly by direct calculation.

Let D be the *decomposition matrix* for the Δ-modules (ordered in any way consistent with $\lambda > \mu$ if $|\lambda| > |\mu|$). It follows that D is upper unitriangular. It also follows that the *Cartan decomposition matrix* C is $C = DD^T$.

1.11 Structure theorem for T_n over $\mathbb{C}$

To introduce our approach to the analysis and results for P_n it will be convenient to begin with T_n. Here we give a quick illustrative summary of T_n. For details and alternative approaches to T_n see Ch.11 and references therein.

Step 1: construct, and show to be isomorphic, various classes of 'useful modules'. Each construction has distinct useful properties, so the isomorphism means that they — the 'Δ-modules' — have all the useful properties. Roughly speaking the classes are of

Specht modules (modules defined integrally, and generically simple, as useful for π-modular systems);

global-standard modules (images of simple modules under globalisation functors); and possibly some others such as *standard modules* (for a quasihereditary algebra — indec. projective modules for certain special quotient algebras).

Step 2 is to state the simple composition factors for the Δ-modules (NB this assumes we know them, or have a conjecture!). By the Specht property and Brauer reciprocity this determines the Cartan decomposition matrix.

Step 3 is to set up an inductive proof using the global-standard property to move data directly up the ranks, and Frobenius reciprocity (induction and restriction between ranks) and the block decomposition to build 'translation functors' that determine the remaining data.

(**1.11.1**) Set $\Delta^T_l(l) = k$ (the trivial T_l-module) and

$$\Delta^T_n(l) = G_{\mathsf{e}_1}\Delta^T_{n-2}(l)$$

(**1.11.2**) EXAMPLE. $G_{\mathsf{e}_1}\Delta^T_{n-2}(n-2) = T_n\mathsf{e}_1 \otimes_{T_{n-2}} \Delta^T_{n-2}(n-2)$ (using the isomorphism to confuse $T_{n-2} \cong \mathsf{e}_1 T_n \mathsf{e}_1$). Noting $T_n\mathsf{e}_1 = k\mathsf{T}_{n,n-2} \otimes \cap$ (as in (1.2.9)); this is spanned by $\mathsf{T}_{n,n-2} \otimes_{T_{n-2}} 1_{n-2}$, where $\{1_{n-2}\}$ is acting as a basis for $\Delta^T_{n-2}(n-2)$. Note that $\mathsf{T}_{n,n-4,n-2} \otimes_{T_{n-2}} 1_{n-2} = 0$, so a basis is $\mathsf{T}_{n,n-2,n-2} \otimes_{T_{n-2}} 1_{n-2}$.

pr:DelD (**1.11.3**) LEMMA. *For* $l \in \Lambda^T_n = \{n, n-2, ..., 1/0\}$

$$\Delta^T_n(l) \cong D^{\text{TL}}_n(l)$$

Proof. As above, a basis of $\Delta^T_n(l)$ is $\mathsf{T}_{n,l,l} \otimes_{T_l} 1_l$. Now cf. (1.10.7) and consider the obvious bijection between bases. The actions of $a \in T_n$ are the same — if (in the T category) $a * b \in k\mathsf{T}_{n,l,l}$ then $ab = a * b$ in both cases; otherwise $ab = 0$ in $\Delta^T_n(l)$ by the balanced map, and in $D^{\text{TL}}_n(l)$ by the quotient. $\square$

...See §**??** for more details and treatment of the $\delta = 0$ case.

TLwallnotation (**1.11.4**) Consider Fig.1.6. Fix $r \in \mathbb{N}$. We give the positive real line two labellings for integral points: the natural labelling (with the origin labelled 0); and the *shifted* labelling. Points of form mr in the natural labelling ($mr - 1$ in the shifted labelling) are called *walls*. The regions between walls are called *alcoves*. Write $\sigma_{(m)} : \mathbb{R} \to \mathbb{R}$ for reflection in the m-th wall. Write

$$\Sigma^{(r)} = \langle \sigma_{(0)}, \sigma_{(1)} \rangle$$

for the group of (affine) reflections. Write $l^{\Sigma^{(r)}}$ for the dominant (non-negative) part of the orbit of point l (in the shifted labelling) under $\Sigma^{(r)}$. Thus for example $0^{\Sigma^{(r)}} = \{0, 2r-2, 2r, 4r-2, ...\}$.

th:TLoverC (**1.11.5**) THEOREM. [84, §7.3 Th.2] (Structure Theorem for T_n over $\mathbb{C}$.) Set $k = \mathbb{C}$ and fix $\delta \in k$; or set $k = \mathbb{C}(\delta)$. The T_n-modules $\{L_n(\lambda) = \text{head } \Delta^T_n(\lambda)\}_{\lambda \in \Lambda^T_n}$ are a complete set of simple T_n-modules. The simple content of the modules $\{\Delta^T_n(\lambda)\}_\lambda$ determines the structure of T_n, and is given depending on δ as follows.

(I) In case there is no $r \in \mathbb{N}$ such that δ is of the form $\delta = q + q^{-1}$ with $q^r = 1$, the Δ-modules are simple, and absolutely irreducible, and T_n is split semisimple.

(II) Fix $r \in \mathbb{N}$ (here we take $r \geq 3$ for now) and let $q \in \mathbb{C}$ be a primitive $2r$-th root of unity. Suppose $\delta = q + q^{-1}$.

For given $\lambda \in \mathbb{N}_0$ determine m and b in $\mathbb{N}_0$ by $\lambda+1 = mr+b$ with $0 \leq b < m$ (so b is the position of $\lambda+1$ in the alcove above mr, in the sense of (1.11.4)). For $b > 0$ set $\sigma_{(m+1)}.\lambda = \lambda + 2m - 2b$ — the image of λ under reflection in the wall above.
1) If $b = 0$ then $\Delta_n^T(\lambda) = L_n(\lambda)$.
2) Otherwise

$$0 \longrightarrow L_n(\lambda + 2m - 2b) \longrightarrow \Delta_n^T(\lambda) \longrightarrow L_n(\lambda) \longrightarrow 0 \qquad (1.40)$$ eq:sesTLa

Here $L_n(\lambda + 2m - 2b)$ is to be understood as 0 if n is too small.

In particular the orbits of the reflection action describe the 'regular' blocks (blocks of points not fixed by any non-trivial reflection); while the singular blocks (of points fixed by a non-trivial reflection) are singletons.

(III) We leave the cases $r = 1, 2$ ($q = 1, -1, i, -i$) as an exercise for now. See §**??**.

(**1.11.6**) An informal way to present Theorem 1.11.5, following [84], is that the simple content of standard T_n-modules (arranged as in Fig.1.2, and cf. Fig.1.6) is indicated by the example in Figure 1.7.

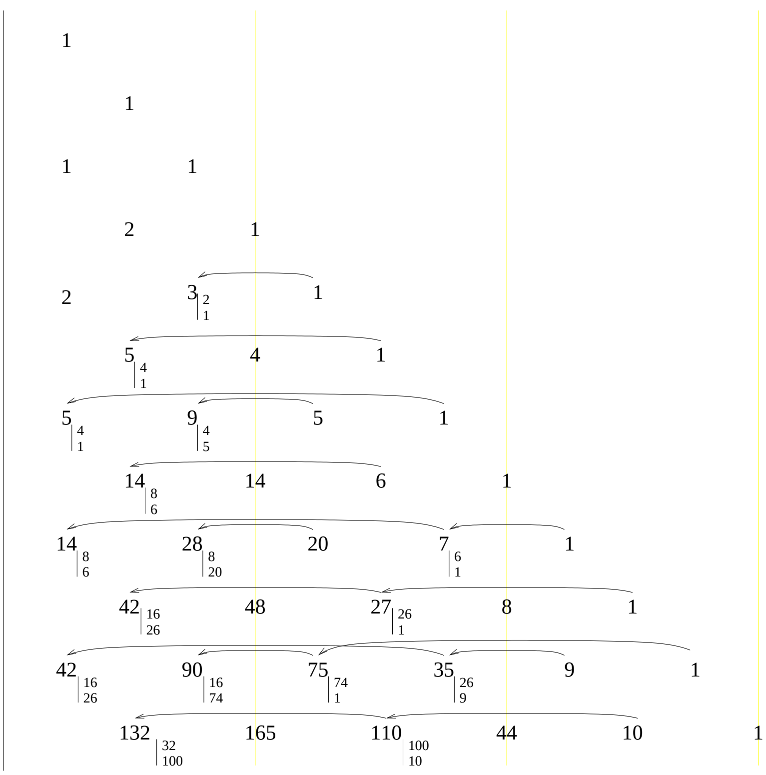


Figure 1.7: Simple content, dimensions and morphisms of standard T_n-modules (in case $k = \mathbb{C}$, $r = l = 4$).
fig:TLbratthop1

ss:decompmatex1

1.11.1 The decomposition matrices of T_n over $\mathbb{C}$

Note that the decomposition matrices (from §1.7 and (1.30)) are determined by the structure Theorem 1.11.5. The standard-decomposition matrix for a single regular block $l^{\Sigma^{(r)}}$ (starting from the low-numbered weight) is of form

$$D_{block} = \begin{pmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ & & & & & 1 \end{pmatrix}$$

(this should be thought of as the n-dependent truncation of a semiinfinite matrix continuing down to the right), that is $\Delta^T(0)$ (say, from the first row) contains $L(0)$ and the next simple in the block, and so on. This gives the block Cartan decomposition matrix:

$$C_{block} = D_{block}^T D_{block} = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{pmatrix}$$

1.11.2 Proof of Theorem: set up the induction — translation functors

Proof. Firstly, by construction the modules $D_n^{\mathrm{TL}}(\lambda)$ give a filtration of the left-regular T_n-module. Thus by Jordan–Holder(III) (1.3.13) every simple module appears in (the head of) some $D_n^{\mathrm{TL}}(l)$. The completeness of $\{L_n(\lambda)\}_\lambda$ follows by, say, 1.10.5 and 1.11.3.

To proceed we will need some lemmas.

(**1.11.7**) Lemma. [Δ-filtration Lemma] *Projective T_n-modules have filtrations by Δ modules; and the corresponding composition multiplicities are well defined.*

Proof. Filtration was proved in Cor.1.10.6. We can see well-definedness in various different ways. For now we note from §1.7 (specifically (1.7.4) and (1.7.1) respectively) that both kinds of modules have lifts to the integral case $k = \mathbb{C}[\delta]$, and hence corresponding modules in the ordinary case $k = \mathbb{C}(\delta)$. But in the ordinary case the Δ-modules are simple, with well-defined multiplicities by Jordan–Holder. ■

lem:wol0 (**1.11.8**) Lemma. *The composition multiplicities*

$$(\Delta_n^T(\mu) : L_n(\lambda)) \;=\; 0 \quad \textit{unless } \lambda \geq \mu$$

(and $(\Delta_n^T(\lambda) : L_n(\lambda)) = 1$*).*

Proof. Otherwise we can localise until $\Delta_m^T(\mu) \cong\; e\Delta_n^T(\mu)$ (some e) is simple and get a contradiction using (1.5.14).

lem:wol (**1.11.9**) LEMMA. [Weight-order Lemma] *Once $n \geq \lambda$, so indecomposable projective $P_n(\lambda)$ is defined, then the multiplicity $(P_n(\lambda) : \Delta_n^T(\lambda)) = 1$; $(P_n(\lambda) : \Delta_n^T(\mu)) = 0$ if $|\mu| \geq |\lambda|$ $(\mu \neq \lambda)$; and otherwise $(P_n(\lambda) : \Delta_n^T(\mu))$ does not depend on n.*

Proof. By (1.7.6) and (1.5.14). Note from 1.11.8 that $(\Delta_n^T(\mu) : L_n(\lambda)) = 0$ unless $\lambda \geq \mu$ (and $(\Delta_n^T(\lambda) : L_n(\lambda)) = 1$). We can express this as saying that the corresponding decomposition matrix is lower-unitriangular. Then apply Brauer reciprocity, as in 1.7.6, in case base ring $R = \mathbb{C}[\delta]$ or $R = \mathbb{C}[q, q^{-1}]$. (NB Reciprocity assumes that T_n is semisimple over the field R_0. This follows from Case (I) in the Theorem.) □

(**1.11.10**) Proof of Theorem in a case of type-(I). Method 1: Note from 1.10.15 that there is always a T_n-module map from a Δ-module to its contravariant dual (so that they have at least one simple factor in common); and that if δ is indeterminate then this map is an isomorphism. Since each Δ contains only one copy of its head-simple (Lem.1.11.8), a single isomorphic factor must be both the head and socle of the cv dual. That is, both modules are simple. If $\delta \in \mathbb{C}$ then this argument shows specifically that $\Delta_n^T(n-2)$ is simple for all n unless $\delta = q + q^{-1}$ with q some root of unity. One may then show that all the other Δs are simple using 1.11.13 and Frobenius reciprocity. (See later.)

Method 2: If $1 \notin q^{\mathbb{N}}$ then the Δs are in different blocks (by 1.10.10) and so none contains a composition factor in common with another. Thus each is simple by (the parenthetical result in) Lem.1.11.8.

(**1.11.11**) Proof in a case of type-(II). We proceed by induction on n. Let $\mathsf{A}(n)$ denote the proposition that the Theorem holds in level n and below. In case (I) we assume $\mathsf{A}(mr-1)$, i.e. we assume level $n = mr-1$ and below. (And will work through a 'cycle' $n = mr, mr+1, ..., mr+r-1$. That is, the inductive step is from m to $m+1$.) It is an exercise to check the base cases. By $\mathsf{A}(mr-1)$ we have $\Delta_{mr-1}^T(mr-1) = L_{mr-1}(mr-1) = P_{mr-1}(mr-1)$.

(Thus, if $n' \equiv mr-1$ mod.2, we have $\Delta_{n'}^T(mr-1) = L_{n'}(mr-1) = P_{n'}(mr-1)$.

Why? Firstly, we have some organisational Lemmas.)

(**1.11.12**) Remark: By Lem.1.11.9 if $\Delta_{n=mr-1}^T(mr-1) = P_{n=mr-1}(mr-1)$ this identification holds for all higher n. (NB this does not of itself guarantee that the module is *simple* for all n.)

pr:resDeTL (**1.11.13**) PROPOSITION. [Δ-restriction Lemma] Let $\psi : T_{n-1} \hookrightarrow T_n$ and $\mathrm{Res} = \mathrm{Res}_\psi$. We have

$$0 \longrightarrow \Delta_{n-1}^T(l-1) \longrightarrow \mathrm{Res}\Delta_n^T(l) \longrightarrow \Delta_{n-1}^T(l+1) \longrightarrow 0$$

Proof. Hint: consider Fig.1.2. ■

pr:indresG (**1.11.14**) PROPOSITION. *The functors Ind_ψ and $Res_\psi G$ are naturally isomorphic.*

Proof. $\mathrm{Ind}-$ is $T_{n+1} \otimes_{T_n} -$ while $G-$ is $k\mathsf{T}(n+2, n) \otimes_{T_n} -$. But $T_{n+1} = k\mathsf{T}(n+1, n+1)$ and $k\mathsf{T}(n+2, n)$ are isomorphic as left-T_{n+1} right-T_n-modules (by the 'disk bijection', which draws partitions on a disk instead of a rectangular frame). ■

(**1.11.15**) By 1.11.13 and 1.11.14 (and the definition of $\Delta^T(l)$) we have

$$\mathrm{Ind}\,\Delta^T(l) = \Delta^T(l+1) + \Delta^T(l-1),$$

So for example if the inductive assumption holds we have

$$\operatorname{Ind} P(mr-1) = \Delta^T(mr) + \Delta^T(mr-2). \tag{1.41}$$ eq:PDD-2

On the other hand, by Lem.1.11.9,

lem:Phwt1 (**1.11.16**) LEMMA. *Any projective T_n-module is a direct sum of indecomposable projectives including those with the highest shifted label among those appearing in its Δ^T factors.* □

(**1.11.17**) Define Pr_l as the projection functor onto the block of $L(l)$. (This is to be considered formally for the moment — we make no intrinsic assumptions about which other simples lie in this block.) Define the 'translation functor' $\operatorname{Ind}_l - \;=\; Pr_l \operatorname{Ind} -$.

We have for example $\operatorname{Ind}_l P(l-1) \;=\; P(l) + Q$, where Q is a (possibly zero) 'lower' projective in the block of l.

1.11.3 Starting the induction

We now proceed with the induction. The first step is to show that $\mathsf{A}(mr-1)$ implies $\mathsf{A}(mr)$. For this it is sufficient to 'compute' the Δ-content of $P(mr)$.

(**1.11.18**) By (1.41) (i.e. by the inductive assumption) and (1.11.16) we have that $\operatorname{Ind} P(mr-1)$, which is projective since $\operatorname{Ind}-$ preserves projectivity (Prop.1.10.30), contains $P(mr)$ as a direct
pa:step0 summand.

Suppose (for a contradiction) that $P(mr) = \Delta^T(mr)$. Then in particular (i) $P_{mr}(mr) = \Delta^T_{mr}(mr) = L_{mr}(mr)$ and the module would be in a simple block here. Next note that the remaining factor in $\operatorname{Ind} P(mr-1)$ would also be projective, so (again by 1.11.16) $P(mr-2) = \Delta^T(mr-2)$. But this would imply (ii) $\Delta^T(mr-2) = L(mr-2)$ by the argument in the proof of (1.11.9), since the only other possible factor is $L(mr)$, but the working assumptions place this in a different block. Finally this contradicts the fact (iii) from 1.10.14 that the gram determinant $||\Delta^T_{mr}(mr-2)|| = [mr] = 0$ when $q^{2r} = 1$, which implies that $\Delta^T_{mr}(mr-2)$ has a submodule in this case.

Thus $\operatorname{Ind} P(mr-1) = P(mr)$. Thus $P(mr) = \Delta^T(mr) + \Delta^T(mr-2)$.

REMARK. Appart from case $m = 1$ the supposition above (specifically the implication $P(mr-2) = \Delta^T(mr-2)$) also contradicts the inductive assumption. That is, we only strictly need the argument above in case $m = 1$.

(**1.11.19**) Next (to verify $\mathsf{A}(mr+1)$) we need to compute $P(mr+1)$. We have

$$\operatorname{Ind} P(mr) = \Delta^T(mr+1) + \Delta^T(mr-1) + \Delta^T(mr-1) + \Delta^T(mr-3)$$

Again this contains $P(mr+1)$ and the game is to determine which of the factors are in $P(mr+1)$.

Step 1: If $\Delta^T(mr-1)$ is in $P(mr+1)$ then $L(mr+1)$ would be in $\Delta^T(mr-1)$ by modular reciprocity (necessarily in the socle); in particular $\Delta^T_{mr+1}(mr-1)$ would have a submodule, which would imply a degenerate unique contravariant form, and hence $||\Delta^T_{mr+1}(mr-1)|| = 0$ — a contradiction since by (1.37) $||\Delta^T_{mr+1}(mr-1)|| = [mr+1] = 1$ when $q^{2r} = 1$.

REMARK. Alternatively it is very easy to show using Schur's Lemma and a suitable central element of T_n (such as the image in T_n of the double-twist braid) that indecomposables $\Delta^T(mr-1)$ and $P(mr+1)$ are not in the same block — see (1.36).

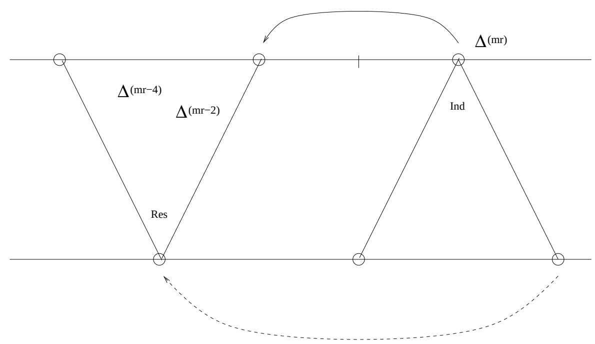


Figure 1.8: Δ-module maps by Frobenius reciprocity. fig:FRTL1

de:TL901 (**1.11.20**) Step 2: Next we will show by a contradiction that $P(mr+1) = \Delta^T(mr+1)+\Delta^T(mr-3)$. Suppose this sum splits. Then this would imply $P(mr-3) = \Delta^T(mr-3)$ and hence $L(mr-3) = \Delta^T(mr-3)$, arguing as in (1.11.18)(I-II). However, for a contradiction consider the following (method for avoiding computing the analogue of (1.11.18)(III) by hand!).

de:TL902 (**1.11.21**) By Frobenius reciprocity (7.5.16) we have

$$\mathrm{Hom}(\mathrm{Ind}\, A, B) \cong \mathrm{Hom}(A, \mathrm{Res} B)$$

in particular in the case in Fig.1.8: [5]

$$\mathrm{Hom}(\mathrm{Ind}\, \Delta^T_{ml}(ml),\ \Delta^T_{ml+1}(ml-3)) \ \cong\ \mathrm{Hom}(\Delta^T_{ml}(ml),\ \mathrm{Res}\Delta^T_{ml+1}(ml-3))$$

Note that $\mathrm{Res}\Delta^T_{ml+1}(ml-3) = \Delta^T_{ml}(ml-2) \oplus \Delta^T_{ml}(ml-4)$ (a direct sum by the block assumption in $\mathsf{A}(ml)$, unless $r = 2$), so that the RHS is nonzero by assumption (noting, say, (1.40)). Thus the LHS is nonzero. There is no map from $\Delta^T(ml+1)$ to $\Delta^T(ml-1)$, as already noted in Step 1, so there is a map from $\Delta^T(ml+1)$ to $\Delta^T(ml-3)$. This demonstrates the contradiction needed in 1.11.20. Thus

$$P(mr+1) = \Delta^T(mr+1) + \Delta^T(mr-3)$$

(**1.11.22**) Step 2 (alternate approach): Suppose again for a contradiction that $\mathrm{Ind}_{mr+1}P(mr) = \Delta^T(mr+1) \oplus \Delta^T(mr-3)$. This would imply $\mathrm{Ind}\,\mathrm{Ind}_{mr+1}P(mr) = \ (\Delta^T(mr+2) + \Delta^T(mr)) \oplus (\Delta^T(mr-2)+\Delta^T(mr-4))$. This would imply that either $P(mr+2) = \Delta^T(mr+2)$ and $P(mr) = \Delta^T(mr)$ — contradicting $\mathsf{A}(mr)$ — or $P(mr+2) = \Delta^T(mr+2) + \Delta^T(mr)$. The latter would imply $(\Delta^T(mr) : L(mr+2)) = 1$ by Brauer reciprocity, but $\Delta^T_{mr+2}(mr)$ is simple (unless $r = 2$) by the determinant calculation (1.37) (which gives determinant $[mr+2] = \frac{q^{mr+2}-q^{-mr-2}}{q-q^{-1}} = \pm[2]$ when $q^{2r} = 1$) — a contradiction.

(**1.11.23**) Next we have to verify $\mathsf{A}(mr+2)$. We have

$$\mathrm{Ind}\, P(mr+1) = \Delta^T(mr+2) + \Delta^T(mr) + \Delta^T(mr-2) + \Delta^T(mr-4)$$

[5] caveat: $l = r$!!!

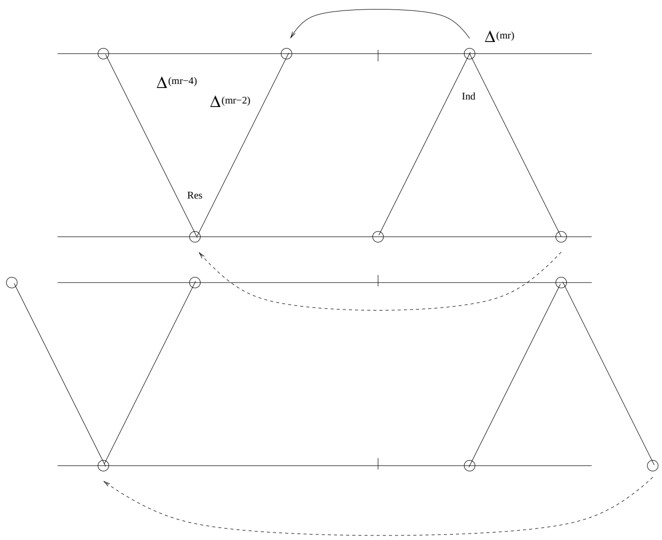


Figure 1.9: $\boxed{\texttt{fig:FRTL3}}$

We have $P(mr+2) = \Delta^T(mr+2) + ...$ The question is, which of the factors above should be included? If we include $\Delta^T(mr)$ then $L(mr+2)$ is in $\Delta^T(mr)$ by modular reciprocity. We can eliminate this possibility in a couple of ways. For example, we can compute a central element of T_n and show using this that the two shifted labels are in different blocks. Alternatively we can compute $||\Delta^T_{mr+2}(mr)||$ and check that it is nonzero in this case.

So far, then, we have that $\mathrm{Ind}\, P(mr+1) = P(mr+2) \oplus P(mr) \oplus ....$ However since $P(mr) = \Delta^T(mr) + \Delta^T(mr-2)$ we have $P(mr+2) = \Delta^T(mr+2) + X$ where $X = \Delta^T(mr-4)$ or zero.

In the latter case we would have $P(mr-4) = \Delta^T(mr-4)$. This contradicts the inductive assumption for every m value except $m=1$. For $m=1$ (or in general) we note instead that

$$\mathrm{Hom}(\mathrm{Ind}\, \Delta^T_{mr+1}(mr+1), \Delta^T_{mr+2}(mr-4)) \cong \mathrm{Hom}(\Delta^T_{mr+1}(mr+1), \mathrm{Res}\Delta^T_{mr+2}(mr-4))$$

and that the RHS is nonzero (for $r > 3$) by the inductive assumption (indeed we just showed this in 1.11.21 above) — see also the schematic in Fig.1.9. Thus the LHS is nonzero. But there is no map $\Delta^T(mr) \to \Delta^T(mr-4)$ by the inductive assumption, so there is a map $\Delta^T(mr+2) \to \Delta^T(mr-4)$. This provides the required contradiction, so $X \neq$ zero. That is

$$P(mr+2) \ = \ \Delta^T(mr+2) + \Delta^T(mr-4) \ \ = \ \ \Delta^T(mr+2) + \Delta^T(\sigma_{(m)}(mr+2))$$

(**1.11.24**) We may continue in the same way until we come to show $\mathsf{A}(mr+r-1)$, by stepping up from $P(mr+(r-2)) = \Delta^T(mr+(r-2)) + \Delta^T(mr-r)$. Thus $\mathrm{Ind}\, P(mr+(r-2)) = P(mr+r-1) \oplus ... = \Delta^T(mr+(r-1)) + \Delta^T(mr+(r-3)) + \Delta^T(mr-r+1) + \Delta^T(mr-r-1)$. Analogously to before we rule out $\Delta^T(mr+(r-3))$ from $P(mr+r-1)$ by modular reciprocity and $||\Delta^T_{mr+r-1}(mr+(r-3))|| = [mr+r-1] \neq 0$ (and hence also rule out $\Delta^T(mr-r+1)$). But

this time we can also rule out $\Delta^T(mr-r-1)$ by modular reciprocity (if it exists, i.e. if $m>1$), since this is simple by the inductive assumption.
(REMARK. At this point $\mathrm{Res}\Delta^T(mr-r-1)$ is not a direct sum (indeed it is indecomposable projective) and the argument for a nonzero RHS in Frobenius reciprocity fails. This tells us that this time there is not necessarily map on the LHS. Indeed we have just shown that there is no map. This then tells us that $P(mr-r)$ has simple socle. In fact it is cv self-dual and injective. See **??**.)
So $P(mr+(r-1)) = \Delta^T(mr+(r-1))$ and we have completed the main inductive step. □

1.11.4 Odds and ends

(**1.11.25**) By 1.5.14 and 1.7.6 the $\Delta_n(l)$ content of $P_n(m)$ does not depend on n (once n is big enough for these modules to make sense). Thus $P_n(0)=\Delta_n(0)$; $P_n(1)=\Delta_n(1)$.

For $P_n(2)$ we have $\mathrm{Ind}\, P_n(1) = \Delta_n(0)+\Delta_n(2)$; and $\mathrm{Ind}\, P_n(1)$ contains $P_n(2)$ as a direct summand. If this is a proper direct sum then this is true in particular at $n=2$ and there is a primitive idempotent decomposition of 1 in T_2. It is easy to see that this depends on δ, but it true unless $\delta=0$. *(We shall assume for now that $k=\mathbb{C}$ for definiteness.)*

Another way to look at the decomposition of $\mathrm{Ind}\, P_n(1)$ is as follows. If it does not decompose then by **??** there is a homomorphism $\Delta(2)\rightarrow\Delta(0)$, so that the gram matrix of $\Delta(0)$ must be singular.

Let us assume $\delta\neq 0$. Proceeding to $P_n(3)$ we have $\mathrm{Ind}\, P_n(2) = \Delta_n(1)+\Delta_n(3)$. Again this splits if and only if the gram matrix for $\Delta(1)$ is singular.

(**1.11.26**) TO DO:
Grothendieck group

1.12 Lie algebras

ss:Liealg0

We include a brief discussion of Lie algebras here,
(a) to provide some contrast with and hence context for our 'associative' algebras; and
(b) as a certain partner notion to the special case of (associative) finite group algebras.
See 19.3.1 for a more detailed exposition. Here k is a field.

(**1.12.1**) A Lie algebra A over field k is a k-vector space and a bilinear operation $A\times A\rightarrow A$ denoted $[a,b]$ such that $[a,a]=0$ and

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0 \qquad \text{('Jacobi identity')}$$

(**1.12.2**) From an associative algebra T we obtain a Lie algebra $Lie(T)$ by $[a,b]=ab-ba$.

(**1.12.3**) In particular, for V a vector space, the space of endomorphisms, sometimes denoted $gl(V)$, is a Lie algebra with $[a,b]=ab-ba$ (where ab is the composition of endomorphisms).

A representation of a Lie algebra A over k is a Lie algebra morphism $\rho : A \to gl(V)$ for some V.

An A-module is a space V and a map $A \times V \to V$ with

$$[a,b]v = a(bv) - b(av).$$

(**1.12.4**) Let V, V' be A-modules. Then the tensor product $V \otimes_k V'$ has a 'diagonal' action of A:

$$a(v \otimes v') = av \otimes v' + v \otimes av'$$

that makes $V \otimes_k V'$ an A-module.

Check: $[a,b](v \otimes v') = [a,b]v \otimes v' + v \otimes [a,b]v' = (a(bv) - b(av)) \otimes v' + v \otimes (a(bv') - b(av')) = ...$

(**1.12.5**) The tensor algebra of Lie algebra A is the vector space

$$\tau = \bigoplus_{n \geq 0} A^{\otimes n}$$

with multiplication given by $(a \otimes b)(c \otimes d) = a \otimes b \otimes c \otimes d$ and so on. Set H to be the ideal in τ generated by the elements of form $a \otimes b - b \otimes a - [a,b]$, with $a, b \in A$. Define

$$U_A = \tau / H$$

(**1.12.6**) A *universal enveloping algebra* (UEA) of Lie algebra A is an associative algebra U together with a Lie algebra homomorphism $I : A \to Lie(U)$ such that every Lie algebra homomorphism of form $h : A \to Lie(B)$ has a unique 'factorisation through $Lie(U)$', that is, a unique morphism of associative (unital) algebras $f : U \to B$ such that $h = f \circ I$.

(**1.12.7**) U_A is a UEA for A, with the homomorphism $I : A \to Lie(U_A)$ given by $a \mapsto a + H$. It is unique as such up to isomorphism.

(**1.12.8**) There is a vector space bijection

$$Hom_{Lie}(A, Lie(B)) \cong \mathrm{Hom}(U, B).$$

(**1.12.9**) Let V be an A-module and $\rho : A \to gl(V)$ the corresponding representation. Then ρ extends to a representation of a UEA U. This lifts to an 'isomorphism' of the categories of A-modules and U-modules (as subcategories of the category of vector spaces).

(**1.12.10**) Theorem. *(Poincare–Birkoff–Witt) Let $J = \{j_1, j_2, ...\}$ be an ordered basis of A. Then the monomials of form $I(j_{i_1})I(j_{i_2})...I(j_{i_n})$ with $i_1 \leq i_2 \leq ...$ and $n \geq 0$ are a basis for U_A.*

(**1.12.11**) Recalling that k is fixed here, write sl_n for the Lie algebra of traceless $n \times n$ matrices. For example, sl_2 has k-basis:

$$x^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad x^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These obey $[x^+, x^-] = h$, $[h, x^+] = 2x^+$, $[h, x^-] = -2x^-$.

1.13 Eigenvalue problems

—

(1.13.1) Operators acting on a space; their eigenvectors and eigenvalues.
Here we remark very briefly and generally on the kind of Physical problem that can lead us into representation theory.

A typical Physical problem has a linear operator Ω acting on a space H, with that action given by the action of the operator on a (spanning) subset of the space. One wants to find the eigenvalues of Ω.

The eigenvalue problem may be thought of as the problem of finding the one-dimensional subspaces of H as an $\langle\Omega\rangle$-module, where $\langle\Omega\rangle$ is the (complex) algebra generated by Ω. That is, we want to find elements h_i in H such that:

$$\Omega h_i = \lambda_i h_i$$

— noting only that, usually, the object of primary physical interest is λ_i rather than h_i. If H is finite dimensional then (the complex algebra generated by) Ω will obey a relation of the form

$$\prod_i (\Omega - \lambda_i)^{m_i} = 0$$

Of course the details of this form are *ab initio* unknown to us. But, proceeding formally for a moment, if any $m_i > 1$ (necessarily) here, so that $S = \prod_i (\Omega - \lambda_i) \neq 0$, then S generates a non-vanishing nilpotent ideal (we say, the algebra has a radical). Obviously any such nilpotent object has 0-spectrum, so two operators differing by such an object have the same spectrum. In other words, the image of Ω in the quotient algebra by the radical has the same spectrum $\{\lambda_i\}$. An algebra with vanishing radical (such as the quotient of a complex algebra by its radical) has a particularly simple structural form, so this is a potentially useful step.

However, gaining *access* to this form may require enormously greater arithmetic complexity than the original algebra. In practice, a balance of techniques is most effective, even when motivated by physical ends. This balance can often be made by analysing the regular module (in which every eigenvalue is manifested), and thus subquotients of projective modules, but not more exotic modules. (Of course Mathematically other modules may well also be interesting — but this is a matter of aesthetic judgement rather than application.)

It may also be necessary to find the subspaces of H as a module for an algebra generated by a set of operators $\langle\Omega_i\rangle$. A similar analysis pertains.

A particularly nice (and Physically manifested) situation is one in which the operators Ω_i (whose unknown spectrum we seek to determine) are known to take the form of the representation matrices of elements of an abstract algebra A in some representation:

$$\Omega_i = \rho(\omega_i)$$

Of course any reduction of Ω_i in the form of (1.10) reduces the problem to finding the spectrum of $R_1(\omega_i)$ and $R_2(\omega_i)$. Thus the reduction of ρ to a (not necessarily direct) sum of irreducibles:

$$\rho(\omega_i) \cong +_\alpha \, \rho_\alpha(\omega_i)$$

reduces the spectrum problem in kind. In this way, Physics drives us to study the representation theory of the abstract algebra A.

1.14 Notes and references

ss:refs

The following texts are recommended reading: Jacobson[61, 62], Bass[6], Maclane and Birkoff[79], Green[52], Curtis and Reiner[30, 32], Cohn[24], Anderson and Fuller[3], Benson[7], Adamson[2], Cassels[20], Magnus, Karrass and Solitar[80], Lang[75], and references therein. .

1.15 Exercises

exe:gr01 (**1.15.1**) Let R be a commutative ring and S a set. Then RS denotes the 'free R-module with basis S', the R-module of formal finite sums $\sum_i r_i s_i$ with the obvious addition and R action. Show that this is indeed an R-module.

exe:gr1 (**1.15.2**) Let R be a commutative ring and G a finite group. Show that the multiplication in (1.12) makes RG a ring.

Hints: We need to show associativity. We have

$$\left(\left(\sum_i r_i g_i\right)\left(\sum_j r'_j g_j\right)\right)\left(\sum_k r''_k g_k\right) = \left(\sum_{ij}(r_i r'_j)(g_i g_j)\right)\left(\sum_k r''_k g_k\right) = \sum_{ijk}((r_i r'_j)r''_k)((g_i g_j)g_k) \tag{1.42}$$

groupalgmult2

and

$$\left(\sum_i r_i g_i\right)\left(\left(\sum_j r'_j g_j\right)\left(\sum_k r''_k g_k\right)\right) = \left(\sum_i r_i g_i\right)\left(\sum_{jk}(r'_j r''_k)(g_j g_k)\right) = \sum_{ijk}(r_i(r'_j r''_k))(g_i(g_j g_k)) \tag{1.43}$$

groupalgmult3

These are equal by associativity of multiplication in R and G separately.

(**1.15.3**) Show that RG is still a ring as above if G is a not-necessarily finite monoid and RG means the free module of finite support as above.

Hints: Multiplication in monoid G is also associative.

1.15.1 Radicals

ss:radical0001

Write J_R for the radical of ring R.

(**1.15.4**) A ring is *semiprime* if it has no nilpotent ideal.

(**1.15.5**) Theorem. A ring is left-semisimple if and only if every left ideal is a direct summand of the left regular module. ■

Show:

(**1.15.6**) Theorem. If S a subring of ring R such that, regarded as an S-bimodule, R contains S as a direct summand, then R left-semisimple implies S is left-semisimple.

Hint:

Let S' be an S-bimodule complement of S in R: that is, $R = S \oplus S'$ as an S-bimodule. (For example if $R = \mathbb{C}$ and $S = \mathbb{R}$ then we can take $S' = \mathbb{R}z$ for any $z \in \mathbb{C} \setminus \mathbb{R}$.) If I is any left ideal of S then it is in particular a subset of R and RI makes sense as a left R-module, and hence as a left S-module by restriction. We claim $RI = (S \oplus S')I = SI \oplus S'I = S \oplus S'I$ as a left S-module. Now RI is a direct summand of ${}_RR$ by left-semisimplicity, so I is a direct summand of ${}_SS$.

(**1.15.7**) Let G be a finite group of automorphisms of ring R. Write r^g for the image of $r \in R$ under $g \in G$. Show that

$$R^G := \{r \in R \mid r^g = r \;\forall\, g \in G\}$$

is a subring of R.

Show:

(**1.15.8**) THEOREM. Suppose that $|G|$ is invertible in R. If R is semisimple Artinian (e.g. a semisimple algebra over a field) then R^G is semisimple Artinian. ■

Hints:

Show that $J_R \cap R^G \subseteq J_{R^G}$.

1.15.2 What is categorical?

ss:whatcat

(**1.15.9**) Prove: THEOREM. Let A be an Artinian algebra and I an ideal. Then A/I non-semisimple implies A non-semisimple. ■

Solution: (There are many ways to prove this. Here is one close to the idea of indecomposable matrix representations.) If A/I non-semisimple then not every module is a direct sum of simple modules (by definition), so there are a pair of modules with a non-split extension between them. That is, there is a short exact sequence

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{p} M'' \longrightarrow 0$$

such that there is no sequence with the arrows reversed. This sequence, indeed any sequence involving these modules, is also 'in' A-mod via $\psi : A \to A/I$. Now suppose (for a contradiction) that there is a sequence in A-mod involving the images of these modules but with the arrows reversed. This means that some $N \subset M$ obeys $N \cong M''$ as an A-submodule of M, i.e. $AN = N$ (keep in mind that the action of A on M and hence N comes by $am = \psi(a)m$, and the A/I-module property of M). But ψ is surjective, so every $x \in A/I$ is $\psi(a)$ for some a, so $(A/I)N = AN = N$ so N is also an A/I-submodule. This is a contradiction. Thus the original sequence is non-split in A-mod. □

(**1.15.10**) Write $\mathrm{Res}_\psi : A/I - \mathrm{mod} \to A - mod$ for the functor associated to $\psi : A \to A/I$.

Let B be any algebra. Note that given a sequence of B-module maps

$$L \xrightarrow{f} M \xrightarrow{g} N$$

there is, trivially, an underlying sequence of maps of these objects as abelian groups. The exactness property at M, $im(f) = ker(g)$, is defined at the level of abelian groups. Thus the sequence is exact for any B if and only if it is exact at the level of abelian groups.

Use this to show that Res_ψ is exact.