Prediction notes

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These are the notes that we predict we will need to accompany the lecture that we predict we will give on April 27th 2027. (This prediction is being made on April 26th 2017.)

These notes accompany the slides on the prediction topic. Many of the slides are self explanatory, but some need further information. Under normal cicurmstances this would be discussed in the lecture, and we predict that circumstances will be normal this year (2027). But you can't be too careful... ...So, Here we add some of this information, or give links to investigate. This document is organised according to the slide number.

1. The topic is entitled 'Prediction' and will explore a little the difference between randomness and determinism. The main idea is that seemingly unpredictable outcomes do not require randomness or chance — this is the mathematics of chaos theory.

2. Niels Bohr was a famous physicist, G.H. Hardy a famous mathematician, and Paul Gascoigne a famous footballer. What things can you predict to the next minute, day, week, year?

3. Some of these sequences seem easy to continue, and others seem difficult. Have a go, and then read our solutions.

    (a) 1, 3, 5, 7, 9, 11, 13 (these are just the odd numbers in ascending order starting from 1)

    (b) 1, 2, 4, 7, 11, 16, 22, 29 (each increase is increasing by 1)

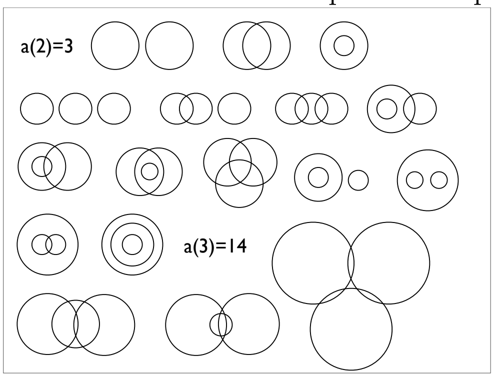
    (c) 1, 1, 2, 3, 5, 8, 13, 21, 34 (this is the Fibonacci sequence, where each number is the sum of the previous two)

    (d) 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 (these are prime numbers, which have no other factors than themselves and 1. Mathematicians have no way to predict the next prime number given the ones we already know. This fact lies at the heart of things like modern cryptography and internet security.)

    (e) 2, 4, 16, 256, 65536, 4294967296 (each number is the square of the previous. These numbers get very big very quickly. Perhaps an order of magnitude estimate is more useful?)

(f) 1, 2, 4, 8, 16, 31 (It might seem like 32 is the obvious next number, but I'm not thinking of the sequence which doubles each number. I'm thinking of 'circle division by chords' `http://mathworld.wolfram.com/CircleDivisionbyChords.html`. The point here is that predicting with insufficient data is difficult and dangerous.

(g) 1, 1, 3, 14, 173, 16951, unknown! (This is the number of 'generic arrangements' of n circles in the plane, starting from $n = 0$. That is, there is 1 way of arranging zero circles. There is 1 way of arranging 1 circle (note that we don't distinguish different positions or sizes of circles intrinsically — the circle is of generic size, in generic position). There are 3 ways of arranging 2 circles (they could be separate; one could be inside the other; they could be linked [tangential contacts are not allowed, since this is special rather than generic]) — see below. There are 14 ways of arranging 3 circles. Nobody knows how many ways of arranging 6 circles! Even apparently mathematical questions are sometimes hard or impossible to predict!



4. Think about these patterns and come up with your own.

5. Often patterns turn up in physical experiments, and this helps to formulate general principles about the real world.

6. Mathematicians like to 'explain' these patterns using symbols and logic. The equation is not the important thing here - what is important is the basic idea of observation, model, theory, equation, solution. There are many examples of 'success' in this sense, where scientists and mathematicians develop a theory which allows them to be confident about what will happen next (e.g., the position of the pendulum at some point in the future).

7. This is really the big idea in the mathematical version of science.

8. Here's an example of it working perfectly - the discovery of the planet Neptune.

9. But we know that predictions based on observation and an understanding of science are often wrong - weather forecasts are an obvious example. We can still write

down relevant equations, which might be harder than the pendulum equation, but maybe this is not enough. Why not?

10. For a long time philosophers have thought about the idea that if we knew everything about our current physical state, and the equations which govern the next state, we could surely predict the future... What about free will? To what extent are our decisions a product of atoms in the brain, and just governed by a bunch of chemical/physical equations? How many atoms are there in the universe anyway? (Did we discuss this already?) Around 10^{80}. Can you estimate how many calculations Laplace's demon would have to make? (One for the mathematicians perhaps). How many calculations per second would the demon have to make to finish the calculation before the death of the universe? But this is supposed to be 'prediction'. How many calculations does the demon have to do in 1 second, in order to predict 2 seconds ahead? And what about thermodynamics? Newton's equations go backwards or forwards, but the heat equation does not. And what about quantum mechanics? Roughly, Heisenberg's Uncertainty Principle tells us that you *can't* simultaneously make the measurements you need (position *and* momentum). Essentially things are "indeterminate" (the Copenhagen interpretation). There is a huge amount to think about, all beyond the scope of this module! `http://www.informationphilosopher.com/freedom/laplaces_demon.html`

11. Regardless of the thought experiment about Laplace's demon, there are clearly things that are hard to predict. Think about what randomness means in this context. Are any of these examples 'random'? Or just governed by complicated sets of equations?

12. We might imagine that prediction is difficult either because equations are difficult to solve, or stuff is just too complicated.

13. But here's a simple looking example which turns out to be much more unpredictable than you might expect. A double pendulum is just one pendulum hanging from another. How will it behave when set swinging? We usually bring one to our lecture, but look at this example: `https://www.youtube.com/watch?v=AwT0k09w-jw`

14. This looks totally unpredictable, even though I can write down the equations precisely, and can even solve them (with a computer)

15. For the rest of this lecture, we'll look at very simple systems, called iterated maps, which we will see can produce very complicated and unpredictable behaviour.

16. An iterated map simply gives some rule for generating the next number in a sequence. All we need is the rule, and the first number in the sequence, to generate as many as we like. This slide has a very simple example, where each number is the square of the previous. This is entirely predictable, as we know that the nth number in the sequence is just 2^n.

17. You might like to do this task, either with a pen and paper, probably using a calculator to help, or you may want to programme a computer to help with this (slightly tedious) task. You could do this with Excel, or Python, or anything else. The question is: do these sequences end up stuck at a particular point? (The solutions are that the first sequence ends up at 0, the second at 1/3. The third eventually alternates between two values, around 0.513 and 0.799. The fourth ends up cycling between four different values, while the fifth cycles between eight values).

18. Cobwebs diagrams need a little explanation. This is a graphical way of performing the calculations we just did. We draw the function defining the iterated map (for example, the function $f(x) = 3.2 \times x \times (1 - x)$, and on the same graph draw the diagonal line $y = x$. Given a point x, the next point in the sequence is just $f(x)$, so graphically this is simply starting at some x value of the horizontal axis, and drawing a line up to the curve $f(x)$. Next we want to plug the value $f(x)$ back into the function itself. Graphically, this means moving horizontally to the line $y = x$, and then vertically to the curve $f(x)$ again. We continue this procedure of moving horizontally (left or right) to the diagonal line, and then vertically (up or down) to the curve $f(x)$. This slide demonstrates this ending with the alternating between two values that we found in the previous slide.

19. In the lecture we would have tried this out. Feel free to do so, but for accurate results, you have to be very careful with your drawing! This process is called a *cobweb diagram*. See for example `https://en.wikipedia.org/wiki/Cobweb_plot`, or `https://www.desmos.com/calculator/unan9xh0og` for an interactive plot to play with (there they use a to represent the number in $f(x)$ we have so far chosen to be $0.8, 1.5, 3.2$ etc.

20. The next few slides do a bit of mathematical analysis. It's not really crucial to follow these details to understand the main ideas in this lecture, but some of you may find it interesting. Finding fixed points can be done algebraically, as described on this slide...

21. ... but can also be understood graphically. Plotting the curve of $f(x)$ and the diagonal line $y = x$ again, the fixed points are just the intersections between these two lines. (Can you see why?) For example, the first question has $x = 0$ and $x = 1/2$ as the fixed points of $x \to 2x(1 - x)$.

22. Similarly, stability of a fixed point can be found algebraically, as described on this slide...

23. ... but we can also do this graphically. Roughly speaking, a fixed point x is stable if the slope of the $f(x)$ at x is shallower than the diagonal, but unstable if the slope is steeper. Play around with `https://www.desmos.com/calculator/unan9xh0og` to see why.

24. This summary slide reminds us that we've only really seen predictable behaviour (fixed points or periodic, alternating behaviour) so far in this iterated map.

25. The next few slides gives a motivation for the map we've been looking at. It's a very (very very) simple model of a population which has a net birth/death rate of r, but the population cannot simply grow without bound (for example it has a maximum population which can be supported by the city).

26. Now we'll choose r to be bigger than the 3.55 we already looked at. Looking at the cobweb diagram for these values of $r > 3.55$ (especially $r = 4$ you should find intricate, complicated dynamics. This complexity turns out to be very difficult to predict. So far, it hasn't mattered which value of x was the initial value — (essentially) all initial x have ended up at the same fixed point or periodic point (for a particular r value). But when $r = 4$, the initial value certainly does matter. If you start at a particular x value (say $x = 0.1$, and plot the first few (say 30) values, and then do the same for a slightly different initial x value (say $x = 1.001$), you will see the two plots diverge after a few iterates of the sequence. Even if we start incredibly close to the original x value, the two sequences will soon behave entirely differently. There is nothing random about this iterated map, but the inherent complexity in the apparently simple equation creates an unpredictability, and an *apparent* randomness. This is the fundamental idea behind the subject of *chaos theory.*

27. This plot is called a *bifurcation diagram* and goes some way to explaining where the complexity comes from.To make this plot, we have the parameter r on the horizontal axis, and plot any eventually stable behaviour on the vertical axis. So, for $r < 3.0$ we have a single line, representing the fixed point we find for the sequence given by $x \to rx(1 - x)$. Remember that for $r = 3.2$, the sequence ended up alternating in between two points — this is shown on the graph by the two lines which seem to grow from the single line at $r = 3.0$. You can see these two solutions 'growing' from the fixed point at `https://www.desmos.com/calculator/unan9xh0og` by changing the parameter (a at that site) gradually from just below 3.0 to just above. At $r \approx 3.5$ the two solutions 'bifurcate' into 4, and as r increases, we get 8, then 16, then 32, etc, stable solutions. Remarkably, this 'period doubling' ends at $r \approx 3.5699$, and some chaotic behaviour begins, but for some values of r we slip back into periodic, alternating behaviour. For example, around $r \approx 3.828$ we have a 'period-3 window, where the sequence alternates between three values. Eventually, at $r = 4$ you can see all values of x between 0 and 1 are possible for 'end-points' of the sequence, and the map is fully chaotic. This is already complicated, but the real complexity in the system comes from *unstable* behaviour. When the curve of stable fixed points in this graph first bifurcates into two curves at $r = 3$, the original single line of fixed points continues (not plotted in the graph) as a curve of unstable fixed points. When the two period-2 curves bifurcate into 4, the period-2 lines continue as unstable curves. Eventually all this instability gets 'mixed up' with stable solutions, creating the chaos. Mathematicians understand

a lot about this type of behaviour, but that doesn't necessarily make it easy to control.

28. Perhaps the most famous way of describing the basic idea of chaos (as in the film *Jurassic Park*) is the famous quote about the butterfly effect by one of the parents[1] of chaos theory, Edward Lorenz.

[1] In 2020 the word used here was 'fathers', but this has now been decolonialised.