Chapter 5

Reflection groups and geometry

A number of important tools in representation theory can be characterised as *geometrical* or *combinatorial*. Here we set up some of the basic underlying machinery. We assume the reader knows linear algebra, what is a Euclidean space, a topological space (see §3.3), and so on.

In §5.1 we consider some basic geometry. In §5.2 we consider *reflection group*, *Coxeter group* and *chamber geometry* basics. (In particular we construct several specific groups here. However they appear here in a geometrical context, and not yet as objects of study in their own right.) In §5.4 we look at *parabolic subgroups* (needed later for representation theory) and *alcove geometry*.

In §5.6 we look at the basics of Kazdhan–Lusztig polynomials (considered combinatorially). In §5.7,§5.8 we change tack and look at Young diagram combinatorics (and *its* connection to alcove geometry).

Background reading examples: Humphreys [70], Lefschetz [?] (I am privileged to have inherited Goldie's copy at Leeds :-)).

5.1 Some basic geometry

ss:reflect0

Recall the definition of a topological space from §3.3, and that $\mathbb{R}^n$ is a topological space via the metric topology. In particular $\mathbb{R}$ is a topological space with open sets generated by the fundamental set of open intervals.

A topological space T is *Hausdorff* if every pair of distinct points are contained in distinct open sets of T (see e.g. [?, §6]).

5.1.1 Affine spaces and simplicial complexes

Fix any $n \in \mathbb{N}$. An *affine subspace* of $\mathbb{R}^n$ is a subset of form $A = v + S$ where S is a subspace and v is a vector not in S. Note that if $w \in A$ then $A - w = S$, so A determines the underlying space S. The *dimension* of A is the dimension of S. Note that every affine space defines a fibre of affine spaces over its underlying space, and that this fibre is a partition of $\mathbb{R}^n$.

A subspace of codimension 1 is a *hyperplane*, $H = v + H_0$ say. A hyperplane in $\mathbb{R}^n$ defines a partition of $\mathbb{R}^n$ into three: H itself; the part of the fibre 'above' H relative to v (the union of hyperplanes of form $H_x = xv + H_0$ with $x > 1$ — note that we have used the ordered field property

of $\mathbb{R}$ here); and the part below. (The designation above/below depends on v, but the partition does not.) These unions are called *open half-spaces*. The union of hyperplanes of form $H_x = xv + H_0$ with $x \geq 1$ is a closed half-space. Thus we have a pair of closed half-spaces, whose intersection is the hyperplane.

A hyperplane *supports* a subset T of $\mathbb{R}^n$ if T lies entirely in one of its closed half-spaces.

(5.1.1) The *affine hull* of a set of points in $\mathbb{R}^n$ is the intersection of all affine spaces containing the set.

A set of $t+1$ points in $\mathbb{R}^n$ is *affinely independent* if every proper subset has a strictly smaller affine hull. Thus the vertices of a triangle are affinely independent, but the vertices of a square (or plane quadrilateral) are not.

(5.1.2) A *t-simplex* is the convex hull of an affinely independent set of $t+1$ points in $\mathbb{R}^n$.

A hyperplane supports a simplex if it supports it as a subset of the underlying space. A *face* of a simplex is the intersection with a supporting hyperplane.

Note that a face is a simplex (of dimension not greater than that of the original simplex). Note that there are in general infinitely many supporting hyperplanes for a simplex. If the simplex is of maximal dimension then there is precisely one hyperplane defining each face of codimension 1. The lower-dimensional faces may then be characterised as the intersections of the simplex with more than one of these defining hyperplanes.

(5.1.3) A *simplicial complex* (see e.g. [57, Ch.15]) is a set Z of simplices that is (i) closed under including faces; (ii) if $A, B \in Z$ then $A \cap B$ is a face of both.

A complex Z is *pure* if the maximal faces (faces that are not faces of simplices in Z other than themselves) are all of the same dimension.

Given a complex Z we write Z^i for the subset of i-dimensional faces; and $Z^{\leq i}$ for the *i-skeleton* (the obvious union; which is a subcomplex).

5.1.2 Hyperplane geometry, polytopes etc

Let V be a euclidean space and $H \subset V$ a hyperplane. This H separates V into three connected components: H and the two open components of $V \setminus H$. The complement of one of these two open components is called a half-space (this includes affine half-spaces; to emphasise a non-affine case we can say 'linear half-space').

de:chyp

(5.1.4) Let $\mathbb{H}$ be a set of hyperplanes in V (not necessarily closed under the reflection group they generate). We assume that $\mathbb{H}$ is finite (or in a suitable sense locally finite, so that the set of points of V not lying on any hyperplane is euclidean-topology-open; a generic point on a hyperplane does not lie on any other hyperplane, and so on — we will give not-strictly-finite examples of this later).

The $\mathbb{H}$-singularity of $v \in V$ is

$$s_{\mathbb{H}}(v) = \#\{H \in \mathbb{H} \mid v \in H\}$$

The set $\mathbb{H}$ separates V into various maximal connected components of fixed singularity, called *facets*. Singularity zero facets are *chambers of* $\mathbb{H}$. We write $\mathcal{C}_{\mathbb{H}}$ for the set of chambers.

(5.1.5) For f a facet, the euclidean-space closure $\bar{f}$ is the union of certain facets. Besides f, all the other facets in $\bar{f}$ are of higher singularity.

(5.1.6) Write $\bar{\mathbb{H}}$ for the closure of $\mathbb{H}$ under the reflection group it generates.

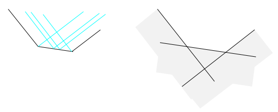


Figure 5.1: Polyhedron as (a) $H + C$; (b) intersection of half-spaces (unshaded region). fig:polyh1

(**5.1.7**) Fix $n \in \mathbb{N}$. Given a set $P \subset \mathbb{R}^n$ of points, its *hull* is the smallest convex set containing P. Given a finite set $P \subset \mathbb{R}^n$ of vectors (NB points define vectors relative to 0), its *cone* is the set of $\mathbb{R}^{\geq 0}$-linear combinations. Note that a cone necessarily contains 0, but a hull does not.

A *(convex) polytope* is the hull of a finite set of points. The dimension of a polytope Π is the smallest dimension of an affine space containing Π.

A *polyhedron* is a subset of $\mathbb{R}^n$ of form $\Pi = H + C$ where H is a finitely generated hull and C is a cone.

Example: Fig.5.1

A *ray* is a polyhedron $\Pi = H + C$ where H and C are each generated by a single point/vector. A polyhedron is *bounded* if it contains no ray. Thus a bounded polyhedron is a polytope.

(**5.1.8**) Theorem. *(I) A subset $\Pi \subset \mathbb{R}^n$ is a polytope (a finitely-generated hull) if and only if it is a bounded intersection of half-spaces, that is: $\Pi = \{x \mid Mx \leq c\}$ for some matrix M and vector c; and Π contains no ray.*
(II) A subset $\Pi \subset \mathbb{R}^n$ is a polyhedron if and only if it is an intersection of half-spaces, that is: $\Pi = \{x \mid Mx \leq c\}$ for some matrix M and vector c.

Proof. See e.g. Ziegler [154, §1.1]. □

(**5.1.9**) Using the theorem we have the following definition.

A *proper face* of a polyhedron (or polytope) Π is the intersection with any hyperplane defining a half-space entirely containing Π. For example, any hyperplane which misses Π entirely defines the *empty face*. The vertices and edges are also faces. A *face* is either a proper face or is the polyhedron itself.

Note that a face is itself a polyhedron (resp. polytope).

(**5.1.10**) A *polyhedral (polytopal) complex* is a (locally) finite set Z of polyhedra (resp. polytopes) such that (i) $A, B \in Z$ implies $A \cap B$ is a face of both; (ii) $A \in Z$ implies all its faces are in Z; (iii) $\emptyset \in Z$.

A complex is *pure* if the maximal faces are all of the same dimension. We will also assume that they are of maximal dimension (i.e. of dimension n in $\mathbb{R}^n$).

(**5.1.11**) Example. Any locally finite set $\mathbb{H}$ of hyperplanes in $V = \mathbb{R}^n$ defines a complex (typically polyhedral) $Z(\mathbb{H})$. The (closures of the) chambers of $V \setminus \mathbb{H}$ are the maximal faces.

de:dual graph (**5.1.12**) The *dual graph* of a pure polyhedral complex in $\mathbb{R}^n$ has a vertex for each polyhedron of dimension n (codimension 0) and an edge (A, B) whenever $A \cap B$ is a face of dimension $n-1$ (codimension 1).
(See e.g. Kalai [57, §17.3].)

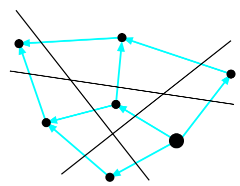


Figure 5.2: Directed rooted dual graph $D(\mathbb{H})$ of a hyperplane complex $Z(\mathbb{H})$. fig:polyh2

(**5.1.13**) Define $D_-(\mathbb{H})$ as the dual graph of hyperplane complex $Z(\mathbb{H})$. Given a 'root' chamber/maximal-face, this graph becomes a directed graph $D(\mathbb{H})$: every hyperplane, hence codimension 1 face, has one half-space containing the root; the edge is directed towards the half-space not containing the root.

Example: Fig.5.2

(**5.1.14**) The digraph $D(\mathbb{H})$ is clearly acyclic. Indeed it has a length function on vertices: $l(v)$ is the number of hyperplanes between v and the root, so that every directed edge corresponds to increasing l by 1. It follows here that any two directed paths from any vertex A to B have the same number of edges.

Note that the dual graph of an arbitrary pure complex does not have a length function since its faces do not necessarily extend to separate the underlying space.

5.2 Reflections, hyperplanes and reflection groups

ss:reflect

Here we follow the approach of Humphreys [70] (cf. Bourbaki) in significant part. See also Jacobson [71], Fulton–Harris [52, §21.1], Moise [128], Ziegler [154], and various chapters from Goodman–O'Rourke [57].

A *reflection* in $\mathbb{E}^n$ is a linear map that fixes a hyperplane pointwise (or the corresponding map for an affine hyperplane).

de:refsex1 (**5.2.1**) Examples of reflections acting on a Euclidean space ($\mathbb{R}^n$ with the usual dot product):

$$(ij) : \mathbb{R}^n \to \mathbb{R}^n \quad (5.1) \quad \text{eq:ref00ij}$$
$$(x_1, x_2, ..., x_i, ..., x_j, ...x_n) \mapsto (x_1, x_2, ..., x_j, ..., x_i, ...x_n) \quad (5.2)$$

with reflection hyperplane

$$H_{(ij)} = \{x \in \mathbb{R}^n \mid x_i = x_j\};$$

$$(i)_- : \mathbb{R}^n \to \mathbb{R}^n \quad (5.3) \quad \text{eq:ref00i-}$$
$$(x_1, x_2, ..., x_i, ..., x_j, ...x_n) \mapsto (x_1, x_2, ..., -x_i, ..., x_j, ...x_n) \quad (5.4)$$

with reflection hyperplane

$$H_{(i)_-} = \{x \in \mathbb{R}^n \mid x_i = 0\}.$$

These reflections make sense in $\mathbb{R}^n$ for any $n \in \mathbb{N}$, and also in $\mathbb{R}^{\mathbb{N}}$, the space of infinite sequences (the dot product is not defined here for arbitrary pairs of vectors, but it is defined for an arbitrary vector with one of the perpendicular vectors for the given reflections).

(**5.2.2**) In the examples (ij) and $(i)_-$ the hyperplane contains the origin of coordinates, i.e. they are subspaces. A hyperplane not containing the origin is an *affine* hyperplane — a translate of a non-affine hyperplane by some vector. Any non-affine hyperplane may be characterised as the subspace of vectors perpendicular to a given non-zero vector v. For example $H_{(i)_-}$ is perpendicular to e_i. Any affine hyperplane can then be characterised by a perpendicular vector and a distance along that vector.

If v is the vector defining H the corresponding reflection is the linear map

$$y \mapsto y - \frac{2y.v}{v.v}v \tag{5.5}$$

eq:reflect

(which obviously fixes H and sends v to $-v$). If H is affine with vector v and distance d we have reflection

$$y \;\mapsto\; ((y-d\bar{v}) - \frac{2(y-d\bar{v}).v}{v.v}v) + d\bar{v} \;=\; y - \frac{2(y-d\bar{v}).v}{v.v}v$$

where $\bar{v}$ is the unit vector. Defining $\check{v} = \frac{2}{v.v}v$ this is

$$y \;\mapsto\; y - (y.v - d\bar{v}.v)\check{v}.$$

A reflection group is a group generated by reflections (affine and or non-affine).

(**5.2.3**) A *non-affine reflection group on a space* E is a subgroup of the group of orthogonal transformations on E that is generated by reflections.

Any set $\mathbb{H}$ of hyperplanes H_t in a space E specifies a set of reflections t and hence a reflection group — the group $W_{\mathbb{H}}$ generated by reflection in these hyperplanes. Thus if $\mathbb{H} = \{H_t \mid t \in S\}$, say, we have, for some set of relations $\sim$, an abstract presentation

$$W_{\mathbb{H}} \;\cong\; \langle t \in S \mid \sim\rangle \tag{5.6}$$

eq:precox

where the relations $\sim$ include $t^2 = 1$ but otherwise depend on the details (see §5.2.2).

(**5.2.4**) Some other elements of the group $W_{\mathbb{H}}$ may also be reflections. Indeed the image of any hyperplane H_s in another, H_t say, is a hyperplane H_{tst} corresponding to a reflection tst in $W_{\mathbb{H}}$ (NB for t, s reflections then $(tst)(tst) = 1$). We write $\bar{\mathbb{H}}$ for the closed set of hyperplanes obtained from $\mathbb{H}$.

(**5.2.5**) Example. Consider the group W_S generated by reflection set $S = \{(12), (23), (34)\}$ in $\mathbb{R}^4$. We have $(12)(23)(12) = (13)$ (indeed $(23)(12)(23) = (13)$). Also $(12)(34)(12) = (34)$, $(34)(12)(34) = (12)$, and $(34)(13)(34) = (14)$. Meanwhile neither $(12)(34)$ nor $(12)(23)$ in W_S are reflections. See §11.1.4 for more examples.

(**5.2.6**) Remark. We might say $W_{\mathbb{H}}$ on space V is 'locally finite' if only finitely many elements of $\bar{\mathbb{H}}$ pass through a finite interval of V. (We will effectively restrict ourselves to such groups here, along with certain finitary versions. The study of *Coxeter systems* (see 5.2.10) gives a way of selecting locally finite cases from the more general context, but we shall not focus on this aspect of their use here.)

5.2.1 Reflection group root systems

de:root system

(**5.2.7**) Let W be a finite reflection group on space V and $\mathbb{H}$ its set of hyperplanes in V. Recall that a (non-affine) hyperplane can be characterised by a perpendicular vector (any vector in the same direction will do). Reflection of a vector in a hyperplane produces a vector for the corresponding reflected hyperplane. Closing this process, however, produces at least two vectors per hyperplane, since reflecting vector v in its own hyperplane gives $-v$. (We can characterise a specific half-space by a vector into this half-space, so then we have two vectors for the two half-spaces associated to a hyperplane.)

A *root system* Φ is a finite set of non-zero vectors closed under the reflections it defines, and minimal as such in that it contains precisely two vectors on each of its lines, v and $-v$.

Notes: The combination of closure and finiteness strongly constrains the possibilities.
It is of interest (for example in Lie theory, see later) to impose the further 'crystallographic' constraint that $n_{v,v'} = 2v.v'/(v'.v') \in \mathbb{Z}$ for all $v, v' \in \Phi$. (This amounts to saying that W fixes some lattice of integral combinations of a basis in V.)
Note that

$$n_{v,v'}n_{v',v} = 4\cos^2(\theta)$$

where θ is the angle between the vectors. (One can think of the constraints in terms of this angle.) Such an angle *may* be defined even between pairs of affine reflections — excepting of course those reflections that are parallel.

: simple system

(**5.2.8**) We can characterise a root system by giving the numbers $n_{v,v'}$. This amounts to a description of the system via certain subgroups, each generated by a certain pair of reflections $s_v, s_{v'}$. (Finiteness implies $(s_v s_{v'})^m = 1$ for some m.) It is enough to give this data for one of each pair $v, -v$ (since $s_v = s_{-v}$), and indeed for a spanning subset of Φ. (Exercise: why is this enough?)

A *positive root system* Π is a choice of one of $v, -v$ over all Φ (a transversal of the partition of Φ into $v, -v$ pairs). A *simple root system* Δ is a subset that is a basis of $\mathbb{R}\Phi$. (It is not meant to be obvious that such a thing exists.)

(**5.2.9**) For given Δ the *Cartan matrix* of Φ is $(n_{v,v'})_{v,v' \in \Delta}$. The constraints limit the possibilities for this matrix to a countable collection of 'types'. Evidently $n_{v,v} = 2$, but also off-diagonal entries are restricted (in strictly finite cases) to $\{0, -1, -2, -3\}$. The zero entries appear symmetrically, and otherwise at least one of each opposing pair is -1. For example:

$$C_{A_2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \qquad C_{A_3} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad C_{B_4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -2 & 2 \end{pmatrix} \tag{5.7}$$

eq:Cartan mat eg

The angle θ for a pair 0,0 is $\theta = \pi/2$. It follows that the product st of the reflections involved acts as a rotation by $2\theta = \pi$; so the pair of reflections obeys $stst = 1$, i.e. $(st)^2 = 1$.
The angle θ for a pair -1,-1 is $\theta = 2\pi/3$ (so the pair of reflections obeys $ststst = 1$, i.e. $(st)^3 = 1$). In fact, in a suitable sense, almost every case is of the 0,0 form (see later), and almost every other of the -1,-1 form.
For a pair -1,-2 the angle is $\theta = 3\pi/4$ (so the pair of reflections obeys $stststst = 1$, i.e. $(st)^4 = 1$); for $-1, -3$ is it $\theta = 5\pi/6$.

The singular case of parallel walls s, t gives a translation st rather than a reflection.

ss:coxeter1

5.2.2 Coxeter systems and reflection groups by presentation

Recall that presentations for groups are useful things, for various reasons. A systematic approach to the presentation problem (5.6) — finding abstract presentations for reflection groups — is given by Coxeter systems.

de:Coxeter

(**5.2.10**) A *Coxeter system* is a pair (W,S) consisting of a group W, and a set of generators $S \subset W$, that satisfy the *Coxeter-generators condition*. This means that there is a symmetric matrix M indexed by S such that $M_{s,s} = 1$ and $M_{s,s'} \geq 2$ (with every entry in $\mathbb{N} \cup \{\infty\}$), defining relations by

$$(ss')^{M_{s,s'}} = 1$$

(here $M_{s,s'} = \infty$ means no relation; $M_{s,s'} = 3$ means $(ss')^3 = 1$, i.e. $ss's = s'ss'$).

(**5.2.11**) Example. 1. Every $(W,\{s\})$ has $M = (1)$ and group W of order 2.
2. The system defined (up to isomorphism) by

$$M = M_{\tilde{A}_1} := \begin{pmatrix} 1 & \infty \\ \infty & 1 \end{pmatrix}$$

has group W of infinite order. (See (5.2.16) for a reflection group realisation.)

(**5.2.12**) Remark. The $\tilde{A}_1$ above refers to the 'type' classification of certain integer matrices. See e.g. [70] and (5.2.7) *et seq.*

de:parab

(**5.2.13**) A *parabolic subsystem* of (W,S) is a system generated by some $I \subset S$.

Coxeter systems are closely related to reflection groups. Indeed (see [70, §6.4]):

th:coxeterr

(**5.2.14**) Theorem. *Coxeter group W is finite if and only if it is a finite reflection group.*

And for our present purposes their main role is as an organisational device for the study of reflection groups. A consequence of the definition is that the generating set S is minimal; and that there is a reflection group realisation of S as a set of reflections (thus corresponding to a set of hyperplanes). Thus we can realise (W,S) as some pair $(W_{\mathbb{H}}, \mathbb{H})$.

de:Clength

(**5.2.15**) Fix a Coxeter system (W,S). The *length* $l(w)$ of $w \in W$ is the smallest number of factors in an expression of w as a product of generators.

pa:hyper1

(**5.2.16**) Let V be a Euclidean space. We want to think for a moment about the reflection groups on V, and Coxeter systems (W,S) with an action generated by reflections on V, that coincide with these reflection groups.

Any single hyperplane in V generates a reflection group of order 2. Starting with this example, one may add in a second hyperplane and ask about the group G generated by reflections in the two hyperplanes.

The nature of this group G depends on the relationship of the two hyperplanes; and the possibilities for this relationship depend on the dimension of V. In dimension 2 or greater reflection in one hyperplane may fix the other; or they may be parallel; or they may generate a finite number of other hyperplanes; or (if the second hyperplane is in generic position with respect to the first) an infinite number. (Here we are interested in all but the generic position case.)

In dimension 1 of course, any two distinct hyperplanes (i.e. points) are necessarily parallel. The associated Coxeter system is necessarily $M = M_{\tilde{A}_1}$.

(**5.2.17**) Let (W,S) be a Coxeter system. The Coxeter relation matrix M can be represented by a graph, with a vertex for each generator and $M_{ss'} - 2$ edges between s and s'. (As far as the Coxeter system is concerned these edges are undirected, although there is a related role in Lie theory where more data is used.) Some of these graphs have names, or 'types'.

5.2.3 Some finite and hyperfinite examples and exercises

ex:Dgroup

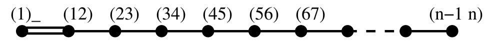
(**5.2.18**) Example. Fix $n \in \mathbb{N}$ and consider the reflections in $\mathbb{R}^n$ defined in (5.2.1).
(1) The reflection group $\mathcal{D}_+ = \mathcal{D}_+(n)$ is the group generated by the set of reflections $S_{A_n} := \{(i\; i{+}1) : i = 1,2,...,n-1\}$. We have $\mathcal{D}_+ \cong S_n$.

Proposition. The generators S_{A_n} obey Coxeter relations of type-A. ■

(2) The reflection group $\mathcal{D}^-$ generated by $S_{B_n} := \{(1)_-, (i\; i{+}1) : i = 1,2,...,n-1\}$ obeys Coxeter relations including:

$$(1)_-(12)(1)_-(12) = (12)(1)_-(12)(1)_-$$

of type-B:

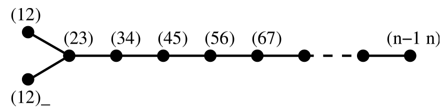


Let $(12)_-$ denote reflection in the hyperplane with $x_1 = -x_2$. Define $(ij)_-$ similarly. Note that $(12)_- = (1)_-(12)(1)_-$. Note that $(12)(1)_-(12) = (2)_-$. Thus $\mathcal{D}^-$ includes all the reflections of form $(ij), (i)_-$ and $(ij)_-$.

(3) The reflection group $\mathcal{D}$ generated by $S_{D_n} := \{(12)_-, (i\; i{+}1) : i = 1,2,...,n-1\}$ obeys relations including

$$(12)_-(12)(12)_-(12) = 1 \qquad \text{and} \qquad (12)_-(23)(12)_- = (23)(12)_-(23)$$

of type-D:



(**5.2.19**) Note that $\mathcal{D}_+ \subset \mathcal{D} \subset \mathcal{D}^-$. Note that $\mathcal{D}_+(n) \hookrightarrow \mathcal{D}_+(n{+}1)$ (take the map which corresponds to inclusion in the corresponding abstract Coxeter system), so we may also define a $n \to \infty$ limit.

(**5.2.20**) Exercise. By construction an automorphism of the Coxeter graph induces an automorphism of the corresponding group that fixes the set of Coxeter generators. It can be shown that the subgroup of elements fixed by the automorphism is also a Coxeter group. For example, the graph for $\mathcal{D}_+$ is a chain, which has a chain order reversal automorphism, call it γ. Construct the fixed point group $\mathcal{D}_{+\gamma}$.

Hint: We represent permutations as diagrams, lets say drawn from top to bottom. Then an element is fixed under γ if it is left-right symmetric. (The resultant Coxeter group is of type-B.)

(**5.2.21**) Exercise. Consider the inner automorphism of $\mathcal{D}^-$ given by $w \mapsto (1)_- w (1)_-$. This interchanges (12) and $(12)_-$ (not a Coxeter generator of $\mathcal{D}^-$) but leaves the other Coxeter generators alone. This map then passes to an outer automorphism ω (say) of $\mathcal{D}$, corresponding to the obvious automorphism of the type-D Coxeter graph. 1. Compute the subgroup $\mathcal{D}_\omega$ of elements fixed by this outer automorphism. 2. Show that this is again a Coxeter group (see e.g. Nanba [?]).

1. This subgroup includes (and is generated by) the Coxeter generators excluding (12) and $(12)_-$, together with the element $(12)(12)_- = (1)_-(2)_-$.
2. Evidently $(1)_-$ commutes with this fixed group and with $(2)_-$, giving

$$(1)_-(2)_-(23)(1)_-(2)_-(23) = (2)_-(23)(2)_-(23) = (23)(2)_-(23)(2)_- = (23)(1)_-(2)_-(23)(1)_-(2)_-$$

so there is a map given by $(1)_-(2)_- \mapsto (1)_-$ and $(i\ i+1) \mapsto (i-1\ i)$ (for $i = 2, 3, ...$) that is a bijection between the fixed group $\mathcal{D}_\omega(n)$ and $\mathcal{D}^-(n-1)$.

Remark: Since $\mathcal{D}(n-1) \subset \mathcal{D}^-(n-1)$ this gives an injection of $\mathcal{D}(n-1) \subset \mathcal{D}(n)$ different from the obvious inclusion. Specifically one takes $(12) \mapsto (23)$ and $(12)_- \mapsto (12)(12)_-(23)(12)(12)_- = (1)_-(2)_-(23)(1)_-(2)_- = (23)_-$ and then $(23) \mapsto (34)$ and so on. (Of course $(23)_-$ is not a Coxeter generator on the right, but otherwise this still seems fairly obvious and uninteresing...)

D--ex **(5.2.22)** It will be clear that the point

$$v_- := (-1, -2, -3, ...) \in \mathbb{Z}^n \subset \mathbb{R}^n$$

($n \in \mathbb{N}$ or $n = \infty$) is not fixed by any nontrivial element of $\mathcal{D}^-$ (it is a *regular* point). The orbit of this point is the set of signed perms of $(-1, -2, -3, ...)$ (with finitely many +'s).

D--ex2 **(5.2.23)** It will be useful later to note that there are several descending ('dominant') elements, such as $(-1, -2, -3, ...)$, $(1, -2, -3, ...)$, $(2, -1, -3, ...)$, $(2, 1, -3, ...)$, $(3, -1, -2, ...)$, ... in the orbit $\mathcal{D}^- v_-$. We denote the set of these by $\mathcal{D}^-(-1, -2, -3, ...)^A$ (this is a transversal of the partition of $\mathcal{D}^-(-1, -2, -3, ...)$ into $\mathcal{D}_+$-orbits). A convenient representation of these descending cases is to list the positive entries (the order in the list does not matter).

A pair of these signed perms are adjacent (on either side of a reflection hyperplane, with no other hyperplane between them) if they are related by 'right' action by a generator.

See Fig.5.5 for the subgraph of descending elements and edges labelled by the corresponding generator.

(5.2.24) On the other hand certain points such as $(0, -2, -3, ...)$ lie on reflection hyperplanes of $\mathcal{D}^-$ (in this case the point lies on the hyperplane corresponding to $(1)_-$), and so are fixed by the corresponding reflections.

Do the orbits of such points have different looking such graphs?

We have $\mathcal{D}^-(0, -2, -3, ...)^A = \{(0, -2, -3, ...), (2, 0, -3, ...), (3, 0, -2, -4, ...), (3, 2, 0, -4, ...), (4, 0, -2, -3, ...), (5, 0, -2, -3, -4, ...), (4, 2, 0, -3, ...), (6, 0, -2, -3, -4, -5, ...), \}$

We will return to this example in §5.5.

(5.2.25) Exercise. Do something similar to (5.2.23) for type-A, restricting to the subgraph of elements descending in all but possible the first position: $(-1, -2, -3, ...)$, $(-2, -1, -3, ...)$, $(-3, -1, -2, ...)$, and so on. (Note that this is much simpler.)

5.3 Reflection group chamber geometry

ss:Hum

We continue with W a reflection group on space V. By (5.2.14) we may assume there is an associated Coxeter system (W, S). The group W and its action defines a closed set of hyperplanes $\mathbb{H} = \bar{\mathbb{H}}$ in V. This is irrespective of S. However the closed set of hyperplanes $\bar{\mathbb{H}}$ will be generated by the set $\mathbb{H}_S$ corresponding to S.

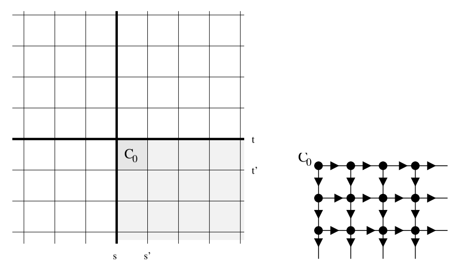


fig:Youngreflect
ig:Youngreflect2

Figure 5.3: (a) Chamber geometry of type $\tilde{A}_1 \times \tilde{A}_1$ with $S = \{s, s', t, t'\}$, and a fundamental region for the $W_{\{s,t\}}$ parabolic. (b) The digraph G_a for $W' = W_{\{s,s',t,t'\}}$ and $W = W_{\{s,t\}}$ from Figure 5.3(a).

(5.3.1) Let H_s be the reflection hyperplane of $s \in S$, or indeed of any reflection $s \in W$ generated by these. For T any subset of S let $[T] \subset W$ be the set of reflections generated by T. Set

$$\mathbb{H}_T = \bigcup_{t \in [S] \setminus [T]} H_t$$

(5.3.2) A *chamber* of W on V is a chamber of $\mathbb{H}$ or $\mathbb{H}_\emptyset$ as in (5.1.4) — a maximal connected component of $V \setminus \mathbb{H}_\emptyset$. Write $\mathcal{C}_W$ for the set of chambers here. That is, $\mathcal{C}_W = \mathcal{C}_{\mathbb{H}}$.

(5.3.3) Example. In our $\tilde{A}_1$ example in (5.2.16) above the chambers are simply a collection of disjoint open intervals of the line.

For an example generalising this we may think of the group generated by two parallel lines in $\mathbb{R}^2$, together with two other parallel lines, perpendicular to the first two:

$$M = \begin{pmatrix} 1 & \infty & 2 & 2 \\ \infty & 1 & 2 & 2 \\ 2 & 2 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix}$$

The chambers here are a collection of disjoint open rectangles (which may be arranged to be squares without loss of generality here). See Figure 5.3.

(5.3.4) Note that the set $H_t \setminus \mathbb{H}_{\{t\}}$ is the subset of hyperplane H_t that intersects no other hyperplane — or equivalently the subset of H_t of elements with singularity 1. This set may similarly be broken up into connected components. At most one of these components intersects any given chamber closure $\overline{C}$. If H_t intersects $\overline{C}$ in this way it is called a *wall* of C. The intersection itself is a *wall-facet* of C. That is, singularity-1 facets (in the sense of (5.1.4)) are wall-facets.

lem:coxeter gens

(**5.3.5**) Lemma. *Let W be a reflection group acting on V. For any given chamber $C \in \mathcal{C}_W$, the set of hyperplanes, hence reflections t, that make up its walls functions as a choice of a Coxeter generating set S in W.*

(**5.3.6**) The choice of a prefered chamber C_0 corresponds to the choice of a simple system (in the sense of (5.2.8)) in V, and the associated reflections are *simple* reflections. (Given a non-commuting pair of these, the conjugate of one by the other is also a reflection, but not 'simple' in this choice.)

(**5.3.7**) NB The choice of C_0 determines S, but S does not necessarily determine C_0. We will usually fix C_0 along with W. Thus we may consider a *reflection system* (W, C_0), which determines a Coxeter system.

(**5.3.8**) A reflection s in W is *simple for chamber B* if its hyperplane H_s makes a wall of B (NB 'simple for B' is not the same as simple, unless $B = C_0$). For our purposes it will be convenient to think specifically of the intersection of the hyperplane with the chamber closure (i.e. this facet) as the wall (thus we distinguish the walls of distinct chambers in general, even if they come from the same hyperplane).

para:reflectx

(**5.3.9**) Lemma. *The reflection action of W on V acts to permute $\mathcal{C}_W$. This action on $\mathcal{C}_W$ is transitive (i.e. for any $C \in \mathcal{C}_W$ the orbit $WC = \mathcal{C}_W$) and indeed regular (simply transitive) (i.e. for $C, D \in \mathcal{C}_W$ there is exactly one $w \in W$ such that $wC = D$).*

Proof. See for example [70, §1.12]. □

(**5.3.10**) Note that W does *not* act transitively on V, or specifically, on the set of walls.

le:orb1

Lemma. The wall-facets of any chosen chamber C_0 are representatives for the W orbits of the set of all wall-facets.

pa:zeta

(**5.3.11**) Regularity of $\mathcal{C}_W$ as a W-set says that for each choice of chamber C_0 we may make an identification

$$\zeta_{C_0} : \mathcal{C}_W \xrightarrow{\sim} W,$$

so that the action of W becomes the left-action on itself. In particular we identify the chosen facet C_0 with $1 \in W$ and write

$$A = w_A C_0 \tag{5.8}$$

idW

to identify the chamber A with group element $\zeta_{C_0}(A) = w_A$.

de:right-act

(**5.3.12**) Note that it follows from the identification ζ_{C_0} that there is another commuting action of W on $\mathcal{C}_W$, corresponding to the right-action of W on itself.

(**5.3.13**) With W and C_0 given, define a length function on $\mathcal{C}_W$: $l_W(A)$ as the number of hyperplanes separating A from C_0. (If W is clear from context we shall write simply $l = l_W$.)

(**5.3.14**) Lemma. *This geometric length function l_W on $\mathcal{C}_W$ for (W, C_0) agrees with the Coxeter length function on W for Coxeter system $(W, \mathbb{H}_{C_0})$ from (5.2.15) via ζ.*

Proof. Firstly $l_W(C_0) = l(1) = 0$. Then for $a \in \mathcal{C}_W$, $l_W(sa) = l_W(a) \pm 1$, since we reflected over a hyperplane, while $l(s\zeta(a)) = l(\zeta(a)) \pm 1$ by definition. It remains to check the signs. Exercise: finish! ???

5.3.1 S_n as a reflection group, permutahedra, etc

See §11.1.4.

5.3.2 Cayley and dual graphs, Bruhat order

para:r-action

(**5.3.15**) We define a digraph $G(W, S)$ with vertex set $\mathcal{C}_W$ by (A, B) an edge if $B = tA$ with t simple for A and $l(B) = l(A) + 1$.

How does this depend on S?

(**5.3.16**) We call t the *left-action label* of edge (A, tA).

By (5.8) the edge (A, tA) may also be written $(w_A C_0, t w_A C_0)$. The image under w_A of a particular 'initial' edge (C_0, sC_0) $(s \in S)$ is

$$(w_A C_0, w_A s C_0) = (w_A C_0, w_A s w_A^{-1} w_A C_0) = (A, w_A s w_A^{-1} A)$$

Our t is such a $w_A s w_A^{-1}$, by Lem.5.3.10. Using the right-action this can be expressed as

$$(w_A C_0, w_A s C_0) = (w_A C_0, w_A C_0 s) = (A, As)$$

We call this s the right-action label of the edge.

(**5.3.17**) CLAIM: With this right-action label the graph $G(W, S)$ is essentially the directed ('right') Cayley graph $\Gamma(W, S)$, and s is the 'colour' label.

Proof. ... directed ...

de:Bruhat

(**5.3.18**) Evidently $G(W, S)$ is a rooted acyclic digraph, with root C_0. The partial order so defined is the *Bruhat order* on chambers/elements of W.

(**5.3.19**) CLAIM: Let (W, S) be a Coxeter system and let $\mathbb{H}$ be the set of hyperplanes of W realised as a reflection group on V, so that W acts regularly on chambers, the fundamental chamber C_0 may be identified with $1 \in W$ and the walls of this chamber give S. That is, $(W, S) = (W_{\mathbb{H}}, \mathbb{H}_{C_0})$. Then as directed graphs

$$G(W, S) = D(\mathbb{H})$$

where the root of $D(\mathbb{H})$ is C_0.

Proof. Both graphs have vertex set $W = W_{\mathbb{H}}$ The edge sets also agree: the edge (a, ta) corresponds by definition to a face between these chambers.

claim: $(a, ta = as)$, some s, and corresponds to a face in the W-orbit of the face between 1 and s.

HOW DO WE KNOW THIS? By the argument above. □

(**5.3.20**) Let $v \in C_0$, and let Wv be the W-orbit of v in V. In the same way as above we may associate a graph to this orbit. It will be evident that this graph is isomorphic to $G(W, S)$, for any such v.

5.4 Coxeter/Parabolic systems (W', W) and alcove geometry

ss:CoxeterPara1

Let (W', S') be a Coxeter system containing (W, S) as a parabolic subsystem (i.e. $S \subset S'$ as in (5.2.13)), with both acting on V. With regard to this pairing, the chambers of W' are then called *alcoves*. Thus the alcoves are a further subdivision of the chambers of W. Write $\mathcal{A} = \mathcal{C}_{W'}$ for the set of alcoves, and $\mathcal{C} = \mathcal{C}_W$ for the set of chambers of W.

Recall that for any Coxeter group W we choose a chamber $C_0 \in \mathcal{C}_W$. This serves three purposes. (1) It determines a choice of Coxeter generators S by Lemma 5.3.5. (2) It determines a bijection $\zeta_{C_0} : \mathcal{C}_W \to W$. (3) It defines a length function and a direction on graph $G(W, S)$ (which should more properly be called $G(W, C_0)$).

(**5.4.1**) Consider the choice of C', the prefered alcove of (W', S'), in the W', W setup. We can no longer freely choose a C' in $\mathcal{A}$ to specify S' *and* a C_0 to specify S (or even a specific W), and expect $S \subset S'$. Since C' determines S', while we also require $S \subset S'$, if a specific W is required then C' must be chosen with the inclusion in mind. Alternatively we may choose C' freely, then define C_0 (and hence W) by removing some walls from the wall set $\mathbb{H}_{C'}$ (producing a chamber that contains but is bigger than C' — the Euclidean closure of C' in the complex defined by W). Then $C_0 \supset C'$ is the prefered chamber of (W, S).

(**5.4.2**) Write $\mathcal{A}^+$ for the set of alcoves lying in C_0. Thus $\mathcal{A}^+$ is a representative set for the W-orbits of $\mathcal{A}$. (In this setting we will call any point $v \in C_0$ *dominant*.)

By (5.3.15) the digraph $G(W', S')$ has vertex set $\mathcal{A}$, and (A, B) an edge if $B = sA$ with s simple for A and $l(B) = l(A) + 1$. And $G(W', S')$ is a rooted acyclic digraph, with root C'. The edges of $G(W', S')$ are in correspondence with the set of walls, and may thus be partitioned into W'-orbits, labelled by the walls of C'.

(**5.4.3**) Let $G_a = G_a(W', W)$ denote the full subgraph of $G(W', S')$ with vertex set $\mathcal{A}^+$.

See Figure 5.3 for an example. See Figures 5.4 and 5.5 for more examples (the construction for the latter is described in §5.5).

(**5.4.4**) The subgraph G_a is again rooted with root C'. Thus any alcove $A \in \mathcal{A}^+$ may be reached from C' by a sequence of simple reflections, always remaining in $\mathcal{A}^+$.

de:res Bruhat

(**5.4.5**) We shall denote the poset defined by the acyclic digraph G_a as $(\mathcal{A}^+, <)$. Note that this coincides with the restriction of the Bruhat order (5.3.18).

(**5.4.6**) Here let $\mathbb{H}$ denote the hyperplanes of W' and $\mathbb{H}_+$ the hyperplanes of W. That is, $W' = W_{\mathbb{H}}$ and $W = W_{\mathbb{H}}$. Via $\zeta_{C'} : \mathcal{A} \to W_{\mathbb{H}}$, we have that $\mathcal{A}^+$ is a transversal of right coset space $W_{\mathbb{H}_+} \setminus W_{\mathbb{H}}$. The right action of $\{w \in W_{\mathbb{H}_+}\}$ on $W_{\mathbb{H}_+} \setminus W_{\mathbb{H}}$ thus gives a closed action on $\{a \in \mathcal{A}^+\}$. Denote this action by

$$a \mapsto a\langle w \rangle.$$

This unpacks as follows. We use $\zeta_{C'}$ to consider alcoves and group elements interchangeably. Write $[a]$ for the coset containing a (i.e. $[a] = W_{\mathbb{H}_+} a$, as it were). Then for $w \in W_{\mathbb{H}}$ we have aw for the usual right action on $a \in \mathcal{A}$ (i.e. on $W_{\mathbb{H}}$ itself), and $a\langle w \rangle = b$ where b is the unique element of $[aw] \cap \mathcal{A}^+$. Note that here it can happen that $a = a\langle w \rangle$.

(**5.4.7**) Under certain special circumstances a in $\mathcal{A}^+$ is taken to $aw \in \mathcal{A}^+$ by the ordinary right action (examples shortly). Indeed if $w \in \mathbb{H}_{C'}$ here then (a, aw) is an edge in G_a with right-label

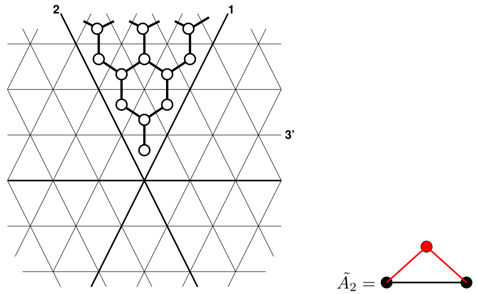


Figure 5.4: Constructing an example of a dominant dual graph G_a: case G_a(affine-A_2, A_2). fig:A2dual

w. More often, though, this action of w does not preserve $\mathcal{A}^+$. Later we will be interested in characterising $a\langle w \rangle$ in case w a reflection element (not necessarily in S), while $a \neq a\langle w \rangle$.

Indeed we will stretch notation slightly by setting $\langle w \rangle a$ to be $a\langle w \rangle$ when $a \neq a\langle w \rangle$; and to be *undefined* otherwise.

5.5 Exercises and examples

ss:Dex1

We return to the examples $\mathcal{D}^- \supset \mathcal{D}_+$ from (5.2.18). Our objective here is to make some observations about this case that will be useful later.

(**5.5.1**) We encode elements of $\mathcal{D}^-$ as alcoves via regularity, and then encode alcoves using the regular orbit $\mathcal{D}^- v_-$ where $v_- = (-1, -2, -3, ...)$.

5.5.1 Constructing $G_a(\mathcal{D}^-, \mathcal{D}_+)$ and $G_a(\mathcal{D}, \mathcal{D}_+)$, and beyond

See Fig.5.5 for the beginning of $G_a(\mathcal{D}^-, \mathcal{D}_+)$. The vertex set (of dominant alcoves) is labelled by dominant elements of $\mathcal{D}v_-$ (descending signed perms of $(-1, -2, -3, ...)$), written in $\mathtt{P}(\mathbb{N})$ notation. An edge (a, b) with label 1 means $b = a(1)_-$; otherwise label x means that $b = a(x)$, where these are the *right*-reflection actions of Coxeter generators.

Here we consider a couple of exercises. (0) Produce a version showing the left-action labels.

(1) This graph can be considered as the graph of orbit $\mathcal{D}v_- \cap C_0$. Is there something similar for $\mathcal{D}(0, -2, -3, ...) \cap C_0$ say? This case is not regular so there is no right action, but there is a left action.

(2) How does this compare with $G_a(\mathcal{D}, \mathcal{D}_+)$? In this case we may label with $\mathcal{D}v_-$, which is the subset of descending signed perms of $(-1, -2, -3, ...)$ with an even number of positive terms — and hence with the corresponding subset of $\mathtt{P}(\mathbb{N})$.

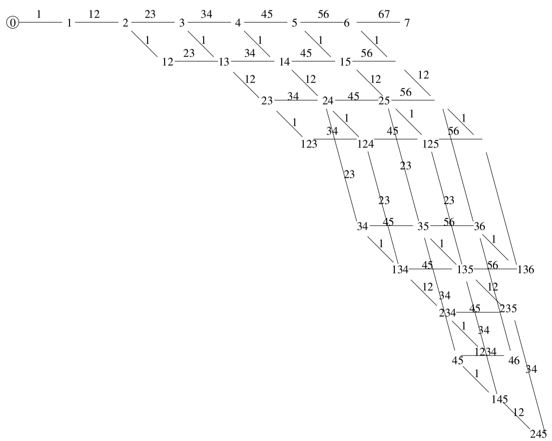
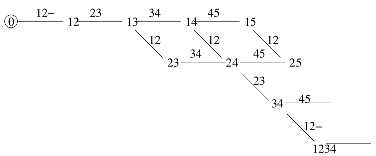


Figure 5.5: Beginning of the type-B/A dominant dual graph (directed from left to right). `fig:KL-HS3`

(3) These are *graphs with extended structure* ('graphes'), in the sense that the edge labels give a partition of the edges. What happens to the comparison when extended structure is noted?

(**5.5.2**) (2): We can describe a vertex set homomorphism h as follows. In what follows a vertex is considered as a string S (an unpunctuated list of natural numbers, so $1S$, say, is a longer string); and given such a string S then $S+1$ denotes the string obtained by adding 1 to each element.
The vertex map is $h(S) = S+1$ if $|S|$ even; and $h(S) = 1(S+1)$ if $|S|$ odd. (Note that this is well-defined.)

$G_a(\mathcal{D}, \mathcal{D}_+) =$



(**5.5.3**) Theorem. *The map $h : G_a(\mathcal{D}^-, \mathcal{D}_+) \to G_a(\mathcal{D}, \mathcal{D}_+)$ is a graph isomorphism; but not a graphe isomorphism.*

Proof. First we need to check that h passes to a graph homomorphism. We do this by checking the 'images' of all possible edge types:

Edges of form $S \xrightarrow{(1)_-} T = 1S$ are taken to $1(S+1) \xrightarrow{(12)} 2(S+1)$ if $|S|$ odd; and to $S+1 \xrightarrow{(12)_-} 12(S+1)$ if $|S|$ even.

Edges of form $S \xrightarrow{(ij)} T$ with $i, j > 1$ are taken to $1(S+1) \xrightarrow{(12)} 1(T+1)$ if $|S|$ odd; and to $S+1 \xrightarrow{(12)_-} T+1$ if $|S|$ even.

Edges of form $1S \xrightarrow{(12)} 2S$ (hence with $1, 2 \notin S$) are taken to $2(S+1) \xrightarrow{(23)} 3(S+1)$.
The inverse is as follows: First remove 1 if present; then $S \to S-1$. We check that this is a graph homomorphism. The most interesting cases are:
Edges of form $S \xrightarrow{(12)_-} 12S$ are taken to $(S-1) \xrightarrow{(1)_-} 1(S-1)$.
Edges of form $1S \xrightarrow{(12)} 2S$ are taken to $(S-1) \xrightarrow{(1)_-} 1(S-1)$.
Note that this is injective because the sets S in the two cases necessarily have different parities.
Finally note from this that h is not a graphe isomorphism. Two components in the edge partition in $G_a(\mathcal{D}, \mathcal{D}_+)$ are taken into one. □

(**5.5.4**) (0,1): ...

5.5.2 Right cosets of $\mathcal{D}_+$ in $\mathcal{D}^-$

Consider the right cosets of $\mathcal{D}_+$ in $\mathcal{D}^-$. Every coset contains one strictly descending ('dominant') sequence, and every descending sequence in $\mathcal{D}^- v_-$ arises this way, so we may choose the strictly descending sequences as a set of representatives. A descending sequence in $\mathcal{D}^- v_-$ is determined by its positive elements, and every subset of $\mathbb{N}$ arises, so we may index the cosets by $\mathtt{P}(\mathbb{N})$.
As for any group and subgroup the set of right cosets (the 'coset space') forms a right $\mathcal{D}^-$-set, so we have in this way a right action of $\mathcal{D}^-$ on $\mathtt{P}(\mathbb{N})$. For example:
since
$$(-1,-6,...)(16) = ((26)(-1,-2,...,-6,...))(16) = (26)(16)(-1,-2,...,-6,...)$$
$$= (26)(-6,-2,...,-1,...) = (-6,-1,...)$$
we have $\emptyset(16) = \emptyset$;
since
$$(-1,6,...)(16) = ((26)_-(-1,-2,...,-6,...))(16) = (26)_-(16)(-1,-2,...,-6,...)$$
$$= (26)_-(-6,-2,...,-1,...) = (-6,1,...)$$
we have $\{6\}(16) = \{1\}$;
since $(-1,6,...)(6)_- = (-1,-6,...)$ we have $\{6\}(6)_- = \{\}$;
since $(-1,-6,...)(16)_- = (6,1,...)$ we have $\emptyset(16)_- = \{1,6\}$;
since $(1,-6,...)(16)_- = (-6,1,...)$ we have $\{1\}(16)_- = \{1\}$.

(**5.5.5**) Note the following. Let $\sigma_{ij} : \mathbb{N} \to \mathbb{N}$ denote the i,j-transposition function. The right action of (ij) on $a \in \mathtt{P}(\mathbb{N})$ is given by
$$a(ij) = \sigma_{ij}(a)$$
while $a(i)_-$ is obtained from a by toggling the presence of i.

(**5.5.6**) Let us compare the right action on cosets with the right action restricted to dominant elements as in Fig.5.5.

A small subset of the right actions of $\mathcal{D}^-$ on $\mathcal{D}^- v_-$ take a dominant element to a dominant element (a necessarily different dominant element, of course, unless we act with the identity).

If we restrict attention to the action of reflections ((ij), $(i)_-$ and $(ij)_-$), then a requirement is that the element acting is either of form $(i\,i{+}1)$ or $(1)_-$ (or $(12)_-$) ...

(This is not sufficient. There are also conditions on the dominant element, as we shall see in the proof.)

To see this consider first (ij) acting on a dominant element v. A dominant element is a descending sequence of positive terms then a descending sequence of negative terms, thus precisely one of i, j must be positive. Else both are same sign, and they cannot be descending (as a pair) before *and* after transposition. On the other hand if they are not adjacent then there is $i+1$ (say) between them. Either this positive in v with j before it or i after it; or it is negative with $-i$ before it or $-j$ after. In any case after the transposition, taking account of signs, one of i, j is the wrong side of $i+1$.

Now consider $(i)_-$ on v. Since this changes the sign of i without changing its position, it must either be the last positive or the first negative element, and must be 1.

The argument for $(ij)_-$ is similar. □

If we further require that the action only changes length l by 1 then even $(12)_-$ is excluded in this case.

5.5.3 On connections of reflection groups with representation theory

There are many direct and indirect connections between properties of reflection groups and Coxeter systems and representations theory. Later we will look at Lie groups and algebras; Hecke algebras; and also at the combinatorial structures called Kazhdan–Lusztig polynomials.

5.6 Combinatorics of Kazhdan–Lusztig polynomials

ss:KLP01

KL polynomials can be defined as straightening coefficients between bases for representations of Hecke algebras (of certain Coxeter systems — see §**??** or [88, 70], corresponding to certain parabolic subsystems); or as polynomials encoding Ext data for generalised Verma modules over simple modules for Lie algebras over parabolic subalgebras (see e.g. [11]); or as polynomials whose evaluation at $u = 1$ determines the relationship between Verma characters and simple characters; or indeed in some other ways involving representation theory (see later).

Combinatorially they can be defined as the solutions to certain recursions defined on reflection group alcove geometries. The well-definedness of such definitions may depend on the equivalence with the representation theoretic definitions, but otherwise they are self-contained, and we can look at them without requiring any further machinery. A recursion definition [39] is described next.

5.6.1 The recursion for polynomial array $P(W'/W)$

ss:pKLp def

Let $W' \supset W$ be Coxeter systems as in §5.4 above. Let the set X^+ of dominant alcoves be totally ordered by any order consistent with the Bruhat order (5.4.5). The array $P = P(W'/W)$ is a (generally semiinfinite) lower unitriangular matrix of polynomials in v, with row and column positions indexed by the ordered set X^+, whose remaining entries we describe below.

Write $P = (p_{AB})_{A,B \in X^+}$. It is natural to organise this data into rows p_A (although it is also of interest to organise it into columns). These rows are thus 'finite' (i.e. of finite support), while the columns are not in general.

(**5.6.1**) The recursion for rows of $P(W', W)$ above the root in the poset (acyclic digraph/Bruhat) order may be given as follows (see [145] for equivalent constructions). To compute the row p_A for alcove A, assuming all lower cases known, we first compute another polynomial for each alcove D, p'_{AD}, also denoted $p'_A(D)$ as follows. (Actually $p'_A(D)$ can depend on the choice made next in the computation, but p_A does not and we supress this dependence in notation.)
(**I**) Pick an edge (B, A) in G_a ending at A (so p_B is known). For each alcove D let $\Gamma^\pm_D = \Gamma^\pm_D(B, A)$ be the set of alcoves D' of G_a such that (D', D) (resp. (D, D')) is an edge in the orbit of the edge (B, A). That is,

$$\Gamma^+_D = \Gamma^+_D(B, A) \;=\; \{D' \mid (D', D) \sim (B, A)\}$$

which says that if Γ^+_D non-empty then it contains a D' obtained by reflecting 'down' from D; and complementarily for Γ^-_D. (By (5.3.15) we can express $(B, A) = (B, Bs)$, $s \in S'$, whereupon any such D' must obey $(D', D) = (D', D's) = (Ds, D)$ (respectively $(D, D') = (D, Ds)$).) Then

$$p'_A(D) = \sum_{D' \in \Gamma^+_D} (v^{-1} p_B(D) + p_B(D')) + \sum_{D' \in \Gamma^-_D} (v p_B(D) + p_B(D')) \tag{5.9}$$

eq:pKL recursion

$$= \begin{cases} v^{-1} p_B(D) + p_B(D') & \exists D' = Ds < D \\ v p_B(D) + p_B(D') & \exists D' = Ds > D \\ 0 & \text{otherwise} \end{cases}$$

(As noted there is at most one edge in the orbit of (B, A) involving any alcove D. Thus at most one of these sums is non-trivial, and that contains only one entry.)
REMARK. Another way to read this is that we get p_A from p_B as follows. Keeping in mind that $(B, A) = (B, Bs)$, then whenever there is an entry $p_B(D)$ and a pair D, Ds then there are entries $p_A(D)$ and $p_A(Ds)$, given as follows. If $Ds > D$ (i.e. the s-labelled reflection wall reflects 'up' from D) then $p_A(Ds) = p_B(D)$ and $p_A(D) = v p_B(D)$. If $Ds < D$ (i.e. the s-labelled reflection wall reflects 'down' from D) then $p_A(Ds) = p_B(D)$ and $p_A(D) = v^{-1} p_B(D)$.

Examples: In particular (B, A) is in its own orbit, so $\Gamma^+_A = B$ and $\Gamma^-_A = \emptyset$ and so $p'_A(A) = v^{-1} p_B(A) + p_B(B) = 0 + 1$; and $\Gamma^+_B = \emptyset$ and $\Gamma^-_B = A$ and so $p'_A(B) = v p_B(B) + p_B(A) = v + 0$.

(**II**) To obtain the row of P that we want from p'_A it is then necessary to perform a subtraction in case the evaluation $p'_A(D)(v = 0)$ is non-zero for any $D < A$:

$$p_A = p'_A - \sum_{D<A} p'_A(D)(v = 0)\; p_D \tag{5.10}$$

eq:pKL recurse2

(**5.6.2**) THEOREM. *Let W' be a Coxeter system and W a parabolic as above. Then for $A \in \mathcal{A}^+$ row p_A of $P(W'/W)$ does not depend on the choice of B such that $A = Bs$.*

Proof. See [145] and references therein. □

(**5.6.3**) In order to work with this recursion rule in any given alcove geometry it is necessary to be able to manipulate the graph G_a and its edge orbits efficiently. In Section 21.2 we set up the requisite machinery for the case $\mathcal{D}/\mathcal{D}_+$, where $\mathcal{D}_+ = \langle (ij) \rangle_{i,j \in \mathbb{N}}$ (a reflection group on $\mathbb{R}^{\mathbb{N}}$) and $\mathcal{D} = \langle (ij), (ij)_- \rangle_{ij}$.

5.6.2 Example

(**5.6.4**) Example. Here we look at the case $\mathcal{D}^-/\mathcal{D}_+$ (what one might call 'type-B'). As already noted, Fig.5.5 gives the graph G_a in this case. We will totally order alcoves by the binary count (of the binary sequence corresponding to the alcove via its $\mathtt{P}(\mathbb{N})$ label), thus 0, 1, 2, 12, 3, 13, 23, 123, 4, ... and so on. We first run the recursion explicitly, and then look at solutions.

(**5.6.5**) We have $p_{00} = 1$. That is, $p_0 = (1, 0, 0, ...)$. We have $(1)_-0 = 0(1)_- = 1$ so

$$p_{11} = 1 \qquad \text{and} \qquad p_{10} = u$$

(Caveat: we get our rows and columns mixed up!) That is

$$p_1 = (u, 1, 0, 0, ...)$$

Comparing with the general recursion we are taking $B = 0$ here, then since it only has one non-zero entry we need only consider edges out of 0. Our next step up is again forced: $1(12) = 2$. Of the alcove labels for non-zero entries in p_1 only 1 has an edge (12), so

$$p_2 = (0, u, 1, 0, 0, ...)$$

From here we can get p_{12} and p_3. For p_{12} we have $2(1)_- = 12$. Two of the alcoves with non-zero entries in p_2 have $(1)_-$-edges. Specifically, fixing $(B, A) = (2, 12)$ we have $\Gamma_0^+ = 1$, since $(1, 0)$ (note the reverse order) is in the orbit of $(2, 12)$ via $12(1)_-(12)(1)_-(12) = 0$ and $2(1)_-(12)(1)_-(12) = 1$. Note that $\Gamma_0^+ = \emptyset$, so $p'_{12}(0) = up_2(0) + p_2(1) = u$. Meanwhile $\Gamma_1^- = 0$, so $p'_{12}(1) = u^{-1}p_2(1) + p_2(0) = 1$. Overall we get

$$p'_{12} = (u, 1, u, 1, 0, 0, ...) \qquad \text{so} \qquad p_{12} = (0, 0, u, 1, 0, 0, ...)$$

Similarly $p_3 = (0, 0, 0, u, 1, 0, 0, ...)$.

For p_{13} we have $12(23) = 13$. Setting $(B, A) = (12, 13)$ we have $(2, 3)$ in the same orbit via $2(1)_- = 12$ and $3(1)_- = 13$. That is, $(2, 2(23)) \sim (12, 12(23))$. Thus $\Gamma_2^+ = \emptyset$, $\Gamma_2^- = 3$, giving

$$p_{13}(2) = up_{12}(2) + p_{12}(3) = u^2 + 0,$$

and $\Gamma_3^+ = 2$, $\Gamma_2^- = \emptyset$, giving $p_{13}(3) = up_{12}(3) + p_{12}(2) = 0 + u$ and so on, giving $p_{13} = (0, 0, u^2, u, u, 1, 0, 0, ...)$. The beginning of the array is in Fig. 5.6.

(**5.6.6**) Exercise. Check p_{23} and p_{123}. We have $23 = 13(12)$.

(**5.6.7**) Our exercise in continuing this will be to bring together the existing treatments in Boe [11] (which computes inverse KL polynomials — i.e. the entries in the inverse array to P) and [112], which requires slightly different notation from the former, in order to treat the large n limit. [1] Our characterisation of dominant elements as subsets of $\mathbb{N}$ gives us a binary sequence representation (write 101 if 1 and 3 are positive, etc. — this is equivalent to an '$\alpha\beta$' notation of Lascoux–Schutzenberger [94]).

(**5.6.8**) The *transpose* or reverse of a binary sequence w is the sequence written in reverse order. The *star* of w is w with occurences of 0 and 1 swapped. The *conjugate* of a binary sequence w is

[1] We consider descending sequences in $(-1, -2, ...)$ rather than $(n, n-1, ..., 1)$ as in Boe. We write binary sequences $w = w_1w_2...$ rather than $w = w_nw_{n-1}...w_1$ as in Boe. We write $w_i = 0$ if $-i$ appears in the descending sequence.

	–	1	2	12	3	13	4	23	14	5	123
	0	1	01	11	001	101	0001	011	1001	00001	111
0	1										
1	u	1									
01		u	1								
11			u	1							
001			u		1						
101			u²	u	u	1					
0001					u		1				
011		u²	u			u		1			
1001					u²	u	u		1		
00001							u			1	
111	u²	u	u²	u	u	1		u			1

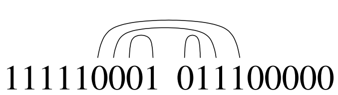
Figure 5.6: Matrix of KL polynomials. fig:KL-HS3matrix

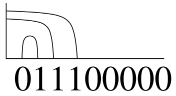
the star-transpose sequence $w^\dagger$. If w is a binary sequence we write w^t for the reverse sequence, so that $w^t w$ is a (possibly doubly infinite) palindromic sequence.

(**5.6.9**) We associate a TL 'half-diagram' $T(w)$ to each binary sequence $w^\dagger w$ (each '*-palindromic' sequence) as follows. We join the elements of each 01 subsequence with an arc (note that every 01 sequence in w begets an 01 sequence in $w^\dagger$, so that this construction is left-right symmetric). Note that if $w_1 = 1$ then in particular we join this with its image 0 in $w^\dagger$. Now ignoring the subsequences covered by arcs so far, we arc each 01 subsequence in (the remainder of) $w^\dagger w$. (Some of these may be parts of matching pairs left and right. Some may arc over *from* left to right.)

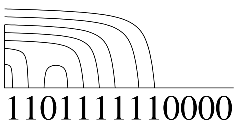
In the infinite finitary case every w has a tail of 0s on the right, so $w^\dagger$ has a tail of 1s on the left. These are not arc'ed.

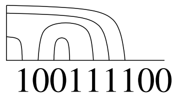
(**5.6.10**) Example.



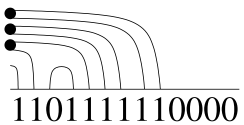


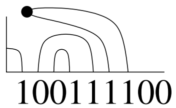
We claim that these TL diagrams are palindromic. Therefore it is enough to draw only the right-hand side, the 'quarter-diagram' as indicated in the figure. Some more examples:





This is Martin's construction in [112]. (Strictly speaking, Martin's construction applies the 'blob' map [119, 115] to symmetric arcs, paired from the left end of the right-half diagram. For our present purposes this is a 'type-D' move, whereas we want a 'type-B' move. So we should leave the first 1 on the left alone, and pair from the left thereafter:





and so on.) To make contact with Boe [11] we should take the dual graph in the sense of (5.1.12), which for TL diagrams is a rooted tree.

(**5.6.11**) A binary sequence is called a *Catalan* binary sequence if the running total of 1s exceeds the running total of 0s. In the case of finitary large n, where every *-palindromic sequence can be considered to begin with many 1s, every sequence trivially has the property that the running total of 1s exceeds the running total of 0s. The *-palindromic sequences are thus a subset of the Catalan sequences. The combinatorics of 'type-A' binary sequences (Catalan binary sequences in general)

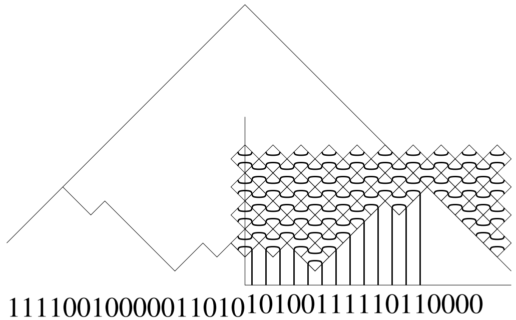


Figure 5.7: *-palindromic binary sequence, Dyck path, and TL diagram via tile map. fig:tile1

covers well-known bijections with many other sets (see Stanley's Catalan notes for example [?], or Marsh–Martin [?]).

Boe uses the correspondence with Dyck paths on $\mathbb{Z} \times \mathbb{Z}$ (Stanley's sequence 17(g), and see below), for example. As Boe notes, this leads to a simple statement of the Bruhat order (**??**) on dominant alcoves ($w > y$ if y can be obtained from w by removing diamond tiles). However it also leads to a simple construction of the associated TL diagrams (Stanley's sequence 17(j)), via the tile map as in Fig.5.7.

Note that the upper envelope Dyck path in the figure is the path for ...111000... which is the *-palindrome of 000..., which is the sequence for the Bruhat-lowest element. Thus we have 0100... > 1000... > 0000... and so on.

(**5.6.12**) Exercise. Recast Boe's construction for inverse Kazhdan–Lusztig polynomials using TL diagrams.

(**5.6.13**) Next we construct hypercubes associated with dominant alcoves (i.e. with sequences) as follows. For each sequence w we define the set of sequences obtained by flipping 01 to 10 across an arc involved in the TL quarter-diagram of $T(w)$. This includes the 'generator' arcs associated to the right actions of $(1)_-$ and $(i\ i{+}1)$ (the lowest arcs), and also any arcs that cover these. Note that the $(1)_-$ arc flip appears to flip an initial 1 to 0 if only the original sequence (the right-hand half) is considered.

What happens to long symmetric arcs (i.e. after the first symmetric arc, which might be in position 1, or might be later if there are 01 pairs to the left in the quarter-diagram)??? Do we have 11 to 00? I guess so. Consider our direct calculation of p_{123} for example.

We perform all possible subsets of flips, so that, if there are r arcs (not counting any odd long-symmetric arc; and only counting paired arcs as one) then it is an r-dimensional hypercube, with 2^r vertices.

We claim that the hypercube of alcove A gives the row p_A in the obvious way (if B occurs in level i then $p_A(B) = u^i$, otherwise it is zero).

Example:

...

The matrix is shown in Fig.5.6.

5.6.3 Alternative constructions: wall-alcove

We may associate arrays of Kazhdan–Lusztig polynomials to other orbits of reflection group actions. In particular we may define an array $P^\omega = P^\omega(W', W)$ of polynomials with rows and columns indexed by dominant walls (the singularity-1 /codimension-1 facets). Each such wall is a wall to two dominant alcoves, and they inherit a corresponding partial order. We can compute the arrays P and P^ω by an interlocking recursion that is in some ways simpler than the original (see e.g. [122]).

Start again from $p_{C'} = (1, 0, 0, 0, ...)$. Moving up the order, we compute row p^ω_w of P^ω, for w a wall above alcove A (say), from p_A as follows. Let s be such that A and As are the alcoves on either side of w. We first compute another row of polynomials $p^{\omega'}_w$. The entry $p^{\omega'}_w(x)$ is non-zero only if x is an s-wall (a wall in the orbit of s), whereupon it is obtained from p_A by $p^{\omega'}_w(x) = p_A(B) + v^{-1}p_A(Bs)$, where $B < Bs$ are the alcoves around x. That is, we 'throw' the polynomial from alcove B up onto the wall, and throw the (modified) polynomial from alcove Bs down onto the wall.

To compute p^ω_w we then make the analogous subtraction to (5.10).

Next, to compute p_{As} we start with p^ω_w. We throw each polynomial in p^ω_w up into the adjacent alcove above, and throw v times each polynomial down into the adjacent alcove below (note that these two throws are never both into the same alcove, since no alcove has more than one wall of a given 'colour').

Again it is not clear that this procedure is well-defined (in a number of regards).

(**5.6.14**) Example. To follow.

ss:Young diagram

5.7 Young graph combinatorics

5.7.1 Young diagrams and the Young lattice

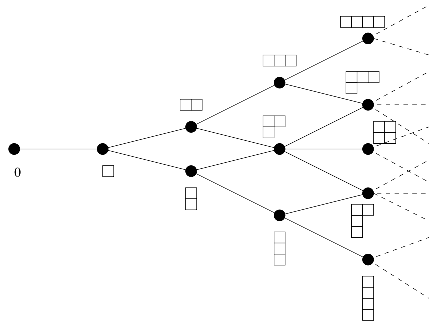
See James–Kerber [79, §1.4,§2.3] for standard terminology for Young diagrams and so on. See also Fulton [51, pp.1-4].

(**5.7.1**) Recall that Λ is the set of all integer partitions and Λ_n the set of integer partitions of degree n. The subpartition order $(\Lambda, \supset)$ is defined by $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for all i.

(**5.7.2**) The subpartition order $(\Lambda, \supset)$ is a lattice (the *Young lattice*) — meet is partition intersection and join is union.
In this case λ covers μ (in the sense of (3.4.4)) if the skew λ/μ (see (5.7.8)) is a single box. See Figure 5.8.

de:young graph

(**5.7.3**) The *Young directed graph* $\mathcal{Y}^+$ is the Hasse graph of the Young lattice. The *Young graph* $\mathcal{Y}$ is the underlying (undirected) graph of $\mathcal{Y}^+$.



Younggraph1

Figure 5.8: The Young graph (covering DAG of the Young lattice, increasing from left to right).

(**5.7.4**) The *Young matrix* is the (semiinfinite) adjacency matrix of the underlying (undirected) graph of the Hasse graph of the Young lattice. We have

$$A(\mathcal{Y}) = A(\mathcal{Y}^+) + A(\mathcal{Y}^+)^t$$

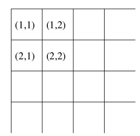
de:yyy

(**5.7.5**) Define a map $\mathtt{y} : \Lambda \to \mathtt{P}(\mathbb{N}^2)$ by $\mathtt{y}(\lambda) = \cup_i\{(i, 1), (i, 2), ..., (i, \lambda_i)\}$. Example: $\mathtt{y}(3, 1, 1) = \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1)\}$.

de:lc1

(**5.7.6**) We visualise an integer partition $\lambda = (\lambda_1, \lambda_2, ...) \in \Lambda$ as a *Young diagram* as follows.

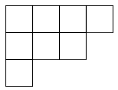
One first pictures the lower right quadrant of the plane partitioned into unit boxes. It is useful to label the boxes by their row and column position in this arrangement — in matrix labelling, as it were; so that the top-left box has label $(1, 1)$:



Note that the row-column label only coincides loosely with (rotated) Cartesian coordinates, in which $(0, 0)$ would be the top-left corner; and $(1, 1)$ would be the bottom-right corner of the box with label $(1, 1)$. (We will be vague for now about which box edges are open or closed - in a *diagram* such differences are generally undetectable. For definiteness we might take $(0, 1] \times (0, 1]$ and so on, noting that the first coordinate increases down; and the second increases to the right.) The set of all these boxes is thus in bijection with $\mathbb{N}^2$. We write $\mathfrak{b}(\mathbb{N}^2)$ for this set; and $\mathfrak{b}(i,j)$ for the corresponding single box. A Young diagram is a certain subset of these boxes, as we now explain.

The Young diagram for partition λ is $\mathfrak{b}(\mathsf{y}(\lambda))$, i.e. it consists of the first λ_i boxes in the i-th row, for each $i = 1, 2, \ldots$. (Hereafter we generally identify λ with its Young diagram.)

Example:

$(4,3,1) \mapsto$ 

(**5.7.7**) We associates to each such box $b \in \mathfrak{b}(\mathbb{N}^2)$ the rectangle of boxes $r(b)$ between it and the top-left box. We define a *light-cone order* on the set of boxes by $b \geq b'$ if $b' \in r(b)$. If $b \geq b'$ we say b *pins* b'.

For B a set of boxes, define

$$r(B) = \bigcup_{b \in B} r(b)$$

A Young diagram is a subset of boxes such that if a box is included, then every box in its rectangle is included. That is, B is a Young diagram if $B = r(B)$.

Note that the identification of a partition λ with the subset of $\mathfrak{b}(\mathbb{N}^2)$ whose i-th row has length λ_i is indeed a Young diagram.

de:skew (**5.7.8**) A *skew* (or skew Young diagram, or skew diagram, or skew shape [51]) of a pair $\lambda \supset \mu \in \Lambda$ is the subset of $\mathfrak{b}(\mathbb{N}^2)$ given by $\lambda \setminus \mu$. This skew is denoted λ/μ.

(**5.7.9**) For any box b there is a minimal $\lambda \in \Lambda$ containing this box, and this coincides with $r(b)$. Given a partition μ and a box, there is a minimal $\lambda \in \Lambda$ containing both. Given a partition μ and a box b, and hence a container λ, the skew λ/μ is called the skew over μ *pinned* by b.

lem:pin1 (**5.7.10**) Lemma. *Fix a diagram μ. For a set of boxes γ to be a skew λ/μ it must not intersect μ, and must not pin any box outside $\gamma \cup \mu$.*

(**5.7.11**) The *rim* of a diagram λ is the subset of its boxes with the property that the box immediately to the SE is *not* in λ.

Note that the rim is a skew and, in the obvious sense, *connected.*

A *rim hook* of λ is a subset of the rim that is connected. We extend this notion slightly (cf. [79]): A *rim* or *rim hook* in general is any collection of boxes that is a rim hook for some λ.

(**5.7.12**) For each pair of Young diagrams λ, μ there is a skew diagram

$$\lambda \setminus \mu := \lambda/(\lambda \cap \mu)$$

(i.e. such that a box is in $\lambda \setminus \mu$ if it occurs in λ but not in μ); and a skew diagram

$$\lambda \Delta \mu := (\lambda \setminus \mu) \cup (\mu \setminus \lambda) = \lambda \cup \mu/(\lambda \cap \mu)$$

(**5.7.13**) The *dominance order* on Λ_n is the partial order

$$\lambda \leq \mu \iff (\sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j \qquad \forall i)$$

ss:YDO2

5.8 Young graph via alcove geometry on $\mathbb{Z}^{\mathbb{N}}$

Here we investigate the appearance of the Young graph, and certain subgraphs, within a more homogeneous structure. (This will be useful in representation theory later on.)

5.8.1 Nearest-neighbour graphs on $\mathbb{Z}^n$

(**5.8.1**) We define $\mathbb{Z}^n_{\mathfrak{g}}$ as the graph with vertex set $\mathbb{Z}^n = \mathbb{Z}\{e_1, e_2, ..., e_n\}$ and an edge whenever

$$x - x' = \pm e_i$$

Here $\{e_i\}_i$ could be any set of degree n. If we embed $\mathbb{Z}^n$ in $\mathbb{E}^n$ such that $\langle e_i, e_j \rangle = \delta_{ij}$ then $\mathbb{Z}^n_{\mathfrak{g}}$ is also known as the simple hypercubical lattice in n-dimensions. The edges are between all the nearest neighbour pairs of vertices.

Note that $\mathbb{Z}^n_{\mathfrak{g}}$ is a bipartite and connected graph (in the sense of 3.5.22).

(**5.8.2**) Consider the symmetric group S_n action on $\mathbb{R}^n$ and hence $\mathbb{Z}^n$ given by (5.1) — in which the generator (ij) acts by swapping the corresponding coordinates. This induces an action on $\mathbb{Z}^n_{\mathfrak{g}}$. Note that if there is an edge (x, x') then there is an edge $((ij)x, (ij)x')$. Hence (ij) is a graph automorphism.

(Indeed the set of points in the linear interval between x and x' is taken to the set of points between $(ij)x$ and $(ij)x'$.)

(**5.8.3**) The orbits of the S_n action on $\mathbb{R}^n$ may be represented by the set of (not necessarily strictly) descending sequences.

Alternatively, if we take the euclidean embedding of $\mathbb{Z}^n$ in $\mathbb{E}^n = \mathbb{R}^n$ as above, we may partition $\mathbb{Z}^n$ by the hyperplanes of the reflection group action $\mathcal{D}^n_+ = \langle (ij) \rangle_{ij}$ on $\mathbb{R}^n$ (this is simply S_n realised as a reflection group). The hyperplanes partition $\mathbb{R}^n$ into chambers and facets as in (5.1.4).

(**5.8.4**) The strictly descending sequences of $\mathbb{R}^n$ lie in a single 'dominant' chamber. All descending sequences lie in the closure of that chamber.

For example, with $n = 4$, $(0,0,0,0)$, $(1,0,0,0)$ and $(0,0,0,-1)$ lie in the closure of the dominant chamber.

de:Areg (**5.8.5**) Given the action of $\mathcal{D}^n_+$, a sequence in $\mathbb{Z}^n$ is called *regular* (or A-regular, since $\mathcal{D}_+$ is type-A in the classification of finite crystallographic reflection groups, or *Weyl groups* [70, Ch.2]) if it is not fixed by any group element except the identity.

If a sequence is regular then so is every element of its orbit. The regular orbits may be represented by the set of strictly descending sequences.

For example $(4,3,-3,-6)$ is a representative element of a regular S_4 orbit in $\mathbb{Z}^4$.

(**5.8.6**) Lemma. *$v \in \mathbb{Z}^n$ is regular if and only if it is singularity-zero.*

(**5.8.7**) We define $\mathbb{Z}^{nA}_{\mathfrak{g}}$ as the following graph. The vertices are the orbits of the S_n action on $\mathbb{Z}^n_{\mathfrak{g}}$ (or equivalently the integer sequences in the closure of the dominant chamber); and there is an edge $([x], [x'])$ in $\mathbb{Z}^{nA}_{\mathfrak{g}}$ if there are representatives x, x' connected by an edge in $\mathbb{Z}^n_{\mathfrak{g}}$.

(**5.8.8**) We define $\mathbb{Z}^{nA}_{\mathfrak{g}\,reg}$ as the full subgraph of $\mathbb{Z}^{nA}_{\mathfrak{g}}$ on the regular orbits (or equivalently on the integer sequences in the open dominant chamber).

For example $((4,3,-3,-6),(4,3,-2,-6))$ represents an edge in $\mathbb{Z}^{nA}_{\mathfrak{g}\,reg}$.

(**5.8.9**) We CLAIM that $\mathbb{Z}^{nA}_{\mathfrak{g}\,reg}$ is isomorphic to the full subgraph of $\mathbb{Z}^n_{\mathfrak{g}}$ on vertices in the open dominant chamber.

(**5.8.10**) A walk on $\mathbb{Z}^n_{\mathfrak{g}}$ is A-regular if it visits only A-regular vertices.
For example $(4,3,2,0)-(4,3,2,-1)-(5,3,2,-1)$ is A-regular.

5.8.2 Graphs on $\mathbb{Z}^{\mathbb{N}}$

ss:GrZN

de:RN (**5.8.11**) Define $\mathbb{R}^{\mathbb{N}}$ as the space of real sequences $(x_1,x_2,...)$. Define $\mathbb{R}^f$ as the subspace of finitary elements. Define $\mathbb{Z}^{\mathbb{N}}$ ('integral points') and $\mathbb{Z}^f$ similarly.
Set $e_i = (0,0,...,0,1,0,...) \in \mathbb{Z}^f$ and

$$w = \frac{-1}{2}(1,1,1,...)$$

which is not finitary or integral (if $x+w$ is integral we say x is *half-integral*). We say an element of $\mathbb{R}^{\mathbb{N}}$ is *dominant* if it is strictly descending.

(**5.8.12**) Define graph $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ with vertices $\mathbb{Z}^{\mathbb{N}}$ and an edge (v,v') whenever $v-v'=\pm e_i$ for some i.

(**5.8.13**) Note that the dot product is not defined for arbitrary pairs in $\mathbb{R}^{\mathbb{N}}$, but is defined for any element of $\mathbb{R}^{\mathbb{N}}$ with any element of $\mathbb{R}^f$. Thus the permutation (ij) may be realised as reflection in a hyperplane; as may the signed permutation $(ij)_-$.

(**5.8.14**) An *integral walk* in $\mathbb{R}^{\mathbb{N}}$ is a sequence of points $x^1,x^2,...\in\mathbb{R}^{\mathbb{N}}$ such that $x^i - x^{i+1} = \pm e_i$.

(**5.8.15**) Define $\mathfrak{Z}$ as the graph with vertex set $\mathbb{R}^{\mathbb{N}}$ and an edge whenever $x-y=\pm e_i$ for some i. For $x\in\mathbb{R}^{\mathbb{N}}$ define $\mathfrak{Z}(x)$ as the connected component of $\mathfrak{Z}$ containing x. Thus $\mathfrak{Z}(x)\cong\mathfrak{Z}(x')$ and

$$\mathfrak{Z}(0) = \mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}.$$

(**5.8.16**) NB, $y,y'\in\mathfrak{Z}(x)$ implies $y-y'\in\mathbb{Z}^f$.

(**5.8.17**) Here define reflection group actions $\mathcal{D}_+ = \langle\{(ij)\}_{i,j\in\mathbb{N}}\rangle$ and

$$\mathcal{D} = \langle\{(ij),(ij)_-\}_{i,j\in\mathbb{N}}\rangle$$

on $\mathbb{R}^{\mathbb{N}}$. Note that the respective sets $\mathbb{H}_+,\mathbb{H}_\pm$ of hyperplanes corresponding to the given sets of reflections are closed (albeit infinite) in each case.

There follows in (5.8.27) a brief discussion of the hyperfinite and possibly larger reflection groups on $\mathbb{R}^{\mathbb{N}}$. However the characterisation above will be sufficient for our purposes.

(**5.8.18**) CLAIM: If there is an edge (x,x') in $\mathfrak{Z}$ then there is an edge $((ij)x,(ij)x')$, thus $\mathfrak{Z}$ is fixed by $\mathcal{D}_+$. $\mathfrak{Z}(x)$ is fixed by $\mathcal{D}_+$ if and only if every $x_i - x_j\in\mathbb{Z}$.
SOMETHING
$\mathfrak{Z}(0)$ and $\mathfrak{Z}(w)$ are fixed by $\mathcal{D}$.

(**5.8.19**) Define the chamber of $\mathcal{D}_+$ (the chamber of $\mathbb{H}_+$) containing

$$\rho = (0,-1,-2,...)$$

as the *dominant* chamber.

Note that the set of dominant points is the set of points in the dominant chamber.

We call the chambers of $\mathcal{D}$ (the chambers of $\mathbb{H}_\pm$) alcoves. Define the *fundamental* alcove as the chamber of $\mathcal{D}$ containing $(0,-1,-2,...)$.

(**5.8.20**) Here a point $x\in\mathbb{R}^{\mathbb{N}}$ is called *regular* (or A-regular, by analogy with (5.8.5)) if it is not fixed by any element of $\mathcal{D}_+$, i.e. if $x_i=x_j$ implies $i=j$.

Note: The set of points in chambers is the set of regular points. No finitary point is regular.

— Is it possible to step (on an integral walk) from a regular integral point to another in a different chamber? How about from a half-integral point?

(**5.8.21**) Define $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}$ (similarly to $\mathbb{Z}^{nA}_{\mathfrak{g}\,reg}$) as the full subgraph of $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ on vertices in the dominant chamber.

For v a vertex in the dominant chamber write $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}(v)$ for the connected component of $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}$ containing v.

(**5.8.22**) Our other notation: $\mathfrak{Z}_+ = \mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}$ is the full subgraph of $\mathfrak{Z}$ on vertices in the dominant chamber. For v a vertex in the dominant chamber $\mathfrak{Z}_+(v) = \mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}(v)$ is the connected component of $\mathfrak{Z}_+$ containing v.

(**5.8.23**) CLAIM: Define $w,\rho\in\mathbb{R}^{\mathbb{N}}$ by $-2w=(1,1,1,...)$ and $-\rho=(0,1,2,...)$.
(i) Every sequence of form $\rho_\delta=\delta w+\rho$ is A-regular.
(ii) The slowest integral descent (in the obvious sense) from any initial integer γ is $\rho_{-2\gamma}$.
(iii) If $\delta\in 2\mathbb{Z}$ then ρ_δ is a vertex of $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}$. No two distinct ρ_δs are in the same connected component of $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}$.
(iv) For each $\delta\in 2\mathbb{Z}$,

$$\mathfrak{Z}^+(\rho_\delta)\cong\mathcal{Y}$$

i.e. $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}\,reg}(\rho_\delta)$ is isomorphic to the Young graph (as defined in (5.7.3)). The isomorphic image of the Young graph is given by $x\mapsto x+\rho_\delta$ for each vertex x in the Young graph.

Proof. (i-iii) are clear. (iv): note that every such $x+\rho_\delta$ is strictly descending and in the connected component. On the other hand every strictly descending element in the component must differ from ρ_δ by a finitary (non-strictly) descending sequence, and hence by an integer partition. □

(**5.8.24**) We define the 'natural' inclusion of $\mathbb{Z}^n\hookrightarrow\mathbb{Z}^{n+1}$ by $(x_1,x_2,...,x_n)\mapsto(x_1,x_2,...,x_n,0)$. We extend this to a natural inclusion of $\mathbb{Z}^n_{\mathfrak{g}}$ in $\mathbb{Z}^{n+1}_{\mathfrak{g}}$.

(**5.8.25**) We define $\mathbb{Z}^f_{\mathfrak{g}}$ as the natural inverse limit of the graphs $\mathbb{Z}^n_{\mathfrak{g}}$.

The 'finitary' graph $\mathbb{Z}^f_{\mathfrak{g}}$ is still bipartite and connected, since for any pair of vertices there is an embedded $\mathbb{Z}^n_{\mathfrak{g}}$ in which they both lie.

(**5.8.26**) We define $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ as the 'infinite' version. For this the vertex set is all integer sequences.

Note that $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ has infinitely many connected components. For example the vertices $(0,0,0,...)$ and $(1,1,1.,,,)$ are not in the same component.

The connected component of $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ containing $\mathbb{Z}^f_{\mathfrak{g}}$ is the component containing $(0,0,0,...)$. Note that every other connected component is isomorphic to this one.

rem:infrg

(**5.8.27**) Remark. Consider generalising the partition of $\mathbb{Z}^n$ by orbits/hyperplanes of the S_n action to the case $\mathbb{Z}^{\mathbb{N}}$. Here an example of a descending sequence is $-(0,1,2,3,...)$. A sequence formally in the same orbit is $-(1,0,3,2,...)$ (transpose in pairs). Is $-(1,2,3,...)$ in the same orbit (formally, an infinite cyclic shift)? There are some choices to be made about how one defines the orbits of the reflection group action (one could restrict orbits to sequences related by reflection group elements of finite length, say, so that $-(0,1,2,3,...)$ and $-(1,0,3,2,...)$ are not in the same orbit).

On the other hand, the partition into chambers generalises relatively usefully.

We note that two sequences are only in the same connected component of $\mathbb{Z}^{\mathbb{N}}_{\mathfrak{g}}$ if they are different in finitely many places. This means that if they are in the same connected component *and* in the same orbit then they are necessarily related by a *hyperfinite* group element (i.e. an element in the hyperfinite subgroup of the infinite reflection group — the inductive limit of finite group inclusions). This will be sufficient for our purposes (even though we do not entirely restrict to the hyperfinite graph).

(**5.8.28**) Remark. We might like to try to define $\mathbb{Z}^{\mathbb{N}A}_{\mathfrak{g}}$ similarly to $\mathbb{Z}^{nA}_{\mathfrak{g}}$. Vertices would be integral points in the closure of the dominant chamber.

What about edges?:

The points $(0,0,0,...)$ and $(-1,0,0,0,...)$ are connected in the underlying graph, but $(-1,0,0,0,...)$ does not lie in the closure of the dominant chamber. The image (y, say) of $(-1,0,0,0,...)$ in the closure of the dominant chamber is not a finitary sequence. That is, $(0,0,0,...)-y$ is not polynomial. Further y is not adjacent to $(0,0,0,...)$. That is, there is no $i \in \mathbb{N}$ such that $(0,0,0,...) - y = \pm e_i$.

We shall not need to resolve this obstruction.