Montreal '22 {Recent $P \cap \gamma \sigma \mu$ Rep. Thy. $\cap$ comp. Stat. mech.

Preamble 0–4

1. Currently 2 postdoc positions at Leeds

4.

overview 1. Recall / review one possible unifying context for some of our talks ($\cap$, but not recent).

2. use this to introduce a classification theorem for spin-chain braid reps. (w. E. Rowell) (if time!)

$V_N = k^N$

- various mathematical miracles

$$\begin{tikzcd}
 & U_q \arrow[d, rightsquigarrow] \arrow[dr] & & S^N_n(q) \arrow[dl] \arrow[d, rightsquigarrow] & \boxed{B_n} \arrow[l] \\
 & GL(V_N) \arrow[r] \arrow[d, phantom, "\cup"] & V_N^{\otimes n} & S^N_n \arrow[l] \arrow[d, phantom, "\cap"] & S_n \arrow[l] \\
U_q \arrow[r, rightsquigarrow] & O(V_N) \arrow[ur] \arrow[d, phantom, "\cup"] & & \mathrm{Braver}(N) \arrow[ul] \arrow[d, phantom, "\cap"] & \mathrm{BMW} \arrow[l, rightsquigarrow] \\
 & S_N \arrow[uur] & & P(N) \arrow[uul] &
\end{tikzcd}$$

q-def.
natural/
miraculous
/very
miraculous

no reason
for
canonical
q-def.
?

2d classical SM

1+1d q-spin chain (Jimbo)

lives in both worlds
but q-version lives
in 1+1d.

mean field

high-d SM

(3d is sub.)

"Models": A model is ...

planar and other geometrically defined subcats (including different realisations of S^2 (TL))

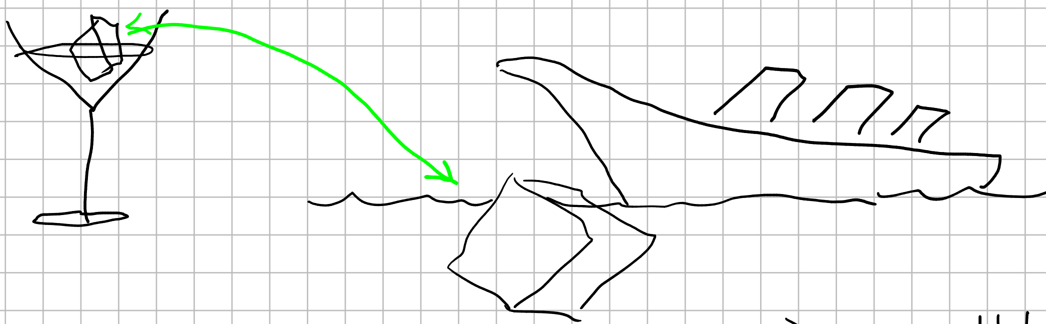
doping,
TQC
blobs

A 'model' is a collection of algebras / Hamiltonians "for all n".

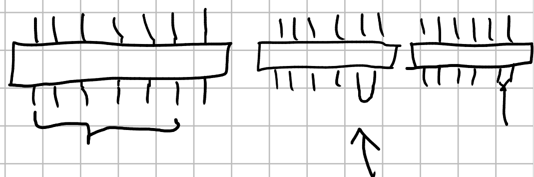
- Engineering addresses large n - "behaviours" stable at large n

"thermodynamic limit" [Q. What means algebraically?
A. Embedding of Rep. categories

- so, ~ fixing N fixes model - which then includes all n.



cvitanovic
Kuperberg
Westbury



How vary n? ...

Algebraically (approx.) Ind / Res
glob / loc.

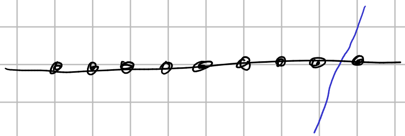
(concept.) physically : freeze (or heat) a bond

glob / loc. adjunction says there is n-limit.

(Then Ind / Res "geometry" says pKL governs classical RT.)

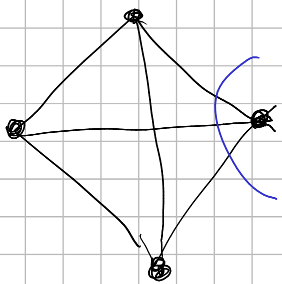
How vary n?

— advantageous to have natural way to vary n (ideally in small steps)



spin-chain or transfer matrix

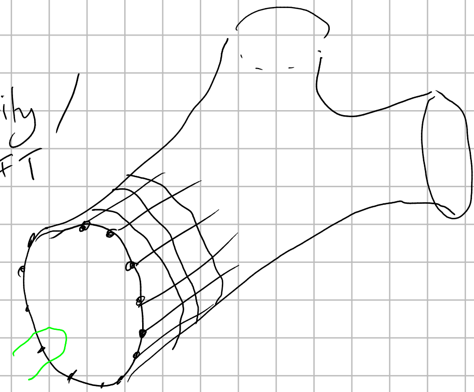
fusion



complete graph

gravity / TQFT

blob algebras / categories

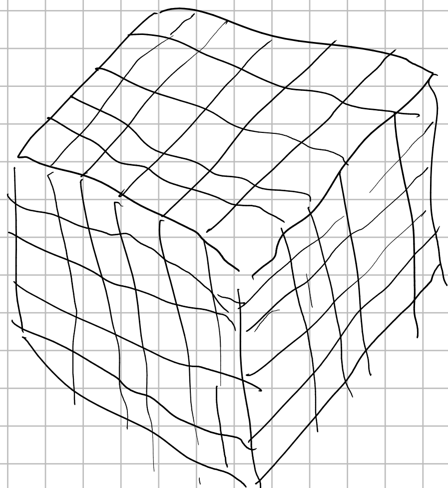


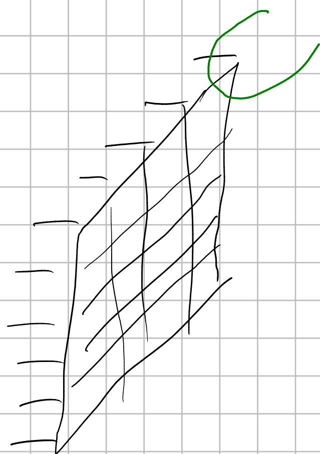
3d algebra is subalgebra of partition algebra .... but sub is weak in RT ☹

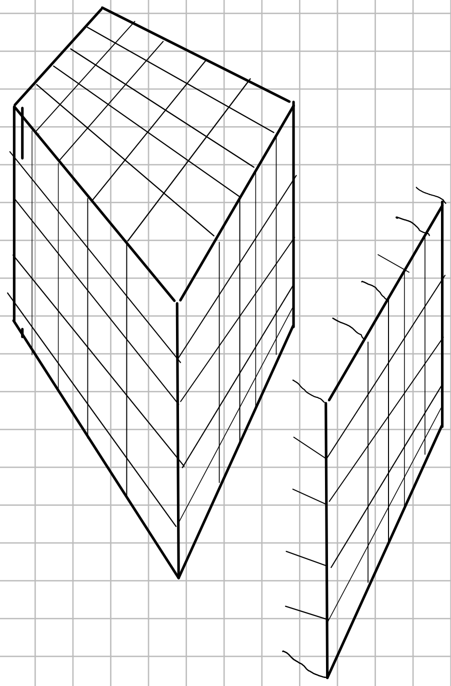
ice cube

3d

2+1d physics







$V_N^{\otimes n}$

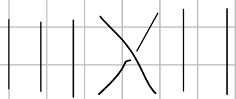
Spin-chain braid reps.

$B = (\mathbb{N}, \text{braids}, \cdot, \otimes, 0)$

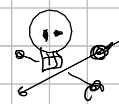
$F : B \to \mathrm{Vect}$

obj $1 \mapsto F_0(1) = V$

$\sigma \mapsto F_1(\sigma) \in \mathrm{End}(V^{\otimes 2})$

 $\mapsto 1 \otimes 1 \otimes 1 \otimes \sigma \otimes 1 \otimes 1$

translates

Classify reps? — have N^6 cubic eqns in N^4 unknowns 

$N = \dim(V)$

OK for $N=2$ (Hietarinta '92) Others open ... so need paradigm shift.

$F : B \longrightarrow$ ~~Vect~~

~~Mat~~

Mat^N - subcat monoidally generated by V_N

So, what nice monoidal subcats of Mat^N are there?

$F : \mathcal{B} \to \mathrm{Mat}^N$ $\qquad F(\sigma) \quad \alpha = (\alpha_{ij})$ $\quad$ i, j words in $\{1, 2, \ldots, N\}$

row

col.

monoidal subcat $\cup$ Match^N

$\alpha_{ij} = 0$ unless $j = w(i)$

some perm w

q-spin chain world:

6-vertex model $\quad$ "charge conserving"

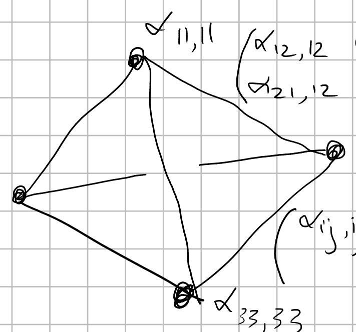
e.g.

$$\begin{array}{c|cccc} & 11 & 12 & 21 & 22 \\ \hline 11 & * & & & \\ 12 & & * & * & \\ 21 & & * & * & \\ 22 & & & & * \end{array}$$

General N

non-zero entries

decorate K_N



$\begin{pmatrix} \alpha_{12,12} & \alpha_{12,21} \\ \alpha_{21,12} & \alpha_{21,21} \end{pmatrix}$ $\rightsquigarrow$ element of $\mathrm{Match}^N(2,2)$

!

$\begin{pmatrix} \alpha_{ij,ij} & \end{pmatrix}$

<u>Lemma</u> $\quad \mathrm{Match}^N$ is monoidal subcat.

$\left(\begin{array}{l} \text{natural?} \\ \text{lucky?} \\ \text{Physics} \checkmark \end{array} \right)$

can this be the paradigm changing target ?

Aim Classify (and construct) $\mathrm{Tens}(\mathcal{B}, \mathrm{Match}^N)$ $\forall N$.
obj.

$\mathrm{Tens}(\mathcal{C}, \mathrm{match}^N) \sim$ (higher version of $\mathrm{Rep}\,G$ for mon. cat. $\mathcal{C}$)

← underlying target is Vect
— an abelian cat with additivity, Jordan–Hölder, Krull–Schmidt, …
Artin–Wedderburn, …
… Artinian rep. theory.

What does a classification theorem look like? ☺

"$\boxed{\text{Irreps}}$ of $\mathbb{C}S_n$ are indexed $\boxed{\text{up to isomorphism}}$ by $\boxed{\text{integer partitions of } n}$ (And every rep. is isom. to direct sum.)"

Recast in a suitably general framework …

$(\mathrm{Mat}^N, \mathrm{Mat}^N) \supset GL_N$
$\cup$
$(\mathrm{match}^N, \mathrm{Match}^N) \supset S_N$

[Thm Functors F in $\mathrm{Tens}(\mathcal{B}, \mathrm{Match}^N)$ / R-matrices $F(\sigma)$ take the following form:
$F(\sigma) \in \mathcal{R}(w)$ for some $w \in \mathcal{J}_N(\mathbb{Z}^\times)$ defined next.]

Forest combinatorics

If Ω a set and $\Psi \in \mathrm{Set}(\Omega, \mathbb{N})$

allowed mult. 0 on any/every.

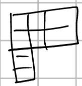
$$J_N(\Omega,\Psi) = J_N(\Omega) = \Big\{ f \in \mathrm{Set}(\Omega, \mathbb{N}_0) \;\Big|\; \sum_{w\in\Omega} f(w)\,\Psi(w) = N \Big\}$$

multiset of total degree N

eg. for S a set, $S^* = \bigsqcup_{N\in\mathbb{N}} \mathrm{Set}(\underline{N}, S)$ — sequences of any non-0 length

$\Psi(w) = |w| = \mathrm{length}(w)$

$\Lambda_N = J_N(\underline{1}^*)$



$= \begin{array}{l} 111 \mapsto 2 \\ 1 \mapsto 3 \\ w \mapsto 0 \quad \text{o/w} \end{array}$

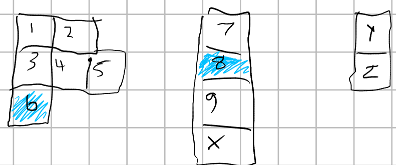
Exercise: $\Gamma_N = J_N(\underline{2}^*)$ — multisets of integer compositions.

We use $J_N(\underline{3}^*)$ [Thm. Every $f(\sigma)$ is in $R(w)$ ...]

$J_N(\underline{3}^*)$ = multisets of 2-coloured compositions

(admissible)

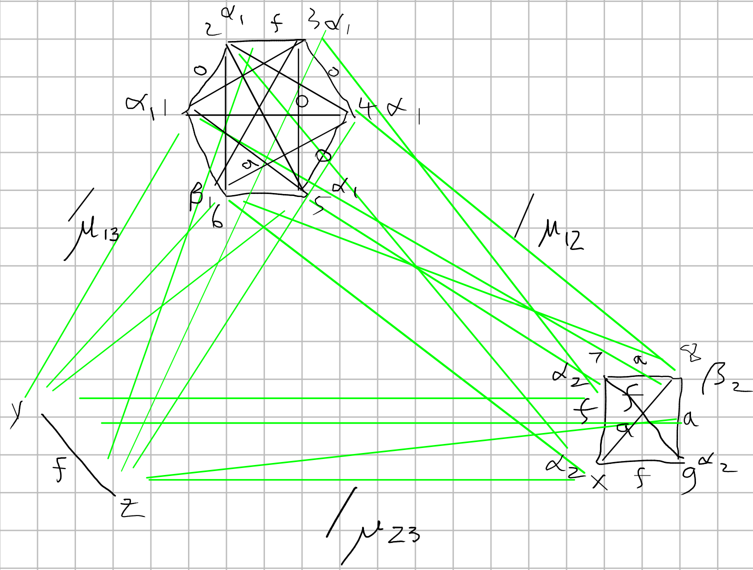
eg.

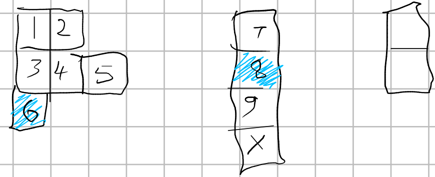


$\xrightarrow{R\,?}$ Higgs colouring of K_{12} $\left[\begin{array}{l} GL(1) \text{ on vertices} \\ GL(2) \text{ on edges} \end{array}\right.$

$\boxed{R}$

K_{12}





Key

$\alpha \xrightarrow{O} \alpha \qquad O = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ trivial

$\alpha \xrightarrow{\mu} \beta \qquad \mu = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}$ classical [MG1]

$\alpha \xrightarrow{f} \alpha \qquad f = \begin{pmatrix} \alpha+\beta & \beta \\ -\alpha & 0 \end{pmatrix}$ $U_q sl_2$

$\alpha \xrightarrow{\underline{f}} \alpha \qquad \underline{f} = \begin{pmatrix} 0 & \beta \\ -\alpha & \alpha+\beta \end{pmatrix}$ $U_q sl_2$

$\alpha \xrightarrow{a} \beta \qquad a = \begin{pmatrix} \alpha+\beta & \beta \\ -\alpha & 0 \end{pmatrix}$ $U_q sl_{1|1}$

$\alpha \xrightarrow{\underline{a}} \beta$ $U_q gl_{1|1}$

[Here it is not obvious that conjugation, even by diagonal, is isomorphism. But it is.] (by diagonal)

Proof: There is calculus for constraint generation. Then:

$N=2$ is brute force

$N=3$ finitely many classes of solns once restrictions to $N=2$ enforced. End up with 10 admissible types with beautiful properties ...