

# Zeros of the partition function for the triangular lattice three-state Potts model II

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## 1 Introduction

The overarching aim here is to study the physics and in particular the critical phenomena associated to various simple but physically-representative lattice models, such as the 3d Ising model. In particular one is interested in the way (and the extent to which) critical phenomena are manifested in statistical mechanical partition functions. Very few of these models are integrable, so one either uses perturbative approximations; or looks for early evidence of limit behaviour on sequences of small lattices (as in monte carlo simulations) or very small lattices (as in exact brute-force transfer matrix calculations).

Fixing a model, and hence the ensemble of finite lattice systems from among which sequences approaching the thermodynamic/large-lattice limit can be chosen, the first question is what ‘behaviour’ of such a sequence might have a stable limit? For example, if a model has a single critical point (in the ferromagnetic region, say), then the specific heat curve might have a single maximum in all cases in the sequence, and the sequence of values of the position of this maximum would be expected to have a limit: the critical temperature. In practice, of course, there is no guarantee that a stable limit would manifest itself over a given finite subsequence of lattices. On the other hand, such direct thermodynamic observables are not the only ones available to us computationally. One could look for evidence of limit behaviour in more analytically sophisticated observations.

This paper is concerned with the distribution of zeros of the partition function of finite lattice models such as Potts models, when regarded as a function of temperature. Since the original flurry of activity in this area [?, ?, 5, ?], anticipated advances in the understanding of these important physical models through more sophisticated techniques in integrability and conformal field theory have perhaps been

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the predominant focus of attention. However these advances have not fully materialised (work on applying these techniques has been concentrated where they are most effective, rather than on models of direct physical interest [?]). In contrast, as computers get more powerful the exact finite lattice methods, while relatively crude, can provide a surprising amount of information on the location and nature of phase transitions, analytic structure and good variables.

## 1.1 Draft of follow-up to Maillard paper

About 15 years ago we [6] used a supercomputer to analyse the distribution of zeros of the partition function for the triangular lattice three-state Potts model. We noted that the intriguing and beautiful results already obtained for the square lattice [4] were obfuscated by the severity of finite size effects in the antiferromagnetic region, and that the finite lattice effects should be much less problematic on the triangular lattice (cf. [3]). This expectation was born out by the results, which provided good (although not irrefutable) evidence to support a claim that all the *real* zeros of the partition function for this model could lie at the roots of two simple cubic equations in  $a = \exp \beta$  [2, 8, 3].

That paper concluded with a call for studies on larger finite lattices, to settle a number of interesting questions about the possible constraint of the distribution of zeros to algebraic varieties (probably in contrast to the square lattice case). At the time, no such work was forthcoming — possibly due to the fact that this work already used the state-of-the-art in both hardware (processor power and memory limits were both saturated) and computer science to achieve its results. A brief comment on this technical obstruction is in order here, in as much as the progress we can now report might provide encouragement to further development in the area. Recall first that the partition function is given by

$$Z = \sum_{\text{configurations } \sigma} \exp \left( \beta \sum_{\text{links } (i,j)} \delta_{\sigma_i, \sigma_j} \right)$$

where each  $\sigma_i$  takes values from  $\{1, 2, 3\}$ , so that, at least nominally, the number of computations to be done triples with every additional lattice site.

The results reported *then* were for lattices of size up to 108 sites. Thus a 50 percent increase in the number of sites could be met simply (!) by a  $3^{50}$ -fold increase in computational power (given a commensurate increase in memory capacity). In fact the increase required is much less than this, but it is still encouraging to note that the results we present here, which achieve that 50 percent increase, were obtained in a few hours on an ordinary laptop computer. (The newest state-of-the-art in computer science is again required, but this is not the place to report on that.)

Figure 1 contains a plot of the distribution of zeros for a 156 site triangular lattice three-state Potts model, plotted in  $a$ . Figures 2 and 3 blow up parts of this distribution to show agreement with the predicted algebraic criticality conditions —  $a^3 - 3a - 1 = 0$  (ferromagnetic) and  $a^3 + 6a^2 + 3a - 1 = 0$  (antiferromagnetic). (For brevity we omit the arguments for these predictions, which are given in [6].)

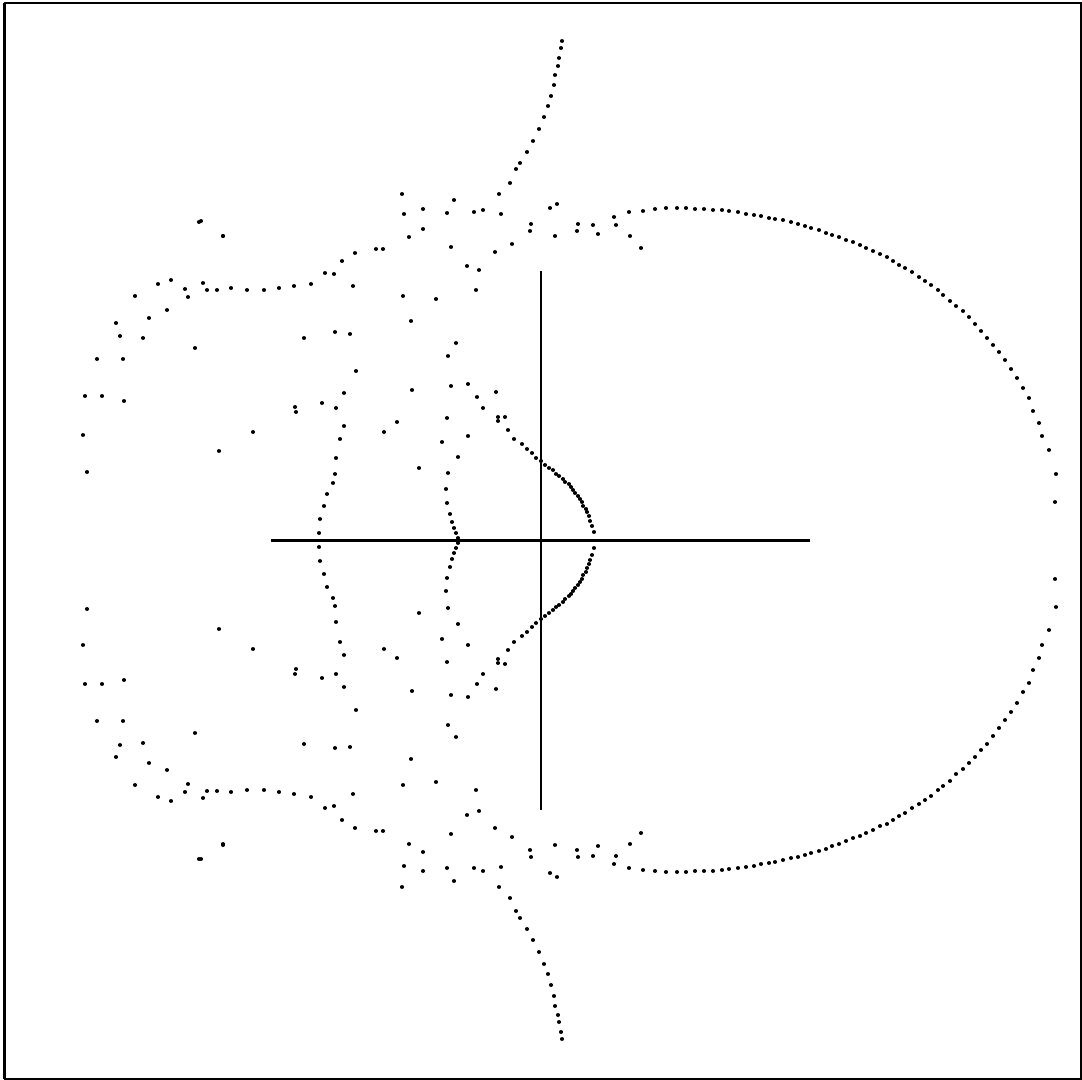


Figure 1: Zeros of the partition function in the complex  $a$  plane for a 156 site three-state Potts model. Axis are of unit length.

For comparison we include, in figure 4, the result for a 12x13 square lattice three-state model (an equivalent advance on the square lattice result given [6]).

## References

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- [3] J M Maillard and R Rammal, J Phys A **16** (1983), 353.

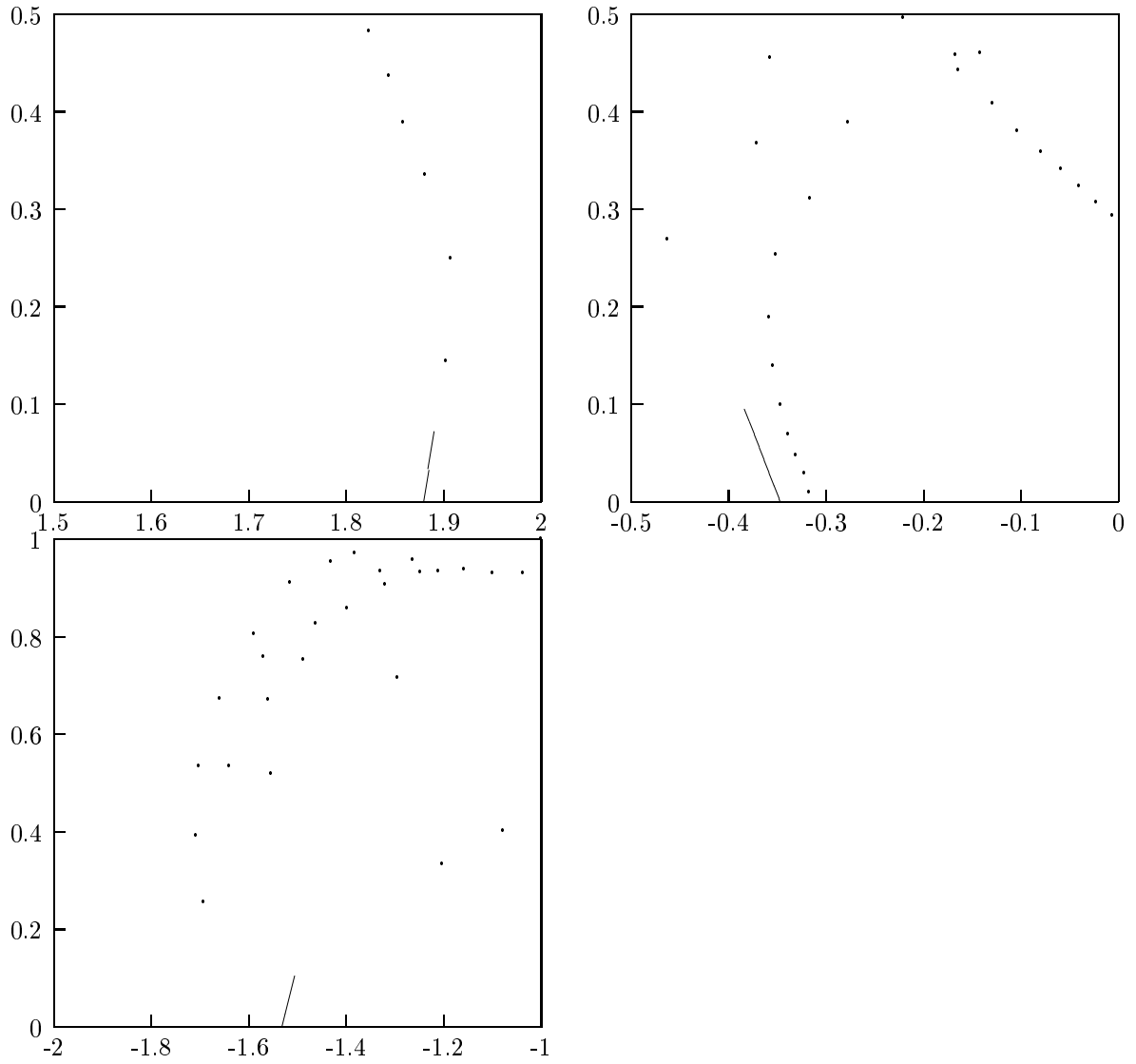


Figure 2: Blowups of figure 1 showing complex neighbourhoods of the real axis around where  $a^3 - 3a - 1$  has a root; and corresponding sections of the real function  $a^3 - 3a - 1$  showing these roots.

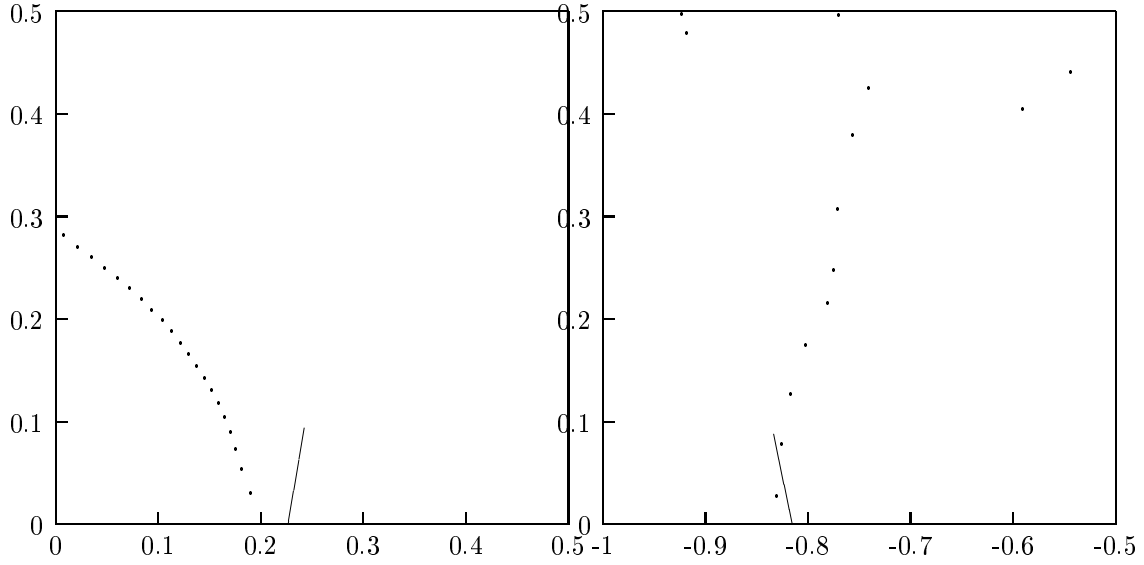


Figure 3: Blowups of figure 1 showing complex neighbourhoods of the real axis around where  $a^3 + 6a^2 + 3a - 1$  has a root; and corresponding sections of the real function  $a^3 + 6a^2 + 3a - 1$  showing these roots. (The third root is at  $a \sim -5.41$ .)

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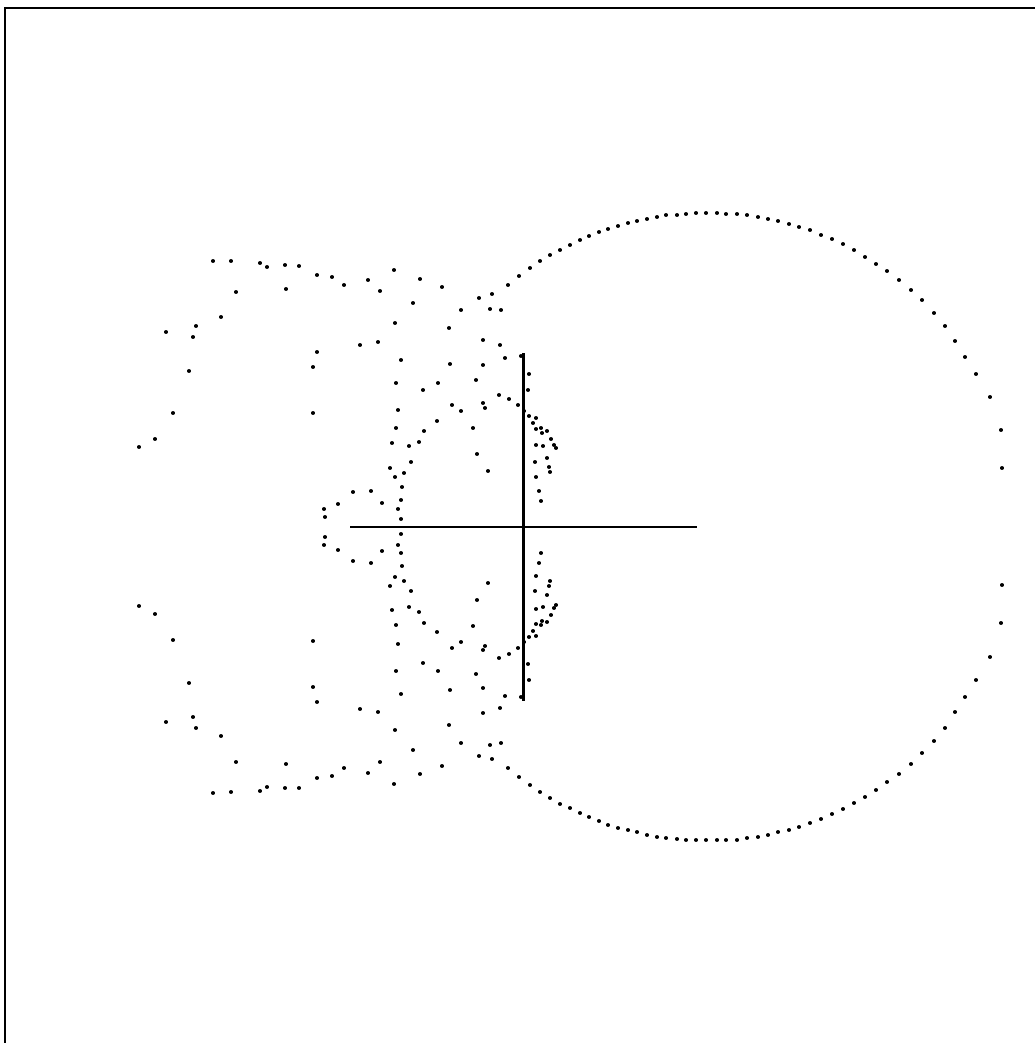


Figure 4: Zeros of the partition function of a 12x13 square lattice model in the complex  $a$  plane. Axis are of unit length.