GAMBA'S CHARACTER FORMULA AND CORRESPONDING STABILITY PROPERTIES FOR THE SYMMETRIC GROUPS REVISITED

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Abstract. We prove rules determining terms in Gamba's character formula for symmetric group representations either directly or in terms of other characters. We hence compute most terms in Gamba's character formula in terms of known characters. We discuss computation of the remaining terms. We show how to use our closed forms to compute composition multiplicities in several classes of representations.

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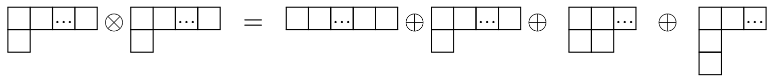
1. Introduction

There is a character formula for irreducible/Specht representations of symmetric groups, due to Gamba [2], well-suited to investigating certain stability properties of symmetric group Kronecker products. In this paper we study and determine rules for computing Gamba's formula. Our description of Gamba's formula is adapted from Boerner [1] (confer e.g. Hamermesh [3] and for example applications such as in [8]), via an 'updated' Lie theoretic/geometric perspective.

[put this remark... somewhere?] It is reasonable to suppose that the accessible subset of (modular) representation theory will remain measure-zero in an absolute sense. No 'modern perspective' change this. But it is an important exercise to continue to understand the 'right' measure-zero subsets, which might be measure-one relative to the needs of some specific important problem (exactly as in the partition category paradigm, which is naively measure-zero for the symmetric group but possibly universal for computational statistical mechanics - and we will give other examples shortly).

Stability properties in the spirit of the thermodynamic limit [?, ?, 8] play an important role in a number of areas of current interest in algebra and related areas of combinatorics and geometry [?, ?, 11, ?, ?, ?, ?] and computational physics [?, 8, ?, ?]. A first paradigm for this is the following. Consider the Kronecker product of symmetric group representations of the form

{eq:n-112}
$$(n-1,1)\otimes(n-1,1) \;=\; (n,0) \;+\; (n-1,1) \;+\; (n-2,2)+(n-2,1^2) \tag{1}$$



$(n > 3)$, where each partition represents the corresponding class of complex irreducible representations (in the Grothendieck ring). Omitting the 'top rows' from (1) we get

$$(-,1)\otimes(-,1) \;=\; (-,0) \;+\; (-,1) \;+\; (-,2) \;+\; (-,1^2)$$

so long as n is large enough for all terms to make sense (here $n > 3$). That is, we have an (almost) all n formula, cast so that the formula is independent of n. From a Schur–Weyl duality perspective this can be seen as the fact underlying the appearance of the partition algebra (replace the irreducible with the corresponding Young representation - the defining representation - for the classical duality). The marginalised dependence on n allows us to replace n with a parameter in the partition algebra. But it is also tantalising in that problems such as the modular representation theory of the symmetric groups remain largely open, and at least part of the obstruction is the lack of representation-theoretically useful stabilities in properties of $\mathfrak{S}_n$ as n varies (indeed an even more classical treatment uses Schur–Weyl duality on the other side, bla bla cf. Lie theory bla bla...) [?].

The partition algebra approach itself has its origins in another manifestation of stability - the thermodynamic limit in statistical mechanics. Here we find highly reducible representations with stability properties, such as the Potts representation (treated from this perspective in [8]). We will return to this representation later. A partial unification of perspectives can be achieved with the completed Schur–Weyl duality diagram

{eq:SWdual}
$$\begin{tikzcd} GL_Q \arrow[r] & V^n & \mathfrak{S}_n \arrow[l] \arrow[d, hook] \\ \mathfrak{S}_Q \arrow[u, hook] \arrow[ur] & & P_n(Q) \arrow[ul] \end{tikzcd} \tag{2}$$

where $V = \mathbb{C}^Q$ (see below). These various directly-observed stabilities all imply corresponding character phenomena, starting with Gamba's formula itself.

Outline of paper:
In Sec.2 we give necessary notation, recast in alcove geometric formalism as natural for the partition algebra (which turns out to be a natural setting for Gamba's approach).
In Sec.3 we state Gamba's formula and the Gamba Theorem.
In Sec.4 we derive the main results of the paper — rules for computing terms in Gamba's formula.
In Sec.5 we use the rules to illustrate computation of some composition multiplicities. ...

We now pass on to a more technical exposition. For this we recall some convenient facts about, and notations for, the symmetric groups $\mathfrak{S}_n$. (For example from [9, 3, 1], but see also our companion paper on hypergeometric stabilities [6] and references therein.)

{de:Lambdan} (**1.1**) Here Λ_n denotes the set of integer partitions of n.

(**1.2**) Let $i_1 i_2 ... i_k$ be an ordered subset of $\{1, 2, ..., n\}$ (i.e. a word or sequence without repetition). We may write $(i_1 i_2 ... i_k) \in \mathfrak{S}_n$ for the element that permutes cyclically as indicated: $(i_1 i_2 ... i_k)(i_j) = i_{j+1}$ (with $i_{k+1} = i_1$) and $(i_1 i_2 ... i_k)(i) = i$ otherwise.

Observe that if the notation $(i_1 i_2 ... i_k) \in \mathfrak{S}_n$ makes sense for some n then it makes sense for every larger n.

Two cycle-elements of $\mathfrak{S}_n$ as above commute if the underlying sets of their ordered subsets are disjoint. Every element s of $\mathfrak{S}_n$ is a product of such disjoint cycles. (Observe that the 'singleton cycles' are a notational vestige - each of them is the identity element.) Note that the set of sizes of these disjoint cycles in s (including the singletons) is a partition of n, so we have a map

$$\mathsf{c} : \mathfrak{S}_n \to \Lambda_n$$

The partition $\mathsf{c}(s)$ is called the cycle-structure of s, and the fibres of c are the conjugacy classes of $\mathfrak{S}_n$.

We may write $\sigma_1 = (12) \in \mathfrak{S}_n$ for the elementary transposition; and similarly for $\sigma_i = (i\; i{+}1)$ up to σ_{n-1}. Elements σ_i and σ_{i+j} commute for $j > 1$. We have $\mathsf{c}(\sigma_1 = (12)) = 21^{n-2}$
$\mathsf{c}((123)) = \mathsf{c}(\sigma_1\sigma_2) = 31^{n-3}$
$\mathsf{c}((1234)) = \mathsf{c}(\sigma_1\sigma_2\sigma_3) = 41^{n-4}$
$\mathsf{c}((123)(45)) = \mathsf{c}(\sigma_1\sigma_2\sigma_4) = 321^{n-5}$ and so on.

(**1.3**) For given $Q \in \mathbb{N}$ let the symbol set $\underline{Q} = \{1, 2, ..., Q\}$ be a basis for $V = \mathbb{C}^Q$. Then the set $\underline{Q}^n$ is the set of words of length n in $\underline{Q}$, a basis for $V^n = V \otimes V \otimes ... \otimes V$. The symmetric group $\mathfrak{S}_n$ acts on $\underline{Q}^n$ by permuting the factors. For example for $Q = 2$ and $n = 3$ we have the corresponding representation

$$\mathsf{r} : \sigma_1 \mapsto \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix} \otimes 1_2, \qquad\qquad \sigma_2 \mapsto 1_2 \otimes \begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$$

(NB the symbol $\otimes$ occurs here for the Kronecker product of matrices, not to be confused with the inner/Kronecker product of representations, or the tensor product of left and right ring-modules!) and so on, with the obvious extension to any n.

(**1.4**) This action $\mathsf{r}(\mathfrak{S}_n)$ on V^n commutes with the diagonal action of $GL(V) = GL_Q$ on V^n (the GL action is recorded diagrammatically in case $Q = 2$ by

$$\square \otimes \square = \square\!\square \oplus \emptyset, \qquad\qquad \square \otimes \square \otimes \square \;=\; (\square \otimes \square) \otimes \square \;=\; \square\!\square\!\square \oplus 2.\square \tag{3}$$

$$, \qquad\qquad \square^{\otimes 4} \;=\; \square\!\square\!\square\!\square \oplus 3.\square\!\square \oplus 2.\emptyset \tag{4}$$

and so on, where $\square$ denotes the defining representation, $\emptyset$ the trivial representation, and so on). Indeed the actions are dual (they are each other's commutants) so, over $\mathbb{C}$, the multiplicities of irreducible representations ('irreps') on one side are the dimensions of irreducibles on the other. The fact that there is an action of $GL(V)$ on V^n independently of n (the details of the action of course depend on n) mean that individual $GL(V)$ irreps can recur as n varies. Thus this itself leads to a form of stability phenomenon. However, in this construction the phenomenon depends on Q - a parameter coming from r rather than intrinsic to $\mathfrak{S}_n$. In the $Q = 2$ case the GL representation theory is indexed by one-row partitions (and in the general fixed Q case by $Q-1$-row partitions), in other words by $\mathbb{N}_0$. This says that we can use these to index a certain subset of $\mathfrak{S}_n$ irreps for all n. Specifically (for $Q = 2$) for $m \in \mathbb{N}_0$ congruent to n mod.2 we have

$$m \mapsto ((n+m)/2, (n-m)/2)$$

(two-part partition). This scheme has a useful geometric context related to that for the partition algebra, but 'in a different dimension'. In this setting we see Q sets of fibres of stability as n varies — in case $Q = 2$ the even n values form one fibre and the odd values the other. This is different from the phenomenon controlled by the partition algebra, as we shall see, but both can be treated. Each giving a different way to work with characters.

(**1.5**) The character of a representation is the map from the group to scalars given by the trace. For a representation λ, say, we write $\chi^\lambda(s) = Trace(\lambda(s))$. Since these characters are fixed within a conjugacy class we write $\chi^\lambda_c = \chi^\lambda(s)$ for any s with $\mathsf{c}(s) = c$.

In general computing characters is hard. But there are some striking exceptions. Observe that for any n (still with $Q = 2$ for now):

$$\chi^{\mathsf{r}}(1) = \chi^{\mathsf{r}}_{1^n} = (\chi^{\mathsf{r}}_1)^n \ = 2^n$$
$$\chi^{\mathsf{r}}(\sigma_1) = \chi^{\mathsf{r}}_{21^{n-2}} = \chi^{\mathsf{r}}_2 (\chi^{\mathsf{r}}_1)^{n-2} \ = 2^{2-1}2^{n-2} = \ 2^{n-1}$$
$$\chi^{\mathsf{r}}(\sigma_{l_1}\sigma_{l_2}...\sigma_{l_k}) = 2^{n-k}$$

for any $l_1 < l_2 < ... < l_k$ (see e.g. [8] — the verification is elementary: for any Q the cycle $\sigma_1\sigma_2...\sigma_{k-1} \sim (1, 2, ..., k)$ fixes, in $\mathfrak{S}_k$, only the Q words of form $qq...q$ so contributes a factor Q out of a possible Q^k here, i.e. $Q = Q^{k-(k-1)}$ as claimed, and similarly for each cycle factor from c). We will write c_i for the exponent of i in c, i.e. the partition is $1^{c_1}2^{c_2}3^{c_3}...$. Thus the number of σ_i factors is

$$\sum_{i=1}^{n}(i-1)c_i \ = \ 0.c_1 + 1.c_2 + 2.c_3 + ...,$$

so we can express the general character here as

$$\chi^{\mathsf{r}}_c = Q^{n-\sum_{i=1}^{n}(i-1)c_i} \tag{5}$$

Clearly, from above, this gives the complete character.

Later we will use our results to confirm the multiplicities of irreducible representations in this representation r. (The multiplicities here may also be computed using more sophisticated methods such as idempotent localisation; or Schur–Weyl duality. We will also return to this point later.)

(**1.6**) Simple modules of the symmetric group $\mathfrak{S}_n$ are indexed by integer partitions of n. If $\lambda \vdash n$ and c is a class of $\mathfrak{S}_n$ then let χ^λ_c (conform this notation! - done! I think!) denote the corresponding irreducible character. Perhaps familiar from various textbooks, such as Hamermesh [3], will be the tables of specific characters in low ranks.

On the other hand, quoting from [8] (cf. e.g. Problem 3 of Sec.7.4 in [3]) we have, for example,

$$\chi^{n-5,5}_c = \frac{c_1(c_1-1)(c_1-2)(c_1-3)(c_1-9)}{5!} \ + \ (c_1-1)\frac{c_2(c_2-1)}{2}$$
$$+ \ \frac{c_1(c_1-1)(c_1-5)c_2}{3!} \ + \ \frac{c_1(c_1-3)}{2}c_3 \ + \ c_2c_3 \ + \ (c_1-1)c_4 \ + \ c_5$$

- recall $\chi^\lambda_c = \chi^\lambda_{1^{c_1}2^{c_2}3^{c_3}...}$. A much simpler one:

{eq:n-1,1}
$$\chi^{n-1,1}_c = c_1 - 1 \tag{6}$$

These character formulae appear (and are) highly non-trivial. But observe that here it has been possible to give a ‘fibre’ of characters for all n. Shortly we will explain the fibres for which this is possible. The characters are then determined in principle by Gamba’ Theorem, as we will review. The Theorem itself does not give a usable closed form per se, but we will largely resolve this issue here.

(**1.7**) Let us give an example of the utility of the above character formulae. Recall the orthogonality relation for irreducible characters:

$$\sum_{s\in G}\chi^\lambda(s)\chi^\mu(s^{-1}) = |G|\delta_{\lambda,\mu}$$

(recall that for $\mathfrak{S}_n$ inverse elements are in the same class) and the formula for Kronecker products

$$\chi^{r\times r'}(s) = \chi^r(s).\chi^{r'}(s)$$

for any representations r and r'. In particular then the multiplicity of $\mathfrak{S}_n$ irrep μ in $r \times (n-1, 1)$ is:

$$\sum_c \frac{1}{\prod_i i^{c_i}c_i!}\chi^{r\times(n-1,1)}_c\chi^\mu_c \ = \ \sum_c \frac{1}{\prod_i i^{c_i}c_i!}(c_1-1)\chi^r_c\chi^\mu_c$$

Taking $r = \lambda$ irreducible we have multiplicity of irrep μ in $\lambda \times (n-1, 1)$ given by

$$-\delta_{\lambda\mu} + \sum_c \frac{1}{\prod_i i^{c_i}c_i!}c_1\chi^\lambda_c\chi^\mu_c \ = \ -\delta_{\lambda\mu} + \sum_{c|c_1\neq 0}\frac{1}{(c_1-1)!\prod_{i>1}i^{c_i}c_i!}\chi^\lambda_c\chi^\mu_c$$

The usual $\mathfrak{S}_{n-1} \hookrightarrow \mathfrak{S}_n$ restriction rule gives the branching relation:

{eq:branchi}
$$\chi^{\mu}_{1^{c_1+1}2^{c_2}\dots} = \sum_i \chi^{\mu-e_i}_{c} \tag{7}$$

where the sum is over 'rows with removable boxes' - this is partial data on the character χ^μ. Plugging in we get

$$-\delta_{\lambda\mu} + \sum_{c|c_1\neq 0} \frac{1}{(c_1-1)!\prod_{i>1} i^{c_i} c_i!} \left(\sum_i \chi^{\lambda-e_i}_{1^{c_1-1}2^{c_2}\dots}\right)\left(\sum_j \chi^{\mu-e_j}_{1^{c_1-1}2^{c_2}\dots}\right) = -\delta_{\lambda\mu} + \sum_i \sum_j \delta_{\lambda-e_i,\mu-e_j}$$

Thus $\lambda \times (n-1,1)$ contains μ once for each $\mu \neq \lambda$ obtained by first removing then adding a box to the diagram for λ; and $m-1$ copies of λ, where m is the number of rows of λ from which a box can be removed. See for example (1).

(**1.8**) Aside. Here are the dimensions for the representations in (1):

$$(n-1).(n-1) = 1+(n-1) + \frac{n!}{(n-1)(n-2)(n-4)!2} + \frac{n!}{n.(n-3)!2} = n + \frac{n(n-3)}{2} + \frac{(n-1)(n-2)}{2}$$

as an aid to the eye. For $n = 5$ we have $4.4 = 1+4+5+6$.

(**1.9**) Some (random and incomplete) refs.: [10, 9, 7, 4]; some subsequent revisits: [**?**, **?**, 13, 12]. And some more random ones to chase on Kronecker aspect (for which there is a huge literature, encompassing several different drivers) etc.: Regev [**?**, **?**, **?**].

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2. Notation via geometric formulation {ss:geo}

In order to state and work with Gamba's Theorem, Thm.3, it is useful to be able to manipulate certain formal characters. Classical tools for this are essentially combinatorial (see e.g. Boerner [1]). But we will show that these manipulations have a natural setting in the context of alcove geometry (in particular as arising in the context of the partition algebra [**?**, **?**], but confer also [5, **?**]). We describe this next.

2.1. **Group orbits in the space of weights.**

Consider the space of 'weights'

$$\mathbb{R}^f \quad = \quad \{x = (x_1, x_2, \dots) | \; x \in \mathbb{R}^{\mathbb{N}} \text{ of finite support } \}$$

('finitary' elements of $\mathbb{R}^{\mathbb{N}}$). The *depth* of $x \in \mathbb{R}^f$ is the last non-zero entry.

It will also be useful to specify two non-finitary elements of $\mathbb{R}^{\mathbb{N}}$:

$$\varrho = (1,1,1,\dots), \qquad \tau = (1,2,3,\dots)$$

We say $\lambda \in \mathbb{R}^{\mathbb{N}}$ is integral if all $x_i \in \mathbb{Z}$ and *dominant* if $x_i \geqslant x_{i+1}$ for all i.

(**2.1**) Note that since support is finite in $\mathbb{R}^f$, a dominant weight in $\mathbb{R}^f$ is non-negative, indeed consisting of a leading positive part and then a tail of 0s.

In writing a dominant weight we may omit the tail. Indeed we may omit an infinite tail of zeros in writing any weight $x \in \mathbb{R}^f$. (Thus $() = (0) = (0,0)$ and so on.)

For $x \in \mathbb{R}^f$, write depth(x) for the position of the last non-zero entry.

(**2.2**) A 'naive reflection group' action on $\mathbb{R}^{\mathbb{N}}$, that closes on $\mathbb{R}^f$, is:

$$\underline{\sigma}_i(x_1, x_2, \dots, x_i, x_{i+1}, x_{i+2}, \dots) = (x_1, x_2, \dots, x_{i+1}, x_i, x_{i+1}, \dots)$$

for any $i \in \mathbb{N}$.

(This $\underline{\sigma}_i$ action corresponds to reflection in the $x_i = x_{i+1}$ hyperplane. This, and indeed any, hyperplane path-separates the space. A regular x [-meaning what?! that no two entries are same?

(so no finitary is regular.) regular used below for something else!? maybe say 0-regular here?] and its image are in distinct parts.)

(**2.3**) Let $\underline{\mathfrak{S}} = \langle \underline{\sigma}_i \rangle_i$. The $\underline{\mathfrak{S}}$-orbit of weight x is its orbit under the action *generated* by $\underline{\mathfrak{S}}$.
Write $[x]\_$ for the $\underline{\mathfrak{S}}$-orbit of x. The orbit has a transversal of dominant weights (non-increasing sequences).

(**2.4**) The shifted reflection group action on $\mathbb{R}^f$ is defined by

{eq:reflect1}
$$\sigma_i(x_1, x_2, ..., x_i, x_{i+1}, ...) = (x_1, x_2, ..., x_{i+1} - 1, x_i + 1, ...) \tag{8}$$

That is
$$\sigma_i x = \underline{\sigma}_i(x - \tau) + \tau$$

(**2.5**) Let $\mathfrak{S} = \langle \sigma_i \rangle_i$. The $\mathfrak{S}$-orbit of weight x is its orbit under the action *generated* by $\mathfrak{S}$ (a group of such reflections together with composites of reflections, some of which are reflections, such as $\sigma_1\sigma_2\sigma_1$, given by $(\sigma_1\sigma_2\sigma_1)(x) = \sigma_1(\sigma_2(\sigma_1(x)))$; and some are not, such as $\sigma_1\sigma_2$).

(**2.6**) A weight is *singular* if any element of its $\mathfrak{S}$-orbit is fixed by any σ_i.
A weight is *regular* otherwise.

(**2.7**) Examples. Singular: $\sigma_1(0,1) = (0,1)$; $\sigma_1(-2,1) = (0,-1) = \sigma_2(0,-1)$.
Regular: $\sigma_2() = (0,-1,1)$, $\sigma_1(-2,1,1) = (0,-1,1)$.

{pa:x-tau} (**2.8**) Note that a weight x is singular if and only if two entries in $x - \tau$ are the same.
Examples. Observe that $(-2,1) - \tau = (-3,-1,-3,-4,...)$, so $(-2,1)$ is singular.
Observe that a dominant weight is never singular.

(**2.9**) Consider an integral weight $x \in \mathbb{R}^f$ and put $m = \text{depth}(x)$ and $d = x_m$. An x with $d = x_{\text{depth}(x)} < 0$ is singular. (Proof. We have $(x-\tau)_i = -i$ for $i > m$ since then $x_i = 0$. Thus $(x-\tau)_m = d - m = (x-\tau)_{m-d}$. Now use (2.8).)

{cl:1} (**2.10**) Prop./Claim: Let λ be an integral weight. If the sum from some entry λ_j to the right (i.e. $\sum_{i=j}^{\infty} \lambda_i$) is negative then λ is singular. Indeed, if $-\lambda_i > \text{depth}(\lambda) - i$ for some i then λ is singular.

Proof. The first claim follows from the second. For the second consider (2.18). Set $m = \text{depth}(\lambda)$. We see that $(\sigma_{m-1}\sigma_{m-2}...\sigma_i\lambda)|_{m,m+1} = (-1,0)$, which is manifestly singular. [FIX THIS! OK NOW? BUT move the proof back there! -Actually this proof can easily be adjusted to use only the above.] □

{de:dom} (**2.11**) Write Λ for the set of dominant integral weights:
$$\Lambda = \{(), (1), (2), (1,1), (3), (2,1), (1,1,1), ...\}$$
For $\lambda = (\lambda_1, \lambda_2, ...) \in \mathbb{R}^f$ define $|\lambda| = \sum_i \lambda_i$. In this context define $\Lambda_n = \{\lambda \in \Lambda \mid |\lambda| = n\}$ (observe that this is essentially consistent with the definition in (1.1) - we return to this point in Sec.2.3). Define $\Lambda^n = \Lambda^{\leqslant n} = \bigcup_{i \leqslant n} \Lambda_i$.

Since we identify Λ with the set of integer partitions (and hence of Young diagrams), we may write λ^t for the conjugate partition. Thus:
$$(3,1)^t = (2,1^2)$$

Let $\{e_1, e_2, ...\}$ be the standard ordered basis of $\mathbb{R}^f$.

(**2.12**) Note that the usual dot product $a.b = \sum_i a_i b_i$ is well-defined between any element of $\mathbb{R}^f$ and any element of $\mathbb{R}^{\mathbb{N}}$.
Let $|x| = \sum_i x_i$. Observe that $|\sigma_i x| = |x|$.

(**2.13**) Define $r_\Sigma : \mathbb{R}^f \to \mathbb{R}^{\mathbb{N}}$ by
$$r_\Sigma(x) = \left(x_1, x_1 + x_2, \sum_{i=1}^{3} x_i, \sum_{i=1}^{4} x_i, ... \right)$$
Define a partial order on $\mathbb{R}^{\mathbb{N}}$ by $x \geqslant y$ if $x - y$ has all non-negative entries.

Observe that if we apply σ_i at a point with an underhang of 2 or more (i.e. $x_i - x_{i+1} \leqslant -2$) then $r_\Sigma(\sigma_i x) > r_\Sigma(x)$.

Example:

$$\begin{array}{ccc} 1\,3\,5\,7\,2 & \stackrel{\sigma_1}{\rightsquigarrow} & 2\,2\,5\,7\,2 \\ r_\Sigma(x) = (1,4,9,16,18,...) & & r_\Sigma(\sigma_1 x) = (2,4,9,16,18,...) \end{array}$$

Since all weights x end in a tail of zeroes, every $r_\Sigma(x)$ attains a corresponding constant tail at the *terminal value* $|x|$.

(**2.14**) Recall that an underhang of 1 gives a singular weight. Thus every regular weight x is either dominant or has an underhang of 2 or more at some $(..., x_i, x_{x+1}, ...)$. Applying σ_i at any such point increases r_Σ. But iteration of this process must terminate, since r_Σ cannot exceed the sequence given by immediate attainment of the terminal value. Thus a dominant weight must be reached.

(**2.15**) We claim ! that no two elements of Λ are in the same $\mathfrak{S}$-orbit. Exercise: prove or disprove. (Intuitively if two are in same orbit then there would be at least one reflection hyperplane between...)

Perhaps every regular integral orbit intersects Λ, so we have a transversal? Not quite... Prove the correctly axiomatised version using SS2.2. (-now I seem to have done this... without condition... so have a think!!)

{ss:strait}

2.2. **Straightening a weight.**

The procedures in this section are motivated by their facilitation of formal character calculations to be described in Sec.2.4.

The process of finding out if an integral weight is regular, and if so finding the orbit representative of the integral weight, involves a procedure called 'straightening'. Here we illustrate how to apply reflections to try to straighten a non-dominant weight.

Examples varying non-trivially across only 2 or 3 dimensions can be illustrated directly as reflections projected on the plane. To view 3d we project onto 2d along the 111 direction. First we can consider naive reflections, as in Fig.1.

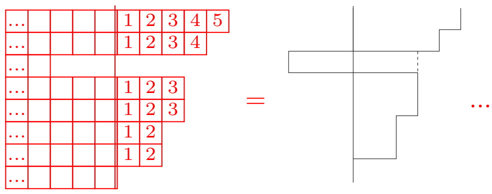
We can visualise shifted reflections either by shifting the reflection hyperplanes on the picture, or correspondingly by shifting the coordinates, as in Fig.2.

Thus for example

$$\begin{aligned} \sigma_1\, 000 &= \underline{\sigma}_1(000-123)+123 = -2\,-1\,-3+123 = -110 \\ \sigma_1\, 210 &= \underline{\sigma}_1(210-123)+123 = -11\,-3+123 = 030 \\ \sigma_1\, 010 &= \underline{\sigma}_1(010-123)+123 = -1\,-1\,-3+123 = 010 \end{aligned}$$

For more general cases it is convenient to realise reflections as moves on (generalised) Young diagrams. The diagram of an integral weight is a picture like this:

$\ldots (5,4,-3,3,3,2,2) \mapsto$



(in the middle we indicate but omit an infinite sea of negative boxes). Thus in particular the diagram of a dominant integral weight is essentially a Young diagram. For use below, we will mainly need to straighten weights that differ from dominant integral weights λ by $\lambda \rightsquigarrow \lambda - le_i$ for some l and i. And the following example is of this type. (It will be clear that every integral

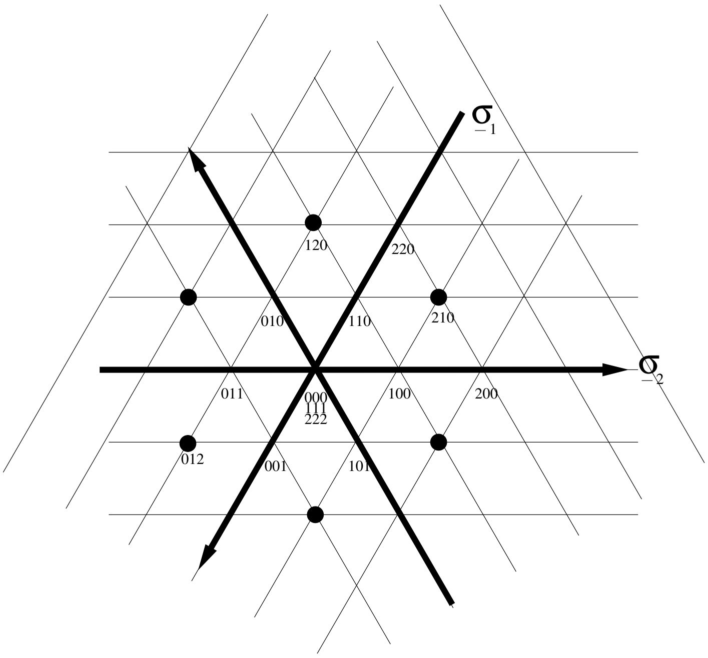


fig:reflectjust3-3
FIGURE 1. Weights and reflection hyperplanes visualised projected along the 111 direction.

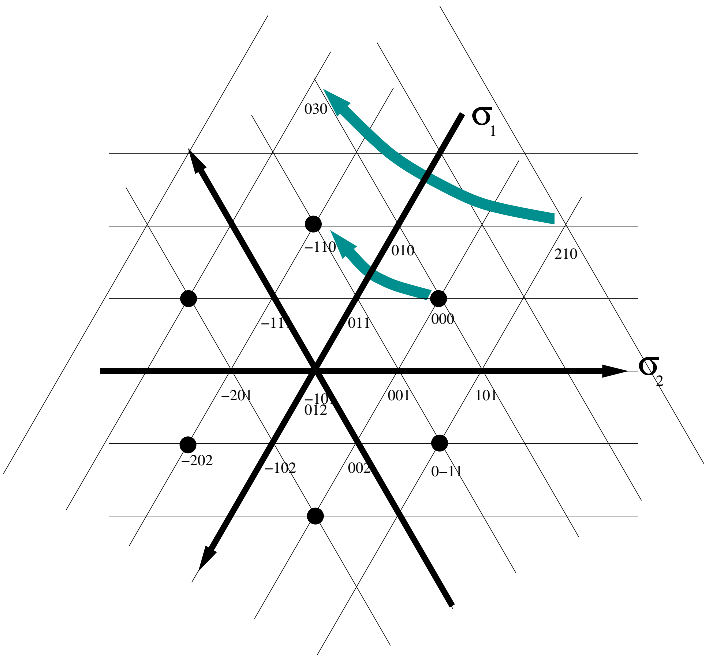


fig:reflectjust3-3s
FIGURE 2. Weights and shifted reflection hyperplanes visualised projected along the 111 direction.

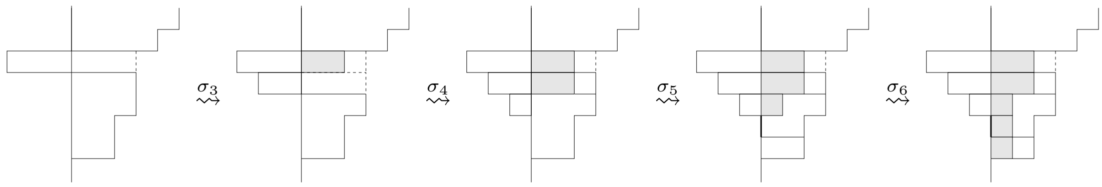
weight can be constructed by iterating such subtractions starting from some dominant weight. We can consider our procedure as applying to weights of the $\lambda - le_i$ type, or for an arbitrary weight applying to the part of it from the lowest negative entry and below — note that if the lowest negative entry is the last non-zero entry then the weight is necessarily singular.)

Observe that here the weight obtained by deleting row-1 to row-i is a dominant weight. - We say that a weight ω is dominant below i if deleting rows 1-i makes it dominant.

(**2.16**) Consider the non-dominant weight

$$\omega = \ (5,4,-3,3,3,2,2) = (5,4,3,3,3,2,2) - 6e_3 = (5,4,4,3,3,2,2) - 7e_3$$

By default we will consider such weights expressed in the form $\lambda - le_i = (5,4,3,3,3,2,2) - 6e_3$ - taking the λ so that the l is minimal (that is, so $\lambda_i = \lambda_{i+1}$). (Although note that for later use this may not be the form arising in the Gambda computation.) This ω is illustrated by the leftmost figure here:



$$5\ 4\ \underline{-3\ 3}\ 3\ 2\ 2 \ \rightsquigarrow\ 5\ 4\ \underline{2\ \ -2}\ \underline{3}\ 2\ 2 \ \rightsquigarrow\ 5\ 4\ 2\ \underline{2\ \ -1}\ \underline{2}\ 2 \ \rightsquigarrow\ 5\ 4\ 2\ 2\ \underline{1\ 0}\ \underline{2} \ \rightsquigarrow\ 5\ 4\ 2\ 2\ 1\ \underline{1\ 1}$$

The leftmost weight fails to be dominant due to the third 'row' as shown not covering or overhanging the fourth row. This initial obstruction is rectified here by applying σ_3 — which engenders an obstruction in the fourth row, so we continue with σ_4, and so on. In the last move we pass, in this case, to a dominant weight.

Observe that

$$\sigma_i \omega = (\omega_1, ..., \omega_{i-1},\ \omega_{i+1} - 1,\ \omega_i + 1,\ \omega_{i+2}, ...).$$

Expressing ω in the 'minimal' $\lambda - le_i$ form as above, then $\lambda_i = \lambda_{i+1}$. We have

$$\omega = \lambda - le_i = (\lambda_1, ..., \lambda_{i-1},\ \ \lambda_i - l,\ \ \ \underbrace{\lambda_{i+1},}_{=\lambda_i}\ \ \ \ \lambda_{i+2}, ...)$$

so

$$\sigma_i \omega = (\lambda_1, ..., \lambda_{i-1},\ \lambda_i - 1,\ \lambda_i - l + 1,\ \lambda_{i+2}, ...) = \lambda - e_i - (l-1)e_{i+1}$$

Note that this is not necessarily of the same dominant-minus form, since the $-e_i$ may mean row-i is underhanging later rows as well as row-$i+1$ (see the example above). And also it is not necessarily $l-1$ that is the minimal subtraction if it *is* in dominant-minus form. However, what we do see is that the largest underhang is at most $l-1$. So let us iterate. ...

Observe that a run of reflections like $\sigma_6\sigma_5\sigma_4\sigma_3$ has the effect of pushing rows 4-7 up to be rows 3-6, each shifted -1; and pushing row-3 to row-7 shifted +4. If the initial weight is ω and we apply $\sigma_{i+j}\sigma_{i+j-1}...\sigma_{i+1}\sigma_i$ then the new weight is

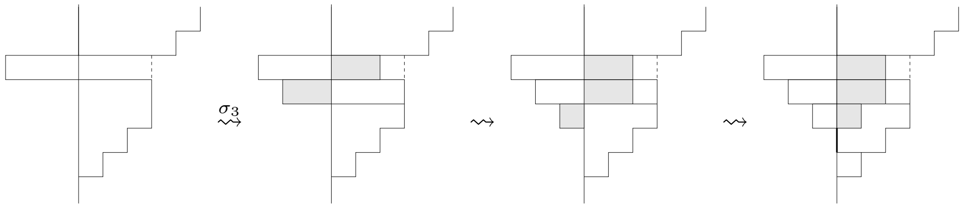
$$(\omega_1, ..., \omega_{i-1},\ \omega_{i+1} - 1, \omega_{i+2} - 1, ..., \omega_{i+j} - 1,\ \omega_i + j,\ \omega_{i+j+1}, ...).$$

If $\omega = \lambda - le_i$, and hence is dominant below i, then the new weight is dominant at least from row-1 to row-$(i+j-1)$; and from row-$(i+j+1)$ onwards. Since row lengths in ω do not increase as we go down, by making j large enough we can make row-$(i+j)$ greater or equal to row-$(i+j+1)$ - so we have dominance from row-$(i+j)$ onwards. So the dominance or otherwise of $\sigma_{i+j}\sigma_{i+j-1}...\sigma_{i+1}\sigma_i(\lambda - le_i)$ for some j depends on what happens as the underhang "gets smaller" ...

Putting it another way, for a weight of form $\omega = \lambda - le_i$ that is non-dominant (i.e. λ is dominant and l is large enough to create an underhang in row i) we apply σ_i. If the underhang is larger than 1 then we will have a weight of form $\omega' = \sigma_i\omega = \lambda' - l'e_{i+1}$, with necessarily smaller underhang in row $i+1$. If the underhang is 1 then $\sigma_i\omega = \omega$ and ω is singular.

We are addressing a couple of questions here. One is whether a weight of form $\omega = \lambda - le_i$ is singular or not. And the other is whether, if non-singular, the weight has a dominant weight in its orbit.

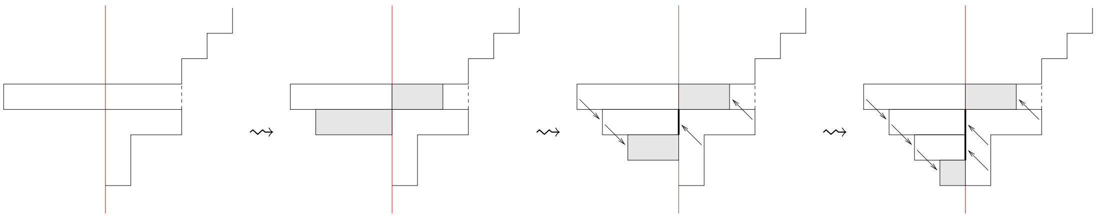
(**2.17**) Consider the following example:



$$5\,4\,\underline{-3\;3}\,3\,2\,1 \;\rightsquigarrow\; 5\,4\,\underline{2\;-2}\,\underline{3}\,2\,1 \;\rightsquigarrow\; 5\,4\,2\,\underline{2\;-1}\,\underline{2}\,1 \;\rightsquigarrow\; 5\,4\,2\,2\,\underline{1\,0}\,\underline{1} \;\rightsquigarrow\; 5\,4\,2\,2\,1\,\underline{0\,1}$$

The leftmost weight fails to be dominant and we start as before. The final picture is fixed under σ_6, and so shows that this orbit is singular.

{pa:x1} (**2.18**) It will be useful later to have considered one further possible situation.



$$5\,4\,3\;-4\,3\,1\,1 \;\rightsquigarrow\; 5\,4\,3\,2\;-3\,1\,1 \;\rightsquigarrow\; 5\,4\,3\,2\,0\;-2\,1 \;\rightsquigarrow\; 5\,4\,3\,2\,0\,0\;-1 \;\rightsquigarrow$$

Note that this is singular.

(**2.19**) Suppose that λ is a non-dominant weight. It is a weight, so it is eventually zero ($\lambda_i = 0$ for large enough i), and then (since non-dominant) it has a last i where $\lambda_i < \lambda_{i+1}$ (row $i+1$ underhangs row i) - thus with $\lambda_{i+1} \geqslant 0$, since i is the last underhung row. Applying σ_i we get $(\sigma_i\lambda)_i = \lambda_{i+1} - 1$ and $(\sigma_i\lambda)_{i+1} = \lambda_i + 1$. If $\lambda_i = \lambda_{i+1} - 1$ then observe that $\lambda = \sigma_i\lambda$ is singular. If $\lambda_i = \lambda_{i+1} - k$ with $k > 1$ then we now have ...

{ss:YoungD}

2.3. **Summary of some generalities around Young diagrams and integer partitions.** In the Introduction we used the set of integer partitions of n to index conjugacy classes of $\mathfrak{S}_n$. An integer partition of n is a multiset of positive integers whose sum is n.

Since multisets are unordered, we can write an integer partition as a non-increasing sequence. That is, dominant integral weights, as in (2.11), are in natural bijection with integer partitions - the map is to remove the tail of 0's. As noted, we abuse notation and identify these sets, thus writing

$$\Lambda \cong \bigcup_n \Lambda_n$$

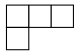
for both.

(**2.20**) We assume familiarity with Young diagrams as representations of integer partitions, and hence dominant integral weights. Young diagrams are certain sets of unit boxes in the positive quadrant of the plain (here the positive quadrant is drawn 'matrix style', i.e. to the right and down). Recall [-from where? is it true!? what is the map? true for dom non-negative int, but is this what we have in mind?] that dominant integral weights are in natural bijection with integer partitions - multisets of positive integers.

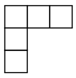
(**2.21**) Write Y for the set of Young diagrams. Write $\lambda \subseteq \mu$ if diagram λ is contained in μ as a subset. A *skew* diagram is any subset of the form $\mu \setminus \lambda$. A skew diagram is a *ribbon* (or rim-hook) if it is connected and has no 2x2 subdiagram.

Observe that the representations of dominant integral weights in Sec.2.2 are pictures of the underlying sets of Young diagrams (flattening out the set of boxes to a simple area).

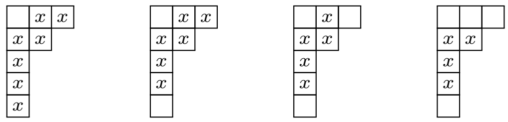
(**2.22**) We have a map $\mathfrak{Y} : \Lambda \to \mathsf{Y}$, exemplified by

$(3,1) \mapsto$ 

and so on. Similary we have $\mathfrak{Y}^{\cdot} : \Lambda^{\cdot} \to \mathsf{Y}$ exemplified by

$1^2 2^0 3^1 \mapsto$ 

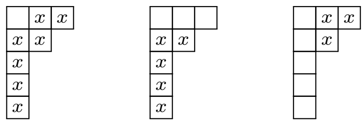
(**2.23**) Recall that a boundary component of a Young diagram is a connected subset of boxes all of which have at least one point touching the exterior of the diagram in the positive quadrant. Example: {pa:boundar}



Observe that a boundary component is a ribbon.

A subset of boxes in a Young diagram is *removable* if its complement is a Young diagram. (Thus only the first example above is removable.)

Recall that a *rim* of a Young diagram is a removable boundary component. Example:



(**2.24**) With Table 1 in mind it can be useful to consider total orders on Λ_n. Reverse lexicographic order is useful. [-but we appear to use something else. what!? certainly other prescribed orders exist. Exercise: give a prescription consistent with the part of an order appearing in the Table.]

(**2.25**) We write $\lambda\mu$ for the partition corresponding to the 'pointwise addition of exponents', which is also the composition $\lambda \oplus \mu$ (with $\oplus$ being the obvious concatenation of compositions). [2023: -not sure I understand this in general now! its ok if the composition is a partition... maybe this is all we need?] For example $(41)(1) = (41^2)$. We will write $\lambda 1$ for $\lambda\mu$ when $\mu = (1)$.

2.4. **Formal characters and Gamba differentials.** {ss:diff}

Notation 1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be an integral weight, such as a partition. A formal character is a symbol $[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_m]$ or a formal sum, or $\mathbb{Z}$-linear combination, of such symbols, where we set

$$[\sigma_i \lambda] = -[\lambda].$$

Thus $[\lambda] = 0$ if λ is singular; and $[\lambda], [\lambda']$ are not independent if they are in the same $\mathfrak{S}$-orbit. I.e. a formal character is an element of $\mathbb{Z}[\Lambda]$.

(**2.26**) Given an integer $r > 0$, denote by $[\lambda]_r$ the formal sum {de:rdiff}

$$[\lambda_1, \ldots, \lambda_m]_r = \sum_{i=1}^{\infty} [\lambda - re_i] \tag{9}$$ {eq:diffr}

$$= [\lambda_1 - r, \lambda_2, \ldots, \lambda_m] + [\lambda_1, \lambda_2 - r, \ldots, \lambda_m] + \cdots + [\lambda_1, \lambda_2, \ldots, \lambda_m - r].$$

We write each non-zero term as the (signed) orbit representative from Λ.

Remarks: Consider one such summand $[\lambda'] = [\lambda_1, \ldots, \lambda_k - r, \lambda_{k+1}, \ldots, \lambda_m]$. If $\lambda_{k+1} = \lambda_k - r + 1$ or $k = m$ and $\lambda_m - r < 0$, then $\lambda - re_k$ is singular, so $[\lambda - re_k] = 0$ and we omit this summand.

Examples are in Fig.3. (1) Throw in a few pics, Paul. :-) (2) We are pointedly not talking

$$[1]_1 = [0]$$

$$[2]_1 = [1]$$
$$[2]_2 = [0]$$
$$[1^2]_1 = [1] + [0,1] = [1]$$
$$[1^2]_2 = [-1,1] + [1,-1] = -[0]$$

$$[2,1]_1 = [1,1] + [2]$$
$$[2,1]_2 = [0,1] + [2,-1] = 0$$
$$[2,1]_3 = -[0]$$
$$[1^3]_1 = [0,1,1] + [1,0,1] + [1,1] = [1,1]$$
$$[1^3]_2 = [-1,1,1] + [1,-1,1] + [1,1,-1] = -[1]$$
$$[1^3]_3 = [0]$$

$$[3,1]_1 = [2,1] + [3]$$
$$[3,1]_2 = [1,1]$$
$$[3,1]_3 = 0$$
$$[3,1]_4 = -[0]$$
$$[2^2]_1 = [1,2] + [2,1] = [2,1]$$
$$[2^2]_2 = -[1,1] + [2]$$
$$[2^2]_3 = -[1]$$
$$[2^2]_4 = 0$$
$$[2,1^2]_1 = [1,1,1] + [2,1]$$
$$[2,1^2]_2 = -[2]$$
$$[2,1^2]_3 = 0$$
$$[2,1^2]_4 = [0]$$
$$[1^4]_1 = [1^3]$$
$$[1^4]_2 = -[1^2]$$
$$[1^4]_3 = [1]$$
$$[1^4]_4 = -[0]$$

$$[3,1^2]_1 = [2,1^2] + [3,1]$$
$$[3,1^2]_2 = [1^3] - [3]$$
$$[3,1^2]_3 = 0$$
$$[3,1^2]_4 = 0$$
$$[3,1^2]_5 = [0]$$

$$[4,1]_1 = [3,1] + [4]$$
$$[4,1]_2 = [2,1]$$
$$[4,1]_3 = [1^2]$$
$$[4,1]_4 = 0$$
$$[4,1]_5 = -[0]$$

$$[2^2,1]_1 = [2,1^2] + [2^2]$$
$$[2^2,1]_2 = -[1^3]$$
$$[2^2,1]_3 = -[2]$$
$$[2^2,1]_4 = [1]$$
$$[2^2,1]_5 = 0$$

$$[3,2]_1 = [2^2] + [3,1]$$
$$[3,2]_2 = [3]$$
$$[3,2]_3 = -[1^2]$$
$$[3,2]_4 = -[1]$$
$$[3,2]_5 = 0$$

$$[2,1^3]_1 = [1^4] + [2,1^2]$$
$$[2,1^3]_2 = -[2,1]$$
$$[2,1^3]_3 = [2]$$
$$[2,1^3]_4 = [-2,1^3] = -[0,-1,1^2] = 0$$
$$[2,1^3]_5 = [-3,1^3] = -[0,-2,1^2] = -[0]$$
$$[1^5]_1 = [1^4]$$
$$[1^5]_2 = -[1^3]$$
$$[1^5]_3 = [-2,1^4] + [1,-2,1^3] + [1^2,-2,1^2] = [1^2]$$
$$[1^5]_4 = [-3,1^4] + [1,-3,1^3] = -[1]$$

FIGURE 3. fig:fig Evaluating r-differentials $[\lambda]_r$, as in (2.26), up to rank-5 (omitting easy cases such as $[(n)]_m = [(n-m)]$ for some $n > 2$).

about regular boundary parts.

(**2.27**) We claim that $[-] \rightsquigarrow [-]_r$, $[-] \rightsquigarrow [-]_s$ are commutative linear operations on formal sums. We can thus extend this procedure (called "differentiation" in [1]) to allow multiple subscripts in the following way:

$$\begin{aligned}[\lambda]_{rs} &= [\lambda_1 - r, \lambda_2, \ldots, \lambda_m]_s + [\lambda_1, \lambda_2 - r, \ldots, \lambda_m]_s + \cdots + [\lambda_1, \lambda_2, \ldots, \lambda_m - r]_s \\ &= [\lambda_1 - s, \lambda_2, \ldots, \lambda_m]_r + [\lambda_1, \lambda_2 - s, \ldots, \lambda_m]_r + \cdots + [\lambda_1, \lambda_2, \ldots, \lambda_m - s]_r.\end{aligned}$$

In particular, we can define $[\lambda]_{1^{c_1}2^{c_2}\ldots}$ for any vector $c = (c_1, c_2, ...)$ of integers $c_i \geqslant 0$.

(**2.28**) We write

$$\begin{aligned}\Lambda^{\cdot} &= \{0, 1, 2, 1^2, 3, 21, 1^3, 4, 31, 2^2, 21^2, 1^4, ...\} \\ &= \{\tau^{(0,0,...)}, \tau^{(1,0,0,...)}, \tau^{(0,1,0,...)}, \tau^{(2,0,0,...)}, \tau^{(0,0,1,0,...)}, \tau^{(1,1,0,0,...)}, ...\}\end{aligned}$$

for the set of cycle structures $\tau^c = 1^{c_1}2^{c_2}\ldots$ — i.e. of partitions expressed in the 'exponential' notation.

(**2.29**) Consider the matrix representation of the operator r_c taking $[\lambda]$ to $[\lambda]_c$. The case r_1 starts:

$$\mathsf{r}_1 \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1^2 \\ 3 \\ 21 \\ 1^3 \\ 4 \\ 31 \\ 22 \\ 211 \\ 1^4 \\ \vdots \end{pmatrix} = \left(\begin{array}{c|c|cc|ccc|ccccc|c} 0 &&&&&&&&&&&& \\ \hline 1 & 0 &&&&&&&&&&& \\ \hline 0 & 1 & 0 &&&&&&&&&& \\ 0 & 1 & 0 & 0 &&&&&&&&& \\ \hline 0 & 0 & 1 & 0 & 0 &&&&&&&& \\ 0 & 0 & 1 & 1 & 0 & 0 &&&&&&& \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 &&&&&& \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 &&&&& \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 &&&& \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 &&& \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 && \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 &&&&&& \ddots \end{array}\right) \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1^2 \\ 3 \\ 21 \\ 1^3 \\ 4 \\ 31 \\ 22 \\ 211 \\ 1^4 \\ \vdots \end{pmatrix}$$

Such a block once-below-diagonal matrix can be represented by giving the non-zero blocks:

$$\mathsf{r}_1 \mapsto \left(\begin{array}{c} 0 \\ \hline 1 \end{array}\right) \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \oplus \ldots$$

Similarly we have a block twice-below-diagonal matrix

$$\mathsf{r}_2 \mapsto \left(\begin{array}{c} 0 \\ 0 \\ \hline 1 \\ -1 \end{array}\right) \oplus \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \oplus \ldots$$

$$\mathsf{r}_3 \mapsto \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \hline 1 \\ -1 \\ 1 \end{array}\right) \oplus \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \ldots$$

and so on. check me! and work out where signs go.

2.5. **On computation of differentials and Boerner-Murnaghan-Nakayama.**

Some rules:

{eq:nmm}
$$[(n)]_m = [(n-m)]. \tag{10}$$

For $\lambda \vdash n$, $[\lambda]_1$ coincides with the usual restriction rule to $S_{n-1} \hookrightarrow S_n$.
Proof: this is what (9) says once we note that if $\lambda_i = \lambda_{i+1}$ then $[..., \lambda_i - 1, \lambda_i, ...] = 0$ ...

Should probably write down the linear operator of r-differentiation for $r = 1$ ...

3. The Gamba theorem

{ss:theo}

First we will need some notation.

A 'long cycle' in $\mathfrak{S}_n$ is an element with the cycle structure given by $k = n^1$, such as $\omega = (12...n)$. We define a function which evaluates the square bracket characters (and their derivatives) on long cycles.

{de:ff}

Definition 2. (I) For $\lambda \in \Lambda$

$$\mathsf{f}(\lambda) = \begin{cases} (-1)^l & \text{if } \lambda = 1^l \text{ for some } l \\ 0 & \text{otherwise} \end{cases}$$

We extend this linearly to the domain $\mathbb{Z}[\Lambda]$ (of formal characters).

Examples: $\mathsf{f}((1)) = -1$; $\mathsf{f}((2)) = 0$; $\mathsf{f}((1^2)) = 1$; $\mathsf{f}((2) - (1^2)) = -1$; ...

{de:f}

(**3.1**) Define a function $\mathsf{f}(-)_- : \mathbb{Z}[\Lambda] \times \Lambda^{\cdot} \to \mathbb{Z}$ as follows. For $\lambda \in \Lambda$

$$\mathsf{f}(\lambda)_{1^{c_1}2^{c_2}\ldots} = \mathsf{f}([\lambda]_{1^{c_1}2^{c_2}\ldots}) \tag{11}$$

and again [needed?- (cf below)] extend linearly.

(**3.2**) Examples: $\mathsf{f}(4)_0 = \mathsf{f}(4) = 0$; $\mathsf{f}(4)_1 = \mathsf{f}(3) = 0$; $\mathsf{f}(4)_2 = \mathsf{f}(2) = 0$; $\mathsf{f}(4)_{1^2} = \mathsf{f}(3)_1 = \mathsf{f}(2) = 0$;
$\mathsf{f}(4)_3 = \mathsf{f}(1) = -1$;
$\mathsf{f}(4)_{21} = \mathsf{f}(4-2)_1 = \mathsf{f}(2-1) = -1$; ...

(**3.3**) For $\lambda = (\lambda_1, \lambda_2, ...)$ define

$$\lambda^- = (\lambda_2, \lambda_3, ...)$$

For example $(3,1)^- = (1)$ and more generally $(\lambda_1, 1)^- = (1)$.

[ideas: still not happy quite with the formulation of Thm. Want to capture somehow that output is fn depending only on λ^-, even though, in (some) use, c would not vary entirely freely, since it should 'match' λ,...... Perhaps its a fibre thing? A bit like later, only for row-1 instead of col-1?
We will have a 'Gamba' function that takes λ as input - really λ^- - and produces a polynomial in indeterminates $c_1, c_2, ...$ as output. ... Then the Thm is about how this polynomial gives various characters. Maybe call the function X^μ for $\lambda^- = \mu$? Then $\mathsf{X}^\mu \rightsquigarrow \chi^{\lambda_1\mu}$ for any $\lambda_1 \geqslant \mu_1$.]

(**3.4**) Fix a set $c_1, c_2, ...$ of indeterminates. The Gamba function is a function $\mathsf{X} : \Lambda \to \mathbb{Z}[c_1, c_2, ...]$ given by

$$\mathsf{X}(\mu) \;=\; \sum_{k\in\Lambda^{\cdot}} \mathsf{f}(\mu)_k \prod_{i=1}^{\infty} \binom{c_i}{k_i} \;=\; \sum_{\substack{k_1,\ldots,k_p\in\mathbb{Z}_{\geqslant 0}\\ \sum ik_i\leqslant p}} \mathsf{f}(\mu)_{1^{k_1}2^{k_2}\ldots p^{k_p}} \binom{c_1}{k_1}\cdots\binom{c_p}{k_p}.$$

where $p = |\mu|$.

The second form follows from the first simply because $\mathsf{f}(\mu)_k = 0$ if $|k| > |\mu|$, where $|k| = \sum_i ik_i$.

(**3.5**) Examples.

Note first that if $k = (0) = 1^0 2^0 3^0 ... \in \Lambda^{\cdot}$ then $\prod_{i=1}^{\infty}\binom{c_i}{k_i} = \prod_{i=1}^{\infty}\binom{c_i}{0} = 1$; while $k = (1) = 1^1 2^0 3^0 ...$ gives $\binom{c_1}{1}\binom{c_2}{0}\binom{c_3}{0}... = c_1$. Meanwhile $\mathsf{f}(0)_0 = 1$, $\mathsf{f}(1)_2 = 0$. Then $\mathsf{X}(0) = 1$,

$$\mathsf{X}(1) = \sum_{k\in\Lambda^{\cdot}} \mathsf{f}(1)_k \prod_{i=1}^{\infty}\binom{c_i}{k_i} = \mathsf{f}(1)_0 \;+\; \mathsf{f}(1)_1\, c_1 \;+ \mathsf{f}(1)_2\, c_2 + ... = \mathsf{f}(1)_0 \;+\; \mathsf{f}(1)_1\, c_1 \;+ 0 = -1 + c_1$$

$$\mathsf{X}(2) = \mathsf{f}(2)_0 \;+\; \mathsf{f}(2)_1\, c_1 \;+\; \mathsf{f}(2)_2\, c_2 \;+ \mathsf{f}(2)_{1^2}\binom{c_1}{2} + ... \;=\; 0 - c_1 + c_2 + \frac{c_1(c_1-1)}{2} + 0$$

$$\mathsf{X}(1^2) = \mathsf{f}(1^2)_0 \;+\; \mathsf{f}(1^2)_1\, c_1 \;+\; \mathsf{f}(1^2)_2\, c_2 \;+\; \mathsf{f}(1^2)_{1^2}\binom{c_1}{2} + ...$$

$$= 1 - c_1 + f[-1,1]c_2 + \frac{c_1(c_1-1)}{2} + 0 = 1 - c_1 - c_2 + \frac{c_1(c_1-1)}{2}$$

$$\mathsf{X}(3) = \mathsf{f}(3)_0 \;+\; \mathsf{f}(3)_1\, c_1 \;+\; \mathsf{f}(3)_2\, c_2 + \;\mathsf{f}(3)_{1^2}\binom{c_1}{2} + \;\mathsf{f}(3)_3\binom{c_3}{1} \;+\; \mathsf{f}(3)_{2,1}\, c_1 c_2 \;+\; \mathsf{f}(3)_{1^3}\binom{c_1}{3}$$
$$= 0 - 0.c_1 - c_2 - \frac{c_1(c_1-1)}{2} + c_3 + c_1c_2 + \frac{c_1(c_1-1)(c_1-2)}{6}$$

{thm:gamba} **Theorem 3** (Gamba [2])**.** *Let $\lambda \in \Lambda$ (and hence $\lambda^- \in \Lambda$). Let $p = n - \lambda_1 = |\lambda^-|$. Then the $\mathfrak{S}_n$ complex irreducible character χ^λ is given by*

$$\chi^\lambda_c \;=\; \mathsf{X}(\lambda^-)|_c$$

- that is, the λ character evaluates at class c by evaluating $\mathsf{X}(\lambda^-)$ at the c_i values given by class c. (Several examples follow.)

Need to settle notation so λ in $\mathsf{f}(\lambda)$ refers to below first row only, OR, fix usage later!!

(**3.6**) Remarks. Note from Def.3.1 and the first formulation that, in the sense that $\mathsf{X}(\lambda^-)$ is used, Theorem 3 does not depend on λ_1. No it doesn't. But it looks like it does in the second form, through the sum. Something to think about! The point is that the sum can be taken over all $k \in \mathbb{N}_0^{\mathbb{N}}$. By (2.10) the ones in the given range are simply a superset of the non-vanishing ones.

(**3.7**) We illustrate the Theorem with some examples. In the table 1 (or table 2) we compute, for each λ, the summands in the Gamba formula.

Here are some individual examples as a warm-up.

$$\chi_c^{(\lambda_1,1)} = \mathsf{X}(1) \;=\; \sum_{k\in\Lambda^{\cdot}} \mathsf{f}(1)_k \prod_{i=1}^{\infty}\binom{c_i}{k_i} = \mathsf{f}(1)_0 \;+\; \mathsf{f}(1)_1\, c_1 \;+\mathsf{f}(1)_2\, c_2 + ... = \mathsf{f}(1)_0 \;+\; \mathsf{f}(1)_1\, c_1 \;+0 = -1 + c_1$$
$$\chi_c^{(\lambda_1,2)} = \mathsf{f}(2)_0 \;+\; \mathsf{f}(2)_1\, c_1 \;+\; \mathsf{f}(2)_2\, c_2 \;+\mathsf{f}(2)_{1^2}\binom{c_1}{2} \;=\; 0 - c_1 + c_2 + \frac{c_1(c_1-1)}{2}$$
$$\chi_c^{(\lambda_1,1^2)} = \mathsf{f}(1^2)_0 \;+\; \mathsf{f}(1^2)_1\, c_1 \;+\; \mathsf{f}(1^2)_2\, c_2 \;+\; \mathsf{f}(1^2)_{1^2}\binom{c_1}{2}$$
$$= 1 - c_1 + f[-1,1]c_2 + \frac{c_1(c_1-1)}{2} = 1 - c_1 - c_2 + \frac{c_1(c_1-1)}{2}$$
$$\chi_c^{(\lambda_1,3)} = \mathsf{f}(3)_0 \;+\; \mathsf{f}(3)_1\, c_1 \;+\; \mathsf{f}(3)_2\, c_2 + \;\mathsf{f}(3)_{1^2}\binom{c_1}{2} + \;\mathsf{f}(3)_3\binom{c_3}{1} \;+\; \mathsf{f}(3)_{2,1}\, c_1 c_2 \;+\; \mathsf{f}(3)_{1^3}\binom{c_1}{3}$$
$$= 0 - 0.c_1 - c_2 - \frac{c_1(c_1-1)}{2} + c_3 + c_1c_2 + \frac{c_1(c_1-1)(c_1-2)}{6}$$

For example the first of these gives the $\lambda = (5,1)$ row of the $\mathfrak{S}_6$ character table (reproduced in (12) here). We have $\chi_6^{(5,1)} = -1+0 = -1$, $\chi_{51}^{(5,1)} = -1+1 = 0$, $\chi_{42}^{(5,1)} = -1+0 = -1$, $\chi_{41^2}^{(5,1)} = -1+2 = 1$, ..., $\chi_{21^4}^{(5,1)} = -1+4 = 3$, $\chi_{1^6}^{(5,1)} = -1+6 = 5$.

k	0	1	2	1^2	3	21	1^3	4	31	2^2	21^2	1^4	5	41	32	31^2	2^21	21^3	1^5
$\lambda \setminus \prod$	1	c_1	c_2	$\binom{c_1}{2}$	c_3	c_1c_2	$\binom{c_1}{3}$	c_4	c_1c_3	$\binom{c_2}{2}$	$\binom{c_1}{2}c_2$	$\binom{c_1}{4}$	c_5	c_1c_4	c_2c_3	$\binom{c_1}{2}c_3$	$c_1\binom{c_2}{2}$	$\binom{c_1}{3}c_2$	$\binom{c_1}{5}$
0	1																		
1	−1	1																	
2	0	−1	1	1															
1^2	1	−1	−1	1															
3	0	0	−1	−1	1	1	1												
2, 1	0	1	0	−2	−1	0	2												
1^3	−1	1	1	−1	1	−1	1												
4	0	0	0	0	−1	−1	−1	1	1	1	1	1							
3, 1	0	0	1	1	0	−1	−3	−1	0	−1	1	3							
2^2	0	0	−1	1	1	0	−2	0	−1	2	0	2							
$2, 1^2$	0	−1	0	2	0	1	−3	1	0	−1	−1	3							
1^4	1	−1	−1	1	−1	1	−1	−1	1	1	−1	1							
5	0	0	0	0	0	0	0	−1	−1	−1	−1	−1	1	1	1	1	1	1	1
41	0	0	0	0	1	1	1	0	−1	0	−2	−4	−1	0	−1	1	0	2	4
32	0	0	0	0	−1	0	2	1	1	−1	−1	−5	0	−1	1	−1	1	1	5
31^2	0	0	−1	−1	0	1	3	0	0	2	0	−6	1	0	0	0	−2	0	6
2^21	0	0	1	−1	0	−1	3	−1	1	−1	1	−5	0	1	−1	−1	1	−1	5
21^3	0	1	0	−2	0	−1	3	0	−1	0	2	−4	−1	0	1	1	0	−2	4
1^5	−1	1	1	−1	1	−1	1	1	−1	−1	1	−1	1	−1	−1	1	1	−1	1
6	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	−1
5, 1	0	0	0	0	0	0	0	1	1	1	1	1	0	−1	0	−2	−1	−3	−5
4, 2	0	0	0	0	0	0	0	−1	0	−1	1	3	1	1	0	0	−1	−3	−9
41^2	0	0	0	0	−1	−1	−1	0	1	0	2	4	0	0	1	−1	2	−2	−10
3^2	0	0	0	0	0	0	0	0	−1	2	0	2	0	1	−1	1	−1	−1	−5
321	0	0	0	0	1	0	−2	0	−1	0	0	8	−1	0	0	2	0	0	−16
2^3	0	0	0	0	−1	1	−1	1	0	−1	−1	3							
31^3	0	0	1	1	0	−1	−3	0	0	−2	0	6							
2^21^2	0	0	−1	1	0	1	−3	0	0	2	−2	6							
21^4	0	−1	0	2	0	1	−3	0	1	0	−2	4							
1^6	1	−1	−1	1	−1	1	−1	−1	1	1	−1	1	−1	1	1	−1	−1	1	−1
7	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
61	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
52	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	−1	1	0	2	4
51^2	0	0	0	0	0	0	0	−1	−1	−1	−1	−1	0	1	0	2	1	3	5
43	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	1	−1	1	1	5
421	0	0	0	0	0	0	0	1	0	1	−1	−3	0	−1	0	0	−1	3	15
3^21	0	0	0	0	0	0	0	0	1	−2	0	−2	0	0	0	−2	2	0	10
32^2	0	0	0	0	0	0	0	−1	0	1	1	−3	1	1	−1	−1	−1	−1	11
41^3	0	0	0	0	1	1	1	0	−1	0	−2	−4	0	0	−1	1	−2	2	10
321^2	0	0	0	0	−1	0	2	0	1	0	0	−8	0	0	1	−1	0	−2	20
2^31	0	0	0	0	1	−1	1	0	−1	0	2	−4	−1	1	0	0	1	−3	9
31^4	0	0	−1	−1	0	1	3	0	0	2	0	−6							
2^21^3	0	0	1	−1	0	−1	3	0	0	−2	2	−6							
21^5	0	1	0	−2	0	−1	3	0	−1	0	2	−4							
1^7	−1	1	1	−1	1	−1	1	1	−1	−1	1	−1							
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
71	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

TABLE 1. All summands in the Gamba formula for each λ up to (1^7) - tabulating the coefficients $\mathsf{f}(\lambda)_k$ rather than the full terms $\mathsf{f}(\lambda)_k \prod_k$. The first block of omitted entries are obtained by Cf. ordinary character tables. The second block by alt-stability: Sec.4.3!! For more commentary see also Sec.3.1. needs checking! tab:12

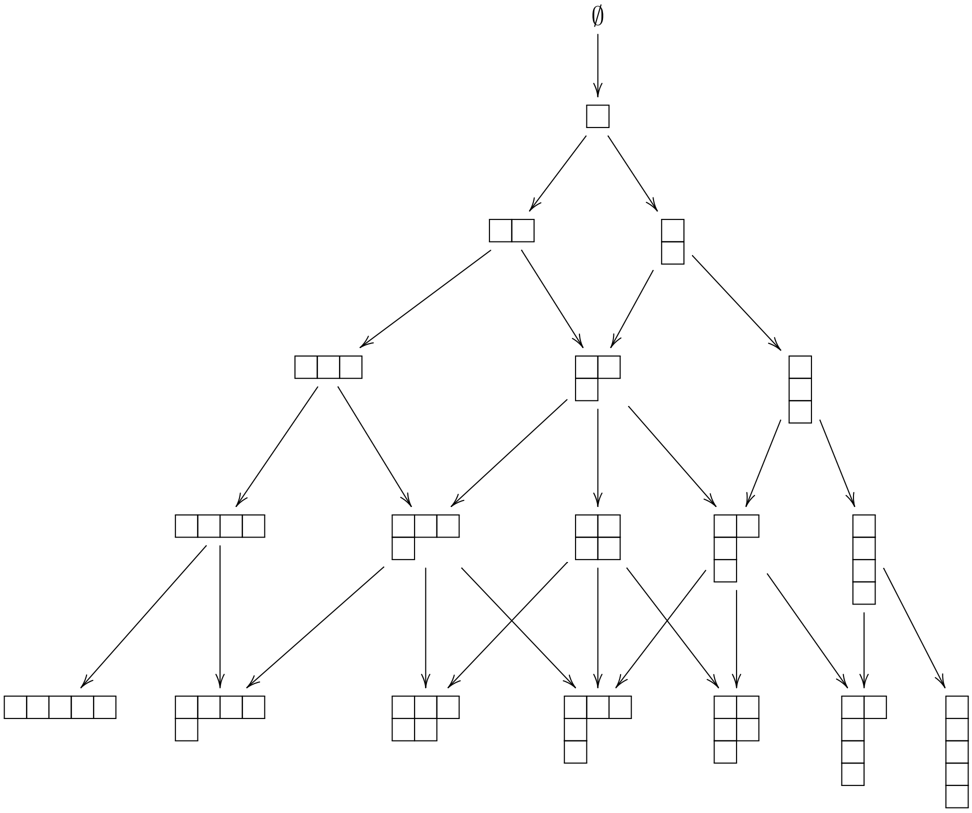


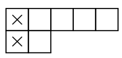
Figure 4. Start of the Young lattice. fig:younglattice

{ss:comment}

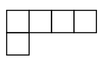
3.1. **On computing the table, Table 1.**

We will show in Sec.4 that most entries can be computed using the vanishing rules and alt-stability. Here we treat some examples not covered in this way.

(**3.8**) Consider $\mathsf{f}(52)(5)$. (And other cases with λ not ending in 1.) Note that for each k we need to appropriately tile



and that this is the same problem as tiling

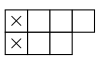
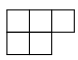


Thus

$$\mathsf{f}(52)(5) = \mathsf{f}(41)(5)$$

which is the 41 character. In particular consider $\mathsf{f}(52)_{1^5}$. This is given by the number of paths on the Young lattice, Fig.4, from 41 to $\emptyset$ (which is also the dimension of the 41 Specht module) which is 4.

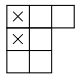
(**3.9**) Similarly for $\mathsf{f}(43)(5)$ we have

 which gives the same problem as tiling 

thus

$$\mathsf{f}(43)(5) = \mathsf{f}(32)(5)$$

(**3.10**) The remaining rank-7 case not of form $\lambda 1$ is 322. Thus $\mathsf{f}(322)(5)$ gives



and must be computed slightly differently. We have

$$\mathsf{f}(322)(5) = \mathsf{f}(31^2)(5) + \mathsf{f}(221)(5)$$

3.2. **A few more pump-priming/bootstrap calculations.**

Here let us recall the standard character table for S_6. On the one hand we can use it as input for the relevant fibre. And it is gives the content of the first block for which there is insufficient room in Table 1.

$\lambda \setminus \mu$	6	51	42	411	33	321	222	3111	2211	21^4	1^6
6	1	1	1	1	1	1	1	1	1	1	1
51	−1	0	−1	1	−1	0	−1	2	1	3	5
42	0	−1	1	−1	0	0	3	0	1	3	9
411	1	0	0	0	1	−1	−2	1	−2	2	10
33	0	0	−1	−1	2	1	−3	−1	1	1	5
321	0	1	0	0	−2	0	0	−2	0	0	16
222	0	0	−1	1	2	−1	3	−1	1	−1	5
3111	−1	0	0	0	1	1	2	1	−2	−2	10
2211	0	−1	1	1	0	0	−3	0	1	−3	9
21^4	1	0	−1	−1	−1	0	1	2	1	−3	5
1^6	−1	1	1	−1	1	−1	1	−1	1	−1	1

(12) {eq:chi6-11}

But we can also use the previous entries in the $\mathsf{f}(-)_-$ table, Table 1, to compute it, as follows. (In principle we can also use Murnaghan-Nakayama, as in Sec.A.1. But of course this eventually becomes hard.)

For $\lambda = 2^3$ (or indeed $\lambda = \lambda_1 2^2$ for $\lambda_1 \geqslant 2$, so $\lambda^- = 2^2$) the Gamba formula gives:

$$\chi_c^{2^3} = \sum_k \mathsf{f}(2^2)_k \prod_i \binom{c_i}{k_i} = -c_2 + \binom{c_1}{2} + c_3 - 2\binom{c_1}{3} - c_1 c_3 + 2\binom{c_2}{2} + 2\binom{c_1}{4}$$

(reading directly from the Table 1) so in particular

$$\chi_6^{2^3} = 0, \qquad \chi_{51}^{2^3} = 0, \qquad \chi_{42}^{2^3} = -1, \qquad \chi_{411}^{2^3} = 1, \qquad \chi_{321}^{2^3} = -1+1-1 = -1,$$

$$\chi_{33}^{2^3} = 2, \qquad \chi_{222}^{2^3} = -3{+}6 = 3, \qquad \chi_{2211}^{2^3} = -2{+}1{+}2 = 1, \qquad \chi_{3111}^{2^3} = 3{+}1{-}2{-}3 = -1,$$

$$\chi_{21111}^{2^3} = -1+6-8+2 = -1, \qquad \chi_{1^6}^{2^3} = +15-40+30 = 5$$

...

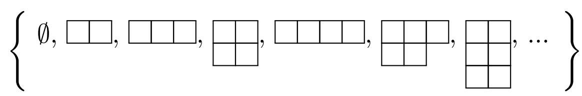
4. General rules

{ss:meo}

Firstly some notation.

(**4.1**) Write $\Lambda_{||} \subset \Lambda$ for the subset of partitions in which (regarded as Young diagrams) the first two columns have the same length.

$\Lambda_{||} =$



(**4.2**) We write $r^|(\lambda)$ or $\lambda^|$ for $\lambda \in \Lambda$ (viewed as a Young diagram) with the first *column* removed. For example $(3,2)^| = (2,1)$.

(**4.3**) Define $a^| : \Lambda \to \Lambda$ by $\lambda \mapsto \lambda_1 \lambda$. Observe that this is injective and $a^|(\Lambda) = \Lambda_{||}$. Furthermore {de:a}

$$r^|(a^|(\lambda)) = \lambda.$$

(**4.4**) The $r^{|}$-fibre *over* $\lambda \in \Lambda$ is $\mathsf{F}(\lambda) = r^{|^{-1}}(\lambda)$. For example

$$\mathsf{F}(\emptyset) = \left\{ \emptyset, \ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1,1}, \dots \right\}$$

(this fibre is unique in containing its argument);

{eq:FF1}
$$\mathsf{F}((1)) = r^{|^{-1}}((1)) = \{(2,1),(2,1^2),(2,1^3),...\} \quad = \quad \mathsf{F}(\ydiagram{1}) = \left\{ \ydiagram{2}, \ydiagram{2,1}, \ydiagram{2,1,1}, \dots \right\} \tag{13}$$

$$\mathsf{F}((2,1)) = \; r^{|^{-1}}((2,1)) = \{(3,2),(3,2,1),(3,2,1^2),(3,2,1^3),...\} = \left\{ \ydiagram{3,2}, \ydiagram{3,2,1}, \ydiagram{3,2,1,1}, \dots \right\} \tag{14}$$
{eq:FF21}

and in general
$$\mathsf{F}(\mu) = \{\mu_1\mu, (\mu_1+1)\mu, (\mu_1+2)\mu, ...\}$$
Note that every fibre is infinite, but in natural bijection with $\mathbb{N}_0$. The rank of the first element of $\mathsf{F}(\mu)$ is given by $\mu_1\mu \vdash |\mu| + \mu_1$ and each subsequent partition has rank increased by 1.

Thus Λ is partitioned into fibres. So every integer partition is contained in some fibre. But NB the fibre over λ does not contain λ in general. For example the fibre containing $(2,1)$ is $\mathsf{F}((1))$ as in (13).

The '*sub*fibre *of* λ' is the subset of integer partitions μ in the fibre containing λ such that $\mu \supseteq \lambda$. For example the subfibre of $(3,2,1)$ is $\{(3,2,1),(3,2,1^2),(3,2,1^3),...\}$. (Thus 'subfibre of' subsets of Λ are not the same as the subsets in the fibre partition.)

(**4.5**) By (4.3) a transversal of the $r^{|}$-fibre partition of Λ is $\Lambda_{||}$.

(**4.6**) We will write $\mathsf{f}(\lambda)_k(m)$, or simply $\mathsf{f}(\lambda)(m)$, to denote the whole subvector of $\mathsf{f}(\lambda)$ of components $\mathsf{f}(\lambda)_k$ with $|k| = m$. For example
$$\mathsf{f}(3)_k(3) = \mathsf{f}(3)(3) = (\mathsf{f}(3)_3, \mathsf{f}(3)_{21}, \mathsf{f}(3)_{1^3}) = (1,1,1)$$

The next bits are for the moment somewhat out of order, or anachronisms. But should be more or less readable.

4.1. **Vanishing rules.**

{rule:1} (**4.7**) Let $k \in \Lambda^{\cdot}$ and $\lambda \in \Lambda$. If $|k| < |\lambda^{|}|$ then $\mathsf{f}(\lambda)_k = 0$.

Proof. Write $\mu \in [\lambda]_k$ to mean that for $\mu \in \Lambda$, $[\mu]$ has non-zero coefficient in the differential. Then

{eq:11}
$$\mu \in [\lambda]_k \;\Rightarrow\; |\mu| = |\lambda| - |k|. \tag{15}$$
by the construction and (8); and

{eq:22}
$$\mu \in [\lambda]_k \;\Rightarrow\; \mu \subseteq \lambda \tag{16}$$
since every row i is either left alone, or shorter, or becomes $\lambda_{i+1} - 1$, where $\lambda_{i+1} \leqslant \lambda_i$. By (15), if $\mu = 1^l$ for some l (the condition for $\mathsf{f}(\mu)$ to be *non*-zero) then $l = |\lambda| - |k|$. But then $l = \mu_1^t \leqslant \lambda_1^t$ by (16) so $|\lambda| - |k| \leqslant \lambda_1^t$, so $|\lambda| - \lambda_1^t \leqslant |k|$, so $|\lambda^{|}| \leqslant |k|$. □

OLD: If the condition applies then no box deletions by differentiation can prevent a row with two boxes. There could be a shorter row before a 2-box row in the unstraightened version, but then there must be another 2-or-more-box row. □

Applications: $\mathsf{f}(\lambda)_k = 0$ unless $\lambda^{|}$ is a partition of order $\leqslant |k|$. That is:
$\mathsf{f}(\lambda)_0 = 0$ unless λ of form 1^n (i.e. $\lambda^{|} = 0$).
$\mathsf{f}(\lambda)_1 = 0$ unless λ of form 1^n or 21^{n-2} (i.e. $\lambda^{|} = 0$ or (1)).
$\mathsf{f}(\lambda)_2 = 0$ unless λ of form 1^n or 21^{n-2} or 2^21^n or 31^n (i.e. $\lambda^{|} = 0$, (1), (2) or 1^2).
$\mathsf{f}(\lambda)_{1^2} = 0$ unless λ of form 1^n or 21^{n-2} or 2^21^n or 31^n (i.e. $\lambda^{|} = 0$, (1), (2) or 1^2).

{rule:-1} (**4.8**) If $|k| > |\lambda|$ then $\mathsf{f}(\lambda)_k = 0$.
Proof. Consider the definition (3.1). By (2.26) every summand $[\mu]$ of $[\lambda]_k$ has $|\mu| = |\lambda| - |k|$. Now see (2.10). □

4.2. **Alt-stability/magic Murnaghan–Nakayama-tree.**

In this section we prove a Theorem that identifies the finitely many entries in each column of the $\mathsf{f}(\lambda)_k$ array that must be computed before the remainder are determined by our rules ('alt-stability' and vanishing rules).

(**4.9**) Consider $\lambda \in \Lambda$. Then $\lambda - le_i$ is regular when it is possible to remove a rim (as in (2.23)) of length l from λ starting in row i; and, if it is possible, then the orbit representative, denoted $\overline{\lambda - le_i}$, is the partition λ with the rim removed. The sign of $[\lambda - le_i] = \pm[\overline{\lambda - le_i}]$ is $(-1)^{d-1}$ where d is the number of rows visited by the rim. {pa:rim}

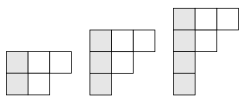
Proof. Note from Sec.2.2 that if $\lambda - le_i$ is regular then this rim removal coincides with 'straightening'; while otherwise the straightening will terminate prematurely with a fixed pair of rows. □

(**4.10**) Consider $\lambda \in \Lambda$ and $k \in \Lambda^{\cdot}$. Note that non-zero contributions to $\mathsf{f}(\lambda)_k$ occur in summands of the differential of form 1^x where {pa:rom}

$$x = |\lambda| - |k|. \tag{17}$$

(**4.11**) Combining these observations (4.9) and (4.10), note that we can determine $\mathsf{f}(\lambda)_k$ by looking at certain 'tilings' of the skew $\lambda \setminus 1^x$ by sets of rims of length given by k.

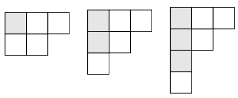
(**4.12**) For example, consider some cases with λ in the fibre $F((2,1))$. If $|k| = 3$ then we tile the unshaded regions shown here in each case:



...; (18) {eq:k=3}

Note that in this sequence the shape to be tiled is actually fixed. Note also for example that $k = 1^1 2^1$ gives zero in all these cases.

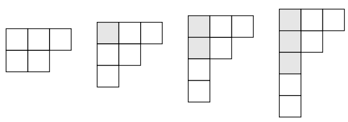
Staying with the fibre $F((2,1))$, for $|k| = 4$ the regions to be tiled are:



... (19) {eq:k=4}

Here the skew to be tiled varies at first, but then settles down to a fixed pair of disconnected component shapes. In the first of the $|k| = 4$ cases we can remove a rim of any length 1,2,3,4 from the skew, and all the $\mathsf{f}(32)_k$ ($|k| = 4$) are non-zero. In the second, we cannot remove a rim of length 4, or of length 2, so several $\mathsf{f}(321)_k$ are zero. Note in particular that the skew is not connected, so we can treat the combinatorics of the two connected components essentially separately, up to a choice of how to partition the cycle structure. In the third case the connected components are similar to the second case, except for a relative shift. We deduce $\mathsf{f}(3211)_k = -\mathsf{f}(321)_k$ (NB $|k| = 4$ here), and indeed $\mathsf{f}(3211^i)_k = (-1)^i \mathsf{f}(321)_k$.

Next consider $|k| = 5$. For $F((2,1))$ we have:

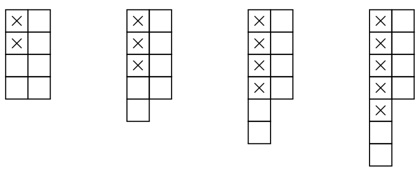


... (20) {eq:k=5}

Here we note that the skew is connected for 32 and 321, but not for 321^i with $i > 1$. Thus $\mathsf{f}(32)_k \neq \mathsf{f}(321)_k \neq \mathsf{f}(3211)_k$ in general. But after that the connected components are similar among the 321^i so we have $\mathsf{f}(321^2 1^i)_k = (-1)^i \mathsf{f}(321^2)_k$.

How exactly does the disconnected combinatorics work out? some kind of product??

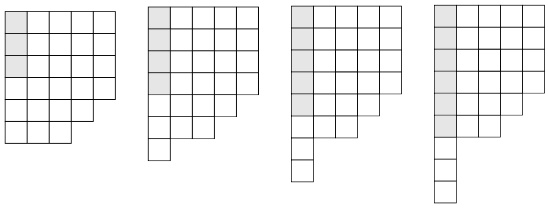
(**4.13**) Example: Set $|k| = 6$ and conside the $r^{|}$-fibre over 1^4. The first case is $\lambda = 2^4$ with $x = 2$:



...

-go ahead and work this out please man!! ...

(**4.14**) Example: Consider some (or all) k with $|k| = 24$. Consider $\lambda^{|} \in \Lambda^{24}$ [-fix me! in some sense it is ok. just needs to be big enough.] given by $\lambda^{|} = (4^4 32)$, with $|\lambda^{|}| = 21$. The first weight in this fibre is $\lambda = (5^4 43)$ with $|\lambda| = 27$. The first few weights in the fibre are



where we have drawn $\lambda \setminus 1^x$ for this $|k|$. (Note that the final case is determined by ...; and that subsequent ones are determined by this one and (21).)

(**4.15**) Starting to restate the bit below as a theorem...

(**4.16**) Theorem. *In each k-column of the array $\mathsf{f}(\lambda)_k$ (with column entries indexed by Λ; and $k \in \Lambda^{\cdot}$ given) - i.e. each column as in Table 1, all entries are determined (using our formula (21) below and vanishing rules) given only finitely many initial entries. Indeed this number of initial entries is bounded by $\frac{|k|}{2}.|\Lambda^k|$.*

In the 'k-column' for given k: (we have infinitely many infinite $r^{|}$-fibres, since we have all $\lambda \in \Lambda$ but) we have only $|\Lambda^k|$ infinite fibres (i.e. finitely many) with $\mathsf{f}(\lambda)_k \neq 0$ somewhere in the fibre (indeed it is the fibres over Λ^k); and the maximum number of fibre members for fibre $F(\lambda)$ before 'alt-stability' is λ_1^t (i.e. finitely many). Indeed this number is $\leqslant \frac{|k|}{2}$. Subsequent entries in the fibre are determined by the 'alt-stability formula'

{eq:alts} $$\mathsf{f}(\lambda 1)_k = -\mathsf{f}(\lambda)_k \tag{21}$$

For the Theorem: Consider the array $\{\mathsf{f}(\lambda)_k\}_{\lambda \in \Lambda, k \in \Lambda^{\cdot}}$. We organise row index set Λ into $r^{|}$-fibres $F(\mu)$, so these subsets can be labelled by $\lambda \in \Lambda_{||}$ or by $a^{|}(\lambda)$, $\lambda \in \Lambda$. Then $r^{|}(a^{|}(\lambda)) = \lambda \in \Lambda$ so $\mathsf{f}(\lambda)_k$ entire subset zero if $|\lambda| > |k|$. So for fixed k all but finitely many subsets have $\mathsf{f}(F(\lambda))_k = \{0\}$. Specifically $\mathsf{f}(F(\mu))_k = \{0\}$ if $|\mu| > |k|$. So $\mathsf{f}(F(\mu))_k = \{0\}$ unless $\mu \in \Lambda^k = \cup_{i \leqslant k} \Lambda_i$
Every fibre $F(\mu)$ is infinite, but in natural bijection with $\mathbb{N}_0$

$$F(\mu) = \{\mu_1 \mu, (\mu_1 + 1)\mu, (\mu_1 + 2)\mu, ...\}$$

and every term after the jth term (with $j = ...$) is determined by the jth term.

What is being restated/proof:

Note that if we fix k, since the vanishing rule (4.7) bounds the order of the skew to the right of the first column, since the minimum order needed to 'connect' the parts to the right of and below the 1^x grows with x, we always have, for $\mathsf{f}(\lambda)_k \neq 0$, a disconnection into two components for sufficiently large λ. We may thus form a fibre of case where the right component is the same, and the below component is the same up to a shift. Thus for every shape of right component we have a fibre where

$$\mathsf{f}(\lambda 1)_k = -\mathsf{f}(\lambda)_k.$$

Let us call this *column stability*. It is routine to determine which λs occur.

So the next question is: which cases do we actually need to compute? I.e. which skews $\lambda \setminus 1^x$ do not fall into two parts?

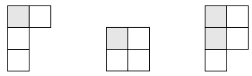
Examples: for $|k| = 4$ the largest 'new case' is $\mathsf{f}(2^4)_k$. All the others are already in our table. (NB the lower rank cases have already been done in 4.4.)

Indeed the last new case in each rank is $2^{|k|}$. In fact these cases are not completely new. The skew is just a shift of an earlier skew, so there is nothing new to compute.

So, what is left to compute that presents an interesting challenge!?...

The first case nominally not determined by 4.17 and 4.18 is $\mathsf{f}(22)_k(2)$. This is of the type $2^{|k|}$ just discussed, so in fact it is determined.

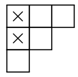
Next is $\mathsf{f}(221)(3)$. Here is one little illustration:



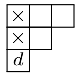
Ignore 211 for a moment, then the next two are the nominal start of the 221^i sequence, with the first being determined by (4.18), except of course that these are not similar (the skew $22 \setminus 1$ is connected), so 221 is the start of the stable sequence. Nominally we have to compute this. However the connected components are similar to those for $\mathsf{f}(211)_k(3)$, which *are* determined by (4.18).

Next is $\mathsf{f}(2^3)(3)$, but we already dealt with this form. So next is $\mathsf{f}(42)(4)$. The calculations for this are the same as $\mathsf{f}(31)(4)$; and $\mathsf{f}(3^2)_{(}4) = \mathsf{f}(2^2)_{(}4)$. So next we have $\mathsf{f}(321)(4)$. This appears to be the first one that actually needs to be calculated.

The picture here is



Clearly there is no rim of length 4 or 2; and precisely one of length 3; leaving only the 1^4 case to compute. There are 4 ways to choose which length-1 'rim' goes into the isolated box:



and for each of these then two ways of completing. Altogether:

$$\begin{aligned}\mathsf{f}(321)(4) &= (\mathsf{f}(321)_4, \mathsf{f}(321)_{31}, \mathsf{f}(321)_{22}, \mathsf{f}(321)_{211}, \mathsf{f}(321)_{1^4}) \\ &= \;\; (0, \qquad -1, \qquad 0, \qquad 0, \qquad 8)\end{aligned}$$

4.3. **Block-diagonal rules / Checking.** {ss:ch}

(**4.17**) We claim that when the order of the differential k equals $p = |\lambda|$ then the coefficient $\mathsf{f}(\lambda)_k$ is the corresponding character χ^λ_k. (NB *not* $\chi^{(n-p,\lambda)}_k$, which is undefined.) {rule0}
Proof: Compare the definition with the Murnaghan-Nakayama rule (as recalled in SSA.1).

(**4.18**) When the order of the differential k is $p-1$ then $\mathsf{f}(\lambda)_k = -\chi^\lambda_{1k}$, where $1k$ means k with the exponent in 1^{k_1} increased by 1. {rule1}
Proof: Any surviving differential summand in this calculation is the partition (1). Applying the $r = 1$-differential to this completes the character calculation as above, and $\mathsf{f}(1)_0 = -1$.

(**4.19**) Something interesting is going on in the other blocks too. :-)

...

{ss:stab}

4.4. Stability rules.

(**4.20**) If $|k| = |\lambda^{|}|$ then ...

We do not yet have a clean statement of the general 'stability' result, so there follow some illuminating notes and excursions.

(Recall from (4.17) and (4.18) that if $|k| = |\lambda|$ or $|k| = |\lambda| - 1$ then $\mathsf{f}(\lambda)_k$ is known. Thus in particular the block under each diagonal block in our table is known. For the row of λ in the diagonal there is a row of $\lambda 1$ in the block below. But these are not directly 'related' — by our rules they come from characters of different symmetric groups. On the other hand, we have the following.)

(**4.21**) Claim:
If $|k| < 2$ then all is known.
If $|k| = 2$ and $|\lambda| > 3$ then $\mathsf{f}(\lambda 1)_k = -\mathsf{f}(\lambda)_k$. Indeed for $|\lambda| > 4$ every non-zero entry is obtained as an image of a lower case in this way. I.e. every non-zero entry has λ of form $\lambda = \mu 1^l$ with $\mu \in \{31, 2^2, 21^2, 1^4\} \subset \Lambda_4$. Some of these are already 'derived' from the subdiagonal block above by $\lambda \rightsquigarrow \lambda 1$. The exception is 2^2, which obviously has no preimage.

If $|k| = 3$ and $|\lambda| > 4$ then

$$\mathsf{f}(\lambda 1)_k = -\mathsf{f}(\lambda)_k.$$

The non-zero entries are again of form $\lambda = \mu 1^l$ with $\mu \in \{41^2, 321, 2^3, 31^3, 2^2 1^2, 21^4, 1^6\} \subset \Lambda_6$. Here note that 221 is not given by the formula, even though it has a formal preimage. And 2^3 is the last 'new' non-zero case to appear.

If $|k| = 4$ ...things seem to be heading in an analogous direction... NB 2221 is new even though it has preimage. Non-zero seed set might include $\{32^2\}$.

With certain exceptions, if $|k| < |\lambda|$ then $\mathsf{f}(\lambda 1)_k = -\mathsf{f}(\lambda)_k$.
NB False in general! Consider
$[22]_3 \rightsquigarrow 1$ and $[221]_3 \rightsquigarrow 0$, and also unequal for $k = 21, 111$.
$[32]_4 \rightsquigarrow 1$ and $[321]_4 \rightsquigarrow 0$.
$[221]_4 \rightsquigarrow -1$ and $[2211]_4 \rightsquigarrow 0$.
$[222]_4 \rightsquigarrow 1$ and $[2221]_4 \rightsquigarrow 0$.
So... why working sometimes?...

Proof. True? LOL. Ideas... Cf. $[\lambda]_k$ and $[\lambda 1]_k$. First in particular consider $k = (l)$ for $l = 1, 2, 3, ...$:

$$[\lambda]_{k=(l)} = \sum_i \lambda - le_i = \sum_{i=1}^{\text{depth}(\lambda)} \lambda - le_i$$

and hence consider straightening $\lambda - le_i$ cf. $\lambda 1 - le_i$...

$$[\lambda 1]_{k=(l)} = \sum_{i=1}^{\text{depth}(\lambda)+1} \lambda 1 - le_i$$

Remark: something funny here! The formal case $l = 0$, if understood as $[\lambda]_0 = [\lambda]$, is elementary, since $\lambda^{|} = (\lambda 1)^{|}$, $\mathsf{f}(\lambda) = 0$ unless $\lambda^{|} = 0$, and hence $\mathsf{f}(\lambda 1) = -\mathsf{f}(\lambda)$.

The case $l = 1$ can be checked by brute force.

In general there are 4 cases to consider in comparing singularity of λ with that of $\lambda 1$. In all cases, as we apply the straightening process, the row that is the image of the ith row under the sequence of reflections moves to the right, while the row that 'moves up' under reflection moves, relatively speaking, to the left. The key step comes when these two motions 'meet'. Examples:

...

Case 1a: $\lambda - le_i$ singular with 'meet' of form $\overline{\lambda - le_i} = ... - 1(0)$
In this case $\lambda 1$ gives $... - 1, 1$, which is amenable to one more straightening giving ...00, which is regular.

Example: $[111]_2 = ((111) - 2e_1) + ((111) - 2e_2) + ((111) - 2e_3)$ is NOT AN EXAMPLE!..., so $\mathsf{f}(111)_2 = 1$; meanwhile $[1111]_2 = ((1111) - 2e_1) + ((1111) - 2e_2) + () + ()$.

Case 1b: $\lambda - le_i$ singular not of form $\overline{\lambda - le_i} = ... - 1(0)$ — Here we claim $\lambda 1 - le_i$ is also singular. If we reach a 'meet' of form $... - 2(0)$ or indeed $... - m(0)$ with $m > 1$, then the $\lambda 1$ version is $... - m1$ and both are singular.
If we reach $..., +m, +m+1, ...$ then the $\lambda 1$ version is $..., +m, +m+1, ..., 1$ and both are singular.

Case 2a: $\lambda - le_i$ not singular and $\overline{\lambda - le_i}$ shorter than λ.

Case 2b: ...

In case 1a λ is singular but $\lambda 1$ is not.

Draw up my nice pictures!...

5. Towards applications: Orthogonality and composition multiplicities {ss:leo}

Here we show briefly how to apply our results. (Here we only verify some known results, but the method illustrates the use of our new results.)

Consider the class of representations of symmetric groups obtained by fixing a space V, of dimension Q say, and then allowing S_n to act on V^n by permuting tensor factors.

We can compute characters for these representations for all n simultaneously. Hence in principle we can compute composition factors for all n simultaneously by computing orthogonality formulae with n as a variable. The character formula for these representations is

$$\chi(\sigma_{l_1}\sigma_{l_2}...\sigma_{l_j}) = Q^{n-j}$$

for $l_1 < l_2 < ... < l_j$ (see e.g. [?]). (Note that every class has an element of this form.)

Our first task is to set up the problem in a variable-n way.

Rather than use the full orthogonality theorems we will use the fact that the multiplicities are known, and simply verify some of the corresponding identities. For example, the dimension identity for $Q = 2$ is

$$2^n = \sum_{i=0}^{n/2} \chi_{n,0,0,...,0}^{n-i,i} . (n + 1 - 2i).$$

To verify this note the following. Firstly by (10) we have

$$\chi_{n,0,0,...,0}^{n-i,i} = \binom{n}{i} - \binom{n}{i-1}$$

and hence

$$\sum_{i=0}^{m} \chi_{n,0,0,...,0}^{n-i,i} = \binom{n}{m}$$

Consider $Q = 2$

... ...

5.1. **Checking.**

4. Promote 'checking' stuff to here.

6. More examples

Example 4. We will calculate the character polynomial for $\chi^{(\lambda_1,3,1,1)}$. We must therefore determine $[3,1,1]_{1^{k_1}2^{k_2}3^{k_3}4^{k_4}5^{k_5}}$ where $\sum_i ik_i \leqslant 5$. The cases for $k=(k_1,k_2,...)$ are

$$\{(),(1),(2),(0,1),(3),(1,1),(0,0,1),...,(5),(3,1),(1,2),(2,0,1),(0,1,1),(1,0,0,1),(0,0,0,0,1)\}$$

A few of these are given below. Observe, from the rim rule (4.9) or otherwise, that $[311]_3 = \mathsf{r}_3[311\rangle = 0$ and $[311]_4 = 0$.

$$\begin{aligned}
[3,1,1]_1 &= [2,1,1]+[3,0,1]+[3,1,0] \\
&= [2,1,1]+[3,1], \\
[3,1,1]_2 &= [1,1,1]+[3,-1,1]+[3,1,-1] \\
&= [1,1,1]-[3] \\
[3,1,1]_{1^2} &= [2,1,1]_1+[3,1]_1 \qquad = [1,1,1]+[2,0,1]+[2,1]+[2,1]+[3] \\
&= [1,1,1]+2[2,1]+[3] \\
[3,1,1]_3 &= [0,1,1]+[3,-2,1]+[3,1,-2] \\
&= 0, \\
[3,1,1]_{12} &= [1,1,1]_1-[3]_1 \qquad = [1,1]+[1,0,1]+[0,1,1]-[2], \\
&= [1,1]-[2], \\
[3,1,1]_{1^3} &= [2,1,1]_{1^2}+[3,1]_{1^2} \text{ (using the } [311]_1 \text{ above)} \\
&= [1,1,1]_1+[2,0,1]_1+2[2,1,0]_1+[3]_1 \text{ (or using the } [311]_{11} \text{ above)} \\
&= [0,1,1]+[1,0,1]+[1,1,0]+2[1,1]+2[2]+[2] \\
&= 3[1,1]+3[2], \\
[3,1,1]_4 &= 0 \\
[3,1,1]_{31} &= 0 \\
[3,1,1]_{22} &= [111]_2-[3]_2 = -[1]-[1] \\
[3,1,1]_{211} &= [1]-[1] = 0 \\
[3,1,1]_{1^4} &= 6[1] \\
[3,1,1]_{1^12^2} &= [2,1,1]_{2^2}+[3,1]_{2^2} \\
&= [0,1,1]_2+[2,-1,1]_2+[2,1,-1]_2+[1,1]_2+[3,-1]_2 \\
&= -[2,0,0]_2+[1,1]_2 \\
&= -[0]+[-1,1]+[1,-1] \\
&= -2[0].
\end{aligned}$$

Specifically for $\langle 1^2]\mathsf{r}_1\mathsf{r}_1\mathsf{r}_1[311\rangle$, the coefficient of $[1^2]$ in $[311]_{1^3}$, we have contributions as indicated by:

$$\begin{array}{|c|c|c|}\hline \ & b & a \\ \hline \ & \multicolumn{2}{c}{} \\ \cline{1-1} c & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline \ & c & a \\ \hline \ & \multicolumn{2}{c}{} \\ \cline{1-1} b & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline \ & c & b \\ \hline \ & \multicolumn{2}{c}{} \\ \cline{1-1} a & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

From the above, then, $\mathsf{f}(3,1,1)_1 = 0$, $\mathsf{f}(3,1,1)_{1^3} = 3$ and $f(\lambda_1,3,1,1)(1^12^2) = -2$. Completing the rest gives

$$\chi_c^{(\lambda_1,3,1,1)} = -\binom{c_1}{2} + 3\binom{c_1}{3} - 6\binom{c_1}{4} + 6\binom{c_1}{5} + c_1c_2 - 2c_1\binom{c_2}{2} - c_2 + 2\binom{c_2}{2} + c_5.$$

In particular, this shows that the character polynomial can always be written in the required form for use in Hamermesh's method.

Appendix A. Appendix

{ss:MN}

A.1. **Classical Murnaghan–Nakayama examples.**

Here, purely as a warm-up exercise, we recall some examples of the classical Murnaghan-Nakayama rule which can be used to compute the character for a *given* representation.

(**A.1**) For p a set of rims (or regular/removable rim parts) - then l_p is the total number of vertical steps in all these rims.

For example if p is a set of single row rims then $l_p = 0$. Meanwhile for $p = \{21, 31\}$ (in the obvious shorthand) then $l_p = 1 + 1 = 2$.

(**A.2**) Consider partitions $\lambda \supset \mu \in \Lambda$ and hence skew $\lambda \setminus \mu$ of degree $m = |\lambda| - |\mu|$. Consider cycle structure $k = 1^{k_1} 2^{k_2} ... \in \Lambda^{\cdot}_m$, and $a = a_1 a_2 ...$ a perm of the symbol sequence corresponding to k - with k_1 1's; k_2 2's and so on. Then $P_a^{\lambda\setminus\mu}$ is the set of ways of first deleting a regular rim part (as in ..., here just called a rim) of length a_1 from $\lambda \setminus \mu$; then a_2 from the remnant and so on. {de:PPa}

(**A.3**) Example.

$$P_{32}^{321\setminus 1} = \left\{ \begin{array}{|c|c|c|}\hline \times & a_2 & a_1 \\ \hline b_1 & a_3 \\ \cline{1-2} b_2 \\ \cline{1-1}\end{array} \;,\qquad \begin{array}{|c|c|c|}\hline \times & b_2 & b_1 \\ \hline a_2 & a_1 \\ \cline{1-2} a_3 \\ \cline{1-1}\end{array} \right\}, \qquad\qquad P_{23}^{321\setminus 1} = \emptyset$$

- here the a_is indicate a single rim; and then the b_is indicate a rim for what remains.

(**A.4**) This is related to the r-differential calculus. Consider the coefficient of $\mu = (1) = \square$ in $[\lambda]_k = [321]_{32}$. We have

$$[321]_2 = [0, 2, 1] + [3, -1, 1] = -[1, 1, 1] - [3]$$

so

$$[3, 2, 1]_{32} = -[1, 1, 1]_2 - [3]_2 = -([-1, 1, 1] + [1, -1, 1]) - [1] = [1] - [1] = 0$$

Meanwhile

$$[3, 2, 1]_2 = [1, 2, 1] + [3, 0, 1] = 0, \qquad \text{so} \qquad [3, 2, 1]_{23} = 0$$

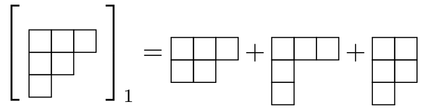
- omitting some terms that are clearly zero.

(**A.5**) The *Murnaghan–Nakayama rule* says that the character χ^λ, for $\lambda \in \Lambda_n$ say, evaluated on cycle structure $k = 1^{k_1} 2^{k_2} ... \in \Lambda^{\cdot}_n$ is

$$\chi_k^\lambda = \sum_{p \in P_a^{\lambda\setminus\emptyset}} (-1)^{l_p} \tag{22}$$ {eq:MN}

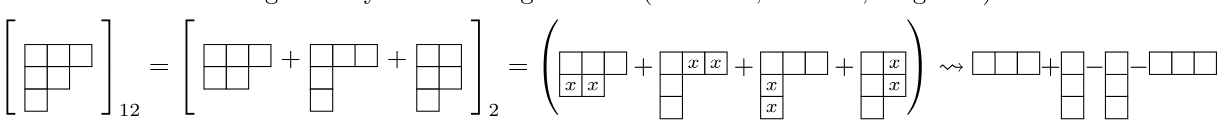
(independently of which a is chosen).

(**A.6**) Examples.
Firstly $\chi^{321}_{2211} = \chi^{321}((12)(34)) = 0$, because if $\lambda = 321$ and $a = 2211$, or indeed $a = 2121$ or $a = 2112$, then $P_a = \emptyset$ because no 2-rim can be removed from 321.
Next we can check the same character with $a = 1212$. We have

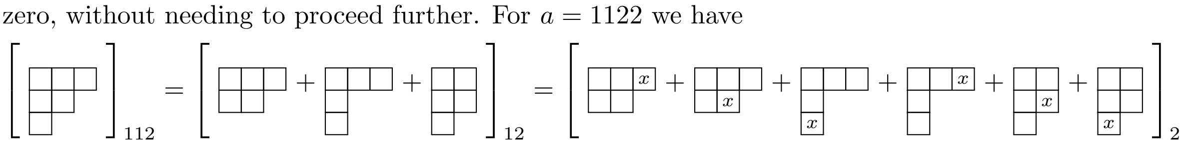


(23) {eq:321-1}

- a shorthand indicating the ways of removing one box (from one, or more, diagrams). Thus



again in an appropriate shorthand, where we continue to remove a regular rim part of length 2 and then (at the last) we keep track of the signs so far. Since the signed diagrams cancel here, we can already see that the character computations from (22) for $a = 1212$ and 1221 will indeed give zero, without needing to proceed further. For $a = 1122$ we have



k	0	1	2	1^2	3	21	1^3	4	31	2^2	21^2	1^4	5	41	32	31^2	2
$\lambda \setminus \prod$	1	c_1	c_2	$\binom{c_1}{2}$	c_3	c_1c_2	$\binom{c_1}{3}$	c_4	c_1c_3	$\binom{c_2}{2}$	$\binom{c_1}{2}c_2$	$\binom{c_1}{4}$	c_5	c_1c_4	c_2c_3	$\binom{c_1}{2}c_3$	
0	1																
1	-1	c_1															
2	0	$-c_1$	c_2	$\binom{c_1}{2}$													
1^2	1	$-c_1$	$-c_2$	$\binom{c_1}{2}$													
3	0	0	$-c_2$	$-\binom{c_1}{2}$	c_3	c_1c_2	$\binom{c_1}{3}$										
$2,1$	0	c_1	0	$-2\binom{c_1}{2}$	$-c_3$	0	$2\binom{c_1}{3}$										
1^3	-1	c_1	$-c_2$	$-\binom{c_1}{2}$	c_3	$-c_1c_2$	$\binom{c_1}{3}$										
4	0	0	0	0	$-c_3$	$-c_1c_2$	$-\binom{c_1}{3}$	c_4	c_1c_3	$\binom{c_2}{2}$	$\binom{c_1}{2}c_2$	$\binom{c_1}{4}$					
$3,1$	0	0	c_2	$\binom{c_1}{2}$	0	$-c_1c_2$	$-3\binom{c_1}{3}$	$-c_4$	0	$-\binom{c_2}{2}$	$\binom{c_1}{2}c_2$	$3\binom{c_1}{4}$					
2^2	0	0	$-c_2$	$\binom{c_1}{2}$	c_3	0	$-2\binom{c_1}{3}$	0	$-c_1c_3$	$2\binom{c_2}{2}$	0	$2\binom{c_1}{4}$					
$2,1^2$	0	$-c_1$	0	$2\binom{c_1}{2}$	0	c_1c_2	$-3\binom{c_1}{3}$	c_4	0	$-\binom{c_2}{2}$	$-\binom{c_1}{2}c_2$	$3\binom{c_1}{4}$					
1^4	1	$-c_1$	c_2	$\binom{c_1}{2}$	$-c_3$	c_1c_2	$-\binom{c_1}{3}$	$-c_4$	c_1c_3	$\binom{c_2}{2}$	$-\binom{c_1}{2}c_2$	$\binom{c_1}{4}$					
5	0	0	0	0	0	0	0	$-c_4$	$-c_1c_3$	$-\binom{c_2}{2}$	$-\binom{c_1}{2}c_2$	$-\binom{c_1}{4}$	c_5	c_1c_4	c_2c_3	$\binom{c_1}{2}c_3$	
41	0	0	0	0	c_3	c_1c_2	$\binom{c_1}{3}$	0	$-c_1c_3$	0	$-2\binom{c_1}{2}c_2$	$-4\binom{c_1}{4}$	$-c_5$				
32	0	0	0	0	$-c_3$												
31^2	0	0	$-c_2$	$-\binom{c_1}{2}$	0												
2^21	0	0															
21^3	0	c_1															
1^5	-1	c_1															

TABLE 2. All summands in the Gamba formula for each λ. (might be better to only write the coefficients rather than the full terms?) check vs ordinary character tables!! tab:1

The reader will readily check that again everything cancels at the next step.

Similarly any a starting with 4 or 6 gives zero with $\lambda = 321$, so we have now verified most of the $\chi^{\lambda=321}$ row of the relevant character table. Looking at (23) we see that $\chi^{321}_{51} = 1$. And so on. For $\chi^{321}_{33} = -2$ ...

Now compare with (4.18) et seq; and also the character table in (12).

A.2. **Some odds and ends.** (Aside: Once I complete the last rows here I can check with [8, **?**], then continue...)

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References

[1] Hermann Boerner. *Representations of groups. With special consideration for the needs of modern physics.* Translated from the German by P. G. Murphy in cooperation with J. Mayer-Kalkschmidt and P. Carr. Second English edition. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1970.

[2] A Gamba. Sui caratteri delle rappresentazioni del gruppo simmetrico. *Atti Accad. naz. Lincei, Rend.*, 12:167–169, 1952.

[3] M. Hamermesh. *Group theory and its application to physical problems.* Addison-Wesley Series in Physics. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London, 1962.

[4] G. James and A. Kerber. *The representation theory of the symmetric group*, volume 16 of *Encyclopedia of Mathematics and its Applications.* Addison-Wesley Publishing Co., Reading, Mass., 1981. With a foreword by P. M. Cohn, With an introduction by Gilbert de B. Robinson.

[5] J. C. Jantzen. *Representations of algebraic groups.* Academic Press, 1987.

[6] O King and P P Martin. Kronecker products for the hyperoctahedral group. *notes/in preparation*, 2017.

[7] D. E. Littlewood. Products and plethysms of characters with orthogonal, symplectic and symmetric groups. *Canad. J. Math.*, 10:17–32, 1958.

[8] P P Martin. Temperley–Lieb algebra, group theory and the Potts model. *J Phys A*, 21:577–591, 1988.

[9] F. D. Murnaghan. The Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group. *Amer. J. Math.*, 60(3):761–784, 1938.

[10] F D Murnaghan. *The theory of group representations.* Johns Hopkins 1938/Dover 1963, 1938.
[11] R. C. Orellana. On the algebraic decomposition of a centralizer algebra of the hyperoctahedral group. In *Algebraic structures and their representations*, volume 376 of *Contemp. Math.*, pages 345–357. Amer. Math. Soc., Providence, RI, 2005.
[12] M. Pelletier. A Geometric Approach to the stabilisation of certain sequences of Kronecker coefficients. arXiv:1606.05475.
[13] J. C. H. Wilson. $\mathrm{FI}_{\mathcal{W}}$-modules and stability criteria for representations of classical Weyl groups. *J. Algebra*, 420:269–332, 2014.

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