SOME BRIEF NOTES ON CATALAN COMBINATORICS: CLUSTERS AND NON-CROSSING PARTITIONS

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Notes in support of the figure at:
www1.maths.leeds.ac.uk/∼ppmartin/IMAGES/xfig-lib.html
(Joint work with R Marsh. Cf. [4].)

The figure is reproduced in Fig.1 here. For a minimal bibliography cf. [1, 2, 3, 5, 4, 6, 7].

1. Recollections on Catalan combinatorics

Abbreviations: noncrossing pair partitions (NCPP); noncrossing partitions (NCP); c-sortable (or 231 avoiding) permutations (CSP).

Here we construct bijections: clusters $\leftrightarrow$ CSP $\leftrightarrow$ NCP.

1.1. **Preliminaries.** Fix a Dynkin diagram of type A_{n-1} and a corresponding root system Φ (see e.g. [3]). The *almost positive* roots $\Phi_{\geq -1}$ are the positive roots $\alpha_{ij} = \alpha_i + \alpha_{i+1} + ... + \alpha_j$ together with the negative simple roots $-\alpha_i$.

Set $\underline{n} = \{1, 2, ..., n\}$. For S a set, P_S denotes the set of set partitions of S, and $P_n = P_{\underline{n}}$.

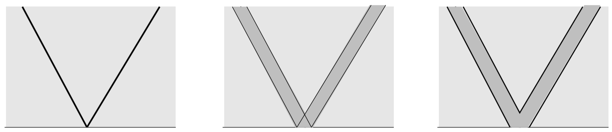
1.2. **Pictures of set partitions.** A graph on vertex set V describes an element of P_V — the partition into connected components. Any finite graph with ordered vertices may be represented as a 'picture', a subset of the upper half-plane, as follows. A vertex is drawn as a point on the boundary (the x-axis). Collectively the vertex images are distinct and respect the order along the axis in the natural way. An edge is drawn as a 'line': the image of an embedding of the interval $[0, 1]$ in the half-plane, terminating at the appropriate vertex points. (It is not possible to avoid line crossings in general in such a picture, but) Any pair of embeddings that cross at a point are required to have different tangent directions at the crossing.

Note that a picture contains a union of images of embeddings, not the embeddings themselves. However the different-tangent condition implies that such pictures of graphs (and hence partitions) are unambiguous. The idea works equally well on an oriented disk with a marked point, say.

Example: Pictures for all the partitions in P_4 are given in the third column in Fig.1.

1.3. **Non-crossing partitions.** A partition q in $P_{\underline{n}}$ is said to be *non-crossing* if there exists a picture of q that is non-crossing in the obvious sense. The subset of non-crossing partitions in P_n is denoted NCP_n.

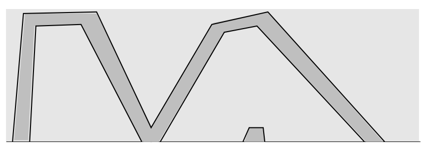
Consider a non-crossing picture of a partition, and consider the part of this picture containing the lines of a single connected component and their immediate neighbourhood (the union of small balls around points in the various lines involved, small enough not to include any other lines). Within this neighbourhood we may thicken each line to a ribbon; and each vertex point to a disk (or rather a half-disk). A ribbon may be thought of as two parallel lines (the lines describing its boundary components). In particular a ribbon defines two points either side of each vertex endpoint of the underlying line. These new lines do not touch between different ribbons except in case the underlying lines meet at a vertex, in which case there is necessarily a crossing like in (b) in the illustrations (a,b,c) of the neighbourhood of a vertex here:



We resolve this as shown in (c). A vertex point without lines becomes a half-disk, and hence the boundary line of that half-disk, again a line between two points either side of the original vertex.

Lemma 1.1. *This construction defines a bijection $\beta : NCP_n \rightarrow NCPP_{2n}$ (the range is non-crossing pair partitions since every (new) vertex now has exactly one incident line).*

Example. Here the partition $\{\{1,2,4\},\{3\}\}$ in NCP_4 becomes the pair-partition $\{\{1,8\},\{2,3\},\{4,7\},\{5,6\}\}$:



Homework: What happens to this construction if there *are* crossing lines.

1.4. **The χ-map.** Fix n. Associated to each permutation p of $123...n$ is the set $\chi'(p)$ of its *cover reflections*, the pairs (i,j) such that $i > j$ and there exists k such that $p_k = i$ and $p_{k+1} = j$. Each $\chi'(p)$ is a relation on $\{1,2,...,n\}$. Its RST closure $\chi(p)$ is an equivalence relation, and hence gives a partition.

Example: See columns 1,2,3 of Figure 1.

1.5. **c-sortability.** Fix in S_n the Coxeter element

$$c_1 := s_1 s_2 ... s_n$$

Then every element w of $W = S_n$ can be written as a word in the Coxeter generators obtained by starting with some finite number of repeats of this 'block' (the above defining word for c_1):

$$(c_1)^x \;=\; s_1 s_2 ... s_n \; s_1 s_2 ... s_n \; ... \; s_1 s_2 ... s_n$$

and deleting factors. The generalisation to any Coxeter element c will be clear. We call this the c-sorted form of w, denoted $f_c(w)$ (note well that this form is a sequence dependent on c as well as w).

For examples see column 4 of Fig.1.

Consider in $f_c(w)$ the set of factors kept from each block. If these nest (i.e. no factor is kept in the i-th block that is not kept in the $i-1$-th block), then w is said to be c-sortable.

CLAIM: χ is a bijection from the set S_n^c of c-sortable perms to NCP.

1.6. **The r-map.** For any product $s_{i_1} s_{i_2} ... s_{i_{k-1}} s_{i_k}$ we define

$$ref(s_{i_1} s_{i_2} ... s_{i_{k-1}} s_{i_k}) = s_{i_1} s_{i_2} ... s_{i_{k-1}} s_{i_k} s_{i_{k-1}} ... s_{i_2} s_{i_1}$$

Note that $ref(x)$ gives a pair permutation for any x (since it is the conjugate of an elementary pair permutation).

Now consider the form $f_{c_1}(w)$. For each i we can write out $f_{c_1}(w)$ from left to right, stopping at the *last* (i.e. rightmost) occurence of s_i. Call this truncation

perm	chi	partition	123−sort	123−cluster
1234	–		–	–
1243	43		3	3
1324	32		2	2
1342	42		23	2 23
1423	42		3/2 ×	
1432	43 32		23/2	3 23
2134	21		1	1
2143	21 43		13	1 3
2314	31		12	1 12
2341	41		123	1 12 123
2413	41		13/2 ×	
2431	43 31		123/2	1 3 123
3124	31		×	
3142	31 42		×	
3214	32 21		12/1	2 12
3241	32 41		123/1	12 123 2
3412	41		×	
3421	42 21		123/12	123 2 23
...			×	
4321	43 32 21		123/12/1	123 23 3

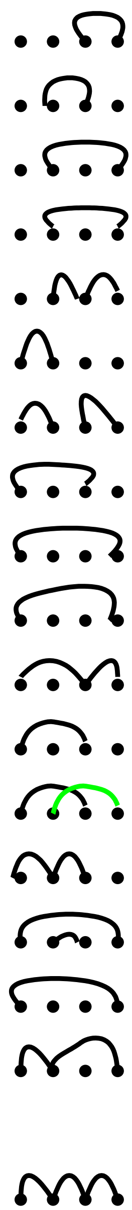


FIGURE 1. Maps from permutations (showing perms in dictionary order, excluding some non-c-sortable cases) to: (I) partitions; (II) 123-sorts (i.e. sorts with $c = s_1 s_2 s_3$; notation is $ab/c = s_a s_b | s_c$); (III) 123-clusters (notation is that all $-\alpha_i$ are omitted, and $a...b = \alpha_a + ... + \alpha_b$).

$f_{c_1}(w)_i$. We define a subset of $\Phi_{\geq -1}$ denoted $r_{c_1}(w)$ for each c_1-sortable perm w as follows. There is a root for each i. If s_i does not appear in $f_{c_1}(w)$ then the root is $-\alpha_i$. If s_i appears then the root is the positive root associated with the pair permutation given by $ref(f_{c_1}(w)_i)$.

That is,

$$r_{c_1}(w) \;=\; \{\alpha_{ref(f_{c_1}(w)_i)} \mid i = 1, 2, ..., n-1\}$$

where $\alpha_x = \alpha_{jk}$ if x gives elementary transposition (jk). See Fig.1 column 5 for examples.

CLAIM: r_{c_1} is a bijection from c_1-sortable perms to c_1-clusters.

2. Clusters and triangulations (and Tagged cluster arrays)

Fix n and let Φ be the root system of type A_{n-1} (corresponding to the Weyl group S_n, with generators $s_1, s_2, ..., s_{n-1}$); Φ^+ be the set of positive roots, $\Phi_- = \{-\alpha_i \mid i = 1, 2, ..., n-1\}$ the set of negative simple roots, and

$$\Phi^k \;:=\; \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_k \;:\; 1 \leq i \leq k\}$$

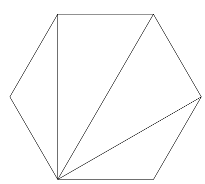
Thus $\Phi = \Phi^+ \cup (-\Phi^+)$, and $\Phi^+ = \sqcup_{k=1}^{n-1} \Phi^k$. For $i \leq j$ define

$$\alpha_{ij} = \alpha_i + \alpha_{i+1} + \cdots + \alpha_j$$

Write $\alpha \subseteq \alpha'$ when $\alpha = \alpha_{ij}$ and $\alpha' = \alpha_{i'j'}$ and $i \leq i' \leq j' \leq j$.

Let $\mathbb{T}_n$ be the set of triangulations of an $(n+2)$-gon. Recall that an A_{n-1} c-cluster may be defined as follows [6].
(1) Draw an $(n+2)$-gon and choose a suitable 'base' triangulation, depending on c.

Aside: Note well the role of the rank $n-1$. Consider $A_3 = A_{4-1}$ (with Weyl group S_4) for a moment. Here is a 6-gon:



Note the triangulation giving 3 interior edges in a 'fan'.

For example $c^1 = s_{n-1}s_{n-2}...s_1$ requires a 'fan' base triangulation: a triangulation in which all diagonals emanate from the same vertex, and are labelled $n-1, n-2, .., 1$ clockwise. For another example, $c = s_2 s_1 s_3$ requires a 'snake' base triangulation. A 'snake' is a triangulation such that the added edges form a directed chain, with edge labels $1, 2, ..., n-1$.
(2) Now, for fixed base, consider an arbitrary triangulation t. Each edge either coincides with an edge in the base, or crosses some subset of the base edges. We associate a set $s_c(t)$ of roots to t, one for each of its edges. The root corresponding to edge e is either $-\alpha_i$ if e is snake edge i, or $\alpha_{i_1} + ... + \alpha_{i_k}$ if e crossed snake edges $i_1, ..., i_k$.

Then the set $s_c(\mathbb{T}_n)$ is the set of c-clusters.

CLAIM: s_c is injective.

Remark: One can move through the set $\mathbb{T}_n$ (and hence $s_c(\mathbb{T}_n)$) from any initial triangulation t by a sequence of steps. In each step we erase one (suitably chosen) edge, to create a square, and then re-complete the triangulation of this square in the complementary way.

At this point we are done. We continue a little in the direction of [4, §3.6], just for entertainment ...

2.1. **Root clusters.**

There follow a few notes on the direct definition of root clusters corresponding to the Coxeter element c. See p22/23 of [6].

For each $i = 1, 2, \ldots, n$ let σ_i be the involution of $\Phi_{\geq -1}$ given by the formula:

$$\sigma_i(\alpha) := \begin{cases} -\alpha, & \text{if } \alpha = \pm\alpha_i, \\ \alpha, & \text{if } \alpha = -\alpha_j,\ j \neq i, \\ s_i(\alpha), & \text{otherwise.} \end{cases}$$

For $s = s_i \in S$ we may write $\sigma(s)$ for σ_i.

Definition 2.1. *Define a family of binary relations $||_c$ on the set $\Phi_{\geq -1}$, unique satisfying:*
(I) For any $-\alpha_i$ and any $\alpha \in \Phi^+$, $-\alpha_i||_c\alpha$ if and only if α_i does not occur in α (see 7.2(i) in [6]).
(II) For any $\alpha, \beta \in \Phi_{\geq -1}$ and any initial letter s (equiv. final letter) of c, $\alpha||_c\beta$ if and only if $\sigma(s)(\alpha)||_{scs}\sigma(s)(\beta)$.

To check c-compatibility of $\alpha, \beta \in \Phi_{\geq -1}$, just apply $\sigma(s)$ for appropriate s, using (II) to reduce α to a negative simple root, then apply (I).

Definition 2.2. *A c-cluster is a maximal c-compatible subset of $\Phi_{\geq -1}$.*

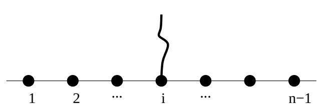
We define $\mathcal{C}_c(n)$ to be the set of c-clusters of type A_{n-1}.

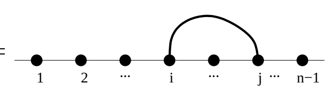
CLAIM: $s_c(T_n) = \mathcal{C}_c(n)$.

If c is fixed we shall simply refer to *clusters* for c-clusters.

2.2. **arc-diagrams.**

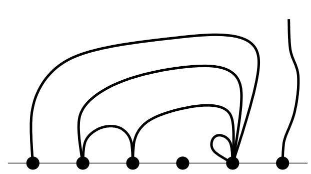
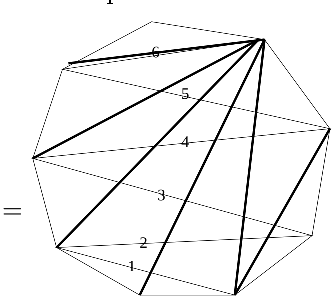
For expository convenience we depict roots by graphs of $n-1$ vertices (together with an undrawn 'vertex at infinity') drawn in the upper half-plane, as follows:

$-\alpha_i =$ 

$\alpha_{ij} =$ 

Note that an edge in such a graph partitions the upper half-plane into two parts. In case $\alpha = \alpha_{ij}$, one of these parts is bounded, and we call this the part *covered* by α. We say that a root α_{ij} with $i \leq j$, or its corresonding edge or 'line', *begins* at i and *terminates* at j.

A cluster may be drawn in the same way. Here is a snake example:

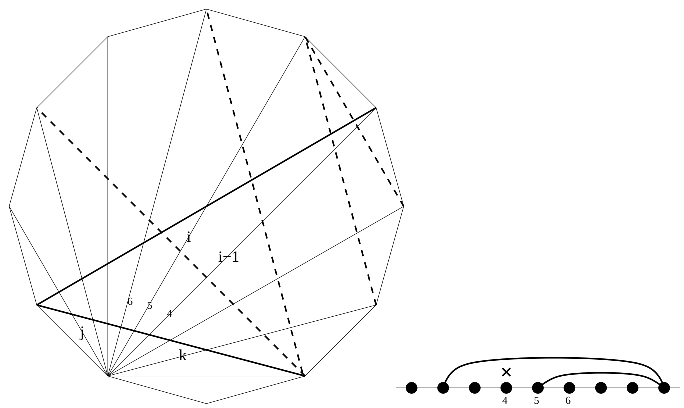
$\{\alpha_{15}, \alpha_{23}, \alpha_{25}, \alpha_{35}, \alpha_5, -\alpha_6\} =$  $=$ 

We call these arc-diagrams.

CLAIM: (IS FALSE FOR GENERAL c!!) In the fan case, $c = c^1$, the cluster rules ensure that such a graph may be drawn without crossing lines, and indeed that if $\alpha \subseteq \alpha'$ then the part of the plane covered by α may be chosen to be a subset of the part covered by α' (as illustrated - albeit not in the fan case).

Lemma 2.3. *Note that if $\alpha_{ij}, \alpha_{kj} \in X$ and $i - k > 1$ then the only possible line with an endpoint at $i - 1$ has its other endpoint at j.*

Proof: α_{ij}, α_{kj} define an irregular polygonal partition of the interior of the triangulated polygon (see picture below), which partition must be extended to a triangulation by X. Thus in particular any α with an endpoint at $i-1$ must lie in the 'interior' of the irregular polygon, and hence must terminate at j. This is well illustrated by the case below:



Here the solid lines illustrate the hypothesis, and the dashed lines illustrate how other possibilities involving $i-1$ (in this case $i = 5$) are blocked.

Thus:

Lemma 2.4. *If two or more lines in a cluster X terminate at j, so that $Y_j = \{\alpha_{i_1 j}, ..., \alpha_{i_l j} \mid i_1 < ... < i_l\}$, and there is no $\alpha_{ab} \supset \alpha_{i_1 j}$ in X, then Y_j can be replaced by*

$$Y_j^- := \{-\alpha_{i_2 - 1}, ..., -\alpha_{i_l - 1}, -\alpha_j\}$$

without breaking the cluster property.

3. Outro

See [4] and reference therein for the continuation.

References

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