

RIMS-782

On Schur-Weyl Duality,  $A_n$  Hecke Algebras and  
Quantum  $sl(N)$  on  $\otimes^{n+1} \mathbb{C}^N$

By

Paul Purdon MARTIN

August 1991



京都大学 数理解析研究所

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES  
KYOTO UNIVERSITY, Kyoto, Japan

RIMS91 PROJECT "INFINITE ANALYSIS"

June 1 – August 31, 1991

Contributed Papers No.10

ON SCHUR-WEYL DUALITY,  $A_n$  HECKE ALGEBRAS AND  
QUANTUM  $sl(N)$  ON  $\otimes^{n+1}\mathbb{C}^N$

By

Paul Purdon MARTIN

---

This project is supported in part by a grant of the Com-memorative Association for the Japan World Exposition (1970).

ON SCHUR-WEYL DUALITY,  $A_n$  HECKE ALGEBRAS AND  
QUANTUM  $sl(N)$  ON  $\otimes^{n+1} \mathbb{C}^N$

PAUL PURDON MARTIN

RIMS, Kyoto University,  
Kyoto 606, Japan,  
and  
Mathematics Department,  
City University,  
London EC1V 0HB, UK.

**Abstract** We prove that an  $N$ -state vertex model representation of the  $A_n$  Hecke algebra quotient  $NH_n(q)$  is faithful for all  $q$ . We use the result to examine the indecomposable content of these representations, and hence the structure of the centraliser algebra, which is generically a quotient of  $U_q sl(N)$ , at  $q$  a root of unity. We achieve a complete analysis in the case  $N = 2$ , finding a number of Morita self-dual algebraic structures.

## 1 Introduction

It is well known, and a straightforward combinatorial proof exists, that a given quotient of the  $A_n$  type Hecke algebra over the complex numbers  $H_n(q)$  (called  $NH_n(q)$  - see section 2) and an appropriate quotient of the so called quantum group  $U_q sl(N)$  (Jimbo 1985, Drinfeld 1986) are in Schur-Weyl duality on  $\otimes^{n+1} \mathbb{C}^N$  if  $q$  is not a root of unity. We will briefly review these results shortly. For an introductory review see, for example, Martin (1991).

The simple proof does not work for the physically crucial (and mathematically most interesting) case of  $q$  a root of unity, even though the actions of both algebras remain well defined (care must be taken with the definition of  $U_q sl(N)$  - see appendix A). The key issue in this case is the faithfulness of the representation of  $NH_n(q)$ . It is important to know if this result holds in order to compare our partial knowledge of the structures of the two algebras (see e.g. Dipper and James 1989, Lusztig 1989). Our main result in this paper (section 3) is that it does!

In order to illustrate the importance of this result explicitly we then consider some aspects of the mechanism of its application (section 4). In the process we find that, in all the cases we can check, i.e. all of  $N = 2$ , the 2 algebras remain Schur-Weyl dual for all  $q$ , and furthermore are Morita equivalent (for a general

reference on Morita equivalence try Pierce 1982). This is a trivial result for all  $N$  for generic  $q$ , but is very strong, if true, in general.

The faithfulness result, and the techniques involved in its proof, are also crucial for the computation of the spectrum of quantum spin chains at  $q$  a root of unity, and in providing the means to analyse  $H_n(q)$  through other sequences of quotients besides  $NH_n(q)$ , as we will see in a subsequent paper (Martin and Rittenberg 1991).

Let  $A^\circ$  be an associative algebra over the complex numbers and  $M$  a finite dimensional left  $A^\circ$ -module. Then  $B = End_{A^\circ}(M)$  is the centraliser algebra of  $A^\circ$  on  $M$  (that is, the algebra of endomorphisms of  $M$  which commute with the action of  $A^\circ$ ), and  $A = End_B(M)$  and  $B$  are said to be in Schur-Weyl duality on  $M$ . That is

$$B = End_{End_B(M)}(M).$$

In this case  $A$  and  $B$  are sometimes called a dual pair (see e.g. Zelevinskii 1987) or a Howe pair on  $M$ .

Let  $K$  be the smallest double sided ideal of  $A^\circ$  such that  $M$  is a faithful  $A^\circ/K$  module (i.e.  $K = ann_A M$ ). Then  $A \supset A^\circ/K$ , and in general we have a composite morphism

$$A^\circ \rightarrow A^\circ/K \rightarrow A$$

For example, if  $A^\circ/K$  gives all upper triangular matrices on  $M$  then  $B$  is just scalars and  $A$  is all matrices on  $M$ .

We say that  $A^\circ$  itself has a Schur-Weyl dual on  $M$  if and only if

$$A^\circ/K = A.$$

A sufficient, but not necessary, condition for this is that  $A^\circ/K$  is semi-simple (this happens in our case when  $q$  is not a root of unity). We will discuss necessary conditions in section 4.

If  $A^\circ = H_n(q)$  and  $M$  is the usual  $N$ -state vertex model representation (see section 2), then  $B$  is generically a certain ( $n$ -dependent) quotient of  $U_q sl(N)$  (section 4.2), called  $U_q^n sl(N)$ , and  $A^\circ/K = A = NH_n(q)$ . The representation  $M$  of  $NH_n(q)$  is block diagonal for all  $q$ , with the blocks being  $q$ -permutation modules in the sense of a  $q$  deformation of the symmetric group permutation modules (see section 4). This means that the indecomposable content of these modules is known generically, and in particular that some of them are themselves faithful representations of the algebra (see Robinson 1962 for example). It is the multiplicities and morphisms of the indecomposables which tell us the structure of the centraliser algebra in general, so these are crucial results. We can construct a counterexample (see section 4) showing that the faithfulness of generic blocks is not necessarily preserved in the case  $q$  a root of unity! The faithfulness of the whole representation is thus by no means a trivial result,

giving an important clue to its indecomposable and irreducible content (which will be discussed, and in some cases determined, using this property).

In the next section we define  $NH_n(q)$  and the action of  $H_n(q)$  on  $M = \otimes^{n+1} \mathbb{C}^N$ . We then establish our main result - that this gives a faithful representation of the algebra  $NH_n(q)$ . In the following section we illustrate the use of these results with some applications. We determine the structure of the  $U_q sl(2)$  quotient from that of  $2H_n(q)$ , and note in particular a Morita equivalence. In appendix A we define (from Jimbo 1985, Drinfeld 1986, Lusztig 1989) the appropriate action of  $U_q sl(N)$  on  $\otimes^{n+1} \mathbb{C}^N$ , and note commutativity with the representation of  $NH_n(q)$ . Finally appendix B contains a technical remark.

## 2 Hecke algebra

The main result of this section is a theorem given on page 7.

Following existing notation (for a recent introductory review and full references see, for example, Martin 1991 - hereafter called I, or Westbury 1990) we have:

**Definition 1 (Hecke algebra)** For  $n$  a positive integer,  $q \in \mathbb{C} - \{0\}$  and

$$\sqrt{Q} = q + q^{-1}$$

$H_n(q)$ , or simply  $H_n$ , is the unital associative algebra over  $\mathbb{C}$  defined by generators  $\{U_i : i = 1, 2, \dots, n\}$  and relations

$$U_i U_i = \sqrt{Q} U_i \quad (1)$$

$$U_i U_{i \pm 1} U_i - U_i = U_{i \pm 1} U_i U_{i \pm 1} - U_{i \pm 1} \quad (2)$$

$$U_i U_{i+j} = U_{i+j} U_i \quad (j \neq 1) \quad (3)$$

**Proposition 1** There is an isomorphism of right  $H_{n-1}$  modules

$$H_n \cong H_{n-1} \oplus \bigoplus_{m=1}^n H_{n-1} \left( \prod_{i=1}^m U_{n+1-i} \right).$$

*Proof:* By induction: The proposition holds at level  $n = 1$ . Now suppose the proposition is true at level  $n$ , then trivially there is a natural mapping

$$H_n \rightarrow H_{n-1} \oplus H_{n-1} U_n H_{n-1}$$

(remark: this is an isomorphism of  $H_{n-1}$  bimodules) so, using this and the defining relations,

$$H_n U_{n+1} H_n U_{n+1} H_n \rightarrow H_n U_{n+1} H_n + H_n$$

whereupon

$$H_{n+1} \rightarrow H_n \oplus H_n U_{n+1} H_n.$$

This may now be combined with the proposition at level  $n$  to give the inductive step to level  $n+1$ .

**Corollary 1.1** *The proposition also holds with  $U_i$  replaced by  $U'_i = \alpha + \beta U_i$ , for any complex number  $\alpha$  and non-zero complex number  $\beta$ .*

*Proof:* as above.

We immediately obtain a well known result:

**Corollary 1.2** *The dimension of  $H_n(q)$  is  $(n+1)!$ , and if  $\{b_i : i = 1, 2, \dots, n!\}$  is a basis for  $H_{n-1}$  then*

$$\{b_i, b_i U'_n, b_i U'_n U'_{n-1}, \dots, b_i U'_n U'_{n-1} \dots U'_1 : i = 1, 2, \dots, n!\}$$

*is a basis for  $H_n$ .*

**Definition 2** *Let  $B_n(U')$  be the basis of words in  $\{U'_i\}$  obtained by iterating this process from  $B_0(U') = \{1\}$ .*

**Remark 1** *The words in  $B_n(U')$  cannot be written as shorter words in  $\{U'_i\}$  by applying the defining relations.*

*Outline proof:* Suppose true for  $n-1$ . The only relation which can shorten every word in the output is the first, but each new word has at most one factor of  $U_n$ , and at least one factor of  $U_i$  between each factor of  $U_{i-1}$  by construction.

**Corollary 1.3** *Words in  $B_n(U')$  of length  $l$  or less span all such words in  $\{U'_i\}$ .*

For many purposes we break the analysis of  $H_n(q)$  down through a sequence of quotients  $NH_n(q)$ . To define these we need first to define some special elements of  $H_n(q)$  (which will generate the kernels of the these quotients).

## 2.1 Special elements of $H_n(q)$

This subsection is taken from Martin and Westbury (1991) and references therein.

**Definition 3 (Idempotents)** *For each  $m = 1, 2, 3, \dots, n+2$  define an idempotent  $E_m \in H_n(q)$  by*

$$E_1 = E_2 = 1$$

*and then*

$$E_m \in H_{m-2}(q) \subset H_n(q)$$

*and*

$$E_m E_m = E_m$$

*and for  $i = 1, 2, \dots, m-2$*

$$E_m U_i = U_i E_m = 0.$$

There can be at most one such element, since if  $E_m, E'_m \in H_{m-2}(q)$  both have the above properties then  $E_m E'_m = E_m = E'_m$ .

Let us consider the existence of such an element. We need

**Definition 4** *For each positive integer  $n$  define  $k_n$ , a function of  $Q$ , by  $k_1 = 0$  and*

$$k_{n+1} = 1/(\sqrt{Q} - k_n).$$

**Definition 5** *For  $s$  an integer and  $q$  given*

$$[s] = \frac{q^s - q^{-s}}{q - q^{-1}}$$

*and for  $N$  a positive integer*

$$[N]! = \prod_{s=1}^N [s].$$

For example, with  $\sqrt{Q} = q + q^{-1}$  as before:

$$[0] = 0$$

$$[1] = 1$$

$$[2] = \sqrt{Q}$$

$$[3] = Q - 1$$

and

$$k_n = \frac{[n-1]}{[n]}$$

from definition 4.

**Definition 6** *Define  $I[m-2] \in H_{m-2}(q)$  by  $I[0] = 1$  and*

$$I[m-2] = I[m-3](1 - k_{m-1} U_{m-2})I[m-3]$$

The existence of  $I[m-2]$  for a given value of  $q$  is guaranteed unless some  $k_n$  required in its construction has a pole at that point.

**Proposition 2 (see 1)** *If  $I[m-2]$  exists then*

$$E_m = I[m-2]$$

Under the automorphism  $D : H_n(q) \rightarrow H_n(q)$  defined by

$$U_i \mapsto \sqrt{Q} - U_i$$

we have another idempotent

$$D(E_m) = F_m.$$

For  $X \in H_n(q)$  we define  $X^{(t)} \in H_{n+t}(q)$  by the translation

$$U_i^{(t)} = U_{i+t}.$$

**Definition 7** For  $n \geq b \geq a > 0$  and  $c = 3 - a + b$  define  $F_{ab} \in H_n(q)$  by

$$F_{ab} = F_c^{(a-1)}.$$

Consequently, if  $a \leq i \leq b$  then

$$U_i F_{ab} = F_{ab} U_i = \sqrt{Q} F_{ab}. \quad (4)$$

For example,

$$\begin{aligned} F_{11} &= \frac{U_1}{\sqrt{Q}} \\ F_{12} &= \frac{U_1 U_2 U_1 - U_1}{\sqrt{Q}(Q-1)}. \end{aligned}$$

With  $c = 3 - a + b$  again, we similarly define

$$E_{ab} = E_c^{(a-1)}$$

so that

$$U_i E_{ab} = E_{ab} U_i = 0$$

if  $a \leq i \leq b$ . For example  $E_{1-1} = 1$ .

Note that, as with  $E_{ij}, F_{ij}$  may not be well defined for all  $Q$  (consider our examples). However, note the following

**Definition 8** For  $n$  a positive integer and  $Y_1 = 1$

$$Y_{n+1} = -[n-1] Y_n + Y_n \left( \sum_{m=1}^n [m-2] (U_r U_{n-1} \dots U_m) \right).$$

The element  $Y_{n+1} \in H_n$  is clearly finite for all  $Q$ , with the coefficient of the longest word (c.f. proposition 1) equal to one. It is established in Martin and Westbury (1991) that

$$Y_n = [n] F_{n+1}.$$

## 2.2 The quotient algebras $NH_n(q)$

We define a sequence of quotient algebras of  $H_n(q)$  as follows. The quotient  $NH_n(q)$ , or simply  $NH_n$ , is obtained by imposing the quotient relations

$$Y_{N+1} = 0.$$

For example, with  $Q \neq 0$  then  $1H_n(q)$  can be obtained by putting  $F_{11} = U_1/\sqrt{Q} = 0$ , whilst with  $Q = 0$  the quotient relation is  $\sqrt{Q} F_{11} = U_1 = 0$  (the first expression is, of course, purely formal at  $Q = 0$ ). Note that the case  $N = 2$  corresponds to the Temperley-Lieb algebra. The case  $NH_n(1)$  is the quotient of the group algebra of the symmetric group on  $n+1$  objects to exclude irreducible representations with Young diagrams of more than  $N$  rows (c.f. Robinson 1962).

**Proposition 3** For  $q$  a nonzero complex number the dimension of  $NH_n(q)$  equals the dimension of  $NH_n(1)$ .

We prove this proposition in the course of proving our main theorem.

### 2.3 Representations

Let  $V_N = \{1, 2, \dots, N\}$  be shorthand for the standard ordered basis for  $\mathbb{C}^N$ , and  $I_N$  be the  $N \times N$  identity matrix, and  $M$  the  $N^2 \times N^2$  matrix with action on  $V_N^2$  given by

$$M(a, b) = 0 \quad \text{if } a = b \quad (6)$$

and otherwise, with  $p = \text{sign}(b - a)$ ,

$$M(a, b) = q^p (a, b) + (b, a). \quad (7)$$

Then for  $N < n$  and  $V$  the space spanned by  $V_N^{n+1}$  we can check by direct computation that there is a representation  $R_N : H_n(q) \mapsto \text{End}_{\mathbb{C}}(V)$  given by

$$R_N(U_i) = I_N \otimes I_N \otimes \dots \otimes M \otimes \dots \otimes I_N$$

where  $M$  appears in the  $i^{\text{th}}$  position in the product.

Note that the objects  $R_{PM}(U_i)$ , with  $P + M = N$ , defined by replacing equation 6 by

$$M(a, b) = q + q^{-1}(a, b) \quad \text{if } a = b > P \quad (8)$$

if  $a = b > P$  (so  $R_{N0} = R_N$ ), provide a host of useful representations, which are studied in Martin and Rittenberg 1991.

**Theorem 1 (Main theorem)**  $R_N H_n(q)$  is a faithful representation of  $NH_n(q)$  for all  $q$ .

### 3 Proof of the Theorem

#### 3.1 Part 1: $R_N$ a representation

We need to show that  $R_N(F_{N+2}) = 0$  when  $q$  is an indeterminate (this is sufficient, since  $R_N(kF_{N+2}) = kR_N(F_{N+2})$  and  $k = [N+1]$  is well defined in any specialisation). We proceed by induction.

The result is true for  $N = 1$  by equation 6. If true for  $N = m$  then for  $a \in V_{m+1}^{n+1}$

$$R_{m+1}(F_{m+3}) a = R_{m+1}(F_{m+3}F_{m+2}) a = R_{m+1}(F_{m+3})R_{m+1}(F_{m+2}) a \quad (9)$$

can only possibly be non-vanishing if  $a_1a_2\dots a_{m+1}$  are distinct (and  $a_2a_3\dots a_{m+2}$  distinct, by symmetry), so  $a_1 = a_{m+2}$ . We then have

$$R_{m+1}(F_{m+3}) a = R_{m+1}(F_{m+3}U_1/\sqrt{Q}) a$$

which, using equation 7 becomes

$$R_{m+1}(F_{m+3}) \left( \frac{q}{\sqrt{Q}} a + a_2a_1a_3a_4\dots a_{m+1}a_1\dots \right).$$

Since the second through  $(m+2)^{\text{th}}$  components of the latter vector are no longer distinct it is killed by the  $F_{m+3}$  ( $= F_{m+3}F_{m+2}^{(1)}$  as above) leaving

$$R_{m+1}(F_{m+3}) a = R_{m+1}(F_{m+3}) \frac{q}{\sqrt{Q}} a = 0.$$

Q.E.D.

#### 3.2 Part 2: $R_N$ faithful

We construct a basis for  $NH_n(q)$  and prove explicitly that every element is distinct in  $R_N$ .

We will need some properties of  $R_N(F_{N+1})$ . Note from equations 6 and 7 that each  $U_i$  mixes between basis vectors in  $V$  with a fixed number of 1's, 2's, ...,  $N$ 's appearing as its components. Consequently the representation  $R_N$  is block diagonal up to permutations of the basis, and equivalence classes of the direct summand representations may be associated with a subset of the partitions (see e.g. Macdonald 1979, James and Kerber 1981) of  $n+1$ .

**Definition 9** Define the set  $\mathcal{D}_n^N$ , of partitions of  $n+1$  into at most  $N$  parts, as  $N$ -tuples of non-negative integers

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$$

with the properties

$$\alpha_i > \alpha_j \Rightarrow j > i$$

and

$$\sum_i \alpha_i = n+1.$$

Define a total order on  $\mathcal{D}_n^N$  by  $\alpha > \beta$  if there exists integer  $j$  such that

$$\alpha_i = \beta_i$$

for  $N \geq i > j$ , and

$$\alpha_j - \beta_j < 0.$$

This is the  $L'_n$  order defined in Macdonald 1979. For example, the  $N$ -tuple

$$\nu_N = (1, 1, 1, \dots, 1)$$

is the least element and  $(N, 0, 0, \dots, 0)$  the greatest in this order in the set of all partitions of  $N$ ,  $\mathcal{D}_N^{N-1}$ .

We will regard  $\mathcal{D}_n^M \subset \mathcal{D}_n^N$  for  $M < N$  by extending the  $M$ -tuples to  $N$ -tuples by adding zeros on the right.

For given  $n$  and  $N$ , and  $a \in \mathcal{D}_n^N$ , we define  $R_N^a$ , or simply  $R^a$ , as the representation on subspace with basis vectors containing  $a_1$  1's,  $a_2$  2's, and so on. For example,  $R_3^{(1,1,1)}$  has basis  $\{123, 213, 132, 312, 231, 321\}$ .

**Proposition 4** The matrix elements of  $R_N^{\nu_N}(Y_N) = R_N^{\nu_N}([N]F_{N+1})$  are all integer powers of  $q$  (and hence finite, i.e. non-zero, for all non-zero  $q$ ).

*Proof:*

It follows from the defining relations 1–3 and corollary 1.2 that  $R_N^{\nu_N}$  is the regular representation of  $H_{N-1}(q)$  (see I for details). The relations

$$U_i [N]F_{N+1} = [N]F_{N+1} U_i = \sqrt{Q} [N]F_{N+1}$$

then imply

$$\text{rank}(R_N^{\nu_N}([N]F_{N+1})) = 1$$

(consider the action of  $[N]F_{N+1}$  on each element of the algebra). The matrix  $R_N^{\nu_N}([N]F_{N+1})$  is also symmetric by construction, so there exists some row vector  $h$  such that

$$R_N^{\nu_N}(kF_{N+1}) = h^t h.$$

The relations above then further imply that in this representation

$$U_i h^t = \sqrt{Q} h^t.$$

Now  $U_i$  mixes basis vectors in pairs so this may be broken up into sub-equations of the form

$$\begin{pmatrix} q & 1 \\ 1 & q^{-1} \end{pmatrix} \begin{pmatrix} s_i \\ s_j \end{pmatrix} = \sqrt{Q} \begin{pmatrix} s_i \\ s_j \end{pmatrix}$$

which implies

$$s_i = qs_j.$$

Using all the  $U_i$ 's all the components of the vector are connected in this way. Finally, from definition 8 note that  $[N]F_{N+1} \neq 0$ . This completes the proof of proposition 4.

To review, we have seen that with  $p_{ij} \subset V$  the subset of basis vectors  $\{p \in V_N^{n+1} : p_i, p_{i+1}, \dots, p_j \text{ distinct elements of } V_N\}$ , and introducing bra and ket notation to make row vectors easier to spot, then

$$R_N(kF_{ij})|a\rangle = 0 \quad \text{if } a \in V_N^{n+1} - p_{ij} \quad (10)$$

so  $R_N$  is a representation of  $NH_n$ , and from proposition 4 that

$$R_N([j-i+2]F_{ij})|p\rangle = \sum_{r \in p_{ij}} q^{f(p,r)} |r\rangle \quad \text{if } p \in p_{ij} \quad (11)$$

where  $f$  is a finite integer. We do not need to know this integer for our purposes, it is given in appendix B.

In what follows we will use the notion of standard tableau (see e.g. James and Kerber 1981). These are the insertions of numbers  $1, 2, \dots, n+1$  into the Young diagrams of partitions of  $n+1$  such that each row and column has the natural order.

**Definition 10** For  $\alpha$  a partition define  $\mathcal{D}(\alpha)$  as the set of standard tableau of shape  $\alpha$ .

If  $\{s\}$  is a standard tableau with  $i$  in a lower row than  $i+1$ , and the tableau obtained by interchanging  $i, i+1$  in  $\{s\}$  is standard, then this tableau may be called  $\{s^i\}$ .

**Definition 11** Define a partial order on standard tableau  $\{s\}$  in  $\mathcal{D}(\alpha)$  by

$$\{s\} \leq \{t\}$$

if and only if  $\{t\}$  can be obtained from  $\{s\}$  by a sequence of moves of the form

$$\{s\} \rightarrow \dots \rightarrow \{v\} \rightarrow \{v^i\} \rightarrow \dots \rightarrow \{t\}$$

The lowest sequence in the partial order in  $\mathcal{D}(\alpha)$ , call it  $\{e_\alpha\}$ , is the unique standard tableau with unit increases down the columns. The poset is a lattice.

**Definition 12** Define a partial order on all standard tableau of  $n+1$  boxes by the total order on partitions if the shapes are different, and by definition 11 otherwise.

**Definition 13** Define  $P(\alpha)$  as the set of pairs of standard tableau  $(\{s\}, \{t\})$  for all  $\{s\}, \{t\} \in \mathcal{D}(\alpha)$ .

**Definition 14** Define  $P^N(n)$  as the disjoint union of sets  $P(\alpha)$  over all values of  $\alpha \in \mathcal{D}_n^N$ .

### 3.2.1 A basis for $NH_n(q)$

Let us adopt the convention that  $F_{ij}$  may be regarded as a word in the generators of  $H_n(q)$ , and that for  $W$  a word in  $H_n(q)$  then  $WT$  is obtained by writing the generators in reverse order (note that  $F_{ij} = F_{ij}^T$ ). We need the following definitions:

For each partition  $\alpha$  let us define elements  $(e_\alpha \circ e_\alpha) \in H_n(q)$  iteratively by

$$(e_\alpha \circ e_\alpha) = 1$$

if  $\alpha = (n+1)$ , and then for  $\alpha_+ = \alpha + \nu_{N+1}$  by

$$(e_{\alpha_+} \circ e_{\alpha_+}) = Y_{N+1} \quad (e_\alpha \circ e_\alpha) \overset{\text{translation}}{\underset{\text{of } F_{ij}}{\longrightarrow}} (N+1).$$

This means that

$$U_i(e_\alpha \circ e_\alpha) = \sqrt{Q}(e_\alpha \circ e_\alpha)$$

if  $i$  is above  $i+1$  in the standard tableau  $\{e_\alpha\}$ . For example,

$$(e_{(5,3,1)} \circ e_{(5,3,1)}) = Q\sqrt{Q}(Q-1) F_{12} F_{44} F_{66}$$

$$= (U_1 U_2 U_1 - U_1) U_4 U_6.$$

Note that  $(e_\alpha \circ e_\alpha) = 0$  in  $NH_n(q)$  if  $\alpha$  has more than  $N$  rows.

**Definition 15** If  $(\{s\}, \{t\})$  is in  $P^N(n)$ , the set of pairs of standard tableau of the same shape and at most  $N$  rows, then  $(s \circ t)$  is a word in the generators of  $H_n(q)$  (counting  $F_{ij}$  as a word) obtained iteratively from  $(e_\alpha \circ e_\alpha)$  as follows:

$$(s^i \circ t) = U_i(s \circ t) \quad \text{if } s \text{ is a standard tableau of minimum shape,}$$

and

$$(t \circ s) = (s \circ t)^T. \quad (13)$$

If  $(t \circ s)$  is obtained from  $(e_\alpha \circ e_\alpha)$  in this way we call  $(e_\alpha \circ e_\alpha)$  the root of  $(t \circ s)$ . For example,  $(e_{(2,1)} \circ e_{(2,1)}) = U_1$  is the root of  $(U_1)U_2$ .

**Proposition 5 (see I)** *The set of ‘words’ in definition 15 spans  $NH_n(q)$ .*

In fact it is proved in I that these words form a basis, but we will of necessity prove linear independence in what follows, so it is sufficient to note that this set has order  $\dim(NH_n(1))$ . Since we show linear independence for each  $N$  then in particular the large  $N$  limit (i.e.  $N \geq n+1$ ) shows that overall we have  $(n+1)!$  linearly independent elements and hence a basis for  $H_n(q)$ . But within this full basis  $(n+1)! - \dim(NH_n(1))$  linearly independent elements are manifestly taken to zero under the  $Y_{N+1} = 0$  quotient (by definition 15, and specifically the definition of  $(e_\alpha \circ e_\alpha)$ ). Overall then

$$\dim(H_n Y_{N+1} H_n) \geq (n+1)! - \dim(NH_n(1)).$$

Meanwhile  $\dim(NH_n(1)) = \text{order}(NP(n))$  elements will be shown linearly independent in  $NH_n(q)$  so that  $\dim(NH_n(q)) \geq \dim(NH_n(1))$ . Clearly the bounds are saturated. This also proves proposition 3.

### 3.3 Proof of faithfulness (conclusion)

For given  $N$  and  $n$ , the set of words in  $NH_n(q)$  from definition 15 will be called  $S_W$ . Then with  $w, v \in V_N^{n+1}$  and  $W \in S_W$  we write  $W_{wv}$  for the  $w, v$  matrix element of  $R_N(W)$ . For example, if  $w = v = 1111$  (we suppress all commas and brackets for brevity) then  $W_{wv} = 0$  unless  $W = 1$ .

Note that

$$\langle w | v \rangle = \delta_{wv}$$

and

$$W_{wv} = \langle w | R_N(W) | v \rangle.$$

Then

$$R_N(U_i) | v \rangle = 0 \quad \text{if } v_i = v_{i+1} \quad (14)$$

and with  $v^i = v$  except for

$$v_i^i = v_{i+1}$$

and

$$v_{i+1}^i = v_i$$

we have

$$R_N(U_i) | v \rangle = q^{\pm 1} | v \rangle + | v^i \rangle \quad \text{if } v_i < v_{i+1}. \quad (15)$$

**Definition 16** Define a function

$$F : \mathcal{D}(\alpha) \rightarrow V_N^{n+1}$$

by

$$a \mapsto Fa$$

where  $(Fa)_i$  is the number of the row in which the number  $i$  appears in standard tableau  $a$ .

For example,  $\{e_{(1,1,1)}\}$ , the unique  $(1^3)$  standard tableau, gives  $Fa = 123$ . Note that  $F$  is not surjective unless  $n = 0$  and  $N = 1$ , but is injective. If  $i$  is in a row above  $i+1$  in  $a$  then we say  $Fa$  has a maximum at  $i$ . We also take the partial order on  $\mathcal{D}(\alpha)$  over onto  $F(\mathcal{D}(\alpha))$ .

**Definition 17** We define

$$P_R : S_W \rightarrow V$$

by

$$\begin{aligned} P_R : (a \circ b) &\mapsto Fa \\ P_L : S_W &\rightarrow V \\ P_L : (a \circ b) &\mapsto Fb. \end{aligned}$$

**Definition 18** Define a function

$$\begin{aligned} f : S_W &\rightarrow V \times V \\ \text{by} \\ (a \circ b) &\mapsto (Fa, Fb). \end{aligned}$$

For example, with  $n = 3$ ,  $f F_{12} U_3 = (1231, 1213)$ . Note that  $f$  is injective.

**Definition 19** Let  $(w, v) \in V \times V$ , then define a set

$$J_{(w,v)} = \{W \in S_W : W_{wv} \neq 0\}.$$

**Definition 20** Define a partial order  $(S_W, \geq)$  by

$$\begin{aligned} (a \circ b) &\geq (c \circ d) \\ \text{iff } a &\geq c \text{ and } b \geq d. \end{aligned}$$

Note that this is a lattice.

Here are some illustrative examples of the above ideas:

Example 1.  $n = 1$

Here  $H_1(q)$  is spanned by  $\{1, U_1\}$  and

$$f1 = (11, 11)$$

$$fU_1 = (12, 12)$$

$$J_{f1} = \{1\}$$

$$J_{fU_1} = \{1, U_1\}$$

where  $(Fa)_i$  is the number of the row in which the number  $i$  appears in standard tableau  $a$  (note that  $1 > U_1$ ).

Example 2.  $n = 2$

Here  $H_2(q)$  is spanned by  $\{1 > U_2 > U_1 U_2, U_2 U_1 > U_1 > F_{12}\}$  (we have also given the order) and

$$f1 = (111, 111)$$

**Proposition 8**

$$W' \in J_{fW} \Rightarrow W' \geq W.$$

**Proposition 9**

$$W \in J_{fW}.$$

$$\begin{aligned} fU_2 &= (112, 112) \\ fU_1 U_2 &= (121, 112) \\ fU_2 U_1 &= (112, 121) \\ fU_1 &= (121, 121) \end{aligned}$$

$$fF_{12} = (123, 123)$$

$$\begin{aligned} J_{f1} &= \{1\} \\ J_{fU_2} &= \{1, U_2\} \\ J_{fU_1 U_2} &= \{U_2, U_1 U_2\} \\ \text{so } \alpha_i &= 0 \end{aligned}$$

with some  $\alpha_i$  non-vanishing. Consider in particular a maximal  $W_i$  in the lattice  $(S_W, \geq)$  such that  $\alpha_i \neq 0$ . Then

$$X_{fW_i} \stackrel{\text{prop. 9}}{\geq} \sum_{W_j \geq W_i} \alpha_j (W_j)_{fW_i} \stackrel{\text{maximality}}{=} \alpha_i (W_i)_{fW_i} = 0,$$

by proposition 10, giving a contradiction. Therefore  $S_W$  is linearly independent in our representation.

*Proof of proposition 8:*

There are two cases to consider. Either  $W$  and  $W'$  have the same root, or they do not.

In the latter case the proposition follows from the observation that  $kF_{ij}$  is zero on any basis vector in which the  $i^{th}$  to  $j^{th}$  entries are not all distinct, so that, from their definitions, the roots of all words  $W'$  are already zero on the whole subspace containing  $v = P_R W$  if  $W' < W$ . To see this explicitly first note that each root  $E$  takes the form

$$P_R E = P_L E = 123 \dots a_1 123 \dots a_2 123 \dots a_3 \dots$$

for some  $a_1 \geq a_2 \geq a_3 \geq \dots$ . Making repeated application of proposition 4 (as in equation 11) we see that  $E_{fE}$  is a finite power of  $q$ .

**Proposition 7** Let  $u \in F(\mathcal{D}(\alpha) - \{e_\alpha\})$ , then

$$R_N(E)|u\rangle = 0.$$

*Proof:*

Here  $R_N(E)|u\rangle \neq 0$  implies  $u_1 u_2 \dots u_{a_1}$  distinct, and so on, by equation 10. But  $u \in F(\mathcal{D}(\alpha))$  implies  $u_1 = 1$  and that no number can appear as a component of  $u$  until every lower positive integer has appeared at least once more often. The only possibility for  $u \in \mathcal{D}(\alpha)$  is  $F\{e_\alpha\} = P_R E$ .

More generally we have:

$$v_{a_1} v_{a_1+1} \dots v_{a_2-1}$$

must be distinct (i.e.  $v$  is associated to the partition of  $n + 1$  whose Young diagram has as its left hand edge a column of boxes of length at least  $a_2 - 1$ ). Without loss of generality we can make these components of  $v$

$$1 2 \dots a_2 - 1.$$

The next  $F_{ij}$  in  $E$  tells us that

$$v_{a_2} v_{a_2+1} \dots v_{a_3-1}$$

are distinct. It follows that the partition associated to  $v$  satisfying  $E|v\rangle \neq 0$  will be the lowest possible in the total order if in fact

$$\{v_{a_2}, v_{a_2+1}, \dots, v_{a_3-2}\} \subset \{1, 2, \dots, a_2 - 1\}$$

(i.e.  $v$  has partition with second column at least this length). Iterating we obtain the desired result. In fact we obtain the stronger result that  $R^\lambda(E_\alpha) \neq 0$  only if  $(\lambda, \alpha)$  appears in the *natural* (partial) order of partitions (c.f. Dipper and James 1989).

In the former case we want to prove that  $W' f_W = 0$  for all  $W' \not\geq W$  (but with the same root). We can work by an induction on the sublattice  $(P(\alpha), \geq)$ .

Consider a word  $Y$ , and assume that the proposition is true for all  $W' < Y$  with *all*  $W$  (it is true for  $W' = E$ , the root of any  $W$ , by proposition 7). Without loss of generality we may assume that this includes  $W' = Y < Y$  such that  $Y.U_i = Y$ . We want to establish now that  $Y \in J_{fW}$  implies  $Y \geq W$  for all  $W$ , i.e.  $Y f_W = 0$  if  $Y \not\geq W$ , so it is sufficient to consider  $W \not\geq Y$ . Note that this gives  $W \not\geq Y < Y$ . Then with  $fW = (w, v)$  and writing  $R_N(H)|x\rangle$  as  $H|x\rangle$  for brevity

$$\langle w|Y|v\rangle = 0$$

by assumption. Now we want

$$\langle w|Y|v\rangle = \langle w|Y.U_i|v\rangle$$

which vanishes by equation 10 unless  $v$  has a maximum or minimum at  $i$ . In the latter cases it takes the form

$$\langle w|Y.U_i|v\rangle = q^{\pm 1} \langle w|Y.|v\rangle + \langle w|Y.|v'\rangle = \langle w|Y.|v'\rangle$$

where  $|v'\rangle$  is  $|v\rangle$  with 2 components interchanged. If it was a minimum then  $(w, v') = fZ$  for some  $Z > W$  such that  $Z = WU_i$ ; and since  $Z > W$  and  $W \not\geq Z$ , then  $Z \not\geq Y$ , and the proposition is true by assumption. If it was a maximum then  $(w, v') = fZ$  for some  $Z < W$  such that  $ZU_i = W$ , so  $Y \not\geq W$  implies  $Y \not\geq Z$  and again the proposition is true by assumption. This completes the proof of proposition 8.

for the universal lower bound  $W' = E$  by proposition 6). By symmetry, there is no loss of generality in assuming the proposition true for  $W' = W$ , such that  $W < W$  and  $W.U_i = W$  for some  $i$ . Then with  $fW = (w, v)$  and writing  $W.f_W$  for  $W_{wu}$  we have the inductive assumption

$$W.f_W \neq 0.$$

Noting that  $W = W.U_i$  implies  $|v\rangle = |P_R W\rangle$  has a maximum at  $i$  we have, from equations 14 and 15,

$$W|v\rangle = W.(q|v\rangle + |P_R W\rangle)$$

so that

$$W.f_W = \langle w|W|v\rangle = q\langle w|W|v\rangle + W.f_W.$$

The first term on the right vanishes by proposition 8, so we have

$$W.f_W = W.f_W = E_f E \neq 0$$

where we have used proposition 6 at the last step.

This completes the proof of proposition 9 and the main theorem.

## 4 Applications

To review: with  $A^\circ = H_n(q)$ ,  $M = V$  the  $A^\circ$ -module defined above,  $B = End_{A^\circ}(M)$ , and  $A = End_B(M)$ , we have established that  $A^\circ/K = NH_n(q)$  so  $NH_n(q) \subset A$  for all  $q$ . The next question is ... Are there any other matrices which commute with  $B$  in  $End(M)$ ? Clearly not for  $q$  indeterminate. We will see shortly that, at least for  $N = 2$  there are not for any  $q$ . We do this by computing  $B$  from  $NH_n M$ , and thence  $A$ . The final question is ... Is  $B$  given by  $U_q^n sl(N)$ , i.e. by the quotient of  $U_q sl(N)$  faithfully represented on  $M$ ? The structure of  $U_q^n sl(N)$  is known (see Lusztig 1989 and appendix A), so this will be answered in the process of answering the previous question.

### Some Algebra

The abstract algebraic problems associated with centralisers of non-semisimple algebras are interesting in their own right. We do not wish to get bogged down with what are, in the present context, technical details, so we will merely quote the results applicable here and refer to a companion paper - Martin and McAnally 1991.

Note that if  $P_i$  (resp.  $Q_i$ ) are indecomposable  $A^\circ$  (resp.  $B$ ) modules and  $R_i$  (resp.  $S_i$ ) simple  $A^\circ$  (resp.  $B$ ) modules then

$$A^\circ M = \bigoplus_i m_i P_i$$

*Proof of proposition 9:*  
We again proceed by induction using the  $(P(\alpha), \geq)$  sublattice. Let  $E$  be the root of some  $W > E$  and assume the proposition is true for  $W' < W$  (it is true

Let  $J = \text{rad}(B)$  so  $B/J$  is the maximal semi-simple quotient (as an  $B$ -module) of  $B$ , and  $B/J \subset B$  as an algebra. In the case of  $B$  semi-simple ( $J = 0$ ) then so is  $A^\circ/K$  and  $M$  may be regarded as a  $B \otimes A^\circ/K$  module.

In our case each simple and projective module of the algebra  $B/J$  may be associated to some simple  $B$  module  $S_i$ ; and

$$M = \bigoplus_i S_i \otimes P_i.$$

where  $\dim(P_i)$  is then the multiplicity of each simple module in  $B/JM$  (or  $BM$ , although not in this case as a direct summand). Obviously if  $A^o/K = A = End_B(M)$  there is a similar result with  $A$  and  $B$  interchanged.

These essentially combinatorial results provide the first stage in establishing the structure of the centraliser from that of  $N_{H_n}M$ . What remains is the effect of the internal structure of the indecomposables on the quiver diagram - which can then be computed up to Morita equivalence. We will give explicit examples shortly.

## 4.1 The case $N = 2$

Our main result is that  $NH_n M$  is faithful. Since we know the structure of  $2H_n$  (we will review it now), the content of  $M$  in the case  $N = 2$  can be deduced as follows:

Recall from I that each block (i.e. each connected piece of the quiver diagram) in  $2H_n$  takes the form either of a single simple module or, for some  $m$ , has Loewy structure

$$S_1 \oplus S_2 = \left( \bigoplus_{i=2}^m S_{i-1} \quad S_i \quad S_{i+1} \right) \oplus S_{m+1}$$

From the definition  $M$  is a direct sum of permutation modules (see I - taking the large imaginary limit of  $x$  in the definition there, or Dipper and James 1986,1989), and these may be written as a nested sequence of invariant subspaces  $M \supset M_1 \supset M_2 \dots$  such that  $M_i/M_{i+1}$  is a given Specht module (a module with structure  $s_i$  in the labelling convention above). A faithful representation which has this property must contain at least one copy of each indecomposable projective except possibly the leftmost one above (in order that the glue between copies of  $s_i$  be represented). The faithfulness, the symmetric property of the generators in  $M$ , and the defining relations, ensure that indecomposables must

look the same (in Loewy decomposition) upside down, so no other glue can be omitted.

On the other hand a simple counting argument shows that  $s_{m+1}$  is also a direct summand of  $M$ . This means that there are at least as many inequivalent indecomposable summands in  $M$

The symmetry and Specht properties allow no other indecomposable constituents.

almost all indecomposable projectives (only  $\frac{s_1}{s_2}$  type of multiplicity zero) plus copies of  $s_{m+1}$  type modules, which alone are not projective. Note that these could be quotiented out without destroying the faithfulness property, so that  $M/L$  for some (known) invariant subspace  $L$  is a faithful projective module. These observations determine the  $NH_n$  module content of  $M$  completely. We are now in a position to read off the structure of the centraliser algebra.

## 4.2 Examples

The situation is best illustrated by some examples. We write  $q = e^{i\pi/r}$ . In fact the situation differs in no qualitative way for different rational  $r$  values within  $N = 2$ , so any one well illustrates the procedure. Here are the first few cases for  $N = 2, r = 4$ :

The successive rows of the table below give the generic irreducible dimensions of  $2H_n$  for  $n = 0, 1, 2, \dots, 8$  respectively.

		1	1
	2	2	1
2		3	1
5	5	4	1
14	14	14	5
42	28	20	6
	48		1
	27	7	1
	8		1
	1		1

Next we give the (corresponding) dimensions for the various ‘permutation’ modules in the representation  $R_2$ . The representations down the left hand spine of the diagram occur only once (equal numbers of 1’s and 2’s in the basis vectors), all the others twice in each  $R_2$  ( $\dim(R_2) = 2^{n+1}$ ). Note that each representation generically contains the corresponding irreducible in the table above plus a copy of each irreducible to the right in that row (see e.g. Robinson 1962).

(ignoring multiplicities - see Martin and McAnally 1991)

$$P_2 \xrightarrow{\alpha_2} P_3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_m} P_{m+1} \xrightarrow{\alpha_{m+1}} s_{m+1}$$

where

$$\alpha_i \beta_i = \gamma_{i+1} \neq 0 \quad i = 2, 3, \dots, m$$

$\beta_i \alpha_i = \gamma_i \neq 0$   
and all other composite morphisms are zero. Altogether this glues the simples in  $B$  together as follows (with the multiplicities now forced)

The following table gives the indecomposable content of each of the blocks above for  $r = 4$ , as forced by the symmetry, faithfulness and Specht conditions:

1 + 1	2 + 1	1	1	3
2 + 1	1	2	1	2
4 + 1	1	4 + 1	1	3
5 + 1	5 + 1	5 + 1	1	1
4 + 1	4 + 1	4 + 1	1	2
14 + 6 + 1	14 + 6 + 1	14 + 6 + 1	6 + 1	1
20 + 8 + 1	20 + 8 + 1	20 + 8 + 1	1	1
48 + 8 + 1	48 + 8 + 1	8 + 8 + 1	1	1
26	26	26	1	1
48 + 16 + 8 + 1	48 + 16 + 8 + 1	8 + 8 + 1	1	1
26	26	26	1	1

Note that the  $n = 7$  spine block is not faithful, and that quotienting by the trivial representation in the larger algebra followed by a vertical move is identical to the action of the standard Morita equivalence functor from  $2H_{n+2} \rightarrow 2H_n$  (from Martin and Westbury 1991).

Finally, in the following table we form the centraliser structure corresponding to that in the diagram above (in the form (multiplicity).(dim. of indecomposable)). The dimensions of simples come from the multiplicities of indecomposables in the diagram above using the results quoted in the previous section (not forgetting that each block has multiplicity 2 unless on the left hand spine), and so on. The structures of the indecomposables come from the observation that the quiver blocks of  $NH_n M$  from above contain indecomposables with Loewy structure

$$\left( \bigoplus_{i=2}^m s_{i-1} \otimes s_{i+1} \right) \oplus s_m \oplus s_{m+1}.$$

We thus read off the non-trivial morphisms between the indecomposables  $P_i \subset 2H_n M$  (which commute with the action of  $2H_n$  and are in the radical of  $B$ ) as

### Acknowledgements

I would like to thank Bruce Westbury and Fred Goodman for useful conversations. This work is supported in part by a grant of the Commemorative Association for the Japan World Exposition (1970). It has also been greatly helped by grants from the Nuffield Foundation and the Royal Society.

## References

- C W Curtis and I Reiner, Representation theory of finite groups and associative algebras (Wiley, New York, 1962).
- E Date, M Jimbo, M Miki and T Miwa, RIMS preprint 1990.
- R Dipper and G James, Proc.London Math.Soc.(3)52 (1986) 20.
- R Dipper and G James, Proc.London Math.Soc.(3)59 (1989) 23.
- V G Drinfeld, in ICM proceedings, Berkeley, (1986) 798-820.
- G James and A Kerber, The Representation Theory of the Symmetric Group (Addison-Wesley, Reading Ma., 1981).
- M Jimbo, Lett. Math.Phys. 10 (1985)63.
- G Lusztig, 1989 unpublished (see also 1988 Advances in Math.70,237).
- I G Macdonald, Symmetric Functions and Hall Polynomials (OUP, Oxford, 1979).
- P P Martin, Potts Models and related problems in Statistical Mechanics (World Scientific, Singapore, 1991).
- P P Martin and D McAnalley, On Schur-Weyl duality and non-semisimple algebras, in preparation (1991).
- P P Martin and V Rittenberg, A Template for Quantum Spin Chain Spectra, RIMS preprint (1991), submitted to Int.J.Mod.Phys.A.
- P P Martin and B W Westbury, A Morita Equivalence theorem for Centraliser Algebras of Quantum Linear Groups, Warwick preprint 34/1991 (submitted to J. Algebra).
- R S Pierce, Associative Algebras (Springer-Verlag, New York, 1982).
- G de B Robinson, Representation theory of the Symmetric Group (University of Toronto, Toronto, 1961).
- B W Westbury, Cyclic Cohomology, University of Manchester Ph.D. thesis (1990).
- A V Zelevinskii, Funct. Anal. 21 (1987) 152.

**Appendix A: Quantum  $sl(N)$**

Here we briefly review the properties of  $U_q sl(N)$  for comparison with the results of section 4.

**Definition 21** For  $N$  a positive integer and  $q$  an indeterminate define  $U_q^g sl(N)$  as a unital associative bialgebra over  $\mathbb{C}[q^{\pm 1}]$  with generators  $1, e_i, f_i, k_i^{\pm 1}$  ( $i = 1, 2, \dots, N-1$ ) in the following way:

where it is to be understood that  $r$  is taken to its specialisation after reducing the ratio to its lowest form.

$$e_i v_j = \delta_{ij-1} v_{j-1}$$

$$f_i v_{j-1} = \delta_{ij-1} v_j$$

$$k_i v_j = (\delta_{ij} q + \delta_{ij-1} q^{-1}) v_j.$$

All finite dimensional indecomposable representations appear as constituents of those generated from this one by use of the coassociative comultiplication, which is given by

$$m(e_i) = e_i \otimes 1 + k_i \otimes e_i$$

$$m(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i$$

$$m(k_i) = k_i \otimes k_i.$$

Truncating this procedure at the  $n^{th}$  comultiplication gives the quotient algebra  $U_q^n sl(N)$ .

For example, for  $N=2$ , the complete list of finite irreducible representations is as follows. There is a one dimensional representation  $e = f = 0, k = 1$ , and then one of each dimension,  $p$ , called  $r_{p-1}$ , with the action on a basis  $\{v_1, v_2, \dots, v_p\}$  given by

$$(e) \quad v_i = [i-1] v_{i-1}$$

$$(f) \quad v_i = [p-i] v_{i+1}$$

$$(k) \quad v_i = q^{p+1-2i} v_i.$$

Note that  $r_1$  is our defining representation. It follows that

$$r_1 \otimes r_m = r_{m-1} \oplus r_{m+1} \tag{16}$$

and the content of the various comultiplications of  $r_1$  can be deduced from this.

The above definitions hold for  $q$  specialised to any non-zero complex number other than a root of unity. There are some inequivalent choices available for the definition of  $U_q sl(N)$  in the specialisation to  $q$  a root of unity. The one appropriate for its roll as a centraliser algebra (but which excludes the so called cyclic representations, c.f. Date et al 1990) is the following.

**Definition 22** For  $q = e^{i\pi/r}$  and  $r$  integer we define  $U_q sl(N)$  as before, except to include additional generators

$$e_i^{(r)} = e_i^r / [r]!$$

and

$$f_i^{(r)} = f_i^r / [r]!$$

where it is to be understood that  $r$  is taken to its specialisation after reducing the ratio to its lowest form.

Note that in the specialisation to  $r$  integer equation 16 does not generally hold. However  $U_q s\ell(N)$  is in fact a Hopf algebra, which ensures that the product of any module with a projective module is projective.

**Proposition 10** *In the specialisation to  $r$  integer the representations  $r_p$  remain well defined and indecomposable, but develop an irreducible invariant subspace with basis*

$$\{v_{i+1} : i = 0, 1, 2, \dots, (p-1)_{mod,r} \bmod r\}$$

*and irreducible quotient. For  $p+1 = mr + k$  the quotient is isomorphic to the invariant subspace for  $p+1 = mr - k$ .*

*Proof:* In this case it follows from the definition of  $r_p$  and  $[s]$  that all actions are zero except

$$\begin{aligned} e v_i &\propto v_{i-1} & (i \neq 1 \bmod r) \\ f v_i &\propto v_{i+1} & (p-i \neq 0 \bmod r) \\ e^{(r)} v_{i+4} &\propto v_{i-1} & (i = 1 \bmod r) \\ f^{(r)} v_i &\propto v_{i+4} & (p-i = 0 \bmod r). \end{aligned}$$

The representations are thus block upper triangular in two blocks when written with the specified basis elements first. The irreducibility and indecomposability properties follow from continuity with the generic case.

For example, with  $r=4$  we have the following table: The top line gives the generic irreducible dimensions and below are the non-generic Loewy decompositions into simple modules:

$$p+1 : 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ \dots \ mr \ mr+1 \ mr+2 \ \dots$$

$$r_p : \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 3 & 2 & 1 & 8 & 6 \end{array} \dots \begin{array}{c} mr \\ m(r-1) \\ m+1 \end{array} \begin{array}{c} mr \\ m(r-2) \\ 2(m+1) \end{array} \dots$$

where the top representation is isomorphic to the first one of that dimension found by moving to the left, and the invariant subspace is not isomorphic to any representation to the left.

Note that  $r_{mr}$  ( $m$  a natural number) remains irreducible in proposition 10, so it will also be projective.

It is the representation on  $M = \otimes^{n+1} \mathcal{G}^N$  (obtained, up to isomorphism, by repeating the comultiplication in definition 21  $n$  times) which commutes with the action of  $H_n(q)$ . Strictly speaking the action of  $H_n(q)$  required is the isomorphic action obtained from equation 7 by replacing  $1 \rightarrow -1$  in the off-diagonal elements. This may be readily verified by direct computation. In

particular note that the additional generators in definition 22 must be included in the centraliser algebra  $B$ .

## Appendix B: The vector $h$

**Definition 23** *For each non-negative integer  $m$  define an  $m+1$  component row vector*

$$h_m = (q^m, q^{m-1}, \dots, q^0).$$

*and then  $s \in \mathcal{G}^{N!}$  by*

$$s = \otimes_{m=1}^{N-1} h_m.$$

**Remark 2** *There is an ordering of the  $\nu_N$  basis such that  $R_N^{\nu_N}(Y_N) = s^t s$ .*

To see this introduce a basis  $B$  for the  $N!$  dimensional space  $L = \otimes_{m=1}^N \mathcal{G}_m$  (isomorphic to the  $\nu_N$  subspace of  $V$ , which we will call  $V^{\nu_N}$ ) as follows:

$$B = \{(a_1 a_2 \dots a_N) : a_j \in \{0, 1, 2, \dots, j-1\}\}.$$

The isomorphism is given by

$$J : V^{\nu_N} \rightarrow L$$

defined on the given bases by

$$J : \alpha \rightarrow a$$

where

$$a_i = \text{No.}(\alpha_j <_i \text{ s.t. } \alpha_j > \alpha_i).$$

**Definition 24** *Define a length function*

$$l : B \rightarrow \mathbb{Z}$$

$$by \quad l(a) = N - \sum_i a_i.$$

We will also write  $l(\alpha)$  for  $\alpha \in V^{\nu_N}$  to mean  $l(J(\alpha))$ .

Then for  $a \in B$  and  $i = 1, 2, \dots, N-1$  define matrices  $M_i \in \text{End}(L)$  by

$$M_i a = \overbrace{qa + (a_1 a_2 \dots a_{i-1} \ a_{i+1} \ a_i + 1 \dots a_N)}^b \quad ((J^{-1}a)_{i+1} > (J^{-1}a)_i).$$

$$M_i b = q^{-1}b + \overbrace{(b_1 b_2 \dots b_{i-1} \ b_{i+1} - 1 \ b_i \dots b_N)}^a.$$

It follows by direct computation that

$$R_N^{\nu_N} : H_{N-1}(q) \rightarrow End_{\mathbb{G}}(L)$$

is given by

$$R_N^{\nu_N}(U_i) = M_i$$

and the remark follows from this.

#### Corrections to RIMS-782

Page 10:

The partial order in Definition 11 should be denoted  $\triangleleft$  and not  $<$ .

We use another partial order. Let a sequence of moves from  $s$  to  $t$  in Definition 11 be recorded by  $t = s^{i_1 i_2 \dots i_m}$ , i.e.

$$s \rightarrow s^{i_1} \rightarrow s^{i_1 i_2} \rightarrow \dots \rightarrow t = s^{i_1 i_2 \dots i_m}$$

then:

Definition 11b Define a partial order  $(D(\alpha), \leq)$  by  $\{s\} \leq \{t\}$  iff for some list  $i_1 i_2 \dots i_m$  such that  $t = e_\alpha^{i_1 i_2 \dots i_m}$  there exists a sublist, say  $i_1 i_2 \dots i_k$  ( $k \leq m$ ) with  $j_1 < j_2 < \dots < j_k$  and such that  $s = e_\alpha^{i_1 i_2 \dots i_k}$ .

Page 11:

Definition 12 applies to the partial order in Definition 11b.

Section 3.2.1: Clarification of definition of  $(e_\alpha, e_\alpha)$ . For each partition  $\alpha$  of  $k+1$  let us define elements  $(e_\alpha, e_\alpha) \in H_k(q)$  iteratively as follows. Firstly take  $(e_1, e_1) = 1$  and suppose  $(e_{\alpha_-}, e_{\alpha_-})$  defined for all  $\alpha_-$  partitions of  $k$  or less. For each  $\alpha$  a partition of  $k+1$  there is some  $\alpha_-$  a partition of  $l < k+1$  obtained from  $\alpha$  by removing the first column of partition shape  $\alpha$ . This column has length  $m = k+1-l$  (note that  $m \geq (\alpha_-)_1$ , the length of the first column in  $\alpha_-$ ). Then

$$(e_{\alpha_-}, e_{\alpha_-}) = Y_m \left( (e_{\alpha_-}, e_{\alpha_-})^{(m)} \right)$$

(recall that  $X^{(m)}$  denotes the translation of  $X$  by  $U_i \mapsto U_{i+m}$ ).

Definition 15 Equation (12): let  $g_i = 1 - q U_i$ , then

$$(s^i \circ t) = g_i(s \circ t).$$

Correspondingly we must replace  $U_i$  by  $g_i$  in examples on pages 11, 13, 14.

Page 13: Definition 20 no longer gives a lattice. Correspondingly on page 15 (line 8) we have only a poset  $(Sw, \geq)$ .

Page 16: From the modification of Definition 15 we have:

- line 15:  $Y \cdot g_i = Y$
- line 21:  $\langle w|Y|v\rangle = \langle w|Y \cdot g_i|v\rangle$
- line 26:  $Z = W \cdot g_i$
- line 28:  $Z \cdot g_i = W$ ; and the subsequent implication follows from Definition 11b.
- Page 17:
- line 3:  $W \cdot g_i = W$
- line 6:  $W = W \cdot g_i$
- line 8:  $W|v\rangle = W((1 - q^2)|v\rangle - q|P_R W\rangle)$
- line 10: similarly;
- line 12:  $W_{JW} = -q W_J W \neq 0$