# CODING THEORY: SOLUTIONS 5

(1) (a) $G$ is the generator matrix for $C$ if $G \cdot H^t = 0$, so we compute and get,

$$G \cdot H^t = \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as required.

(b) In our representation of the alphabet in $\mathbb{Z}_3^3$ $P$ is represented by 121. To encode multiply 121 by $G$ to get,

$$(121) \cdot \begin{pmatrix} 2 & 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 2 & 1 \end{pmatrix} = 661221 \cong 001221$$

(c) As $d = 2(1) + 1$ we know that *all* weight 1 vectors in $\mathbb{Z}_3^6$ are coset leaders (see problems 4 or Thm 3.69). To calculate their syndromes we multiply each vector by $H^t$. We therefore get,

$$\begin{aligned} S(100000) &= 100, & S(200000) &= 200 \\ S(010000) &= 010, & S(020000) &= 020 \\ S(001000) &= 110, & S(002000) &= 220 \\ S(000100) &= 210, & S(000200) &= 120 \\ S(000010) &= 001, & S(000020) &= 002 \\ S(000001) &= 101, & S(000002) &= 202 \end{aligned}$$

(d) We first note that for any $a, b, c \in \mathbb{Z}_3$ if we encode $abc$ using $G$ we get,

$$(abc) \cdot G = (2a + b + 2c, 2a + 2b, \underline{a}, \underline{b}, 2c, \underline{c}),$$

and so we can decode the encoded codeword by looking at its $3^{rd}$, $4^{th}$ and last digits. The decoding process then works as follows:

- For each codeword $x$ calculate $x \cdot H^t$,
- If $x \cdot H^t = 000$ then we may read off the representation of the letter from the $3^{rd}, 4^{th}$ and last digits of $x$.
- If $x \cdot H^t \neq 000$ then we need to find a coset leader $e$ such that $e \cdot H^t = x \cdot H^t$. We then read off the representation of the letter from the $3^{rd}, 4^{th}$ and last digits of $x - e$.

We therefore get,

- $212012 \cdot H^t \cong 000$, thus $212012 \Rightarrow 202 \Rightarrow T$.
- $012212 \cdot H^t \cong 220$, so we need coset leader $e$ such that $e \cdot H^t = 220$. Looking this up in our table of syndromes we have $S(002000) = 220$ so $e = 002000$ will do. Then we decode $012212 - 002000 = 010212 \Rightarrow 022 \Rightarrow H$.
- $220112 \cdot H^t \cong 000$, thus $220112 \Rightarrow 012 \Rightarrow E$.
- $112100 \cdot H^t \cong 210$. As $S(000100) = 210$ we decode $112100 - 000100 = 112000 \Rightarrow 200 \Rightarrow R$.
- $220112 \Rightarrow E$ by above.
- $000000 \cdot H^t = 0$, thus $000000 \Rightarrow 000 \Rightarrow$ *space*.
- $200021 \cdot H^t \cong 000$, thus $200021 \Rightarrow 001 \Rightarrow A$.
- $112000 \Rightarrow R$ by above.
- $220112 \Rightarrow E$ by above.
- $000000 \Rightarrow$ *space*, by above.
- $022021 \cdot H^t \cong 010$. As $S(010000) = 010$ we decode $022021 - 010000 = 012021 \Rightarrow 201 \Rightarrow S$.
- $221000 \cdot H^t \cong 000$, thus $221000 \Rightarrow 100 \Rightarrow I$.
- and so on.

(2) (a) $C_1$ is not cyclic as it is not closed under cyclic shifts, e.g. $11011 \in C_1$ but the cyclic shift $11101 \notin C_1$

(b) $C_2$ is both linear and closed under cyclic shifts of codewords, it is therefore cyclic.

(c) $C_3$ not cyclic for although it is closed under cyclic shifts of codewords, it is not linear, e.g. $100, 001 \in C_3$ but $100 + 001 = 101 \not\in C_3$

(d) $C_4$ is both linear and closed under cyclic shifts, it is therefore cyclic.

(3) (a) The parity check matrix of $Ham(\mathbb{Z}_2^4)$ takes as its columns all of the non-zero vectors of $\mathbb{Z}_2^4$. We can arrange these in any order but as we also want to find the generator matrix it makes most sense to arrange it in standard form.

Therefore,

$$H_{Ham(\mathbb{Z}_2^4)} = \begin{pmatrix} 1&1&1&1&0&1&0&0&1&1&0&1&0&0&0 \\ 1&1&1&0&1&1&0&1&0&0&1&0&1&0&0 \\ 1&1&0&1&1&0&1&1&0&1&0&0&0&1&0 \\ 1&0&1&1&1&0&1&0&1&0&1&0&0&0&1 \end{pmatrix}$$

$$G_{Ham(\mathbb{Z}_2^4)} = \begin{pmatrix} 1&0&0&0&0&0&0&0&0&0&0&1&1&1&1 \\ 0&1&0&0&0&0&0&0&0&0&0&1&1&1&0 \\ 0&0&1&0&0&0&0&0&0&0&0&1&1&0&1 \\ 0&0&0&1&0&0&0&0&0&0&0&1&0&1&1 \\ 0&0&0&0&1&0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&1&0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&1&1 \\ 0&0&0&0&0&0&0&1&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&0&1&0&0&1&0&0&1 \\ 0&0&0&0&0&0&0&0&0&1&0&1&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&1&0&1 \end{pmatrix}$$

(b) The parity check matrix of $Ham(\mathbb{Z}_7^2)$ takes as its columns representatives of each of the projective equivalence classes of $\mathbb{Z}_7^2$. As above we arrange this in standard form to get,

$$H_{Ham(\mathbb{Z}_7^2)} = \begin{pmatrix} 1&1&1&1&1&1&1&0 \\ 1&2&3&4&5&6&0&1 \end{pmatrix}$$

$$G_{Ham(\mathbb{Z}_7^2)} = \begin{pmatrix} 1&0&0&0&0&0&-1&-1 \\ 0&1&0&0&0&0&-1&-2 \\ 0&0&1&0&0&0&-1&-3 \\ 0&0&0&1&0&0&-1&-4 \\ 0&0&0&0&1&0&-1&-5 \\ 0&0&0&0&0&1&-1&-6 \end{pmatrix}$$