Prediction

MATH1210

21 February 2017

Predicting the future

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in the next minute? tomorrow? next year?

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But mathematics (and science) oftens attempts the future, by recognising and understanding patterns.

> *"A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with ideas" — G. H. Hardy*

Prediction is sometimes easy, and sometimes hard.

> *"I never predict anything, and I never will." — Paul Gascoigne*

Predict the next number...

What comes next, and why?

$$1, 3, 5, 7, 9, 11, \ldots$$

$$1, 2, 4, 7, 11, 16, 22, \ldots$$

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

$$2, 3, 5, 7, 11, 13, 17, 23, \ldots$$

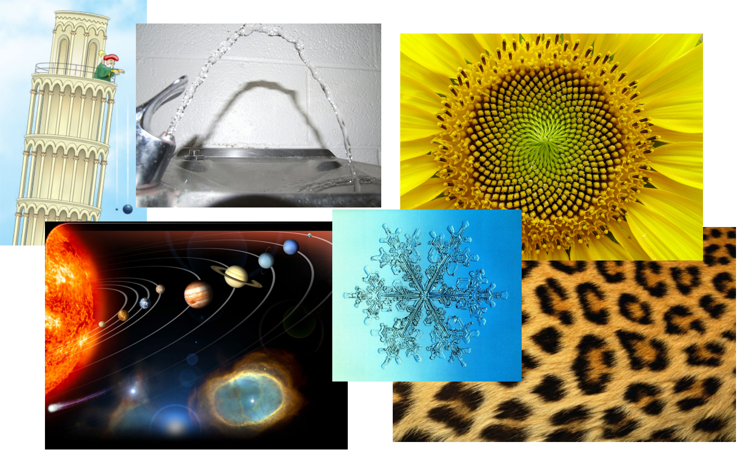
$$2, 4, 16, 256, 65536, \ldots$$

$$1, 2, 4, 8, 16, \ldots$$

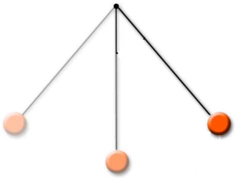
$$1, 1, 3, 14, 173, \ldots$$

Patterns and Order in the real world

Examples of natural patterns



Example: The pendulum

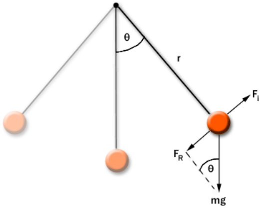


In the early 17th century Galileo studied the pendulum:

He found that the swing time is constant and does not depend on

- the starting position
- the initial push
- when the experiment is being done.

Newton!



In 1687 Isaac Newton published his theory of mechanics and showed that the motion of the pendulum can be described by his laws of motion.

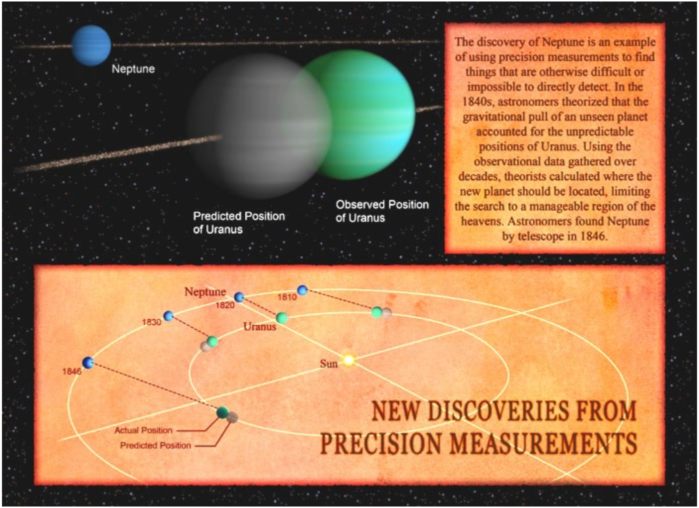
$$l\frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + g\sin\theta = 0.$$

$\Rightarrow$ Order and patterns in motion follow from maths

The big idea

- Write down the equations describing a physical system
- Solve them
- Predict what happens to the system.

Does it work? It certainly does.



(Scientific American, December 2004)

Weather forecast

Navier-Stokes equation:

$$u_t + u.\nabla u = -\nabla P + \frac{1}{Re}\nabla T$$

This seems to be much harder...

Determinism and Laplace's demon

We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at a certain moment would know all forces that set nature in motion, and all positions of all items of which nature is composed, if this intellect were also vast enough to submit these data to analysis, it would embrace in a single formula the movements of the greatest bodies of the universe and those of the tiniest atom; for such an intellect nothing would be uncertain and the future just like the past would be present before its eyes. - Pierre Simon Laplace, 1814.

That is,

> *If we knew the exact position and speed of every particle in the universe, then we could predict the future.*

Hard to predict

Find examples of natural events, which seem hard or impossible to predict!

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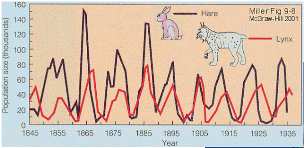
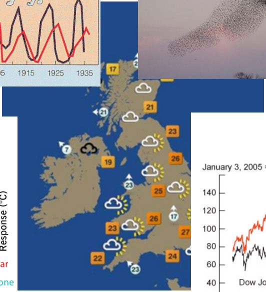
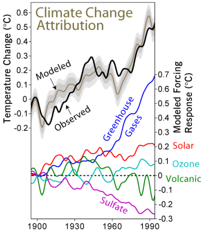
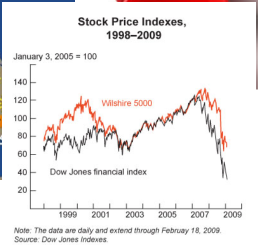








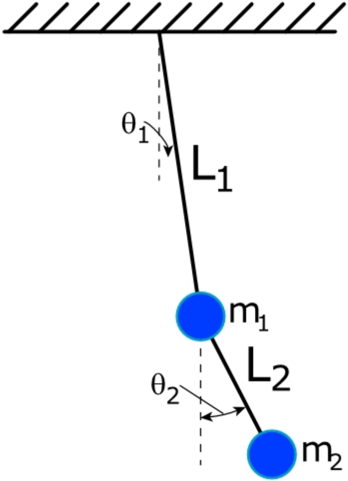
The big question

Does complex behaviour arise, because

- nature is complicated and unpredictable?
- the equations are too difficult?
- we just don't know enough?

Or do Newton's laws themselves give rise to unpredictable behaviour?

Example: The double pendulum



Equations and types of motion

$$\begin{aligned}\theta_1'' &= [-g(2m_1+m_2)\sin\theta_1 - m_2 g\sin(\theta_1-2\theta_2) \\ &\quad -2\sin(\theta_1-\theta_2)m_2(\theta_2'^2 L_2+\theta_1'^2 L_1\cos(\theta_1-\theta_2))] \,/ \\ &\quad\quad [L_1(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))]\end{aligned}$$

$$\begin{aligned}\theta_2'' &= \big[2\sin(\theta_1-\theta_2)(\theta_1'^2 L_1(m_1+m_2)+g(m_1+m_2)\cos\theta_1 \\ &\quad +\theta_2'^2 L_2 m_2\cos(\theta_1-\theta_2))\big] \,/ \\ &\quad\quad [L_2(2m_1+m_2 m_2\cos(2\theta_1-2\theta_2))]\end{aligned}$$

Motion can be

- periodic in phase or out of phase (predictable)
- irregular and ‘chaotic’ (unpredictable)

The simplest class of models

We have ‘seen’ that Newton’s laws naturally give rise to complex motion in physical systems.

$\Rightarrow$ It seems that we cannot ignore ‘chaos’.

But Newton’s laws still involve differential equations.

As a much simpler model, we will consider **iterated maps** in the following.

We will see that even the simplest rules can lead to very complicated and unpredictable behaviour.

$\Rightarrow$ Chaos!

Iterated Maps

A map is a relation between one number and another: For example, the map $f : x \mapsto x^2$ maps the number x onto its square $f(x) = x^2$.

An iterated map means performing the same relation again and again: With $x \mapsto x^2$ and starting at $x = 2$, we get

$$2 \mapsto 4,\ \ 4 \mapsto 16,\ \ 16 \mapsto 256,\ \ \text{etc.}$$

If we write this as a list of numbers, we can set $x_0 = 2$, then $x_1 = 4$, $x_2 = 16$, $x_3 = 256\ldots$.

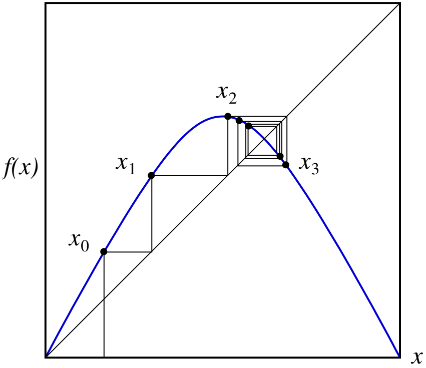
$\Rightarrow$ An iterated map gives rise to a sequence of numbers.

Task 1

Decide where the iterations of the following maps finally end up, starting with first values as suggested:

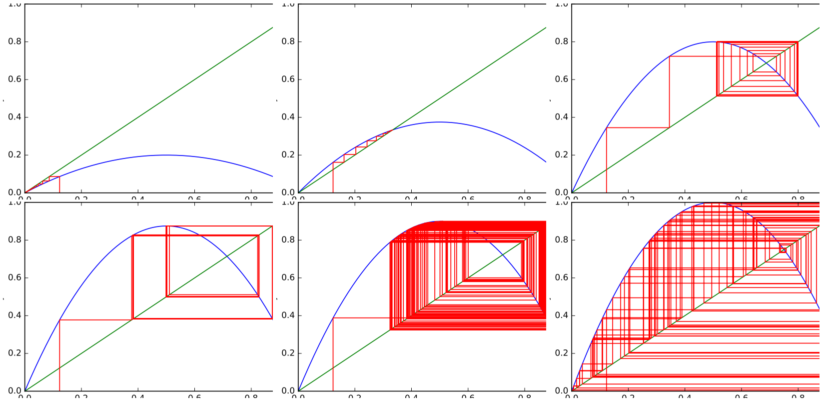
- $x \mapsto 0.8x(1-x)$ with a first value between 0 and 1
- $x \mapsto 1.5x(1-x)$ with a first value between 0 and 1
- $x \mapsto 3.2x(1-x)$ with a first value between 0 and 1
- $x \mapsto 3.5x(1-x)$ with a first value between 0 and 1
- $x \mapsto 3.6x(1-x)$ with a first value between 0 and 1
- $x \mapsto 4.0x(1-x)$ with a first value between 0 and 1

Graphical iteration: Cobweb diagrams

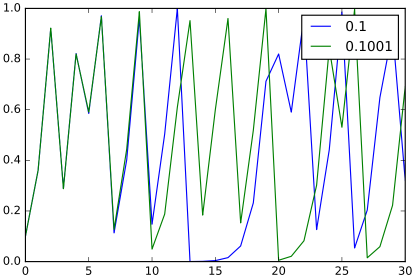


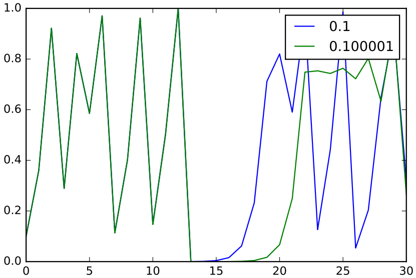
Cobweb diagram for $f(x) = 3.2 \cdot x(1-x)$.

Results



The case $r = 4$ — sensitive dependence on initial conditions





Chaos for $r = 4$

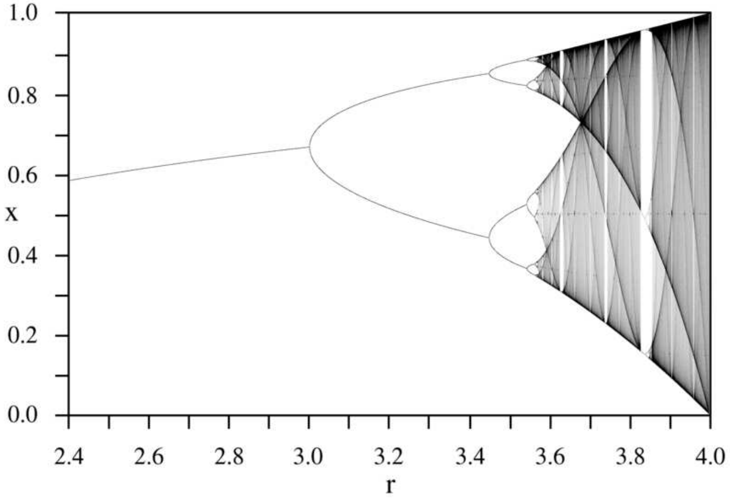
We see that when $r = 4$, the iteration gives rise to sequences which do not seem to approach any number.

The sequences do not seem to follow any patterns, and look pretty irregular in the diagrams.

Moreover, even if the first values of two sequences are close to together, the sequences look completely different after a certain number of iterations. ('Sensitive dependence on initial conditions')

All of this makes the iterations unpredictable, and we call this map chaotic.

Where does the chaos come from?



The Lorenz attractor and the butterfly effect

"One meteorologist remarked that if the theory were correct, one flap of a sea gull's wings would be enough to alter the course of the weather forever. The controversy has not yet been settled, but the most recent evidence seems to favor the sea gulls." — Edward Lorenz

Chaos: When the present determines the future, but the approximate present does not approximately determine the future." — Edward Lorenz