HW5 José A Melchor Soto

1

a) $G = \ln S^2$

$$\frac{\partial G}{\partial S} = \frac{2}{S} \qquad \frac{\partial^2 G}{\partial S} = -\frac{2}{S^2} \qquad \frac{\partial G}{\partial t} = \emptyset$$

$$dS = S\mu dt + S\sigma dW \Rightarrow \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}dS + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}dS^2$$

$$= \cancel{\emptyset}dt + \frac{2}{S}dS - \cancel{\frac{1}{2}}\cancel{\frac{2}{S^2}}^{\frac{1}{S^2}} dS^2$$

$$= \frac{2}{S}dS - \frac{1}{S^2}dS^2$$

$$= \frac{2}{\cancel{S}}(\cancel{S}\mu dt + \cancel{S}\sigma dW) - \frac{1}{S^2}(S\cancel{\mu dt}^{0} + S\sigma dW)^2$$

$$= 2\mu dt + 2\sigma dW - \frac{1}{S^2}(S\sigma dW)^2$$

$$= 2\mu dt + 2\sigma dW - \frac{1}{\cancel{S^2}}(\cancel{S^2}\sigma^2 dt)$$

$$= 2\mu dt + 2\sigma dW - \sigma^2 dt$$

$$= 2\sigma dW + (2\mu - \sigma^2)dt$$

- $E[dG] = E[(2\mu-\sigma^2)dt + 2\sigma dW]$

$$E[dG] = (2\mu - \sigma^2)dt$$

- $V[dG] = V[\cancel{(2\mu-\sigma^2)dt}^{0} + 2\sigma dW]$

$$= V[4\sigma^2 dW]$$

b) $\mu = 0.3 \quad \sigma = 0.1 \Rightarrow P(dG > 0)$

- $dS = S(0.3)dt + S(0.1)dW$

$$E[dG] = (2\mu - \sigma^2)dt = (2(0.3) - (0.1)^2 dt) = 0.59$$

$$V[dG] = (4(0.1)^2 dW) = 0.04$$

- $P(dG>0) = \frac{0 - E[dG]}{\sqrt{V[dG]}} = \frac{0 - 0.59}{\sqrt{0.04}} = -2.95 = 99.84\ \%$

- $\ln(S_{t+1}) = \ln(S_t) + [\mu - \frac{1}{2}\sigma^2]dt + \sigma dW$

$$= \ln(S_t) + [0.3 - \tfrac{1}{2}(0.1)^2]dt + 0.1\,dW$$

$$= \ln(S_t) + 0.305\,dt + 0.1\sqrt{T}\,\epsilon$$

$$E[\ln(S_{t+1})] = \ln(S_t) + 0.295\,dt$$

$$V[\ln(S_t+1)] = (0.1\sqrt{T})^2 = 0.01$$

$$P(\ln(S_t+1) > \ln(S_t)) = \frac{P(z > \ln(S_t) - \ln(S_t) - 0.295)}{\sqrt{0.01}}$$

$$P(z > -2.95) = 99.84\ \%$$

Ambos probabilidades son igualmente probables, tanto el cambio de precio del derivado, como que el precio aument el próximo año. Con un 99.84%.

(2) $dZ(t) = 0.05Z(t)dt + 0.20\,Z(t)dW(t)$

$\mu = 0.05 \quad \sigma = 0.2$

$x(t) = \ln Z(t) \qquad Z(0) = 100$

$\frac{\partial x}{\partial t} = \emptyset \qquad \frac{\partial x}{\partial z} = \frac{1}{z} \qquad \frac{\partial^2 x}{\partial z^2} = -\frac{1}{z^2}$

$\frac{\partial x}{\partial t}dt + \frac{\partial x}{\partial z}dz + \frac{1}{2}\frac{\partial^2 x}{\partial z^2}dz^2$

$\emptyset\,dt + \frac{1}{z}dz - \cancel{\frac{1}{2}\cdot\frac{1}{z^2}}^{\frac{1}{2z^2}} = \frac{1}{z}dz - \frac{1}{2z^2}$

$dZ(t) = 0.05Z(t)dt + 0.20\,Z(t)dW(t)$

$dz(t) = \frac{1}{\cancel{z}}(0.05\cancel{z}dt + 0.20\cancel{z}dW) - \frac{1}{2z^2}(\cancel{0.05z\,dt}^{0} + 0.20z\,dW)^2$

$= 0.05\,dt + 0.20\,dW - \frac{1}{2}\cancel{z^2}(0.04\cancel{z^2}dt)$

$= 0.05\,dt + 0.20\,dW - \cancel{\frac{0.04}{2}}^{0.2}dt = 0.03\,dt + 0.20\,dW$ ✓

$dx = 0.03\,dt + 0.20\,dW \qquad E[X[0]] = E[\ln(100)] = \ln 100$

- $\int_0^1 E[dx] = \int_0^1 E[0.03\,dt + \cancel{0.20\,dW}^{0}] = \int_0^1 0.03\,dt = 0.03 = 0.03 + \ln 100 = E[X(1)]$ ✓
- $\int_0^t V[dx] = \int_0^t V[\cancel{0.03\,dt}^{0} + 0.20\,dW] = \int_0^t 0.04\,dW = 0.04t$ ✓

(3) $dS(t) = S(t)\mu\,dt + S\sigma\,dW(t)$

$S(t) = S_0\,e^{(\mu - \frac{1}{2}\sigma^2)dt}e^{\sigma dW}$ ✓

$\text{Var}[S(t)] = \text{Var}[S_0 e^{(\mu-\frac{1}{2}\sigma^2)dt\,\sigma dW}] = S_0^2 e^{2(\mu-\frac{1}{2}\sigma^2 dt)}\text{Var}[e^{\sigma dW}]$

$= S_0^2 e^{2(\mu-\frac{1}{2}\sigma^2 dt)}[e^{2(0)+2(\sigma^2 dt)} - e^{2(0)+\sigma^2 dt}]$

$= S_0^2 e^{2(\mu-\sigma^2/2)dt}[e^{2\sigma^2 dt} - e^{\sigma^2 dt}]$

$= S_0^2 e^{2\mu dt}\cdot e^{-\sigma^2 dt}[e^{2\sigma^2 dt} - e^{\sigma^2 dt}] = S_0^2 e^{2\mu dt}[e^{\sigma^2 dt} - 1]$ ✓

④ $dS(t) = 0.5\,S(t)\,dt + 0.25\,S(t)\,dW(t)$

$$E[S_t] = E[S_0 e^{(\mu - \frac{\sigma^2}{2})dt} e^{\sigma dW}] = S_0 e^{\mu dt}$$

- $S_T = S_0 e^{(0.5 - \frac{1}{2}(0.25)^2)dt} e^{0.25dW}$

$$S_T = S_0 e^{0.5dt} e^{0.25dW}$$

$$E[S_T] = S_0 e^{0.5dt}$$

$$V[S_T] = S_0^2 e^{2(0.5)dt} [e^{0.25^2 dt} - 1]$$

- $\ln S_t = \ln S_0 + [0.5 - \frac{0.25^2}{2}]dt + 0.25\,dW$

$$E[\ln S_t] = \ln S_0 + [0.5 - \frac{0.25^2}{2}]dt = \ln S_0 + 0.46875\,dt$$

$$V[\ln S_t] = (0.25)^2 dt = 0.0625\,dt$$

- $P(S_t > 1.3 S_0)$

$$= P(\ln S_t > \ln(1.3 S_0)) = P\left(z > \frac{0.262 - 0.46875}{0.25}\right) = P(z > -0.827)$$

$$= P(S_t > 1.3 S_0) = 79.5\%.$$

⑤ $dZ(t) = K\mu Z(t)\,dt + \sigma Z(t)\,dW(t)$

- $dX(t)$, $E[dX(t)]$, $V[dX(t)]$

$$X = Z^2$$

$$\frac{\partial x}{\partial t} = \varnothing \quad \frac{\partial x}{\partial z} = 2z \quad \frac{\partial^2 x}{\partial z^2} = 2$$

ITO

$$dx = 2z\,dz + dz^2$$

$$dx = 2z(K\mu z\,dt + \sigma z\,dW) + (\cancel{K\mu z\,dt} + \sigma z\,dW(t))^2$$

$$dx = 2z^2 K\mu\,dt + 2z^2\sigma\,dW + \sigma^2 z^2 dt$$

$$dx = z^2 dt\,(2K\mu + \sigma^2) + 2z^2\sigma\,dW$$

$$E[dx] = E[z^2 dt(2K\mu+\sigma^2) + 2z^2\sigma dW] = z^2 dt(2K\mu+\sigma^2)$$

$$V[dx] = V[z^2 dt(2K\mu+\sigma^2) + 2z^2\sigma dW] = 4z^4\sigma^2 dw$$

⑥ $$dx = a(x_0 - x)dt + sx\,dW$$

$$z = e^{-x(T-t)}$$

$$\frac{\partial z}{\partial t} = e^{-x(T-t)}x \qquad \frac{\partial z}{\partial x} = -(T-t)e^{-x(T-t)} \qquad \frac{\partial^2 z}{\partial z} = (T-t)^2 e^{-x(T-t)}$$

$$dz = e^{-x(T-t)}x\,dt - (T-t)e^{-x(T-t)}dx + \frac{1}{2}(T-t)^2 e^{-x(T-t)}dx^2$$

$$= e^{-x(T-t)}x\,dt - (T-t)e^{-x(T-t)}dx\,(a(x_0-x)dt + sx\,dW) + \frac{1}{2}((T-t)^2 e^{-x(T-t)}\,a(x_0-x)dt + sx\,dW)^2$$

$$= z\,(x - (T-t)\,a(x_0 - x) + \frac{1}{2}((T-t)^2 s^2 x^2)dt - z(T-t)S_x dW$$

⑦ $$dS = S\mu dt + \sigma S dW$$

- $y = 2S$

$$\frac{\partial Y}{\partial t} = \emptyset \qquad \frac{\partial Y}{\partial S} = 2 \qquad \frac{\partial^2 Y}{\partial S^2} = \emptyset$$

$$dy = [0 + 2S\mu + \tfrac{1}{2}(\sigma)(S\sigma)^2]dt + 2S\sigma dW$$

$$dy = 2S\mu\,dt + 2S\sigma dW$$

$$dy = y\mu\,dt + y\sigma\,dW$$

- $y = e^S$

$$\frac{\partial y}{\partial t} = \emptyset \qquad \frac{\partial y}{\partial S} = e^S \qquad \frac{\partial^2 y}{\partial S^2} = e^S$$

$$dy = [0 + e^S S\mu + \tfrac{1}{2}e^S(\sigma S)^2]dt + e^S S\sigma dW$$

$$= e^S S\mu dt + \frac{e^S(\sigma S)^2}{2}dt + e^S\sigma S\,dW$$

$$= [y S\mu + \tfrac{1}{2}y(\sigma S)^2]\,dt + y\sigma S\,dW$$

$$y = \frac{e^{r(T-t)}}{S}$$

$$\frac{\partial y}{\partial t} = \frac{1}{S}\left(e^{r(T-t)}\right) \qquad \frac{\partial y}{\partial S} = -\frac{e^{r(T-t)}}{S^2} \qquad \frac{d^2y}{dS^2} = \frac{e^{r(T-t)}}{S^3}$$

$$= -\frac{r\,e^{r(T-t)}}{S}$$

$$dy = \left[-\frac{r e^{r(T-t)}}{S} - \frac{e^{r(T-t)}}{S^{\cancel{2}}}(\cancel{S}\mu) + \frac{e^{r(T-t)}}{2S^3}(S\sigma^2)\right]dt - \frac{e^{r(T-t)}}{\cancel{S^2}}\cancel{S\sigma}\,dW$$

$$dy = [-ry - \mu y + \sigma^2 y]dt - \sigma y\, dW$$

8

$\mu = 5\%$ $\quad \sigma = 2\%$ $\quad S_0 = 17$

$P(S_{t+1} > 19) \to 6$ meses

$dt = 6$

$(\ln S_t + 1 > \ln 19)$

$$\ln(S_{t+1}) = \ln S_0 + (\mu - \tfrac{1}{2}\sigma^2)dt + \sigma\, dW$$

$$= \ln 17 + (0.05 - \tfrac{1}{2}(0.02)^2)6 + 0.02\,\xi\sqrt{6}$$

$$= 3.132 + 0.0489\,\xi$$

$E[\ln(S_{t+1})] = 3.132$ $\qquad V[\ln(S_{t+1})] = 2.39121 \times 10^{-3}$

$$P(\ln S_{t+1} > \ln 19) = P\left(Z > \frac{\ln 19 - 3.132}{0.0489}\right) = -3.83$$

$P(\ln S_{t+1} > \ln 19) = 99.9\%$