Contents

1 Computed values for the gradient dynamics of the Motzkin polynomial

This section presents the values of the computed matrices and function for the gradient dynamics of the Motzkin polynomial.

$$f_{11} = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), \ F_{12} = \left(\begin{smallmatrix} 1&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{smallmatrix}\right) \tag{1a}$$

$$f_{21} = \begin{pmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\-4\\0\\6\\0\\0\\0\\-2\\0\\0\\-4\\6\\-2 \end{pmatrix}, \ F_{22} = \left(\begin{smallmatrix}
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{smallmatrix}\right) \tag{1b}$$

$$\Delta(x) = \mathrm{blkdiag}(x_1 I_{13}, x_2 I_{13}) \tag{1c}$$

$$\pi_{\mathrm{lfr}}(x) = \begin{pmatrix} 1\\ -4x_1^3x_2^2\\ -4x_1^2x_2^2\\ -4x_1x_2^2\\ 6x_1x_2^2\\ -2x_1x_2^4\\ -4x_1^2x_2^3\\ -4x_1x_2^3\\ 6x_1^2x_2\\ 6x_1x_2\\ -2x_1^4x_2\\ -2x_1^3x_2\\ -2x_1^2x_2\\ -2x_1x_2\\ -4x_2^2\\ -4x_2\\ 6x_2^2\\ 6x_2\\ -2x_2^4\\ -2x_2^3\\ -2x_2^2\\ -2x_2\\ -4x_2^3\\ -4x_2^2\\ -4x_2\\ 6x_2\\ -2x_2 \end{pmatrix}, \ S = \left(\begin{smallmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1.5&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&-0.3333&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-0.3333&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1.5&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-1.5&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0.5&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0.5&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-1.5&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0.5&0&0
\end{smallmatrix}\right), \ \pi(x) = \begin{pmatrix} 1\\ -4x_1^3x_2^2\\ -4x_1^2x_2^2\\ -4x_1x_2^2\\ -2x_1x_2^4\\ -4x_1^2x_2^3\\ -4x_1x_2^3\\ 6x_1^2x_2\\ 6x_1x_2\\ -2x_1^4x_2\\ -2x_1^3x_2\\ -4x_2^2\\ -4x_2\\ -2x_2^4\\ -2x_2^3 \end{pmatrix}, \ N(x) = \left(\begin{smallmatrix}
0&1&0&0&0&0&0&0&0&0&-2x_2&0&0&0&0\\
0&0&1&0&0&0&0&\frac{2x_2}{3}&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&\frac{2x_2}{3}&0&0&0&0&0&0\\
0&0&0&0&1&0&-\frac{x_2}{2}&0&0&0&0&0&0&0&0\\
0&0&-x_2&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&-x_2&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&-x_1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&\frac{3x_1}{2}&0&0\\
0&0&0&0&0&0&0&0&0&1&-x_1&0&0&0&0\\
0&0&0&0&0&0&0&\frac{x_1}{3}&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&-x_2&0&0\\
4x_2&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&-x_2\\
0&0&0&0&0&0&0&0&0&0&0&-\frac{x_2}{2}&0&0&1\\
0&x_1&0&0&0&0&0&0&0&-2x_2&0&0&0&0&0\\
0&0&x_1&0&0&0&0&0&0&0&-2x_2&0&0&0&0\\
0&0&0&x_1&0&0&0&\frac{2x_2}{3}&0&0&0&0&0&0&0\\
0&0&0&0&x_1&-\frac{x_2}{2}&0&0&0&0&0&0&0&0&0\\
0&-x_2&0&0&0&x_1&0&0&0&0&0&0&0&0&0\\
0&0&-x_2&0&0&0&x_1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&\frac{2x_2}{3}&0&0&x_1&0&0&0\\
0&0&0&0&0&0&-\frac{x_2}{2}&0&0&0&0&0&0&x_1&0\\
0&0&0&-\frac{x_2}{2}&0&0&0&0&0&0&0&0&0&0&x_1
\end{smallmatrix}\right) \tag{2}$$

$$S_{f,1} = \begin{pmatrix}
1&0&0&0&0\\
0&1&0&0&0\\
0&-0.5&0&0&0\\
0&0.25&0&0&0\\
0&0&1&0&0\\
0&0&0&1&0\\
0&0&0&-0.5&0\\
0&0&0&0&1\\
0&0&0&0&-0.5\\
0&0&0&0&-1.3333\\
0&0&0&0&0.6667\\
0&-0.125&0&0&0\\
0&0&0&0&-0.1667\\
0&0&-0.5&0&0\\
0&0&0&0.125&0
\end{pmatrix}, \ \pi_{f,1}(x) = \begin{pmatrix} 1\\ -4x_1^3x_2^2\\ -2x_1x_2^4\\ -4x_1^2x_2^3\\ 6x_1^2x_2 \end{pmatrix}, \ N_{f,1}(x) = \begin{pmatrix}
\frac{x_1}{2}+1&0&0&0&0\\
0&1&0&0&-\frac{4x_2}{3}\\
0&0&1&\frac{x_2}{4}&0\\
0&\frac{x_2}{2}&0&1&0\\
-24x_2&0&0&0&1\\
0&x_1&0&0&\frac{8x_2}{3}\\
0&0&x_1&-\frac{x_2}{2}&0\\
0&-x_2&0&x_1&0\\
48x_2&0&0&0&x_1
\end{pmatrix} \tag{3}$$

$$S_{f,2} = \begin{pmatrix}
1&0&0&0&0\\
0&1&0&0&0\\
0&0.5&0&0&0\\
0&0.25&0&0&0\\
0&0&1&0&0\\
0&0&0&1&0\\
0&0&0&0.5&0\\
0&0&0&0&1\\
0&0&0&0&0.5\\
0&0&0&0&-1.3333\\
0&0&0&0&-0.6667\\
0&0.125&0&0&0\\
0&0&0&0&-0.1667\\
0&0&0.5&0&0\\
0&0&0&0.125&0
\end{pmatrix}, \ \pi_{f,2}(x) = \begin{pmatrix} 1\\ -4x_1^3x_2^2\\ -2x_1x_2^4\\ -4x_1^2x_2^3\\ 6x_1^2x_2 \end{pmatrix}, \ N_{f,2}(x) = \begin{pmatrix}
1-\frac{x_1}{2}&0&0&0&0\\
0&1&0&0&\frac{4x_2}{3}\\
0&0&1&-\frac{x_2}{4}&0\\
0&-\frac{x_2}{2}&0&1&0\\
-24x_2&0&0&0&1\\
0&x_1&0&0&\frac{8x_2}{3}\\
0&0&x_1&-\frac{x_2}{2}&0\\
0&-x_2&0&x_1&0\\
-48x_2&0&0&0&x_1
\end{pmatrix} \tag{4}$$

$$S_{f,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0.75 & 0 & 0 \\ 0 & 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -0.25 & 0 & 0 & 0 \\ -16 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 \\ -32 & 0 & 0 & 0 & 0 \\ 16 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \pi_{f,3}(x) = \begin{pmatrix} 1 \\ -4x_1^3x_2^2 \\ -4x_1^2x_2^2 \\ -4x_1x_2^2 \\ -2x_1^4x_2 \end{pmatrix}, \ N_{f,3}(x) = \begin{pmatrix} \frac{x_2}{2}+1 & 0 & 0 & 0 & 0 \\ 0 & \frac{x_2}{2}+1 & 0 & 0 & 0 \\ 0 & 0 & \frac{x_2}{2}+1 & 0 & 0 \\ 0 & 0 & 0 & \frac{x_2}{2}+1 & 0 \\ 0 & 0 & 0 & 0 & \frac{x_2}{2}+1 \\ x_1 & 0 & 0 & -\frac{x_2}{32} & 0 \\ 0 & x_1 & 0 & 0 & -2x_2 \\ 0 & \frac{x_2}{2} & x_1 & 0 & 0 \\ 0 & 0 & \frac{x_2}{2} & x_1 & 0 \end{pmatrix} \tag{5}$$

$$S_{f,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -0.75 & 0 & 0 \\ 0 & 0 & 0 & -0.75 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0.25 & 0 & 0 & 0 \\ -16 & 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 & 0 \\ -32 & 0 & 0 & 0 & 0 \\ -16 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \pi_{f,4}(x) = \begin{pmatrix} 1 \\ -4x_1^3x_2^2 \\ -4x_1^2x_2^2 \\ -4x_1x_2^2 \\ -2x_1^4x_2 \end{pmatrix}, \ N_{f,4}(x) = \begin{pmatrix} 1-\frac{x_2}{2} & 0 & 0 & 0 & 0 \\ 0 & 1-\frac{x_2}{2} & 0 & 0 & 0 \\ 0 & 0 & 1-\frac{x_2}{2} & 0 & 0 \\ 0 & 0 & 0 & 1-\frac{x_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 1-\frac{x_2}{2} \\ x_1 & 0 & 0 & \frac{x_2}{32} & 0 \\ 0 & x_1 & 0 & 0 & -2x_2 \\ 0 & -\frac{x_2}{2} & x_1 & 0 & 0 \\ 0 & 0 & -\frac{x_2}{2} & x_1 & 0 \end{pmatrix} \tag{6}$$

$$S_{f,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ \pi_{f,5}(x) = \begin{pmatrix} 1 \\ -4x_2^2 \\ -4x_2 \\ -2x_2^4 \\ -2x_2^3 \end{pmatrix}, \ N_{f,5}(x) = \begin{pmatrix} 0 & 1 & -x_2 & 0 & 0 \\ 4x_2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -x_2 \\ 0 & -\frac{x_2}{2} & 0 & 0 & 1 \\ x_1 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & x_1 \end{pmatrix} \tag{7}$$

$$S_{f,6} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \pi_{f,6}(x) = 1, \ N_{f,6}(x) = x_2 \tag{8}$$

$$P = \begin{pmatrix} 1.0205 & 0.0198 & -0.0409 & 0.1611 & 0.1048 & -0.1341 & 0.1372 & 0 & 0 & 0 & 0 & -0.0707 & 0 & -0.0315 & -0.0186 \\ 0.0198 & -0.0643 & -0.0067 & -0.0009 & -0.0007 & -0.0065 & 0.0042 & -0.0023 & -0.0042 & 0 & 0.0191 & 0.0007 & -0.0519 & 0.006 & 0.0257 \\ -0.0409 & -0.0067 & -0.0035 & 0.005 & -0.0051 & -0.0012 & -0.0061 & -0.0055 & -0.0179 & -0.0191 & 0.0087 & -0.0101 & -0.0604 & -0.0133 & 0.0165 \\ 0.1611 & -0.0009 & 0.005 & 0.0031 & 0.0094 & -0.0051 & 0.0018 & -0.014 & -0.0022 & -0.0155 & 0.0079 & 0.0103 & -0.0031 & -0.0007 & 0.0035 \\ 0.1048 & -0.0007 & -0.0051 & 0.0094 & -0.001 & 0 & -0.0005 & -0.0002 & 0.0016 & 0.0262 & -0.0267 & 0.0023 & 0.0127 & 0 & 0.0017 \\ -0.1341 & -0.0065 & -0.0012 & -0.0051 & 0 & 0.0042 & -0.0004 & -0.002 & -0.0003 & 0.0669 & -0.0169 & 0.0284 & 0.0522 & 0.0005 & 0.0022 \\ 0.1372 & 0.0042 & -0.0061 & 0.0018 & -0.0005 & -0.0004 & -0.0091 & 0.0024 & -0.011 & 0.0304 & -0.0792 & 0.0041 & -0.0181 & -0.0008 & -0.0016 \\ 0 & -0.0023 & -0.0055 & -0.014 & -0.0002 & -0.002 & 0.0024 & 0.0809 & 0.0176 & -0.514 & -0.0013 & -0.0157 & 0.0234 & 0.0496 & 0.0199 \\ 0 & -0.0042 & -0.0179 & -0.0022 & 0.0016 & -0.0003 & -0.011 & 0.0176 & -0.0609 & 0.0013 & -0.092 & -0.0249 & -0.0269 & 0.0163 & -0.028 \\ 0 & 0 & -0.0191 & -0.0155 & 0.0262 & 0.0669 & 0.0304 & -0.514 & 0.0013 & 0.0042 & 0 & -0.0369 & -0.2118 & 0.0145 & 0.0834 \\ 0 & 0.0191 & 0.0087 & 0.0079 & -0.0267 & -0.0169 & -0.0792 & -0.0013 & -0.092 & 0 & -3.1572 & 0.0083 & -0.0693 & -0.0071 & -0.0103 \\ -0.0707 & 0.0007 & -0.0101 & 0.0103 & 0.0023 & 0.0284 & 0.0041 & -0.0157 & -0.0249 & -0.0369 & 0.0083 & 0.0039 & -0.0023 & -0.0035 & 0.0007 \\ 0 & -0.0519 & -0.0604 & -0.0031 & 0.0127 & 0.0522 & -0.0181 & 0.0234 & -0.0269 & -0.2118 & -0.0693 & -0.0023 & -0.0433 & -0.0007 & 0.0049 \\ -0.0315 & 0.006 & -0.0133 & -0.0007 & 0 & 0.0005 & -0.0008 & 0.0496 & 0.0163 & 0.0145 & -0.0071 & -0.0035 & -0.0007 & 0.005 & 0 \\ -0.0186 & 0.0257 & 0.0165 & 0.0035 & 0.0017 & 0.0022 & -0.0016 & 0.0199 & -0.028 & 0.0834 & -0.0103 & 0.0007 & 0.0049 & 0 & -0.0195 \end{pmatrix} \tag{9}$$

2 Computed values for the genetic toggle switch in Escherichia coli

This section presents the values of the computed matrices and function for the genetic toggle switch dynamics.

$$f_{11} = \begin{pmatrix} 1.3 \\ 1 \end{pmatrix}, \; F_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$f_{21} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \; F_{22} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\Delta = \mathrm{blkdiag}(x_1 I_{11}, x_2 I_4)$$

$$\pi_{\mathrm{lfr}}(x) = \pi(x) = \begin{pmatrix} 1 \\ -x_1 \\ \frac{{x_1}^{10}}{{x_1}^{10}+1} \\ \frac{x_1^9}{{x_1}^{10}+1} \\ \frac{x_1^8}{{x_1}^{10}+1} \\ \frac{x_1^7}{{x_1}^{10}+1} \\ \frac{x_1^6}{{x_1}^{10}+1} \\ \frac{x_1^5}{{x_1}^{10}+1} \\ \frac{x_1^4}{{x_1}^{10}+1} \\ \frac{x_1^3}{{x_1}^{10}+1} \\ \frac{x_1^2}{{x_1}^{10}+1} \\ \frac{x_1}{{x_1}^{10}+1} \\ \frac{x_2^3}{x_2^3+1} \\ \frac{x_2^2}{x_2^3+1} \\ \frac{x_2}{x_2^3+1} \\ -x_2 \end{pmatrix}, \; S = I_{16}$$

$$S_{f,1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-0.5 & 0 & 0 & 0 & 0 \\
0.001 & 0 & 0 & 0 & 0 \\
0.002 & 0 & 0 & 0 & 0 \\
0.0039 & 0 & 0 & 0 & 0 \\
0.0078 & 0 & 0 & 0 & 0 \\
0.0156 & 0 & 0 & 0 & 0 \\
0.0312 & 0 & 0 & 0 & 0 \\
0.0624 & 0 & 0 & 0 & 0 \\
0.1249 & 0 & 0 & 0 & 0 \\
0.2498 & 0 & 0 & 0 & 0 \\
0.4995 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \; \pi_{f,1} = \begin{pmatrix} 1 \\ \frac{x_2^3}{x_2^3+1} \\ \frac{x_2^2}{x_2^3+1} \\ \frac{x_2}{x_2^3+1} \\ -x_2 \end{pmatrix}, \; N_{f,1} = \begin{pmatrix}
1-2x_1 & 0 & 0 & 0 & 0 \\
0 & 1 & -x_2 & 0 & 0 \\
0 & 0 & 1 & -x_2 & 0 \\
-x_2 & x_2 & 0 & 1 & 0 \\
x_2 & 0 & 0 & 0 & 1 \\
0 & x_1 & -\frac{x_2}{2} & 0 & 0 \\
0 & 0 & x_1 & -\frac{x_2}{2} & 0 \\
-\frac{x_2}{2} & \frac{x_2}{2} & 0 & x_1 & 0 \\
\frac{x_2}{2} & 0 & 0 & 0 & x_1
\end{pmatrix}$$

$$S_{f,2} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-1.5 & 0 & 0 & 0 & 0 \\
0.983 & 0 & 0 & 0 & 0 \\
0.6553 & 0 & 0 & 0 & 0 \\
0.4369 & 0 & 0 & 0 & 0 \\
0.2912 & 0 & 0 & 0 & 0 \\
0.1942 & 0 & 0 & 0 & 0 \\
0.1294 & 0 & 0 & 0 & 0 \\
0.0863 & 0 & 0 & 0 & 0 \\
0.0575 & 0 & 0 & 0 & 0 \\
0.0384 & 0 & 0 & 0 & 0 \\
0.0256 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}, \; \pi_{f,2} = \begin{pmatrix} 1 \\ \frac{x_2^3}{x_2^3+1} \\ \frac{x_2^2}{x_2^3+1} \\ \frac{x_2}{x_2^3+1} \\ -x_2 \end{pmatrix}, \; N_{f,2} = \begin{pmatrix}
1-\frac{2x_1}{3} & 0 & 0 & 0 & 0 \\
0 & 1 & -x_2 & 0 & 0 \\
0 & 0 & 1 & -x_2 & 0 \\
-x_2 & x_2 & 0 & 1 & 0 \\
x_2 & 0 & 0 & 0 & 1 \\
0 & x_1 & -\frac{3x_2}{2} & 0 & 0 \\
0 & 0 & x_1 & -\frac{3x_2}{2} & 0 \\
-\frac{3x_2}{2} & \frac{3x_2}{2} & 0 & x_1 & 0 \\
\frac{3x_2}{2} & 0 & 0 & 0 & x_1
\end{pmatrix}$$

$$S_{f,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -0.0059 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0326 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.1811 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \pi_{f,3} = \begin{pmatrix} 1 \\ -x_1 \\ \frac{x_1^{10}}{x_1^{10}+1} \\ \frac{x_1^9}{x_1^{10}+1} \\ \frac{x_1^8}{x_1^{10}+1} \\ \frac{x_1^7}{x_1^{10}+1} \\ \frac{x_1^6}{x_1^{10}+1} \\ \frac{x_1^5}{x_1^{10}+1} \\ \frac{x_1^4}{x_1^{10}+1} \\ \frac{x_1^3}{x_1^{10}+1} \\ \frac{x_1^2}{x_1^{10}+1} \\ \frac{x_1}{x_1^{10}+1} \end{pmatrix}, \ N_{f,3} = \begin{pmatrix} \frac{50x_2}{9}+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{50x_2}{9}+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{50x_2}{9}+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{50x_2}{9}+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{50x_2}{9}+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9}+1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9}+1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9}+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9}+1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9}+1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9}+1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9}+1 \\ x_1 & -\frac{50x_2}{9} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{50x_2}{9} & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{50x_2}{9} \\ 0 & 0 & \frac{50x_2}{9} & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{50x_2}{9} & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{50x_2}{9} & x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9} & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9} & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9} & x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9} & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9} & x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{50x_2}{9} & x_1 \end{pmatrix}$$

$$S_{f,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0.6334 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5279 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4399 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \pi_{f,4} = \begin{pmatrix} 1 \\ -x_1 \\ \frac{x_1^{10}}{x_1^{10}+1} \\ \frac{x_1^9}{x_1^{10}+1} \\ \frac{x_1^8}{x_1^{10}+1} \\ \frac{x_1^7}{x_1^{10}+1} \\ \frac{x_1^6}{x_1^{10}+1} \\ \frac{x_1^5}{x_1^{10}+1} \\ \frac{x_1^4}{x_1^{10}+1} \\ \frac{x_1^3}{x_1^{10}+1} \\ \frac{x_1^2}{x_1^{10}+1} \\ \frac{x_1}{x_1^{10}+1} \end{pmatrix}, \ N_{f,4} = \begin{pmatrix} 1-\frac{5x_2}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\frac{5x_2}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\frac{5x_2}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\frac{5x_2}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-\frac{5x_2}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-\frac{5x_2}{6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{5x_2}{6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{5x_2}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{5x_2}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{5x_2}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{5x_2}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{5x_2}{6} \\ x_1 & \frac{5x_2}{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{5x_2}{6} & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{5x_2}{6} \\ 0 & 0 & -\frac{5x_2}{6} & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{5x_2}{6} & x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5x_2}{6} & x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{5x_2}{6} & x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5x_2}{6} & x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5x_2}{6} & x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5x_2}{6} & x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5x_2}{6} & x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{5x_2}{6} & x_1 \end{pmatrix}$$

$$S_{f,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0.8866 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ \pi_{f,5} = \begin{pmatrix} 1 \\ -x_1 \\ \frac{x_1^{10}}{x_1^{10}+1} \\ \frac{x_1^9}{x_1^{10}+1} \\ \frac{x_1^8}{x_1^{10}+1} \\ \frac{x_1^7}{x_1^{10}+1} \\ \frac{x_1^6}{x_1^{10}+1} \\ \frac{x_1^5}{x_1^{10}+1} \\ \frac{x_1^4}{x_1^{10}+1} \\ \frac{x_1^3}{x_1^{10}+1} \\ \frac{x_1^2}{x_1^{10}+1} \\ \frac{x_1}{x_1^{10}+1} \\ \frac{x_2^3}{x_2^3+1} \\ \frac{x_2^2}{x_2^3+1} \\ \frac{x_2}{x_2^3+1} \end{pmatrix}, \ N_{f,5} = \begin{pmatrix} \frac{8807x_2}{7808} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{8807x_2}{7808} & 0 & 0 & 0 \\ -\frac{8807x_2}{7808} & 0 & \frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -x_2 \\ -x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 1 \\ x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1-\frac{8807x_2}{7808} \end{pmatrix}$$

$$P = \begin{pmatrix} 65.1394 & 42.2162 & -34.8158 & 22.3949 & 24.4583 & -17.6864 & -26.8477 & -61.821 & -8.0911 & 89.9776 & 96.7132 & -97.611 & 2.0871 & -22.6982 & -5.2142 & 2.9387 \\ 42.2162 & 27.5742 & -12.3709 & 34.7505 & -34.2573 & -48.9653 & -3.0919 & -14.3045 & 25.3861 & 11.4539 & 73.8419 & -63.3605 & 22.1027 & -20.727 & -36.2502 & -39.6294 \\ -34.8158 & -12.3709 & 30.1875 & 5.4484 & -24.7694 & -8.2696 & -9.2531 & 33.4806 & 24.7396 & -16.8181 & 4.2453 & 14.997 & 89.8425 & -30.4036 & -44.7064 & -76.8154 \\ 22.3949 & 34.7505 & 5.4484 & 31.7056 & -76.9501 & -45.7517 & -5.041 & 19.1884 & 36.1357 & 14.5209 & -11.646 & -35.864 & -31.1718 & 2.3427 & -15.0105 & 4.8888 \\ 24.4583 & -34.2573 & -24.7694 & -76.9501 & 19.7012 & 7.4838 & 53.4006 & 19.1714 & 5.3898 & -29.0471 & -28.7647 & -17.7078 & -40.6794 & 14.5715 & 7.6 & 33.1098 \\ -17.6864 & -48.9653 & -8.2696 & -45.7517 & 7.4838 & 41.8226 & 42.1296 & -6.566 & -62.0578 & -45.6941 & -18.6213 & 41.7698 & -15.6784 & 9.9814 & 23.531 & 23.6955 \\ -26.8477 & -3.0919 & -9.2531 & -5.041 & 53.4006 & 42.1296 & 25.3456 & -31.1007 & -62.2103 & -48.0422 & 16.0926 & 65.8572 & 4.5791 & -0.3026 & 16.7549 & -9.738 \\ -61.821 & -14.3045 & 33.4806 & 19.1884 & 19.1714 & -6.566 & -31.1007 & 39.7442 & -12.4576 & 14.5291 & 24.7449 & 14.0434 & 42.2717 & -11.0346 & -4.6005 & -33.1091 \\ -8.0911 & 25.3861 & 24.7396 & 36.1357 & 5.3898 & -62.0578 & -62.2103 & -12.4576 & 39.5729 & 37.2141 & 42.2166 & -41.0033 & 25.2799 & -5.4981 & -20.3807 & -20.7611 \\ 89.9776 & 11.4539 & -16.8181 & 14.5209 & -29.0471 & -45.6941 & -48.0422 & 14.5291 & 37.2141 & 60.1508 & -29.2235 & -73.6347 & -6.821 & 11.4941 & -31.3459 & 19.8702 \\ 96.7132 & 73.8419 & 4.2453 & -11.646 & -28.7647 & -18.6213 & 16.0926 & 24.7449 & 42.2166 & -29.2235 & 23.5009 & -23.7131 & -52.3148 & -8.2372 & 33.8011 & 41.9339 \\ -97.611 & -63.3605 & 14.997 & -35.864 & -17.7078 & 41.7698 & 65.8572 & 14.0434 & -41.0033 & -73.6347 & -23.7131 & 51.5343 & 61.8672 & -7.0911 & -40.4226 & -72.1123 \\ 2.0871 & 22.1027 & 89.8425 & -31.1718 & -40.6794 & -15.6784 & 4.5791 & 42.2717 & 25.2799 & -6.821 & -52.3148 & 61.8672 & 9.5181 & 4.2156 & 12.9434 & -11.737 \\ -22.6982 & -20.727 & -30.4036 & 2.3427 & 14.5715 & 9.9814 & -0.3026 & -11.0346 & -5.4981 & 11.4941 & -8.2372 & -7.0911 & 4.2156 & -41.9394 & 7.7923 & 35.1646 \\ -5.2142 & -36.2502 & -44.7064 & -15.0105 & 7.6 & 23.531 & 16.7549 & -4.6005 & -20.3807 & -31.3459 & 33.8011 & -40.4226 & 12.9434 & 7.7923 & 12.6124 & -15.7859 \\ 2.9387 & -39.6294 & -76.8154 & 4.8888 & 33.1098 & 23.6955 & -9.738 & -33.1091 & -20.7611 & 19.8702 & 41.9339 & -72.1123 & -11.737 & 35.1646 & -15.7859 & -1.778 \end{pmatrix}$$

3 Computed values for the periodic ring dynamics

This section presents the values of the computed matrices and function for the periodic ring dynamics.

$$N(x)=\begin{pmatrix} x_1 & -1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & -1 & 0 & 0 & 0 \\ 0 & x_1 & 0 & -1 & 0 & 0 \\ 0 & x_2 & 0 & 0 & -1 & 0 \\ 0 & 0 & x_1 & 0 & -1 & 0 \\ 0 & 0 & x_2 & 0 & 0 & -1 \\ 0 & 0 & 0 & x_2 & -x_1 & 0 \\ 0 & 0 & 0 & 0 & x_2 & -x_1 \end{pmatrix},\ N_g(x)=\begin{pmatrix} x_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_1 \end{pmatrix}$$

$$S_{f,1}=\begin{pmatrix} 1 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & 1 & 0 \\ 9 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \pi_{f,1}=\begin{pmatrix} 1 \\ x_2 \\ x_2^2 \end{pmatrix},\ N_{f,1}=\begin{pmatrix} \frac{x_1}{3}+1 & 0 & 0 \\ -x_2 & 1 & 0 \\ 0 & -x_2 & 1 \\ 3x_2 & x_1 & 0 \\ 0 & 3x_2 & x_1 \end{pmatrix}$$

$$S_{g,1}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -27 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 81 & 0 & 0 & 0 & 0 \\ 0 & -27 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ \pi_{g,1}=\begin{pmatrix} 1 \\ x_2 \\ x_2^2 \\ x_2^3 \\ x_2^4 \end{pmatrix},\ N_{g,1}=\begin{pmatrix} \frac{x_1}{3}+1 & 0 & 0 & 0 & 0 \\ -x_2 & 1 & 0 & 0 & 0 \\ 0 & -x_2 & 1 & 0 & 0 \\ 0 & 0 & -x_2 & 1 & 0 \\ 0 & 0 & 0 & -x_2 & 1 \\ 3x_2 & x_1 & 0 & 0 & 0 \\ 0 & 3x_2 & x_1 & 0 & 0 \\ 0 & 0 & 3x_2 & x_1 & 0 \\ 0 & 0 & 0 & 3x_2 & x_1 \end{pmatrix}$$

$$S_{f,2}=\begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \\ 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \pi_{f,2}=\begin{pmatrix} 1 \\ x_2 \\ x_2^2 \end{pmatrix},\ N_{f,2}=\begin{pmatrix} 1-\frac{x_1}{3} & 0 & 0 \\ -x_2 & 1 & 0 \\ 0 & -x_2 & 1 \\ -3x_2 & x_1 & 0 \\ 0 & -3x_2 & x_1 \end{pmatrix}$$

$$S_{g,2}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 27 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 81 & 0 & 0 & 0 & 0 \\ 0 & 27 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},\ \pi_{g,2}=\begin{pmatrix} 1 \\ x_2 \\ x_2^2 \\ x_2^3 \\ x_2^4 \end{pmatrix},\ N_{g,2}=\begin{pmatrix} 1-\frac{x_1}{3} & 0 & 0 & 0 & 0 \\ -x_2 & 1 & 0 & 0 & 0 \\ 0 & -x_2 & 1 & 0 & 0 \\ 0 & 0 & -x_2 & 1 & 0 \\ 0 & 0 & 0 & -x_2 & 1 \\ -3x_2 & x_1 & 0 & 0 & 0 \\ 0 & -3x_2 & x_1 & 0 & 0 \\ 0 & 0 & -3x_2 & x_1 & 0 \\ 0 & 0 & 0 & -3x_2 & x_1 \end{pmatrix}$$

$$S_{f,3}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 0 \\ 9 & 0 & 0 \end{pmatrix},\ \pi_{f,3}=\begin{pmatrix} 1 \\ x_1 \\ x_1^2 \end{pmatrix},\ N_{f,3}=\begin{pmatrix} \frac{x_2}{3}+1 & 0 & 0 \\ 0 & \frac{x_2}{3}+1 & 0 \\ 0 & 0 & \frac{x_2}{3}+1 \\ x_1 & \frac{x_2}{3} & 0 \\ 0 & x_1 & \frac{x_2}{3} \end{pmatrix}$$

$$S_{g,3}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ -27 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & -27 & 0 & 0 & 0 \\ 81 & 0 & 0 & 0 & 0 \end{pmatrix},\ \pi_{g,3}=\begin{pmatrix} 1 \\ x_1 \\ x_1^2 \\ x_1^3 \\ x_1^4 \end{pmatrix},\ N_{g,3}=\begin{pmatrix} \frac{x_2}{3}+1 & 0 & 0 & 0 & 0 \\ 0 & \frac{x_2}{3}+1 & 0 & 0 & 0 \\ 0 & 0 & \frac{x_2}{3}+1 & 0 & 0 \\ 0 & 0 & 0 & \frac{x_2}{3}+1 & 0 \\ 0 & 0 & 0 & 0 & \frac{x_2}{3}+1 \\ x_1 & \frac{x_2}{3} & 0 & 0 & 0 \\ 0 & x_1 & \frac{x_2}{3} & 0 & 0 \\ 0 & 0 & x_1 & \frac{x_2}{3} & 0 \\ 0 & 0 & 0 & x_1 & \frac{x_2}{3} \end{pmatrix}$$

$$S_{f,4}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \\ 9 & 0 & 0 \end{pmatrix},\ \pi_{f,4}=\begin{pmatrix} 1 \\ x_1 \\ x_1^2 \end{pmatrix},\ N_{f,4}=\begin{pmatrix} 1-\frac{x_2}{3} & 0 & 0 \\ 0 & 1-\frac{x_2}{3} & 0 \\ 0 & 0 & 1-\frac{x_2}{3} \\ x_1 & -\frac{x_2}{3} & 0 \\ 0 & x_1 & -\frac{x_2}{3} \end{pmatrix}$$

$$S_{g,4}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 27 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 27 & 0 & 0 & 0 \\ 81 & 0 & 0 & 0 & 0 \end{pmatrix},\ \pi_{g,4}=\begin{pmatrix} 1 \\ x_1 \\ x_1^2 \\ x_1^3 \\ x_1^4 \end{pmatrix},\ N_{g,4}=\begin{pmatrix} 1-\frac{x_2}{3} & 0 & 0 & 0 & 0 \\ 0 & 1-\frac{x_2}{3} & 0 & 0 & 0 \\ 0 & 0 & 1-\frac{x_2}{3} & 0 & 0 \\ 0 & 0 & 0 & 1-\frac{x_2}{3} & 0 \\ 0 & 0 & 0 & 0 & 1-\frac{x_2}{3} \\ x_1 & -\frac{x_2}{3} & 0 & 0 & 0 \\ 0 & x_1 & -\frac{x_2}{3} & 0 & 0 \\ 0 & 0 & x_1 & -\frac{x_2}{3} & 0 \\ 0 & 0 & 0 & x_1 & -\frac{x_2}{3} \end{pmatrix}$$

$$P=\begin{pmatrix} 0.8934 & 0 & 0 & 0.0994 & 0.0454 & -0.1659 \\ 0 & -0.5977 & -0.0454 & 0 & -0.0033 & -0.0011 \\ 0 & -0.0454 & -0.067 & 0.0033 & 0.0011 & 0 \\ 0.0994 & 0 & 0.0033 & 0.0499 & 0 & -0.0195 \\ 0.0454 & -0.0033 & 0.0011 & 0 & 0.1386 & 0 \\ -0.1659 & -0.0011 & 0 & -0.0195 & 0 & 0.0499 \end{pmatrix}$$

4 Computed values for the Van der Pol oscillator

This section presents the values of the computed matrices and function for the Van der Pol oscillator:

$$
A_g = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \qquad (10)
$$

5 Sum of squares decomposition of the translated Motzkin polynomial

Consider function

$$
W(x) = \underbrace{x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 + 1}_{\text{Motzkin polynomial [1]}} \quad + \quad 0.1 = \pi^\top(x)\, P\, \pi(x) \qquad (11)
$$

where

$$
P = \begin{pmatrix}
1.1 & 0 & 0 & 0 & 0.1132 & 0 & -0.0413 & 0 & 0.6667 & -0.1601 \\
0 & 0 & -0.1132 & 0.0413 & 0 & -0.3333 & 0 & 0.1068 & -1.8997 & 0.0338 \\
0 & -0.1132 & 0 & 0 & -0.3333 & 0.1601 & -0.0534 & 0.1776 & -0.0461 & -0.0185 \\
0 & 0.0413 & 0 & 0 & -0.0534 & 0.1776 & 0 & 0 & 0 & 0 \\
0.1132 & 0 & -0.3333 & -0.0534 & 0.0888 & 0.0124 & 0 & 0 & 0 & 0 \\
0 & -0.3333 & 0.1601 & 0.1776 & 0.0124 & 0.037 & 0 & 0 & 0 & 0 \\
-0.0413 & 0 & -0.0534 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\
0 & 0.1068 & 0.1776 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 \\
0.6667 & -1.8997 & -0.0461 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\
-0.1601 & 0.0338 & -0.0185 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0
\end{pmatrix}, \quad
N(x) = \begin{pmatrix}
x_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_2 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & x_2 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & x_1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & x_2 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & x_1 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & x_2 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_2 & -x_1
\end{pmatrix}, \quad
\pi^\top(x) = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_1^2 \\ x_1 x_2 \\ x_2^2 \\ x_1^3 \\ x_1^2 x_2 \\ x_1 x_2^2 \\ x_2^3 \end{pmatrix}. \qquad (12)
$$

Furthermore, P satisfies

$$
P + \mathrm{He}\{LN(x)\} \succeq 0 \text{ for all } x \in [-2,2] \times [-2,2], \qquad (13)
$$

where

$$
L = \begin{pmatrix}
0 & 0 & 2.51 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2.18 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.336 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.2 & 0 & 0 & -0.107 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.0194 & -0.0194 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.107 & 0 & 0 & -0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.0607 & 0 & 0 & 0 & 0 & 0 & -0.0881 & 0 & 0 & 0.0062 & 0.0197 & 0 & 0 & 0 & 0 \\
0 & -0.414 & 0 & 0 & 0 & 0 & 0 & 0.0963 & -0.0037 & 0 & 0 & 0.284 & 0 & 0 & 0 \\
-0.414 & 0 & 0 & 0 & 0 & 0 & 0.284 & 0 & 0 & -0.0037 & 0.0963 & 0 & 0 & 0 & 0 \\
0 & 0.0607 & 0 & 0 & 0 & 0 & 0 & 0.0197 & 0.0062 & 0 & 0 & -0.0881 & 0 & 0 & 0
\end{pmatrix} \qquad (14)
$$

References

[1] Theodore Samuel Motzkin. The arithmetic-geometric inequality. *Inequalities (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965)*, pages 205–224, 1967.