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This section presents the values of the computed matrices and function for the gradient dynamics of the Motzkin polynomial.

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$$S_{f,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 & -1.3333 \\ 0 & 0 & 0 & 0 & 0.6667 \\ 0 & -0.125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1667 \\ 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.125 & 0 \end{pmatrix}, \quad \pi_{f,1}(x) = \begin{pmatrix} 1 \\ -4x_1^3x_2^2 \\ -2x_1x_2^4 \\ -4x_1^2x_2^3 \\ 6x_1^2x_2 \end{pmatrix}, \quad N_{f,1}(x) = \begin{pmatrix} \frac{x_1}{2} + 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{4x_2}{3} \\ 0 & 0 & 1 & \frac{x_2}{4} & 0 \\ 0 & \frac{x_2}{2} & 0 & 1 & 0 \\ -24x_2 & 0 & 0 & 0 & 1 \\ 0 & x_1 & 0 & 0 & \frac{8x_2}{3} \\ 0 & 0 & x_1 & -\frac{x_2}{2} & 0 \\ 0 & -x_2 & 0 & x_1 & 0 \\ 48x_2 & 0 & 0 & 0 & x_1 \end{pmatrix} \quad (3)$$

$$S_{f,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & -1.3333 \\ 0 & 0 & 0 & 0 & -0.6667 \\ 0 & 0.125 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1667 \\ 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.125 & 0 \end{pmatrix}, \quad \pi_{f,2}(x) = \begin{pmatrix} 1 \\ -4x_1^3x_2^2 \\ -2x_1x_2^4 \\ -4x_1^2x_2^3 \\ 6x_1^4x_2 \end{pmatrix}, \quad N_{f,2}(x) = \begin{pmatrix} 1 - \frac{x_1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{4x_2}{3} \\ 0 & 0 & 1 & -\frac{x_2}{4} & 0 \\ 0 & -\frac{x_2}{2} & 0 & 1 & 0 \\ 0 & x_1 & 0 & 0 & \frac{8x_2}{3} \\ 0 & 0 & x_1 & -\frac{x_2}{2} & 0 \\ 0 & -x_2 & 0 & x_1 & 0 \\ -48x_2 & 0 & 0 & 0 & x_1 \end{pmatrix} \quad (4)$$

$$S_{f,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0.75 & 0 & 0 \\ 0 & 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -0.25 & 0 & 0 & 0 \\ -16 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 \\ -32 & 0 & 0 & 0 & 0 \\ 16 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \pi_{f,3}(x) = \begin{pmatrix} 1 \\ -4x_1^3x_2^2 \\ -4x_1^2x_2^2 \\ -4x_1x_2^2 \\ -2x_1^4x_2 \end{pmatrix}, \quad N_{f,3}(x) = \begin{pmatrix} \frac{x_2}{2} + 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{x_2}{2} + 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{x_2}{2} + 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{x_2}{2} + 1 & 0 \\ x_1 & 0 & 0 & 0 & \frac{x_2}{2} + 1 \\ 0 & x_1 & 0 & -\frac{x_2}{32} & 0 \\ 0 & x_1 & 0 & 0 & -2x_2 \\ 0 & \frac{x_2}{2} & x_1 & 0 & 0 \\ 0 & 0 & \frac{x_2}{2} & x_1 & 0 \end{pmatrix} \quad (5)$$

$$S_{f,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -0.75 & 0 & 0 \\ 0 & 0 & 0 & -0.75 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0.25 & 0 & 0 & 0 \\ -16 & 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & 0 & 0 \\ -32 & 0 & 0 & 0 & 0 \\ -16 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \pi_{f,4}(x) = \begin{pmatrix} 1 \\ -4x_1^3x_2^2 \\ -4x_1^2x_2^3 \\ -4x_1x_2^4 \\ -2x_1^4x_2 \end{pmatrix}, \quad N_{f,4}(x) = \begin{pmatrix} 1 - \frac{x_2}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 - \frac{x_2}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 - \frac{x_2}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 - \frac{x_2}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 - \frac{x_2}{2} \\ x_1 & 0 & 0 & \frac{x_2}{32} & 0 \\ 0 & x_1 & 0 & 0 & -2x_2 \\ 0 & -\frac{x_2}{2} & x_1 & 0 & 0 \\ 0 & 0 & -\frac{x_2}{2} & x_1 & 0 \end{pmatrix} \quad (6)$$

$$S_{f,5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \pi_{f,5}(x) = \begin{pmatrix} 1 \\ -4x_2^2 \\ -4x_2 \\ -2x_2^4 \\ -2x_2^3 \end{pmatrix}, \quad N_{f,5}(x) = \begin{pmatrix} 0 & 1 & -x_2 & 0 & 0 \\ 4x_2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -x_2 \\ 0 & -\frac{x_2}{2} & 0 & 0 & 1 \\ x_1 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & x_1 & 0 & 0 \\ 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & x_1 \end{pmatrix} \quad (7)$$

$$S_{f,6} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \pi_{f,6}(x) = 1, \quad N_{f,6}(x) = x_2 \quad (8)$$

$$P = \begin{pmatrix} 1.0205 & 0.0198 & -0.0409 & 0.1611 & 0.1048 & -0.1341 & 0.1372 & 0 & 0 & 0 & -0.0707 & 0 & -0.0315 & -0.0186 \\ 0.0198 & -0.0643 & -0.0067 & -0.0009 & -0.0007 & -0.0065 & 0.0042 & -0.0023 & -0.0042 & 0 & 0.0191 & 0.0007 & -0.0519 & 0.006 \\ -0.0409 & -0.0067 & -0.0035 & 0.005 & -0.0051 & -0.0012 & -0.0061 & -0.0055 & -0.0179 & -0.0191 & 0.0087 & -0.0101 & -0.0604 & -0.0133 & 0.0165 \\ 0.1611 & -0.0009 & 0.005 & 0.0031 & 0.0094 & -0.0051 & 0.0018 & -0.014 & -0.0022 & -0.0155 & 0.0079 & 0.0103 & -0.0031 & -0.0007 & 0.0035 \\ 0.1048 & -0.0007 & -0.0051 & 0.0094 & -0.001 & 0 & -0.0005 & -0.0002 & 0.0016 & 0.0262 & -0.0267 & 0.0023 & 0.0127 & 0 & 0.0017 \\ -0.1341 & -0.0065 & -0.0012 & -0.0051 & 0 & 0.0042 & -0.0004 & -0.002 & -0.0003 & 0.0669 & -0.0169 & 0.0284 & 0.0522 & 0.0005 & 0.0022 \\ 0.1372 & 0.0042 & -0.0061 & 0.0018 & -0.0005 & -0.0004 & -0.0091 & 0.0024 & -0.011 & 0.0304 & -0.0792 & 0.0041 & -0.0181 & -0.0008 & -0.0016 \\ 0 & -0.0023 & -0.0055 & -0.014 & -0.0002 & -0.002 & 0.0024 & 0.0809 & 0.0176 & -0.514 & -0.0013 & -0.0157 & 0.0234 & 0.0496 & 0.0199 \\ 0 & -0.0042 & -0.0179 & -0.0022 & 0.0016 & -0.0003 & -0.011 & 0.0176 & -0.0609 & 0.0013 & -0.092 & -0.0249 & -0.0269 & 0.0163 & -0.028 \\ 0 & 0 & -0.0191 & -0.0155 & 0.0262 & 0.0669 & 0.0304 & -0.514 & 0.0013 & 0.0042 & 0 & -0.0369 & -0.2118 & 0.0145 & 0.0834 \\ 0 & 0.0191 & 0.0087 & 0.0079 & -0.0267 & -0.0169 & -0.0792 & -0.0013 & -0.092 & 0 & -3.1572 & 0.0083 & -0.0693 & -0.0071 & -0.0103 \\ -0.0707 & 0.0007 & -0.0101 & 0.0103 & 0.0023 & 0.0284 & 0.0041 & -0.0157 & -0.0249 & -0.0369 & 0.0083 & -0.0039 & -0.0023 & -0.0035 & 0.0007 \\ 0 & -0.0519 & -0.0014 & -0.0604 & 0.0127 & -0.0031 & 0.0522 & -0.0181 & -0.0269 & -0.0218 & -0.0023 & -0.0077 & -0.0023 & 0.0007 & 0.0049 \\ -0.0315 & 0.006 & -0.0133 & -0.0007 & 0 & 0.0005 & -0.0008 & 0.0496 & 0.0163 & 0.0145 & -0.0071 & -0.0035 & -0.0007 & 0.005 & 0 \\ -0.0186 & 0.0257 & 0.0165 & 0.0035 & 0.0017 & 0.0022 & -0.0016 & 0.0199 & -0.0028 & 0.0834 & -0.0103 & 0.0007 & 0.0049 & 0 & -0.0195 \end{pmatrix} \quad (9)$$

2 Computed values for the genetic toggle switch in Escherichia coli

This section presents the values of the computed matrices and function for the genetic toggle switch dynamics.

[illegible]

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$$\Delta = \text{blkdiag}(x_1 I_{11}, x_2 I_4)$$

$$\pi_{\text{fr}}(x) = \pi(x) = \left(\begin{array}{c} -x_1 \\ x_1^{10} \\ x_1^{10+1} \\ x_1^9 \\ x_1^{10+1} \\ x_1^8 \\ x_1^{10+1} \\ x_1^7 \\ x_1^{10+1} \\ x_1^6 \\ x_1^{10+1} \\ x_1^5 \\ x_1^{10+1} \\ x_1^4 \\ x_1^{10+1} \\ x_1^3 \\ x_1^{10+1} \\ x_1^2 \\ x_1^{10+1} \\ x_1 \\ x_1^{10+1} \\ x_1^3 \\ x_2^3+1 \\ x_2^2 \\ x_2^3+1 \\ x_2 \\ x_2^3+1 \\ -x_2 \end{array} \right), \quad S = I_{16}$$

$$S_{f,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0 & 0 \\ 0.001 & 0 & 0 & 0 & 0 \\ 0.002 & 0 & 0 & 0 & 0 \\ 0.0039 & 0 & 0 & 0 & 0 \\ 0.0078 & 0 & 0 & 0 & 0 \\ 0.0156 & 0 & 0 & 0 & 0 \\ 0.0312 & 0 & 0 & 0 & 0 \\ 0.0624 & 0 & 0 & 0 & 0 \\ 0.1249 & 0 & 0 & 0 & 0 \\ 0.2498 & 0 & 0 & 0 & 0 \\ 0.4995 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \pi_{f,1} = \begin{pmatrix} 1 \\ \frac{x_3}{x_2^3+1} \\ \frac{x_2^2}{x_2^3+1} \\ \frac{x_2}{x_2^3+1} \\ -x_2 \end{pmatrix}, \quad N_{f,1} = \begin{pmatrix} 1-2x_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -x_2 & 0 & 0 \\ 0 & 0 & 1 & -x_2 & 0 \\ -x_2 & x_2 & 0 & 1 & 0 \\ x_2 & 0 & 0 & 0 & 1 \\ 0 & x_1 & -\frac{x_2}{2} & 0 & 0 \\ 0 & 0 & x_1 & -\frac{x_2}{2} & 0 \\ -\frac{x_2}{2} & \frac{x_2}{2} & 0 & x_1 & 0 \\ \frac{x_2}{2} & 0 & 0 & 0 & x_1 \end{pmatrix}$$

$$S_{f,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1.5 & 0 & 0 & 0 & 0 \\ 0.983 & 0 & 0 & 0 & 0 \\ 0.6553 & 0 & 0 & 0 & 0 \\ 0.4369 & 0 & 0 & 0 & 0 \\ 0.2912 & 0 & 0 & 0 & 0 \\ 0.1942 & 0 & 0 & 0 & 0 \\ 0.1294 & 0 & 0 & 0 & 0 \\ 0.0863 & 0 & 0 & 0 & 0 \\ 0.0575 & 0 & 0 & 0 & 0 \\ 0.0384 & 0 & 0 & 0 & 0 \\ 0.0256 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \pi_{f,2} = \begin{pmatrix} 1 \\ \frac{x_2^3}{x_2^3+1} \\ \frac{x_2^2}{x_2^2+1} \\ \frac{x_2}{x_2+1} \\ -x_2 \end{pmatrix}, \quad N_{f,2} = \begin{pmatrix} 1 - \frac{2x_1}{3} & 0 & 0 & 0 & 0 \\ 0 & 1 & -x_2 & 0 & 0 \\ 0 & 0 & 1 & -x_2 & 0 \\ -x_2 & x_2 & 0 & 1 & 0 \\ x_2 & 0 & 0 & 0 & 1 \\ 0 & x_1 & -\frac{3x_2}{2} & 0 & 0 \\ 0 & 0 & x_1 & -\frac{3x_2}{2} & 0 \\ -\frac{3x_2}{2} & \frac{3x_2}{2} & 0 & x_1 & 0 \\ \frac{3x_2}{2} & 0 & 0 & 0 & x_1 \end{pmatrix}$$

[illegible]

3 Computed values for the periodic ring dynamics

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$$N(x) = \begin{pmatrix} x_1 & -1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & -1 & 0 & 0 & 0 \\ 0 & x_1 & 0 & -1 & 0 & 0 \\ 0 & x_2 & 0 & 0 & -1 & 0 \\ 0 & 0 & x_1 & 0 & -1 & 0 \\ 0 & 0 & x_2 & 0 & 0 & -1 \\ 0 & 0 & 0 & x_2 & -x_1 & 0 \\ 0 & 0 & 0 & 0 & x_2 & -x_1 \end{pmatrix}$$

$$S_{f,1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 0 & 0 \\ 0 & 1 & 0 \\ 9 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_{g,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ -27 & 0 & 1 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 81 & 0 & 0 & 0 & 0 \\ 0 & -27 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$S_{f,2} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \\ 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \pi$$

$$S_{g,2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 27 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 81 & 0 & 0 & 0 & 0 \\ 0 & 27 & 0 & 0 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \pi_{g,2} = \begin{pmatrix} 1 \\ x_2 \\ x_2^2 \\ x_2^3 \\ x_2^4 \end{pmatrix}$$

$$S_{f,3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 0 \\ 9 & 0 & 0 \end{pmatrix}$$

$$S_{g,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ -27 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 9 & 0 & 0 \\ 0 & -27 & 0 & 0 & 0 \\ 81 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S_{f,4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \\ 9 & 0 & 0 \end{pmatrix}, \quad \pi$$

$$S_{g,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 9 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 27 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 9 & 0 \\ 0 & 27 & 0 & 0 & 0 \\ 81 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \pi_{g,4} = \begin{pmatrix} 1 \\ x_1 \\ x_1^2 \\ x_1^3 \\ x_1^4 \end{pmatrix}$$

$$P = \begin{pmatrix} 0.8934 & - & - \\ 0 & - & - \\ 0 & - & - \\ 0.0994 & - & - \\ 0.0454 & - & - \\ -0.1659 & - & - \end{pmatrix}$$

4 Computed values for the Van der Pol oscillator

This section presents the values of the computed matrices and function for the Van der Pol oscillator:

[illegible]