Today's Agenda:-

- Modular Arithmetic Introduction
- Count pairs whose sum mod m is 0
- Introduction to GCD
- Properties of GCD
- Delete One

$\underbrace{A \% B}_{\to [0 \quad B-1]} \longrightarrow$ Remainder when A is divided by B.

$x \% 6 \longrightarrow [0 \quad 5]$

$30 \% 7 \longrightarrow 2$

$40 \% 9 \longrightarrow 4$

$5 \% 5 \longrightarrow 0$

$1 \% 2 \longrightarrow$

$$\begin{array}{r} 2 \,\big)\, 1 \,\big(\, 0 \\ 0 \\ \hline 1 \end{array}$$

$30 \% 7 = 30 - 7 = 23 - 7 = 16 - 7 = 9 - 7 \Rightarrow \underline{2}.$

$A \% B \Rightarrow$ keep subtracting B from A, till $A \geq B$.

Why do we need mod

$-\infty$

$\% 10 \longrightarrow [0 \text{ to } 9]$

To limit our range.

$-\infty$

Divisor) Dividend (Quotient

remainder

Remainder = Dividend − greatest multiple of divisor <= dividend

$30 \% 7 \Rightarrow 30 - 28 \Rightarrow \underline{2}$.

$61 \% 5 \Rightarrow 61 - 60 \Rightarrow \underline{1}$

$-7 \% 3 \Rightarrow -7 - (-9) \Rightarrow \underline{2}$.

$-30 \% 7 \Rightarrow -30 - (-35) \Rightarrow \underline{5}$

$-1 + 3$ $\quad -2 + 7$

$a \% m \longrightarrow (x + m)$ if $a < 0$.

Rules of Modular arithmetic

1 $\quad (a + b) \% m \longrightarrow \underbrace{(\underbrace{a \% m}_{(0 \; m-1)} + \underbrace{b \% m}_{(0 \; m-1)})}_{(0 \quad 2m-1)} \% m \rightarrow \underline{0 \; m-1}$.

$a = 9, \quad b = 8, \quad m = 5$.

LHS

$(9 + 8) \% 5 \Rightarrow 17 \% 5 \Rightarrow 2$

RHS

$a \% m$

$(9 \% 5 + 8 \% 5) \% 5$

$(4 + 3) \% 5$

$\Rightarrow \underline{2}$.

Let's say our no' range is till 10 only.

2) $(a * b) \% m \longrightarrow (a\%m * b\%m) \% m$.

3) $\underbrace{(a+m) \% m} \longrightarrow \underline{\underline{a \% m}}$

$(a\%m + \underbrace{m\%m}_{0}) \% m$

$\underbrace{(a\%m)\%m}$

4) $(a-b) \% m \Rightarrow (a\%m - b\%m + m) \% \underline{m}$.

$a=10, \; b=8, \; m=9$

LHS.

$(10-8)\%9$

$2\%9 \Rightarrow 2$

RHS

$(10\%9 - 8\%9)\%9$

$(1-8)\%9$

$\underbrace{-7\%9}$

$\underline{-ve}$, $+\underline{m}$.

5) $a^b \% m = (a\%m)^b \% m$

$(\underbrace{a*a*a*a* \cdots \cdots a}_{b \text{ times}})\%m \longrightarrow (\underbrace{a\%m * a\%m \cdots a\%m}_{b\text{ times}})\%m$

Quiz

$$(37^{103} - 1) \% 12$$

$$(37^{103} \% 12 - 1 \% 12 + 12) \% m$$

$$(\underbrace{(37 \% 12)}_{1}{}^{103} \% 12 - 1 + 12) \% m$$

$$(1^{103} \% 12 - 1 + 12) \% 12$$

$$(1 \% 12 - 1 + 12) \% 12$$

$$(1 - 1 + 12) \% 12$$

$$(0 + 12) \% 12$$

$$\Rightarrow \quad 12 \% 12 \Rightarrow 0$$

Ques) Given N array elements,

find pairs (i, j) s.t.,

(arr[i] + arr[j]) % m = 0, i != j.

A[] = {4, 3, 6, 3, 8, 12}

m = 6.

Brute Force Approach :-

Two for loops.

Checking all pairs.

T.C → O(N²).

idea 2

$(a+b)\ \%\ m = \underline{0}$.

$\downarrow$

$(\underbrace{a\ \%\ m}_{[0\ \ m-1]} + \underbrace{b\ \%\ m}_{[0\ \ m-1]})\ \%\ m = \underline{0}$.

$[1 + (m-1)]\ \%\ m$

$[2 + (m-2)]\ \%\ m$

$[3 + (m-3)]\ \%\ m$

$[0 + 0]\ \%\ m$

	0	1	2	3	4	5	6	7	8	9	10	11
arr[12] =	6	7	5	11	19	20	9	15	14	13	12	23
arr[12] % 5 =	1	2	0	1	4	0	4	0	4	3	2	3

M = 5

remainder		freq
0	$\longrightarrow$	3
1	$\longrightarrow$	2
2	$\longrightarrow$	2
3	$\longrightarrow$	2
4	$\longrightarrow$	3

6

4

$\longrightarrow 3 * \frac{(3-1)}{2} \Rightarrow \underline{3}$

$\frac{n*(n-1)}{2}$ $\quad {}^{n}\underline{C_2}$.

no. of ways to pick 2 items out of $\underline{n}$

	0	1	2	3	4	5	6	7	8	9	10	11
arr[12] :-	6	7	5	11	19	20	9	15	14	13	12	23

$\boxed{M = 6}$

	0	1	2	3	4	5	6	7	8	9	10	11
arr[12] % 6 :-	0	1	5	5	1	2	3	3	2	1	0	5

rem		freq	
0	→	2	→ $2*\frac{(2-1)}{2}$
1	→	3	
2	→	2	
3	→	2	→ $2*\frac{(2-1)}{2} = 1$
4	→	0	
5	→	3	

9

0

given arr[N] & M,

hashmap<int, int> hm;

→ Insert all elements in hMap.

↳ arr[i] % M

c=0;

Pairs of 0 {

x = hm[0]; // freq of 0

$c = c + \frac{x*(x-1)}{2}$

if (m % 2 == 0) {

$x = hm[\frac{m}{2}]$

$c = c + x * \frac{(x-1)}{2}$

}

m = 5

for (i = ~~1~~ 1 ; i < $\frac{(m+1)}{2}$; i++) {

c = c + hm[i] * hm[m-i];

hm[2] * hm[3]

}

T.C → O(N + M)

S.C →

Case - 1

N = 100

M = 10,

S.C → O(M)

Case. 2

N = 10

M = 100

S.C → O(N)

S.C → O(min(N, M))

Break 8:34 Am – 8:44 am

← GCD → Maths

↳ Greatest Common divisor

↳ HCF → Highest common factor

$\gcd(a, b) = x$,

↳ x is the greatest no',

$a \% x = 0$

$b \% x = 0$.

$\gcd(15, 25) \Rightarrow 5$

15: 1, 3, 5, 15

25: 1, 5, 25

$\gcd(10, -25) \Rightarrow 5$

10: 1, 2, 5, 10

-25: -25, -5, -1, 1, 5, 25

$\gcd(12, 30) \Rightarrow 6$

12: 1, 2, 3, 4, 6, 12

30: 1, 2, 3, 5, 6, 15, 30

$\gcd(0, 8) \Rightarrow 8$.

0: 0, 1, 2, 3, ..., ∞

8: 1, 2, 4, 8

$\gcd(0, -10) = 10$.

0: 0, 1, 2, 3, ..., ∞

-10: -10, -5, -2, -1, 1, 2, 5, 10

$\gcd(0, 0) \Rightarrow \infty$.

0: 0, 1, 2, 3, ..., ∞

0: 0, 1, 2, 3, ..., ∞

$\gcd(-2, -3) \Rightarrow 1$.

-2: -2, -1, 1, 2

-3: -3, -1, 1, 3

$\gcd(0, -5) \Rightarrow \underline{5}$. $\gcd(0,5) = \underline{5}$.

$0, 1, 2, \ldots, \infty$ $-5, -1, 1, 5$

$\gcd(0, x) = \underline{|x|}$. $\underline{(x \neq 0)}$

Properties of gcd,

1) $\gcd(a, b) = \gcd(b, a)$

2) $\gcd(0, x) = \underline{|x|}$.

3) $\gcd(A, B, C) = \gcd(A, \gcd(B, C)) =$

$\gcd(B, \gcd(A, C)) =$

$\gcd(C, \gcd(A, B))$

Special Property :-

$A, B \underline{>} 0$, $(A > \underline{= B})$

$\gcd(a, b) = \gcd(a-b, \underline{b})$

$\gcd(10, 5) = \gcd(5, 5) = \gcd(0, 5)$

Proof $\gcd(a, b) = \gcd(a-b, b)$

$\gcd(a,b) = d$

$a \% d = 0$, $b \% d = 0$,

$a = d * k_1$, $b = d * k_2$

$\Rightarrow (a-b) = d(k_1 - k_2)$

$(a-b) \% d = 0$,

$\gcd(a,b) = d$, & m is also a factor of a & $\underline{b}$.

$d >\underline{=} m$.

$\gcd(a-b, b) = m$

$(a-b) \% m = 0$, & $b \% m = 0$.

$a - b = k_3 m$, & $b = k_4 m$

$+$

$a = m(k_3 + k_4)$

$a \% m = 0$

$\gcd(a-b,b) = m$ & d is a factor of $(\underline{a-b})$ & $\underline{b}$.

$m >\underline{=} d$.

$\underline{m = d}$

$\gcd(a,b) = \gcd(a-b, \underline{b})$.

$\gcd(23,5) = \gcd(18,5) = \gcd(13,5) = \gcd(8,5)$

$\downarrow$

$\gcd(3,5)$

$$\gcd(a,b) = \gcd(a-b,b)$$

$$\downarrow$$

$$\gcd(a-2b,b)$$

$$\downarrow$$

$$\gcd(a-3b,b)$$

$$\vdots$$

$$\gcd(a-yb,b)$$

$$\Rightarrow \gcd(a \% b, b)$$

Property :-

gcd (a, b) = gcd (a % b, b)

gcd (24, 16) = gcd (8, 16) = gcd (8, 16) = gcd (8, 16)
— - - - - .
∞

gcd (a, b) = gcd (b, a % b)

gcd (24, 16) = gcd (16, 8) = gcd (8, 0)
||
8.

gcd (14, 24) = gcd (24, 14) = gcd (14, 10)
= gcd (10, 4)
= gcd (4, 2)
= gcd (2, 0) => 2 only.

gcd (|a|, |b|) ;

```
int   gcd ( a, b ) {
        if (b = = 0) { return a }
        return  gcd ( b, a % b );
}
```

T.C → O(log (max(a, b)))

Ques) Given arr →, delete one element, s.t, gcd of remaining elements become maxm.

$$\begin{array}{ccccccc} & & 0 & 1 & 2 & 3 & 4 \\ ar[\,] = & \{ & 24 & 16 & 18 & 30 & 15 \} \end{array}$$

gcd

$$\begin{array}{cccccc} & 0 & 1 & 2 & 3 & 4 \\ \{ & \cancel{24} & 16 & 18 & 30 & 15 \} \end{array} \qquad 1$$

$$\begin{array}{ccccccc} & & 0 & 1 & 2 & 3 & 4 \\ ar[\,] = & \{ & 24 & \cancel{16} & 18 & 30 & 15 \} \end{array} \qquad 3$$

$$\begin{array}{ccccccc} & & 0 & 1 & 2 & 3 & 4 \\ ar[\,] = & \{ & 24 & 16 & \cancel{18} & 30 & 15 \} \end{array} \qquad 1$$

$$\begin{array}{ccccccc} & & 0 & 1 & 2 & 3 & 4 \\ ar[\,] = & \{ & 24 & 16 & 18 & \cancel{30} & 15 \} \end{array} \qquad 1$$

$$\begin{array}{ccccccc} & & 0 & 1 & 2 & 3 & 4 \\ ar[\,] = & \{ & 24 & 16 & 18 & 30 & \cancel{15} \} \end{array} \qquad 2$$

Brute force

$n \times (n * \log \max(arr[i]))$

delete an arr[i] element, calculate gcd of remaining elements & get overall max.

idea 2 :-

	0	1	2	3	4
	24	16	18	30	15
pfgcd →	24	8	2	2	1
sfgcd →	1	1	3	15	15

```
int deleteOne (int[] arr, int n) {
        pfgcd [N];   } Todo.
        sfgcd [N];
        ans = max(sfgcd[1], pfgcd [N-2]);
        for (i=1; i<n-1 ; i++) {
                // deleting ith element.
                left = pfgcd[i-1];
                right = sfgcd [i+1];
                val = gcd (left, right);
                ans = max (ans, val);
```

}

return ans;

}

T.C → O(n log max(arr[i]))

S.C → O(n)

// a > b

$gcd(a, b) = gcd(a \% b, b)$ $< \frac{a}{2}$

Case-1	Case-2	Case-3
$b < \frac{a}{2}$	$b = \frac{a}{2}$	$b > \frac{a}{2}$
$\underbrace{a \% b}_{0 \text{ to } b-1} < b < \frac{a}{2}$	$\underbrace{a \% b}_{0 \text{ to } b-1} < b = \frac{a}{2}$	→ $2b > a$
		→ $2b - a > 0$
	$a \% b < \frac{a}{2}$	// (-) both sides.
$a \% b < \frac{a}{2}$		$a - 2b < 0$
		adding a on both sides.
		$2a - 2b < a$
		$(a-b) < \frac{a}{2}$

$\boxed{a \% b = a - b}$

a - 2b

a - 3b

$\boxed{a \% b < \frac{a}{2}}$

a

$-30 \% 7 = -2$ (Java follows the rule of sign of remainder will be same as sign of dividend)

$+7$

$$
\begin{array}{r|l}
7 & 30 \quad (4 \\
 & \underline{28} \\
 & \underline{2}
\end{array}
$$

$-30 \% 7 = -30 - (-35) = \underline{5}$.

$10^9 + 7 \Rightarrow$

$\approx \underline{10^9}$, Prime $\underline{no}$.

$\underline{(37 \% 2)}^{\wedge 103} \approx 1^{103}$

$\downarrow$

$1 \, \underline{^{\wedge 103}}$