Homework 3

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October 24, 2025

Problem 1.

(a) The dual of each model are as follows:

$$
\begin{aligned}
\text{(i)} \quad \max_{y} \quad & 19y_1 + 55y_2 + 7y_3 \\
\text{s.t.} \quad & y_1 + y_3 = 5 \\
& y_1 + 4y_2 + 6y_3 \leq 1 \\
& y_1 - y_3 \leq -4 \\
& y_1 + 8y_2 \geq 0 \\
& y_1 \geq 0, y_2 \leq 0, y_3 \text{ free}
\end{aligned}
$$

$$
\begin{aligned}
\text{(ii)} \quad \min_{y} \quad & 20y_1 + 35y_2 + 15y_3 + 10y_4 + 6y_5 + 2y_6 \\
\text{s.t.} \quad & 2y_1 + 7y_2 + 4y_3 + y_4 \geq 2 \\
& -3y_1 + 2y_2 + 5y_3 + y_5 \geq -7 \\
& -5y_1 + 6y_2 - 3y_3 + y_6 \geq 6 \\
& -4y_1 - 2y_2 - 2y_3 \geq 5 \\
& y_1 \geq 0, y_2 \text{ free}, y_3 \leq 0, y_4 \geq 0, y_5 \geq 0, y_6 \leq 0
\end{aligned}
$$

(b) The KKT optimality conditions for each model are as follows:

(i) Primal feasibility:

$$
\begin{aligned}
& x_1 + x_2 + x_3 + x_4 \geq 19 \\
& 4x_2 + 8x_4 \leq 55 \\
& x_1 + 6x_2 - x_3 = 7 \\
& x_1 \text{ free}, x_2 \geq 0, x_3 \geq 0, x_4 \leq 0
\end{aligned}
$$

Complementary slackness conditions:

$$x_1(y_1 + y_3 - 5) = 0$$
$$x_2(y_1 + 4y_2 + 6y_3 - 1) = 0$$
$$x_3(y_1 - y_3 + 4) = 0$$
$$x_4(y_1 + 8y_2) = 0$$
$$y_1(x_1 + x_2 + x_3 + x_4 - 19) = 0$$
$$y_2(4x_2 + 8x_4 - 55) = 0$$
$$y_3(x_1 + 6x_2 - x_3 - 7) = 0$$

(ii) Primal feasibility:

$$2x_1 - 3x_2 - 5x_3 - 4x_4 \leq 20$$
$$7x_1 + 2x_2 + 6x_3 - 2x_4 = 35$$
$$4x_1 + 5x_2 - 3x_3 - 2x_4 \geq 15$$
$$0 \leq x_1 \leq 10, 0 \leq x_2 \leq 5, x_3 \geq 2, x_4 \geq 0$$

Complementary slackness conditions:

$$x_1(2y_1 + 7y_2 + 4y_3 + y_4 - 2) = 0$$

$$x_2(-3y_1 + 2y_2 + 5y_3 + y_5 + 7) = 0$$

$$x_3(-5y_1 + 6y_2 - 3y_3 + y_6 - 6) = 0$$

$$x_4(-4y_1 - 2y_2 - 2y_3 - 5) = 0$$

$$y_1(2x_1 - 3x_2 - 5x_3 - 4x_4 - 20) = 0$$

$$y_2(7x_1 + 2x_2 + 6x_3 - 2x_4 - 35) = 0$$

$$y_3(4x_1 + 5x_2 - 3x_3 - 2x_4 - 15) = 0$$

$$y_4(x_1 - 10) = 0$$

$$y_5(x_2 - 5) = 0$$

$$y_6(x_3 - 2) = 0$$

Problem 2.

(a) Let the decision variables be x_1, x_2, x_3, representing the number of first-class, bussiness-class and coach-fare seats respectively.

And n_t^s represent the number of t-class tickets sold in s-scenarios, where $t = 1$ representing first-class, $t = 2$ representing bussiness-class and $t = 3$ representing coach-fare; $s = 1$ representing scenarios (i), $s = 2$ representing scenarios (ii) and $s = 3$ representing scenarios (iii).

We let the profit of coach-fare ticket be 1 unit, thus the model is as follows:

$$
\begin{aligned}
\max_{n} \quad & \frac{1}{3}\sum_{s=1}^{3}(3n_1^s + 2n_2^s + n_3^s) \\
\text{s.t.} \quad & 2x_1 + 1.5x_2 + x_3 \le 200 \\
& n_t^s \le x_t \quad \forall s \in \{1,2,3\}, t \in \{1,2,3\} \\
& n_1^1 \le 20, \quad n_2^1 \le 50, \quad n_3^1 \le 200 \\
& n_1^2 \le 10, \quad n_2^2 \le 25, \quad n_3^2 \le 175 \\
& n_1^3 \le 5, \quad n_2^3 \le 10, \quad n_3^3 \le 150 \\
& x_1, x_2, x_3, n_t^s \ge 0 \quad \forall s \in \{1,2,3\}, t \in \{1,2,3\}
\end{aligned}
$$

(b) Optimal partition: $x^* = (10, 20, 150)$

Scenario (i): Sell first-class:10, business-class:20, coach-fare:150

Scenario (ii): Sell first-class:10, business-class:20, coach-fare:150

Scenario (iii): Sell first-class:5, business-class:10, coach-fare:150

Optimal profit: 208.33

(c) The shadow price of the constraint is $y_1^* = 0.8889$.

(d) The new optimal value $z^* = z^*_{original} + y_1^* \cdot (201 - 200) = 209.2222$.

(e) The linear program's new optimal value is 209.2222, which is consistent with the previous calculated value z^*.

(f) The shadow prices of the constraints are shown in Table 1:

Table 1: Shadow prices of the constraints

Scenario	first-class	business-class	coach-fare
i	0	0	0
ii	0.2222	0	0
iii	1	0.6667	0.1111

For example, in scenario (i), the shadow price of first-class ticket is 0 because the constraint $n_1^1 \leq 20$ is not tight at optimality (we only sell 10 first-class tickets), thus increasing the limit of first-class tickets would not increase the overall profit.

Problem 3.

(a) The dual is as follows:

$$\begin{aligned}\min_{y} \quad & 12y_1 + 10y_2 + 10y_3 \\ \text{s.t.} \quad & 2y_1 + y_2 + 3y_3 \geq 4 \\ & 3y_1 + 4y_2 + y_3 \geq 2 \\ & y_1 + 2y_2 + y_3 \geq 3 \\ & y_1, y_2, y_3, y_4 \geq 0\end{aligned}$$

(b) The complementary slackness conditions are as follows:

$$\begin{aligned} & x_1(2y_1 + y_2 + 3y_3 - 4) = 0 \\ & x_2(3y_1 + 4y_2 + y_3 - 2) = 0 \\ & x_3(y_1 + 2y_2 + y_3 - 3) = 0 \\ & y_1(2x_1 + 3x_2 + x_3 - 12) = 0 \\ & y_2(x_1 + 4x_2 + 2x_3 - 10) = 0 \\ & y_3(3x_1 + x_2 + x_3 - 10) = 0 \end{aligned}$$

Since we have $(x_1^*, x_2^*, x_3^*) = (2, 0, 4)$, we can deduce that

$$\begin{cases} 2y_1^* + y_2^* + 3y_3^* - 4 = 0 \\ y_1^* + 2y_2^* + y_3^* - 3 = 0 \\ y_1^* = 0 \end{cases} \Rightarrow \begin{cases} y_1^* = 0 \\ y_2^* = 1 \\ y_3^* = 1 \end{cases}$$

(c) Primal objective value is $4 \times x_1^* + 2 \times x_2^* + 3 \times x_3^* = 20$.

While dual objective value is $12 \times y_1^* + 10 \times y_2^* + 10 \times y_3^* = 20$.

Problem 4.

(a) Let the decision variables be x_{ij} representing whether to go the route from hole i to hole j, and d_{ij} representing the distance from hole i to hole j. The model is as follows:

$$
\begin{aligned}
\min_{x} \quad & \sum_{i=1}^{5}\sum_{j=1}^{5} d_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{j=1}^{5} x_{ij} = 1 \quad \forall i \in \{1,2,\dots,5\} \\
& \sum_{i=1}^{5} x_{ij} = 1 \quad \forall j \in \{1,2,\dots,5\} \\
& \sum_{i\in S}\sum_{j\in S} x_{ij} \leq |S| - 1 \quad \forall S \subseteq \{1,2,\dots,5\}, |S| \leq 4 \\
& x_{ij} \in \{0,1\} \quad \forall i,j \in \{1,2,\dots,5\}
\end{aligned}
$$

(b) The optimal route is: $1 \to 3 \to 2 \to 4 \to 5 \to 1$, with the optimal distance being 35.

Problem 5.

(a) Let the decision variables be x_i representing whether to choose the course i. The model is as follows:

$$\begin{aligned}
\min_{x} \quad & \sum_{i=1}^{7} x_i \\
\text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 + x_7 \geq 2 \\
& x_2 + x_4 + x_5 + x_7 \geq 2 \\
& x_3 + x_5 + x_6 \geq 2 \\
& x_3 \leq x_6 \\
& x_4 \leq x_1 \\
& x_5 \leq x_6 \\
& x_7 \leq x_4 \\
& x_i \in \{0,1\} \quad \forall i \in \{1,2,\ldots,7\}
\end{aligned}$$

(b) The optimal courses to choose are course 2, course 3, course 5 and course 6 (Operations Research, Data Structures, Computer Simulation, Intro to Programming), with the optimal number of courses being 4.