DDA 5002 Optimization Fall 2025
Homework # 2
Due: Sunday, October 12 at 11:59pm

1. Reformulate the following two problems as a linear program:

   (a)
   $$\begin{aligned} \min_{x} \quad & c^\top x + f(d^\top x) \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

   where $x, c, d \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and

   $$f(\alpha) = \max\{\alpha,\, 0,\, 2\alpha - 4\} \quad \text{for } \alpha \in \mathbb{R}.$$

   (b)
   $$\begin{aligned} \min_{x_1, x_2, x_3} \quad & 2x_2 + |x_1 - x_3| \\ \text{s.t.} \quad & |x_1 + 2| + |x_2| \leq 5, \\ & x_3^2 \leq 1 \end{aligned}$$

2. Consider the following linear programs:

$$\begin{aligned} \max_{x} \quad & 5x_1 + x_2 - 4x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_4 \geq 19 \\ & 4x_2 + 8x_4 \leq 45 \\ & x_1 + 6x_2 - x_3 = 7 \\ & x_1 \text{ unrestricted}, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_4 \leq 0 \end{aligned}$$

and

$$\begin{aligned} \min_{x} \quad & 2x_1 - 7x_2 + 6x_3 + 5x_4 \\ \text{s.t.} \quad & 2x_1 - 3x_2 - 5x_3 - 4x_4 \leq 20 \\ & 7x_1 + 2x_2 + 6x_3 - 2x_4 = 35 \\ & 4x_1 + 5x_2 - 3x_3 - 2x_4 \geq 15 \\ & 0 \leq x_1 \leq 10,\ 0 \leq x_2 \leq 8,\ x_3 \geq 2,\ x_4 \geq 0 \end{aligned}$$

Rewrite each of these models in *standard form.* That is, reformulate each linear program as a "min" with equality constraints and nonnegative decision variables.

3. Solve the following 3-dimensional linear optimization problem using the graphical method:

$$\begin{aligned}
\max_{x_1,x_2,x_3} \quad & x_3 \\
\text{s.t.} \quad & x_1 + x_2 - x_3 = 0, \\
& -x_1 + x_2 \leq 2.5, \\
& x_1 + 2x_2 \leq 9, \\
& 0 \leq x_1 \leq 4, \\
& 0 \leq x_2 \leq 3
\end{aligned}$$

Which constraints are active at your optimal solution? Also list all the vertices (extreme points) of the feasible region.
Hint: You can reduce one dimension in order to apply the graphical method.

4. Consider the following linear program:

$$\begin{aligned}
\max_{x_1,x_2} \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 3x_2 \geq 15, \\
& 2x_1 + x_2 \geq 10, \\
& x_1 + 2x_2 \leq 40, \\
& 3x_1 + x_2 \leq 60, \\
& x_1, x_2 \geq 0.
\end{aligned}$$

Rewrite the model in standard form (a "min" objective with equality constraints and non-negative decision variables). Then, solve the model using the simplex method by the linear algebra derivation way. For the initial basis, use variables x_1 and x_2 along with the slack variables for the third and fourth constraints (you can call those slack variables x_5 and x_6). Sketch the feasible region, and indicate the solution at each iteration on the figure.

5. Assume we want to apply the two-phase simplex method to solve the following linear program:

$$\begin{aligned}
\min_{x} \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{s.t.} \quad & x_1 - x_2 + 2x_3 \geq 2, \\
& x_2 - x_3 + 2x_4 \leq 4, \\
& 2x_1 + 3x_3 - x_4 = 2, \\
& x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}$$

(a) Write down the auxiliary LP of Phase I (use variables s as slack variables in the standard form, variables y as auxiliary variables in Phase I, and order the variables as $(x; s; y)$);

(b) For the auxiliary LP of Phase I, consider the basis associated with variables x_1, x_4, and the auxiliary variable corresponding to the second constraint (i.e., y_2). Compute the associated basic feasible solution by solving the resultant linear system. Is this solution optimal for the auxiliary LP of Phase I? Why?

(c) With the above basis, compute the inverse of the basis matrix (i.e., A_B^{-1}), all basic directions, and their reduced costs. Are all reduced costs nonnegative? If not, derive the next simplex iteration by following the smallest index rule.

6. Consider an LP in its standard form and the corresponding constraint set

$$\min_x \; c^\top x \quad \text{s.t.} \quad Ax = b, \; x \geq 0.$$

Suppose that the matrix $A \in \mathbb{R}^{m\times n}$ with $m < n$ and its rows are linearly independent. For each of the following statements, state whether it is true or false. Please explain your answers (if not true, please show a counterexample).

(a) The set of all optimal solutions (assuming existence) must be bounded.

(b) At every optimal solution, no more than m variables can be positive.

(c) If there is more than one optimal solution, then there are infinity many optimal solutions.

(d) Every optimal solution of the LP is a basic feasible solution.