Homework 1

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Problem 1.

(a) The Linear program as follows:

$$
\begin{aligned}
\min_{x,z} \quad & c^T x + z \\
\text{s.t.} \quad & z \geq d^T x \\
& z \geq 0 \\
& z \geq 2d^T x - 4 \\
& Ax \geq b \\
& x \in \mathbb{R}^n, z \in \mathbb{R}
\end{aligned}
$$

(a) The Linear program as follows:

$$
\begin{aligned}
\min_{x_2,z} \quad & 2x_2 + x_4 \\
\text{s.t.} \quad & x_1 - x_3 \leq x_4 \\
& x_1 - x_3 \geq -x_4 \\
& x_1 + 2 \leq x_5 \\
& x_1 + 2 \geq -x_5 \\
& x_2 \leq x_6 \\
& x_2 \geq -x_6 \\
& x_5 + x_6 \leq 5 \\
& x_3 \leq 1 \\
& x_3 \geq -1 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \in \mathbb{R}
\end{aligned}
$$

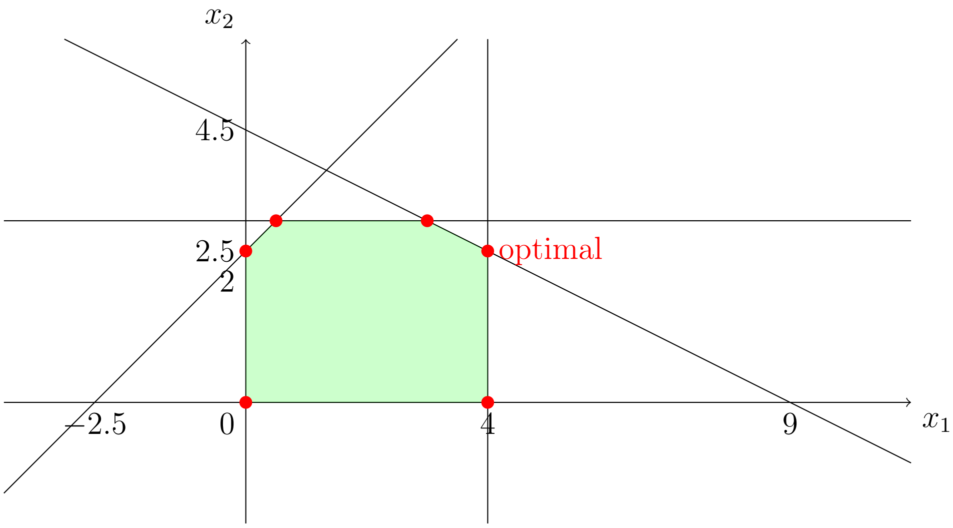
Problem 2. The standard form of the two linear program is as follows:

$$\begin{aligned}
\min_{x} \quad & 5x_1^- - 5x_1^+ - x_2 + 4x_3 \\
\text{s.t.} \quad & x_1^+ - x_1^- + x_2 + x_3 - x_5 - x_6 = 19 \\
& 4x_2 - 8x_5 + x_7 = 45 \\
& x_1^+ - x_1^- + 6x_2 - x_3 = 7 \\
& x_1^+, x_1^-, x_2, x_3, x_5, x_6, x_7 \geq 0
\end{aligned}$$

and

$$\begin{aligned}
\min_{x} \quad & 2x_1 - 7x_2 + 6x_3 + 5x_4 \\
\text{s.t.} \quad & 2x_1 - 3x_2 - 5x_3 - 4x_4 + x_5 = 20 \\
& 7x_1 + 2x_2 + 6x_3 - 2x_4 = 35 \\
& 4x_1 + 5x_2 - 3x_3 - 2x_4 - x_6 = 15 \\
& x_1 + x_7 = 10 \\
& x_2 + x_8 = 8 \\
& x_3 - x_9 = 2 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \geq 0
\end{aligned}$$

Problem 3. The graphic is as follows:



All the vertices of the feasible region are:

(0, 0), (0, 2.5), (0.5, 3), (3, 3), (4, 2.5), (4, 0).

The optimal solution is $(4, 2.5, 6.5)$, while the active constraints are

$x_1 + x_2 - x_3 = 0$, $-x_1 + 2x_2 \leq 2.5$ and $x_1 + 2x_2 \leq 9$.

Problem 4. The standard form of the linear program is as follows:

$$
\begin{aligned}
\min_{x} \quad & -x_1 - x_2 \\
\text{s.t.} \quad & x_1 + 3x_2 - x_3 = 15 \\
& 2x_1 + x_2 - x_4 = 10 \\
& x_1 + 2x_2 + x_5 = 40 \\
& 3x_1 + x_2 + x_6 = 60 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{aligned}
$$

Simplex method process by the linear algebra derivation way is as follows:

Iteration 1: We choose basis $\{1, 2, 5, 6\}$

$$
A_B = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}, \quad A_B^{-1} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{5} & 0 & 0 \\ \frac{2}{5} & -\frac{1}{5} & 0 & 0 \\ -\frac{3}{5} & -\frac{1}{5} & 1 & 0 \\ \frac{1}{5} & -\frac{8}{5} & 0 & 1 \end{bmatrix}
$$

$$
x_B = A_B^{-1} b = \begin{bmatrix} 3 \\ 4 \\ 29 \\ 47 \end{bmatrix}
$$

$$
\bar{c}_3 = c_3 - c_B^T A_B^{-1} A_3 = -\frac{1}{5}
$$

$$
\bar{c}_4 = c_4 - c_B^T A_B^{-1} A_4 = -\frac{2}{5}
$$

So we choose x_4 to enter the basis.

$$d_B = -A_B^{-1}A_4 = \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \\ -\frac{1}{5} \\ -\frac{8}{5} \end{bmatrix}$$

$$\theta^* = \min_{i \in B, d_i < 0} \frac{-x_i}{d_i} = \frac{-x_2}{d_2} = 20$$

Therefore, x_2 exits the basis.

Iteration 2: We choose basis $\{1, 4, 5, 6\}$

$$A_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}, \quad A_B^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}$$

$$x_B = A_B^{-1}b = \begin{bmatrix} 15 \\ 20 \\ 25 \\ 15 \end{bmatrix}$$

$$\bar{c}_2 = c_2 - c_B^T A_B^{-1} A_2 = 2$$

$$\bar{c}_3 = c_3 - c_B^T A_B^{-1} A_3 = -1$$

So we choose x_3 to enter the basis.

$$d_B = -A_B^{-1}A_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}$$

$$\theta^* = \min_{i \in B, d_i < 0} \frac{-x_i}{d_i} = \frac{-x_6}{d_6} = 5$$

Therefore, x_6 exits the basis.

Iteration 4: We choose basis $\{1, 2, 3, 4\}$

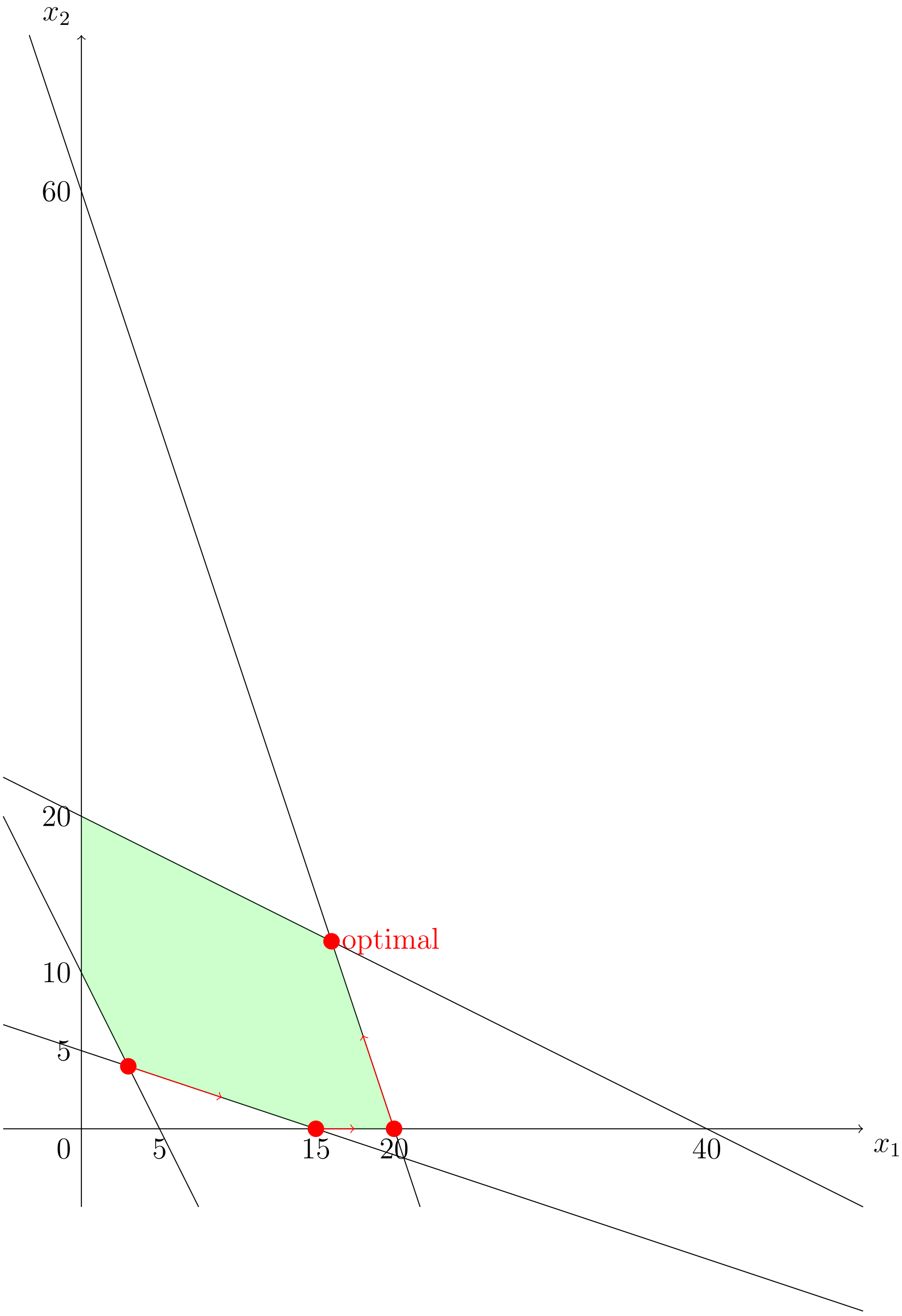
$$A_B = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 2 & 1 & 0 & -1 \\ 1 & 2 & 0 & 0 \\ 3 & 1 & 0 & 0 \end{bmatrix}, \quad A_B^{-1} = \begin{bmatrix} 0 & 0 & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & \frac{3}{5} & -\frac{1}{5} \\ -1 & 0 & \frac{8}{5} & -\frac{1}{5} \\ 0 & -1 & \frac{1}{5} & \frac{3}{5} \end{bmatrix}$$

$$x_B = A_B^{-1} b = \begin{bmatrix} 16 \\ 12 \\ 37 \\ 34 \end{bmatrix}$$

$$\bar{c}_5 = c_5 - c_B^T A_B^{-1} A_5 = \frac{2}{5}$$

$$\bar{c}_6 = c_6 - c_B^T A_B^{-1} A_6 = \frac{1}{5}$$

So the current BFS is optimal. Optimal solution is $(16, 12, 37, 34, 0, 0)$ and optimal value is -28.



x_2
60
20
10
5
0
5
15
20
40
x_1
optimal

Problem 5.

(a) The auxiliary LP of Phase I is as follows:

$$
\begin{aligned}
\min_{y} \quad & y_1 + y_2 \\
\text{s.t.} \quad & x_1 - x_2 - 2x_3 - s_1 + y_1 = 2 \\
& x_2 - x_3 + 2x_4 + s_2 = 4 \\
& 2x_1 + 3x_3 - x_4 + y_2 = 2 \\
& x_1, x_2, x_3, x_4, s_1, s_2, y_1, y_2 \geq 0
\end{aligned}
$$

(b) With the basis of $\{x_1, x_4, y_2\}$:

$$
A_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}, \quad A_B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & \frac{1}{2} & 1 \end{bmatrix}
$$

$$
\begin{bmatrix} x_1 \\ x_4 \\ y_2 \end{bmatrix} = A_B^{-1} b = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}
$$

While $y_1 = y_2 = 0$, this solution is optimal for the auxiliary LP of Phase I.

(c) Follow (b), we have:

$$
\begin{aligned}
\bar{c}_{x_2} &= c_{x_2} - c_B^T A_B^{-1} A_{x_2} = -\frac{5}{2} \\
\bar{c}_{x_3} &= c_{x_3} - c_B^T A_B^{-1} A_{x_3} = \frac{3}{2} \\
\bar{c}_{s_1} &= c_{s_1} - c_B^T A_B^{-1} A_{s_1} = -2 \\
\bar{c}_{s_2} &= c_{s_2} - c_B^T A_B^{-1} A_{s_2} = -\frac{1}{2} \\
\bar{c}_{y_1} &= c_{y_1} - c_B^T A_B^{-1} A_{y_1} = 3
\end{aligned}
$$

So we choose x_2 to enter the basis.

$$d_B = -A_B^{-1} A_2 = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ -\frac{5}{2} \end{bmatrix}$$

$$\theta^* = \min_{i \in B, d_i < 0} \frac{-x_i}{d_i} = \frac{-y_2}{d_{y_2}} = 5$$

Therefore, y_2 exits the basis, and the new basis in the next iteration is $\{x_1, x_2, x_4\}$.

Problem 6.

(a) False. Assume a LP as follows:

$$\begin{aligned} \min_x \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 - x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The optimal solution is $\{x_1, x_2, x_3 | x_1 + x_2 = 1\}$, which is unbounded.

(b) False. As LP in (a), one of the optimal solution is $(0, 1, 1)$, while $m = 1$. However, it has more than m variables being positive.

(c) True.

(d) False. As LP in (a), one of the optimal solution $(0, 0.5, 0.5)$ is not a basic feasible solution.