For simplicity, let $n = q - 1$.
Observe that, when $k \neq 0$,

$$\sum_{i=0}^{n-1} \alpha^{ik} = 0, \forall \alpha \in \mathbb{Z}_q^*,$$

and when $k = 0$, obviously

$$\sum_{i=0}^{n-1} \alpha^{ik} = \sum_{i=0}^{n-1} \alpha^0 = n, \forall \alpha \in \mathbb{Z}_q^*.$$

Therefore we have

$$
\begin{aligned}
f(i) &= \frac{1}{n} \sum_{k,j} f(j) \cdot \alpha^{k(j-i)} \\
&= \frac{1}{n} \sum_{k,j} f(j) \cdot \alpha^{kj-ki} \\
&= \frac{1}{n} \sum_{k,j} f(j) \cdot \alpha^{kj} \cdot \alpha^{-ik} \\
&= \frac{1}{n} \sum_{k} F(k) \cdot \alpha^{-ik},
\end{aligned}
$$

where $F(k) \stackrel{def}{=} \sum_j f(j) \cdot \alpha^{kj}$.

We can compute $F(k)$ recursively —

$$\begin{aligned} F^{(0)}(k) &= \sum_{j=0}^{n-1} f(j) \cdot \alpha^{kj} \\ &= \sum_{l=0}^{\frac{n}{2}-1} f(2l) \cdot \alpha^{k \cdot 2l} + \alpha^k \cdot \sum_{l=0}^{\frac{n}{2}-1} f(2l+1) \cdot \alpha^{k \cdot 2l} \\ &= F_0^{(1)}(k \bmod \frac{n}{2}) + \alpha^k \cdot F_1^{(1)}(k \bmod \frac{n}{2}) \end{aligned}$$

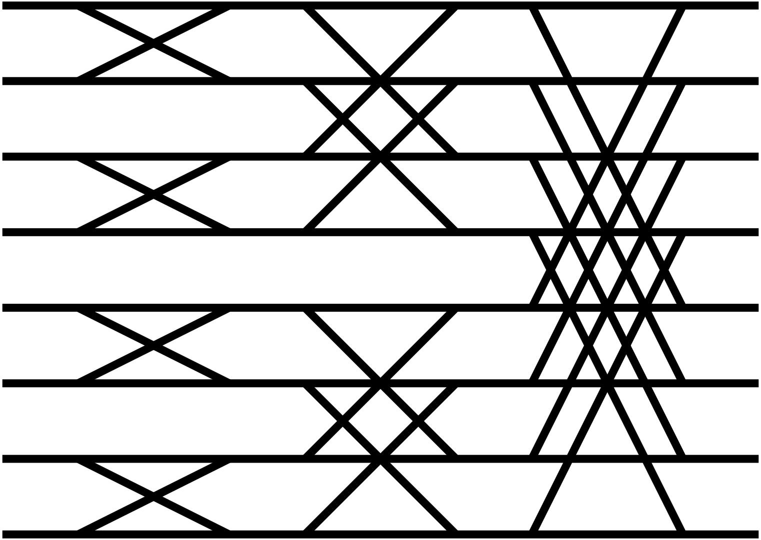
where $F_0^{(1)}$ and $F_1^{(1)}$ are the NTT's of with halved parameter n.

We can compute $F(k)$ recursively —

$$\begin{aligned} F^{(0)}(k) &= F_0^{(1)}(k') + \alpha^k \cdot F_1^{(1)}(k') \\ &= F_0^{(2)}(k' \bmod \frac{n}{4}) + \alpha^{k'} \cdot F_1^{(2)}(k' \bmod \frac{n}{4}) \\ &\quad + \alpha^k \cdot F_2^{(2)}(k' \bmod \frac{n}{4}) + \alpha^{k+k'} \cdot F_3^{(2)}(k' \bmod \frac{n}{4}) \\ &= \cdots , \end{aligned}$$

where $k' = k \bmod \frac{n}{2}$.

Hence the butterfly structure —



Hence the butterfly structure —