For simplicity, let $n = q - 1$.
Observe that, when $k \neq 0$,

$$\sum_{i=0}^{n-1} \alpha^{ik} = 0, \forall \alpha \in \mathbb{Z}_q^*,$$

and when $k = 0$, obviously

$$\sum_{i=0}^{n-1} \alpha^{ik} = \sum_{i=0}^{n-1} \alpha^{0} = n, \forall \alpha \in \mathbb{Z}_q^*.$$

Therefore we have

$$\begin{aligned} f(i) &= \frac{1}{n} \sum_{k,j} f(j) \cdot \alpha^{k(j-i)} \\ &= \frac{1}{n} \sum_{k,j} f(j) \cdot \alpha^{kj-ki} \\ &= \frac{1}{n} \sum_{k,j} f(j) \cdot \alpha^{kj} \cdot \alpha^{-ik} \\ &= \frac{1}{n} \sum_{k} F(k) \cdot \alpha^{-ik}, \end{aligned}$$

where $F(k) \stackrel{def}{=} \sum_j f(j) \cdot \alpha^{kj}$.

We can compute $F(k)$ recursively —

$$\begin{aligned} F^{(0)}(k) &= \sum_{j=0}^{n-1} f(j)\cdot\alpha^{kj} \\ &= \sum_{l=0}^{\frac{n}{2}-1} f(2l)\cdot\alpha^{k\cdot 2l} + \alpha^k\cdot\sum_{l=0}^{\frac{n}{2}-1} f(2l+1)\cdot\alpha^{k\cdot 2l} \\ &= F_0^{(1)}(k \bmod \frac{n}{2}) + \alpha^k\cdot F_1^{(1)}(k \bmod \frac{n}{2}), \end{aligned}$$

where $F_0^{(1)}$ and $F_1^{(1)}$ are the NTT's of with halved parameter n.

We can compute $F(k)$ recursively —

$$\begin{aligned} F^{(0)}(k) &= \sum_{j=0}^{n-1} f(j) \cdot \alpha^{kj} \\ &= \sum_{l=0}^{\frac{n}{2}-1} f(2l) \cdot \alpha^{k \cdot 2l} + \alpha^k \cdot \sum_{l=0}^{\frac{n}{2}-1} f(2l+1) \cdot \alpha^{k \cdot 2l} \\ &= F_0^{(1)}(k \bmod \frac{n}{2}) + \alpha^k \cdot F_1^{(1)}(k \bmod \frac{n}{2}). \end{aligned}$$

Notice the computation of $F_0^{(1)}$ uses $f(j)$'s for **even** j's, and that of $F_1^{(1)}$ uses $f(j)$'s for **odd** j's.

We can compute $F(k)$ recursively —

$$\begin{aligned} F^{(0)}(k) &= F_0^{(1)}(k') + \alpha^k \cdot F_1^{(1)}(k') \\ &= F_0^{(2)}(k' \bmod \frac{n}{4}) + \alpha^{k'} \cdot F_1^{(2)}(k' \bmod \frac{n}{4}) \\ &\quad + \alpha^k \cdot F_2^{(2)}(k' \bmod \frac{n}{4}) + \alpha^{k+k'} \cdot F_3^{(2)}(k' \bmod \frac{n}{4}) \\ &= \cdots, \end{aligned}$$

where $k' = k \bmod \frac{n}{2}$.

We can compute $F(k)$ recursively —

$$\begin{aligned}
F^{(0)}(k) &= F_0^{(1)}(k') + \alpha^k \cdot F_1^{(1)}(k') \\
&= F_0^{(2)}(k' \bmod \frac{n}{4}) + \alpha^{k'} \cdot F_1^{(2)}(k' \bmod \frac{n}{4}) \\
&\quad + \alpha^k \cdot F_2^{(2)}(k' \bmod \frac{n}{4}) + \alpha^{k+k'} \cdot F_3^{(2)}(k' \bmod \frac{n}{4}) \\
&= \cdots .
\end{aligned}$$

The computation of $F_0^{(2)}, F_1^{(2)}$ uses $f(j)$'s for even j's, and that of $F_2^{(2)}, F_3^{(2)}$ uses $f(j)$'s for odd j's.

We can compute $F(k)$ recursively —

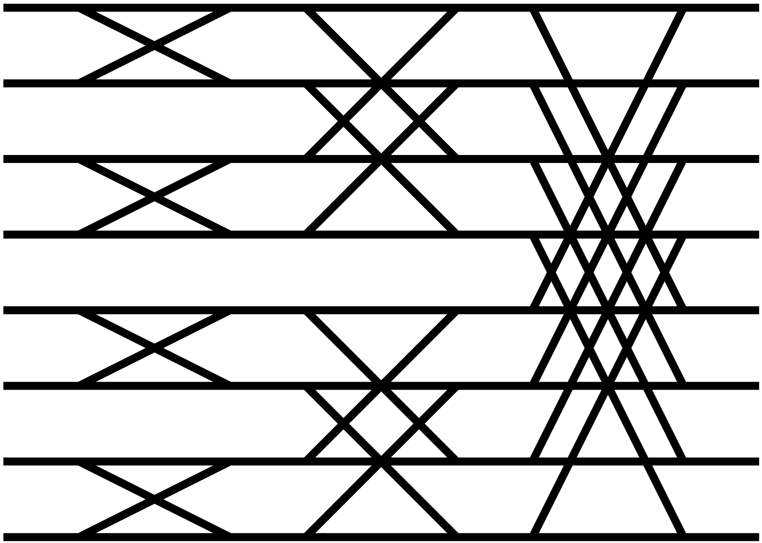
$$\begin{aligned}
F^{(0)}(k) &= F_0^{(1)}(k') + \alpha^k \cdot F_1^{(1)}(k') \\
&= F_0^{(2)}(k' \bmod \frac{n}{4}) + \alpha^{k'} \cdot F_1^{(2)}(k' \bmod \frac{n}{4}) \\
&\quad + \alpha^k \cdot F_2^{(2)}(k' \bmod \frac{n}{4}) + \alpha^{k+k'} \cdot F_3^{(2)}(k' \bmod \frac{n}{4}) \\
&= \cdots .
\end{aligned}$$

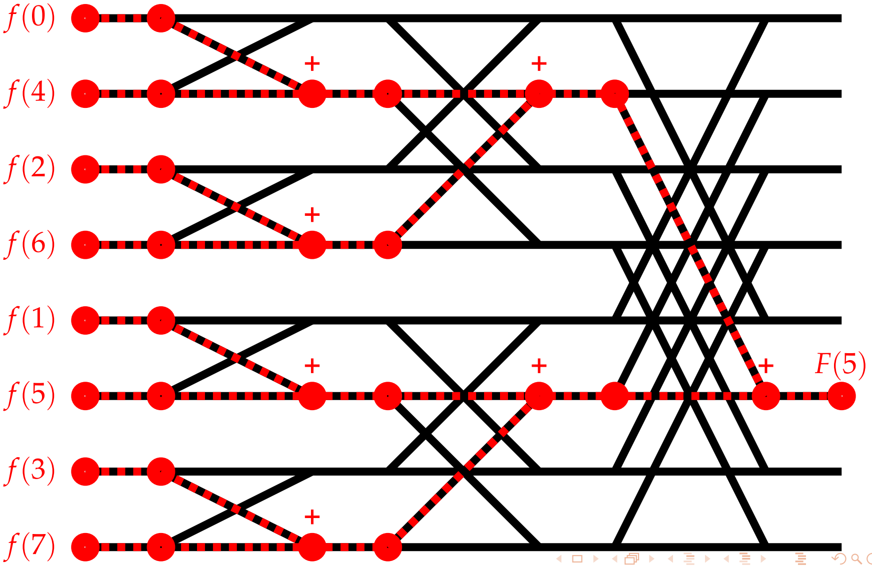
The computation of $F_0^{(2)}, F_1^{(2)}$ uses $f(j)$'s for even j's, and that of $F_2^{(2)}, F_3^{(2)}$ uses $f(j)$'s for odd j's. The computation of $F_0^{(2)}, F_2^{(2)}$ uses $f(j)$'s for the j's s.t. $(j \bmod 2) \bmod 4 = 0$, and that of $F_1^{(2)}, F_3^{(2)}$ uses $f(j)$'s for the j's s.t. $(j \bmod 2) \bmod 4 \neq 0$.

We say an index is odder than another if it has 1 in a lower bits. We place the j's following the setting (which is the origin of bits-flipping):

The more even — the upper;
The odder — the lower.

Hence the butterfly structure —




$f(0)$
$f(4)$
$f(2)$
$f(6)$
$f(1)$
$f(5)$
$f(3)$
$f(7)$
+
+
+
+
+
+
+
$F(5)$

Terms from odd branches will be multiplied by α^k before addition.