For simplicity, let $n = q - 1$.
Observe that, when $k \neq 0$,

$$\sum_{i=0}^{n-1} \alpha^{ik} = 0, \forall \alpha \in \mathbb{Z}_q^*,$$

and when $k = 0$, obviously

$$\sum_{i=0}^{n-1} \alpha^{ik} = \sum_{i=0}^{n-1} \alpha^0 = n, \forall \alpha \in \mathbb{Z}_q^*.$$

Therefore we have

$$\begin{aligned} f(i) &= \frac{1}{n} \sum_{k,j} f(j) \alpha^{k(j-i)} \\ &= \frac{1}{n} \sum_{k,j} f(j) \alpha^{kj-ki} \\ &= \frac{1}{n} \sum_{k,j} f(j) \alpha^{kj} \alpha^{-ik}. \end{aligned}$$

Let $F(k) \stackrel{def}{=} \sum_j f(j) \alpha^{kj}$, then

$$f(i) = \frac{1}{n} \sum_{k} F(k) \alpha^{-ik}.$$