Report of Assignment 3

Zhihan Li

1600010653

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Question 1

Answer Consider the shear matrix

$$\widetilde{\mathbf{S}} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \tag{1}$$

first, where $u = \cot\theta$. Using singular eigenvalue decomposition, we may obtain

$$\widetilde{\mathbf{S}} = \widetilde{\mathbf{P}}\widetilde{\mathbf{D}}\widetilde{\mathbf{Q}}^{\mathrm{T}}, \tag{2}$$

where $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}$ are orthogonal and $\widetilde{\mathbf{D}}$ is non-negative diagonal. Explicitly, a choice for the matrices is

$$\widetilde{\mathbf{P}} = \begin{bmatrix} -\frac{2}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} & -\frac{u+\sqrt{u^2+4}}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} \\ \frac{u+\sqrt{u^2+4}}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} & -\frac{2}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} \end{bmatrix}, \tag{3}$$

$$\widetilde{\mathbf{Q}} = \begin{bmatrix} -\frac{u+\sqrt{u^2+4}}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} & -\frac{2}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} \\ \frac{2}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} & -\frac{u+\sqrt{u^2+4}}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} \end{bmatrix}, \tag{4}$$

$$\widetilde{\mathbf{D}} = \begin{bmatrix} \frac{4\sqrt{u^2+4}}{\left(u+\sqrt{u^2+4}\right)^2+4} & 0 \\ 0 & \frac{u^3+u^2\sqrt{u^2+4}+4u+2\sqrt{u^2+4}}{u^2+u\sqrt{u^2+4}+4} \end{bmatrix}, \tag{5}$$

where $\det \widetilde{\mathbf{P}} = \det \widetilde{\mathbf{Q}} = 1$. Because $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}^{\mathrm{T}}$ are orthogonal with determinant 1, therefore they represent rotations. Because $\widetilde{\mathbf{D}}$ is positive diagonal, therefore it stands for scaling transform. Consequently, the original scaling matrix can be decomposed into

$$\mathbf{S} = \begin{bmatrix} \widetilde{\mathbf{S}} & 0 & 0 \\ 0 & 1 & 0 \\ & & 1 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{P}} & 0 & 0 \\ 0 & 1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{D}} & 0 & 0 \\ 0 & 1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{Q}}^{\mathrm{T}} & 0 & 0 \\ 0 & 1 & 0 \\ & & 1 \end{bmatrix}, \tag{6}$$

where the last three matrices represent rotation, translation, rotation respectively. Symbolic verification can be seen in `Problem1.ipynb`.

Question 2

Answer The composed matrix is

$$\begin{aligned}
\mathbf{R} &= \mathbf{R}_x(\theta_x)\,\mathbf{R}_y\left(\theta_y\right)\,\mathbf{R}_z(\theta_z) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ \sin\theta_x\sin\theta_y & \cos\theta_x & -\sin\theta_x\cos\theta_y & 0 \\ -\cos\theta_x\sin\theta_y & \sin\theta_x & \cos\theta_x\cos\theta_y & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_y\cos\theta_z & -\cos\theta_y\sin\theta_z & \sin\theta_y & 0 \\ \sin\theta_x\sin\theta_y\cos\theta_z + \cos\theta_x\sin\theta_z & -\sin\theta_x\sin\theta_y\sin\theta_z + \cos\theta_x\cos\theta_z & -\sin\theta_x\cos\theta_y & 0 \\ -\cos\theta_x\sin\theta_y\cos\theta_z + \sin\theta_x\sin\theta_z & \cos\theta_x\sin\theta_y\sin\theta_z + \sin\theta_x\cos\theta_z & \cos\theta_x\cos\theta_y & 0 \\ & & & 1 \end{bmatrix}.
\end{aligned} \tag{7}$$

Question 3

Answer This is possible. For example, we try to devise translation matrix $\mathbf{T}'$, such that

$$\mathbf{M} = \mathbf{TRS} = \mathbf{RST}'. \tag{8}$$

Assume

$$\mathbf{R} = \begin{bmatrix} \widetilde{\mathbf{R}} & 0 \\ & 1 \end{bmatrix}, \mathbf{S} = \begin{bmatrix} \widetilde{\mathbf{S}} & 0 \\ & 1 \end{bmatrix}, \tag{9}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{x} \\ & 1 \end{bmatrix}, \mathbf{T}' = \begin{bmatrix} \mathbf{I} & \mathbf{x}' \\ & 1 \end{bmatrix}. \tag{10}$$

From (8), we deduce that

$$\begin{bmatrix} \widetilde{\mathbf{R}}\widetilde{\mathbf{S}} & \mathbf{x} \\ & 1 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{R}}\widetilde{\mathbf{S}} & \widetilde{\mathbf{R}}\widetilde{\mathbf{S}}\mathbf{x}' \\ & 1 \end{bmatrix}. \tag{11}$$

Since $\mathbf{R}$ and $\mathbf{S}$ are invertible and so are $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{S}}$, therefore if we let

$$\mathbf{x}' = \widetilde{\mathbf{S}}^{-1}\widetilde{\mathbf{R}}^{-1}\mathbf{x}, \tag{12}$$

then $\mathbf{T}'$ is a appropriate translation matrix and (8) is satisfied.

In conclusion, different order of transformations (with some transformations modified) may yield the same result.

Question 4

Answer Quaternions of the two rotations are

$$q_x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}, \tag{13}$$

$$q_y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{j} \tag{14}$$

respectively. Therefore,

$$\begin{aligned} q_x q_y &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}\right)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{j}\right) \\ &= \frac{1}{2} + \frac{1}{2}\mathrm{i} + \frac{1}{2}\mathrm{j} + \frac{1}{2}\mathrm{k}. \end{aligned} \tag{15}$$

Therefore, composition of two rotations is a rotation about $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ by $\frac{2}{3}\pi$.

Question 5

Answer By compositing transformations, the model view matrix should be

$$\mathbf{T}\,(\mathbf{disp})\,\mathbf{R}_y\,(yaw)\,\mathbf{R}_x\,(pitch)\,\mathbf{R}_z\,(roll)\,, \tag{16}$$

where **disp** is the displacement vector from the view point to the object.