Report of Assignment 3

Zhihan Li

1600010653

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Question 1

Answer Consider the shear matrix

$$\widetilde{\mathbf{S}} = \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \tag{1}$$

first, where $u = \cot\theta$. Using singular eigenvalue decomposition, we may obtain

$$\widetilde{\mathbf{S}} = \widetilde{\mathbf{P}}\widetilde{\mathbf{D}}\widetilde{\mathbf{Q}}^{\mathrm{T}}, \tag{2}$$

where $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}$ are orthogonal and $\widetilde{\mathbf{D}}$ is non-negative diagonal. Explicitly, a choice for the matrices is

$$\widetilde{\mathbf{P}} = \begin{bmatrix} -\frac{2}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} & -\frac{u+\sqrt{u^2+4}}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} \\ \frac{u+\sqrt{u^2+4}}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} & -\frac{2}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} \end{bmatrix}, \tag{3}$$

$$\widetilde{\mathbf{Q}} = \begin{bmatrix} -\frac{u+\sqrt{u^2+4}}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} & -\frac{2}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} \\ \frac{2}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} & -\frac{u+\sqrt{u^2+4}}{\sqrt{\left(u+\sqrt{u^2+4}\right)^2+4}} \end{bmatrix}, \tag{4}$$

$$\widetilde{\mathbf{D}} = \begin{bmatrix} \frac{4\sqrt{u^2+4}}{\left(u+\sqrt{u^2+4}\right)^2+4} & 0 \\ 0 & \frac{u^3+u^2\sqrt{u^2+4}+4u+2\sqrt{u^2+4}}{u^2+u\sqrt{u^2+4}+4} \end{bmatrix}, \tag{5}$$

where $\det \widetilde{\mathbf{P}} = \det \widetilde{\mathbf{Q}} = 1$. Because $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{Q}}^{\mathrm{T}}$ are orthogonal with determinant 1, therefore they represent rotations. Because $\widetilde{\mathbf{D}}$ is positive diagonal, therefore it stands for scaling transform. Consequently, the original scaling matrix can be decomposed into

$$\mathbf{S} = \begin{bmatrix} \widetilde{\mathbf{S}} & 0 & 0 \\ 0 & 1 & 0 \\ & & 1 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{P}} & 0 & 0 \\ 0 & 1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{D}} & 0 & 0 \\ 0 & 1 & 0 \\ & & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{Q}}^{\mathrm{T}} & 0 & 0 \\ 0 & 1 & 0 \\ & & 1 \end{bmatrix}, \tag{6}$$

where the last three matrices represent rotation, translation, rotation respectively. Symbolic verification can be seen in `Problem1.ipynb`.

Question 2

Answer The composed matrix is

$$\begin{aligned}
\mathbf{R} &= \mathbf{R}_x(\theta_x)\, \mathbf{R}_y\left(\theta_y\right) \mathbf{R}_z(\theta_z) \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ \sin\theta_x \sin\theta_y & \cos\theta_x & -\sin\theta_x\cos\theta_y & 0 \\ -\cos\theta_x\sin\theta_y & \sin\theta_x & \cos\theta_x\cos\theta_y & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_y\cos\theta_z & -\cos\theta_y\sin\theta_z & \sin\theta_y & 0 \\ \sin\theta_x\sin\theta_y\cos\theta_z + \cos\theta_x\sin\theta_z & -\sin\theta_x\sin\theta_y\sin\theta_z + \cos\theta_x\cos\theta_z & -\sin\theta_x\cos\theta_y & 0 \\ -\cos\theta_x\sin\theta_y\cos\theta_z + \sin\theta_x\sin\theta_z & \cos\theta_x\sin\theta_y\sin\theta_z + \sin\theta_x\cos\theta_z & \cos\theta_x\cos\theta_y & 0 \\ & & & 1 \end{bmatrix}.
\end{aligned} \tag{7}$$

Question 3

Answer This is possible. For example, we try to devise translation matrix $\mathbf{T}'$, such that

$$\mathbf{M} = \mathbf{TRS} = \mathbf{RST}'. \tag{8}$$

Assume

$$\mathbf{R} = \begin{bmatrix} \widetilde{\mathbf{R}} & 0 \\ & 1 \end{bmatrix}, \mathbf{S} = \begin{bmatrix} \widetilde{\mathbf{S}} & 0 \\ & 1 \end{bmatrix}, \tag{9}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{x} \\ & 1 \end{bmatrix}, \mathbf{T}' = \begin{bmatrix} \mathbf{I} & \mathbf{x}' \\ & 1 \end{bmatrix}. \tag{10}$$

From (8), we deduce that

$$\begin{bmatrix} \widetilde{\mathbf{R}}\widetilde{\mathbf{S}} & \mathbf{x} \\ & 1 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{R}}\widetilde{\mathbf{S}} & \widetilde{\mathbf{R}}\widetilde{\mathbf{S}}\mathbf{x}' \\ & 1 \end{bmatrix}. \tag{11}$$

Since $\mathbf{R}$ and $\mathbf{S}$ are invertible and so are $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{S}}$, therefore if we let

$$\mathbf{x}' = \widetilde{\mathbf{S}}^{-1}\widetilde{\mathbf{R}}^{-1}\mathbf{x}, \tag{12}$$

then $\mathbf{T}'$ is a appropriate translation matrix and (8) is satisfied.

In conclusion, different order of transformations (with some transformations modified) may yield the same result.

(More precisely, because

$$\begin{bmatrix} \mathbf{I} & \mathbf{x} \\ & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{R}} & 0 \\ & 1 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{R}} & 0 \\ & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \widetilde{\mathbf{R}}^{-1}\mathbf{x} \\ & 1 \end{bmatrix}, \tag{13}$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{x} \\ & 1 \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{S}} & 0 \\ & 1 \end{bmatrix} = \begin{bmatrix} \widetilde{\mathbf{S}} & 0 \\ & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & \widetilde{\mathbf{S}}^{-1}\mathbf{x} \\ & 1 \end{bmatrix}, \tag{14}$$

$$\tag{15}$$

therefore rotation, and scaling as well, may be exchangable with a translation (with some modification to the trnaslation). Therefore, compositions **TRS**, **RTS** and **RST** have identical ranges. Moreover, **TSR**, **STR** and **SRT** also have identical ranges. Note that the 3×3 principle sub-matrix of these two classes are $\widetilde{\mathbf{R}}\widetilde{\mathbf{S}}$ and $\widetilde{\mathbf{S}}\widetilde{\mathbf{R}}$ respectively. If we have

$$\mathbf{PD} = \mathbf{EQ}, \mathbf{PDQ}^{\mathrm{T}} = \mathbf{E}, \tag{16}$$

where $\mathbf{P}$, $\mathbf{Q}$ are orthogonal and $\mathbf{D}$, $\mathbf{E}$ are diagonal, then using the uniqueness of singular value decomposition yields that absolute values of diagonal entries of **D** coincides that of **E** and that **P**, **Q** $\neq$ **I** iff **D** have repeated entries regardless of signs. Therefore, for most cases, (16) cannot be established for most cases except the one mentioned above, which means ranges of the two class do not coincide. In conclusion, a instance transformation can be achieved by **TRS**, **RTS** and **RST** but not (always) by **TSR**, **STR** and **SRT**.)

Question 4

Answer Quaternions of the two rotations are

$$q_x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}, \tag{17}$$

$$q_y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{j} \tag{18}$$

respectively. Therefore,

$$\begin{aligned} q_x q_y &= \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}\right)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{j}\right) \\ &= \frac{1}{2} + \frac{1}{2}\mathrm{i} + \frac{1}{2}\mathrm{j} + \frac{1}{2}\mathrm{k}. \end{aligned} \tag{19}$$

Therefore, composition of two rotations is a rotation about $\left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ by $\frac{2}{3}\pi$. Moreover,

$$q_y q_x = \frac{1}{2} + \frac{1}{2}\mathrm{i} + \frac{1}{2}\mathrm{j} - \frac{1}{2}\mathrm{k}. \tag{20}$$

Question 5

Answer According to Section 4.3.4 and Figure 4.19 of (6th edition, or Section 5.3.4 and Figure 5.19 in the 7th edition), by compositing transformations, the model view matrix should be

$$\mathbf{T}(\mathbf{disp})\,\mathbf{R}_y(yaw)\,\mathbf{R}_x(pitch)\,\mathbf{R}_z(roll), \tag{21}$$

where **disp** is the displacement vector from the view point to the object.