An Efficient Incremental Algorithm for Generating All Maximal Independent Sets in Hypergraphs of Bounded Dimension*

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Abstract

We show that for hypergraphs of bounded edge size, the problem of extending a given list of maximal independent sets is NC-reducible to the computation of an arbitrary maximal independent set for an induced sub-hypergraph. The latter problem is known to be in RNC. In particular, our reduction yields an incremental RNC dualization algorithm for hypergraphs of bounded edge size, a problem previously known to be solvable in polynomial incremental time. We also give a similar parallel algorithm for the dualization problem on the product of arbitrary lattices which have a bounded number of immediate predecessors for each element.

1 Introduction

Let $\mathcal{A} \subseteq 2^V$ be a hypergraph (set family) on a finite vertex set V. A vertex set $I \subseteq V$ is called *independent* if I contains no hyperedge of $\mathcal{A}$. Let $\mathcal{I}(\mathcal{A}) \subseteq 2^V$ denote the family of all maximal independent sets of $\mathcal{A}$. We assume that $\mathcal{A}$ is given by a list of its hyperedges and consider the problem of incrementally generating $\mathcal{I}(\mathcal{A})$:

$MIS(\mathcal{A},\mathcal{I})$: *Given a hypergraph $\mathcal{A}$ and a collection $\mathcal{I} \subseteq \mathcal{I}(\mathcal{A})$ of maximal independent sets for $\mathcal{A}$, either find a new maximal independent set $I \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{I}$, or prove that the given collection is complete: $\mathcal{I} = \mathcal{I}(\mathcal{A})$.*

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Our objective in this note is to show that for hypergraphs of bounded dimension,

$$\dim(\mathcal{A}) \stackrel{\text{def}}{=} \max\{|A| \ : \ A \in \mathcal{A}\} \leq const,$$

problem $MIS(\mathcal{A}, \mathcal{I})$ can be efficiently solved in parallel:

Theorem 1 *$MIS(\mathcal{A}, \mathcal{I}) \in NC$ for $dim(\mathcal{A}) \leq 3$, and $MIS(\mathcal{A}, \mathcal{I}) \in RNC$ for $dim(\mathcal{A}) = 4, 5, \ldots$*

The statements of Theorem 1 were previously known [4, 23] only for $\mathcal{I} = \emptyset$, when $MIS(\mathcal{A}, \mathcal{I})$ turns into the classical problem of computing a single maximal independent set for $\mathcal{A}$ (see [1, 12, 15, 16, 20, 21, 22, 25]). We show that conversely, $MIS(\mathcal{A}, \mathcal{I})$ can be reduced to the above special case.

Theorem 2 *If $dim(\mathcal{A}) \leq const$, then problem $MIS(\mathcal{A}, \mathcal{I})$ is NC-reducible[1] to problem $MIS(\mathcal{A}', \emptyset)$, where $\mathcal{A}'$ is some induced partial hypergraph of $\mathcal{A}$.*

(Given a hypergraph $\mathcal{A} \subseteq 2^V$, a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ is called a *partial hypergraph* of $\mathcal{A}$, while $\{A \cap U | A \in \mathcal{A}'\}$ for some $U \subseteq V$ is called an *induced partial hypergraph* of $\mathcal{A}$.)

Note that if $I \in \mathcal{I}(\mathcal{A})$ is an independent set, the complement $B = V \setminus I$ is a *transversal* to $\mathcal{A}$, i.e. $B \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, and vice versa. Hence $\{B \mid B = V \setminus I, I \in \mathcal{I}(\mathcal{A})\} = \mathcal{A}^d$, where $\mathcal{A}^d \stackrel{\text{def}}{=} \{B \mid B \text{ minimal transversal to } \mathcal{A}\}$ is the *transversal* or *dual* hypergraph of $\mathcal{A}$. For this reason, $MIS(\mathcal{A}, \mathcal{I})$ can be equivalently stated as the *hypergraph dualization problem*:

$DUAL(\mathcal{A}, \mathcal{B})$: *Given a hypergraph $\mathcal{A}$ and a collection $\mathcal{B} \subseteq \mathcal{A}^d$ of minimal transversals to $\mathcal{A}$, either find a new minimal transversal $B \in \mathcal{A}^d \setminus \mathcal{B}$ or show that $\mathcal{B} = \mathcal{A}^d$.*

The hypergraph dualization problem has applications in combinatorics [29], graph theory [19, 24, 30, 31], artificial intelligence [13], game theory [17, 18, 28], reliability theory [10, 28], database theory [2, 6, 7, 27, 32], integer programming [6, 7], and learning theory [3]. It is an open question whether problem $DUAL(\mathcal{A}, \mathcal{B})$, or equivalently $MIS(\mathcal{A}, \mathcal{I})$, can be solved in polynomial time for arbitrary hypergraphs. The fastest currently known algorithm [14] for $DUAL(\mathcal{A}, \mathcal{B})$ is quasi-polynomial and runs in time $O(nm) + m^{o(\log m)}$, where $n = |V|$ and $m = |\mathcal{A}| + |\mathcal{B}|$. However, as shown in [13, 5], for hypergraphs of bounded dimension problem $DUAL(\mathcal{A}, \mathcal{B})$ can be solved in polynomial time. Theorem 1 strengthens this result by implying that $DUAL(\mathcal{A}, \mathcal{B}) \in NC$ for $\dim(\mathcal{A}) \leq 3$ and $DUAL(\mathcal{A}, \mathcal{B}) \in RNC$ for $\dim(\mathcal{A}) = 4, 5, \ldots$. As mentioned above, Theorem 1 is a corollary of Theorem 2 and the results of [4, 23].

A vertex set S is called a *sub-transversal* of $\mathcal{A}$ if $S \subseteq B$ for some minimal transversal $B \in \mathcal{A}^d$. Our proof of Theorem 2 makes use of a characterization of sub-transversals suggested in [5]. Even though it is NP-hard in general to test whether a given set $S \subseteq V$ is a sub-transversal of $\mathcal{A}$, for $|S| \leq const$ the sub-transversal criterion of [5] is in NC. This turns out to be sufficient for the proof of Theorem 2.

[1] In fact, our reduction is in AC_0

The remainder of the paper is organized as follows. In Sections 2 and 3 we recall the sub-transversal criterion of [5] and prove Theorem 2. Then in Section 4 we discuss a generalization of the sub-transversal criterion and Theorem 2 for the dualization problem on the Cartesian products of n lattices. More precisely, given n lattices $\mathcal{P}_1, \ldots, \mathcal{P}_n$ and a set $\mathcal{A} \subseteq \mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$, we consider the problem of generating all maximal elements in $\mathcal{P} \setminus \mathcal{A}^+$, where $\mathcal{A}^+$ is the (upper) ideal generated by $\mathcal{A}$. If $\mathcal{P} = \{0,1\}^n$ is the product of n chains $\{0,1\}$, then this problem is equivalent to the generation of the transversal hypergraph for $\mathcal{A}$. In general, when $\mathcal{A}$ is a set in $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$, we define $\dim(\mathcal{A}) = \max\{|\mathrm{Supp}(a)| : a \in \mathcal{A}\}$, where $\mathrm{Supp}(a)$ is the *support* of $a \in \mathcal{P}$, i.e., the set of all non-minimal components of a. Then we show that for $\dim(\mathcal{A}) \leq const$, the dualization problem on $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ is NC-reducible to the maximal independent set problem for some hypergraphs of dimension at most $\dim(\mathcal{A})$, provided that the number of immediate predecessors of any element in each factor-lattice $\mathcal{P}_i$ is also bounded by a constant.

2 Characterization of Sub-transversals to a Hypergraph

Given a hypergraph $\mathcal{A} \subseteq 2^V$, a subset $S \subseteq V$, and a vertex $v \in S$, let $\mathcal{A}_v(S) = \{A \in \mathcal{A} \mid A \cap S = \{v\}\}$ denote the family of all hyperedges of $\mathcal{A}$ whose intersection with S is exactly v. Let further $\mathcal{A}_0(S) = \{A \in \mathcal{A} \mid A \cap S = \emptyset\}$ denote the partial hypergraph consisting of the hyperedges of $\mathcal{A}$ disjoint from S. A selection of $|S|$ hyperedges $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ is called *covering* if there exists a hyperedge $A \in \mathcal{A}_0(S)$, such that $A \subseteq \bigcup_{v \in S} A_v$. Proposition 1 below states that a non-empty set S is a sub-transversal of $\mathcal{A}$ if and only if there is a non-covering selection for S.

Proposition 1 (cf. [5]) *Let $S \subseteq V$ be a non-empty vertex set in a hypergraph $\mathcal{A} \subseteq 2^V$.*

(i) If S is a sub-transversal for $\mathcal{A}$, then there exists a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S.
(ii) Given a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S, we can extend S to a minimal transversal of $\mathcal{A}$ by solving problem $MIS(\mathcal{A}', \emptyset)$ for the induced partial hypergraph

$$\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}_0(S)\} \subseteq 2^U, \tag{1}$$

where $U = V \setminus \bigcup_{v \in S} A_v$.

Proof. Let us start with the following observations:

(a) If $S \subseteq B \subseteq V$, then $\mathcal{A}_v(B) \subseteq \mathcal{A}_v(S)$ holds for all $v \in S$.

(b) If B is a transversal to $\mathcal{A}$, then B is minimal if and only if $\mathcal{A}_v(B) \neq \emptyset$ for all $v \in B$.

Observation (a) follows directly from the definitions of $\mathcal{A}_v(S)$ and $\mathcal{A}_v(B)$. To see (b), note that if $\mathcal{A}_v(B) = \emptyset$ for some $v \in B$, then $B \setminus \{v\}$ is still a transversal to $\mathcal{A}$.

Proof of (i) Suppose that $\emptyset \neq S \subseteq B$, where $B \in \mathcal{A}^d$ is a minimal transversal. By observations (a) and (b), we have $\emptyset \neq \mathcal{A}_v(B) \subseteq \mathcal{A}_v(S)$ for each $v \in S$. Consider then a selection of the form $\{A_v \in \mathcal{A}_v(B) \mid v \in S\}$. If it covers a hyperedge $A \in \mathcal{A}_0(S)$, then A would be disjoint from B, contradicting the fact that $B \in \mathcal{A}^d$.

Proof of (ii) Suppose we are given a non-covering selection $\{A_v \in \mathcal{A}_v(S) | \; v \in S\}$. If $\mathcal{A}_0(S) = \emptyset$, then S is obviously a transversal to $\mathcal{A}$. Hence by (b), S itself is a minimal transversal to $\mathcal{A}$. Let us assume now that $\mathcal{A}_0(S) \neq \emptyset$ and consider the hypergraph $\mathcal{A}'$ as defined in (1). Since the given selection is non-covering and $\mathcal{A}_0(S) \neq \emptyset$, we conclude that the vertex and edge sets of $\mathcal{A}'$ are not empty, and $\mathcal{A}'$ contains no empty edges. Let T be a minimal transversal to $\mathcal{A}'$. (Such a transversal can be computed by letting $T = U \setminus I$, where $I = output(MIS(\mathcal{A}', \emptyset))$.) It is easy to see that $S \cup T$ is a transversal to $\mathcal{A}$. Moreover, $S \cup T$ is minimal, since if we delete a vertex $v \in S$, then $A_v \cap [(S \setminus \{v\}) \cup T] = \emptyset$, while deleting a vertex $v \in T$ results in an empty intersection with some $A \in \mathcal{A}_0(S)$. □

Unfortunately, if the cardinality of S is not bounded, finding a non-covering selection for S (equivalently, testing if S is a sub-transversal) is NP-hard. In fact, this is so even for $\dim(\mathcal{A}) = 2$ (i.e., for graphs and transversals $\equiv$ vertex covers).

Proposition 2 *Given an undirected graph $G = (V, E)$ and a vertex set $S \subseteq V$, it is NP-complete to determine whether S can be extended to a minimal vertex cover.*

Proof. We use a polynomial transformation from the satisfiability problem. Let $C = C_1 \wedge \ldots \wedge C_m$ be a conjunctive normal form, and let us consider the graph $G = (V, E)$, where V is the set of all clauses and literals of C, and where E consists of the pairs $(x, \overline{x})$ of mutually negating literals, and the pairs (C_i, u) formed by a clause and one of its literals. Then the set $S = \{C_1, \ldots, C_m\}$ can be extended to a minimal vertex cover of G if and only if C is satisfiable. □

We close this section with the observation that if the size of S is bounded by a constant, then there are only polynomially many selections $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S. All of these selections, including the non-covering ones, can be easily enumerated in parallel.

Corollary 1 *For any fixed c, there is an NC algorithm which, given a hypergraph $\mathcal{A} \subseteq 2^V$ and a set S of at most c vertices, determines whether S is a sub-transversal to $\mathcal{A}$ and if so, finds a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$.*

Note that Corollary 1 holds for hypergraphs $\mathcal{A}$ of arbitrary dimension.

3 Proof of Theorem 2

We prove the theorem for the equivalent problem $DUAL(\mathcal{A}, \mathcal{B})$, i.e. show that for $\dim(\mathcal{A}) \leq const$, problem $DUAL(\mathcal{A}, \mathcal{B})$ is NC-reducible to $MIS(\mathcal{A}', \emptyset)$, for some induced partial hypergraph $\mathcal{A}'$ of $\mathcal{A}$. Our reduction consists of several steps.

Step 1. Delete all hyperedges of $\mathcal{A}$ that contain other hyperedges of $\mathcal{A}$. Clearly, this does not change the minimal transversals to $\mathcal{A}$. We assume in the sequel that no hyperedge of $\mathcal{A}$ contains another hyperedge of $\mathcal{A}$, i.e., that

$$\mathcal{A} \text{ is Sperner.} \tag{2}$$

Note that the dual hypergraph $\mathcal{A}^d$ is Sperner by definition, and hence $\mathcal{B} \subseteq \mathcal{A}^d$ is Sperner as well.

Step 2 (optional). Delete all vertices in V that are not covered by some $A \in \mathcal{A}$ so that we have $V = \bigcup_{A \in \mathcal{A}} A$. If $\bigcup_{B \in \mathcal{B}} B$ is a proper subset of V, a new minimal transversal in $\mathcal{A}^d \setminus \mathcal{B}$ can be found as follows:

- Pick a vertex $u \in V \setminus \bigcup_{B \in \mathcal{B}} B$.
- The set $S = \{u\}$ is a sub-transversal to A. In view of (2), any hyperedge $A_u \in \mathcal{A}$ such that $u \in A_u$ is a non-covering selection for S.
- Let $u \in T \in \mathcal{A}^d$, then $T \notin \mathcal{B}$, because none of the transversals in $\mathcal{B}$ contains u. By Proposition 1, the problem of extending $S = \{u\}$ to a minimal transversal T is equivalent to that of computing a maximal independent set for hypergraph (1) with $U = V \setminus A_u$.

We can thus assume without loss of generality that $\bigcup_{A \in \mathcal{A}} A = \bigcup_{B \in \mathcal{B}} B = V$.

Step 3. By definition, each set $B \in \mathcal{B}$ is a minimal transversal to $\mathcal{A}$. This implies that each set $A \in \mathcal{A}$ is transversal to $\mathcal{B}$. Check whether each $A \in \mathcal{A}$ is a *minimal* transversal to $\mathcal{B}$. Suppose that some $A^o \in \mathcal{A}$ is not minimal, i.e. there is a vertex $u \in A^o$ such that $A^* = A^o \setminus \{u\}$ is still transversal to $\mathcal{B}$. Then we can proceed as follows.

- Let $\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}\}$, where $U = V \setminus A^*$.
- By (2), we have $A \cap U \neq \emptyset$ for each hyperedge $A \in \mathcal{A}$. Hence any minimal transversal T to $\mathcal{A}'$ is also a minimal transversal for $\mathcal{A}$.
- It easy to see that $T \notin \mathcal{B}$. This is because any set $B \in \mathcal{B}$ intersects A^* whereas T is disjoint from A^*. This reduces the computation of a new element in $\mathcal{A}^d \setminus \mathcal{B}$ to problem $MIS(\mathcal{A}', \emptyset)$.

In the sequel we assume in addition to (2) that each set in $\mathcal{A}$ is a minimal transversal to $\mathcal{B}$:

$$\mathcal{A} \subseteq \mathcal{B}^d. \tag{3}$$

Before proceeding to the next step of the reduction, we pause to make some observations. Clearly, $(\mathcal{A}^d)^d = \mathcal{A}$ for any Sperner hypergraph $\mathcal{A}$. Therefore, if $B \neq \mathcal{A}^d$ then $\mathcal{A} \neq \mathcal{B}^d$. By (3), we then have $\mathcal{B}^d \setminus \mathcal{A} \neq \emptyset$. Hence we arrive at the following duality criterion: $\mathcal{A}^d \setminus \mathcal{B} \neq \emptyset$ if and only if there is a sub-transversal S to $\mathcal{B}$ such that

$$|S| \leq \dim(\mathcal{A}), \quad \text{and} \tag{4}$$

$$S \not\subseteq A \quad \text{for all} \quad A \in \mathcal{A}. \tag{5}$$

The "if" part is obvious and holds even without assumption (3). To show the "only if" part, consider an arbitrary minimal transversal $T \in \mathcal{B}^d \setminus \mathcal{A}$. Clearly, T satisfies (5). Let S be a minimal subset of T that still satisfies (5) and let v be an arbitrary vertex in S. Since $S \setminus \{v\}$ does not satisfy (5) we have $S \setminus \{v\} \subseteq A$ for some $A \in \mathcal{A}$.

Assuming $|S| > \dim(\mathcal{A})$, we obtain $A = S \setminus \{v\}$ by (5). Hence $A \subset S \subseteq T$. However, both A and T are minimal transversals to $\mathcal{B}$. This contradiction shows (4).

So far, we have not relied on the assumption that $\dim(\mathcal{A})$ is bounded. We need this assumption to guarantee that the next step of our reduction is in NC.

Step 4 (Duality test.) For each set S satisfying (4), (5) and the condition that

$$A \not\subseteq S \text{ for all } A \in \mathcal{A}, \tag{6}$$

check whether or not

$$S \text{ is a sub-transversal to } \mathcal{B}. \tag{7}$$

Recall that by Proposition 1, S satisfies (7) if and only if there is a selection

$$\{B_v \in \mathcal{B}_v(S) \mid v \in S\} \tag{8}$$

which covers no set $B \in \mathcal{B}_0(S)$. Here as before, $\mathcal{B}_0(S) = \{B \in \mathcal{B} \mid B \cap S = \emptyset\}$ and $\mathcal{B}_v(S) = \{B \in \mathcal{B} \mid B \cap S = \{v\}\}$ for $v \in S$.

If conditions (4), (5),(6) and (7) cannot be met, we conclude that $\mathcal{B} = \mathcal{A}^d$ and halt.

Step 5. Suppose we have found a non-covering selection (8) for some set S satisfying (4), (5), (6) (and hence (7)). We claim that the set

$$Z = S \bigcup \left[V \setminus \bigcup_{v \in S} B_v \right]$$

is independent in $\mathcal{A}$. Suppose to the contrary that $A \subseteq W$ for some $A \in \mathcal{A}$. By (5), there is a vertex $u \in S$ such that $u \notin A$. Then $A \cap B_u = \emptyset$, yielding a contradiction. Note also that Z is transversal to $\mathcal{B}$ because selection (8) is non-covering.

Let $\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}\}$, where $U = V \setminus Z$, and let T be a minimal transversal to $\mathcal{A}'$. (As before, we can let $T = U \setminus output(MIS(\mathcal{A}', \emptyset))$.) Since Z is an independent set of $\mathcal{A}$, we have $T \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, i.e., T is transversal to $\mathcal{A}$. Clearly, T is minimal, i.e. $T \in \mathcal{A}^d$. It remains to argue that T is a *new* minimal transversal to $\mathcal{A}$, i.e., $T \notin \mathcal{B}$. This follows from the fact that Z is transversal to $\mathcal{B}$ and disjoint from T. $\square$

Remark Theorems 1 and 2 can be extended to so-called *fairly independent sets*. Let $\mathcal{A} \subseteq 2^V$ be a hypergraph and let $t \in \{0, 1, \ldots, |\mathcal{A}| - 1\}$ be a given threshold. A vertex set $I \subseteq V$ is called fairly independent or t-independent if I contains at most t hyperedges of $\mathcal{A}$. For $t = 0$ each fairly independent set is thus an independent set of $\mathcal{A}$. Let us call a vertex set $U \subseteq V$ a *t-union* if U contains at least t hyperedges of $\mathcal{A}$, and let $\mathcal{A}_{u_t}$ denote the hypergraph of all minimal t-unions. It is not difficult to see that a vertex set $I \subseteq V$ is t-independent in $\mathcal{A}$ if and only if I is a standard independent set of $\mathcal{A}_{u_{t+1}}$. Furthermore, if t and $\dim(\mathcal{A})$ are both bounded, then $\mathcal{A}_{u_{t+1}}$ can be constructed in NC and the dimension of $\mathcal{A}_{u_{t+1}}$ is bounded as well. Hence for $t \leq const$, all maximal t-independent sets in a hypergraph of bounded dimension can be incrementally generated by an RNC algorithm.

The sub-transversal criterion of Proposition 1 is also extendable to t-independent sets. Call a vertex set $T \subseteq V$ a *t-transversal* to $\mathcal{A}$ if T is disjoint from at most t hyperedges of $\mathcal{A}$. Note that T is a t-transversal if and only if $I = V \setminus T$ is t-independent in $\mathcal{A}$. By definition, a vertex set S is a *t-sub-transversal* if S is a subset of some minimal t-transversal T. Let $\mathcal{B}$ be a selection of $|S|$ *subfamilies* of hyperedges $\{\mathcal{B}_v \subseteq \mathcal{A}_v(S) \mid v \in S\}$ and let $k_v = |\mathcal{B}_v|$. Denote by l the number of hyperedges in $\mathcal{A}_0$ covered by the union of all sets in $\mathcal{B}$. Call $\mathcal{B}$ a *t-selection* if $l \leq t < l + k_v$ for all $v \in S$. Proposition 1 can be generalized as follows: *A non-empty vertex set S is a t-sub-transversal of $\mathcal{A}$ if and only if there exists a t-selection for S.* Note that in this criterion, we can consider only those selections $\mathcal{B}$ for which $k_v \leq t+1$ for all $v \in S$. In particular, for $t = 0$ we obtain $l = 0$ and $k_v \equiv 1$, which is equivalent to the definition of non-covering selections introduced in Section 2.

4 Maximal Independent Sets in Products of Lattices

In this section, we discuss a generalization of problem $MIS(\mathcal{A},\mathcal{I})$ in which the input hypergraphs $\mathcal{A}$ and $\mathcal{I}$ are replaced by two subsets of a partially ordered set $\mathcal{P}$. Given a subset $\mathcal{A} \subseteq \mathcal{P}$, let $\mathcal{A}^+ = \{x \in \mathcal{P} \mid x \succcurlyeq a, \quad a \in \mathcal{A}\}$ and $\mathcal{A}^- = \{x \in \mathcal{P} \mid x \preccurlyeq a, \quad a \in \mathcal{A}\}$ denote the ideal and the filter generated by $\mathcal{A}$. Any element in $\mathcal{P} \setminus \mathcal{A}^+$ is called *independent of* $\mathcal{A}$. Let $\mathcal{I}(\mathcal{A})$ be the set of all maximal independent elements for $\mathcal{A}$, then

$$\mathcal{A}^+ \cap \mathcal{I}(\mathcal{A})^- = \emptyset \quad \text{and} \quad \mathcal{A}^+ \cup \mathcal{I}(\mathcal{A})^- = \mathcal{P}.$$

Consider the following problem:

$MIS(\mathcal{P},\mathcal{A},\mathcal{B})$: *Given a set $\mathcal{A}$ in a poset $\mathcal{P}$ and a collection of maximal independent elements $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$, either find a new maximal independent element $x \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$, or prove that $\mathcal{B} = \mathcal{I}(\mathcal{A})$.*

Clearly, the above problem can be efficiently solved in parallel for any explicitly given poset, i.e, when $\mathcal{P}$ is represented by the list of its elements and their precedence graph. If $\mathcal{P}$ is the product of n chains $\{0,1\}$ and $\mathcal{A} \subseteq \mathcal{P} = \{0,1\}^n$ is (the set of characteristic vectors of the hyperedges of) a hypergraph on n vertices, we obtain problem $MIS(\mathcal{A},\mathcal{I})$ stated in the introduction. We are interested in the more general case where $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ for explicitly given posets $\mathcal{P}_1, \ldots, \mathcal{P}_n$. For instance, partially ordered sets as attribute values arise in many data analysis applications, e.g., chains or products of chains in [9, 11, 26], lattices and products of lattices in [8]. Frequently a partially defined monotone binary function $f(x_1, \ldots, x_n)$ is sought to explain the data, where the variables $x_1, \ldots, x_n$ represent some attributes ranging over such posets. In many applications, $f : \mathcal{P} \to \{0,1\}$ is defined by sets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ of positive and negative samples, i.e. $f(x) = 1$ for $x \in \mathcal{A}$ and $f(x) = 0$ for $x \in \mathcal{B}$ is assumed. Due to monotonicity, we can assume without loss of generality that $\mathcal{A}$ and $\mathcal{B}$ are both antichains, and that $\mathcal{A}^+ \cap \mathcal{B}^- = \emptyset$. Now, determining whether f is totally defined and if not, finding a point $x \in \mathcal{P} \setminus (\mathcal{A}^+ \cap \mathcal{B}^-)$, is easily seen to be equivalent to problem $MIS(\mathcal{P},\mathcal{A},\mathcal{B})$.

In what follows, we assume that each poset $\mathcal{P}_i$ has a unique minimum element 0_i, and let $\mathrm{Supp}(x) = \{i \mid x_i \succ 0_i\}$ denote the set of non-minimal components of $x = (x_1, \ldots, x_n) \in \mathcal{P}$. As mentioned in the introduction, we define $\dim(\mathcal{A}) = \max\{|\mathrm{Supp}(a)| : a \in \mathcal{A}\}$. We also denote by $x^\perp$ the set of immediate predecessors of x, i.e., $x^\perp = \{y \in \mathcal{P} \mid z \preccurlyeq x, \ z \neq x \ \Rightarrow \ z \preccurlyeq y \text{ for some } y \in x^\perp\}$, and let $\text{in-deg}(\mathcal{P}) = \max\{|x^\perp| : \ x \in \mathcal{P}\}$. Clearly, $\text{in-deg}(\mathcal{P}) = \sum_{i=1}^n \text{in-deg}(\mathcal{P}_i)$ for

$\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$. If $\mathcal{P}$ is a lattice, we let $x \vee y$ and $x \wedge y$ denote the maximum and minimum of $x, y \in \mathcal{P}$.

Theorems 1 and 2 admit the following generalizations.

Theorem 1′ *Let $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$, where each poset $\mathcal{P}_i$ is a lattice of in-degree $\leq$ const, and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$ be two given sets such that $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$. Then $MIS(\mathcal{P}, \mathcal{A}, \mathcal{B}) \in NC$ for $dim(\mathcal{A}) \leq 3$, and $MIS(\mathcal{P}, \mathcal{A}, \mathcal{B}) \in RNC$ for $dim(\mathcal{A}) = 4, 5, \ldots$*

Theorem 2′ *Under the assumptions of Theorem 1′, $MIS(\mathcal{P}, \mathcal{A}, \mathcal{B})$ is NC-reducible to $MIS(\mathcal{P}', \mathcal{A}', \emptyset)$, where $\mathcal{P}' = \{z\}^+$ for some $z \in \mathcal{P}$ and $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$.*

Note that for any $z = (z_1, \ldots, z_n) \in \mathcal{P}$, we have $\mathcal{P}' = \{z\}^+ = \{z_1\}^+ \times \ldots \times \{z_n\}^+$, i.e. $\mathcal{P}'$ is still the product of n lattices $\mathcal{P}'_i = \{z_i\}^+$ whose in-degrees are bounded by the in-degrees of the original lattices $\mathcal{P}_i$. Moreover, we have $0'_i = z_i$ in $\mathcal{P}'_i$, and for this reason the dimension of $\mathcal{A}' \subseteq \mathcal{P}'$ does not exceed the dimension of $\mathcal{A}$ in $\mathcal{P}$. In addition, it is easy to see that Theorem 2′ is indeed a generalization of Theorem 2. If $\mathcal{P} = \{0, 1\}^n$, then z is the characteristic vector of some set $Z \subseteq V = \{1, \ldots, n\}$ and $\{Z\}^+$ is the family of all supersets of Z. Furthermore, each element $a \in \mathcal{A}$ is then the characteristic vector of some hyperedge $A \subseteq V$. Under this interpretation, $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$ can be regarded as the hypergraph $\{Z \cup A \mid A \in \mathcal{A}\}$. Problem $MIS(\mathcal{P}', \mathcal{A}', \emptyset)$ calls for computing a set $X \subseteq V$ such that $Z \subseteq X$ and X is a maximal independent set for $\{Z \cup A \mid A \in \mathcal{A}\}$. Letting $U = V \setminus Z$, the latter problem is easily seen to be equivalent to computing a maximal independent set for the induced hypergraph $\{A \cap U \mid A \in \mathcal{A}\}$, as stated in Theorem 2.

In addition to Theorems 1′ and 2′, we show that if each poset $\mathcal{P}_i$ has a unique minimum element, then problem $MIS(\mathcal{P}, \mathcal{A}, \emptyset)$ can be reduced to the maximal independent set problem for some hypergraphs.

Theorem 3 *For each fixed c, there is an NC-algorithm which, given posets $\mathcal{P}_1, \ldots, \mathcal{P}_n$ with unique minimum elements $0_i \in \mathcal{P}_i$, $i = 1, \ldots, n$, and a set $\mathcal{A} \subseteq \mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ such that $dim(\mathcal{A}) \leq c$, reduces $MIS(\mathcal{P}, \mathcal{A}, \emptyset)$ to the maximal independent set problem for some hypergraphs of dimension at most c.*

Note that Theorem 3 holds for the posets $\mathcal{P}_i$ of arbitrarily large in-degrees and does not require that these posets be lattices. It is also clear that Theorem 1′ is a corollary of Theorems 3 and 2′.

4.1 Characterization of sub-minimal elements of an ideal

Our proof of Theorem 2′ makes use of an analogue of Proposition 1. This analogue, Proposition 3 below, assumes that each of the posets $\mathcal{P}_i$, $i \in V = \{1, \ldots, n\}$, is a lower semi-lattice, i.e., for any two elements $x, y \in \mathcal{P}_i$ there is a unique minimum element $x \wedge y$. As before, we denote by $x^\perp$ the set of immediate predecessors of x. Note that if $x = (x_1, \ldots, x_n) \in \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$, then any element $y \in x^\perp$ has the form $y = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$, where $y_i \in x_i^\perp$ is an immediate predecessor of x_i in $\mathcal{P}_i$ and $i \in \text{Supp}(x)$.

Given a set $\mathcal{B} \subseteq \mathcal{P}$ and a vector $s \in \mathcal{P}$, we say that s is *sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$* if $s \preccurlyeq x$ for some minimal element x of the ideal $\mathcal{P} \setminus \mathcal{B}^-$. We call a subset $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$ a *majorant for $s^\perp$* if for any $y \in s^\perp$ there is an element $b \in \widetilde{\mathcal{B}}$ such that $b \succcurlyeq y$.

Proposition 3 *Let $\mathcal{B}$ be a given set in $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$, where each poset $\mathcal{P}_i$ is a lower semi-lattice. A vector $s \in \mathcal{P}$ is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$ if and only if there is a majorant $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$ for $s^\perp$ and a vector $z \in \{s\}^+ \cap (\mathcal{P} \setminus \mathcal{B}^-)$ such that*

(a) *$z[S]$ is minimal in $\mathcal{P}[S] \setminus \widetilde{\mathcal{B}}[S]^-$,*

(b) *$z_i = \wedge\{b_i \mid b = (b_1, \ldots, b_n) \in \widetilde{\mathcal{B}}\}$ for all $i \in V \setminus S$, and*

(c) *$|\widetilde{\mathcal{B}}| \leq \sum\{in\text{-}deg(\mathcal{P}_i) \mid i \in S\}$,*

where $S = Supp(s)$ and $z[S], \widetilde{\mathcal{B}}[S]$ are the restrictions respectively, of z and $\widetilde{\mathcal{B}}$ on S.

Let us note that if $|\text{Supp}(s)|$ and all poset in-degrees $\max\{|x^\perp| : x \in \mathcal{P}_i\}$ are bounded, then $|\widetilde{\mathcal{B}}| \leq const$ and hence there are only polynomially many sets $\widetilde{\mathcal{B}}$ satisfying condition (c). In addition, (b) and the boundedness of $|\text{Supp}(S)|$ imply that for each $\widetilde{\mathcal{B}}$, there are only polynomially many candidate vectors z that can satisfy (a). It is clear that all such sets $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$ and vectors $z \in \mathcal{P}$ can be generated and tested efficiently in parallel. We shall also make use of the following fact.

Proposition 4 *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$ such that $\mathcal{A}^+ \cap \mathcal{B}^- = \emptyset$. Let us assume further that $s \notin \mathcal{A}^-$ is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$, and let $z \in \mathcal{P}$ be the vector proving this, as in Proposition 3. Then, $z \notin \mathcal{A}^+$.*

Proof of Proposition 3. To show the "only if" part, suppose that s is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$, and let x be a minimal element in $\mathcal{P} \setminus \mathcal{B}^-$ such that $s \preccurlyeq x$. Denote by $Y \subseteq \mathcal{P}$ the set of all elements of the form $y = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$, where $i \in S = \text{Supp}(s)$ and $y_i \in x_i^\perp$. Clearly, $Y \subseteq x^\perp$ and $|Y| \leq \sum\{\text{in-deg}(\mathcal{P}_i) \mid i \in S\}$. By the minimality of x in $\mathcal{P} \setminus \mathcal{B}^-$, for each $y \in Y$ we can find an element $b = b(y) \in \mathcal{B}$ such that $b \succcurlyeq y$. Since $s \preccurlyeq x$, it follows that any immediate predecessor of s can be majorized by some $y \in Y$. Hence we conclude that $\widetilde{\mathcal{B}} = \bigcup\{\ b(y) \mid y \in Y\}$ is a majorant for $s^\perp$. By definition, $\widetilde{\mathcal{B}}$ satisfies (c). Now letting

$$z_i = \begin{cases} x_i & \text{if} \quad i \in S \\ \wedge\{b_i \mid b = (b_1, \ldots, b_n) \in \widetilde{\mathcal{B}}\} & \text{if} \quad i \in V \setminus S, \end{cases} \tag{9}$$

we readily obtain (b). To prove (a), let us first show that $x[S] \notin \widetilde{\mathcal{B}}[S]^-$. Suppose, to the contrary, that $b[S] \succcurlyeq x[S]$ for some $b \in \widetilde{\mathcal{B}}$. By the definition of $\widetilde{\mathcal{B}}$, we have

$$b \succcurlyeq y = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \tag{10}$$

for some $i \in S$ and $y_i \in x_i^\perp$. We cannot have $b_i \succcurlyeq x_i$ because this would imply $x \in \mathcal{B}^-$. Hence $b_i \not\succcurlyeq x_i$ for some $i \in S$, and consequently, $b[S] \not\succcurlyeq x[S]$. We have thus shown that $x[S] \in \mathcal{P}[S] \setminus \widetilde{\mathcal{B}}[S]^-$. Now it is easy to see that $x[S]$ is minimal in $\mathcal{P}[S] \setminus \widetilde{\mathcal{B}}[S]^-$, because any immediate predecessor of $x[S]$ in $\mathcal{P}[S]$ can be majorized by the restriction $b[S]$ of some vector $b \in \widetilde{\mathcal{B}}$, see (10). By the first line of (9), we have $z[S] = x[S]$ and (a) follows. It remains to show that $z \in \{s\}^+ \cap (\mathcal{P} \setminus \mathcal{B}^-)$. To this end, note that $x \preccurlyeq z$, because $x[S] = z[S]$ and for any $b = (b_1, \ldots, b_n) \in \widetilde{\mathcal{B}}$ and $i \in V \setminus S$ we have $x_i \preccurlyeq b_i$, see (10) and the second line of (9). The inequality $x \preccurlyeq z$

implies that $z \in \{s\}^+ \cap (\mathcal{P} \setminus \mathcal{B}^-)$, because on the one hand $s \preccurlyeq x$, and on the other hand $x \notin \mathcal{B}^-$.

The “if” part of the proof does not require condition (c). Since $z \notin \mathcal{B}^-$, there is a vector $x \notin \mathcal{B}^-$, minimal in $\mathcal{P} \setminus \mathcal{B}^-$, such that $x \preccurlyeq z$. We have $x[S] = z[S]$ for this vector, since by (a) and (b), any decrease of z in a coordinate $i \in S$ would yield a vector majorized by some $b \in \widetilde{\mathcal{B}} \subseteq \mathcal{B}$. From $x[S] = z[S]$ and $z \succcurlyeq s$ it follows that $x \succcurlyeq s$, thus proving that s is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$. □

Proof of Proposition 4. Suppose, to the contrary, that $z \succcurlyeq a$ for some $a \in \mathcal{A}$. Let $b \in \widetilde{\mathcal{B}}$. By (b), $b[V \setminus S] \succcurlyeq z[V \setminus S]$. This implies $b[V \setminus S] \succcurlyeq a[V \setminus S]$. Since $\mathcal{A}^+ \cap \mathcal{B}^- = \emptyset$, it follows that $b[S] \not\succcurlyeq a[S]$ for all $b \in \widetilde{\mathcal{B}}$, i.e., $a[S] \notin \widetilde{\mathcal{B}}[S]^-$. On the other hand, $z[S] \succcurlyeq a[S]$ by our assumption that $z \succcurlyeq a$. Now the minimality of $z[S]$ in $\mathcal{P}[S] \setminus \widetilde{\mathcal{B}}[S]^-$ implies that $z[S] = a[S]$. However, Proposition 3 also says that $z \succcurlyeq s$ and hence $a[S] \succcurlyeq s[S]$. Recalling that $S = \mathrm{Supp}(s)$, we conclude that $a \succcurlyeq s$, i.e., $s \in \mathcal{A}^-$. □

4.2 Proof of Theorem 2′

The proof is analogous to that of Theorem 2. Without loss of generality we can assume that $\mathcal{A}$ is an antichain in $\mathcal{P}$ (cf. Step 1 in Section 3). If there exists a vector $a \in \mathcal{A}$ which is not minimal in $\mathcal{P} \setminus \mathcal{B}^-$, then we can find an element $z \in a^\perp$ such that $z \in \mathcal{P} \setminus \mathcal{B}^-$. This can be done fast in parallel. We can then compute a new maximal independent point $b' \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$ by letting $b' = \mathit{output}(MIS(\mathcal{P}', \mathcal{A}', \emptyset))$, where $\mathcal{P}' = \{z\}^+$ and $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$ (cf. Step. 3 in the proof of Theorem 2′). It is clear that b' is indeed a new maximal independent element for $\mathcal{A}$ because $b' \succcurlyeq z$ and $z \notin \mathcal{B}^-$. (Note that if $\mathcal{P}$ is not an upper semi-lattice then the set $\mathcal{A}'$ of all minimal elements of $\{z\}^+ \cap \mathcal{A}^+$ may be exponentially large.)

Let us assume now that

$$\text{Each } a \in \mathcal{A} \text{ is minimal in } \mathcal{P} \setminus \mathcal{B}^-. \tag{11}$$

If

$$\mathcal{I}(\mathcal{A}) \neq \mathcal{B}, \tag{12}$$

then there is a vector $x \in \mathcal{P} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. Without loss of generality, we may assume that x is minimal in $\mathcal{P} \setminus \mathcal{B}^-$. We have $x \notin \mathcal{A}^-$ because otherwise $x \in \mathcal{B}^-$ by (11). Let s be a minimal element in $\{x\}^-$ such that $s \notin \mathcal{A}^-$, then $|\mathrm{Supp}(s)|$ does not exceed $\dim(\mathcal{A}) + 1$. Thus, (12) implies that

$$\begin{array}{l}\text{There is a vector } s \notin \mathcal{A}^- \text{ such that } |\mathrm{Supp}(s)| \leq \dim(\mathcal{A}) + 1 \\ \text{and } s \text{ is sub-minimal for } \mathcal{P} \setminus \mathcal{B}^-.\end{array} \tag{13}$$

Conversely, (13) implies (12) even without (11) and the assumption that $|\mathrm{Supp}(s)| \leq \mathit{dim}(\mathcal{A}) + 1$. To see this, observe that if $s \notin \mathcal{A}^-$ is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$, then by Proposition 4 we can find a vector $z \notin \mathcal{A}^+$ which satisfies the conditions of Proposition 3. In particular, $z \notin \mathcal{B}^-$, which implies (12).

As mentioned in Section 4.1, Proposition 3 gives an NC test for (13) provided that the dimension of $\mathcal{A}$ and the in-degrees of all posets $\mathcal{P}_i$ are bounded (cf. Step 4 in

Section 3). Moreover, if we find an s satisfying (13), then, according to Propositions 3 and 4, we also obtain an element $z \in \mathcal{P} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. Letting $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$, any solution to $MIS(\{z\}^+, \mathcal{A}', \emptyset)$ yields a new element in $\mathcal{I}(\mathcal{A})$. $\square$

4.3 Proof of Theorem 3

Consider the following problem:

$MIS(\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset)$: *Given $2n$ non-empty finite posets $\mathcal{R}_i \subseteq \mathcal{P}_i$, $i \in V = \{1, \dots, n\}$, each of which has a unique minimum element, and a set $\mathcal{A} \subseteq \mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$, find a maximal $\mathcal{A}$-independent element x* in $\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_n$.

Denoting by 0_i and r_i the minimum elements of $\mathcal{P}_i$ and $\mathcal{R}_i$, respectively, we shall assume without loss of generality that

$$r = (r_1, \dots, r_n) \notin \mathcal{A}^+, \tag{14}$$

for otherwise $\mathcal{R}$ contains no $\mathcal{A}$-independent element. As before, we let $\mathrm{Supp}(a) = \{i \in V \mid a_i \succ 0_i\}$ and let $\dim(\mathcal{A}) = \max\{|\mathrm{Supp}(a)| : a \in \mathcal{A}\}$ denote the dimension of $\mathcal{A}$. Our goal is to show that for $\dim(\mathcal{A}) \leq c$, problem $MIS(\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset)$ is NC-reducible to the maximal independent set problem for some hypergraphs of dimensions $\leq c$. This will prove Theorem 3 because $MIS(\mathcal{P}, \mathcal{A}, \emptyset)$ is a special case of $MIS(\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset)$ for $\mathcal{R} = \mathcal{P}$. Our reduction iteratively decreases $|\mathcal{A}|$ and the maximum cardinality of the posets $\mathcal{R}_i$.

Step 1. If $\max\{|\mathcal{R}_i| : i \in V\} = 1$, return $x = r$ and halt.

Step 2. If $\mathcal{A} = \emptyset$, return any maximal point in $\mathcal{R}$ and halt.

Step 3. Let $\mathrm{Supp}_{\mathcal{R}}(a) = \{i \in V \mid a_i \succ r_i\}$. In view of (14), we have $|\mathrm{Supp}_{\mathcal{R}}(a)| \geq 1$ for all $a \in \mathcal{A}$. Remove all points $a \in \mathcal{A}$ with $|\mathrm{Supp}_{\mathcal{R}}(a)| = 1$ and reduce $\mathcal{R}$ accordingly:

$$\mathcal{R}_i \leftarrow \mathcal{R}_i \setminus \bigcup\{\, a_i^+ \mid \mathrm{Supp}_{\mathcal{R}}(a) = \{i\},\ a \in \mathcal{A}\}, \quad \mathcal{A} \leftarrow \{a \in \mathcal{A} :\ |\mathrm{Supp}_{\mathcal{R}}(a)| \geq 2\}.$$

Step 4. For each $i \in V = \{1, \dots, n\}$, topologically sort poset $\mathcal{R}_i$, i.e., find a one-to-one mapping $\phi_i : \mathcal{R}_i \to \{1, \dots, |\mathcal{R}_i|\}$ such that $\phi_i(x) < \phi_i(y)$ whenever $x \prec y$ in $\mathcal{R}_i$. Let $\mathcal{R}_i^u = \{x \in \mathcal{R}_i \mid \phi_i(x) \geq \lceil |\mathcal{R}_i|/2 \rceil\}$ and let $\mathcal{Q}_i$ denote the antichain consisting of all minimal elements of $\mathcal{R}_i^u$. Note that $\mathcal{R}_i^u$ and hence $\mathcal{Q}_i$ are not empty for all $i \in V$.

Step 5. Let $U = \bigcup_{i=1}^n \mathcal{Q}_i$, and let $\mathcal{H} \subseteq 2^U$ be the hypergraph whose hyperedges are:
1) all pairs of the form $\{x, y\}$, where $x \neq y$ and $x, y \in \mathcal{Q}_i$ for some $i \in V$, and
2) all collections H of at most $c = \dim(\mathcal{A})$ elements of U such that H contains at most one element from each $\mathcal{Q}_i$ and $\pi(H) \succcurlyeq a$ for some $a \in \mathcal{A}$, where $\pi(H) = (\pi_1(H), \dots, \pi_n(H)) \in \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_n$ is the vector with the following components:

$$\pi_i(H) = \begin{cases} H \cap \mathcal{Q}_i & \text{if } H \cap \mathcal{Q}_i \neq \emptyset \\ r_i & \text{otherwise.} \end{cases}$$

Step 6. Compute a maximal independent set I for $\mathcal{H}$. Note that $I \neq \emptyset$ since $\mathcal{H}$ does not contain singletons. Also, by the definition of $\mathcal{H}$, the independent set I

contains at most one element from each antichain $\mathcal{Q}_i$ and the vector $\pi(I) \in \mathcal{Q} = \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_n \subseteq \mathcal{R}$ is independent of $\mathcal{A}$.

Step 7. Go to Step 1 and compute $MIS(\mathcal{R}' \subseteq \mathcal{P}, \mathcal{A}', \emptyset)$, where $\mathcal{R}' = \mathcal{R}'_1 \times \ldots \times \mathcal{R}'_n$ is defined as follows:

$$\mathcal{R}'_i = \begin{cases} \mathcal{R}_i & \text{if } |\mathcal{R}_i| = 1 \\ \mathcal{R}_i \cap \{\pi_i(I)\}^+ & \text{if } I \cap \mathcal{Q}_i \neq \emptyset \\ \mathcal{R}_i \setminus \mathcal{R}_i^u & \text{otherwise,} \end{cases}$$

and $\mathcal{A}' = \{a \in \mathcal{A} \mid \{a\}^+ \cap \mathcal{R}' \neq \emptyset\}$.

The correctness of the above iterative procedure can be seen from the following observations:

(a) Each poset $\mathcal{R}'_i$ still has a unique minimum element r'_i;

(b) Since $\pi(I)$ is independent of $\mathcal{A}$, the new minimum element $r' = (r'_1, \ldots, r'_n)$ satisfies (14);

(c) Let $x \in \mathcal{R}'$ be a maximal $\mathcal{A}'$-independent element in $\mathcal{R}'$. Then x is a maximal $\mathcal{A}$-independent element of $\mathcal{R} \cap \{\pi(I)\}^+$. Hence x is also a maximal $\mathcal{A}$-independent element of $\mathcal{R}$, i.e., x solves the original problem $MIS(\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset)$.

Since each iteration almost halves the maximum size of the posets $\mathcal{R}_i$, our reduction consists of $O\big(\log(\max\{|\mathcal{R}_i| :\ i \in V\})\big)$ iterations and Theorem 3 follows.

Remark It is essential, in the above result, to assume that each poset $\mathcal{R}_i$ has a unique minimum element, for otherwise problem $MIS(\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset)$ becomes NP-hard even for posets $\mathcal{P}_i$ with only 3 elements. To see this, let each poset $\mathcal{P}_i$ be a "$\bigvee$", i.e. composed of 3 elements $\{u, 0, w\}$, where $0 \leq u$ and $0 \leq w$ are the only relations in $\mathcal{P}_i$. Let $R_i = \{u, w\} \subseteq \mathcal{P}_i$. Now given a disjunctive normal form $D = D_1 \vee \ldots \vee D_m$ in n variables $x_1, \ldots, x_n$, let us associate a vector $a^j \in \mathcal{P}$ with every term D_j as follows: a^j_i takes the value u if variable x_i appears in term D_j, the value w if $\overline{x}_i$ appears in D_j, and the value 0 otherwise. Letting $\mathcal{A} = \{a^j \mid j = 1, \ldots, m\} \subseteq \mathcal{P}$, it is then easy to see that $\mathcal{R} \subseteq \mathcal{A}^+$ if and only if D is a tautology.

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