INCREMENTAL ALGORITHMS FOR ENUMERATING EXTREMAL SOLUTIONS OF MONOTONE SYSTEMS OF SUBMODULAR INEQUALITIES AND THEIR APPLICATIONS

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ABSTRACT OF THE DISSERTATION

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We give a quasi-polynomial-time incremental algorithm for enumerating all minimal solutions to a system of integer-valued monotone submodular inequalities

$$f_i(x_1, \ldots, x_n) \geq t_i, \quad i = 1, \ldots, r,$$

where $x_1, \ldots, x_n$ are integer or binary variables, each function f_i is defined by an evaluation oracle and the right-hand sides $t_1, \ldots, t_r$ are bounded by a quasi-polynomial in the dimension of the system: $t_1, \ldots, t_r \leq 2^{\mathrm{polylog}(nr)}$. The latter condition can be dropped for systems of linear inequalities, but is necessary in general, unless all problems in NP can be solved in quasi-polynomial time. Our enumeration algorithm provides a partial answer to the conjecture of Lawler, Lenstra and Rinnooy Kan (1980) that all minimal integer solutions to a system of non-negative linear inequalities cannot be enumerated efficiently, unless P=NP. It also implies an incremental quasi-polynomial-time algorithm for enumerating all maximal independent sets in r matroids defined by independence oracles on a common ground set. This improves on the previously known exponential bound on the complexity of the matroid intersection problem. Other special cases of our general result include the generation of minimal infrequent sets of a

database (data mining), minimal space-spanning collections of subspaces from a given list, and minimal connectivity ensuring collections of subgraphs from a given list (reliability theory).

Our proofs are based on two key facts. We first show that the number of maximal infeasible solutions to any system of monotone submodular inequalities can be bounded by a quasi-polynomial in n, r, and the number of minimal feasible solutions. We also give stronger bounds for some special cases, as well as some generalizations, of polymatroid functions, including 2-monotonic functions, k-smooth polymatroid functions and functions with non-negative Möbius coefficients. Examples are also given to demonstrate that our bounds are reasonably sharp. Second, we extend the known quasi-polynomial bound on the complexity of the hypergraph dualization problem to families of vectors defined on products of chains (or more generally, on products of lattices of bounded width).

Finally, we show that for hypergraphs of bounded edge size, the problem of extending a given list of transversals (equivalently, maximal independent sets) can be solved efficiently in parallel.

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Dedication

In memory of my mother Aleyya Hanem Abdul-Ghaffar,

I dedicate this work to my father Maamoon Elbassioni

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Chapter 1

Introduction

1.1 Monotonicity, submodularity, oracles, and incremental generation

Monotonicity is a natural property that arises in a variety of applications. For example, the property that a given collection of edges on some vertex set defines a connected graph, the property that an integral vector satisfies a given system of linear inequalities $Ax \geq b$ with non-negative coefficients, and the property that a subset of given vector subspaces spans the entire space, are all monotone properties. In fact, these are examples of *monotone increasing* properties: if a subgraph is connected, then adding an edge to this subgraph will result also in a connected subgraph; if an integral vector satisfies a system of linear inequalities with non-negative coefficients, then increasing any component of this vector will result also in a feasible vector; etc. Similarly, the property that a set intersects at most a specified number of hyperedges of a given hypergraph, and the property that a set is independent in a given list of matroids, are examples of monotone decreasing properties. Throughout this manuscript, we shall usually use "monotone" to refer to monotonically increasing properties, and "anti-monotone" to refer to monotonically decreasing properties.

Given a set of elements satisfying a certain monotone property, it is natural to represent this set by the family of all minimal elements satisfying the same property (the family of all maximal elements in case of anti-monotone properties). For the graph connectivity problem, each such element is a subgraph, the removal of an edge from which will disconnect the graph, that is a spanning tree of the given graph. Clearly, if the list of all elements satisfying the property is given explicitly, then finding the minimal elements is straightforward. However, in many situations, this list is specified by an oracle, i.e. an efficiently executable procedure to check if a given element satisfies the given property. For example, checking graph connectivity is such an oracle. In this case, enumerating the minimal elements, that satisfy the given monotone property, becomes a non-trivial problem. This thesis is concerned with developing efficient algorithms for such enumeration problems.

Formally, let V be a finite set of cardinality $|V| = n$. For a hypergraph $\mathcal{H} \subseteq 2^V$, let us denote by $\mathcal{I}(\mathcal{H})$ the family of its *maximal independent* subsets, i.e. maximal subsets of V not containing any hyperedges of $\mathcal{H}$. The complement of a maximal independent subset is called a *minimal transversal* of $\mathcal{H}$ (i.e. minimal subset of V intersecting all elements of $\mathcal{H}$). The collection $\mathcal{H}^d$ of minimal transversals is also called the *dual* or *transversal* hypergraph for $\mathcal{H}$ (see [9]).

Let us consider a monotone property π over the subsets of V, and let $\mathcal{F}_\pi^+$ denote the corresponding collection of all subsets having property π. We shall assume that π is represented by a *satisfiability oracle* $\mathcal{O}_\pi$, i.e. an algorithm which, given an input description of $\mathcal{O}_\pi$ of size $\|\mathcal{O}_\pi\|$ and a subset $X \subseteq V$, can decide whether or not X has property π, in time (quasi-) polynomial in n and $\|\mathcal{O}_\pi\|$ (where quasi-polynomial means $2^{\mathrm{polylog}(\cdot)}$ for problems with input size $(\cdot)$). Since π is monotone, $Y \supseteq X \in \mathcal{F}_\pi^+$ implies $Y \in \mathcal{F}_\pi^+$. Thus, we may as well represent π by the hypergraph $\mathcal{F}_\pi \subseteq \mathcal{F}_\pi^+$ consisting of all minimal subsets having property π. Let us note that the oracle $\mathcal{O}_\pi$ could equivalently be called a *membership oracle* for the family $\mathcal{F}_\pi^+$, or a *superset oracle* for the family $\mathcal{F}_\pi$, since a subset $X \subseteq V$ has property π if and only if $X \in \mathcal{F}_\pi^+$, which is further equivalent to the existence of a subset $Y \subseteq X$, $Y \in \mathcal{F}_\pi$. Let us also note that the complementary family $\mathcal{I}_\pi = 2^V \setminus \mathcal{F}_\pi^+$ is an *independence system* (i.e. $Y \subseteq X \in \mathcal{I}_\pi$ implies $Y \in \mathcal{I}_\pi$, see e.g. [65]), the maximal elements of which are the maximal independent sets of $\mathcal{F}_\pi$.

Let us finally remark that any *Sperner* hypergraph $\mathcal{H}$ (i.e. one which has no two hyperedges $X, Y \in \mathcal{H}$ for which $X \subseteq Y$ or $Y \subseteq X$) defines uniquely a monotone property π such that $\mathcal{H} = \mathcal{F}_\pi$ holds.

Given a monotone property π, described by a satisfiability oracle $\mathcal{O}_\pi$, we are interested in generating all elements of $\mathcal{F}_\pi$. In general, the size of this set might be exponentially large both in n and the size of the oracle $\|\mathcal{O}_\pi\|$. Therefore, it is natural in such situations to measure the efficiency of the generation algorithm in terms of both the input and output sizes. One can generally distinguish a number of classes of generation algorithms, depending on how the efficiency of the algorithm is measured (see, for instance, [56]):

- *Output efficient* algorithms: an algorithm to generate the set of elements $\mathcal{F}_\pi \subseteq 2^V$, satisfying a certain monotone property π, is said to be output efficient if it generates all the required elements in time (quasi-) polynomial in n, $\|\mathcal{O}_\pi\|$ (the input size) and $|\mathcal{F}_\pi|$ (the output size).

- *Incrementally efficient* algorithms: given a partial list of minimal elements $\mathcal{X} \subseteq \mathcal{F}_\pi$, the algorithm outputs a new element in time (quasi-) polynomial in n, $\|\mathcal{O}_\pi\|$, and the size of the current partial list $\mathcal{X}$.

- *Polynomial delay* algorithms: an algorithm of this type generates the elements of $\mathcal{F}_\pi$ in some order such that, the delay until the first element is output, and thereafter, the delay between two consecutively generated elements is bounded by a polynomial in the size of the input n, $\|O_\pi\|$ only. Algorithms in this class can be further required to satisfy stronger constraints, e.g., to output the elements in some *specific (e.g., lexicographic) order*, to use only polynomial space in the input size regardless of the output size, or to use only linear space in the input size (strong enumerability).

In this thesis, we shall be mostly concerned with the class of incrementally efficient algorithms. Specifically, we consider the following problem of incrementally generating the minimal elements $\mathcal{F}_\pi$ satisfying a monotone property π:

GEN$(\mathcal{F}_\pi, \mathcal{X})$**:** *Given a subfamily $\mathcal{X} \subseteq \mathcal{F}_\pi$, either find a new minimal satisfying set $X \in \mathcal{F}_\pi \setminus \mathcal{X}$, or prove that the given partial list is complete: $\mathcal{X} = \mathcal{F}_\pi$.*

Clearly, the entire family $\mathcal{F}_\pi$ can be generated by initializing $\mathcal{X} = \emptyset$ and iteratively solving the above problem $|\mathcal{F}_\pi| + 1$ times. As stated above, the algorithm will be said to be incrementally efficient, if it can solve problem GEN$(\mathcal{F}_\pi, \mathcal{X})$ efficiently in terms of n, $\|\mathcal{O}_\pi\|$ and the size of the partial list $|\mathcal{X}|$.

All monotone properties considered in this thesis can be described as a system of $r \geq 1$ *monotone submodular* inequalities

$$f_i(X) \geq t_i, \quad i = 1, \ldots, r, \tag{1.1}$$

over the subsets $X \subseteq V$ (or more generally, over the elements of the product of some lattices), where $t_i \in \mathbb{R}$, $i = 1, \ldots, r$, are given thresholds, and $f_i : 2^V \mapsto \mathbb{R}$ are given functions. Thus, we assume that each set-function f_i is *monotone*, i.e. $f(X) \leq f(Y)$ whenever $X \subseteq Y$, *submodular*, i.e.,

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y) \tag{1.2}$$

holds for all subsets $X, Y \subseteq V$, and that f_i is defined via a (quasi-) polynomial-time evaluation oracle $\mathcal{O}_i$. For a subset $X \subseteq V$, let $\pi(X)$ be the property that X satisfies (1.1), then we have a satisfiability oracle for $\pi(X)$ (this requires only checking the feasibility of X for (1.1)). It will be seen in Section 1.3.2 that, under some fairly general assumptions, such submodular systems enjoy a nice property that enables the efficient computation of their minimal feasible sets $\mathcal{F}_\pi$.

1.2 Applications

Before proceeding further, let us motivate the discussion by some examples of monotone properties π which can be defined by systems of monotone submodular inequalities:

- *Monotone systems of linear inequalities in integer variables:* Given a monotone system $Ax \geq b$ of r linear inequalities in n integer variables, where A is a given real $r \times n$-matrix, b is given real r-vector, generate all minimal integer vectors x feasible for the system. This problem has interesting applications in integer and stochastic programming (see, e.g., [84]).

- *Matroid intersections:* Given a set of r matroids on the common ground set V, each described by an independence oracle, generate all maximal subsets of V that are independent with respect to the r matroids, a problem that has some applications in combinatorial optimization [64] and symbolic analysis of electrical circuits [44].

- *Infrequent elements for a database:* Given a database $\mathcal{D} \subseteq 2^V$ with binary attributes, and an integer threshold t, generate all minimal subsets of V that are contained in at most t transactions of $\mathcal{D}$. We shall also consider generalizations of this example to databases with quantitative, hierarchical, and lattice attributes. These problems arise in data mining applications [1, 48, 74, 75, 78, 79, 90, 91, 96].

- *Connectivity ensuring collections of subgraphs:* Given a collection of n graphs on the same vertex set R, generate all minimal sub-collections of graphs which interconnect all vertices in R. This problem has applications in reliability theory [29, 87].

- *Space covering:* Given a collection of n linear subspaces of $\mathbf{F}^r$, for some field $\mathbf{F}$, enumerate all minimal sub-collections that span the entire space $\mathbf{F}^r$.

- *Efficient (inefficient) points of discrete distributions:* Given a discrete probability distribution of a random variable $\xi \in \mathbb{Z}^n$ and a threshold probability $p \in (0,1)$, generate all minimal (maximal) vectors $x \in \mathbb{Z}^n$ such that $\Pr[\xi \leq x] \geq p$ ($\Pr[\xi \leq x] \leq p$). This notion of efficient points comes from Stochastic Programming [32, 84].

- *Packing points into boxes:* Given a set of n-dimensional points $\mathbb{P} \subseteq \mathbb{R}^n$, a coloring $C : \mathbb{P} \mapsto \{1, 2, \ldots, r\}$ of the point set, and integers $0 \leq t_1, t_2, \ldots, t_r \leq |\mathbb{P}|$, generate all maximal n-dimensional boxes that contain at most t_1 points of $\mathbb{P}$ of the first color, at most t_2 points of the second color, ..., and at most t_r points of the r^{th} color. This problem has some applications in quantitative data mining [91].

Using the generation framework, presented in the next section, we shall be able to provide *quasi-polynomial* upper bounds on the incremental time complexity of all the above generation problems. Interestingly, the analogous problem of generating the family of extremal subsets not having property π (i.e. maximal infeasible integer vectors for a monotone system of linear inequalities, minimal subsets that are dependent with respect to at least one of a given set of r matroids, etc.) is NP-hard for all the monotone properties π described above.

1.3 A framework for incremental generation

As we shall see below, it will be convenient to consider, along with problem $\mathrm{GEN}(\mathcal{F}_\pi, \mathcal{X})$, the following *joint generation* problem:

GEN$(\mathcal{F}_\pi, \mathcal{I}(\mathcal{F}_\pi), \mathcal{X}, \mathcal{Y})$**:** *Given two explicitly listed set families $\mathcal{X} \subseteq \mathcal{F}_\pi$ and $\mathcal{Y} \subseteq \mathcal{I}(\mathcal{F}_\pi)$, either find a new set in $(\mathcal{F}_\pi \setminus \mathcal{X}) \cup (\mathcal{I}(\mathcal{F}_\pi) \setminus \mathcal{Y})$, or prove that these families are complete: $(\mathcal{X}, \mathcal{Y}) = (\mathcal{F}_\pi, \mathcal{I}(\mathcal{F}_\pi))$.*

It is clear that for a given monotone property π, represented by a satisfiability oracle $\mathcal{O}_\pi$, we can generate both $\mathcal{F}_\pi$ and $\mathcal{I}(\mathcal{F}_\pi)$ simultaneously by starting with $\mathcal{X} = \mathcal{Y} = \emptyset$ and solving problem GEN$(\mathcal{F}_\pi, \mathcal{I}(\mathcal{F}_\pi), \mathcal{X}, \mathcal{Y})$ for a total of $|\mathcal{F}_\pi| + |\mathcal{I}(\mathcal{F}_\pi)| + 1$ times, incrementing in each iteration either $\mathcal{X}$ or $\mathcal{Y}$ by the newly found subset $S \in (\mathcal{F}_\pi \setminus \mathcal{X}) \cup (\mathcal{I}(\mathcal{F}_\pi) \setminus \mathcal{Y})$, according to the answer of the oracle $\mathcal{O}_\pi$, until we have $(\mathcal{X}, \mathcal{Y}) = (\mathcal{F}_\pi, \mathcal{I}(\mathcal{F}_\pi))$.

As we shall see in the next section, problem GEN$(\mathcal{F}_\pi, \mathcal{I}(\mathcal{F}_\pi), \mathcal{X}, \mathcal{Y})$ can be solved efficiently for any monotone property π described by a satisfiability oracle. Unfortunately, this joint generation may not be an efficient algorithm for solving either of GEN$(\mathcal{F}_\pi, \mathcal{X})$ or GEN$(\mathcal{I}(\mathcal{F}_\pi), \mathcal{Y})$ separately for the simple reason that we do not control which of the families $\mathcal{F}_\pi \setminus \mathcal{X}$ and $\mathcal{I}(\mathcal{F}_\pi) \setminus \mathcal{Y}$ contains each new hyperedge produced by the algorithm. Suppose we want to generate $\mathcal{F}_\pi$ and the family $\mathcal{I}(\mathcal{F}_\pi)$ is exponentially larger than $\mathcal{F}_\pi$. Then, if we are unlucky, we may get hyperedges of $\mathcal{F}_\pi$ with exponential delay, while getting large subfamilies of $\mathcal{I}(\mathcal{F}_\pi)$ (which are not needed at all) in between.

For the above reasons, we shall study in this thesis Sperner hypergraphs $\mathcal{F}_\pi$ (or equivalently, monotone properties π) for which joint generation in fact provides an incrementally efficient framework for generating all hyperedges of $\mathcal{F}_\pi$. Following [19, 20], let us call a Sperner hypergraph $\mathcal{H} = \mathcal{F}_\pi$ (corresponding to a monotone property π) *dual-bounded* if the size of $\mathcal{I}(\mathcal{F}_\pi)$ is (quasi-) polynomially limited in the size of $\mathcal{F}_\pi$, the size of V, and the size of the oracle $\|O_\pi\|$ representing π. Let us further call $\mathcal{H} = \mathcal{F}_\pi$ *uniformly dual-bounded*, if

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_\pi)| \leq (quasi\text{ -})poly(|\mathcal{X}|, |V|, \|\mathcal{O}_\pi\|) \tag{1.3}$$

holds for any non-empty subfamily $\mathcal{X} \subseteq \mathcal{F}_\pi$.

In the next section we shall recall first from [11, 49] that problem GEN$(\mathcal{F}_\pi, \mathcal{I}(\mathcal{F}_\pi), \mathcal{X}, \mathcal{Y})$ can be reduced in polynomial time to the *hypergraph dualization* problem (that is, the generation of the minimal transversals of an explicitly given hypergraph), and that the latter problem can be solved in quasi-polynomial time (see [43]). As a consequence, problem GEN$(\mathcal{F}_\pi, \mathcal{X})$ can be solved in quasi-polynomial time for every subfamily $\mathcal{X} \subseteq \mathcal{F}_\pi$, whenever $\mathcal{F}_\pi$ is uniformly dual-bounded (see [20]). In Section 1.3.2 we state several inequalities proving uniformly dual-boundedness for a number of families of hypergraphs (see [14, 15, 20, 22]).

1.3.1 Dualization and joint generation

Let $\mathcal{H} \subseteq 2^V$ be a Sperner hypergraph given by an explicit list of its hyperedges. For $X \subseteq V$, let $\pi(X)$ be the property that X is a transversal of $\mathcal{H}$, that is, X intersects all hyperedges of $\mathcal{H}$. Then $\mathcal{F}_\pi = \mathcal{H}^d$ is the transversal hypergraph of $\mathcal{H}$ (the complementary hypergraph of maximal independent sets $\mathcal{I}(\mathcal{H})$), the above description provides a satisfiability oracle for π of size at most $n|\mathcal{H}|$, and problem GEN$(\mathcal{F}_\pi, \mathcal{X})$ reduces to the well-known hypergraph transversal, or *dualization* problem:

DUAL$(\mathcal{H}, \mathcal{X})$**:** *Given two explicitly listed Sperner families* $\mathcal{H} \subseteq 2^V$ *and* $\mathcal{X} \subseteq \mathcal{I}(\mathcal{H})$, *either find a new maximal independent set* $X \in \mathcal{I}(\mathcal{H}) \setminus \mathcal{X}$ *or show that* $\mathcal{X} = \mathcal{I}(\mathcal{H})$.

The dualization problem can be efficiently solved for many classes of hypergraphs. For example, if the sizes of all the hyperedges of $\mathcal{H}$ are limited by a constant k, then dualization can be executed in incremental polynomial time, see e.g. [12, 39] (and even stronger, we shall show in Chapter 6 that it can be solved efficiently in parallel [13]). In the quadratic case, i.e. when $k = 2$, there are even more efficient algorithms that run with polynomial delay, i.e. in $poly(|V|, |\mathcal{H}|)$ time (see e.g. [56, 94]). Efficient algorithms also exist for the dualization of 2-monotonic, threshold, matroid, read-bounded, acyclic and some other classes of hypergraphs (see e.g. [10, 24, 30, 35, 39, 69, 72, 73, 82, 83]), and the recent results of [40].

Even though no incremental polynomial time algorithm for the dualization of arbitrary hypergraphs is known, an incremental *quasi-polynomial time* algorithm exists (see [43], and also [49] for more detail):

Theorem 1.1 (Fredman and Khachiyan [43]) *Problem DUAL*$(\mathcal{H}, \mathcal{X})$ *can be solved in* $O(n) + m^{o(\log m)}$ *time, where* $n = |V|$ *and* $m = |\mathcal{H}| + |\mathcal{X}|$.

(Recent improvements on this result, but which are still quasi-polynomial in running time, can be found in [45, 93].)

It was observed independently in [11] and [49] that for any polynomial-time satisfiability oracle O_π problem GEN$(\mathcal{F}_\pi, \mathcal{I}(\mathcal{F}_\pi), \mathcal{X}, \mathcal{Y})$ can be reduced in polynomial time to dualization:

Proposition 1.1 ([11, 49]) *GEN($\mathcal{F}_\pi, \mathcal{I}(\mathcal{F}_\pi), \mathcal{X}, \mathcal{Y}$) can be solved in time $n(poly(|\mathcal{X}|, |\mathcal{Y}|) + T(\|\mathcal{O}_\pi\|)) + T_{dual}$, where $T(\|\mathcal{O}_\pi\|)$ is the worst-case running time of the oracle on any $X \subseteq V$, and T_{dual} denotes the time required to solve the dualization problem with $\mathcal{X}$ and $\mathcal{Y}$.*

In fact, Proposition 1.1 can be even generalized to antichains defined on products of partially ordered sets (see Section 1.4). From Proposition 1.1 it follows easily that joint generation is an incrementally efficient way of generating $\mathcal{F}_\pi$, whenever this hypergraph is uniformly dual bounded:

Corollary 1.1 ([20]) *If the hypergraph $\mathcal{F}_\pi$ is uniformly dual-bounded, then problem GEN($\mathcal{F}_\pi, \mathcal{X}$) is solvable in quasi-polynomial time for every subfamily $\mathcal{X} \subseteq \mathcal{F}_\pi$.*

(see the proof of Corollary 1.2.)

1.3.2 Some uniformly dual-bounded hypergraphs; Polymatroid functions, special cases and generalizations

Consider a system of monotone submodular inequalities (1.1) over subsets $X \subseteq V$. Let $\mathcal{F}$ be the set of all minimal feasible solutions for (1.1), and denote by $\mathcal{F}^+$ the family $\{X \subseteq V \mid X \supseteq Y \text{ for some } Y \in \mathcal{F}\}$. In this section, we list some classes of functions for which the hypergraph $\mathcal{F}$ is uniformly dual-bounded.

Probably the most general example, considered in this thesis, for which $\mathcal{F}$ is uniformly dual-bounded is the class of *polymatroid* functions. A function $f : 2^V \mapsto \mathbb{Z}_+$ is called polymatroid if it is monotone, submodular and if $f(\emptyset) = 0$. One of the main results of this thesis, which will proved in Chapter 2, is that the hypergraph of minimal feasible sets for a system of polymatroid inequalities, is quasi-polynomially dual-bounded, provided that that the right hand sides of (1.1) are bounded by a quasi-polynomial in the dimension of the system: $t_1, \ldots, t_r \leq 2^{\text{polylog}(nr)}$.

Theorem 1.2 *Let $\mathcal{F}$ be the set of all minimal feasible sets for a polymatroid system (1.1), and let $\mathcal{X} \subseteq \mathcal{F}^+$ be an arbitrary family of feasible sets of size $|\mathcal{X}| \geq 1$. Then*[1]

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq \max\{rn, r(n|\mathcal{X}|)^{\log t}\}, \tag{1.4}$$

[1] *In fact, a stronger bound will be proved in Chapter 2, see Theorem 2.2*

where $t = \max\{t_1, \ldots, t_r\}$. *In particular, for* $\mathcal{X} = \mathcal{F}$ *we get* $|\mathcal{I}(\mathcal{F})| \leq \max\{rn, r(n|\mathcal{F}|)^{\log t}\}$.

(Note that (1.4) is stronger than (1.3) in that (1.4) holds for all $\mathcal{X} \in \mathcal{F}^+$ rather than for all $X \in \mathcal{F}$.) Below, we state some special cases of polymatroid functions for which the inequality (1.4) can be strengthened or generalized.

Modular functions. A submodular function $f : 2^V \mapsto \mathbb{Z}_+$ is said to be *modular (linear)* if equality holds in (1.2) for all $X, Y \subseteq V$. It is immediate to see that modular functions are completely defined by their values on singletons and the empty-set, i.e., $f(X) = f(\emptyset) + \sum_{x \in X} f(\{x\})$. Thus if the "weights" $f(\{x\})$ are non-negative for all $x \in V$, then the function f is monotone. In general, a system of modular functions, with arbitrary (not necessarily non-negative) real-valued weights, may define a monotone property. It was shown in [20, 22] that for systems of modular functions, the family of minimal solutions is uniformly dual bounded. Unlike (1.4), the bound in this case is linear in the size of $\mathcal{F}$, and does not depend on the right hand sides of (1.1).

Theorem 1.3 (Boros, Gurvich, Khachiyan, and Makino [20, 22]) *Let* $\mathcal{F}$ *be the set of all minimal feasible solutions for a monotone system of modular inequalities (1.1) and let* $\mathcal{X} \subseteq \mathcal{F}^+$ *be an arbitrary family of feasible subsets of size* $|\mathcal{X}| \geq 1$. *Then*

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq r \sum_{X \in \mathcal{X}} |X|.$$

In particular, $|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq rn|\mathcal{X}|$, *and for* $\mathcal{X} = \mathcal{F}$ *we get* $|\mathcal{I}(\mathcal{F})| \leq rn|\mathcal{F}|$.

2-Monotonic functions. Theorem 1.3 will be generalized in Chapter 3 to systems of modular functions in *integer variables.* Towards this end, we shall consider the more general class of *2-monotonic functions.* A monotone function $f : 2^V \mapsto \mathbb{R}$ is called 2-monotonic, if there exists a permutation $\sigma \in \mathbb{S}_V$ of the ground set V such that $f(X \cup \{v\} \setminus \{u\}) \geq f(X)$ whenever $u \in X \subseteq V$, $v \notin X$ and v precedes u in their σ-order. Clearly, a linear set-function is 2-monotonic with respect to the permutation that puts the weights $f(\{x\})$ in non-increasing order. For a single 2-monotonic inequality in n variables, it is known [30] that the number of maximal infeasible solutions is upper-bounded by n times the number of minimal feasible solutions. In Chapter 3 of this manuscript, it will be illustrated that this result can be extended uniformly for systems of 2-monotonic inequalities:

Theorem 1.4 *If $\mathcal{F}$ is the set of all minimal feasible solutions for a system of 2-monotonic inequalities (1.1) and $\mathcal{X} \subseteq \mathcal{F}^+$ is an arbitrary family of feasible subsets of size $|\mathcal{X}| \geq 1$, then*

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq r \sum_{X \in \mathcal{X}} |X|.$$

In particular, $|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq rn|\mathcal{X}|$, and for $\mathcal{X} = \mathcal{F}$ we get $|\mathcal{I}(\mathcal{F})| \leq rn|\mathcal{F}|$.

In fact, the above inequality will be proved for systems of 2-monotonic inequalities over products of chains, implying in particular, that minimal feasible solutions for monotone systems of linear inequalities in integer variables can be generated in quasi-polynomial time.

k-Smooth polymatroid functions. Let $f : 2^V \mapsto \mathbb{Z}_+$ be an integer-valued monotone function. For an integer $k \in \mathbb{Z}_+$, the function f is said to be *k-smooth* [67] if for any $v \in V$ and any $X \subseteq V$, we have

$$f(X \cup \{v\}) - f(X) \leq k.$$

The following theorem will be proved in Chapter 3:

Theorem 1.5 *Let $\mathcal{F}$ be the set of all minimal feasible solutions for a system of k-smooth polymatroid inequalities (1.1) and let $\mathcal{X} \subseteq \mathcal{F}^+$ be an arbitrary subset of $\mathcal{F}^+$ of size $|\mathcal{X}| \geq 1$. Then*

$$\sum_{Y \in \mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})} |V \setminus Y| \leq r \sum_{X \in \mathcal{X}} |X| \cdot |V \setminus X|^{k-1}.$$

In particular, we have $|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq rn^k|\mathcal{X}|$, which for $\mathcal{X} = \mathcal{F}$ gives $|\mathcal{I}(\mathcal{F})| \leq rn^k|\mathcal{F}|$.

Transversal functions. Given a (not necessarily) Sperner hypergraph $\mathcal{H} \subseteq 2^V$ and a non-negative real weight $w(H) \in \mathbb{R}_+$ associated with each hyperedge $H \in \mathcal{H}$, let us call the function $f : 2^V \mapsto \mathbb{R}_+$, defined by

$$f(X) = \sum \{w(H) \mid X \cap H \neq \emptyset,\ H \in \mathcal{H}\}$$

a *(weighted) transversal function.* Note that such a function is submodular, but not generally polymatroid since the weights are not necessarily integral. In contrast to the bound of (1.4), the following stronger bound is known [22] for transversal functions.

Theorem 1.6 (Boros, Gurvich, Khachiyan, and Makino [22]) *Let $f_1, \ldots, f_r : 2^V \mapsto \mathbb{R}_+$ be r transversal functions defined by r hypergraphs $\mathcal{H}_1, \ldots, \mathcal{H}_r$. Let $\mathcal{F}$ be the set of all minimal feasible solutions for the system of inequalities (1.1) and let $\mathcal{X} \subseteq \mathcal{F}^+$ be an arbitrary subset of $\mathcal{F}^+$ of size $|\mathcal{X}| \geq 1$. Then, regardless of the weights,*

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq \sum_{i=1}^{r} \sum_{X \in \mathcal{X}} |\{H \in \mathcal{H}_i \mid H \cap X \neq \emptyset\}|. \tag{1.5}$$

In particular, it follows that $|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq r|\mathcal{H}_{max}| \cdot |\mathcal{X}|$, which for $\mathcal{X} = \mathcal{F}$ gives $|\mathcal{I}(\mathcal{F})| \leq r|\mathcal{H}_{max}| \cdot |\mathcal{F}|$, where $\mathcal{H}_{max}$ is the hypergraph with largest size.

Given a system (1.1), it will be enough to prove a bound of the form (1.3) for each inequality $f_i(X) \geq t_i$ individually. Then proving the required bound for the whole system becomes an easy corollary:

Proposition 1.2 *Let $\mathcal{F}_i$ be the family of all minimal feasible sets for the inequality $f_i(X) \geq t_i$, for $i = 1, \ldots, r$. Suppose it is known that $|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_i)| \leq \kappa_i(|\mathcal{X}|, |V|, \|\mathcal{O}_i\|)$, for some (quasi-) polynomials $\kappa_1, \ldots, \kappa_r$, and any $\emptyset \neq \mathcal{X} \subseteq \mathcal{F}_i^+$. Then for any non-empty subset $\mathcal{X}$ of the family of feasible sets for the system (1.1), we have*

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq \sum_{i=1}^{r} \kappa_i(|\mathcal{X}|, |V|, \|\mathcal{O}_i\|).$$

Proof. Consider any $Y \in \mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})$. Then $Y \in \mathcal{I}(\mathcal{F})$ implies that $Y \in \mathcal{I}(\mathcal{F}_i)$ for some $i \in \{1, \ldots, r\}$. Also $\mathcal{X} \subseteq \mathcal{F}^+$ means that any $X \in \mathcal{X}$ is feasible for the system and therefore is feasible for every inequality of the system, i.e., $\mathcal{X} \subseteq \mathcal{F}_i^+$. Thus we conclude that

$$\begin{aligned} |\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| &\leq |\mathcal{I}(\mathcal{X}) \cap \bigcup_{i=1}^{r} \mathcal{I}(\mathcal{F}_i)| \\ &\leq \sum_{i=1}^{r} |\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_i)| \leq \sum_{i=1}^{r} \kappa_i(|\mathcal{X}|, |V|, \|\mathcal{O}_i\|), \end{aligned}$$

from which the uniformly dual-boundedness of the whole system follows. □

Combining the above dual-bounding inequalities with Corollary 1.1, we obtain the following theorem on the complexity of generating minimal feasible solutions for monotone systems (1.1).

Theorem 1.7 *Consider a system of monotone inequalities (1.1) in which each function f_i is either polymatroid with (quasi-) polynomial range (in the number of variables), 2-monotonic, or transversal, and is specified by a (quasi-) polynomial-time evaluation oracle. Let $\mathcal{F}$ be the set of minimal feasible solutions of (1.1). Then problem $GEN(\mathcal{F}, \mathcal{X})$ can be solved in quasi-polynomial time, and hence all minimal feasible sets or the system (1.1) can be enumerated in incremental quasi-polynomial time.*

In contrast to Theorem 1.7, we have the following negative result:

Proposition 1.3 *There exist polymatroid inequalities $f(X) \geq t$, with polynomial-time computable left-hand side, for which problem $GEN(\mathcal{F}, \mathcal{X})$ is NP-hard for exponentially large t.*

Proof. The result follows from the following reduction from the so-called *relay cuts* enumeration problem in a relay circuit with two terminals. Let $G(U, E)$ be a graph with two distinguished vertices s and t. To each edge in E, is assigned a relay $i \in V$ from a given set of relays V (two or more distinct edges may be assigned identical relays). Let $\mathcal{F}$ be the family of all minimal *s-t relay cuts*, i.e., minimal subsets of relays that disconnect s and t. It is known that the problem of incrementally generating $\mathcal{F}$ is NP-hard, see [49]. For $X \subseteq V$, define $\phi(X)$ to be the number of (not necessarily simple) s-t paths of length $k = |U|$ in the circuit when only the relays in X are on. Then it is not difficult to see that $\phi(X)$ is a *supermodular* function (i.e., $\phi(X \cup Y) + \phi(X \cap Y) \geq \phi(X) + \phi(Y)$ holds for all $X, Y \subseteq V$), and hence the function $f(X) \stackrel{\text{def}}{=} \phi(V) - \phi(V \setminus X)$ is polymatroid of range at most k^k. Given $X \subseteq V$, we can compute $\phi(X)$ in polynomial time (this requires only computing the k^{th} power of the adjacency matrix of the graph obtained from G by deleting all edges whose relays $i \notin X$). This gives a polynomial time evaluation oracle for f. Finally, note that minimal s-t relay cuts are exactly the minimal feasible solutions $\mathcal{F}$ of the polymatroid inequality $f(X) \geq \phi(V)$. $\square$

1.4 Generalization to products of partially ordered sets

It will be necessary for some of the applications listed in Section 1.2 (for example, minimal integer solutions of a monotone system of linear inequalities, minimal infrequent elements of a database of quantitative or hierarchical attributes, p-inefficient points, packing points into boxes) to also consider natural generalizations of uniformly dual-bounded hypergraphs and the dualization problems to ideals defined over products of partially ordered sets (posets). Specifically, let $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ be the product of n posets. Let us use $\preceq$ to denote the precedence relation in $\mathcal{P}$ and also in $\mathcal{P}_1, \ldots, \mathcal{P}_n$, i.e., if $p = (p_1, \ldots, p_n) \in \mathcal{P}$ and $q = (q_1, \ldots, q_n) \in \mathcal{P}$, then $p \preceq q$ in $\mathcal{P}$ if and only if $p_1 \preceq q_1$ in $\mathcal{P}_1$, $p_2 \preceq q_2$ in $\mathcal{P}_2$, $\ldots$, and $p_n \preceq q_n$ in $\mathcal{P}_n$. For $\mathcal{A} \subseteq \mathcal{P}$, denote by $\mathcal{A}^+ = \{x \in \mathcal{P} \mid x \succeq a, \text{ for some } a \in \mathcal{A}\}$ and $\mathcal{A}^- = \{x \in \mathcal{P} \mid x \preceq a, \text{ for some } a \in \mathcal{A}\}$, the ideal and filter generated by $\mathcal{A}$. Any element in $\mathcal{P} \setminus \mathcal{A}^+$ is called *independent of* $\mathcal{A}$. Let $\mathcal{I}(\mathcal{A})$ be the set of all maximal independent elements for $\mathcal{A}$:

$$\mathcal{I}(\mathcal{A}) \stackrel{\text{def}}{=} \{p \in \mathcal{P} \mid p \notin \mathcal{A}^+ \text{ and } (q \in \mathcal{P},\ q \succeq p,\ q \neq p \ \Rightarrow\ q \in \mathcal{A}^+)\}.$$

Analogously, define $\mathcal{I}^{-1}(\mathcal{A})$ to be the set of minimal elements in $\mathcal{P} \setminus \mathcal{A}^-$. Note that for any $\mathcal{A} \subseteq \mathcal{P}$, we have the decomposition:

$$\mathcal{A}^+ \cap \mathcal{I}(\mathcal{A})^- = \emptyset, \qquad \mathcal{A}^+ \cup \mathcal{I}(\mathcal{A})^- = \mathcal{P}. \tag{1.6}$$

Call a subset $\mathcal{A} \subseteq \mathcal{P}$ an *antichain* if no two elements are comparable in $\mathcal{A}$. Given a monotone property π over elements of $\mathcal{P}$, described by an efficiently computable satisfiability oracle $\mathcal{O}_\pi$, we are again interested in incrementally generating the antichain $\mathcal{F}_\pi$ of minimal elements of $\mathcal{P}$ satisfying π:

GEN$(\mathcal{P}, \mathcal{F}_\pi, \mathcal{X})$**:** *Given an antichain $\mathcal{X} \subseteq \mathcal{F}_\pi \subseteq \mathcal{P}$, either find a new minimal satisfying element $x \in \mathcal{F}_\pi \setminus \mathcal{X}$, or prove that the given partial list is complete: $\mathcal{X} = \mathcal{F}_\pi$.*

The corresponding *joint* generation problem GEN$(\mathcal{P}, \mathcal{F}_\pi, \mathcal{I}(\mathcal{F}_\pi), \mathcal{X}, \mathcal{Y})$ can be analogously defined. As in the case of hypergraphs, we shall say that an antichain $\mathcal{F} \subseteq \mathcal{P}$ is *uniformly dual-bounded* if there exists a (quasi-) polynomial κ such that

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_\pi)| \leq \kappa(|\mathcal{X}|, \sum_{i=1}^{n} |\mathcal{P}_i|, \|\mathcal{O}_\pi\|) \tag{1.7}$$

for any nonempty subset $\mathcal{X} \subseteq \mathcal{F}_\pi$. To perform joint generation, we consider the following natural generalization of the hypergraph dualization problem to products of partially ordered sets:

DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$**:** *Given an antichain $\mathcal{A} \subseteq \mathcal{P}$ in a poset $\mathcal{P}$ and a subset of maximal independent elements $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$, either find a new maximal independent element $p \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$, or prove that $\mathcal{A}$ and $\mathcal{B}$ form a dual pair: $\mathcal{B} = \mathcal{I}(\mathcal{A})$.*

In Chapter 4, we consider a number of classes of posets for which this dualization problem can be solved in quasi-polynomial time. Specifically, we shall prove the following theorem.

Theorem 1.8 *Problem DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$ can be solved in poly$(n) + m^{o(\log m)}$ time if $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ is either the product of chains (totally ordered sets), the product of posets with acyclic precedence graphs of bounded degrees, or the product of lattices with bounded width, where $m = |\mathcal{A}| + |\mathcal{B}|$.*

Proposition 1.1 and Corollary 1.1 also generalize to antichains defined over products of partially ordered sets.

Proposition 1.4 *Problem GEN$(\mathcal{P}, \mathcal{F}_\pi, \mathcal{I}(\mathcal{F}_\pi), \mathcal{X}, \mathcal{Y})$ can be solved in time $\sum_{i=1}^{n} |\mathcal{P}_i|(poly(|\mathcal{A}|, |\mathcal{B}|) + T(\|\mathcal{O}_\pi\|)) + T_{dual}$ for any monotone property π defined by a satisfiability oracle $\mathcal{O}_\pi$, where $T(\|\mathcal{O}_\pi\|)$ is the worst-case running time of the oracle on any $x \in \mathcal{P}$, and T_{dual} denotes the time required to solve problem DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$.*

Proof. Let first consider two subroutines which can be defined for any antichain $\mathcal{F}_\pi \subseteq \mathcal{P}$. The first of these subroutines takes as input a vector $x \in \mathcal{F}_\pi^+$ and returns a minimal vector x^* in $\mathcal{F}_\pi^+ \cap \{x\}^-$. Such a vector $x^* = \min_{\mathcal{F}_\pi}(x)$ can, for instance, be computed by coordinate decent:

$$x_1^* \leftarrow \min\{y_1 \in \mathcal{P}_1 \mid (y_1, y_2, \ldots, y_{n-1}, y_n) \in \mathcal{F}_\pi^+ \cap \{x\}^-\},$$
$$x_2^* \leftarrow \min\{y_2 \in \mathcal{P}_2 \mid (x_1^*, y_2, \ldots, y_{n-1}, y_n) \in \mathcal{F}_\pi^+ \cap \{x\}^-\},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$x_n^* \leftarrow \min\{y_n \in \mathcal{P}_n \mid (x_1^*, x_2^*, \ldots, x_{n-1}^*, y_n) \in \mathcal{F}_\pi^+ \cap \{x\}^-\},$$

where $\min\{y_i \in \mathcal{P}_i \mid \cdots\}$ returns an arbitrary minimal element with the specified property. The second subroutine is to compute, for a given vector $x \in \mathcal{I}(\mathcal{F}_\pi)^-$, a maximal vector $x^* \in \mathcal{I}(\mathcal{F}_\pi)^- \cap x^+$. Similarly, this problem can be done by coordinate decent. It is clear that these two subroutines can be executed in at most $O(\sum_{i=1}^{n} |\mathcal{P}_i| \cdot T(\|\mathcal{O}_\pi\|))$ time.

Now the result stated in the proposition is obtained via the following algorithm.

Algorithm $\mathcal{J}$

Step 1. Check whether $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$. Since $\mathcal{B} \subseteq \mathcal{I}(\mathcal{F}_\pi)$ and $\mathcal{A} \subseteq \mathcal{F}_\pi$, each vector $x \in \mathcal{B}$ is independent of $\mathcal{A}$ and we only need to check the maximality of x for $\mathcal{I}(\mathcal{A})$. In other words, we have to check whether or not $y \geq \mathcal{A}^+$ for every immediate successor y of x. Since both $\mathcal{A}$ and $\mathcal{B}$ are explicitly given, this check can be done in $poly(\sum_{i=1}^n |\mathcal{P}_i|, |\mathcal{A}|, |\mathcal{B}|)$ comparisons. If there is an $x \in \mathcal{B} \setminus \mathcal{I}(\mathcal{A})$, then $x \notin \mathcal{F}_\pi^+$ because $x \in \mathcal{B} \subseteq \mathcal{I}(\mathcal{F}_\pi)$. This and the inclusion $\mathcal{A} \subseteq \mathcal{F}_\pi$ imply that $x \notin \mathcal{A}^+$. Since $x \notin \mathcal{I}(\mathcal{A})$, we can find an immediate successor $y \notin \mathcal{A}^+$ of x. By the maximality of x in $\mathcal{P} \setminus \mathcal{F}_\pi^+$, y belongs to $\mathcal{F}_\pi^+$. Now letting $y^* = \min_{\mathcal{F}_\pi}(y)$ we conclude that $y^* \in \mathcal{F}_\pi \setminus \mathcal{A}$, i.e., y^* is a new minimal vector in $\mathcal{F}_\pi$.

Step 2 is similar to the previous step: we check whether $\mathcal{A} \subseteq \mathcal{I}^{-1}(\mathcal{B})$. If $\mathcal{A}$ contains an element that is not minimal in $\mathcal{P} \setminus \mathcal{B}^-$, we can find a new vector in $\mathcal{I}(\mathcal{F}_\pi) \setminus \mathcal{B}$ and halt.

Step 3. Suppose that $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$ and $\mathcal{A} \subseteq \mathcal{I}^{-1}(\mathcal{B})$. Then

$$(\mathcal{A}, \mathcal{B}) = (\mathcal{F}_\pi, \mathcal{I}(\mathcal{F}_\pi)) \quad \Longleftrightarrow \quad \mathcal{B} = \mathcal{I}(\mathcal{A}).$$

To see this, assume that $\mathcal{B} = \mathcal{I}(\mathcal{A})$, and suppose on the contrary that there is an $x \in \mathcal{F}_\pi \setminus \mathcal{A}$. Since $x \notin \mathcal{A} = \mathcal{I}^{-1}(\mathcal{B})$ and $x \notin \mathcal{B}^- \subseteq \mathcal{I}(\mathcal{F}_\pi)^-$, there must exist a $y \in \mathcal{A} \subseteq \mathcal{F}_\pi$ such that $y \preceq x$. Hence we get two distinct elements $x, y \in \mathcal{F}_\pi$ such that $y \preceq x$, which contradicts the definition of $\mathcal{F}_\pi$. The existence of an $x \in \mathcal{I}(\mathcal{F}_\pi) \setminus \mathcal{B}$ leads to a similar contradiction.

To check the stopping criterion $\mathcal{B} = \mathcal{I}(\mathcal{A})$, we solve problem DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$. If $\mathcal{B} \neq \mathcal{I}(\mathcal{A})$, we obtain a new point $x \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$. By (1.6), either $x \in \mathcal{F}_\pi^+$, or $x \in \mathcal{I}(\mathcal{F}_\pi)^-$ and we can decide which of these two cases holds by calling the satisfiability oracle $\mathcal{O}_\pi$ on x. In the first case, we conclude that $x^* = \min_{\mathcal{F}_\pi}(x)$ is a new vector in $\mathcal{F}_\pi \setminus \mathcal{A}$. In the second case, we can extend $\mathcal{I}(\mathcal{F}_\pi) \setminus \mathcal{B}$ by using the second subroutine to compute a maximal vector in $x^+ \cap \mathcal{I}(\mathcal{F}_\pi)^-$. $\square$

Remark. Note that, if $c \in \mathbb{Z}_+$ is an integer vector and $\mathcal{P} = \{x \in \mathbb{Z}^n \mid 0 \leq x \leq c\}$ is an integral box, each of the n coordinate steps in the above procedures can be reduced via binary search to at most $\log(\|c\|_\infty + 1)$ membership tests for $\mathcal{F}_\pi^+$. If, furthermore, $\mathcal{F}_\pi = \mathcal{F}_{A,b,c}$ is the set of minimal integer solutions for an explicitly given monotone system (1.1) of linear inequalities, then both of the above coordinate descends can be clearly performed in $O(nr)$ comparisons and operations $+, -, \times, /, \lfloor\ \rfloor$, regardless of the box size. Thus we obtain a *strongly* polynomial-time reduction in this case.

Corollary 1.2 *Suppose $\mathcal{F}_\pi$ is uniformly dual-bounded as in (1.7) and defined by a (quasi-) polynomial time membership oracle for $\mathcal{F}_\pi^+$. Then problem GEN$(\mathcal{P}, \mathcal{F}_\pi, \mathcal{X})$ is (quasi-) polynomial-time reducible to at most $\kappa(|\mathcal{X}|, \sum_{i=1}^n |\mathcal{P}_i|, \|\mathcal{O}_\pi\|) + 1$ instances of problem DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$.*

Proof. Given a subset $\mathcal{X}$ of $\mathcal{F}_\pi$, we repeatedly run Algorithm $\mathcal{J}$, starting with $\mathcal{A} = \mathcal{X}$ and $\mathcal{B} = \emptyset$, until it either produces a new element in $\mathcal{F}_\pi \setminus \mathcal{X}$ or proves that $\mathcal{X} = \mathcal{F}_\pi$ by generating the entire family $\mathcal{I}(\mathcal{F}_\pi)$. By Step 1, either $\mathcal{B} \subseteq \mathcal{I}(\mathcal{X})$ is maintained during the execution of the algorithm, or a new element $x \in \mathcal{F}_\pi \setminus \mathcal{X}$ can be found. Thus, as long as Algorithm $\mathcal{J}$ outputs elements of $\mathcal{I}(\mathcal{F}_\pi)$, these elements also belong to $\mathcal{I}(\mathcal{X})$, and hence the total number of such elements does not exceed $\kappa(|\mathcal{X}|, \sum_{i=1}^n |\mathcal{P}_i|, \|\mathcal{O}_\pi\|)$. □

1.5 Contribution and organization of the thesis

The main theorem of this thesis is that all the generation problems described in Section 1.2 can be incrementally solved in quasi-polynomial time, providing strong evidence that none of them is likely to be NP-hard. The ingredients of this theorem will be presented in Chapters 2, 3 and 4. In Chapter 5, we study the parallel complexity of the hypergraph dualization problem for a special class of hypergraphs, and its generalization on products of lattices.

Here are the details. In Chapter 2, we prove the uniform dual-boundedness, Theorem 1.2, of the hypergraph of minimal feasible sets of a system of polymatroid inequalities. We complement this result with a lower bound on the size of the maximal infeasible sets showing that, in general (and also in the case of rank functions defined on the subspaces of some linear space), inequality (1.4) is accurate within a factor of 1.81 in the exponent. The inequality will be also generalized to a slightly weaker bound for polymatroid functions defined on products of arbitrary lattices. These results appear also in [14, 18].

In Chapter 3, we prove the stronger inequalities of Theorems 1.4 and 1.5 along with some generalizations. These immediately provide the required inequalities for bounding the number of maximal infeasible solutions of linear systems and the number of minimal feasible sets for matroid intersections respectively. Using an intersection inequality from [22], we then derive the necessary inequalities for the p-inefficient points and the points-boxes packing applications (all of which are in fact monotone submodular functions, but with possibly continuous range). Other examples of polymatroid inequalities will be also given, including the infrequent sets, the space covering, and the graph connectivity problems. We shall also consider some special cases of systems with non-linear inequalities in integer variables. Some of these results appear in [14, 15, 16, 17].

As pointed out in Section 1.4, it is necessary for some of the above applications to consider the dualization problem on products of partially ordered sets. In Chapter 4, we consider this problem and present quasi-polynomial time algorithms for the classes of posets described in Theorem 1.8. These results appear in [41, 42].

Following that, we show in Chapter 5 that a certain special case of the dualization problem, and therefore, of some of the above generation problems, can be solved efficiently in parallel. This special case includes for example, the generation of bounded integer solutions of a monotone system of linear inequalities when each inequality contains only a fixed number of non-zero coefficients, and the generation of t-infrequent elements of a database of binary or lattice attributes when t, the degree of each element in the lattice, and the number of non-essential attributes in each transaction of the database are bounded. The common feature exhibited by both of these examples is that the dual hypergraph (or family) of the minimal satisfying sets is of *bounded dimension* (i.e., hyperedge size), and the common result is that the corresponding generation problems belong to RNC (see also [13]).

Finally, we conclude with some open problems in Chapter 6.

1.6 Some related work

The generation of maximal independent sets in graphs is a well-studied problem [56, 80, 94]. The more general problem of generating all maximal independent sets for an independence system was shown to be NP-hard in [65]. Some special classes of independence systems were also discussed in the same paper, along with some corresponding generation algorithms. In particular, an algorithm to generate maximal feasible solutions to a single knapsack inequality was described, and it was furthermore conjectured that, for linear systems, generating minimal Boolean solutions cannot be done in polynomial time unless P=NP. An algorithm was also given, to generate all maximal independent sets in the intersection of r matroids, whose running time was exponential in r. In contrast, our results imply that these two problems can be solved in quasi-polynomial time.

Other algorithms for listing combinatorial structures, described by polymatroid inequalities (e.g., cycles, paths and spanning trees of a graph) can be found in [88]. Some other related generation problems appear in [57, 85, 86]. The complexity of counting the number of minimal solutions for linear systems (1.1) is studied in [38].

The use of the hypergraph dualization problem to incrementally generate monotone structures in different applications was suggested in several places, e.g. [11, 39, 49]. The work on dual-bounding inequalities originated in [19, 20, 22] which contain the bounds of Theorems 1.3, 1.6 for modular and transversal functions, and prove the quasi-polynomial time generation, in the *Boolean* case, of the feasible solutions of monotone linear systems, and the minimal infrequent sets. The case of a single Boolean 2-monotonic inequality is discussed in [10, 24, 30, 51, 73, 82, 83].

Some related inequalities are given in [33, 34] which characterize minor-closed classes of matroids and graphs, for which, respectively, the number of bases can be polynomially bounded by the size of the ground set, and the number of circuits can be polynomially bounded by the number of edges.

Finally, we remark that the special case of the polymatroid inequality (1.4), when the Sperner hypergraph $\mathcal{F} = E(G)$ is the edge set of a connected graph G and $\mathcal{X} = \mathcal{I}(\mathcal{F})$, can be derived from results by Balas and Yu [6] on the number of maximal independent sets of a graph, (see also [4, 81] and Section 2.2 of the next chapter).

Chapter 2

Dual-bounding Inequalities for Polymatroid Functions

2.1 Introduction

Let V be a finite set of cardinality $|V| = n$, let $f : 2^V \mapsto \mathbb{Z}_+$ be a set-function taking non-negative integral values, and let $r = range(f) = \max\{f(X) \mid X \subseteq V\}$ denote the *range* of f. Recall that a set-function f is called a *polymatroid function* if it is monotone, submodular, i.e.,

$$f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y) \tag{2.1}$$

holds for all subsets $X, Y \subseteq V$, and $f(\emptyset) = 0$. Let us remark that in general, determining the range of a submodular set function may be an NP-hard problem, while $range(f) = f(V)$ holds for polymatroid functions.

Given a polymatroid function $f : 2^V \mapsto \{0, 1, \ldots, r\}$, and an integral threshold $t \in \{1, ..., r\}$ let us denote by $\mathcal{F}_t = \mathcal{F}_t(f)$ the family of all minimal subsets $X \subseteq V$ for which $f(X) \geq t$, and analogously, let us denote by $\mathcal{A}_t = \mathcal{A}_t(f)$ the family of all maximal subsets $X \subseteq V$ for which $f(X) < t$. It is easy to see that $\mathcal{A}_t = \mathcal{I}(\mathcal{F}_t)$, where $\mathcal{I}(\cdot)$ denotes the family of all maximal independent sets for the hypergraph $(\cdot)$. Throughout this chapter we shall use the notation $\alpha = |\mathcal{A}_t(f)|$ and $\beta = |\mathcal{F}_t(f)|$.

The main result of this chapter is an inequality limiting the size of $\mathcal{A}_t(f)$ in terms of n and the size of $\mathcal{F}_t(f)$ for any integral threshold t:

Theorem 2.1 *For every polymatroid function f and threshold $t \in \{1, \ldots, range(f)\}$ such that $\beta \geq 2$ we have the inequality*

$$\alpha \leq \beta^{(\log t)/c(n,\beta)}, \tag{2.2}$$

where $c(n, \beta)$ is the unique positive root of the equation

$$2^c(n^{c/\log\beta} - 1) = 1. \tag{2.3}$$

In addition, $\alpha \leq n$ holds if $\beta = 1$.

Let us first remark that by (2.3), $1 = n^{-c/\log\beta} + (n\beta)^{-c/\log\beta} \geq 2(n\beta)^{-c/\log\beta}$, and hence $\beta^{1/c(n,\beta)} \leq n\beta$. Consequently, for $\beta \geq 2$ (in which case $n \geq 2$ is implied, too) we can replace (2.2) by the simpler but weaker inequality

$$\alpha \leq (n\beta)^{\log t}. \tag{2.4}$$

In fact, (2.4) holds even in case of $\beta = 1$, because if the hypergraph $\mathcal{F}_t$ consists only of a single hyperedge $X \subseteq V$, then $|\mathcal{A}_t| \leq |X| \leq n$ follows immediately by the relation $\mathcal{A}_t = \mathcal{I}(\mathcal{F}_t)$. On the other hand, for large β the bound of Theorem 2.1 becomes increasingly stronger than (2.4). For instance, $c(n, n) = \log(1 + \sqrt{5}) - 1 > .694$, $c(n, n^2) > 1.102$, and $c(n, n^\sigma) \sim \log\sigma$ for large σ.

Let us remark next that the bound of Theorem 2.1 is reasonably sharp. As we shall show in Section 2.3, for any positive integers k and l there exists a polymatroid function f of range $r = 2^k$ for which $n = kl$, $|\mathcal{A}_r| = l^k$, and $|\mathcal{F}_r| = kl(l-1)/2$. Thus, letting $t = r$ and $l = 2^k$, we obtain an infinite family of polymatroid functions for which

$$\alpha \geq \beta^{(.551\log t)/c(n,\beta)} \quad \text{and} \quad \alpha \geq (n\beta)^{(\frac{1}{3}+o(1))\log t}, \tag{2.5}$$

as $t = r \to \infty$, see Section 2.3 for more detail. In Section 2.3 we also show that our lower bounds (2.5) can be achieved within the subclass of rank functions defined on the subsets of some linear space. Namely, we can construct kl subspaces $V_{ij} \subseteq \mathbb{R}^{2^k}$, $i = 1, \ldots, k$, $j = 0, \ldots, l-1$ of dimension 2^{k-1} each, such that for any i and $j \neq j'$ we have $\dim(V_{ij} \cup V_{ij'}) = 2^k$, while for every $(j_1, j_2, ..., j_k) \in \{0, 1, ..., l-1\}^k$ the inequality $\dim(\bigcup_{i=1}^k V_{ij_i}) < 2^k$ holds.

Let us finally note that for many classes of polymatroid functions, β cannot be bounded by a quasi-polynomial estimate of the form $(n\alpha)^{\mathrm{polylog}(r)}$. Let us consider for instance, a graph $G = t \times K_2$ consisting of t disjoint edges, and let $f(X)$ be the number of edges X intersects, for $X \subseteq V(G)$. Then f is a polymatroid function of range $r = t$, and we have $n = 2t$, $\alpha = |\mathcal{A}_t| = t$ and $\beta = |\mathcal{F}_t| = 2^t$.

We will strengthen Theorem 2.1 as follows. Given a non-empty hypergraph $\mathcal{H}$ on the vertex set V, a polymatroid function $f : 2^V \mapsto \mathbb{Z}_+$, and a integral positive threshold t, the pair (f, t) is called a *polymatroid separator* for $\mathcal{H}$ if $f(H) \geq t$ for all $H \in \mathcal{H}$.

Theorem 2.2 *Let (f, t) be a polymatroid separator for a hypergraph $\mathcal{H}$ of cardinality $|\mathcal{H}| \geq 2$. Then*

$$|\mathcal{A}_t(f) \cap \mathcal{I}(\mathcal{H})| \leq |\mathcal{H}|^{(\log t)/c(n,|\mathcal{H}|)}, \tag{2.6}$$

where $\mathcal{I}(\mathcal{H})$ is the family of all maximal independent sets for $\mathcal{H}$.

In particular, if (f, t) is a polymatroid separator for a non-empty hypergraph $\mathcal{H}$, then $|\mathcal{A}_t(f) \cap \mathcal{I}(\mathcal{H})| \leq (n|\mathcal{H}|)^{\log t}$.

Clearly, Theorem 2.1 is a special case of Theorem 2.2 for $\mathcal{H} = \mathcal{F}_t(f)$. Since the right-hand side of (2.6) monotonically increases with $|\mathcal{H}|$, we will only need to prove Theorem 2.2 for *Sperner* hypergraphs $\mathcal{H}$, i.e., under the assumption that none of the hyperedges of $\mathcal{H}$ contains another hyperedge of $\mathcal{H}$. It is also clear that Theorem 1.2 of the previus chapter is a corollary of Theorem 2.2 and Proposition 1.2.

Let $\mathcal{H}$ be a non-empty Sperner hypergraph on the vertex set V. A polymatroid separator (f, t) is called *exact* for $\mathcal{H}$ if $\mathcal{H} = \mathcal{F}_t(f)$ and consequently, $\mathcal{I}(\mathcal{H}) = \mathcal{A}_t(f)$. It is easy to see that any Sperner hypergraph $\mathcal{H}$ has an exact polymatroid separator (f, t) such that $range(f) = t = |\mathcal{I}(\mathcal{H})|$ holds. For instance, defining for a subset $X \subseteq V$ the value $f(X)$ as the number of minimal transversals of $\mathcal{H}$ intersecting X yields a polymatroid function which forms, with $t = range(f) = |\mathcal{I}(\mathcal{H})|$, an exact polymatroid separator for $\mathcal{H}$. It is much less clear how large the range of *any* exact polymatroid separator for $\mathcal{H}$ should be. Theorem 2.2 partially answers this question, since by (2.6), for exact separators (f, t), we must have $range(f) \geq t \geq |\mathcal{I}(\mathcal{H})|^{c(n,|\mathcal{H}|)/\log|\mathcal{H}|}$ whenever $|\mathcal{H}| \geq 2$. Consequently, hypergraphs with many maximal independent sets cannot be defined by polymatroid separators of small range.

Finally, we shall also consider a generalization of Theorem 2.2 above for polymatroid functions over products of lattices, and show that a slightly weaker version of this theorem holds in that case.

2.2 Proper mappings of independent sets into binary trees

Our proof of Theorem 2.2 makes use of a combinatorial construction which may be of independent interest. Theorem 2.2 states that for any polymatroid separator (f,t) of a hypergraph $\mathcal{H}$ we have

$$range(f) \geq t \geq |\mathcal{S}|^{c(n,|\mathcal{H}|)/\log(|\mathcal{H}|)},$$

where $\mathcal{S} = \mathcal{I}(\mathcal{H}) \cap \{X \mid f(X) < t\}$, i.e., the range of f must increase with the size of $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$. Thus, to prove the theorem we must first find ways to provide lower bounds on the range of a polymatroid function. To this end we shall show that the number of independent sets which can be organized in a special way into a binary tree structure provides such a lower bound.

Let $\mathbf{T}$ denote a binary tree, $V(\mathbf{T})$ denote its node set, and let $L(\mathbf{T})$ denote the set of its leaves. Clearly, every leaf is a node of degree 1, while all other nodes are of degree 3, except a unique node s of degree 2, which is called the *root* of $\mathbf{T}$. For every node $v \in V(\mathbf{T})$, let $\mathbf{T}(v)$ be the binary sub-tree rooted at v. Obviously, for every two nodes u, v of $\mathbf{T}$ either the sub-trees $\mathbf{T}(u)$ and $\mathbf{T}(v)$ are disjoint, or one of them is a sub-tree of the other. The nodes u and v are called *incomparable* in the first case, and *comparable* in the second case.

Given a Sperner hypergraph $\mathcal{H}$ and a binary tree $\mathbf{T}$, let us consider mappings $\phi : L(\mathbf{T}) \mapsto \mathcal{I}(\mathcal{H})$ assigning maximal independent sets $I_\ell \in \mathcal{I}(\mathcal{H})$ to the leaves $\ell \in L(\mathbf{T})$. Let us associate furthermore to every node $v \in V(\mathbf{T})$ the intersection $S_v = \bigcap_{\ell \in L(\mathbf{T}(v))} I_\ell$. Let us call finally the mapping ϕ *proper* if it is injective, i.e., assigns different independent sets to different leaves, and if the sets $S_u \cup S_v$ are not independent whenever u and v are incomparable nodes of $\mathbf{T}$. Let us point out that the latter condition means that the set $S_u \cup S_v$, for incomparable nodes u and v, must contain a hyperedge $H \in \mathcal{H}$, as a subset. Since the intersection of independent sets is always independent, it follows, in particular that both S_v and S_u are non-empty independent sets (otherwise their union could not be non-independent.) Finally, since all non-root nodes $u \in V(\mathbf{T})$ have at least one incomparable node $v \in V(\mathbf{T})$, we get that the sets S_u, for $u \in V(\mathbf{T}) \setminus \{s\}$ are all non-empty and independent.

Lemma 2.1 *Let us consider a Sperner hypergraph $\mathcal{H}$ and a polymatroid separator (f,t) of it, and let us denote by $\mathcal{S}$ the subfamily of maximal independent sets, separated by (f,t) from $\mathcal{H}$. Let* *us assume further that* $\mathbf{T}$ *is a binary tree for which there exists a proper mapping* $\phi : L(\mathbf{T}) \mapsto \mathcal{S}$. *Then, we have*

$$range(f) \geq t \geq |L(\mathbf{T})|. \tag{2.7}$$

Let us note that if a proper mapping exists for a binary tree $\mathbf{T}$, then we can associate a hyperedge $H_u \in \mathcal{H}$ to every node $u \in V(\mathbf{T}) \setminus L(\mathbf{T})$ in the following way: Let v and w be the two successors of u. Since v and w are incomparable, the union $S_v \cup S_w$ must contain a hyperedge from $\mathcal{H}$. Let us choose such a hyperedge, and denote it by H_u. Let us observe next that if $\ell \in L(\mathbf{T}(v))$ and $\ell' \in L(\mathbf{T}(w))$, then $S_v \subseteq I_\ell$ and $S_w \subseteq I_{\ell'}$, and thus $H_u \subseteq I_\ell \cup I_{\ell'}$. In other words, to construct a large binary tree for which there exists a proper mapping, we have to find a way of splitting the family of independent sets, repeatedly, such that the union of any two independent sets, belonging to different parts of the split must contain the same hyperedge of $\mathcal{H}$. We shall show next that, indeed, such a construction is possible.

Lemma 2.2 *For every Sperner hypergraph* $\mathcal{H} \subseteq 2^V$, $|\mathcal{H}| \geq 2$, *and for every subfamily* $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ *of its maximal independent sets there exists a binary tree* $\mathbf{T}$ *and a proper mapping* $\phi : L(\mathbf{T}) \mapsto \mathcal{S}$, *such that*

$$|L(\mathbf{T})| \geq |\mathcal{S}|^{c(n,|\mathcal{H}|)/\log|\mathcal{H}|}, \tag{2.8}$$

where $n = |V|$.

Clearly, Lemmas 2.1 and 2.2 imply Theorem 2.2, which in turn implies Theorem 2.1. Lemmas 2.1 and 2.2 will be proved in Section 2.4.

Let us remark that in the special case when the Sperner hypergraph $\mathcal{H} = E(G)$ is the edge set of a connected graph G and $\mathcal{S} = \mathcal{I}(\mathcal{H})$, Lemma 2.2 can be viewed as an extension of a result of Balas and Yu [6] (see also [4, 81]): Theorem 4 of [6] claims that

$$2^p \leq |\mathcal{I}(\mathcal{H})| \leq \delta^p + 1, \tag{2.9}$$

where δ is the number of pairs of vertices in V at distance 2 (in particular, $\delta < n^2/2$), and p is the cardinality of a maximum induced matching in G. Any such matching can be used to construct a proper mapping $\phi : L(\mathbf{T}) \mapsto \mathcal{I}(\mathcal{H})$ for a uniform binary tree $\mathbf{T}$ of depth p, i.e., for which $|L(\mathbf{T})| = 2^p$. Namely, let $e_i = (v_0^i, v_1^i)$ for $i = 1, ..., p$ denote the edges of the induced matching. For each binary vector $x = (x_1, x_2, ..., x_p) \in \{0,1\}^p$ let us associate a subset $\tilde{I}_x$ defined by $\tilde{I}_x = \{v_{x_i}^i \mid i = 1, ..., p\}$. Finally, let $I_x \in \mathcal{I}(\mathcal{H})$ be a maximal independent set of G containing $\tilde{I}_x$, for $x \in \{0,1\}^p$. (Since the edges e_1, ..., e_p form an induced matching, all sets $\tilde{I}_x$, $x \in \{0,1\}^p$ are independent.) Thus, the sets I_x, $x \in \{0,1\}^p$ are pairwise distinct maximal independent sets of G by the above construction, and $e_i \subseteq I_x \cup I_y$ whenever $x_i \neq y_i$. Naturally, the leaves of the binary tree $\mathbf{T}$ of depth p are also encoded by the binary sequences of length p. Thus a mapping ϕ is defined by assigning I_x to the leaf coded with x, and it is not difficult to verify that this mapping is proper (and that the edge e_i will be the associated hyperedge at each node of $\mathbf{T}$ at depth $i-1$, for $i = 1, ..., p$).

By these definitions, $\log |L(\mathbf{T})| = p$ and $\log |\mathcal{I}(\mathcal{H})| \leq p \log \delta$, according to (2.9), and hence

$$|L(\mathbf{T})| \geq |\mathcal{I}(\mathcal{H})|^{1/\log \delta} \geq |\mathcal{I}(\mathcal{H})|^{1/(2\log n)},$$

whereas the bound of Lemma 2.2 gives:

$$|L(\mathbf{T}) \geq |\mathcal{I}(\mathcal{H})|^{c(n,|\mathcal{H}|)/\log |\mathcal{H}|}.$$

For this reason, Lemma 2.2 can be viewed as an extension of Theorem 4 of [6] from graphs to hypergraphs (even though we could not generalize directly the notion of an induced matching).

Let us remark here that according to the above results, the existence of a "large" induced matching in a graph, or more generally, the existence of a "large" binary tree with a proper mapping in a hypergraph can be viewed as the reason for the existence of "many" maximal independent sets. These reasons, however, may not be easy to exhibit for a given graph or hypergraph. The problem of finding the maximum size induced matching in a graph is known to be NP-hard (see e.g. [92]) and it is even hard to approximate it well (see [37]). The complexities of the corresponding problems of finding or approximating well the largest binary tree with a proper mapping are open.

2.3 A lower bound for $|\mathcal{A}_t|$

In this section we demonstrate that inequality (2.2) of Theorem 2.1 is reasonably tight.

2.3.1 A hypergraph example

In our first example, let $\mathcal{H}$ be the edge set of the graph $G = k \times K_l$ consisting of k pairwise disjoint copies of a clique on l vertices. In this (hyper)graph, the number of vertices is $n = |V| = kl$, the number of (hyper)edges is $|\mathcal{H}| = k\binom{l}{2}$, the number of maximal independent sets is $|\mathcal{I}(\mathcal{H})| = l^k$, and we can prove the following statement.

Lemma 2.3 *For the hypergraph $\mathcal{H}$ defined above, there exists an exact polymatroid separator (f, t), such that $t = range(f) = 2^k$.*

Proof. For $X \subseteq V$, define $f(X)$ by

$$f(X) = \begin{cases} 2^k, & \text{if } X \text{ contains an edge of } \mathcal{H} \\ 2^k - 2^{\gamma(X)}, & \text{otherwise,} \end{cases}$$

where $\gamma(X)$ denotes the number of l-cliques of G disjoint from X. In particular, $f(X) = 2^k - 1$ if (and only if) X is a maximal independent set of $\mathcal{H}$, and $f(X) = 2^k$ if X contains an edge of $\mathcal{H}$. Let us also note that f is obviously monotone, by the above definition. Thus, with $t = 2^k$ the pair (f, t) is indeed an exact separator of $\mathcal{H}$.

It remains to show that f is submodular. For this, let X and Y be two arbitrary subsets of the vertex set V. If both X and Y contain an edge of $\mathcal{H}$, then (2.1) holds trivially, since we have 2^{k+1} on the right hand side, and we have $f(Z) \leq 2^k$ for all subsets $Z \subseteq V$ by definition. Furthermore, if one of these sets contains an edge of $\mathcal{H}$, say X, then of course $X \cup Y$ does too, and hence (2.1) reduces to $f(X \cap Y) \leq f(Y)$ which holds again trivially by the monotonicity of f. Let us assume next that neither X nor Y contain an edge of $\mathcal{H}$, but $X \cup Y$ does. In this case $\gamma(X \cap Y) \geq 1 + \max\{\gamma(X), \gamma(Y)\}$ holds, implying $2^{\gamma(X \cap Y)} \geq 2^{\gamma(X)} + 2^{\gamma(Y)}$, from which (2.1) follows. Let us assume finally that $X \cup Y$ does not contain an edge of $\mathcal{H}$. In this case $\gamma(X \cap Y) \geq \max\{\gamma(X), \gamma(Y)\}$ and $\gamma(X \cap Y) + \gamma(X \cup Y) \geq \gamma(X) + \gamma(Y)$ both hold, implying $2^{\gamma(X \cap Y)} + 2^{\gamma(X \cup Y)} \geq 2^{\gamma(X)} + 2^{\gamma(Y)}$, from which (2.1) follows again. $\square$

The above lemma now implies that with $t = range(f) = 2^k$ we have $\beta = |\mathcal{H}| = k\binom{l}{2}$, $\alpha = l^k$ and $n = |V| = kl$. For $l = 2^k$ and $k \to \infty$, we thus obtain $\log n / \log \beta \to 1/2$ and hence $c(n, \beta)$

converges to the root of the equation $2^c(2^{c/2}-1)=1$. This gives $c = 1.102...$, and consequently

$$\alpha > \beta^{(.551 \log t)/c(n,\beta)}$$

for k sufficiently large.

2.3.2 A rank function example

Let us next show that the polymatroid function f defined above can be realized as the rank function of some linear subspaces of the vector space $\mathbf{F}^r$, $r \in \mathbb{Z}_+$ over a (possibly large) field $\mathbf{F}$.

For a positive integer l let $\mathbf{F}$ be a field with $l \leq |\mathbf{F}|$ (we shall use, as customary, $+$ and $\times$ to denote the two field operations, and we write 0 and 1 for the unit elements of these operations, respectively). Furthermore, let $n = kl$, let $G = k \times K_l$ be the graph, as above, and let $\mathcal{H}$ be again the edge set of G. Let us introduce the notations $K = \{1, 2, ..., k\}$ and $L = \{0, 1, ..., l-1\}$, and let us denote the vertex set of G by $V = K \times L$. We shall associate to each vertex $(i,j) \in V$ a linear subspace V_{ij} of $\mathbf{F}^r$, where $r = 2^k$. These subspaces will be chosen in such a way that every two subspaces corresponding to the same clique of G intersect only in the origin (and hence generate the whole space $\mathbf{F}^r$), while the intersection of arbitrary s subspaces ($1 \leq s \leq k$), each corresponding to distinct cliques of G, is of dimension 2^{k-s}.

Let $\{b_x \mid x \in \{0,1\}^k\}$ be an arbitrary basis in $\mathbf{F}^r$, indexed by the $r = 2^k$ elements of the binary cube of dimension k, and let $\lambda_0 = 0$, $\lambda_1 = 1$, $\lambda_2 \ldots, \lambda_{l-1}$ be distinct elements of $\mathbf{F}$ (hence the requirement $l \leq |\mathbf{F}|$). For every $z \in L^k$, and every index vector $x \in \{0,1\}^k$, define the product

$$\Lambda_z(x) = \prod_{i:x_i=1} \lambda_{z_i} \prod_{i:x_i=0} (1-\lambda_{z_i}).$$

It is easily verified that $\sum_{x\in\{0,1\}^k} \Lambda_z(x) = 1$ for all $z \in L^k$, and that for any two binary vectors $x, y \in \{0,1\}^k$, we have $\Lambda_x(y) = 1$ if $x = y$, and $\Lambda_x(y) = 0$ otherwise. Let us now associate a (unique) vector

$$b_z = \sum_{x\in\{0,1\}^k} \Lambda_z(x) b_x \tag{2.10}$$

of $\mathbf{F}^r$ to every $z \in L^k$. Observe that for $z = x \in \{0,1\}^k$, we get a basis element $b_z = b_x$ by our selection of $\lambda_0 = 0, \lambda_1 = 1$.

Let us next define the linear subspace V_{ij}, for $(i,j) \in V$, to be the subspace generated by the vectors $b_z \in \mathbf{F}^r$ whose index vector z has value j in its i-th coordinate:

$$V_{ij} = \langle b_z \mid z_i = j \rangle.$$

We will show below that this construction has the announced properties. To simplify notation, we shall need a few more definitions.

For index vectors $x, y \in L^k$, and a subset $S \subseteq K$, we denote by $x[S]$ the restriction of x to S, by $y[\overline{S}]$ the restriction of y to the complementary set $\overline{S} = K \setminus S$, and by $z = x[S], y[\overline{S}]$ the vector defined by

$$z_j = \begin{cases} x_j & \text{if } j \in S, \\ y_j & \text{if } j \notin S. \end{cases}$$

Let us note that $\Lambda_{a[S],b[\overline{S}]}(x[S], y[\overline{S}]) = \Lambda_{a[S]}(x[S])\Lambda_{b[\overline{S}]}(y[\overline{S}])$ holds by the above definitions, for all $a, b \in L^k$, $x, y \in \{0,1\}^k$ and $S \subseteq K$.

Lemma 2.4 *For any $S \subseteq K$ and $w = (j_i \mid i \in S) \in L^S$, the set of vectors*

$$\left\{ \sum_{y \in \{0,1\}^S} \Lambda_w(y) b_{y,x} \;:\; x \in \{0,1\}^{\overline{S}} \right\} \tag{2.11}$$

forms a basis for the vector space $V_{S,w} \stackrel{\text{def}}{=} \langle b_z \mid z[S] = w \rangle$. In particular, $\dim(V_{S,w}) = 2^{k-|S|}$.

Proof. First, let us observe that for each $x \in \{0,1\}^{\overline{S}}$, the vector $a = \sum_{y \in \{0,1\}^S} \Lambda_w(y) b_{y,x}$ lies in the space $V_{S,w}$ since $a = b_z$ with $z[S] = w$ and $z[\overline{S}] = x$. Let us observe next that these vectors are linearly independent: suppose, on the contrary, that there exist scalars $\mu_x \in \mathbf{F}$, $x \in \{0,1\}^{\overline{S}}$, not all zero, such that

$$0 = \sum_{x \in \{0,1\}^{\overline{S}}} \mu_x \left(\sum_{y \in \{0,1\}^S} \Lambda_w(y) b_{y,x} \right).$$

Then, by the linear independence of the basis $\{b_x \mid x \in \{0,1\}^k\}$ of $\mathbf{F}^r$, we obtain that

$$\mu_x \Lambda_w(y) = 0, \quad \text{for all} \quad x \in \{0,1\}^{\overline{S}} \quad \text{and} \quad y \in \{0,1\}^S. \tag{2.12}$$

But summing up equations (2.12) for a particular $x \in \{0,1\}^{\overline{S}}$ over all $y \in \{0,1\}^S$, and using $\sum_{y \in \{0,1\}^S} \Lambda_w(y) = 1$, we get $\mu_x = 0$, for all $x \in \{0,1\}^{\overline{S}}$, a contradiction, proving that (2.11) is indeed a family of linearly independent vectors. Let us note finally that these vectors span the entire subspace $V_{S,w}$, since any vector b_z with $z[S] = w$ in this subspace can be written as:

$$\begin{aligned} b_z &= \textstyle\sum_{u \in \{0,1\}^k} \Lambda_z(u) b_u \\ &= \textstyle\sum_{u \in \{0,1\}^k} \Lambda_{z[\overline{S}]}(u[\overline{S}]) \Lambda_{z[S]}(u[S]) b_u \\ &= \textstyle\sum_{x \in \{0,1\}^{\overline{S}}} \Lambda_{z[\overline{S}]}(x) \left(\sum_{y \in \{0,1\}^S} \Lambda_w(y) b_{y,x} \right). \end{aligned}$$

The lemma follows from the above observations. □

For $z', z'' \in L^k$, let us denote by $[z', z'']$ the set of all those vectors $z \in L^k$ for which $z_i \in \{z'_i, z''_i\}$ for $i = 1, \ldots, k$.

Lemma 2.5 *Let $z', z'' \in L^k$ be such that $z'_i \neq z''_i$ for all $i \in K$. Then the set $B_{z',z''} \stackrel{\text{def}}{=} \{b_z \mid z \in [z', z'']\}$ forms a basis for $\mathbf{F}^r$.*

Proof. Let $M_{z',z''} \stackrel{\text{def}}{=} (\Lambda_z(x))_{x,z}$ be the $2^k \times 2^k$-matrix whose rows are indexed by the vectors $x \in \{0,1\}^k$, and whose columns are indexed by the vectors $z \in [z', z'']$. To prove that the set $B_{z',z''}$ is linearly independent, it is enough by (2.10) to show that the matrix $M_{z',z''}$ is non-singular. Indeed, we claim that

$$|\det(M_{z',z''})| = \prod_{i=1}^{k} (\lambda_{z'_i} - \lambda_{z''_i})^{2^{k-1}}, \tag{2.13}$$

from which the lemma will follow by the distinctness of $\lambda_0, \lambda_1, \ldots, \lambda_{l-1}$. To prove (2.13), we first observe that the left hand side is a polynomial in $\mathbf{F}[\lambda_{z'_1}, \ldots, \lambda_{z'_k}, \lambda_{z''_1}, \ldots, \lambda_{z'_k}]$, of degree 2^{k-1} in each variable λ_{z_i}. Let $i \in K$ and let $u, v \in L^k$ be such that $u_i = z'_i$, $v_i = z''_i$, and $u[K \setminus \{i\}] = v[K \setminus \{i\}] = w \in L^{K \setminus \{i\}}$. Then for any $x \in \{0,1\}^k$, it is easy to see that $\Lambda_u(x) - \Lambda_v(x) = (-1)^{x_i} (\lambda_{z'_i} - \lambda_{z''_i}) \Lambda_w(x[K \setminus \{i\}])$. In particular, if we subtract the two columns of $M_{z',z''}$ indexed by u, v, we obtain $\lambda_{z'_i} - \lambda_{z''_i}$ as a factor for the determinant expression in (2.13). Repeating this argument for every $i \in K$ and every $w \in L^{K \setminus \{i\}}$, we conclude that the right hand side of (2.13) is a divisor of the left hand side. Since both polynomials are of the same degree in all variables by our earlier observation, and since they attain the same value at, say, $z' = (0, \ldots, 0)$, $z'' = (1, \ldots, 1)$, (2.13) follows. □

Lemma 2.6 *For all $i \in K$, and for all $j, j' \in L$, $j \neq j'$, the subspaces V_{ij} and $V_{ij'}$ span the entire space $\mathbf{F}^r$, i.e., $\dim(V_{ij} \cup V_{ij'}) = 2^k$.*

Proof. Let $z', z'' \in L^k$ be such that $z'_m \neq z''_m$ for all $m \in K$, $z'_i = j$, and $z''_i = j'$. Since the basis set $B_{z',z''}$ is contained in $V_{ij} \cup V_{ij'}$, the lemma follows. $\square$

Lemma 2.7 *Let $z', z'' \in L^k$ be such that $z'_i \neq z''_i$ for all $i \in K$. Then for any $i \in K$, we have $V_{i,z'_i} = \langle b_z \mid z \in [z', z''],\ z_i = z'_i \rangle$.*

Proof. From Lemma 2.4, we have $\dim(V_{i,z'_i}) = 2^{k-1}$, and from Lemma 2.5, the set $\{b_z \mid z \in [z', z''],\ z_i = z'_i\}$ is linearly independent. Since this set is contained in V_{i,z'_i} by definition, the lemma follows. $\square$

Lemma 2.8 *For $S \subseteq K$ and $w = (j_i \mid i \in S) \in L^S$, we have*

$$\dim(\bigcup_{i \in S} V_{i,j_i}) = 2^k - 2^{k-|S|}. \tag{2.14}$$

Proof. Fix $z', z'' \in L^k$ such that $z'_i \neq z''_i$ for all $i \in K$, and $z'[S] = w$, and let $B = B_{z',z''}$ be the basis set defined by these two vectors. For $i \in S$ let $B^i \stackrel{\text{def}}{=} B \cap V_{i,j_i}$, and let $B^{S,w} = \bigcup_{i \in S} B^i$. It is then immediate from the definitions and Lemma 2.7 that $\bigcup_{i \in S} V_{i,j_i} = \langle B^{S,w} \rangle$, and thus it is enough, by Lemma 2.5, to count the number of elements in the set $B^{S,w}$. Using the inclusion-exclusion formula, we obtain

$$\dim(\bigcup_{i \in S} V_{i,j_i}) = |B^{S,w}| = \sum_{\substack{Q \subseteq S \\ Q \neq \emptyset}} (-1)^{|Q|-1} |\bigcap_{i \in Q} B^i|.$$

Therefore, since $|\bigcap_{i \in Q} B^i| = |\{b_z \in B_{z',z''}\ :\ z[Q] = w[Q]\}| = 2^{k-|Q|}$, we get

$$\dim(\bigcup_{i \in S} V_{i,j_i}) = \sum_{\substack{Q \subseteq S \\ Q \neq \emptyset}} (-1)^{|Q|-1} 2^{k-|Q|} = \sum_{m=1}^{|S|} (-1)^{m-1} \binom{|S|}{m} 2^{k-m} = 2^k - 2^{k-|S|},$$

implying the Lemma. $\square$

We are now ready to verify that our construction indeed has the desired properties. For a subset $X \subseteq V = K \times L$ let us define

$$g(X) = \dim\Big(\bigcup_{(i,j)\in X} V_{ij}\Big),$$

and let us set $t = 2^k$. It follows by Lemma 2.6 that if X contains an edge of the graph G, then $g(X) = 2^k$, i.e, that $\mathcal{F}_t(g) = \mathcal{H}$. It also follows by Lemma 2.8 that $g(X) = 2^k - 2^{k-|X|} \leq 2^k - 1$ for any independent set $X \subset V$. i.e., $\mathcal{A}_t(g) = \mathcal{I}(\mathcal{H})$. In other words, g is the same set-function as the function f described in the previous subsection.

2.4 Proofs of main lemmas

In this section we prove Lemmas 2.1 and 2.2, which are the key statements needed to prove our main results.

Proof of Lemma 2.1. Let us recall that (f,t) is a polymatroid separator of the hypergraph $\mathcal{H}$, separating the maximal independent sets $\mathcal{S} = \mathcal{S}(\mathcal{H}, f, t)$ from $\mathcal{H}$, and that to every node v of $\mathbf{T}$ we have associated an independent set $S_v = \bigcap_{\ell\in L(\mathbf{T}(v))} I_\ell$, where $I_\ell \in \mathcal{S}$ denotes the maximal independent set assigned to the leaf $\ell \in L(\mathbf{T})$ by the proper assignment ϕ.

To prove the statement of the lemma, we shall show by induction that

$$f(S_w) \leq t - |L(\mathbf{T}(w))| \tag{2.15}$$

holds for every node w of the binary tree $\mathbf{T}$. Since f is non-negative, it follows that

$$|L(\mathbf{T}(w))| \leq t \leq range(f)$$

which, if applied to the root of $\mathbf{T}$, proves the lemma. To see (2.15), let us apply induction by the size of $L(\mathbf{T}(w))$. Clearly, if $w = \ell$ is a leaf of $\mathbf{T}$, then $|L(\mathbf{T}(\ell))| = 1$, $S_w = I_\ell \in \mathcal{S}$, and (2.15) follows by the assumption that (f,t) is separating $\mathcal{H}$ from $\mathcal{S}$. Let us assume now that w is a node of $\mathbf{T}$ with u and v as its immediate successors. Then $|L(\mathbf{T}(w))| = |L(\mathbf{T}(u))| + |L(\mathbf{T}(v))|$, and $S_w = S_u \cap S_v$. By our inductive hypothesis, and since f is submodular, we have the inequalities

$$\begin{aligned} f(S_u \cup S_v) + f(S_w) &\leq f(S_u) + f(S_v) \\ &\leq t - |L(\mathbf{T}(u))| + t - |L(\mathbf{T}(v))| \\ &= 2t - |L(\mathbf{T}(w))|. \end{aligned}$$

Since ϕ is a proper mapping, the set $S_u \cup S_v$ contains a hyperedge $H \in \mathcal{H}$, and thus $f(S_u \cup S_v) \geq f(H) \geq t$ by the monotonicity of f, and by our assumption that (f, t) is a separator for $\mathcal{H}$. Thus, from the above inequality we get $t + f(S_w) \leq f(S_u \cup S_v) + f(S_w) \leq 2t - |L(\mathbf{T}(w))|$, from which (2.15) follows. □

For a hypergraph $\mathcal{H}$ and a vertex $v \in V = V(\mathcal{H})$ let us denote by $d_{\mathcal{H}}(v)$ the *degree* of vertex v in $\mathcal{H}$, i.e., $d_{\mathcal{H}}(v)$ is the number of hyperedges of $\mathcal{H}$ containing v.

Lemma 2.9 *For every Sperner hypergraph $\mathcal{H} \subseteq 2^V$ on $n = |V| > 1$ vertices, with $m = |\mathcal{H}| \geq n$ hyperedges, there exists a vertex $v \in V$ for which*

$$m\frac{1}{n} \leq d_{\mathcal{H}}(v) \leq m\left(1 - \frac{1}{n}\right).$$

Proof. Let us define

$$X = \{v \in V \mid d_{\mathcal{H}}(v) < m\frac{1}{n}\}$$

and

$$Y = \{v \in V \mid d_{\mathcal{H}}(v) > m(1 - \frac{1}{n})\},$$

and let us assume indirectly that $X \cup Y = V$ forms a partition of the vertex set.

Let us observe first that $|X| < n$ must hold, since otherwise a contradiction

$$m \leq \sum_{H \in \mathcal{H}} |H| = \sum_{v \in X} d_{\mathcal{H}}(v) < n\frac{m}{n} = m,$$

would follow.

Let us observe next that $|X| > 0$ must hold, since otherwise

$$\sum_{H \in \mathcal{H}} |H| = \sum_{v \in V} d_{\mathcal{H}}(v) = \sum_{v \in Y} d_{\mathcal{H}}(v) > nm(1 - \frac{1}{n}) = m(n-1)$$

follows, implying the existence of a hyperedge $H \in \mathcal{H}$ of size $|H| = n$, i.e., $V \in \mathcal{H}$. Since $\mathcal{H}$ is Sperner, $1 = m < n$ would follow, contradicting our assumptions.

Let us observe finally that the number of those hyperedges which avoid some points of Y cannot be more than $|Y|m/n$, and since $|Y| < n$ by our previous observation, there must exist a hyperedge $H \in \mathcal{H}$ containing Y. Thus, all other hyperedges must intersect X, and hence we have

$$m - 1 \leq \sum_{H \in \mathcal{H}} |H \cap X| = \sum_{v \in X} d_{\mathcal{H}}(v) < |X| \frac{m}{n} \leq m \frac{n-1}{n}$$

by our first observation. From this $m < n$ would follow, contradicting again our assumption that $m \geq n$. This last contradiction hence proves X and Y cannot cover V, and thus follows the lemma. □

For a subset $X \subseteq V$ let $\mathcal{H}^X \stackrel{\text{def}}{=} \{H \in \mathcal{H} \mid H \supseteq X\}$, and let us simply write $\mathcal{H}^v$ if $X = \{v\}$.

Lemma 2.10 *Given a hypergraph $\mathcal{H}$ and a subfamily $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ of its maximal independent sets, $|\mathcal{S}| \geq 2$, there exists a hyperedge $H \in \mathcal{H}$ and a vertex $v \in H$ such that*

$$|\mathcal{S}^v| \geq \frac{|\mathcal{S}|}{n} \text{ and } |\mathcal{S}^{H \setminus v}| \geq \frac{|\mathcal{S}|}{n|\mathcal{H}|}.$$

Proof. Let us note first that if $2 \leq |\mathcal{S}| < n$, then the statement is almost trivially true. To see this, let us choose two distinct maximal independent sets S_1 and S_2 from $\mathcal{S}$, and a vertex $v \in S_2 \setminus S_1$. Since $S_1 \cup \{v\}$ is not independent, there exists a hyperedge $H \in \mathcal{H}$ for which $v \in H \cap S_2$ and $H \setminus \{v\} \subseteq S_1$, implying thus that both $|\mathcal{S}^v|$ and $|\mathcal{S}^{H \setminus v}|$ are at least 1, and the right hand sides in the claimed inequalities are not more than 1.

Thus, we can assume in the sequel that $|\mathcal{S}| \geq n$. Let us then apply Lemma 2.9 for the Sperner hypergraph $\mathcal{S}^c \stackrel{\text{def}}{=} \{V \setminus I \mid I \in \mathcal{S}\}$, and obtain that

$$\frac{|\mathcal{S}|}{n} \leq d_{\mathcal{S}^c}(v) \leq |\mathcal{S}|(1 - \frac{1}{n})$$

holds for some $v \in V$, since $|\mathcal{S}| = |\mathcal{S}^c|$ obviously. Thus, from the second inequality we obtain

$$|\mathcal{S}^v| \geq \frac{|\mathcal{S}|}{n}.$$

To see the second inequality of Lemma 2.10, let us note that members of $\mathcal{S}^c$ are minimal transversals of $\mathcal{H}$, and thus for every $T \in \mathcal{S}^c$, $T \ni v$ there exists a hyperedge $H \in \mathcal{H}$ for which $H \cap T = \{v\}$, by the definition of minimal transversals. Thus,

$$\bigcup_{H \in \mathcal{H}: H \ni v} \{T \in \mathcal{S}^c \mid T \cap H = \{v\}\} \supseteq \{T \in \mathcal{S}^c \mid T \ni v\}$$

holds, from which

$$\sum_{H\in\mathcal{H}:H\ni v} |\mathcal{S}^{H\setminus v}| \geq d_{\mathcal{S}^c}(v) \geq \frac{|\mathcal{S}|}{n}$$

follows. Therefore, since $|\{H \in \mathcal{H} \mid H \ni v\}| = d_{\mathcal{H}}(v) \leq |\mathcal{H}|$ holds obviously, there must exist a hyperedge $H \in \mathcal{H}$, $H \ni v$, for which

$$|\mathcal{S}^{H\setminus v}| \geq \frac{|\mathcal{S}|}{n|\mathcal{H}|}$$

holds, implying thus the lemma. □

Proof of Lemma 2.2. Let us denote by $L(\alpha)$ the maximum number of leaves of a binary tree $\mathbf{T}$ with a proper mapping $\phi : V(\mathbf{T}) \to \mathcal{S}$, where $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ is an arbitrary subfamily of maximal independent sets of $\mathcal{H}$. To simplify notation, let us write $\alpha = |\mathcal{S}|$ and $\beta = |\mathcal{H}|$. To prove the statement, we need to show that

$$L(\alpha) \geq \alpha^{c/\log\beta} \tag{2.16}$$

where $c = c(n, \beta)$ is as defined in (2.3).

Let us prove this inequality by induction on α. Clearly, if $\alpha = 1$, then $L(1) = 1$ holds, and we have equality in (2.16).

Let us assume next that we already have verified the claim for all subfamilies of size smaller than α, and let us consider a subfamily $\mathcal{S} \subseteq \mathcal{I}(\mathcal{H})$ of size $\alpha = |\mathcal{S}|$. According to Lemma 2.10, we can choose two disjoint subfamilies $\mathcal{S}', \mathcal{S}'' \subseteq \mathcal{S}$ such that $|\mathcal{S}'| \geq \frac{\alpha}{n}$ and $|\mathcal{S}''| \geq \frac{\alpha}{n\beta}$, and such that for any pair of sets $S' \in \mathcal{S}'$ and $S'' \in \mathcal{S}''$ the union $S' \cup S''$ contains a member of $\mathcal{H}$. Thus, building binary trees with proper mappings separately for $\mathcal{S}'$ and $\mathcal{S}''$, and joining them as two siblings of a common root, we obtain a binary tree with a proper mapping for $\mathcal{S}$. Since the right hand side of our claim is a monotone function of α, we can conclude for the number of leaves in the obtained binary tree that

$$L(\alpha) \geq L(\frac{\alpha}{n}) + L(\frac{\alpha}{n\beta}). \tag{2.17}$$

Applying now our inductive hypothesis, we get

$$\begin{aligned} L(\alpha) &\geq \left(\frac{\alpha}{n}\right)^{c/\log\beta} + \left(\frac{\alpha}{n\beta}\right)^{c/\log\beta} \\ &= \alpha^{c/\log\beta}\left[n^{-c/\log\beta} + (n\beta)^{-c/\log\beta}\right] \\ &= \alpha^{c/\log\beta}, \end{aligned}$$

where the last equality holds by (2.3). This proves (2.16), and hence the lemma follows. □

Note that the right hand side of (2.16) is the least possible solution of the recursion (2.17).

2.5 Polymatroid inequality over products of lattices

In this section, we discuss a generalization of the previous results, which replaces polymatroid set-functions by submodular functions defined on products of lattices. Let $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$ be the product of n lattices. A function $f : \mathcal{L} \mapsto \{0, 1, \ldots, r\}$, where $r \in \mathbb{Z}_+$, is said to be submodular if

$$f(x \vee y) + f(x \wedge y) \leq f(x) + f(y)$$

holds for all $x, y \in \mathcal{L}$, where $\vee$ and $\wedge$ denote, as usual, the join and meet operators over $\mathcal{L}$. As before, f is said to be monotone if $f(x) \leq f(y)$ whenever $x \preceq y$, and is called polymatroid if it is monotone, submodular, and $f(l) = 0$, where $l = (l_1, \ldots, l_n)$ is the minimum element of l.

Given a polymatroid function f with range r and an integral threshold $t \in \{1, ..., r\}$ let us denote by $\mathcal{F}_t = \mathcal{F}_t(f)$ the family of all minimal vectors $x \in \mathcal{L}$ for which $f(x) \geq t$, and by $\mathcal{A}_t = \mathcal{A}_t(f)$ the family of all maximal vectors $x \in \mathcal{L}$ for which $f(x) < t$. It follows that $\mathcal{A}_t = \mathcal{I}(\mathcal{F}_t)$, where $\mathcal{I}(\mathcal{B}) \stackrel{\text{def}}{=} \{\text{maximal } x \in \mathcal{L} \mid x \not\succeq b, \text{for all } b \in \mathcal{B}\}$ is the family of all maximal independent vectors for $\mathcal{B} \subseteq \mathcal{L}$. As in the Boolean case $\mathcal{L} = \{0, 1\}^n$, we shall use the notation $\alpha = |\mathcal{A}_t|$ and $\beta = |\mathcal{F}_t|$.

Given $\mathcal{X} \subseteq \mathcal{L}$, a polymatroid function $f : \mathcal{L} \mapsto \{0, 1, \ldots, r\}$ and a threshold $t \in \{1, \ldots, r\}$, we say that (f, t) is a separator for $\mathcal{X}$ if $f(x) \geq t$ for all $x \in \mathcal{X}$.

Theorem 2.2 admits the following generalization:

Theorem 2.3 *Let (f,t) be a polymatroid separator for a set $\mathcal{X} \subseteq \mathcal{L}$ of size $|\mathcal{X}| \geq 2$. Then*

$$|\mathcal{A}_t(f) \cap \mathcal{I}(\mathcal{X})| \leq |\mathcal{X}|^{(\log t)/c(2Q,|\mathcal{X}|)}, \tag{2.18}$$

where $Q = \sum_{i=1}^n |\mathcal{L}_i|$. In particular, $\alpha \leq \max(Q, \beta^{(\log t)/c(2Q,\beta)})$. If each lattice $\mathcal{L}_i$ is a chain, i.e., $\mathcal{L} = \mathcal{C} \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^n \mid 0 \leq x \leq c\}$ is an integral box, where $c \in \mathbb{Z}_+^n$ is a given integral n-vector, then

$$|\mathcal{A}_t(f) \cap \mathcal{I}(\mathcal{X})| \leq |\mathcal{X}|^{(\log t)/c(2n,|\mathcal{X}|)}. \tag{2.19}$$

The proof of Theorem 2.3 makes use of a generalization of Lemma 2.2. To state this generalization, we need first to extend the notion of proper mappings of maximal independent sets to binary trees. Given a binary tree $\mathbf{T}$, an antichain $\mathcal{B} \subseteq \mathcal{L}$, and a collection $\mathcal{A} \subseteq \mathcal{I}(\mathcal{B})$ of maximal independent elements of $\mathcal{B}$, let us consider again mappings $\phi : L(\mathbf{T}) \mapsto \mathcal{I}(\mathcal{B})$ that assign a maximal independent element $a^\ell \in \mathcal{A}$ to each leaf ℓ of $\mathbf{T}$. To each node v of the tree $\mathbf{T}$, we associate the element $x^v = \bigwedge_{\ell \in L(\mathbf{T}(v))} a^\ell$. The mapping ϕ will be called proper if it assigns different independent elements to different leaves, and if the element $x^u \vee x^v$ is not independent whenever u and v are incomparable nodes of $\mathbf{T}$. The latter condition implies that for every pair of incomparable nodes $u, v \in V(\mathbf{T})$, there exists an element $a \in \mathcal{A}$ for which $a \preceq x^u \vee x^v$.

Lemma 2.11 *Let $\mathcal{B} \subseteq \mathcal{L}$ be an antichain of size $|\mathcal{B}| \geq 2$ and let $\mathcal{A} \subseteq \mathcal{I}(\mathcal{B})$. Then there exists a binary tree $\mathbf{T}$ and a proper mapping $\phi : L(\mathbf{T}) \mapsto \mathcal{A}$, such that*

$$|L(\mathbf{T})| \geq |\mathcal{A}|^{c(2Q,|\mathcal{X}|)/\log|\mathcal{B}|}.$$

If $\mathcal{L} = \mathcal{C}$ is an n-dimensional integral box, then there exists a binary tree $\mathbf{T}$ and a proper mapping $\phi : L(\mathbf{T}) \mapsto \mathcal{A}$, such that

$$|L(\mathbf{T})| \geq |\mathcal{A}|^{c(2n,|\mathcal{B}|)/\log|\mathcal{B}|}.$$

Proof. By induction on $|\mathcal{A}| \geq 2$. We shall make use of the following lemma.

Lemma 2.12 *(i) For every subset $\mathcal{B} \subseteq \mathcal{L}$ and every $\mathcal{A} \subseteq \mathcal{I}(\mathcal{B})$, $|\mathcal{A}| \geq 2$, there exist $b \in \mathcal{B}$ and two antichains $\mathcal{A}', \mathcal{A}'' \subseteq \mathcal{A}$ such that*

$$\left(\bigwedge_{a' \in \mathcal{A}'} a'\right) \vee \left(\bigwedge_{a'' \in \mathcal{A}''} a''\right) \succeq b, \tag{2.20}$$

$$|\mathcal{A}'| \geq \frac{|\mathcal{A}|}{2Q}, \qquad |\mathcal{A}''| \geq \frac{|\mathcal{A}|}{2Q|\mathcal{B}|}. \tag{2.21}$$

(ii) If further $\mathcal{L} = \mathcal{C}$ is an n-dimensional integral box, then we can find $b \in \mathcal{B}$ and $\mathcal{A}', \mathcal{A}'' \subseteq \mathcal{A}$ such that (2.20) holds and

$$|\mathcal{A}'| \geq \frac{|\mathcal{A}|}{2n}, \qquad |\mathcal{A}''| \geq \frac{|\mathcal{A}|}{2n|\mathcal{B}|}. \tag{2.22}$$

Then, to build a binary tree $\mathbf{T}$ on $\mathcal{A}$, let us associate to its root the element $b \in \mathcal{B}$ which satisfies the conditions of Lemma 2.12, and let us use members of $\mathcal{A}'$ to label all left-leaves of $\mathbf{T}$, and use $\mathcal{A}''$ to label all right-leaves. Clearly, $0 < |\mathcal{A}'| < |\mathcal{A}|$, $0 < |\mathcal{A}''| < |\mathcal{A}|$, (since $\mathcal{A}'$ and $\mathcal{A}''$ are disjoint by (2.20)) and both $\mathcal{A}', \mathcal{A}''$ are collections of maximal independent elements of $\mathcal{B}$. Therefore, we can conclude by induction that there exist binary trees $\mathbf{T}'$ and $\mathbf{T}''$ of sufficiently large number of leaves, and proper mappings $\phi' : L(\mathbf{T}') \mapsto \mathcal{A}'$ and $\phi'' : L(\mathbf{T}'') \mapsto \mathcal{A}''$ such that if we join $\mathbf{T}'$ and $\mathbf{T}''$ as the two children of the root of $\mathbf{T}$, we obtain a proper mapping for $\mathbf{T}$. The proof is then completed by the same argument used in the proof of Lemma 2.2. $\square$

Proof of Lemma 2.12. For $b \in \mathcal{B}$ and $i \in V \stackrel{\text{def}}{=} \{1, \ldots, n\}$, let

$$\begin{aligned} \mathcal{A}_b^i &\stackrel{\text{def}}{=} \{a \in \mathcal{A} \mid a_i \succeq b_i\}, \text{ and} \\ \mathcal{A}_b^{V\setminus i} &\stackrel{\text{def}}{=} \{a \in \mathcal{A} \mid a_j \succeq b_j \text{ for all } j \in V \setminus i\}. \end{aligned}$$

(i) Let $\epsilon = 1$, then $|\mathcal{A}| \geq 1 + 1/\epsilon$. For $i = 1, \ldots, n$ and $x \in \mathcal{L}_i$, define $\mathcal{A}_i(x) \stackrel{\text{def}}{=} \{a \in \mathcal{A} \mid a_i = x\}$, and

$$X_i = \{x \in \mathcal{L}_i \ : \ |\mathcal{A}_i(x)| \geq \frac{|\mathcal{A}|}{(1+\epsilon)Q}\}, \qquad \text{and } X = \bigcup_{i=1}^n X_i.$$

Define further

$$\mathcal{A}(X) = \{a \in \mathcal{A} \mid a_i \in X_i \text{ for all } i \in [n]\}.$$

Then we claim that

$$|\mathcal{A}(X)| > |\mathcal{A}| \frac{\epsilon}{1+\epsilon}. \tag{2.23}$$

Indeed, we have

$$\begin{aligned} |\mathcal{A} \setminus \mathcal{A}(X)| &= |\{a \in \mathcal{A} \ : \ a_i \notin X_i \text{ for some } i \in [n]\}| = |\bigcup_{i=1}^n \{a \in \mathcal{A} \ : \ a_i \notin X_i\}| \\ &\leq \sum_{i=1}^n \sum_{x \notin X_i} |\mathcal{A}_i(x)| < \sum_{i=1}^n |\mathcal{L}_i \setminus X_i| \frac{|\mathcal{A}|}{(1+\epsilon)Q} \leq \frac{|\mathcal{A}|}{(1+\epsilon)}, \end{aligned}$$

by the definition of X_i, establishing (2.23).

Let us note next that $|X_i| \geq 2$ for some $i = 1, \ldots, n$. If this was not the case, then since $\mathcal{A}(X)$ is an antichain, it follows that $|\mathcal{A}(X)| = 1$ implying by (2.23) that $|\mathcal{A}| < 1 + 1/\epsilon$, in contradiction to our assumptions.

Thus there exist an $i \in [n]$, and $x, y \in X_i$, such that either $x \prec y$ or x, y are incomparable in $\mathcal{L}_i$. Letting $z = x \vee y$ and noting that $z \succ x$, we conclude that, for every $a \in \mathcal{A}_i(x)$, there exists an element $b \in \mathcal{B}$ such that $b_i \preceq z$, and $b_j \preceq a_j$ for all $j \neq i$, by the maximality of the independent element a, i.e., $\mathcal{A}_i(x) = \bigcup_{b \in \mathcal{B}} \{a \in \mathcal{A}_i(x) \mid a_i \prec z, a_j \succeq b_j \text{ for all } j \neq i\}$. From this, it follows that there must exist an element $b \in \mathcal{B}$ such that

$$|\{a \in \mathcal{A}_i(x) \mid a_i \prec z, a_j \succeq b_j \text{ for all } j \neq i\} \geq |\mathcal{A}_i(x)|/|\mathcal{B}| \geq \frac{|\mathcal{A}|}{(1+\epsilon)Q|\mathcal{B}|}.$$

This immediately gives $|\mathcal{A}_b^{V \setminus i}| \geq \frac{|\mathcal{A}|}{(1+\epsilon)Q|\mathcal{B}|}$, from which we get the second inequality of (2.21) by taking $\mathcal{A}'' \stackrel{\text{def}}{=} \mathcal{A}_b^{V \setminus i}$.

To get the first inequality of (2.21), let $\mathcal{A}' \stackrel{\text{def}}{=} \mathcal{A}_i(y)$, and note that $|\mathcal{A}'| \geq \frac{|\mathcal{A}|}{(1+\epsilon)Q}$, since $y \in X_i$.

It remains to verify (2.20). Letting $w' = \bigwedge\{a' \ : \ a' \in \mathcal{A}'\}$ and $w'' = \bigwedge\{a'' \ : \ a'' \in \mathcal{A}''\}$, we get $w_i' \vee w_i'' = z \succeq b_i$ and $w''[V \setminus i] \succeq b[V \setminus i]$, concluding that (2.20) indeed holds.

(ii) Let $\mathcal{C} \stackrel{\text{def}}{=} \{x \in \mathbb{Z}^n \mid \ 0 \leq x \leq c\}$. Before proving the statement of Lemma 2.12(ii), we shall need one more definition and a generalization of Lemma 2.9.

For $\mathcal{A} \subseteq \mathcal{C}$ and $x \in \mathbb{Z}$, let us define

$$d_{\mathcal{A}}(x) \stackrel{\text{def}}{=} |\{a \in \mathcal{A} \mid a_i \geq x\}|.$$

Lemma 2.13 *Let $\mathcal{A} \subseteq \mathcal{C}$ be an antichain in an n-dimensional integral box $\mathcal{C}$, and let $\epsilon > 0$ be a given constant. If $m = |\mathcal{A}| \geq 1 + 1/\epsilon$, then there exists an $x \in \mathbb{Z}$ such that $0 \leq x \leq$ $\max\{c_1, \ldots, c_n\}$ and*

$$m \frac{1}{(1+\epsilon)n} \leq d_{\mathcal{A}}(x) \leq m \left(1 - \frac{1}{(1+\epsilon)n}\right).$$

Proof. Let us define the vector $y \in \mathcal{C}$ by setting

$$y_i = \max\{x \in \mathbb{Z} \mid 0 \leq x \leq c_i, \ d_{\mathcal{A}}(x) > m(1 - \frac{1}{(1+\epsilon)n})\},$$

for $i = 1, \ldots, n$, and let $Y = \{i \in V \mid y_i = c_i\}$. Assume indirectly that for every $i \in V$, and every $x \in \mathbb{Z}$ such that $y_i < x \leq c_i$, we have $d_{\mathcal{A}}(x) < m/(1+\epsilon)n$. Let us observe first that $Y \neq V$ must hold, since

$$\begin{aligned} |\bigcup_{i=1}^{n}\{a \in \mathcal{A} \mid a_i < y_i\}| &\leq \sum_{i=1}^{n} |\{a \in \mathcal{A} \mid a_i < y_i\}| \\ &= \sum_{i=1}^{n}(m - d_{\mathcal{A}}(y_i)) < n\frac{m}{(1+\epsilon)n} < m, \end{aligned}$$

implying that there exists an $a^o \in \mathcal{A}$ such that $a^o \geq y = c$, if $Y = V$. But this would imply that $m = 1$, since $\mathcal{A}$ is an antichain, contradicting our assumptions. Now since there is an $a^o \in \mathcal{A}$ such that $a^o \geq y$ and $\mathcal{A}$ is an antichain, it follows that, for all other elements $a \in \mathcal{A} \setminus \{a^o\}$, there must exist an $i \in V$ for which $a_i > y_i$. Consequently,

$$m - 1 \leq \sum_{a \in \mathcal{A}} |\{i \in V \mid a_i > y_i\}| = \sum_{i \notin Y} d_{\mathcal{A}}(y_i + 1) < n\frac{m}{(1+\epsilon)n},$$

and therefore we get $m < 1 + 1/\epsilon$, again a contradiction. $\square$

Let us now apply Lemma 2.13 with $\epsilon = 1$ for the antichain $\mathcal{A} \subseteq \mathcal{I}(\mathcal{B})$, and obtain an $i \in V$ and an $0 \leq x \leq c_i$ such that

$$|\mathcal{A}|\frac{1}{2n} \leq d_{\mathcal{A}}(x) \leq |\mathcal{A}|\left(1 - \frac{1}{2n}\right).$$

Let us next note that for every $a \in \mathcal{A}$ for which $a_i < x$, there exists an element $b \in \mathcal{B}$ such that $b_i \leq x$, and $b_j \leq a_j$ for all $j \neq i$, by the maximality of the independent element a. Thus,

$$\frac{|\mathcal{A}|}{2n} \leq |\mathcal{A}| - d_{\mathcal{A}}(x) \leq |\bigcup_{b \in \mathcal{B}}\{a \in \mathcal{A} \mid a_i < x, a_j \geq b_j \text{ for all } j \neq i\}|$$

holds, from which we conclude that there must exist a $b \in \mathcal{B}$ such that

$$|\{a \in \mathcal{A} \mid a_i < x, a_j \geq b_j \text{ for all } j \neq i\}| \geq \frac{|\mathcal{A}|}{2n|\mathcal{B}|}.$$

This gives $|\mathcal{A}_b^{V \setminus i}| \geq |\mathcal{A}|/(2n|\mathcal{B}|)$. On the other hand, since $b_i \leq x$ and $d_{\mathcal{A}}(x) \geq |\mathcal{A}|/(2n)$, we obtain $|\mathcal{A}_b^i| = d_{\mathcal{A}}(b_i) \geq |\mathcal{A}|/(2n)$. Thus letting $\mathcal{A}' \stackrel{\text{def}}{=} \mathcal{A}_b^i$ and $\mathcal{A}'' \stackrel{\text{def}}{=} \mathcal{A}_b^{V \setminus i}$, both (2.20) and (2.22) follow. $\square$

Remark. In general, Lemma 2.13 does not hold for lattices, i.e., we may not always be able to find a coordinate $i \in n$ and an element $x \in \mathcal{L}_i$, such that the two sets $\mathcal{A}' \stackrel{\text{def}}{=} \{a \in \mathcal{A} \mid a_i \preceq x\}$ and

$\mathcal{A}'' \stackrel{\text{def}}{=} \{a \in \mathcal{A} \mid a_i \succ x\}$ are of sufficiently large cardinalities. This can be illustrated by the following simple example. Let $\mathcal{L}_i = \{l_i, u_i, x_i, y_i, z_i\}$, where $l_i \prec x_i \prec u_i$, $l_i \prec y_i \prec u_i$, $l_i \prec z_i \prec u_i$, for $i = 1, \ldots, n$. Consider the antichains $\mathcal{B} = \{(u_1, l_2, \ldots, l_n), (l_1, u_2, l_3, \ldots, l_n), \ldots, (l_1, \ldots, l_{n-1}, u_n)\}$ and $\mathcal{A} \stackrel{\text{def}}{=} \mathcal{I}(\mathcal{B}) = \{(a_1, \ldots, a_n) \mid a_i \in \{x_i, y_i, z_i\}, \text{ for } i = 1, \ldots, n\}$. Then there is no element $x \in \cup_{i=1}^{n} \mathcal{L}_i$ with the desired property.

Chapter 3

Some Polymatroid Separators, Stronger Inequalities, and Applications

3.1 Introduction

In this chapter, we discuss a number of polymatroid separators and corresponding applications of Theorem 2.2 of the previous chapter, as well as some stronger forms of this theorem for some of the specific cases. Figure 3.1 shows a Venn diagram summarizing the relations between the applications considered in this chapter.

We begin the discussion in Section 3.2 with matroids, their parallel extensions, quasi-polynomial generation of minimal solutions to systems of k-smooth polymatroid inequalities, and its applications to matroid intersections. We show also that, for bounded k, if the number of k-smooth polymatroid inequalities is fixed, then the generation of minimal feasible solutions, can be done with *polynomial delay* and *polynomial space*, a generalization of a result, originally obtained in [94] for generating maximal independent sets for graphs, and extended in [65] for hypergraphs. This last result implies, in particular, that the dualization of read-k monotone CNFs can be done with polynomial delay and polynomial space.

Important applications of the parallel extension of matroids include the generation of all minimal collections, of a given list of linear spaces of some vector space, that span the entire space, and the generation of all minimal collections, of a given list of subgraphs on some vertex set, that interconnect all the vertices. These applications will be discussed in sections 3.3 and 3.4 respectively. In Section 3.5, we present a special case of the linear space covering problem, namely the generation of partial transversals, and illustrate that it is equivalent to the generation of minimal infrequent sets for databases. We also consider a generalization of partial transversals where each edge of the transversed hypergraph is assigned a non-negative weight, and show that the quasi-polynomial generation of minimal *Boolean* solutions of a system of linear inequalities with *bounded non-negative* coefficients follows from this generalization. To extend this result to any *monotone* system of linear inequalities in *integer* variables, we consider the more general class of systems of 2-monotonic inequalities, and show that the set of their minimal feasible solutions is uniformly dual-bounded. These results will be given in Section 3.6, along with an illustration that, for systems with bounded number of inequalities, both the minimal feasible and the maximal infeasible solutions can be generated in incremental polynomial time. Generalizing further, one might consider monotone systems of inequalities in non-linear functions. Among these, we recognize, in Section 3.7, that functions with non-negative Möbius coefficients allow for dual-boundedness, if they have integer range, or if they can be expressed as the sum of functions in the same class, but in bounded number of variables. Applications of these classes of functions include the generation of p-efficient and p-inefficient points of discrete probability distributions.

Finally, in Section 3.8, we give an example of polymatroid functions on products of lattices, related to the generation of maximal boxes that contain a specified number of points from a given set of points. It will be shown that the generation of such boxes, as well as some of its generalizations, can be done in quasi-polynomial time.

It is interesting to note that in all of our examples, incrementally generating extremal elements in the other direction, (i.e. in the direction in which dual-boundedness does not hold, e.g., maximal infeasible solutions of a monotone system of linear inequalities) turns out to be NP-hard.

3.2 Matroids

A matroid M is a pair $(E, \mathbb{I})$, where E is a finite set, and $\mathbb{I}$ is a collection of subsets of E satisfying

$(I1)$ $\emptyset \in \mathbb{I}$,

$(I2)$ If $X \in \mathbb{I}$ and $Y \subseteq X$ then $Y \in \mathbb{I}$,

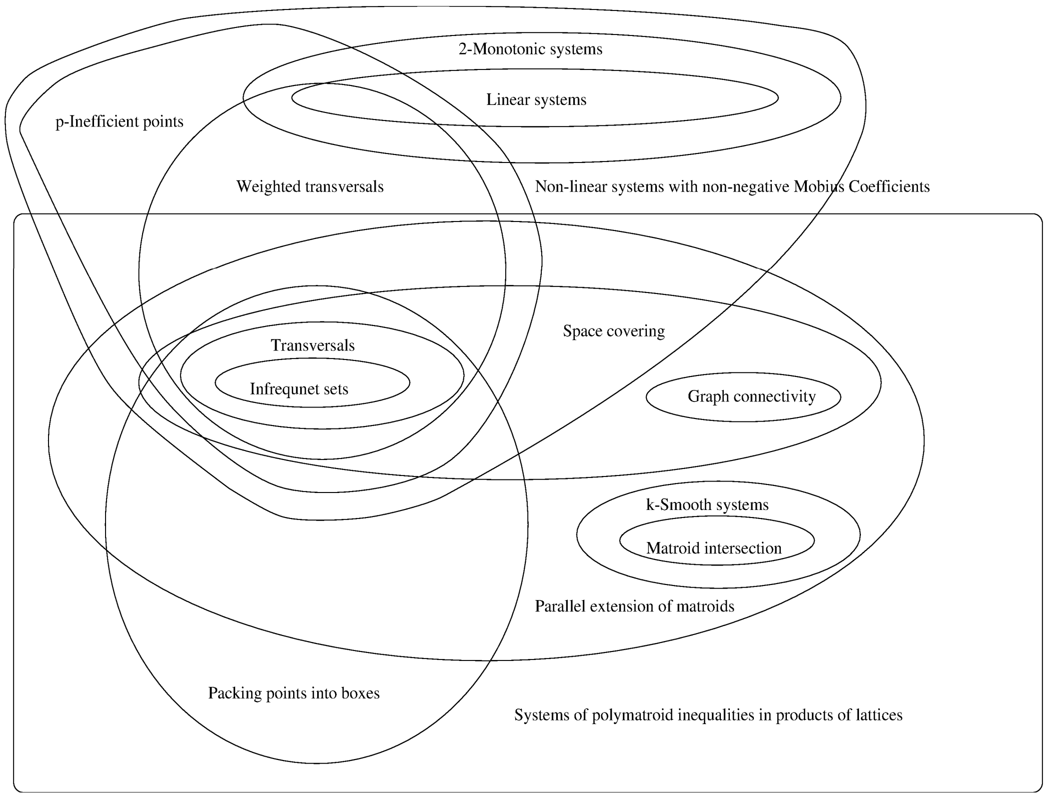


Figure 3.1: Examples of polymatroid separators and generalizations.

$(I3)$ If $X, Y \in \mathbb{I}$ and $|Y| > |X|$ then there exists an element $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathbb{I}$.

The elements of $\mathbb{I} = \mathbb{I}(M)$ are called the *independent sets* of M. The maximal independent subsets of E are called the *bases* of M, denoted by $\mathbb{B}(M)$. The *rank function* of M is a function $\rho : 2^E \mapsto \mathbb{Z}_+$ defined by

$$\rho(X) = \max\{|Y| \ : \ Y \subseteq X, \ Y \in \mathbb{I}\},$$

where $\rho(E)$ is called the rank (or the *dimension*) of M. It follows by (I3) that all maximal independent subsets of X have the same size $\rho(X)$. A *circuit* of M is a minimal dependent set, and the collection of such sets will be denoted by $\mathbb{C}(M)$. A *subspace*, a closed set, or a *flat* of M is as subset $X \subseteq E$ such that for all $y \in E \setminus X$, $\rho(X \cup \{y\}) = \rho(X) + 1$. Finally, a *hyperplane* is a subspace of rank $\rho(E) - 1$, and we let $\mathbb{H}(M)$ denote the set of hyperplanes of the matroid M.

It is well-known, and also easy to see, that the rank function of a matroid is a polymatroid function. In fact, any polymatroid function satisfying $\rho(X) \leq |X|$, for all $X \subseteq E$, is the rank function of a (unique) matroid. Given a matroid M, the complementary set $\{E \setminus X \mid X \in \mathbb{B}(M)\}$

is the bases set of another matroid M^* on the same ground set S, called the *dual matroid* of M. The bases of M^* are called the *co-bases* of M. The circuits of M^* are called the *co-circuits* of M, and their complements are the hyperplanes of M.

In the next subsections, we assume that the matroid M is given by an independence oracle, i.e., an algorithm that, given a set $X \subseteq E$, determines whether or not X is independent.

3.2.1 Parallel extensions of matroids

Several examples of polymatroid functions can be found, for instance, in [67] and [95]. The most general example considered in this chapter is the following. Let $M = (E, \mathbb{I})$ be a matroid with rank function $\rho : 2^E \mapsto \{0, 1, \ldots, |E|\}$, and let $S_1, \ldots, S_n$ be some subsets of E. For each $X \subseteq V \stackrel{\text{def}}{=} \{1, \ldots, n\}$, let $f(X) = \rho(\bigcup_{i \in X} S_i)$. Then f is a polymatroid function (and is obtained by performing *parallel extensions* of the rank function ρ with respect to the sets $S_1, \ldots, S_n$, followed by deletion of the original elements E, see [67]). In fact, every polymatroid function arises by this construction from some matroid, see [55, 77], and also [67]. Clearly, given an independence oracle for M, we get a feasibility oracle for f.

Thus Theorem 1.7 of Chapter 1 implies that, for a given integer threshold t, the family of minimal subsets of $\{S_1, \ldots, S_n\}$, the rank of whose union is at least t, can be listed incrementally in quasi-polynomial time. Since this theorem holds for systems of polymatroid inequalities, we can also consider the generalization of the above example to a number of matroids on the same ground set.

3.2.2 k-Smooth functions

The rank function of a matroid can be generalized in the following way. Let $f : 2^V \mapsto \mathbb{Z}_+$ be an integer-valued monotone function. For an integer $k \in \mathbb{Z}_+$, recall that the function f is said to be *k-smooth* if for any $v \in V$ and any $X \subseteq V$, we have

$$f(X \cup \{v\}) - f(X) \leq k.$$

Note that a polymatroid function $f : 2^V \mapsto \mathbb{Z}_+$ is k-smooth if and only if $f(\{v\}) \leq k$ for all $v \in V$. Thus, 1-smooth polymatroid functions are just matroid rank functions.

Theorem 3.1 *Let $\mathcal{F}$ be the set of all minimal feasible solutions for a k-smooth polymatroid inequality $f(X) \geq t$ and let $\mathcal{X} \subseteq \mathcal{F}^+$ be an arbitrary subset of $\mathcal{F}^+ \stackrel{\text{def}}{=} \{X \subseteq V \mid X \supseteq Y \text{ for some } Y \in \mathcal{F}\}$ of size $|\mathcal{X}| \geq 1$. Then*

$$\sum_{Y \in \mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})} |V \setminus Y| \leq \sum_{X \in \mathcal{X}} |X| \cdot |V \setminus X|^{k-1}. \tag{3.1}$$

In particular, we have $|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq n^k |\mathcal{X}|$, which for $\mathcal{X} = \mathcal{F}$ gives $|\mathcal{I}(\mathcal{F})| \leq n^k |\mathcal{F}|$.

Proof. Let us denote by $\mathcal{Y}$ the set $\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})$. Consider an arbitrary subset $Y \in \mathcal{Y}$ and an arbitrary element $v \in V \setminus Y$. Since $Y \in \mathcal{I}(\mathcal{F})$, we must have $f(Y) \leq t-1$ and $f(Y \cup \{v\}) \geq t$. Also, $Y \cup \{v\}$ must contain a subset $X \in \mathcal{X}$ since $Y \in \mathcal{I}(\mathcal{X})$ is assumed. Then $X \setminus \{v\} \subseteq Y$, and we can obtain Y by appending elements to $X \setminus \{v\}$. Let us do this in w stages, and build successively sets (closures) $X \setminus \{v\} \subseteq X_0 \subseteq X_1 \ldots \subseteq X_w = Y$, where X_0 is maximal with the property that $f(X_0) = f(X \setminus \{v\})$, and X_i is obtained from X_{i-1} by first picking an arbitrary element $y_i \in Y \setminus X_{i-1}$ and then letting $X_{i-1} \cup \{y_i\} \subseteq X_i \subseteq Y$ be a maximal subset of V with the property that $f(X_i) = f(X_{i-1} \cup \{y_i\})$, for $i = 1, \ldots, w$. Note that, if $Y' \subseteq Y$, $y \in V$, and $f(Y' \cup \{y\}) = f(Y')$, then y must be present in Y, since $f(Y \cup \{y\}) - f(Y) \leq f(Y' \cup \{y\}) - f(Y') = 0$, by submodularity. Note also that by this construction, we must have $w \leq k-1$. To see this, observe that the k-smoothness of f gives $f(X_0) \geq t-k$, and thus, the definition of the sets X_i and the monotonicity of f yield

$$t - k \leq f(X_0) \leq f(X_1) - 1 \leq \cdots \leq f(X_w) - w = f(Y) - w \leq t - 1 - w.$$

But how many different subsets $Y \in \mathcal{Y}$ can arise from a given $X \in \mathcal{X}$ by such a construction? In fact not too many, if k is constant. The crucial observation here is that to go from one closure X_i to the next X_{i+1}, we have only at most $|V \setminus X_i|$ choices by the submodularity of the function f. Indeed, the maximality of X_i implies that $f(X_i \cup \{u\}) \geq f(X_i) + 1$ for any $u \in V \setminus X_i$. Suppose we pick u such that $f(X_i \cup \{u\}) = f(X_i) + \delta$, where $\delta \geq 1$ is an integer. Suppose further that this set $X_i \cup \{u\}$ has two different closures X'_{i+1} and X''_{i+1} such that $X_i \cup \{u\} \subseteq X'_{i+1}$, $X_i \cup \{u\} \subseteq X''_{i+1}$ and $f(X'_{i+1}) = f(X''_{i+1}) = f(X_i) + \delta$. Then we claim that $X'_{i+1} \cup X''_{i+1}$ is a third closure, giving a contradiction. To see this claim, observe first that the monotonicity of f implies $f(X'_{i+1} \cap X''_{i+1}) \geq f(X_i \cup \{u\}) = f(X_i) + \delta$ and similarly

$f(X'_{i+1} \cup X''_{i+1}) \geq f(X_i \cup \{u\}) = f(X_i) + \delta$. The submodularity now gives

$$2[f(X_i) + \delta] \leq f(X'_{i+1} \cup X''_{i+1}) + f(X'_{i+1} \cap X''_{i+1}) \leq f(X'_{i+1}) + f(X''_{i+1}) = 2[f(X_i) + \delta],$$

implying that $f(X'_{i+1} \cup X''_{i+1}) = f(X_{i+1}) + \delta$ and establishing the claim.

It remains to count the number of possible Y's that can be obtained from a given $X \in \mathcal{X}$ in the above way. First, we have to pick an element $v \in X$ and the number of such elements is $|X|$. Second, starting from $X \setminus \{v\}$, we have to get $w \leq k-1$ closures $X_1, X_2, \ldots, X_w = Y$, going from one closure X_{i-1} to the next X_i by appending an element $y_i \in V \setminus X_{i-1}$, for $i = 1, \ldots, w$. The number of choices for this is at most $|V \setminus X|^{k-1}$. Finally we sum these numbers over all subsets $X \in \mathcal{X}$, and observe that this sum is at least equal to $\sum_{Y \in \mathcal{Y}} |V \setminus Y|$ since, for each $Y \in \mathcal{Y}$ and each $u \notin Y$, there is at least one subset $X \in \mathcal{X}$ such that $Y \cup \{u\}$ contains X. This readily gives (3.1). $\square$

As we shall see below, the bound of Theorem 3.1 is sharp for $r = k = 1$. On the other hand, there exist examples of 1-smooth polymatroid functions for which the size of $\mathcal{F}$ is exponentially large in the size of $\mathcal{I}(\mathcal{F})$. Despite this, for a single 1-smooth polymatroid inequality, both the sets $\mathcal{F}$ and $\mathcal{I}(\mathcal{F})$ can be generated in polynomial time, and also for constant k and r, the set $\mathcal{F}$ can be generated in polynomial time (see Sections 3.2.4 and 3.2.5 below). In contrast, it can be shown that generating $\mathcal{I}(\mathcal{F})$ is already NP-hard, for *a single* 2-smooth polymatroid inequality, and for a system of *two* 1-smooth polymatroid inequalities [63].

3.2.3 Matroid intersections

Let $M_1, \ldots, M_r$ be r matroids on the common ground set V of cardinality $|V| = n$. In [65] the question of generating the family $\overline{\mathcal{F}}$ of all maximal sets independent in all the matroids $M_1, \ldots, M_r$ was asked, and an $O(n^{r+2}|\overline{\mathcal{F}}| \sum_{i=1}^{r} T_i)$ algorithm was given, where T_i is the time required for independence testing in matroid M_i. In contrast to this, Theorem 3.1 implies that this problem can be solved in $(rn|\overline{\mathcal{F}}|)^{o(\log(rn|\overline{\mathcal{F}}|))}$ oracle time. Indeed, let $\rho_i : 2^V \mapsto \{0, 1, \ldots, n\}$ be the rank function of matroid M_i, for $i = 1, \ldots, r$. Then the function $f_i : 2^V \mapsto \{0, 1, \ldots, n\}$, defined by

$$f_i(X) = \rho_i(V \setminus X) + |X| - \rho_i(V),$$

for $i = 1, \ldots, r$, is 1-smooth polymatroid (and in fact, is the rank function of the dual matroid M_i^*), and a set $X \subseteq V$ is independent in M_i if and only if $f_i(V \setminus X) \geq |V| - \rho_i(V)$. Thus letting $\mathcal{F} \stackrel{\text{def}}{=} \{V \setminus X \mid X \in \overline{\mathcal{F}}\}$, we conclude that $\mathcal{F}$ is the family of minimal solutions for the system of 1-smooth polymatroid inequalities

$$f_i(X) \geq |V| - \rho_i(V), \quad i = 1, \ldots, r,$$

and therefore we get by Theorem 3.1:

$$\sum_{Y \in \mathcal{I}(\mathcal{F})} |V \setminus Y| \leq r \sum_{X \in \mathcal{F}} |X|.$$

This implies the uniformly dual-bundedness, and hence the quasi-polynomial time generation, of $\overline{\mathcal{F}}$:

Theorem 3.2 *Let $M_1, \ldots, M_r$ be r matroids on the common ground set V of cardinality n. Let $\overline{\mathcal{F}}$ and $\mathcal{I}^{-1}(\overline{\mathcal{F}})$ be respectively the family of all maximal sets independent in all the r matroids, and the family of all minimal sets dependent in at least one of the matroids. Then, for any non-empty $\mathcal{X} \subseteq \overline{\mathcal{F}}$,*

$$\sum_{Y \in \mathcal{I}^{-1}(\mathcal{X}) \cap \mathcal{I}^{-1}(\overline{\mathcal{F}})} |Y| \leq r \sum_{X \in \mathcal{X}} |V \setminus X|.$$

where $\mathcal{I}^{-1}(\mathcal{X})$ is the family of minimal subsets, not contained in any hyperedge of $\mathcal{X}$. In particular, we have $|\mathcal{I}^{-1}(\mathcal{X}) \cap \mathcal{I}^{-1}(\overline{\mathcal{F}})| \leq rn|\mathcal{X}|$.

Corollary 3.1 *Let $M_1, \ldots, M_r$ be r matroids on the common ground set V. All maximal sets $X \subseteq V$, independent in the r matroids can be generated in incremental quasi-polynomial time.*

3.2.4 Polynomial delay-polynomial space generation of $\mathcal{F}$ for systems of bounded number of k-smooth polymatroid inequalities

As a special case of Corollary 3.1, and a generalization of the results in [65], we consider in this section the problem of generating the family $\mathcal{F}$ of minimal sets satisfying a bounded number r of k-smooth polymatroid inequalities (3.14).

Theorem 3.3 *All minimal feasible solutions of a system*

$$f_1(X) \geq t_1, \dots, f_r(X) \geq t_r, \qquad X \subseteq V = [n], \tag{3.2}$$

of k-smooth polymatroid inequalities, can be generated with polynomial delay and polynomial space, if k and r are constants.

Proof. Consider the following generalization of an algorithm in [94] for generating maximal independent sets in graphs (see also [65]). Let $\mathcal{F}$ be the family of minimal feasible solutions of the given system. The algorithm works by doing a depth-first search on a rooted-tree whose leaves are the elements of $\mathcal{F}$. The nodes of the tree at level $j \in [n{+}1] \stackrel{\text{def}}{=} \{1, 2, \dots, n, n{+}1\}$ (where the nodes at level $n+1$ are the leaves) are the subsets of V containing $[j : n] \stackrel{\text{def}}{=} \{j, j+1, \dots, n\}$ that are minimal feasible for (3.2) (that is, $X \supseteq [j : n]$ is feasible and $X \setminus \{u\}$ is infeasible for all $u \in X \cap [1 : j-1]$). Clearly, any minimal feasible set Y containing $[j+1 : n]$, must contain a subset $X \setminus \{j\}$, for some minimal feasible set X containing $[j : n]$. In other words, any node Y of the tree at level $j+1$ can be obtained (in polynomially many ways as we shall see below) from some node X at level j, by deleting j from X, and then restoring minimal feasibility by adding some elements from $[1 : j-1]$ to X.

The algorithm only generates nodes of the tree as needed during the search. Given a node $X \subseteq V$ of the tree at level j, the children of X are generated as follows. If $X' \stackrel{\text{def}}{=} X \setminus \{j\}$ is feasible for (3.2), then $Y = X'$ is the only child of X. Otherwise, X has *potentially* a polynomial number of children, the first of which is a copy of X itself, and is *always present.* The other potential children of X are those subsets $Y \subseteq V \setminus \{j\}$, containing $X' \supseteq [j+1 : n]$, that are minimal (with respect to $[1 : j-1]$) feasible for (3.2). Note that, in this case, Y may be a potential child of several nodes of the tree at level j, and to avoid repetition, of all these nodes, Y will be made the child of the lexicographically smallest (finding the lexicographically smallest minimal feasible set X contained in $Y \cup \{j\}$ is easily solvable by the natural greedy algorithm).

Let us now show that the number of children of X is polynomially bounded if k and r are constants. Assume without less of generality that $X' = X \setminus \{j\}$ is infeasible. Then X' is violated by a subset $J \subseteq [r]$ of the constraints of (3.2) (i.e., $f_i(X') \leq t_i - 1$ for $i \in J$). Thus each potential child Y is obtained from X by first finding, for each $i \in J$, the family $\mathcal{X}_i$ of minimal feasible subsets of $V \setminus \{j\}$ for the inequality $f_i(X) \geq t_i$, containing X', and then constructing the family of unions $\mathcal{U} = \{\cup_{i \in J} X_i \mid X_i \in \mathcal{X}_i \text{ for } i \in J\}$. Each subset $X_i \in \mathcal{X}_i$ is obtained by appending elements to X' until feasibly is restored and then checking if the resulting set is minimal (for the i^{th} inequality of (3.2)). Note that the number of appended elements, to obtain each set $X_i \in \mathcal{X}_i$, does not exceed k. This follows from the fact that, first, by k-smoothness $f_i(X') \geq t_i - k$ while X_i is minimal with $f_i(X_i) \geq t_i$, and that, second, by submodularity if we append an element $u \in V$ to the current set $X'' \subseteq X_i$, then u must increase the value of $f_i(X'')$ by at least one, since X_i is minimal and $f_i(X_i) - f_i(X_i \setminus \{u\}) \leq f_i(X'') - f_i(X'' \setminus \{u\})$. We conclude therefore that the degree of each node in the tree is at most n^{kr}, and hence for bounded k and r, we get a polynomial delay, polynomial space generation algorithm. $\square$

As an application, let us consider the dualization of *bounded-degree hypergraphs* (or *read-k CNFs* [35]). Let $\mathcal{H} \subseteq 2^{[n]}$ be a hypergraph of degree at most k, i.e., $|\max\{H \in \mathcal{H} \mid H \ni i\}| \leq k$, for all $i \in [n]$. Then the function $f : 2^{[n]} \mapsto |\mathcal{H}|$ defined by $f(X) = |\{H \in \mathcal{H} \mid H \cap X \neq \emptyset\}|$ is k-smooth polymatroid. Thus, it follows from Theorem 3.3 that if k is bounded, then the minimal transversals of $\mathcal{H}$ can be generated with polynomial delay and polynomial space. In fact, the minimal transversals of a bounded-degree hypergraph $\mathcal{H}$ can be even generated in *lexicographic order* with polynomial delay, see [40, 56].

3.2.5 Bases, flats, hyperplanes, and circuits of a matroid

Let $M = (\mathbb{I}, V)$ be a matroid of rank function ρ on the ground set V. Given an integral threshold $t \in \{0, 1, \ldots, \rho(V)\}$, consider the 1-smooth polymatroid inequality

$$\rho(X) \geq t, \quad X \subseteq V. \tag{3.3}$$

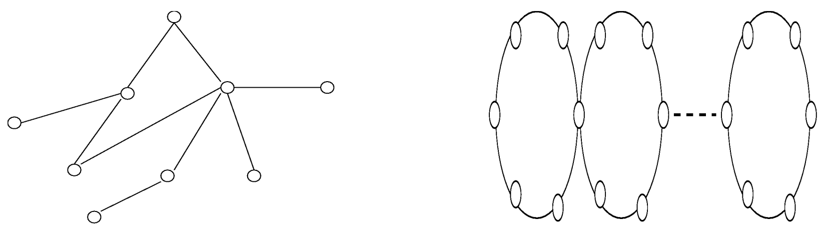
It is easy to see that the families $\mathcal{F}_t$ and $\mathcal{I}(\mathcal{F}_t)$ of minimal feasible and maximal infeasible sets for (3.3), are identified respectively with the families of all independent sets of M of size t and all flats of dimension $t-1$:

$$\mathcal{F}_t = \{X \in \mathbb{I} \ : \ |X| = t\}, \qquad \mathcal{I}(\mathcal{F}_t) = \{\text{maximal } X \in V \ : \ \rho(X) = t-1\}.$$

In particular, for $t = \rho(V)$, $\mathcal{F}_t$ and $\mathcal{I}(\mathcal{F}_t)$ correspond respectively to the bases and hyperplanes of M. By Theorem 3.1, we have the following inequality

$$|\mathcal{I}(\mathcal{X}) \cap \mathbb{H}(M)| \leq \sum_{H \in \mathcal{I}(\mathcal{X}) \cap \mathbb{H}(M)} |E \setminus H| \leq \sum_{B \in \mathcal{X}} |B| = \rho(V) \cdot |\mathcal{X}|, \tag{3.4}$$

for any nonempty $\mathcal{X} \subseteq \mathbb{B}(M)$. This inequality is sharp for the graphical matroid (where independent sets are forests) of a connected graph obtained by adding an edge to a tree as shown in Figure 3.2-a. In particular, for $\mathcal{X} = \mathbb{B}(M)$, we get $|\mathbb{H}(M)| \leq \rho(V) \cdot |\mathbb{B}(M)|$, which when applied to the dual matroid M^* gives the well-known inequality $|\mathbb{C}(M)| \leq (n - \rho(V)) \cdot |\mathbb{B}(M)|$, where $n = |V|$ (see [33, 95]). Note that for any $t \in \{0, 1, \ldots, \rho(V)\}$, the sets $\mathcal{F}_t$ and $\mathcal{I}(\mathcal{F}_t)$ correspond to the bases and hyperplanes of the *truncated* matroid M_t defined by the rank function $\rho_t : 2^V \mapsto \{0, 1, \ldots, t\}$, where $\rho_t(X) = \min\{\rho(X), t\}$. Thus we get $|\mathcal{I}(\mathcal{F}_t)| \leq t|\mathcal{F}_t|$. On the other hand, there are examples for which the number of bases is exponential in both the number of hyperplanes and the number of circuits. Consider for instance, the graphical matroid of the graph composed of k cycles of length l connected serially as shown in Figure 3.2-b. For this matroid, elements of the ground set are edges, bases correspond to spanning trees, circuits correspond to cycles, and complements of hyperplanes correspond to minimal cuts. Hence, $n = lk$, $\rho(V) = (l-1)k$, $|\mathbb{B}(M)| = l^k$, $|\mathbb{C}(M)| = k$, and $|\mathbb{H}(M)| = k\binom{l}{2}$.



a: A tight example for inequality (3.4). b: Reverse inequality does not hold.

Figure 3.2: Graphic matroid examples for inequality (3.4) .

It follows also by Theorem 3.3 that the bases of a matroid (and hence the family of independent sets of a given size) can be generated with polynomial delay and polynomial space. Moreover, incremental polynomial time algorithms, exist for generating circuits, hyperplanes, and flats of a matroid. They are based on the corresponding axioms that are necessary and sufficient for defining the matroid. For instance, it is known that a collection $\mathbb{C}$ of subsets of V is the set of circuits of a matroid on V *if and only if* it satisfies:

$(C1)$ Family $\mathbb{C}$ is Sperner, that is, if $C_1, C_2 \in \mathbb{C}$ and $C_1 \neq C_2$, then $C_1 \not\subseteq C_2$.

$(C2)$ If C_1 and C_2 are distinct members of $\mathbb{C}$ and $z \in C_1 \cap C_2$, then there exists $C_3 \in \mathbb{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus \{z\}$.

Clearly, the first element $C \in \mathbb{C}(M)$ can be found (or $\mathbb{C}(M) = \emptyset$ can be recognized) using $n+1$ calls to the independence oracle. Furthermore, the second circuit can also be identified (or $\mathbb{C}(M) = \{C\}$ can be recognized) in another set of $n+|C|$ oracle calls. Thus we may assume that we have a collection $\mathbb{C}' \subseteq \mathbb{C}(M)$ of at least two elements. To generate a new element of $\mathbb{C}(M)$, we check if axiom $(C2)$ is satisfied by $\mathbb{C}'$. If not, we get a circuit $C \in \mathbb{C}(M) \setminus \mathbb{C}'$. Otherwise, the circuit axioms imply that $\mathbb{C}' = \mathbb{C}(M')$ is the set of circuits of a matroid M' on the ground set V with rank function ρ'. If there is a $C \in \mathbb{C}(M) \setminus \mathbb{C}(M')$, i.e., $M' \neq M$, then $\rho'(V) > \rho(V)$ since for any $x \in C$ there is a $B \in \mathbb{B}(M)$ such that C is the unique circuit contained in $B \cup \{x\}$, implying that $B \cup \{x\} \in \mathbb{I}(M')$. In this case, a new circuit $C \in \mathbb{C}(M) \setminus \mathbb{C}'$ can be obtained by finding a base $B \in \mathbb{B}(M)$ and $x \notin B$ such that $B \cup \{x\} \in \mathbb{I}(M')$. Then the unique circuit $C \in B \cup \{x\}$ belongs to $\mathbb{C}(M) \setminus \mathbb{C}'$.

As mentioned earlier, although, the generation of circuits of a matroid (or, in other words, the maximal infeasible sets for a 1-smooth polymatroid inequality) is easy, this result does not generalize to 2-smooth polymatroid inequalities, and the corresponding generation problem is indeed NP-hard.

3.3 Spanning a linear space by linear subspaces

The transversal hypergraph problem is equivalent to the following set covering problem: Given an r-element ground set $\mathcal{R}$ and a family $\mathcal{V}$ of n subsets of $\mathcal{R}$, enumerate all minimal subfamilies of $\mathcal{V}$ which cover the entire set $\mathcal{R}$. Replacing $\mathcal{R}$ by the vector space $\mathbf{F}^r$ over some field $\mathbf{F}$, and replacing each given subset of $\mathcal{V}$ by a linear subspace of $\mathbf{F}^r$, we arrive at the following *space covering* problem:

Given a collection $\mathcal{V} = \{\mathcal{V}_1, \ldots, \mathcal{V}_n\}$ *of* n *linear subspaces of* $\mathbf{F}^r$, *enumerate all minimal subsets* X *of* $V = \{1, \ldots, n\}$ *such that* $\mathrm{Span}\langle\bigcup_{i \in X} \mathcal{V}_i\rangle = \mathbf{F}^r$.

Generalizing further, consider the polymatroid inequality

$$f(X) = \dim(\bigcup_{i \in X} \mathcal{V}_i) \geq t, \tag{3.5}$$

where $t \in \{1, \ldots, r\}$ is a given threshold. Then the set $\mathcal{F}$ of minimal solutions to (3.5) is the collection of all minimal subsets of $\mathcal{V}$ the dimension of whose union is at least t. Theorem 2.1 then states that for all $t \in \{1, \ldots, r\}$, the size of $\mathcal{I}(\mathcal{F})$ can be bounded by a $\log t$-degree polynomial in n and $|\mathcal{F}|$:

$$|\mathcal{I}(\mathcal{F})| \leq \max(n, |\mathcal{F}|^{(\log t)/c(n,|\mathcal{F}|)}), \tag{3.6}$$

where $c = c(n, \beta)$ is the unique positive root of the equation $2^c(n^{c/\log \beta} - 1) = 1$, and thus all sets in $\mathcal{F}$ can be enumerated in incremental quasi-polynomial time:

Corollary 3.2 *Given a collection $\mathcal{V} = \{\mathcal{V}_1, \ldots, \mathcal{V}_n\}$ of linear subspaces of $\mathbf{F}^r$, all minimal collections that span $\mathbf{F}^r$ can be enumerated in quasi-polynomial time.*

We mention here two special cases of the above space covering problem. First, when each subspace $\mathcal{V}_i$ is spanned by a subset R_i of vectors from some fixed basis of $\mathbf{F}^r$, the value of $f(X)$ is just the size of $\bigcup_{i \in X} R_i$, which is a transversal function (c.f. Section 1.3.2). Hence, by Theorem 1.6, we get the stronger inequality $|\mathcal{I}(\mathcal{F})| \leq r|\mathcal{F}|$. This also shows that incrementally generating all maximal infeasible sets $\mathcal{I}(\mathcal{F})$ for (3.5) is NP-hard.

Second, when the dimension of each input subspace $\mathcal{V}_i$, $i = 1, \ldots, n$, is bounded by some constant d, the function $f(X)$ is d-smooth. Thus by the results of Section 3.2.4, we can enumerate all sets in $\mathcal{F}$ in polynomial time and the size of $\mathcal{I}(\mathcal{F})$ can be bounded by a d-degree polynomial in n and $|\mathcal{F}|$. In particular, when $t = r$ and the given subspaces $\mathcal{V}_i$ are all lines, i.e.,

$$\mathcal{V}_i = \text{Span}\langle b_i \rangle \quad \text{for given vectors } b_i \in \mathbf{F}^r, \quad i = 1, \ldots, n,$$

then the set $\mathcal{F}$ of all column bases of the $r \times n$ matrix $[b_1, \ldots, b_n]$ can be enumerated with polynomial delay and polynomial space.

It is interesting to mention that even though the space covering problem can be solved in incremental quasi-polynomial time for any input subspaces $\mathcal{V}_1, \ldots, \mathcal{V}_n$, the following close modification of the problem is NP-hard: Enumerate all minimal subsets $X \subseteq V$ such that $\mathrm{Span}\langle\bigcup_{i\in X}\mathcal{V}_i\rangle$ contains a given linear subspace $\mathcal{V}_0$. In fact, the above *subspace* covering problem is NP-hard even when $\mathcal{V}_0$ is a line and $\dim(\mathcal{V}_i) = 2$ for all $i = 1, \ldots, n$ [62]. If all the input subspaces are lines, i.e., $\mathcal{V}_i = \mathrm{Span}\langle b_i\rangle$, $i = 0, 1, \ldots, n$, the problem calls for enumerating all minimal dependent sets ($\equiv$ circuits) containing b_0 in the vectorial matroid $M = \{b_0, b_1, \ldots, b_n\}$. The latter problem can be done in incremental polynomial time for any matroid M defined by a polynomial-time independence oracle. (First, we can enumerate all circuits of M in incremental polynomial time as explained in Section 3.2.5. Second, assuming without loss of generality that M is connected, by Lehman's theorem ([95], p.74) any circuit in M can be expressed via two circuits containing b_0, and hence (# circuits of M) $\leq$ (# circuits through $b_0)^2$. More efficient cycle generation algorithms for graphs can be found in [88].)

It is also worth mentioning the NP-hardness of the *conic* variant of the space covering problem: Given a collection of r-dimensional polyhedral cones $K_1, \ldots, K_n$ defined by their rational generators, enumerate all minimal sets $X \subseteq \{1, \ldots, n\}$ such that $\mathrm{Cone}\langle\bigcup_{i\in X} K_i\rangle$ spans the entire space. The following problem is also NP-hard [62]: Given a rational r-vector b and a collection of n dihedral cones $K_i = \mathrm{Cone}\langle a_i, a_i'\rangle$, $i = 1, \ldots, n$, enumerate all minimal sets $X \subseteq \{1, \ldots, n\}$ for which $b \in \mathrm{Cone}\langle\bigcup_{i\in X} K_i\rangle$. Replacing the input dihedral cones by rays we obtain the *vertex enumeration* problem: Enumerate all vertices of a given polyhedron $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$. The incremental complexity of the vertex enumeration problem is not known (see, e.g., [25]). In particular, it is not known whether there exists a quasi-polynomial-time algorithm which, given a rational polytope $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ and a collection Q of the vertices of P, can determine if $P = \mathrm{Conv.hull}\langle Q\rangle$.

3.4 Spanning collections of graphs

Let R be a finite set of r vertices and let $E_1, \ldots, E_n \subseteq \binom{R}{2}$ be a collection of n graphs on R. Given a set $X \subseteq \{1, \ldots, n\}$ define $\delta(X)$ to be the number of connected components in the graph $(R, \bigcup_{i\in X} E_i)$. Then $\delta(X)$ is an anti-monotone supermodular function and hence for any integral threshold t, the inequality

$$f(X) = r - \delta(X) \geq t$$

is polymatroid. In particular, $\mathcal{F}$ is the family of all minimal collections of the input graphs $E_1, \ldots, E_n$ which interconnect all vertices in R. (If the n input graphs are just n disjoint edges, then $\mathcal{F}$ is the set of all spanning trees in the graph $E_1 \cup \ldots \cup E_n$.) Since $\delta(X)$ can be evaluated at any set X in polynomial time, Theorem 1.7 implies that for each $t \in \{1, \ldots, r\}$, all elements of $\mathcal{F}$ can be enumerated in incremental quasi-polynomial time.

Corollary 3.3 *Given a collection of subgraphs $E_1, \ldots, E_n$ of a given graph G, all minimal sub-collections that interconnect the vertices of G can be generated in quasi-polynomial time.*

In particular, given a collection of n *equivalence relations* (partitions) on R, we can enumerate in incremental quasi-polynomial time all minimal subsets of the given relations whose transitive closure puts all elements of R in one equivalence class (or produces at most $r - t$ equivalence classes). Note that this example is a special case of the space covering problem, where $\mathbf{F} = \mathbf{GF}(2)$ and each subspace $\mathcal{V}_i$ is the span of the incidence vectors of the edges of the set E_i, for $i = 1, \ldots, n$.

Interestingly, enumerating all minimal collections of $E_1, \ldots, E_n$ connecting two distinguished vertices $s, t \in R$ turns out to be NP-hard even if the input sets $E_1, \ldots, E_n$ are disjoint and contain at most 2 edges each, see [49]. Needless to say that as before, generating all maximal collections of $E_1, \ldots, E_n$ for which the number of connected components of $(R, \bigcup_{i \in X} E_i)$ exceeds a given threshold remains NP-hard.

3.5 Transversals

3.5.1 Partial unions and transversals of a hypergraph; Fairly independent sets

Let $\mathcal{H}$ be an explicitly given hypergraph on a finite vertex set V. For a set $X \subseteq V$, define

$$f(X) = |\{H \in \mathcal{H} \mid X \cap H \neq \emptyset\}| \tag{3.7}$$

to be the number of hyperedges of $\mathcal{H}$ which have a non-empty intersection with X. The function f is polymatroid and $range(f) = |\mathcal{H}|$. For a given threshold $t \in \{1, \ldots, |\mathcal{H}|\}$, the hypergraph $\mathcal{F}_t$, of minimal feasible sets of the inequality $f(X) \geq t$, consists of all *partial t-transversals*, i.e., all minimal vertex sets which intersect at least t hyperedges of $\mathcal{H}$. (Equivalently, the complement to each partial t-transversal is a *fairly independent set*, i.e., a maximal vertex set containing at most $|\mathcal{H}|-t$ hyperedges of $\mathcal{H}$.) In particular, for $t = |\mathcal{H}|$, the partial transversals are exactly the minimal transversals of the hypergraph $\mathcal{H}$. On the other hand, we can identify the hypergraph $\mathcal{I}(\mathcal{F}_t)$ with the family of all maximal subsets of V that avoid at least $|\mathcal{H}| - t + 1$ hyperedges of $\mathcal{H}$. Equivalently, the complement to each set $X \in \mathcal{I}(\mathcal{F}_t)$ is a *partial* $(|\mathcal{H}| - t + 1)$-*union*, that is a minimal vertex set containing at least $|\mathcal{H}| - t + 1$ hyperedges of $\mathcal{H}$.

This example is exactly the first special case of the space covering problem described after Corollary 3.2 in Section 3.3. Indeed, for $i \in V$, define $R_i = \{H \in \mathcal{H} \mid i \in H\}$. Then it is easy to see that $\bigcup_{i \in X} R_i = \{H \in \mathcal{H} \mid X \cap H \neq \emptyset\}$ for any $X \subseteq V$. As shown in [22], the quasi-polynomial inequality (3.6) can be strengthened for the class of polymatroid functions (3.7) as follows:

$$|\mathcal{I}(\mathcal{F}_t)| \leq t|\mathcal{F}_t| \quad \text{for any } t \in \{1, \dots, |\mathcal{H}|\}. \tag{3.8}$$

Moreover, the above bounds are sharp for arbitrarily large hypergraphs $\mathcal{H}$ and $t \in \{1, \dots, |\mathcal{H}|\}$.

Given a list of hyperedges of $\mathcal{H}$ and a threshold $t \in \{1, \dots, |\mathcal{H}|\}$, we can easily check whether or not a given vertex set X contains a partial t-transversal (we only need to check if X intersects at least t hyperedges of $\mathcal{H}$). This gives a polynomial-time feasibility oracle for $f(X) \geq t$. From Theorem 1.7 we thus conclude that all partial t-transversals $X \in \mathcal{F}_t$ for a given hypergraph $\mathcal{H}$ can be generated in incremental quasi-polynomial time.

Since problem $GEN(\mathcal{F}_t, \mathcal{X})$ can be solved in quasi-polynomial time, it is unlikely to be NP-hard. In contrast to this result, problem $GEN(\mathcal{I}(\mathcal{F}_t), \mathcal{Y})$ is known to be NP-hard. Specifically, it is NP-complete to decide if $\mathcal{X} \neq \mathcal{I}(\mathcal{F}_t)$ for an explicitly given set family $\mathcal{X} \subseteq \mathcal{I}(\mathcal{F}_t)$ (see [71] and also [21] for more detail).

3.5.2 Maximal frequent and minimal infrequent sets (elements) for binary matrices (databases)

The notion of frequent sets in data-mining can be related naturally to the polymatroid separator (3.7) considered above. Let V be a finite set of binary attributes of a database, and let $\mathcal{D}$:

$\mathcal{R} \times V \mapsto \{0,1\}$ be a given $r \times n$ binary matrix representing a set $\mathcal{R}$ of transactions over V. To each subset of columns $X \subseteq V$, we associate the subset $S(X) = S_{\mathcal{D}}(X) \subseteq \mathcal{R}$ of all those rows $i \in \mathcal{R}$ for which $\mathcal{D}(i,j) = 1$ in every column $j \in X$. The cardinality of $S(X)$ is called the *support* of X and is easily seen to be an anti-monotone supermodular function of X. Hence

$$f(X) = r - |S(X)| \tag{3.9}$$

is a polymatroid function of range r. In fact, the above definition is identical to that given in (3.7) if we let $\mathcal{H}$ to be the hypergraph defined by the anti-incidence matrix $\mathcal{D}$.

A column set $X \subseteq V$ is called *t-frequent* if $|S(X)| \geq t$ and otherwise, is said to be *t-infrequent*. Thus the families $\mathcal{F}_{\mathcal{D},t}$ and $\mathcal{I}(\mathcal{F}_{\mathcal{D},t})$ of minimal feasible and maximal infeasible sets for the inequality $f(X) \geq r - t + 1$ correspond, respectively, to the minimal t-infrequent and maximal t-frequent sets for $\mathcal{D}$.

The generation of (maximal) frequent sets of a given binary matrix is an important task of knowledge discovery and data mining, e.g. it is used for mining association rules [1, 48, 74, 75, 78, 79, 96], correlations [26], sequential patterns [3], episodes [76], emerging patterns [36], and appears in many other applications. It follows by (3.8) that the hypergraph of minimal infrequent sets is dual bounded:

$$|\mathcal{I}(\mathcal{F}_{\mathcal{D},t})| \leq (r - t + 1)|\mathcal{F}_{\mathcal{D},t}| \tag{3.10}$$

for all $t \in \{1, \ldots, r\}$. Let us note again that these inequalities are best possible. For instance, they are sharp when $\mathcal{D}$ is an $r \times (r-t+1)$ matrix in which every entry is 1, except the diagonal entries in the first $r - t + 1$ rows, which are 0. In addition, (3.10) stays accurate, up to a factor of $\log r$, even when $r \gg n$ and $|\mathcal{F}_{\mathcal{D},t}|$ and $|\mathcal{I}(\mathcal{F}_{\mathcal{D},t})|$ are arbitrarily large. Let us consider for instance a binary matrix $\mathcal{D}$ with $r = 2^k$ rows and $n = 2k$ columns ($k \geq 1$, integer), such that each row contains exactly one 0 and one 1 in each pair of the adjacent columns $\{1,2\}$, $\{3,4\}$, ..., $\{2k-1, 2k\}$, and in all 2^k possible ways in the $r = 2^k$ rows. Then for $t = 1$, there are 2^k maximal 1-frequent sets, and only k minimal 1-infrequent sets, namely $\{2i-1, 2i\}$ for $i = 1, \ldots, k$. Thus for such examples we have $|\mathcal{I}(\mathcal{F}_{\mathcal{D},t})| = (r/\log r)|\mathcal{F}_{\mathcal{D},t}|$. Examples also exist for which $|\mathcal{F}_{\mathcal{D},t}|$ cannot be bounded by a quasi-polynomial in $|\mathcal{I}(\mathcal{F}_{\mathcal{D},t})|$, n and r. For instance, consider the $k \times 2k$ binary matrix $\mathcal{D}$ where, for $i = 1, \ldots, k,$, row i has ones everywhere except in columns $\{2i-1, 2i\}$. Then, for $t = 1$, the size of $\mathcal{F}_{\mathcal{D},t}$ is k, whereas the size of $\mathcal{I}(\mathcal{F}_{\mathcal{D},t})$ is 2^k. Indeed, it was shown in [21] that problem GEN($\mathcal{I}(\mathcal{F}_{\mathcal{D},t}), \mathcal{Y}$) is NP-hard even if $|\mathcal{Y}| = O(n)$ and $|\mathcal{I}(\mathcal{F}_{\mathcal{D},t})|$ is exponentially large in n, whenever $\mathcal{Y} \neq \mathcal{I}(\mathcal{F}_{\mathcal{D},t})$.

There is a natural generalization of frequent sets for databases defined over products of partially ordered sets (posets). Specifically, let $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ be the product of n posets and consider a database $\mathcal{D} \subseteq \mathcal{P}$ of transactions, each of which is an n-dimensional vector of attribute values over $\mathcal{P}$. For an element $p \in \mathcal{P}$, let us denote by

$$S(p) = S_{\mathcal{D}}(p) \stackrel{\text{def}}{=} \{q \in \mathcal{D} \mid q \succeq p\},$$

the set of transactions in $\mathcal{D}$ that *support* $p \in \mathcal{P}$. Note that, by this definition, the function $|S(\cdot)| : \mathcal{P} \mapsto \{0, 1, \ldots, |\mathcal{D}|\}$ is an anti-monotone function, i.e., $|S(p)| \leq |S(q)|$, whenever $p \succeq q$.

Given $\mathcal{D} \subseteq \mathcal{P}$ and an integer threshold t, let us say that an element $p \in \mathcal{P}$ is t-frequent if it is supported by at least t transactions in the database, i.e., if $|S_{\mathcal{D}}(p)| \geq t$. Conversely, $p \in \mathcal{P}$ is said to be t-infrequent if $|S_{\mathcal{D}}(p)| < t$. Denote by $\mathcal{F}_{\mathcal{D},t}$ the set of all minimal t-infrequent elements of $\mathcal{P}$ with respect to the database $\mathcal{D}$. Then $\mathcal{I}(\mathcal{F}_{\mathcal{D},t})$ is the set of all maximal t-frequent elements.

As in the Boolean case, the separate and joint generation of maximal frequent and minimal infrequent elements of a database can be used for finding association rules in data mining applications. If the database $\mathcal{D}$ contains categorical (e.g., zip code, make of car), or quantitative (e.g., age, income) attributes, and the corresponding posets $\mathcal{P}_i$ are total orders, then the above generation problems can be used to mine the so called *quantitative* association rules [91]. More generally, each attribute a_i in the database can assume values belonging to some partially ordered set $\mathcal{P}_i$. For example, [90] describes applications where items in the database belong to sets of *taxonomies* (or *is-a hierarchies*), see also [52, 53]). Furthermore, many data analysis applications assume data values ranging over lattices of small size, see e.g. [23].

Extending the results stated above for the Boolean case, it easily follows from the intersection lemma (Lemma 3.1), to be stated in Section 3.7.1, that if each poset $\mathcal{P}_i$ is a join semi-lattice (i.e. for every two elements $x, y \in \mathcal{P}_i$, there is a unique upper bound $x \vee y$), then for any integer threshold t, the set of minimal infrequent elements $\mathcal{F}_{\mathcal{D},t}$ of a database $\mathcal{D}$ is uniformly dual-bounded:

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_{\mathcal{D},t})| \leq (|\mathcal{D}| - t + 1)|\mathcal{X}| \tag{3.11}$$

for any nonempty $\mathcal{X} \subseteq \mathcal{F}_{\mathcal{D},t}$. Thus combining (3.11) with the fact that dualization on products of rooted-trees, and bounded-width join semi-lattices, can be solved in quasi-polynomial time (see Chapter 4), we obtain the following result.

Corollary 3.4 *Let $\mathcal{D} \subseteq \mathcal{P}$ be a database whose attributes assume values ranging over the product $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$ of n partially ordered sets, each of which is a join semi-lattice, with either acyclic precedence graph, or bounded-width. Then for any integer threshold t, all minimal infrequent elements for $\mathcal{D}$ can be generated in quasi-polynomial time.*

3.5.3 Weighted transversals

Extending the notion of partial t-transversals to weighted hypergraphs we arrive at weighted transversals [22]. Given a hypergraph $\mathcal{H} \subseteq 2^V$, we assign an r-dimensional non-negative integral weight vector $w = w(H) \in \mathbb{Z}_+^r$ to each hyperedge $H \in \mathcal{H}$ and consider the system of r polymatroid inequalities

$$f(X) = \sum \{w(H) \mid X \cap H \neq \emptyset,\ H \in \mathcal{H}\} \geq t \tag{3.12}$$

where $t \in \mathbb{Z}_+^r$ is a given threshold vector. The minimal solutions $X \subseteq V$ to the above system are called *weighted* transversals for $(\mathcal{H}, w)$. Let $\mathcal{F}_{w,t}$ be the set of all weighted transversals for (3.12) and let $\mathcal{I}(\mathcal{F}_{w,t})$ denote the hypergraph of all maximal infeasible sets for (3.12). Considering that each weighted hypergraph $\mathcal{H}$ can be regarded as a collection of r multi-hypergraphs $(\mathcal{H}_1, \ldots, \mathcal{H}_r)$, where multi-hypergraph $\mathcal{H}_i$ contains $w_i(H)$ copies of edge $H \in \mathcal{H}$, we conclude from (3.8) that $|\mathcal{I}(\mathcal{F}_{w,t})| \leq (t_1 + \cdots + t_r)|\mathcal{F}_{w,t}|$. This bound depends on the threshold vector and can be arbitrarily large with respect to n and r when the range of (3.12) becomes high. However, it was shown in [22] that for any non-negative real-valued weights $w(H)$, and for any non-empty hypergraph $\mathcal{X} \subseteq \mathcal{F}_{w,t}$ we have the inequality

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_{w,t})| \leq r \sum_{X \in \mathcal{X}} |\{H \in \mathcal{H} \mid H \cap X \neq \emptyset\}|, \tag{3.13}$$

see Theorem 1.6. In particular, it follows that $|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_{w,t})| \leq r|\mathcal{H}||\mathcal{X}|$, which for $\mathcal{X} = \mathcal{F}_{w,t}$ gives $|\mathcal{I}(\mathcal{F}_{w,t}| \leq r|\mathcal{H}||\mathcal{F}_{w,t}|$.

This implies the incremental quasi-polynomial generation of all minimal weighted transversals for a given weighted hypergraph $(\mathcal{H}, w)$ and threshold vector t. Generating all maximal infeasible vectors to (3.12) is NP-hard already for scalar unit weights $w(H) \equiv 1$.

3.6 Monotone systems of linear inequalities in integer variables

Consider a system of r linear inequalities in n integer variables

$$Ax \geq b, \quad x \in \mathcal{C} = \{x \in \mathbb{Z}^n \mid 0 \leq x \leq c\}, \tag{3.14}$$

where $A \in \mathbb{R}^{r \times n}$ is a given $r \times n$-matrix, $b \in \mathbb{R}^r$ is a given r-vector, and c is a non-negative integral n-vector some or all of whose components may be infinite. We assume that (3.14) is a monotone system of inequalities: if $x \in \mathcal{C}$ satisfies (3.14) then any vector $y \in \mathcal{C}$ such that $y \geq x$ is also feasible. For instance, (3.14) is monotone if the matrix A is non-negative. Let us denote by $\mathcal{F} = \mathcal{F}_{A,b,c}$ the set of all minimal feasible integral vectors for (3.14). If A is a binary matrix, and b, c are vectors of all ones, then $\mathcal{F}$ is the set of (characteristic vectors of) all minimal transversals to the hypergraph defined by the rows of A. In this case, problem $GEN(\mathcal{F}_{A,b,c}, \mathcal{X})$ turns into the well-known hypergraph dualization problem. More generally, for binary variables, non-negative matrix A, and any vector b, we get a special case of weighted transversals: Let V be the set of columns of A, and let $\mathcal{H}$ be the hypergraph on the vertex set V whose hyperedges are the n singletons (columns). We have $|\mathcal{H}| = n$ and each column of the $r \times n$ matrix A can now be interpreted as a non-negative r-dimensional weight vector assigned to the corresponding hyperedge of $\mathcal{H}$. Under this interpretation, (3.14) is a special case of (3.12) and we can identify the hypergraph $\mathcal{F}$ of weighted transversals with the set of all minimal Boolean solutions to (3.14). Accordingly, the hypergraph $\mathcal{A} = \mathcal{I}(\mathcal{F})$ can be viewed as the set of all maximal infeasible vectors for (3.14). From (3.13) we now conclude that for any non-empty set $\mathcal{X} \subseteq \mathcal{F}$ of minimal feasible solutions to (3.14), with $c = \mathbf{e} \stackrel{\text{def}}{=} (1, 1, \ldots, 1)$, we have the inequalities

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq r \sum_{x \in \mathcal{X}} p(x) \leq nr|\mathcal{X}|, \tag{3.15}$$

where $p(x)$ is number of positive components of x. In particular, for any feasible system (3.14) we obtain

$$|\mathcal{I}(\mathcal{F})| \leq nr|\mathcal{F}|. \tag{3.16}$$

It should be mentioned that the above bounds are sharp for $r = 1$, e.g., for the inequality $x_1 + \ldots + x_n \geq n$ in binary variables. For large r, these bounds are accurate up to a factor poly-logarithmic in r. To see this, let $n = 2k$ and consider the monotone system of $r = 2^k$ inequalities of the form

$$x_{i_1} + x_{i_2} + \cdots + x_{i_k} \geq 1, \; i_1 \in \{1, 2\}, \; i_2 \in \{3, 4\}, \ldots, i_k \in \{2k-1, 2k\},$$

where $x = (x_1, \ldots, x_n) \in \{0, 1\}^n$. This system has 2^k maximal infeasible integral vectors and only k minimal feasible integral vectors, i.e.,

$$|\mathcal{I}(\mathcal{F}_{A,b,c})| = \frac{rn}{2(\log r)^2} |\mathcal{F}_{A,b,c}|.$$

Needless to say that in general, $|\mathcal{F}_{A,b,c}|$ cannot be bounded by a polynomial in r, n, and $|\mathcal{I}(\mathcal{F}_{A,b,c})|$. For instance, for $n = 2k$ the system of k inequalities $x_1 + x_2 \geq 1, \quad x_3 + x_4 \geq 1, \ldots, \; x_{2k-1} + x_{2k} \geq 1$ has 2^k minimal feasible binary vectors and only k maximal infeasible binary vectors.

In the next subsection, we show that, in fact, inequalities (3.15) and (3.16) actually hold for *any monotone* system of linear, or more generally *2-monotonic*, inequalities (3.14) in *integer* variables, i.e., for arbitrary non-negative integral vectors c.

Let $J^* = \{j \mid c_j = \infty\}$ and $J_* = \{1, \ldots, n\} \setminus J^*$ be, respectively, the sets of unbounded and bounded integer variables in (3.14). Consider an arbitrary vector $x = (x_1, \ldots, x_n) \in \mathcal{F}_{A,b,c}$ such that $x_j > 0$ for some $j \in J^*$. Then it is easy to see that

$$x_j \leq \max_{i: a_{ij} > 0} \left\lceil \frac{b_i - \sum_{k \in J_*} \min\{0, a_{ik}\} c_k}{a_{ij}} \right\rceil < +\infty. \tag{3.17}$$

[Indeed, let x' be the vector obtained by decreasing the j^{th} component of x by 1, then $x' \in \mathcal{C}$ is infeasible for (3.14) and hence $b_i - a_{ij} \leq a^i x - a_{ij} = a^i x' < b_i$ for some $i \in \{1, \ldots, r\}$, where a^i is the i^{th} row of A, implying $a_{ij} > 0$. Thus $a^i x < b_i + a_{ij}$. Since $a_{ij}x_j + \sum_{k \in J_*} \min\{0, a_{ik}\}c_k \leq a_{ij}x_j + \sum_{k \in J_*} \min\{0, a_{ik}\}x_k \leq a_{ij}x_j + \sum_{k \in J_*} a_{ik}x_k \leq a^i x$, where the last inequality follows from the non-negativity of the restriction of A on J^*, we have $a_{ij}x_j + \sum_{k \in J_*} \min\{0, a_{ik}\}c_k < b_i + a_{ij}$. This implies (3.17).]

Since the bounds of (3.17) are easy to compute, and since appending these bounds to (3.14) does not change the set $\mathcal{F}_{A,b,c}$, we shall assume in the sequel that all components of the non-negative vector c are finite, even though this may not be the case for the original system. This assumption does not entail any loss of generality and allows us to consider $\mathcal{F}_{A,b,c}$ as a system of integral vectors in a finite box (product of chains). We shall also assume that the input monotone system (3.14) is feasible, i.e., $\mathcal{F}_{A,b,c} \neq \emptyset$. For a finite and non-negative c this is equivalent to $Ac \geq b$. In addition, we say that system (3.14) is non-trivial if $\mathcal{F}_{A,b,c} \neq \mathcal{C}$, i.e., $0 \notin \mathcal{F}_{A,b,c}$.

3.6.1 2-monotonic functions

Let $\mathcal{C} = \{x \in \mathbb{Z}^n \mid 0 \leq x \leq c\}$ be an integral box and let $f : \mathcal{C} \to \mathbb{R}$ be a monotone function over the elements of $\mathcal{C}$. Let $\sigma \in \mathbb{S}_n$ be a permutation of the coordinates and let x, y be two n-vectors. We say that y is obtained from x by a series of *left-shifts* and write $y \succeq_\sigma x$ if the inequalities

$$\sum_{j=1}^{k} y_{\sigma_j} \geq \sum_{j=1}^{k} x_{\sigma_j}$$

hold for all $k = 1, \ldots, n$. Equivalently, y is a left-shift of x, if there are indices $1 \leq i, j \leq n+1$ with $\sigma_i < \sigma_j$ such that $x_{\sigma_k} = y_{\sigma_k}$ for $k \notin \{i, j\}$, $y_{\sigma_i} > x_{\sigma_i}$ and $y_{\sigma_j} \leq x_{\sigma_j}$ (where $x_{n+1} = 1$ for all $x \in \mathcal{C}$, by definition), and we write $y \succeq_\sigma x$ if y can be obtained from x by a series of left-shifts.

A function $f : \mathcal{C} \to \mathbb{R}$ is called *2-monotonic with respect to* σ if $f(y) \geq f(x)$ whenever $y \succeq_\sigma x$ and $x, y \in \mathcal{C}$. Clearly, $y \geq x$ implies $y \succeq_\sigma x$ for any $\sigma \in \mathbb{S}_n$, so that any 2-monotonic function is monotone.

The function f will be called *regular* if it is 2-monotonic with respect to the identity permutation $\sigma = (1, 2, ..., n)$. Any 2-monotonic function can be transformed into a regular one by appropriately re-indexing its variables. To simplify notation, we shall state Theorem 3.4 below for regular functions, i.e., we fix $\sigma = (1, 2, ..., n)$, and use $\preceq$ instead of $\preceq_\sigma$ in this theorem.

For a vector $x \in \mathcal{C}$, and for an index $1 \leq k \leq n$, let the vectors $x(k]$ and $x[k)$ be defined by

$$x_j(k] = \begin{cases} x_j & \text{for } j \leq k, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$x_j[k) = \begin{cases} x_j & \text{for } j \geq k, \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by $\mathbf{e}$ the n-vector of all 1's, let $\mathbf{e}^j$ denote the j^{th} unit vector, $j = 1, \ldots, n$, and let $p(x)$ denote the number of positive components of the vector $x \in \mathcal{C}$.

Theorem 3.4 *Let $\mathcal{F}$ be the set of all minimal feasible solutions for a feasible 2-monotonic inequality $f(X) \geq t$ and let $\mathcal{X} \subseteq \mathcal{F}^+$ be an arbitrary subset of $\mathcal{F}^+$ of size $|\mathcal{X}| \geq 1$. Then*

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq \sum_{x \in \mathcal{X}} p(x). \tag{3.18}$$

Proof. For a vector $y \in \mathcal{C} \backslash \{c\}$ let us denote by $l = l_y$ the index of the last component which is less than c_l, i.e. $l = \max\{j \mid y_j < c_j\} \in \{1, \ldots, n\}$. We claim that for every $y \in \mathcal{Y} \stackrel{\text{def}}{=} \mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})$ there exists an $x \in \mathcal{X}$ such that

$$y = x(l-1] + (x_l - 1)\mathbf{e}^l + c[l+1), \tag{3.19}$$

where $l = l_y$. To see this claim, first observe that $y \neq c$ because $y \in \mathcal{I}(\mathcal{F})$ and $\mathcal{F} \neq \emptyset$. Second, for any j with $y_j < c_j$ we have $y + \mathbf{e}^j \in \mathcal{X}^+$, by the maximality of y in $\mathcal{C} \setminus \mathcal{X}^+$. Hence there exists a minimal feasible vector $x \in \mathcal{X}$ such that $x \leq y + \mathbf{e}^l$ for $l = l_y$. We must have $x(l-1] = y(l-1]$, since if $x_i < y_i$ for some $i < l$, then $y \geq x + \mathbf{e}^i - \mathbf{e}^l \succeq x$ would hold, i.e. $y \succeq x$ would follow, implying $f(y) \geq f(x) \geq t$ by the 2-monotonicity of f, and yielding a contradiction with $f(y) < t$ (which follows from $y \in \mathcal{I}(\mathcal{F})$). Finally, the definition of $l = l_y$ implies that $y[l+1) = c[l+1)$. Hence, our claim and the equality (3.19) follow.

The above claim implies that

$$\mathcal{Y} \subseteq \{x(l-1] + (x_l - 1)\mathbf{e}^l + c[l+1) \mid x \in \mathcal{X},\ x_l > 0\},$$

and hence (3.18) and thus the theorem follow. $\square$

Remark. The above theorem can be easily generalized to 2-monotonic functions in products of partially ordered sets, each of which has a unique maximum element. Let $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ be such a product, let $f : \mathcal{P} \mapsto \mathbb{R}$ be a 2-monotonic function over the elements of $\mathcal{P}$, and let $\mathcal{F}$ be the set of minimal elements x of $\mathcal{P}$ satisfying $f(x) \geq t$ for some real threshold t. Then for any non-empty $\mathcal{X} \subseteq \mathcal{F}^+$, we have the inequality

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F})| \leq \sum_{x \in \mathcal{X}} \sum_{i=1}^{n} |\mathcal{I}(\{x_i\})| \leq n\mu|\mathcal{X}|,$$

where $\mu \stackrel{\text{def}}{=} \max\{|\mathcal{P}_i| \ : \ i \in [n]\}$.

3.6.2 Systems of linear inequalities

It now follows easily from Theorem 3.4 that the set $\mathcal{F}_{A,b,c}$ for a monotone system (3.14) is uniformly dual-bounded. Indeed, given the i^{th} inequality of the system $a_{i1}x_1 + \ldots + a_{in}x_n \geq b_i$, suppose that some of the coefficients $a_{i1}, \ldots, a_{in}$ are negative, say $a_{i1} < 0, \ldots, a_{ik} < 0$ and $a^i[k+1) \geq 0$, where a^i is the i^{th} row of A. Since $a^i x \geq b_i$ for any $x \in \mathcal{F}_{A,b,c}$, and since the system (3.14) is monotone, we have $x \in \mathcal{F} \Rightarrow a^i[k+1)x \geq b_i - a^i(k]c(k]$. By the negativity of the coefficients $a_{i1}, \ldots, a_{ik}$, we also have $\{x \in \mathcal{C} \mid a^i x < b_i\} \subseteq \{x \in \mathcal{C} \mid a^i x < b_i - a^i(k]c(k]\}$. Hence it suffices to prove the inequality for the non-negative weight vector $a^i[k+1)$ and the threshold $b_i - a^i(k]c(k]$. In other words, we can assume without loss of generality that the original row vector a^i is non-negative.

Let $\sigma \in \mathbb{S}^n$ be a permutation such that $a_{k\sigma_1} \geq a_{k\sigma_2} \geq \cdots \geq a_{k\sigma_n}$. Then the function $f : \mathcal{C} \mapsto \mathbb{R}$ defined by $f(X) = a_{k1}x_1 + \ldots + a_{kn}x_n$ is 2-monotonic with respect to σ. Since this statement is true for all inequalities $k = 1, \ldots, r$, we can apply Theorem 3.4 and Proposition 1.2 to arrive at the following result.

Theorem 3.5 *Let $\mathcal{F}_{A,b,c}$ be the set of all minimal feasible solutions of a monotone system of r linear inequalities (3.14). Then for any non-empty $\mathcal{X} \subseteq \mathcal{F}_{A,b,c}$, we have*

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_{A,b,c})| \leq r \sum_{x \in \mathcal{X}} p(x),$$

where $p(x)$ is the number of positive components of x. In particular,

$$|\mathcal{I}(\mathcal{X}) \cap \mathcal{I}(\mathcal{F}_{A,b,c})| \leq rn|\mathcal{X}|,$$

which for $\mathcal{X} = \mathcal{F}_{A,b,c}$ implies the inequality $|\mathcal{I}(\mathcal{F}_{A,b,c})| \leq rn|\mathcal{F}_{A,b,c}|$.

Combining this result with the joint generation procedure on products of chains (or more generally posets) described in Chapter 1, and the fact that the dualization problem on such a product can be solved in *strongly* quasi-polynomial time (see Chapter 4), we obtain the following result.

Corollary 3.5 *All minimal feasible integer solutions for a given monotone system of linear inequalities (3.14) can be generated incrementally in strongly quasi-polynomial time.*

This should be contrasted with the conjecture of [65] that for non-negative A and $c = (1, \ldots, 1)$ this problem cannot be solved in incremental polynomial time unless P=NP. On the other hand, the problem of generating all maximal infeasible binary vectors for (3.14) is NP-hard already for binary matrices A, see [70] and also [20] for more detail.

3.6.3 Bounding the number of minimal feasible solutions for systems with a bounded number of inequalities

Even though generating all maximal infeasible vectors for (3.14) is NP-hard, this problem can be solved efficiently if the number of inequalities in (3.14) is fixed. Specifically, for $r = const$ the size of $\mathcal{F}_{A,b,c}$ can be bounded by a polynomial in n and $|\mathcal{I}(\mathcal{F}_{A,b,c})|$ and consequently, all elements of $\mathcal{I}(\mathcal{F}_{A,b,c})$ can be generated in quasi-polynomial time. In fact, as it will be shown in the next subsection, for $r = const$ both problems $\mathrm{GEN}(\mathcal{F}_{A,b,c}, \mathcal{X})$ and $\mathrm{GEN}(\mathcal{I}(\mathcal{F}_{A,b,c}), \mathcal{Y})$ can be separately solved in incremental polynomial time.

Theorem 3.6 *Suppose that the monotone system (3.14) is non-trivial, i.e., $0 \notin \mathcal{F}_{A,b,c}$. Then for any non-empty subset $\mathcal{Y} \subseteq \mathcal{I}(\mathcal{F}_{A,b,c})$ we have*

$$\left|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{F}_{A,b,c}\right| \leq \Big(\sum_{y \in \mathcal{Y}} q(y)\Big)^r, \tag{3.20}$$

where $\mathcal{I}^{-1}(\mathcal{Y})$ *is the set of all minimal integral vectors of the ideal* $\mathcal{C} \setminus \mathcal{Y}^-$ *and* $q(y)$ *is the number of components* y_l *such that* $y_l < c_l$*. In particular,* $\left|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{F}_{A,b,c}\right| \leq \Big(n|\mathcal{Y}|\Big)^r$*, which for* $\mathcal{Y} = \mathcal{I}(\mathcal{F}_{A,b,c})$ *implies* $|\mathcal{F}_{A,b,c}| \leq \Big(n|\mathcal{I}(\mathcal{F}_{A,b,c})|\Big)^r$*.*

Proof. Let us consider an arbitrary non-empty antichain $\mathcal{Y} \subseteq \mathcal{I}(\mathcal{F}_{A,b,c})$. For any $y \in \mathcal{I}(\mathcal{F}_{A,b,c})$ we can find an index $i = \rho(y) \in [r]$ such that y violates the i^{th} inequality of the system, i.e., $a^i y < b_i$, where a^i and b_i denote the i^{th} row and component of A and b, respectively.

Consider a vector $x \in \mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{F}_{A,b,c}$ and let x_l be a positive component of x. Then there exists a vector $y^l \in \mathcal{Y}$ such that $y^l \geq x - \mathbf{e}^l$. Let $i = \rho(y^l)$ and assume without loss of generality that

$$a_1^i \geq a_2^i \geq \ldots \geq a_n^i. \tag{3.21}$$

We claim that $x(l] = z^l(l]$, where

$$z^l = y^l(l] + \mathbf{e}^l. \tag{3.22}$$

This follows from (3.19) since the maximal feasible solutions of the inequality $a^i y < b_i$ are exactly the minimal feasible solutions of the 2-monotonic inequality $a^i z > a^i c - b_i$, with $z = c - y$. This shows that $x(l] = z^l(l]$ and consequently leads to the representation

$$x = \bigvee_{l \in [n]:\, x_l > 0} z^l, \tag{3.23}$$

where for vectors $v, u \in \mathcal{C}$ we let $v \vee u$ denote the component-wise maximum of v and u.

Not all of the vectors z^l are necessary for this representation. Suppose that $\rho(y^l) = \rho(y^{l'}) = i$ for some positive components x_l and $x_{l'}$ of x, and $l' < l$. Then (3.23) remains valid if we drop $z^{l'}$, the vector with the smaller index l'. In other words, by sorting the i^{th} row of A and then selecting among the vectors $y^l \in \rho^{-1}(i)$ the one with the highest $l = l_i$, we obtain at most r vectors z^{l_i} such that

$$x = \bigvee_{i \in [r]} z^{l_i}. \tag{3.24}$$

The latter representation readily implies (3.20). □

3.6.4 Polynomial generation of $\mathcal{F}_{A,b,c}$ and $\mathcal{I}(\mathcal{F}_{A,b,c})$ for linear systems with a bounded number of inequalities

Theorem 3.6 implies that for $r \leq const$ the antichain $\mathcal{I}(\mathcal{F}_{A,b,c})$ is uniformly dual-bounded and consequently, $\mathcal{I}(\mathcal{F}_{A,b,c})$ can be generated in incremental quasi-polynomial time. In this section we show that for bounded r, the antichains $\mathcal{I}(\mathcal{F}_{A,b,c})$ and $\mathcal{F}_{A,b,c}$ can, in fact, be generated in incremental polynomial time. Since $|\mathcal{F}_{A,b,c}|$ and $|\mathcal{I}(\mathcal{F}_{A,b,c})|$ are (uniformly) polynomially related by Theorems 3.6 and 3.7, the required result will follow from Corollary 1.2, provided that the dualization step, can be done in polynomial time. Thus, it is enough to show that problem DUAL$(\mathcal{C}, \mathcal{A}, \mathcal{B})$ can be solved in polynomial time, if $\mathcal{A} \subseteq \mathcal{F}_{A,b,c}$ is a subset of the minimal solutions of (3.14), with bounded r.

For $i \in [r]$, let $\sigma_{(i)} = (\sigma_{(i)1}, \ldots, \sigma_{(i)n}) \in \mathbb{S}_n$ be a permutation of the coordinates such that

$$a^i_{\sigma_{(i)1}} \geq a^i_{\sigma_{(i)2}} \geq \ldots \geq a^i_{\sigma_{(i)n}}. \tag{3.25}$$

Given $\mathcal{A} \subseteq \mathcal{F}_{A,b,c}$ and $\mathcal{B} \subseteq \mathcal{I}(\mathcal{F}_{A,b,c})$, we may assume that $0 \notin \mathcal{A}$ (otherwise, $\mathcal{F}_{A,b,c} = \{0\}$ and $\mathcal{I}(\mathcal{F}_{A,b,c}) = \emptyset$) and $\mathcal{B} \neq \emptyset$ (otherwise, one can easily generate a point $x \in \mathcal{C} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$, see Proposition 4.3 of the next chapter.) Now we proceed in two basic steps:

Step 1. For every $y \in \mathcal{B}$ and for each pair of indices $\sigma_{(i)j}$ and $\sigma_{(i)l}$ with $y_{\sigma_{(i)j}} > 0$, $y_{\sigma_{(i)l}} < c_{\sigma_{(i)l}}$ and $j < l$, check if there exists a $y' \in \mathcal{B}$ such that

$$y' \geq y - \mathbf{e}^{\sigma_{(i)j}} + \mathbf{e}^{\sigma_{(i)l}}, \tag{3.26}$$

where $i = \rho(y)$ is the index of an infeasible inequality for y, as defined in the previous section. Note that $y - \mathbf{e}^{\sigma_{(i)j}} + \mathbf{e}^{\sigma_{(i)l}}$ is infeasible, and hence is an independent element of $\mathcal{A}$. If no such y' can be found, we generate a new maximal independent vector $y' \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$, satisfying (3.26), and halt.

Step 2. For every collection $(y^i \in \rho_{\mathcal{B}}^{-1}(i) \mid i \in [r])$, where $\rho_{\mathcal{B}}^{-1}(i) \stackrel{\text{def}}{=} \{y \in \mathcal{B} \mid \rho(y) = i\}$, and for every $(l_i \mid i \in [r], y^i_{l_i} < c_{l_i}) \in [n]^r$, construct the vector $x = \bigvee_{i \in [r]} z^{l_i}$, where z^{l_i} is given by (3.22) (according to the permutation $\sigma_{(i)}$ and using $y^l = y^{l_i}$.) If $x \notin \mathcal{A}^+ \cup \mathcal{B}^-$, then a new maximal independent vector can be generated.

Clearly, the above two steps run in $poly(n, m) + O((n|\mathcal{B}|)^r)$ time, which is polynomially bounded for constant r, where $m = |\mathcal{A}| + |\mathcal{B}|$. It is also clear that if the algorithm outputs a point $x \in \mathcal{C}$, then $x \notin \mathcal{A}^+ \cup \mathcal{B}^-$, so it remains to verify that the algorithm indeed outputs such a point if $\mathcal{A}^+ \cup \mathcal{B}^- \neq \mathcal{C}$. To see this, let x be a minimal vector in $\mathcal{C} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. From our assumptions, it follows that $x \neq 0$, and thus, there exists an index l with $x_l > 0$. By the minimality of x, there exists a vector $y \,(= y^l) \in \mathcal{B}$ such that $y \geq x - \mathbf{e}^l$. Let $i = \rho(y)$, assume without loss of generality that (3.21) holds, and consider an index $j < l$. If $y^l_j > x_j$, we get $y^l - \mathbf{e}^j + \mathbf{e}^l \geq x$, and therefore a new maximal independent point $x' \geq x$ must have been output in Step 1 of the algorithm (c.f. (3.26)). On the other hand, if for every $l \in [n]$ such that $x_l > 0$ and for every $y^l \in \mathcal{B}$ such that $y^l \geq x - \mathbf{e}^l$, we have $x(l] = z^l(l]$ (in the ordering implied by $\sigma(i)$, where $i = \rho(y^l)$), then we can conclude that x satisfies (3.24), and consequently, it must have been created in Step 2.

Finally, we mention that all the results of this section remain valid for systems of 2-monotonic inequalities in integer variables. The case of a single Boolean 2-monotonic inequality is discussed in [10, 24, 30, 51, 73, 82, 83].

3.7 Systems of non-linear inequalities; Functions with non-negative Möbius coefficients

Generalizing the results of the previous section, let $f_i : \mathbb{Z}^n \mapsto \mathbb{R}$, $i = 1, \ldots, r$, be monotone real-valued functions, and let us consider the *anti-monotone* system of inequalities

$$\begin{aligned} &f_i(x) \leq b_i, \qquad i = 1, \ldots, r, \\ &l \leq x \leq u, \qquad x \in \mathbb{Z}^n, \end{aligned} \tag{3.27}$$

where b_i, $i = 1, \ldots, r$ are given reals, and l and u are given rational n-vectors. Let $\mathcal{F}$, $\mathcal{I}^{-1}(\mathcal{F})$ denote respectively the set of *maximal feasible* and *minimal infeasible* vectors for (3.27). Let $\mathcal{C}_i = \{l_i, \ldots, u_i\} \subseteq \mathbb{Z}$, and let $\mathcal{C} = \mathcal{C}_1 \times \ldots \times \mathcal{C}_n$ be a product of chains which contains $\mathcal{F}$. Clearly, the linear system (3.14) is a special case of (3.27) if we rewrite it in the form $A(c - x) \leq Ac - b$.

As usual, we shall restrict our discussion only to submodular (or supermodular) functions, and specifically among these, to the class of functions having non-negative *Möbius* coefficients.

For a product of chains $\mathcal{C}$, recall that the Möbius function $\mu : \mathcal{C} \times \mathcal{C} \mapsto \{-1, 0, 1\}$ is given by (see, e.g., [28]):

$$\mu(y, z) = \begin{cases} (-1)^{|I|} & \text{if } z = y - e(I) \text{ for some } I \subseteq [n] \\ 0 & \text{otherwise} \end{cases} \tag{3.28}$$

for $y, z \in \mathcal{C}$, where $e(I) \in \{0, 1\}^n$ is the vector having $e_i(I) = 1$ if and only if $i \in I$. Given a function $f : \mathcal{C} \mapsto \mathbb{R}$ and an $x \in \mathcal{C}$, the *Möbius inversion formula* enables us to express $f(x)$ as the sum of *Möbius coefficients* $\hat{f}(y)$ of all elements $y \leq x$:

$$f(x) = \sum_{l \leq y \leq x} \hat{f}(y) \iff \hat{f}(y) = \sum_{l \leq z \leq y} f(z)\mu(z, y), \tag{3.29}$$

where $l = (l_1, \ldots, l_n)$ is the minimum element of $\mathcal{C}$.

3.7.1 Functions with non-negative Möbius coefficients and integer range

It follows immediately from (3.29) that if $f : \mathcal{C} \mapsto \mathbb{R}$ is a function with non-negative Möbius coefficients, i.e., $\hat{f}(x) \geq 0$ for all $x \in \mathcal{C}$, then f is monotone and supermodular: $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$ for all $x, y \in \mathcal{C}$, where $\vee$ and $\wedge$ denote the join and meet operators over $\mathcal{C}$. If we assume further that f is integer-valued, then the function $range(f) - f(u - x)$ is polymatroid, where $range(f) = f(u)$ is the range of f and u is the maximum element of $\mathcal{C}$. Thus we conclude by Theorem 2.3 of Chapter 2 that the sizes of the sets $\mathcal{F}$ and $\mathcal{I}^{-1}(\mathcal{F})$ of maximal feasible and minimal infeasible vectors for the inequality $f(x) \leq t$ are related by the quasi-polynomial bound: $|\mathcal{I}^{-1}(\mathcal{F})| \leq (2n|\mathcal{F}|)^{\log(range(f))}$. In fact the following stronger bound even holds in this case.

Theorem 3.7 *Let $f : \mathcal{C} \mapsto \mathbb{Z}_+$ be a function with non-negative Möbius coefficients. Given an integer threshold t, let $\mathcal{F}$ be the set of maximal solutions to the inequality $f(x) \leq t$. Then for any non-empty subset $\mathcal{Y} \subseteq \mathcal{F}^- \stackrel{\text{def}}{=} \{x \in \mathcal{C} \mid x \leq y \text{ for some } y \in \mathcal{F}\}$ we have*

$$\left|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F})\right| \leq range(f) \cdot |\mathcal{Y}|. \tag{3.30}$$

Proof. The theorem follows from the following intersection inequality from [22]:

Lemma 3.1 (Intersection Lemma, Boros, Gurvich, Khachiyan, and Makino [22]) *Let $\mathcal{S}, \mathcal{T} \subseteq 2^U$ be two families of subsets of a given set U, and let $w : U \to \mathbb{R}_+$ be a given weight function on U. Suppose $\mathcal{S}$ and $\mathcal{T}$ are threshold separable, i.e., there are real thresholds $t_1 < t_2$ such* *that* $w(T) \leq t_1$, *for all* $T \in \mathcal{T}$, *and* $w(S) \geq t_2$, *for all* $S \in \mathcal{S}$, *where* $w(X) = \sum\{w(v) \mid v \in X\}$ *for* $X \subseteq U$. *Suppose further that* $|\mathcal{S}| \geq 2$ *and* $\mathcal{T}$ *covers all pairwise intersections of* $\mathcal{S}$, *i.e., for all* $S, S' \in \mathcal{S}$, $S \neq S'$, *there exists a* $T \in \mathcal{T}$ *such that* $S \cap S' \subseteq T$. *Then*

(*i*) $|\mathcal{S}| \leq \sum_{T \in \mathcal{T}} |U \setminus T|$

(*ii*) $|\mathcal{S}| \leq \frac{\sum_{v \in U} w(v)}{t_2 - t_1} |\mathcal{T}|$.

Now to prove the theorem, we let $\mathcal{X} \stackrel{\text{def}}{=} \mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F})$, consider the monotonic mapping $\phi : \mathcal{P} \mapsto 2^{\mathcal{C}}$ defined by: $\phi(x) = \{y \in \mathcal{C} \mid y \leq x\}$, and let $\mathcal{S} = \{\phi(x) \mid x \in \mathcal{X}\}$, and $\mathcal{T} = \{\phi(y) \mid y \in \mathcal{Y}\}$. Thus with respect to the set of non-negative weights $\{w(x) = \hat{f}(x) \mid x \in \mathcal{C}\}$, we obtain the threshold separability

$$w(\phi(x)) \geq t + 1, \text{for all } x \in \mathcal{X}; \qquad w(\phi(y)) \leq t, \text{for all } y \in \mathcal{Y},$$

of $\mathcal{S}$ and $\mathcal{T}$. If $|\mathcal{X}| = |\mathcal{S}| = 1$, then inequality (3.30) holds, for otherwise we get the contradiction $t + 1 \leq f(x) \leq range(f) = 0 \leq f(y) \leq t$, for the element $x \in \mathcal{X}$ and any $y \in \mathcal{Y}$.

Let us assume therefore that $|\mathcal{X}| \geq 2$, and observe that $\mathcal{T}$ covers all pairwise intersections of $\mathcal{S}$: for any two distinct elements $x, x' \in \mathcal{X}$, it follows by $x, x' \in \mathcal{I}^{-1}(\mathcal{Y})$ that there is a $y \in \mathcal{Y}$ such that $x \wedge x' \leq y$, and therefore, we get

$$\phi(x) \cap \phi(x') = \phi(x \wedge x') \subseteq \phi(y).$$

Now we apply Lemma 3.1(ii) to get

$$\left|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F})\right| = |\mathcal{S}| \leq \sum_{x \leq u} \hat{f}(x) |\mathcal{T}| = f(u)|\mathcal{Y}| = range(f)|\mathcal{Y}|,$$

proving (3.30). □

As an example for the class of functions considered in Theorem 3.7, let us recall the support function $S(\cdot) = S_{\mathcal{D}}(\cdot)$ with respect to a binary database $\mathcal{D} \subseteq 2^V$ defined by $S(X) = |\{H \in \mathcal{D} \mid H \supseteq X\}$, for $X \subseteq V$. Then the monotone function $f(X) \stackrel{\text{def}}{=} S(V \setminus X)$ has non-negative Möbius coefficients. Indeed, for any $Y \subseteq V$, it is easily seen (and also follows from (3.28) and (3.29)) that $\hat{f}(Y) = |\{H \in \mathcal{D} \mid V \setminus H = Y\}|$. Thus we conclude by Theorem 3.7 that $|\mathcal{I}(\mathcal{F}_{\mathcal{D},t})| \leq |\mathcal{D}| \cdot |\mathcal{F}_{\mathcal{D},t}|$, where as before $\mathcal{F}_{\mathcal{D},t}$ is the set of minimal t-infrequent sets.

If the the function f is not integer-valued, then generally no bound as the one of Theorem 3.7 can be obtained. Take, for instance, the sharp example for inequality 3.10 given in Section 3.5.2 with $|\mathcal{D}| = 2^k$ and $n = 2k$. Then defining $f(X) = S(V \setminus X)/|\mathcal{D}|$, $t = 1/|\mathcal{D}|$ gives a monotone function with non-negative Möbius coefficients and $range(f) = 1$, for which (3.30) is violated. Nevertheless, we shall show next that for a certain class of such real-valued functions, the set of maximal infeasible vectors can still be dual-bounded. This will allow us to efficiently generate minimal p-efficient points for independent random variables.

3.7.2 Sum of functions with non-negative Möbius coefficients and bounded number of variables

Let $\mathcal{C} = \{x \in \mathbb{Z}^n \mid l \le x \le u\}$ and let $f : \mathcal{C} \mapsto \mathbb{R}_+$ be a function that can be expressed as the sum of $k \le \binom{n}{d}$ functions: $f(x) = \sum_{j=1}^{k} f^j(x)$, where each function $f^j : \mathcal{C} \mapsto \mathbb{R}_+$ has non-negative Möbius coefficients and involves at most d variables $x_{i_1}, \ldots, x_{i_d}$ for some constant d. Then we have the following theorem.

Theorem 3.8 *Consider a function $f : \mathcal{C} \mapsto \mathbb{R}_+$ that can be expressed as the sum of k functions, f^j, $j = 1, \ldots, k$, each of which has non-negative Möbius coefficients and involves at most d variables. Let $\mathcal{F}$ be the set of maximal feasible solutions $x \in \mathcal{C}$ of the inequality $f(x) \le t$, where t is a given real threshold. Then for any nonempty subset $\mathcal{Y} \subseteq \mathcal{F}^-$ we have*

$$|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F})| \le k(2|\mathcal{Y}| + 1)^d |\mathcal{Y}|. \tag{3.31}$$

Proof. Let $\mathcal{C}' \stackrel{\text{def}}{=} \mathcal{C}'_1 \times \cdots \times \mathcal{C}'_n$, where $\mathcal{C}'_i$ is the chain composed of those elements that appear in $\mathcal{Y}$ and their successors:

$$\mathcal{C}'_i = \{l_i\} \cup \{y_i : \ y \in \mathcal{Y}\} \cup \{y_i + 1 : \ y \in \mathcal{Y}, \ y_i \ne u_i\}, \tag{3.32}$$

for $i = 1, \ldots, n$. Note that if $x \in \mathcal{I}^{-1}(\mathcal{Y})$, then $x \in \mathcal{C}'$. [If $x \in \mathcal{I}^{-1}(\mathcal{Y})$, $i \in [n]$, and $x_i \ne l_i$ then $x - \mathbf{e}^i \in \mathcal{Y}^-$ and therefore there is a $y \in \mathcal{Y}$ such that $y \ge x - \mathbf{e}^i$. But then y_i must be equal to $x_i - 1$, i.e., $x_i = y_i + 1 \in \mathcal{C}'_i$.] In particular, we have $\mathcal{X} \stackrel{\text{def}}{=} \mathcal{I}^{-1}(\mathcal{F}) \cap \mathcal{I}^{-1}(\mathcal{Y}) \subseteq \mathcal{C}'$.

Let us then consider the restrictions of the functions $f^1, f^2, \ldots, f^k$ on the integral box $\mathcal{C}'$, and observe that these restrictions also satisfy the preconditions of the theorem. [In fact, if

$\{\hat{f}(x) : x \in \mathcal{C}\}$ are the non-negative Möbius coefficients of $f : \mathcal{C} \mapsto \mathbb{R}$, then

$$\left\{ \sum\{\hat{f}(y) : y \in \mathcal{C} \cap x^- \text{ and } y \not\leq \text{any } z \in \mathcal{C}' \cap x^-\} \; : x \in \mathcal{C}' \right\}$$

are the non-negative Möbius coefficients of the restriction of f on $\mathcal{C}'$]. In other words, we may assume without loss of generality that the original functions $f^1, \ldots, f^k$ are defined on $\mathcal{C}'$.

Let $U \stackrel{\text{def}}{=} \bigcup_{j=1}^k \mathcal{C}^j$, where $\mathcal{C}^j \stackrel{\text{def}}{=} \mathcal{C}_1^j \times \cdots \times \mathcal{C}_n^j$, and where for $i = 1, \ldots, n$ and for $j = 1, \ldots, k$, $\mathcal{C}_i^j = \mathcal{C}_i'$ if variable x_i appears in function f^j, and $\mathcal{C}_i^j = \{l_i\}$ otherwise. Note that, since $|\mathcal{C}_i^j| \leq 2|\mathcal{Y}| + 1$ for $j = 1, \ldots, n$ by (3.32), the size of U is bounded by $k(2|\mathcal{Y}| + 1)^d$. Let us now extend each of the functions $f^1, \ldots, f^k$ on U and define the mapping $\phi : U \mapsto 2^U$ by $\phi(x) = \{y \in U \mid y \leq x\}$, and the non-negative weights $w(x) = \sum_{j=1}^k \hat{f}^j(x)$ associated to every $x \in U$. Let further $\mathcal{S} = \{\phi(x) \mid x \in \mathcal{X}\}$, and $\mathcal{T} = \{\phi(y) \mid y \in \mathcal{Y}\}$. Then, for any $x \in \mathcal{C}$,

$$w(\phi(x)) = \sum_{y \in U, y \leq x} w(y) = \sum_{y \in U,\ y \leq x} \sum_{j=1}^k \hat{f}^j(y) = \sum_{j=1}^k \sum_{y \in U,\ y \leq x} \hat{f}^j(y) = \sum_{j=1}^k f^j(x) = f(x),$$

and therefore by the definitions of $\mathcal{X}$ and $\mathcal{Y}$, we get $w(S) > t$, for all $S \in \mathcal{S}$, and $w(T) \leq t$, for all $T \in \mathcal{T}$. Assuming, without loss of generality, that $|\mathcal{X}| \geq 2$, it also follows from the definitions of $\mathcal{X}$ and $\mathcal{Y}$ that $\mathcal{T}$ covers all pairwise intersections of $\mathcal{S}$. Hence, the theorem follows by applying Lemma 3.1(i). □

Corollary 3.6 *Let $f_1, \ldots, f_r : \mathcal{C} \mapsto \mathbb{R}_+$ be real-valued monotone functions, each of which can be expressed as the sum of at most k terms having non-negative Möbius coefficients, and involving at most d variables. Let $\mathcal{F}$ be the set of maximal feasible solutions of the system (3.27), then for any nonempty subset $\mathcal{Y} \subseteq \mathcal{F}$ we have the inequality*

$$|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F})| \leq rk(2|\mathcal{Y}| + 1)^d |\mathcal{Y}|,$$

and consequently, the elements of $\mathcal{F}$ can be incrementally generated in quasi-polynomial time.

3.7.3 Sum of separable functions of bounded number of variables

As an application of Theorem 3.8, consider a function $f : \mathcal{C} \mapsto \mathbb{R}_+$ that can be expressed as the sum of k terms $f^1, \ldots, f^k : \mathcal{C} \mapsto \mathbb{R}_+$, each of which is *$d$-separable*, i.e., can be written as the product of at most d single-variable monotone functions. An example with $k = n$ and $d = 2$ is the function $f(x_1, \ldots, x_n) = x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1$. Consider some term f^j of f and assume without loss of generality that $f^j(x) = \prod_{i=1}^{h} g_i(x_i)$ for some $h \leq d$. Then by (3.28) and (3.29) we have

$$\hat{f}^j(y) = \sum_{I \subseteq [d]} (-1)^{|I|} f^j(y - e(I)) = \sum_{I \subseteq [d]} (-1)^{|I|} \prod_{i=1}^{d} g_i(y - e(I)) = \prod_{i=1}^{d} [g_i(y_i) - g_i(y_i - 1)]$$

for any $y \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_d$, where $g_i(l_i - 1) \stackrel{\text{def}}{=} 0$ for all i. Thus, the monotonicity and non-negativity of the functions $g_1, \ldots, g_d$, imply that the Möbius coefficients for f^j are non-negative. We conclude, therefore, that Theorem 3.8 and Corollary 3.6 apply to sums of separable functions of bounded number of variables.

On the negative side, we have the following proposition.

Proposition 3.1 *Given a system (3.27), where each function f_i is the sum of at most k d-separable terms:*

(i) *Incrementally generating all minimal infeasible vectors for (3.27) is NP-hard, even if $r = 1$, $d = 2$, and each function f_i is integer-valued.*

(ii) *Incrementally generating all maximal feasible vectors for (3.27) is at least as hard as the dualization problem on products of chains.*

Proof. (i) We reduce the problem from the following well-known NP-complete problem: Given a graph $G = (V, E)$ and an integer t, determine if G contains an independent set of size at least t. To do this let us associate a *binary* variable x_i with each vertex $i \in V$, and define the monotone function

$$f(x) = (t - 2) \cdot \sum_{\{i,j\} \in E} x_ix_j + \sum_{i \in V} x_i,$$

over the elements $x \in \{0, 1\}^V$. Let $\mathcal{Y} \subseteq \{0, 1\}^V$ be the set of incidence vectors of the edges of G. Then $\mathcal{Y}$ is a subset of the minimal infeasible vectors for the inequality $f(x) \leq t - 1$, and it is easy to see that there are no other minimal infeasible vectors if an only if there is no independent set of G of size t.

(ii) The dualization problem DUAL$(\mathcal{C}, \mathcal{A}, \mathcal{B})$ on products of chains $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ can be written as problem GEN$(\mathcal{F}, \mathcal{Y})$ of incrementally generating the maximal feasible vectors $\mathcal{F}$ of the monotone system:

$$\sum_{i=1}^{n} \min\{x_i, a_i\} \leq \sum_{i=1}^{n} a_i - 1, \qquad a = (a_1, \ldots, a_n) \in \mathcal{A}, \tag{3.33}$$

since the maximal feasible vectors of (3.33) are exactly the maximal elements of the set $\mathcal{C} \setminus \mathcal{A}^+$.

□

3.7.4 p-Efficient and p-inefficient points of probability distributions

The following application comes from Stochastic Programming [32, 84]. Let $\xi \in \mathbb{Z}^n$ be an n-dimensional integer-valued random variable, defined by its *discrete* probability distribution, i.e., by a finite set of points $\mathbb{P} \subseteq \mathbb{Z}^n$ at which $\Pr[\xi = q] > 0$, for $q \in \mathbb{P}$. Given $p \in [0, 1]$, a point $x \in \mathbb{Z}^n$ is said to be *p-efficient* if it is minimal with the property that $\Pr[\xi \leq x] \geq p$. Let us conversely say that $x \in \mathbb{Z}^n$ is *p-inefficient* if it is maximal with the property that $\Pr[\xi \leq x] < p$. Denote respectively by $\mathcal{F}_p$, $\mathcal{I}^{-1}(\mathcal{F}_p)$ the sets of p-inefficient, and p-efficient points for ξ. Clearly, these sets are finite since, in each dimension $i \in [n]$, we need only to consider the projections $\mathcal{C}_i \stackrel{\text{def}}{=} \{q_i \in \mathbb{Z} \mid q \in \mathbb{P}\}$. In other words, the sets $\mathcal{F}_p$ and $\mathcal{I}^{-1}(\mathcal{F}_p)$ are subsets of a finite integral box of size at most $|\mathbb{P}|$ along each dimension. Let $f(x) = \Pr[\xi \leq x]$, then $f(x) = \sum_{y \leq x} \Pr[\xi = y] = \sum_{q \in \mathbb{P}, q \leq x} \Pr[\xi = q]$, and hence, $f(x)$ is a supermodular function with non-negative Möbius coefficients. It follows then, from Lemma 3.1(ii), that $|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F}_p)| \leq |\mathbb{P}| \cdot |\mathcal{Y}|$ for any non-empty $\mathcal{Y} \subseteq \mathcal{F}_p$, and therefore, the set $\mathcal{F}_p$ can be generated in incremental quasi-polynomial time. On the other hand, the incremental generation of p-efficient points is NP-hard, since it includes the generation of the maximal frequent sets of a binary database $\mathcal{D} \subseteq 2^{[n]}$, if we define $\xi \in \{0, 1\}^n$ by $\Pr[\xi = X] = 1/|\mathcal{D}|$ if and only if $V \setminus X \in \mathcal{D}$.

Corollary 3.7 *p-Inefficient points of a discrete probability distribution can be enumerated in quasi-polynomial time.*

It is interesting to observe that if ξ is an integer-valued finite (or even *infinite*, see [32]) random variable with *independent* coordinates $\xi_1, \ldots, \xi_n$, then the generation of both $\mathcal{F}_p$ and $\mathcal{I}^{-1}(\mathcal{F}_p)$ can still be done in quasi-polynomial time even if the number of points $\mathbb{P}$, defining the distribution of ξ, is *exponential* in n. Indeed, by independence we have $\Pr[\xi \leq x] = \prod_{j=1}^{n} \Pr[\xi_j \leq x_j]$. Defining $f(x) = \ln \Pr[\xi \leq x] = \sum_{j=1}^{n} \ln \Pr[\xi_j \leq x_j]$, we can write $f(x)$ as the sum of 1-separable monotone functions $f^1, \ldots, f^n$. For $i = 1, \ldots, n$, let $l_i = \min\{x_i \in \mathbb{Z} \mid \Pr[\xi_i \leq x_i] > 0\}$, $u_i = \min\{x_i \in \mathbb{Z} \mid \Pr[\xi_i \leq x_i] = 1\}$, and $\mathcal{C}_i = [l_i : u_i]$. Then the function $g^j \stackrel{\text{def}}{=} f^j(x_j) - \ln(\Pr[\xi_j = l_j])$ is non-negative over $x_j \in \mathcal{C}_j$ for $j = 1, \ldots, n$, and the p-inefficient points are the maximal feasible solutions of the inequality $\sum_{j=1}^{n} g^j(x_j) < t \stackrel{\text{def}}{=} \ln p - \ln(\Pr[\xi = l])$. Consequently, for any $\mathcal{Y} \subseteq \mathcal{F}_p$, we get the bound

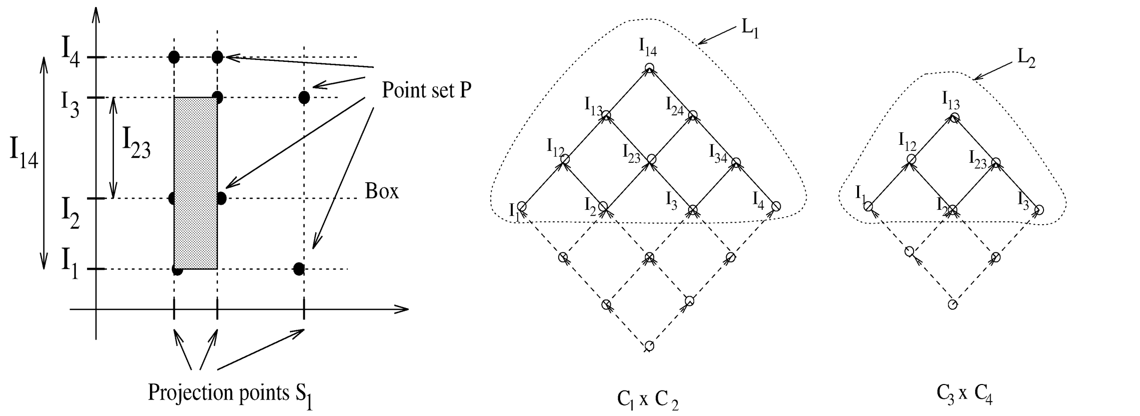
$$|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F})| \leq n(2|\mathcal{Y}| + 1)|\mathcal{Y}|$$

by the results in Section 3.7.3. Thus, if for each $i = 1, \ldots, n$, and for a given $x \in \mathbb{Z}^n$, the value $\max\{y_i \in \mathbb{Z} \mid g^i(y_i) + \sum_{j \neq i} g^j(x_j) < t\}$ is computable in polynomial time, then the set $\mathcal{F}_p$ can be generated in quasi-polynomial time. Finally, since the function $h^j : \mathcal{C}_j \mapsto \mathbb{R}_+$, defined by $h^j(x_j) = \ln(\Pr[\xi_j = u_j]) - f^j(u_j - x_j)$ is monotone in x_j, we get a similar result for the set of p-efficient points $\mathcal{I}^{-1}(\mathcal{F}_p)$.

3.8 Packing points into boxes

We conclude our list of applications by an example of a polymatroid function over products of lattices. Given a set of n-dimensional points $\mathbb{P} \subseteq \mathbb{R}^n$ and a coloring $C : \mathbb{P} \mapsto \{1, 2, \ldots, r\}$ of the point set, suppose that we want to generate all maximal n-dimensional boxes that contain at most t_1 points of $\mathbb{P}$ of the first color, at most t_2 points of the second color, ..., and at most t_r points of the r^{th} color, where $0 \leq t_1, t_2, \ldots, t_r \leq |\mathbb{P}|$ are given integer thresholds. We shall assume without loss of generality that the generated boxes *minimally* bound the points inside them, i.e., there must exist a point of $\mathbb{P}$ on each of the n sides of each generated box.

Interestingly, this problem can be cast as of generating minimal feasible solutions of a system of polymatroid inequalities over a product of $2n$ chains (or more precisely, n join semi-lattices). Indeed, consider the set of projection points $\mathbb{S}_i \stackrel{\text{def}}{=} \{p_i \in \mathbb{R} \mid p \in \mathbb{P}\}$, for $i = 1, \ldots, n$. Clearly, the lower and upper end points a_i, b_i of each candidate box in the i^{th} dimension belong to the set $\mathbb{S}_i$. Thus, letting $\mathcal{C}_{2i-1} \stackrel{\text{def}}{=} \mathbb{S}_i$ and $\mathcal{C}_{2i} \stackrel{\text{def}}{=} \mathbb{S}_i^*$ be the two chains whose elements are $\mathbb{S}_i$, ordered in



a: A 2-dimensional example. b: The corresponding (semi-)lattice of intervals $\mathcal{L}_1 \times \mathcal{L}_2$.

Figure 3.3: Packing points into boxes: each box has at most $t = 4$ points.

increasing and decreasing orders respectively, we conclude that the projection $x_i = [a_i, b_i]$ of each box in the i^{th} dimension belongs to the join semi-lattice $\mathcal{L}_i \stackrel{\text{def}}{=} \{(a_i, b_i) \in \mathcal{C}_{2i-1} \times \mathcal{C}_{2i} \mid a_i \leq b_i\}$. (Correspondingly $\mathcal{L}_i \cup \{l_i\}$ is the *lattice of intervals* whose elements are the different intervals defined by the projection points $\mathbb{S}_i$. The meet of any two intervals is their *intersection*, and the join is their *span*, i.e., the minimum interval containing both of them. The minimum element l_i of $\mathcal{L}_i$ is the empty interval I_0.) A 2-dimensional example is shown in Figure 3.3, where we denote the interval between two projection points $p_i^j, p_i^k \in \mathbb{P}_i$ by I_{jk}. Clearly, the product $\mathcal{L} = \mathcal{L}_1 \times \ldots \times \mathcal{L}_n$ is the set of all candidate boxes, and the problem is to generate the maximal elements of this product that contain at most t_i points from the set $\mathbb{P}$ having color i, for $i = 1, \ldots, r$. Let $\mathbb{P}_i \stackrel{\text{def}}{=} \{p \in \mathbb{P} \mid C(p) = i\}$ for $i = 1, \ldots, r$ and denote by $\mathcal{F}$ the set of maximal boxes containing at most t_i points from $\mathbb{P}_i$, for $i = 1, \ldots, r$. Then $\mathcal{I}^{-1}(\mathcal{F})$ is the set of minimal boxes that contain at least $t_i + 1$ points from $\mathbb{P}_i$, for some $i \in [r]$. Now let us consider the function

$$f_i(x) = \begin{cases} |\{p \in \mathbb{P}_i \mid p \text{ is contained inside the box } x\}|, & \text{if } x \in \mathcal{L} \\ 0, & \text{otherwise,} \end{cases}$$

over the elements of $\mathcal{C} \stackrel{\text{def}}{=} (\mathcal{C}_1 \times \mathcal{C}_2) \times (\mathcal{C}_3 \times \mathcal{C}_4) \times \cdots \times (\mathcal{C}_{2n-1} \times \mathcal{C}_{2n})$, and observe that this function is supermodular (although the very related function $g_i : 2^{\mathbb{P}_i} \mapsto |\mathbb{P}_i|$ defined by $g_i(X) \stackrel{\text{def}}{=} \{p \in \mathbb{P}_i \mid p$ is contained in the minimum bounding box of $X\}$, for $X \subseteq \mathbb{P}_i$ is *not* in general supermodular). It follows then that the function $|\mathbb{P}_i| - f_i(x)$ is polymatroid over the elements x of the *dual lattice*

$\mathcal{C}^*$ (that is, the lattice $\mathcal{C}^*$ with the same set of elements as $\mathcal{C}$, but such that $x \prec y$ in $\mathcal{C}^*$ whenever $x \succ y$ in $\mathcal{C}$). We conclude, therefore, that $\mathcal{F}$ is the set of minimal (with respect to $\mathcal{C}^*$) feasible solutions of the polymatroid system:

$$|\mathbb{P}_i| - f_i(x) \geq t_i, \qquad \text{for } i = 1, \ldots, r,$$

over the elements of $\mathcal{C}^*$ (plus exactly $\sum_{i=1}^n (|\mathbb{S}_i| - 1)$ maximal *artificial* boxes, i.e., boxes with end points $a_i > b_i$). Hence by Theorem 2.3 of Chapter 2, the size of the dual set $\mathcal{I}^{-1}(\mathcal{F})$ is (uniformly dual-) bounded by a quasi-polynomial in the size of $\mathcal{F}$, n, and $|\mathbb{P}|$. In fact, the following stronger and even more general inequality can be obtained. Given, a non-negative weight vector $w : \mathbb{P} \mapsto \mathbb{R}_+$ on the point set $\mathbb{P}$, and a non-negative threshold vector $(t_1, \ldots, t_r) \in \mathbb{R}_+^r$, define a *packing* of the point sets $\mathbb{P}_1, \ldots, \mathbb{P}_r$, with respect to (w, t), to be a maximal box containing a subset of $\mathbb{P}_i$ of total weight at most t_i, for $i = 1, \ldots, r$.

Theorem 3.9 *Let $\mathbb{P}_1, \ldots, \mathbb{P}_r$ be given sets of points in $\mathbb{R}^n$, $w : \mathbb{P} \mapsto \mathbb{R}_+$ be a given non-negative weight vector on the point set $\mathbb{P} = \cup_{i=1}^r \mathbb{P}_i$, and $t_1, \ldots, t_r$ be given non-negative real-thresholds. If $\mathcal{F}$ is the set of packings of the point sets $\mathbb{P}_1, \ldots, \mathbb{P}_r$, with respect to (w, t), then*

$$|\mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F})| \leq \sum_{i=1}^r \sum_{y \in \mathcal{Y}} |\{p \in \mathbb{P}_i \mid \textit{point } p \textit{ lies outside box } y\}|, \tag{3.34}$$

for any $\emptyset \neq \mathcal{Y} \subseteq \mathcal{F}$. In particular, $|\mathcal{I}^{-1}(\mathcal{F})| \leq \sum_{i=1}^r |\mathbb{P}_i| \cdot |\mathcal{F}|$.

Proof. We use again Lemma 3.1. Let us assume without loss of generality that $r = 1$ and $\mathbb{P}_1 = \mathbb{P}$ is the given point set. For $r > 1$, we shall then need to apply proposition 1.2. Let $\mathcal{X} \stackrel{\text{def}}{=} \mathcal{I}^{-1}(\mathcal{Y}) \cap \mathcal{I}^{-1}(\mathcal{F})$ and consider the monotonic mapping $\phi : \mathcal{C} \mapsto 2^{\mathbb{P}}$ defined by: $\phi(x) = \{p \in \mathbb{P} \mid p$ is contained inside the box $x\}$. Let further $\mathcal{S} = \{\phi(x) \mid x \in \mathcal{X}\}$, and $\mathcal{T} = \{\phi(y) \mid y \in \mathcal{Y}\}$. Then, with respect to the set of non-negative weights w, we obtain the threshold separability $w(S) > t$, for all $S \in \mathcal{S}$, and $w(T) \leq t$, for all $T \in \mathcal{T}$. Assuming, without loss of generality, that $|\mathcal{X}| \geq 2$, it is easy to see that $\mathcal{T}$ covers all pairwise intersections of $\mathcal{S}$. Hence, the required result follows by applying Lemma 3.1(i). $\square$

Theorem 4.1 of the next chapter states that the dualization problem in products of chains can be solved in quasi-polynomial time. Combining this with Theorem 3.9, we readily obtain the following:

Corollary 3.8 *All packings of given point sets $\mathbb{P}_1, \ldots, \mathbb{P}_r \subseteq \mathbb{R}^n$, with respect to given non-negative weight and threshold vectors $(w(p) : p \in \cup_{i=1}^r \mathbb{P}_i)$ and $(t_1, \ldots, t_r)$, can be generated in incremental quasi-polynomial time.*

Corollary 3.8 can be complemented with the following negative result.

Proposition 3.2 *Given a set of points $\mathbb{P} \subseteq \mathbb{R}^n$ and an integer threshold t:*

(i) *Incrementally generating all minimal boxes that contain at least t points of $\mathbb{P}$ is NP-hard.*

(ii) *Incrementally generating all maximal boxes that contain at most t points of $\mathbb{P}$ is at least as hard as the hypergraph transversal problem.*

Proof. Both claims of the proposition follow from the following reduction from the maximal frequent and minimal infrequent sets generation problems (see Section 3.5.2). Given a hypergraph $\mathcal{H} \subseteq 2^V$, let us associate a point $p^H \in [0,1]^V$ for each hyperedge $H \in \mathcal{H}$ in the obvious way: $p_i^H = 1$ if $i \in H$ and $p_i^H = 0$ if $i \notin H$. Let $\mathbb{P}$ be the set of points in $[0,1]^V$ consisting of $\{p^H \mid H \in \mathcal{H}\}$ plus $|\mathcal{H}| + 1$ copies of the origin $(0, 0, \ldots, 0)$. Then, for any integer $0 \leq t \leq |\mathcal{H}|$, the set $\mathcal{G}$ of minimal subsets $X \subseteq V$ containing in at least t hyperedges of $\mathcal{H}$ are in one to one correspondence with the set $\mathcal{G}'$ of minimal boxes containing at least $t + |\mathcal{H}| + 1$ points of $\mathbb{P}$. Indeed, given $X \in \mathcal{G}$, define $x \in \mathcal{G}'$ by: $x_i = 0$ if $i \notin X$, and $x_i = [0,1]$ if $i \in X$. Clearly, x contains at least t points of $\mathbb{P}$ corresponding to the hyperedges of $\mathcal{H}$ contained in X. In addition, x also contains $|\mathcal{H}|+1$ copies of the origin, and thus in total, it contains at least $t+|\mathcal{H}|+1$ points of $\mathbb{P}$. Conversely, if $x \in \mathcal{G}'$, then it must include the origin (otherwise, x contains at most $|\mathcal{H}|$ points), which contributes $|\mathcal{H}| + 1$ points inside x. We conclude therefore that each coordinate of x is either 0 or $[0,1]$, and this allows us to define a subset X of V in the obvious way. Clearly, at least t more points of $\mathbb{P}$ are included inside x, corresponding to at least t hyperedges of $\mathcal{H}$ that are contained in X. $\square$

Finally, we mention in closing that all the results stated in this section remain valid if the set of points $\mathbb{P}$ is replaced by a set of *bodies* in $\mathbb{R}^n$.

Chapter 4

Dualization in Products of Chains, Semi-lattices, and Forests

4.1 Introduction

Let $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ be the product of n partially ordered sets. Let $\mathcal{A}^+ = \{x \in \mathcal{P} \mid x \succeq a, \text{ for some } a \in \mathcal{A}\}$ and $\mathcal{A}^- = \{x \in \mathcal{P} \mid x \preceq a, \text{ for some } a \in \mathcal{A}\}$ denote the ideal and filter generated by a set $\mathcal{A} \subseteq \mathcal{P}$. For convenience, we shall use p^+ and p^- to denote $\{p\}^+$ and $\{p\}^-$ for any $p \in \mathcal{P}$. Any element in $\mathcal{P} \setminus \mathcal{A}^+$ is called *independent of* $\mathcal{A}$. Let $\mathcal{I}(\mathcal{A})$ be the set of all maximal independent elements for $\mathcal{A}$, then we have the following decomposition of $\mathcal{P}$

$$\mathcal{A}^+ \cap \mathcal{I}(\mathcal{A})^- = \emptyset, \qquad \mathcal{A}^+ \cup \mathcal{I}(\mathcal{A})^- = \mathcal{P}. \tag{4.1}$$

Call $\mathcal{A}$ *an antichain* if no two elements are comparable in $\mathcal{P}$. In this chapter, we are concerned with the following dualization problem:

DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$**:** *Given an antichain $\mathcal{A} \subseteq \mathcal{P}$ in a poset $\mathcal{P}$ and a collection of maximal independent elements $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$, either find a new maximal independent element $x \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$, or prove that the given collection is complete: $\mathcal{B} = \mathcal{I}(\mathcal{A})$.*

If $\mathcal{P}$ is the Boolean cube, i.e., $\mathcal{P}_i = \{0, 1\}$ for all $i = 1, \ldots, n$, the above dualization problem reduces to the well known *hypergraph transversal* problem, which calls for enumerating all minimal subsets that intersect all edges of a given hypergraph. The complexity of the dualization problem is still an important open question. For the Boolean case, the best known algorithm runs in quasi-polynomial time $poly(n)+m^{o(\log m)}$, where $m = |\mathcal{A}|+|\mathcal{B}|$, see [43], providing strong evidence that the problem is unlikely to be NP-hard. In this chapter, we shall prove Theorem 1.8 which extends this result to a wider class of partially ordered sets. Specifically, it will be shown that the problem can be solved in quasi-polynomial time in the case where each $\mathcal{P}_i$ is

1. a chain, that is, a totally ordered set; or more generally,

2. a join (or meet) semi-lattice with bounded width,

3. a forest, that is a poset with acyclic precedence graph, in which either the in-degree or the out-degree of each element is bounded.

We remark that for case (2), all posets $\mathcal{P}_i$ must be of the same type: either all posets are join semi-lattices, or all of them are meet semi-lattices. Without loss of generality we will only consider join semi-lattices.

Here is a more detailed description of the results in this chapter. For $x \in \mathcal{P}_i$, denote by $x^\perp$ the set of immediate predecessors of x, i.e.

$$x^\perp = \{y \in \mathcal{P}_i \mid y \prec x, \ (\nexists z \in \mathcal{P}_i : y \prec z \prec x)\},$$

and let in-deg$(\mathcal{P}_i) = \max\{|x^\perp| \ : \ x \in \mathcal{P}_i\}$. Similarly, denote by $x^\top$ the set of immediate successors of x, and let out-deg$(\mathcal{P}_i) = \max\{|x^\top| : \ x \in \mathcal{P}_i\}$. Throughout this chapter, denote by $m \stackrel{\text{def}}{=} |\mathcal{A}| + |\mathcal{B}|$, $[n] \stackrel{\text{def}}{=} \{1, \ldots, n\}$, $d \stackrel{\text{def}}{=} \max_{i \in [n]} \min\{\text{in-deg}(\mathcal{P}_i), \text{out-deg}(\mathcal{P}_i)\}$, and $\mu = \mu(\mathcal{P}) \stackrel{\text{def}}{=} \max\{|\mathcal{P}_i| \ : \ i \in [n]\}$. Finally, denote by $W(\mathcal{P}_i)$ the *width* of $\mathcal{P}_i$, i.e. the maximum size of an antichain in $\mathcal{P}_i$ and let $W = W(\mathcal{P}) \stackrel{\text{def}}{=} \max_{i \in [n]}\{W(\mathcal{P}_i)\}$ be the maximum width of the n posets.

Theorem 4.1 *Problem DUAL$(\mathcal{C}, \mathcal{A}, \mathcal{B})$ can be solved in poly$(n) + m^{o(\log m)}$ time if $\mathcal{C}$ is a product of chains.*

Theorem 4.2 *Problem DUAL$(\mathcal{L}, \mathcal{A}, \mathcal{B})$ can be solved in poly$(n, \mu(\mathcal{L})) + m^{\gamma(W(\mathcal{L}))\cdot o(\log m)}$ time, if $\mathcal{L}$ is a product of join semi-lattices, where $\gamma(W) \stackrel{\text{def}}{=} 2W^2 \ln(W+1)$.*

Theorem 4.3 *Problem DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$ can be solved in poly$(n, \mu(\mathcal{P})) + m^{d\cdot o(\log m)}$ time, if $\mathcal{P}$ is a product of forests.*

Clearly, Theorem 4.1 is a special case of Theorems 4.2 and 4.3, for $W = d = 1$. Since, for chains, the algorithm is simpler, and the bound on the running time does not depend on the chain sizes $|\mathcal{C}_i|$, we will present this case in a separate section.

Finally, we note that, for the more general case of products of arbitrary posets, it is not known whether the problem can be solved in quasi-polynomial time, even for posets $\mathcal{P}_i$ of small size.

4.2 General approach

4.2.1 Preliminaries

Given two subsets $\mathcal{A} \subseteq \mathcal{P}$, and $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$, we say that $\mathcal{B}$ is *dual to* $\mathcal{A}$ if $\mathcal{B} = \mathcal{I}(\mathcal{A})$, i.e., if $\mathcal{B}$ contains all the maximal elements of $\mathcal{P} \setminus \mathcal{A}^+$. Let us remark that, by (4.1), this condition is equivalent to $\mathcal{A}^+ \cup \mathcal{B}^- = \mathcal{P}$.

Given any $\mathcal{Q} \subseteq \mathcal{P}$, let us denote by

$$\mathcal{A}(\mathcal{Q}) = \{a \in \mathcal{A} \mid a^+ \cap \mathcal{Q} \neq \emptyset\}, \qquad \mathcal{B}(\mathcal{Q}) = \{b \in \mathcal{B} \mid b^- \cap \mathcal{Q} \neq \emptyset\},$$

the subsets of $\mathcal{A}, \mathcal{B}$ whose ideal and filter intersect $\mathcal{Q}$. A simple but an important observation, which will be used frequently in the algorithms below, is that

$$\mathcal{Q} \subseteq \mathcal{A}^+ \cup \mathcal{B}^- \iff \mathcal{Q} \subseteq \mathcal{A}(\mathcal{Q})^+ \cup \mathcal{B}(\mathcal{Q})^-. \tag{4.2}$$

Note that, for $a \in \mathcal{A}$ and $\mathcal{Q} = \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_n$, $a^+ \cap \mathcal{Q} \neq \emptyset$ if and only if $a_i^+ \cap \mathcal{Q}_i \neq \emptyset$, for all $i \in [n]$. Thus, the sets $\mathcal{A}(\mathcal{Q})$ and $\mathcal{B}(\mathcal{Q})$ can be found in $O(nm\mu)$ time.

To solve problem $DUAL(\mathcal{P}, \mathcal{A}, \mathcal{B})$, we shall use the same general approach used in [43] to solve the hypergraph dualization problem, by decomposing it into a number of smaller subproblems which are solved recursively. In each such subproblem, we start with a subposet $\mathcal{Q} = \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_n \subseteq \mathcal{P}$ (initially $\mathcal{Q} = \mathcal{P}$), and two subsets $\mathcal{A}(\mathcal{Q}) \subseteq \mathcal{A}$ and $\mathcal{B}(\mathcal{Q}) \subseteq \mathcal{B}$, and we want to check whether $\mathcal{A}(\mathcal{Q})$ and $\mathcal{B}(\mathcal{Q})$ are dual in $\mathcal{Q}$, i.e. whether $\mathcal{Q} \subseteq \mathcal{A}(\mathcal{Q})^+ \cup \mathcal{B}(\mathcal{Q})^-$. To estimate the reduction in problem size from one level of the recursion to the next, we measure the change in the "*volume*" of the problem defined as $v = v(\mathcal{A}, \mathcal{B}) \stackrel{\text{def}}{=} |\mathcal{A}||\mathcal{B}|$. Since $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$ is assumed, the following condition holds, by (4.1) for the original problem and all subsequent subproblems:

$$a \not\preceq b, \quad \text{for all } a \in \mathcal{A}, b \in \mathcal{B}. \tag{4.3}$$

It should be be mentioned that if condition (4.3) does not hold, the problem is NP-hard even if $\mathcal{P}$ is just the Boolean cube.:

Proposition 4.1 *Given a finite set V and hypergraphs $\mathcal{H}, \mathcal{F} \subseteq 2^V$, it is NP-complete to decide if there is a subset $S \subseteq V$ such that $S \not\supseteq H$ for all $H \in \mathcal{H}$ and $S \not\subseteq F$ for all $F \in \mathcal{F}$.*

Proof. We use a polynomial transformation from the satisfiability problem. Let $C = C_1 \wedge \ldots \wedge C_m$ be a conjunctive normal form in n variables $x_1, \ldots, x_n$. Let $V \stackrel{\text{def}}{=} \{x_1, \overline{x}_1, \ldots, x_n, \overline{x}_n\}$, $\mathcal{H} \stackrel{\text{def}}{=} \{\{x_1, \overline{x}_1\}, \ldots, \{x_n, \overline{x}_n\}\}$, and let $\mathcal{F}$ be the family of m subsets of V, each of which is the set of literals that do not appear in one of the m clauses $C_1, \ldots, C_m$. Then it is immediate to see that the existence of an $S \subseteq V$, satisfying $S \not\supseteq H$ for all $H \in \mathcal{H}$ and $S \not\subseteq F$ for all $F \in \mathcal{F}$, is equivalent to the existence of a satisfying truth assignment for C. $\square$

In fact, the statement of the above proposition remains valid even if $\mathcal{H} = \mathcal{F}$, in which case the problem is called *the hypergraph saturation problem* (see [39]).

Let $C(\mathcal{A}, \mathcal{B}) = C(v(\mathcal{A}, \mathcal{B}))$ denote the number of subproblems that have to be solved in order to solve the original problem. We assume that $C(\mathcal{A}, \mathcal{B}) \leq R(v(\mathcal{A}, \mathcal{B}))$ where $R(v)$ is a *super-additive* function of v (i.e., $R(v) + R(v') \leq R(v + v')$ for all $v, v' \geq 0$).

We start with three propositions: Proposition 4.2 is useful for decomposing dualization on products of posets with disconnected precedence graphs into a number subproblems in which every poset has a connected precedence graph. Proposition 4.3 provides the base case for recursion. Proposition 4.4 states that a problem, closely related to the dualization problem, is NP-hard.

Proposition 4.2 *Let $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$. Suppose that poset $\mathcal{P}_1$, can be partitioned into two independent posets $\mathcal{P}_1'$ and $\mathcal{P}_1''$ (where two subposets $\mathcal{Q}$ and $\mathcal{R}$ are called independent if $q \not\preceq r$ and $q \not\succeq r$ for all $q \in \mathcal{Q}, r \in \mathcal{R}$). Let $\mathcal{P}' = \mathcal{P}_1' \times \mathcal{P}_2 \times \ldots \times \mathcal{P}_n$, $\mathcal{P}'' = \mathcal{P}_1'' \times \mathcal{P}_2 \times \ldots \times \mathcal{P}_n$, $\mathcal{A}' = \mathcal{A}(\mathcal{P}')$, $\mathcal{B}' = \mathcal{B}(\mathcal{P}')$, $\mathcal{A}'' = \mathcal{A}(\mathcal{P}'')$, and $\mathcal{B}'' = \mathcal{B}(\mathcal{P}'')$. If $C(\mathcal{A}', \mathcal{B}') \leq R(v(\mathcal{A}', \mathcal{B}'))$ and $C(\mathcal{A}'', \mathcal{B}'') \leq R(v(\mathcal{A}'', \mathcal{B}''))$, then $C(\mathcal{A}, \mathcal{B}) \leq R(v(\mathcal{A}, \mathcal{B}))$.*

Proof. We observe by (4.2) and the independence of $\mathcal{P}', \mathcal{P}''$ that

$$\mathcal{P} \subseteq \mathcal{A}^+ \cup \mathcal{B}^- \Longleftrightarrow \mathcal{P}' \subseteq (\mathcal{A}')^+ \cup (\mathcal{B}')^- \text{ and } \mathcal{P}'' \subseteq (\mathcal{A}'')^+ \cup (\mathcal{B}'')^-.$$

Clearly, if $\mathcal{A}' \cup \mathcal{B}' = \emptyset$ (or $\mathcal{A}'' \cup \mathcal{B}'' = \emptyset$) then any element in $\mathcal{P}'$ (respectively, in $\mathcal{P}''$) does not belong to $\mathcal{A}^+ \cup \mathcal{B}^-$. On the other hand, if these unions are not empty, we can proceed by recursively solving the two subproblems DUAL$(\mathcal{P}', \mathcal{A}', \mathcal{B}')$ and DUAL$(\mathcal{P}'', \mathcal{A}'', \mathcal{B}'')$. This gives

$$C(\mathcal{A}, \mathcal{B}) = 1 + C(\mathcal{A}', \mathcal{B}') + C(\mathcal{A}'', \mathcal{B}'') \leq 1 + R(\mathcal{A}', \mathcal{B}') + R(\mathcal{A}'', \mathcal{B}'').$$

Note that $\{\mathcal{A}', \mathcal{A}''\}$ and $\{\mathcal{B}', \mathcal{B}''\}$ form partitions of $\mathcal{A}$ and $\mathcal{B}$ respectively and therefore, we get by the super-additivity of $R(\cdot)$,

$$\begin{aligned} R(v(\mathcal{A}, \mathcal{B})) &= R(v(\mathcal{A}', \mathcal{B}') + v(\mathcal{A}'', \mathcal{B}'') + v(\mathcal{A}', \mathcal{B}'') + v(\mathcal{A}'', \mathcal{B}')) \\ &\geq R(v(\mathcal{A}', \mathcal{B}')) + R(v(\mathcal{A}'', \mathcal{B}'')) + R(v(\mathcal{A}', \mathcal{B}'')) + R(v(\mathcal{A}'', \mathcal{B}')) \\ &\geq R(v(\mathcal{A}', \mathcal{B}')) + R(v(\mathcal{A}'', \mathcal{B}'')) + 1. \end{aligned}$$

implying the proposition. □

Proposition 4.3 *Suppose that* $\min\{|\mathcal{A}|, |\mathcal{B}|\} \leq$ *const,* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$*, then problem DUAL*$(\mathcal{P}, \mathcal{A}, \mathcal{B})$ *is solvable in polynomial time.*

Proof. Let us assume without loss of generality that $\mathcal{B} = \{b^1, \ldots, b^k\}$ for some constant k. Then problem DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$ can be reduced to n^k subproblems of the form DUAL$(\mathcal{P}', \mathcal{A}, \emptyset)$, where $\mathcal{P}' = \mathcal{P}'_1 \times \ldots \times \mathcal{P}'_n$, is obtained from $\mathcal{P}$ by selecting, for each $j \in [k]$, a coordinate $i_j \in [n]$ and setting $\mathcal{P}'_i = \mathcal{P}_i \setminus \bigcup_{j\in[k]} (b^j_{i_j})^-$.

Clearly, $\mathcal{P}' \subseteq \mathcal{A}^+$ if and only if $\mathrm{Min}(\mathcal{P}') \subseteq \mathcal{A}^+$ where $\mathrm{Min}(\mathcal{P}') = \mathrm{Min}(\mathcal{P}'_1) \times \ldots \times \mathrm{Min}(\mathcal{P}'_n)$ and $\mathrm{Min}(\mathcal{P}'_i)$ is the set of minimal elements of $\mathcal{P}'_i$. Now the latter problem is easily seen to be polynomially solvable as follows. Let $\mathrm{Min}(\mathcal{P}'_i) = \{q^1_i, \ldots, q^{k_i}_i\}$, for $i \in [n]$ where $k_i = |\mathcal{P}'_i|$. By construction, only $l \leq k$ of the posets $\mathcal{P}'_i$ satisfy $\mathcal{P}'_i \neq \mathcal{P}_i$. Assume without loss of generality that these posets are $\mathcal{P}'_1 \times \ldots \times \mathcal{P}'_l$, then our problem reduces to finding whether $\{q^{i_1}_1\} \times \ldots \times \{q^{i_l}_l\} \times \mathrm{Min}(\mathcal{P}_{l+1}) \times \ldots \times \mathrm{Min}(\mathcal{P}_n) \subseteq \mathcal{A}^+$ for all $(i_1, \ldots, i_l) \in [k_1] \times \ldots \times [k_l]$. Each such problem is equivalent to determining whether $\mathrm{Min}(\mathcal{P}_{l+1}) \times \ldots \times \mathrm{Min}(\mathcal{P}_n) \subseteq (\mathcal{A}^{i_1,\cdots,i_l})^+$, where $\mathcal{A}^{i_1,\cdots,i_l} = \{(a_{l+1}, \ldots, a_n) \mid a \in \mathcal{A},\ \ a_j \preceq q^{i_j}_j$ for $j = 1, \ldots, l$, and $a^+_j \cap \mathrm{Min}(\mathcal{P}_j) \neq \emptyset$ for $j = l+1, \ldots, n\}$. Note that $\mathcal{A}^{i_1,\cdots,i_l} \subseteq \mathrm{Min}(\mathcal{P}_{l+1}) \times \ldots \times \mathrm{Min}(\mathcal{P}_n)$ since $\mathcal{A} \subseteq \mathcal{P}$ was assumed, and hence, each subproblem of the form $\mathrm{Min}(\mathcal{P}_{l+1}) \times \ldots \times \mathrm{Min}(\mathcal{P}_n) \subseteq (\mathcal{A}^{i_1,\cdots,i_l})^+$ can be solved in $O(W(\mathcal{P})nm)$ as a special case of Proposition 4.2. □

From the above proposition, we see that problem DUAL($\mathcal{P}, \mathcal{A}, \mathcal{B}$) can be solved in $O(n^{k+1} mW(\mathcal{P})^{k+1}\mu)$ if $\min\{|\mathcal{A}|, |\mathcal{B}|\} \le k$. Moreover, if $\mathcal{P}$ is the product of chains, then the problem is solvable in $O(n^{k+1}m)$ since in this case, each chain $\mathcal{C}_i = [l_i : u_i]$ needs only to be represented by its minimum and maximum elements l_i, u_i. Clearly, having found an element $x \in \mathcal{P} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$, it is always possible to extend it to a maximal element with the same property in $O(n^2 m\mu)$ time. For integral boxes $\mathcal{C} = \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$, this can be done in $O(n^2 m)$ time. To see this, note that for $i = 1, \ldots, n$, the i^{th} component of any maximal element in $\mathcal{C} \setminus \mathcal{A}^+$ must belong to the set $\{a_i - 1 \mid a \in \mathcal{A}\} \cup \{u_i\}$. Thus a new element $x' \ge x$ maximal in $\mathcal{C} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$ can be found as follows. For $i = 1, \ldots, n$, we find iteratively the set $\mathcal{A}_i \stackrel{\text{def}}{=} \{a \in \mathcal{A} \mid a_1 \le x'_1, \ldots, a_{i-1} \le x'_{i-1}, a_i > x_i, a_{i+1} \le x_{i+1}, \ldots, a_n \le x_n\}$ in $O(nm)$ time, and then set $x'_i \leftarrow \min(\{a_i - 1 \mid a \in \mathcal{A}_i\} \cup \{u_i\})$.

On the negative side, if we do not insist on the condition $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$ in Proposition 4.3, the problem becomes NP-hard even for $\mathcal{B} = \emptyset$:

Proposition 4.4 *Given a subposet $\mathcal{Q}$ of a poset $\mathcal{P}$ and a subset $\mathcal{A} \subseteq \mathcal{P}$, it is coNP-complete to decide if $\mathcal{Q} \subseteq \mathcal{A}^+$.*

Proof. We again use a polynomial transformation from the satisfiability problem. Let $C = C_1 \wedge \ldots \wedge C_m$ be a conjunctive normal form in n variables $x_1, \ldots, x_n$, and let us consider the poset $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$, where $\mathcal{P}_i = \mathcal{Q}_i \cup \{a_i^1, \ldots, a_i^m\}$, $\mathcal{Q}_i = \{x_i, \overline{x}_i\}$, and where we associate a vector $a^j = (a_1^j, \ldots, a_n^j)$ with each clause C_j, $j = 1, \ldots, m$. The relations in the poset $\mathcal{P}$ are defined as follows: for a literal $l_i \in \mathcal{Q}_i$ and an element $a_i^j \in \mathcal{P}_i \setminus \mathcal{Q}_i$, let $a_i^j \prec l_i$ in $\mathcal{P}_i$ if and only if l_i does not appear in clause C_j in C. Finally, Let $\mathcal{Q} = \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_n$, and $\mathcal{A} = \{a^1, \ldots, a^m\}$. Then $\mathcal{Q} \not\subseteq \mathcal{A}^+$ if and only if C is satisfiable. □

4.2.2 Decomposition

In general, the algorithm will decompose a given problem by selecting an $i \in [n]$ and partitioning $\mathcal{P}_i$ into two subposets $\mathcal{P}'_i$ and $\mathcal{P}''_i$, defining accordingly two poset products $\mathcal{P}'$ and $\mathcal{P}''$. Specifically, let $a^o \in \mathcal{A}$, $b^o \in \mathcal{B}$ be arbitrary elements of $\mathcal{A}, \mathcal{B}$ (in fact the algorithm in Section 4.4.1 will select specific elements $a^o \in \mathcal{A}$ and $b^o \in \mathcal{B}$). By (4.3), there exists an $i \in [n]$, such that $a_i^o \not\preceq b_i^o$. Let us assume, without loss of generality, that $i = 1$ and set $\mathcal{P}'_1 \leftarrow \mathcal{P}_1 \cap (a_1^o)^+$, $\mathcal{P}''_1 \leftarrow \mathcal{P}_1 \setminus \mathcal{P}'_1$ (we may alternatively set $\mathcal{P}_1'' \leftarrow \mathcal{P}_1 \cap (b_1^o)^-$, and $\mathcal{P}_1' \leftarrow \mathcal{P}_1 \setminus \mathcal{P}_1''$, see Section 4.5.2). For brevity, we shall denote by $\overline{\mathcal{P}}$ the product $\mathcal{P}_2 \times \ldots \times \mathcal{P}_n$, and accordingly by $\overline{q}$ the vector $(q_2, \ldots, q_n)$, for an element $q = (q_1, q_2, \ldots, q_n) \in \mathcal{P}$.

Defining $\mathcal{P}' = \mathcal{P}_1' \times \overline{\mathcal{P}}$ and $\mathcal{P}'' = \mathcal{P}_1'' \times \overline{\mathcal{P}}$ to be the two subposets induced by the above partitioning, and letting $\mathcal{A}'' \stackrel{\text{def}}{=} \mathcal{A}(\mathcal{P}'') = \{a \in \mathcal{A} \mid a_1 \not\succeq a_1^o\}$, $\mathcal{A}' \stackrel{\text{def}}{=} \mathcal{A} \setminus \mathcal{A}''$, $\mathcal{B}' \stackrel{\text{def}}{=} \mathcal{B}(\mathcal{P}') = \{b \in \mathcal{B} \mid b_1 \succeq a_1^o\}$, $\mathcal{B}'' \stackrel{\text{def}}{=} \mathcal{B} \setminus \mathcal{B}'$, we conclude by (4.2) that $\mathcal{A}, \mathcal{B}$ are dual in $\mathcal{P}$ *if and only if*

$$\mathcal{A}, \mathcal{B}' \quad \text{are dual in } \mathcal{P}', \quad i.e., \ \mathcal{P}' \subseteq \mathcal{A}^+ \cup (\mathcal{B}')^-, \quad and \tag{4.4}$$

$$\mathcal{A}'', \mathcal{B} \quad \text{are dual in } \mathcal{P}'', \quad i.e., \ \mathcal{P}'' \subseteq (\mathcal{A}'')^+ \cup \mathcal{B}^-. \tag{4.5}$$

Thus we have decomposed the original problem into two new subproblems. Note that the volumes of the resulting problems are strictly less than the volume of the original problem. For lattices and forests, it may be necessary to further decompose the subposet $\mathcal{P}_1''$ in order to maintain a certain nice property (lattice property, connectedness of the precedence graph) which allows for the *projection step* described in the next subsection.

Clearly, there may exist precedence relations between the elements of $\mathcal{P}_1'$ and $\mathcal{P}_1''$ and, therefore, subproblems (4.4) and (4.5) may be not independent. Once we know that (4.4) is satisfied, we gain some information about the solution of subproblem (4.5). The following lemma shows how to utilize such dependence to further decompose (4.5).

Lemma 4.1 *Given $z \in \mathcal{P}_1$, let $\mathcal{P}_1' = \mathcal{P}_1 \cap z^+$, $\mathcal{P}_1'' \subseteq \mathcal{P}_1 \cap z^- \setminus \{z\}$ be two disjoint subsets of $\mathcal{P}_1$. Define*

$$\mathcal{A}'' = \{a \in \mathcal{A} \mid a_1^+ \cap \mathcal{P}_1'' \neq \emptyset\}, \qquad \mathcal{A}' = \{a \in \mathcal{A} \setminus \mathcal{A}'' \mid a_1^+ \cap \mathcal{P}_1' \neq \emptyset\},$$
$$\mathcal{B}' = \{b \in \mathcal{B} \mid b_1^- \cap \mathcal{P}_1' \neq \emptyset\}, \qquad \mathcal{B}'' = \{b \in \mathcal{B} \setminus \mathcal{B}' \mid b_1^- \cap \mathcal{P}_1'' \neq \emptyset\}.$$

Suppose further that $\mathcal{P}_1' \times \overline{\mathcal{P}} \subseteq (\mathcal{A}' \cup \mathcal{A}'')^+ \cup (\mathcal{B}')^-$, then

$$\mathcal{P}_1'' \times \overline{\mathcal{P}} \subseteq (\mathcal{A}'')^+ \cup (\mathcal{B}' \cup \mathcal{B}'')^- \iff \forall a \in \widetilde{\mathcal{A}} : \mathcal{P}_1'' \times (\overline{\mathcal{P}} \cap \overline{a}^+) \subseteq (\mathcal{A}'')^+ \cup (\mathcal{B}'')^-,$$

where $\widetilde{\mathcal{A}} = \{a \in \mathcal{A}' \cup \mathcal{A}'' \mid a_1 \preceq z\}$.

Proof. Suppose first that $\mathcal{P}_1'' \times \overline{\mathcal{P}} \subseteq (\mathcal{A}'')^+ \cup (\mathcal{B}' \cup \mathcal{B}'')^-$. Let $(q_1, \overline{q}) \in \mathcal{P}_1'' \times (\overline{\mathcal{P}} \cap \overline{a}^+)$ for some $a \in \widetilde{\mathcal{A}}$, then $(q_1, \overline{q}) \in (\mathcal{A}'')^+ \cup (\mathcal{B}' \cup \mathcal{B}'')^-$. If $(q_1, \overline{q}) \preceq (b_1, \overline{b}) \in \mathcal{B}'$, then by the definition of

$\mathcal{B}'$, there is a $y \in \mathcal{P}_1'$ such that $y \preceq b_1$. But then, $a \in \widetilde{\mathcal{A}}$, $\overline{q} \in \overline{\mathcal{P}} \cap \overline{a}^+$ and $y \in \mathcal{P}_1'$ imply that $(a_1, \overline{a}) \preceq (z, \overline{q}) \preceq (y, \overline{q}) \preceq (b_1, \overline{b})$, which contradicts the assumed condition (4.3). This shows that $(q_1, \overline{q}) \in (\mathcal{A}'')^+ \cup (\mathcal{B}'')^-$.

For the other direction, let $(q_1, \overline{q}) \in (\mathcal{P}_1'' \times \overline{\mathcal{P}}) \setminus (\mathcal{B}')^-$. Since $x \preceq y$ for all $x \in \mathcal{P}_1''$, $y \in \mathcal{P}_1'$, we must have $(y, \overline{q}) \notin (\mathcal{B}')^-$ for all $y \in \mathcal{P}_1'$, for otherwise we get the contradiction $(q_1, \overline{q}) \preceq (y, \overline{q}) \preceq (b_1, \overline{b})$ for some $b \in \mathcal{B}'$. Now we use our assumption that $\mathcal{P}_1' \times \overline{\mathcal{P}} \subseteq (\mathcal{A}' \cup \mathcal{A}'')^+ \cup (\mathcal{B}')^-$ to conclude that $(y, \overline{q}) \in (\mathcal{A}' \cup \mathcal{A}'')^+$ for all $y \in \mathcal{P}_1'$. In particular, we have $(z, \overline{q}) \succeq (a_1, \overline{a})$ for some $(a_1, \overline{a}) \in \mathcal{A}' \cup \mathcal{A}''$. But this implies that $a \in \widetilde{\mathcal{A}}$ and hence that $(q_1, \overline{q}) \in \mathcal{P}_1'' \times (\overline{\mathcal{P}} \cap \overline{a}^+)$ for some $a \in \widetilde{\mathcal{A}}$. This gives $(q_1, \overline{q}) \in (\mathcal{A}'')^+ \cup (\mathcal{B}'')^-$. $\square$

By considering the dual poset of $\mathcal{P}$ (that is, the poset $\mathcal{P}^*$ with the same set of elements as $\mathcal{P}$, but such that $x \prec y$ in $\mathcal{P}^*$ whenever $x \succ y$ in $\mathcal{P}$), and exchanging the roles of $\mathcal{A}$ and $\mathcal{B}$, we get the following symmetric version of Lemma 4.1.

Lemma 4.2 *Let $\mathcal{P}_1'' = \mathcal{P}_1 \cap z^-$, $\mathcal{P}_1' \subseteq \mathcal{P}_1 \cap z^+ \setminus \{z\}$ be two disjoint subsets of $\mathcal{P}_1$ where $z \in \mathcal{P}_1$. Let $\mathcal{A}'', \mathcal{A}', \mathcal{B}'', \mathcal{B}'$ be defined as in Lemma 4.1, and let $\widetilde{\mathcal{B}} = \{b \in \mathcal{B}' \cup \mathcal{B}'' \mid b_1 \succeq z\}$. Suppose that $\mathcal{P}_1'' \times \overline{\mathcal{P}} \subseteq (\mathcal{A}'')^+ \cup (\mathcal{B}' \cup \mathcal{B}'')^-$, then*

$$\mathcal{P}_1' \times \overline{\mathcal{P}} \subseteq (\mathcal{A}' \cup \mathcal{A}'')^+ \cup (\mathcal{B}')^- \iff \forall b \in \widetilde{\mathcal{B}} : \mathcal{P}_1' \times (\overline{\mathcal{P}} \cap \overline{b}^-) \subseteq (\mathcal{A}')^+ \cup (\mathcal{B}')^-.$$

In the next sections we use the above facts to develop several rules for decomposing a given dualization problem into smaller subproblems. The algorithms will select between these rules in such a way that the total volume is reduced from one iteration to the next.

4.2.3 Elimination

After performing the decomposition step, we end up with two or more subproblems of the form DUAL($\mathcal{P}, \mathcal{A}, \mathcal{B}$), where $\mathcal{P}$ is a subposet of the original poset, and $\mathcal{A}, \mathcal{B}$ are subsets of the antichains we originally started with. It may be the case that for a given subproblem, some of the elements of $\mathcal{A}$ or $\mathcal{B}$ do not intersect $\mathcal{P}$. In that case, (4.2) implies that such elements can be discarded from further consideration. Thus, the algorithm will eliminate all elements $a \in \mathcal{A}$ for which there exists a $k \in [n]$ such that $a_k^+ \cap \mathcal{P}_k = \emptyset$. Similarly the algorithm will eliminate all $b \in \mathcal{B}$ for which there exists a $k \in [n]$ such that $b_k^- \cap \mathcal{P}_k = \emptyset$.

4.2.4 Projection

As seen in Section 4.2.1 (c.f. Proposition 4.4), it is necessary, throughout the algorithm, to maintain the condition $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$, so that when we arrive at the base case, we can apply Proposition 4.3. Clearly, $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$ holds initially, but might not hold after decomposing $\mathcal{P}$. To solve this problem, we project the elements of $\mathcal{A}$ and $\mathcal{B}$ on the poset $\mathcal{P}$, for each newly created subproblem DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$. More precisely, if there is an $a \in \mathcal{A}$, $k \in [n]$ such that $a_k^+ \cap \mathcal{P}_k \neq \emptyset$, but $a_k \notin \mathcal{P}_k$, we replace a by the set of elements $\{(a_1, \ldots, a_{k-1}, x, a_{k+1}, \ldots, a_n) \mid x \in \mathrm{Min}(a_k^+ \cap \mathcal{P}_k)\}$, where $\mathrm{Min}(\cdot)$ is the set of minimal elements of $(\cdot)$. Similarly, if there is an element $b \in \mathcal{B}$, and an index $k \in [n]$ such that $b_k^- \cap \mathcal{P}_k \neq \emptyset$, but $b_k \notin \mathcal{P}_k$, we replace b by the set of elements $\{(b_1, \ldots, b_{k-1}, x, b_{k+1}, \ldots, b_n) \mid x \in \mathrm{Max}(b_k^- \cap \mathcal{P}_k)\}$. Note that condition (4.3) continues to hold after such replacements.

In general, an element of $\mathcal{A}$ or $\mathcal{B}$ may project to a number of elements in $\mathcal{P}$. Thus performing a large number of projection steps, we may end up with an exponential increase in the sizes of $\mathcal{A}, \mathcal{B}$. However, for certain classes of posets, such as lattices and forests with connected precedence graphs (i.e., trees), each element of $\mathcal{A}, \mathcal{B}$ projects on a single element in $\mathcal{P}$, i.e., $|\mathrm{Min}(a_k^+ \cap \mathcal{P}_k)| = |\mathrm{Max}(b_k^- \cap \mathcal{P}_k)| = 1$, for all $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $k \in [n]$. Indeed, if $\mathcal{P}_k$ is a lattice and $\mathrm{Min}(a_k^+ \cap \mathcal{P}_k)$ contains two distinct elements $y, z \in \mathcal{P}_k$ with the property that $y \succ a_k$ and $z \succ a_k$, then $y \wedge z$ is also in $a_k^+ \cap \mathcal{P}_k$, contradicting the minimality of both y and z. Similarly, if the precedence graph of the original poset (of which $\mathcal{P}_k$ is a subposet) is a forest, the precedence graph of $\mathcal{P}_k$ is connected, and there are two distinct minimal elements $y, z \in \mathcal{P}_k$ with the property that $y \succ a_k$ and $z \succ a_k$, then there exists an undirected path between y and z in the precedence graph of $\mathcal{P}_k$, and another path through a_k, forming a cycle, in contradiction to the fact that the original poset is a forest.

Thus, in conclusion, when decomposing a given dualization problem into a number of subproblems, we need to make sure that, in each resulting subproblem DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$, the poset $\mathcal{P}$ is still the product of lattices, or the product of forests with connected precedence graphs. In fact, this is the only place where the algorithms, described below, fail to work for products of general posets.

4.3 Dualization in products of chains

4.3.1 Decomposition rules

Let $\mathcal{C} \stackrel{\text{def}}{=} \mathcal{C}_1 \times \ldots \times \mathcal{C}_n$ be an integer box defined by the product of n chains $\mathcal{C}_i = [l_i : u_i]$ where $l_i, u_i \in \mathbb{Z}$ are, respectively, the lower and upper bounds of chain $\mathcal{C}_i$. Let $\mathcal{A}, \mathcal{B}$ be antichains of $\mathcal{C}$ that satisfy (4.3). As explained in Section 4.2.2, to decompose $\mathcal{C}$, we pick $a^o \in \mathcal{A}$, $b^o \in \mathcal{B}$ arbitrarily, and find and $i \in [n]$ such that such that $a^o_i > b^o_i$. Let us assume, without any loss of generality, that $i = 1$ and set $\mathcal{C}'_1 \leftarrow [a^o_1 : u_1]$, $\mathcal{C}''_1 \leftarrow [l_1 : a^o_1 - 1]$. (Alternatively, we may set $\mathcal{C}''_1 \leftarrow [l_1 : b^o_1]$ and $\mathcal{C}'_1 \leftarrow [b^o_1 + 1 : u_1]$.) Define

$$\begin{array}{rclcrcl} \mathcal{A}'' & = & \{a \in \mathcal{A} \mid a_1 < a^o_1\}, & \qquad & \mathcal{A}' & = & \mathcal{A} \setminus \mathcal{A}'', \\ \mathcal{B}' & = & \{b \in \mathcal{B} \mid b_1 \geq a^o_1\}, & \qquad & \mathcal{B}'' & = & \mathcal{B} \setminus \mathcal{B}'. \end{array}$$

Denoting by $\mathcal{C}' = \mathcal{C}'_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_n$, and $\mathcal{C}'' = \mathcal{C}''_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_n$ the two sub-boxes of $\mathcal{C}$ induced by the above partitioning, it follows that $\mathcal{A}$ and $\mathcal{B}$ are dual in $\mathcal{C}$ *if and only if*

$$\mathcal{A}, \mathcal{B}' \text{ are dual in } \mathcal{C}' \tag{4.6}$$

and

$$\mathcal{A}'', \mathcal{B} \text{ are dual in } \mathcal{C}''. \tag{4.7}$$

This leads to our first decomposition rule for chains:

Rule (i) Solve subproblems (4.6) and (4.7).

Suppose that we have already solved subproblem (4.6). If $\mathcal{A}, \mathcal{B}'$ are not dual in $\mathcal{C}'$, we get a point x maximal in $\mathcal{C}' \setminus [\mathcal{A}^+ \cup (\mathcal{B}')^-]$, and we are done. Otherwise we claim that

$$\mathcal{A}'', \mathcal{B} \text{ are dual in } \mathcal{C}'' \Longleftrightarrow \forall a \in \widetilde{\mathcal{A}} : \mathcal{A}'', \mathcal{B}'' \text{ are dual in } \mathcal{C}''(a), \tag{4.8}$$

where $\widetilde{\mathcal{A}} = \{a \in \mathcal{A} \mid a_1 \leq a^o_1\}$, and $\mathcal{C}''(a) = \mathcal{C}''_1 \times [a_2 : u_2] \times \ldots \times [a_n : u_n]$.

Indeed, take $z \leftarrow a_1^o$, $\mathcal{P}_1' \leftarrow \mathcal{C}_1'$, $\mathcal{P}_1'' \leftarrow \mathcal{C}_1''$, $\mathcal{A}', \mathcal{A}'', \widetilde{\mathcal{A}}, \mathcal{B}', \mathcal{B}''$ as defined above, and apply Lemma 4.1 to obtain (4.8). Thus we arrive at the following rule:

Rule (ii) Solve subproblem (4.6). If it has a solution then we get a vector $x \in \mathcal{C} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. Otherwise, we solve $|\widetilde{\mathcal{A}}|$ subproblems (4.8).

By symmetry, if we solve subproblem (4.7), and discover that $\mathcal{A}'', \mathcal{B}$ are dual in $\mathcal{C}''$, we conclude by Lemma 4.2 that

$$\mathcal{A}, \mathcal{B}' \text{ are dual in } \mathcal{C}' \Longleftrightarrow \forall b \in \widetilde{\mathcal{B}} : \mathcal{A}', \mathcal{B}' \text{ are dual in } \mathcal{C}'(b), \tag{4.9}$$

where $\widetilde{\mathcal{B}} = \{b \in \mathcal{B} \mid b_1 \geq a_1^o - 1\}$, and $\mathcal{C}'(b) = \mathcal{C}_1' \times [l_2 : b_2] \times \ldots \times [l_n : b_n]$.

This gives the following symmetric version of rule (ii):

Rule (iii) Solve subproblem (4.7). If it has a solution then we get a vector $x \in \mathcal{C} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. Otherwise, we solve $|\widetilde{\mathcal{B}}|$ subproblems (4.9).

We are now ready to state the dualization algorithm for chains.

4.3.2 The algorithm

Given subsets of integral vectors $\mathcal{A}, \mathcal{B}$ that satisfy the necessary condition (4.3), we proceed as follows:

Step 1. For each $k \in [n]$:

1. (*eliminate:*) if $a_k > u_k$ for some $a \in \mathcal{A}$ ($b_k < l_k$ for some $b \in \mathcal{B}$), then set $\mathcal{A} \leftarrow \mathcal{A} \setminus a$ (respectively, $\mathcal{B} \leftarrow \mathcal{B} \setminus b$);

2. (*project:*) if $a_k < l_k$ for some $a \in \mathcal{A}$ ($b_k > u_k$ for some $b \in \mathcal{B}$), we set $a_k \leftarrow l_k$ (respectively, $b_k \leftarrow u_k$).

Thus we may assume for next steps that $\mathcal{A}, \mathcal{B} \subseteq \mathcal{C}$.

Step 2. If $\min\{|\mathcal{A}|, |\mathcal{B}|\} \leq 2$, the duality of $\mathcal{A}$ and $\mathcal{B}$ can be tested in polynomial time using Proposition 4.3.

Step 3. Let $a^o \in \mathcal{A}$, $b^o \in \mathcal{B}$. Find an $i \in [n]$ such that $a_i^o > b_i^o$. Let us assume, without any loss of generality, that $i = 1$ and set $\mathcal{C}_1', \mathcal{C}_1'', \mathcal{A}', \mathcal{A}'', \widetilde{\mathcal{A}}, \mathcal{B}', \mathcal{B}'', \widetilde{\mathcal{B}}$ as defined in the previous subsection.

Define further

$$\epsilon^{\mathcal{A}} = \frac{|\mathcal{A}'|}{|\mathcal{A}|}, \qquad \epsilon^{\mathcal{B}} = \frac{|\mathcal{B}''|}{|\mathcal{B}|},$$

and observe that $\epsilon^{\mathcal{A}} > 0$ and $\epsilon^{\mathcal{B}} > 0$ since $a^o \in \mathcal{A}'$ and $b^o \in \mathcal{B}''$.

Step 4. Define $v = v(\mathcal{A}, \mathcal{B}) = |\mathcal{A}||\mathcal{B}|$, and let

$$\epsilon(v) = 1/\chi(v), \qquad \text{where} \quad \chi(v)^{\chi(v)} = v.$$

If $\min\{\epsilon^{\mathcal{A}}, \epsilon^{\mathcal{B}}\} > \epsilon(v)$, we use decomposition rule (i) given above, which amounts to solving recursively two subproblems (4.6), (4.7) of respective volumes:

$$\begin{aligned}
v(\mathcal{A}, \mathcal{B}') &= |\mathcal{A}||\mathcal{B}'| = |\mathcal{A}|(1-\epsilon^{\mathcal{B}})|\mathcal{B}| = (1-\epsilon^{\mathcal{B}})v(\mathcal{A}, \mathcal{B}) \leq (1-\epsilon(v))v,\\
v(\mathcal{A}'', \mathcal{B}) &= |\mathcal{A}''||\mathcal{B}| = (1-\epsilon^{\mathcal{A}})|\mathcal{A}||\mathcal{B}| = (1-\epsilon^{\mathcal{A}})v(\mathcal{A}, \mathcal{B}) \leq (1-\epsilon(v))v.
\end{aligned}$$

This gives rise to the recurrence

$$C(v) \leq 1 + C((1-\epsilon^{\mathcal{B}})v) + C((1-\epsilon^{\mathcal{A}})v) \leq 1 + 2C((1-\epsilon(v))v). \tag{4.10}$$

Step 5. Let us now suppose that $\epsilon^{\mathcal{B}} \leq \epsilon(v)$. In this case, we use rule (ii) which reduces the solution of subproblem (4.7) to solving $|\widetilde{\mathcal{A}}|$ subproblems, each of which has a volume of $v(|\mathcal{A}''|, |\mathcal{B}''|) \leq \epsilon^{\mathcal{B}} v(\mathcal{A}, \mathcal{B})$. Thus we obtain the recurrence

$$C(v) \leq 1 + C((1-\epsilon^{\mathcal{B}})v) + |\mathcal{A}|C(\epsilon^{\mathcal{B}} v) \leq C((1-\epsilon)v) + \frac{v}{2}C(\epsilon v), \tag{4.11}$$

for some $\epsilon \in (0, \epsilon(v)]$, where the last inequality follows from the assumption that $\min\{|\mathcal{A}|, |\mathcal{B}|\} \geq 3$.

Step 6. Finally, if $\epsilon^{\mathcal{A}} \leq \epsilon(v) < \epsilon^{\mathcal{B}}$, we use rule (iii). This reduces our original problem into one subproblem of volume $\leq (1-\epsilon^{\mathcal{A}})v$, plus $|\widetilde{\mathcal{B}}|$ subproblems, each of volume at most $\epsilon^{\mathcal{A}} v$, thus giving the recurrence

$$C(v) \leq 1 + C((1-\epsilon^{\mathcal{A}})v) + |\mathcal{B}|C(\epsilon^{\mathcal{A}} v) \leq C((1-\epsilon)v) + \frac{v}{2}C(\epsilon v), \tag{4.12}$$

for some $\epsilon \in (0, \epsilon(v)]$.

Using induction on $v \geq 9$, it can be shown that recurrences (4.10)–(4.12) imply that $C(v) \leq R(v) \stackrel{\text{def}}{=} v^{\chi(v)}$ (see [43], and also Section 4.4.2 below). As $\chi(m^2) < 2\chi(m)$ and $v(\mathcal{A}, \mathcal{B}) < m^2$, we get $\chi(v) < \chi(m^2) < 2\chi(m) \sim 2\log m/\log\log m$. This establishes the bound stated in Theorem 4.1.

4.4 Dualization in products of join semi-lattices

Let $\mathcal{L} = \mathcal{L}_1 \times \ldots \times \mathcal{L}_n$ where each $\mathcal{L}_i$ is a join semi-lattice with maximum element u_i, and let $\mathcal{A} \subseteq \mathcal{L}$ and $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$.

We begin with the observation that dualization on products of join semi-lattices can be reduced in polynomial time to dualization on products of lattices. Indeed, for each join semi-lattice $\mathcal{L}_i$, let us add a minimum element l_i that precedes every element in $\mathcal{L}_i$. Then it is easy to see that the resulting poset $\mathcal{L}'_i \stackrel{\text{def}}{=} \mathcal{L}_i \cup \{l_i\}$ is a lattice. Given $\mathcal{A}, \mathcal{B} \subseteq \mathcal{L}$ satisfying (4.3), let us obtain a new set $\mathcal{B}' \subseteq \mathcal{L}' \stackrel{\text{def}}{=} \mathcal{L}'_1 \times \ldots \times \mathcal{L}'_n$ by extending $\mathcal{B}$ as follows. For each added minimum element l_i, we define a new element $b \in \mathcal{B}'$ by setting $b_i = l_i$, and $b_j = u_j$ for $j \neq i$. Clearly, condition (4.3) still holds for the pair $(\mathcal{A}, \mathcal{B} \cup \mathcal{B}')$, and $\mathcal{A}^+ \cup \mathcal{B}^- = \mathcal{L}$ if and only if $\mathcal{A}^+ \cup (\mathcal{B} \cup \mathcal{B}')^- = \mathcal{L}'$ by construction. Thus, for the rest of this section, we shall assume without loss of generality that each poset $\mathcal{L}_i$ is a lattice.

Before we prove Theorem 4.2, we show that the simpler algorithm of [43] can also be generalized for lattices (and, indeed, for chains and forests also) to get a weaker bound than that of Theorem 4.2.

4.4.1 Algorithm A

Assume that $\mathcal{A}, \mathcal{B}$ satisfy (4.3), and let us denote, respectively, by l_i, u_i, the minimum and maximum elements of lattice $\mathcal{L}_i$. Define the *support* of $a \in \mathcal{A}$ ($b \in \mathcal{B}$) to be the set of all non-minimal coordinates of a (the set of all non-maximal coordinates of b):

$$\mathrm{Supp}(a) = \{i \in [n] \;:\; a_i \succ l_i\}, \qquad \mathrm{Supp}(b) = \{i \in [n] \;:\; b_i \prec u_i\}.$$

Elements of $\mathrm{Supp}(x)$ will be said to be *essential* for $x \in \mathcal{A} \cup \mathcal{B}$. The following lemma generalizes a known fact for dual Boolean functions (cf. [43]).

Lemma 4.3 *If $\mathcal{A}, \mathcal{B}$ are dual in $\mathcal{L}$, then there exists a point $x \in \mathcal{A} \cup \mathcal{B}$ with a logarithmically small support: $|\mathrm{Supp}(x)| \leq \log m$, where $m = |\mathcal{A}| + |\mathcal{B}|$.*

Proof. Let $z \in \mathcal{L}$ be the vector obtained by picking each coordinate z_i randomly from $\{l_i, u_i\}$, $i = 1, \ldots, n$, and consider the random variable $N(z) \stackrel{\text{def}}{=} |\{a \in \mathcal{A} \mid z \succeq a\}| + |\{b \in \mathcal{B} \mid z \preceq b\}|$. Then the expected value of $N(z)$ is given by

$$\begin{aligned}
\mathbb{E}[N(z)] &= \textstyle\sum_{a \in \mathcal{A}} \Pr\{z \succeq a\} + \sum_{b \in \mathcal{B}} \Pr\{z \preceq b\} \\
&= \textstyle\sum_{a \in \mathcal{A}} \prod_{i \in \mathrm{Supp}(a)} \Pr\{z_i = u_i\} + \sum_{b \in \mathcal{B}} \prod_{i \in \mathrm{Supp}(b)} \Pr\{z_i = l_i\} \\
&= \textstyle\sum_{a \in \mathcal{A}} \left(\frac{1}{2}\right)^{|\mathrm{Supp}(a)|} + \sum_{b \in \mathcal{B}} \left(\frac{1}{2}\right)^{|\mathrm{Supp}(b)|}.
\end{aligned} \tag{4.13}$$

Clearly if $\mathbb{E}[N(z)] < 1$, we can find an $x \in \mathcal{L} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$ in polynomial time as follows: for $i = 1, \ldots, n$, use (4.13) to compute the expectations of $N(x_1, \ldots, x_{i-1}, l_i, z_{i+1}, \ldots, z_n)$ and $N(x_1, \ldots, x_{i-1}, u_i, z_{i+1}, \ldots, z_n)$ over $(z_{i+1}, \ldots, z_n) \in \{l_i, u_i\}^{n-i}$, and select the value of $x_i \in \{l_i, u_i\}$ so as to minimize the corresponding expectation.

Let us therefore assume that $\mathbb{E}[N(z)] \geq 1$, and let $r = \min\{|\mathrm{Supp}(z)| \ : \ z \in \mathcal{A} \cup \mathcal{B}\}$, then (4.13) implies that

$$1 \leq \mathbb{E}[N(z)] \leq (|\mathcal{A}| + |\mathcal{B}|) \left(\frac{1}{2}\right)^r = m \left(\frac{1}{2}\right)^r.$$

The lemma follows. $\square$

Next we show that, for any dual pair $(\mathcal{A}, \mathcal{B})$, an essential coordinate with high frequency exists for either $\mathcal{A}$ or $\mathcal{B}$.

Lemma 4.4 *Let $\mathcal{A}, \mathcal{B}$ be a pair of dual subsets of $\mathcal{L}$ with $|\mathcal{A}||\mathcal{B}| \geq 1$. Then there exists a coordinate $i \in [n]$ and a point $z \in \mathcal{L}_i$, such that either:*

(i) $|\{a \in \mathcal{A} \mid a_i \not\preceq z\}| \geq 1$ and $|\{b \in \mathcal{B} \mid b_i \preceq z\}| \geq \frac{|\mathcal{B}|}{W(\mathcal{L}_i)\log m}$, or

(ii) $|\{b \in \mathcal{B} \mid b_i \not\succeq z\}| \geq 1$ and $|\{a \in \mathcal{A} \mid a_i \succeq z\}| \geq \frac{|\mathcal{A}|}{W(\mathcal{L}_i)\log m}$.

Proof. By Lemma 4.3, $\mathcal{A} \cup \mathcal{B}$ contains an element x with a logarithmically small number of essential coordinates. Suppose without loss of generality that $x \in \mathcal{A}$. From condition (4.3), we know that for every $b \in \mathcal{B}$, there is an $i \in \mathrm{Supp}(b) \cap \mathrm{Supp}(x)$ such that $b_i \not\succeq x_i$. Letting $\mathcal{B}_i^x \stackrel{\text{def}}{=} \{b \in \mathcal{B} \mid b_i \not\succeq x_i\}$ for $i \in \mathrm{Supp}(x)$, we conclude that

$$|\mathcal{B}| = |\bigcup_{i \in \mathrm{Supp}(x)} \mathcal{B}_i^x| \leq \sum_{i \in \mathrm{Supp}(x)} |\mathcal{B}_i^x|,$$

and therefore there is an $i \in [n]$ which is essential for at least $|\mathcal{B}|/|\operatorname{Supp}(x)| \geq |\mathcal{B}|/\log m$ elements of $\mathcal{B}$.

Let us now consider the set $\mathcal{Y} \stackrel{\text{def}}{=} \mathcal{I}(\{x_i\})$ of maximal independent elements in $\mathcal{L}_i \setminus \{x_i\}^+$, and observe that

$$|\mathcal{B}_i^x| = |\bigcup_{z \in \mathcal{Y}} \{b \in \mathcal{B} \mid b_i \preceq z\}|, \tag{4.14}$$

where $i \in [n]$ is a frequent essential coordinate for $\mathcal{B}$. Noting that $|\mathcal{Y}| \leq W(\mathcal{L}_i)$, we conclude, by (4.14), that (i) holds. If x actually was a member of b, then by a similar argument we obtain (ii). $\square$

We are now ready to state the first dualization algorithm for lattices.

Step 1. If $\min\{|\mathcal{A}|, |\mathcal{B}|\} < \delta(W) \stackrel{\text{def}}{=} \sqrt{(W+3)\log(W+2)}$, then the dualization problem can be solved in $O(n^{\delta(W)} W^{\delta(W)} m\mu)$ time using Proposition 4.3.

Step 2. For each $k \in [n]$: if $a_k \notin \mathcal{L}_k$ for some $a \in \mathcal{A}$ ($b_k \notin \mathcal{L}_k$ for some $b \in \mathcal{B}$), set $a_k \leftarrow \bigwedge\{x \mid x \in a_k^+ \cap \mathcal{L}_k\}$ (respectively, set $b_k \leftarrow \bigvee\{x \mid x \in b_k^- \cap \mathcal{L}_k\}$). Thus we may assume that $\mathcal{A}, \mathcal{B} \subseteq \mathcal{L}$.

Step 3. Check if there is an $x \in \mathcal{A} \cup \mathcal{B}$ with $|\operatorname{Supp}(x)| \leq \log m$ essential coordinates. If no such x can be found, a new point in $\mathcal{L} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$ can be obtained as described in the proof of Lemma 4.3. Otherwise, assume without loss of generality, that $x = a^o \in \mathcal{A}$ and find an $i \in \operatorname{Supp}(a^o)$, and a $z \in \mathcal{L}_i$, for which condition (i) of Lemma 4.4 is satisfied. Let us suppose, again without any loss of generality, that $i = 1$ and set $\mathcal{L}_1' \leftarrow \mathcal{L}_1 \setminus z^-$, $\mathcal{L}_1'' \leftarrow \mathcal{L}_1 \cap z^-$. Let $\mathcal{X}$ denote the set of minimal elements in $\mathcal{L}_1'$ and define $\mathcal{L}_1^x = \mathcal{L}_1' \cap x^+$ for $x \in \mathcal{X}$. Let further

$$\mathcal{A}_1'' = \{a \in \mathcal{A} \mid a_1 \preceq z\}, \qquad \mathcal{B}_1' = \{b \in \mathcal{B} \mid b_1 \not\preceq z\},$$

and observe that $|\mathcal{A}_1''| \leq |\mathcal{A}| - 1$ and $|\mathcal{B}_1'| \leq (1 - \frac{1}{W \log m})|\mathcal{B}|$ hold by Lemma 4.3(i).

Step 4. Denoting by $\mathcal{L}^x = \mathcal{L}_1^x \times \mathcal{L}_2 \times \ldots \times \mathcal{L}_n$, for $x \in \mathcal{X}$, and $\mathcal{L}'' = \mathcal{L}_1'' \times \mathcal{L}_2 \times \ldots \times \mathcal{L}_n$ the sub-lattices of $\mathcal{L}$ induced by the above decomposition, then $\mathcal{A}$ and $\mathcal{B}$ are dual in $\mathcal{L}$ if and only if

$$\mathcal{A}, \mathcal{B}_1' \text{ are dual in } \mathcal{L}^x, \quad \text{for all } x \in \mathcal{X}, \tag{4.15}$$

and

$$\mathcal{A}_1'', \mathcal{B} \text{ are dual in } \mathcal{L}'', \tag{4.16}$$

each of which is a dualization problem over lattices. Thus by applying the algorithm recursively to these subproblems, we reduce the computation on a problem of volume $v = |\mathcal{A}||\mathcal{B}|$ to computing the solutions for a set of $|\mathcal{X}| \leq W$ subproblems (4.15) of volume

$$v(\mathcal{A}, \mathcal{B}_1') = |\mathcal{A}||\mathcal{B}_1'| \leq |\mathcal{A}|(1-\epsilon)|\mathcal{B}| = (1-\epsilon)v,$$

each, and another subproblem (4.16) of volume

$$v(\mathcal{A}_1'', \mathcal{B}) = |\mathcal{A}_1''||\mathcal{B}| = (|\mathcal{A}|-1)|\mathcal{B}| \leq v - \delta,$$

where $\epsilon = 1/(W \log m)$ and $\delta = \delta(W)$. This leads to the recurrence

$$C(v) \leq 1 + W \cdot C((1-\epsilon)v) + C(v-\delta).$$

To evaluate this recurrence, we first iterate k times to get $C(v) \leq k + kW \cdot C((1-\epsilon)v) + C(v-k\delta)$. Letting $k = \lceil \frac{v\epsilon}{\delta} \rceil$ yields $C(v) \leq (1 + (W+1)(\frac{v\epsilon}{\delta}+1))C((1-\epsilon)v)$, and hence $C(v) \leq (1 + (W+1)(\frac{v\epsilon}{\delta}+1))^{\log v/\epsilon} = (W + 2 + \frac{W+1}{\delta}v\epsilon)^{\log v/\epsilon}$. Since $\min\{|\mathcal{A}|, |\mathcal{B}|\} \geq \delta$, we have $v \geq \delta^2$ and thus

$$v\left(1 - \frac{W+1}{\delta}\epsilon\right) \geq \delta^2\left(1 - \frac{W+1}{\delta W(1+\log\delta)}\right) \geq W+2, \tag{4.17}$$

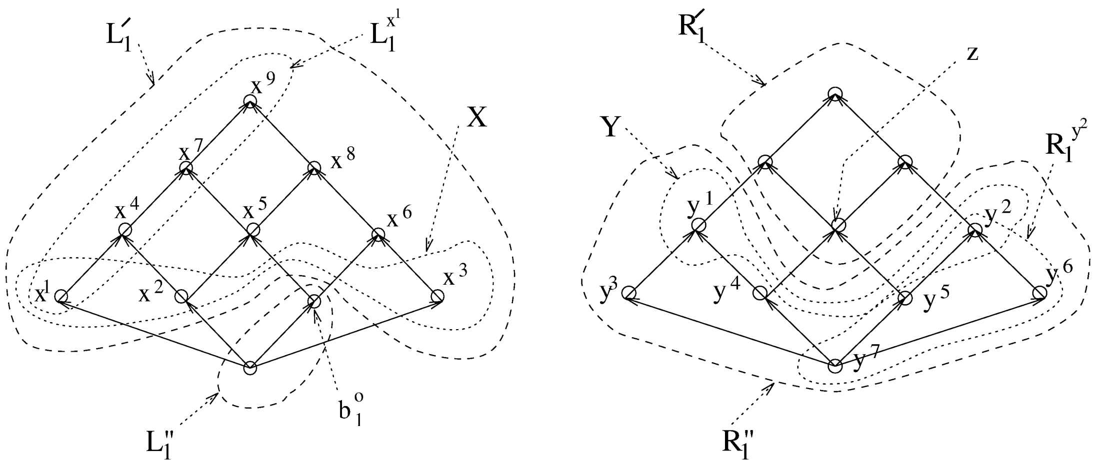
for all $W \geq 1$, by our selection of $\delta(W)$. Since $v \leq m^2$, we get $C(v) \leq v^{\log v/\epsilon}$ by (4.17), implying that the running time of the algorithm is $O(m^{4W\log^2 m+1}(nW)^{\sqrt{(W+3)\log(W+2)}}\mu)$.

4.4.2 Algorithm B

Decomposition rules

As in the case of chains, our second algorithm decomposes $\mathcal{L}$ by selecting $a^o \in \mathcal{A}$, $b^o \in \mathcal{B}$ and an $i \in [n]$, such that $a_i^o \not\preceq b_i^o$. Let us assume, without loss of generality, that $i = 1$ and set $\mathcal{L}_1' \leftarrow \mathcal{L}_1 \setminus (b_1^o)^-$, $\mathcal{L}_1'' \leftarrow \mathcal{L}_1 \cap (b_1^o)^-$. Let $\mathcal{X}$ be the set of minimal elements in $\mathcal{L}_1'$ (see Figure 4.1–a) and define $\mathcal{L}_1^x = \mathcal{L}_1' \cap x^+$ for $x \in \mathcal{X}$. Let further

$$\begin{aligned} \mathcal{A}_1' &= \{a \in \mathcal{A} \mid a_1 \not\preceq b_1^o\}, & \mathcal{A}_1'' &= \mathcal{A} \setminus \mathcal{A}_1', \\ \mathcal{B}_1' &= \{b \in \mathcal{B} \mid b_1 \not\preceq b_1^o\}, & \mathcal{B}_1'' &= \mathcal{B} \setminus \mathcal{B}_1'. \end{aligned}$$



a: Rules (i) and (ii).   b: Rules (i') and (ii').

Figure 4.1: Decomposing the lattice $\mathcal{L}_1$.

Using (4.15) and (4.16), we obtain the first decomposition rule:

Rule (i) Solve $|\mathcal{X}|$ subproblems (4.15) together with subproblem (4.16).

Clearly, subproblems (4.15) and (4.16) are not independent. To use Lemma 4.2 to utilize this dependence, suppose that subproblem (4.16) has no solution (i.e. there is no $q \in \mathcal{L}'' \setminus [(\mathcal{A}_1'')^+ \cup (\mathcal{B}_1' \cup \mathcal{B}_1'')^-])$. We proceed in this case as follows. For $x \in \mathcal{L}_1$, let $\widetilde{\mathcal{A}}(x) = \{a \in \mathcal{A} \mid a_1 \preceq x\}$, $\widetilde{\mathcal{B}}(x) = \{b \in \mathcal{B} \mid b_1 \succeq x\}$, and $\mathcal{A}_1'(x) = \{a \in \mathcal{A}_1' \mid a_1 = x\}$. Let us use $x^1, \ldots, x^k$ to denote the elements of $\mathcal{L}_1'$ and assume, without loss of generality, that they are topologically sorted in this order, that is, $x^j \prec x^h$ implies $j < h$ (see Figure 4.1–a). Let us decompose (4.15) (which is equivalent to checking whether $\mathcal{L}_1' \times \overline{\mathcal{L}} \subseteq \mathcal{A}^+ \cup (\mathcal{B}_1')^-$) further into the k subproblems

$$\{x^j\} \times \overline{\mathcal{L}} \subseteq \left[\left(\bigcup_{y \in (x^j)^\perp} \widetilde{\mathcal{A}}(y)\right) \cup \mathcal{A}_1'(x^j)\right]^+ \cup (\mathcal{B}_1')^-, \quad j = 1, \ldots, k. \tag{4.18}$$

The following lemma will allow us to eliminate the contribution of the set $\mathcal{A}_1''$ in subproblems (4.18) at the expense of possibly introducing at most $|\mathcal{B}|^{W(\mathcal{L})}$ additional subproblems.

Lemma 4.5 *Given $x^j \in \mathcal{L}_i'$, suppose we know that $(y^- \cap \mathcal{L}_1) \times \overline{\mathcal{L}} \subseteq \widetilde{\mathcal{A}}(y)^+ \cup \mathcal{B}^-$ for all $y \in (x^j)^\perp$. Then (4.18) is equivalent to*

$$\{x^j\} \times \left[\overline{\mathcal{L}} \cap \left(\bigwedge_{y \in (x^j)^\perp} \bar{b}(y)\right)^-\right] \subseteq \mathcal{A}_1'(x^j)^+ \cup (\mathcal{B}_1')^-, \tag{4.19}$$

for all collections $\{b(y) \in \widetilde{\mathcal{B}}(y) \mid y \in (x^j)^\perp\}$*. (That is, if* $(x^j)^\perp = \{y^1, \ldots, y^s\}$*, then we consider all collections of the form* $\{b(y^1), \ldots, b(y^s)\}$*, where* $b(y^1) \in \widetilde{\mathcal{B}}(y^1), \ldots, b(y^s) \in \widetilde{\mathcal{B}}(y^s)$*. Note that* $\bigwedge_{y\in(x^j)^\perp} \bar{b}(y)^- = \left(\bigwedge_{y\in(x^j)^\perp} b_2(y)^-, \ldots, \bigwedge_{y\in(x^j)^\perp} b_n(y)^-\right)$*.)*

Proof. We prove by induction on $|Y|$, where $Y \subseteq (x^j)^\perp$, that

$$\begin{aligned}\{x^j\} \times \overline{\mathcal{L}} \subseteq \left[\left(\bigcup\nolimits_{y\in(x^j)^\perp} \widetilde{\mathcal{A}}(y)\right) \cup \mathcal{A}_1'(x^j)\right]^+ \cup (\mathcal{B}_1')^- \qquad \Longleftrightarrow \\ \{x^j\} \times \left[\overline{\mathcal{L}} \cap \left(\bigcap\nolimits_{y\in Y} \bar{b}(y)^-\right)\right] \subseteq \left[\left(\bigcup\nolimits_{y\in(x^j)^\perp\setminus Y} \widetilde{\mathcal{A}}(y)\right) \cup \mathcal{A}_1'(x^j)\right]^+ \cup (\mathcal{B}_1')^-,\end{aligned} \tag{4.20}$$

for all collections $\{b(y) \in \widetilde{\mathcal{B}}(y) \mid y \in Y\}$. This trivially holds for $Y = \emptyset$ and will prove the lemma for $Y = (x^j)^\perp$. To show (4.20), assume that it holds for some $Y \subset (x^j)^\perp$ and let $x \in (x^j)^\perp \setminus Y$. Consider a subproblem of the form

$$\{x^j\} \times \left[\overline{\mathcal{L}} \cap \left(\bigcap_{y\in Y} \bar{b}(y)^-\right)\right] \subseteq \left[\widetilde{\mathcal{A}}(x) \cup \left(\bigcup_{y\in(x^j)^\perp\setminus(Y\cup\{x\})} \widetilde{\mathcal{A}}(y)\right) \cup \mathcal{A}_1'(x^j)\right]^+ \cup (\mathcal{B}_1')^-,$$

for some collection $\{b(y) \in \widetilde{\mathcal{B}}(y) \mid y \in Y\}$. Now we apply Lemma 4.2 with $z \leftarrow x$, $\mathcal{P}_1'' \leftarrow z^- \cap \mathcal{L}_1$, $\mathcal{P}_1' \leftarrow \{x^j\}$, $\mathcal{A}'' \leftarrow \widetilde{\mathcal{A}}(x)$, $\mathcal{A}' \leftarrow \left(\bigcup_{y\in(x^j)^\perp\setminus(Y\cup\{x\})} \widetilde{\mathcal{A}}(y)\right) \cup \mathcal{A}_1'(x^j)$, $\mathcal{B}' \leftarrow \mathcal{B}_1'$, and $\widetilde{\mathcal{B}} \leftarrow \widetilde{\mathcal{B}}(x)$ to get the required result. □

Informally, Lemma 4.5 says that, given $x^j \in \mathcal{L}_1'$, if the dualization subproblems for all sublattices that lie below x^j have been already verified to have no solution, then we can solve subproblem (4.18) by solving at most $\prod_{y\in(x^j)^\perp} |\widetilde{\mathcal{B}}(y)|$ subproblems of the form (4.19). Observe that it is important to check subproblems (4.18) in the topological order $j = 1, \ldots, k$ in order to be able to use Lemma 4.5. Thus we get

Rule (ii) Solve subproblem (4.16). If it has a solution then we get a point $q \in \mathcal{L} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. Otherwise, we solve subproblems (4.35), for all collections $\{b(y) \in \widetilde{\mathcal{B}}(y) \mid y \in (x^j)^\perp\}$, for $j = 1, \ldots, k$ (in the topological order).

Suppose finally that we decompose $\mathcal{L}_1$ by selecting an element $z \in \mathcal{L}_1$, letting $\mathcal{R}_1' \leftarrow \mathcal{L}_1 \cap z^+$, $\mathcal{R}_1'' \leftarrow \mathcal{L}_1 \setminus z^+$, $\mathcal{R}' = \mathcal{R}_1' \times \overline{\mathcal{L}}$, and $\mathcal{R}'' = \mathcal{R}_1'' \times \overline{\mathcal{L}}$. Let $\mathcal{Y}$ denote the set of maximal elements in $\mathcal{R}_1''$ (see Figure 4.1–b), define $\mathcal{R}_1^y = \mathcal{L}_1'' \cap y^-$ for $y \in \mathcal{Y}$, and let

$$\begin{aligned}\mathcal{A}_2' &= \{a \in \mathcal{A} \mid a_1 \succeq z\}, & \mathcal{A}_2'' &= \mathcal{A} \setminus \mathcal{A}_2', \\ \mathcal{B}_2' &= \{b \in \mathcal{B} \mid b_1 \succeq z\}, & \mathcal{B}_2'' &= \mathcal{B} \setminus \mathcal{B}_2'.\end{aligned}$$

By exchanging the roles of $\mathcal{A}$ and $\mathcal{B}$ and replacing $\mathcal{L}$ by its dual lattice $\mathcal{L}^*$ in rules (i), (ii) above, we can also derive the following symmetric versions of these rules:

Rule (i′) Solve the subproblem

$$\mathcal{A}, \mathcal{B}_2' \text{ are dual in } \mathcal{R}', \tag{4.21}$$

and the $|\mathcal{Y}|$ subproblems

$$\mathcal{A}_2'', \mathcal{B} \text{ are dual in } \mathcal{R}_1^y \times \overline{\mathcal{L}}, \quad \text{for all } y \in \mathcal{Y}. \tag{4.22}$$

Rule (ii′) Solve subproblem (4.21), and if it does not have a solution, then solve the subproblems

$$\{y^j\} \times \left[\overline{\mathcal{L}} \cap \left(\bigvee_{x \in (y^j)^\top} \overline{a}(x)\right)^+\right] \subseteq (\mathcal{A}_2'')^+ \cup (\mathcal{B}_2''(y^j))^-, \tag{4.23}$$

for all collections $\{a(x) \in \widetilde{\mathcal{A}}(x) \mid x \in (y^j)^\top\}$, for $j = 1, \ldots, h$, where $y^1, \ldots, y^h$ denote the elements of $\mathcal{R}_1''$ in *reverse* topological order (see Figure 4.1–b), and $\mathcal{B}_2''(x) = \{a \in \mathcal{B}_2'' \mid b_1 = x\}$.

Finally it remains to remark that all the decomposition rules described above result, indeed, in dualization subproblems over lattices.

The algorithm

Given antichains $\mathcal{A}, \mathcal{B} \subseteq \mathcal{L} = \mathcal{L}_1 \times \ldots \times \mathcal{L}_n$ that satisfy (4.3), we proceed as follows:

Step 1. For each $k \in [n]$:

1. if $a_k^+ \cap \mathcal{L}_k = \emptyset$ for some $a \in \mathcal{A}$ ($b_k^- \cap \mathcal{L}_k = \emptyset$ for some $b \in \mathcal{B}$), then set $\mathcal{A} \leftarrow \mathcal{A} \setminus a$ (respectively, $\mathcal{B} \leftarrow \mathcal{B} \setminus b$);

2. if $a_k \notin \mathcal{L}_k$ for some $a \in \mathcal{A}$ ($b_k \notin \mathcal{L}_k$ for some $b \in \mathcal{B}$), set $a_k \leftarrow \bigwedge\{x \mid x \in a_k^+ \cap \mathcal{L}_k\}$ (respectively, set $b_k \leftarrow \bigvee\{x \mid x \in b_k^- \cap \mathcal{L}_k\}$).

Step 2. If $\min\{|\mathcal{A}|, |\mathcal{B}|\} < \delta = 2$, then dualization can be solved in polynomial time.

Step 3. Let $a^o \in \mathcal{A}$, $b^o \in \mathcal{B}$. Find an $i \in [n]$ such that $a_i^o \not\preceq b_i^o$. Assume, without loss of generality, that $i = 1$ and set $\mathcal{L}_1' \leftarrow \mathcal{L}_1 \setminus (b_1^o)^-$, $\mathcal{L}_1'' \leftarrow \mathcal{L}_1 \cap (b_1^o)^-$. Let $\mathcal{X}, \mathcal{A}_1', \mathcal{A}_1'', \mathcal{B}_1'$, $\mathcal{B}_1''$, and $\widetilde{\mathcal{B}}$ be as defined in the previous subsection, and let

$$\epsilon_1^{\mathcal{A}} = \frac{|\mathcal{A}_1'|}{|\mathcal{A}|}, \qquad \epsilon_1^{\mathcal{B}} = \frac{|\mathcal{B}_1''|}{|\mathcal{B}|}.$$

Observe that $\epsilon_1^{\mathcal{A}} > 0$ and $\epsilon_1^{\mathcal{B}} > 0$ since $a^o \in \mathcal{A}_1'$ and $b^o \in \mathcal{B}_1''$.

Step 4. Define $\epsilon(v) = \rho(W)/\chi(v)$, where $v = v(\mathcal{A}, \mathcal{B})$, $\rho(W) \stackrel{\text{def}}{=} \gamma(W)/W = 2W\ln(W+1)$, and $\chi(v)$ is defined to be the unique positive root of the equation

$$\left(\frac{\chi(v)}{\rho(W)}\right)^{\chi(v)} = \frac{v^W}{(1-e^{-\rho(W)})(\delta^W - 1)}$$

and observe that $\epsilon(v) < 1$ for $v \geq \delta^2$, $\delta \geq 2$.

If $\min\{\epsilon_1^{\mathcal{A}}, \epsilon_1^{\mathcal{B}}\} > \epsilon(v)$, we use decomposition rule (i), getting the recurrence

$$\begin{aligned} C(v) &\leq 1 + |\mathcal{X}|C(|\mathcal{A}||\mathcal{B}_1'|) + C(|\mathcal{A}_1''||\mathcal{B}|) \\ &\leq 1 + W \cdot C((1-\epsilon_1^{\mathcal{B}})v) + C((1-\epsilon_1^{\mathcal{A}})v) \\ &\leq 1 + (W+1)C((1-\epsilon(v))v). \end{aligned} \tag{4.24}$$

Step 5. If $\epsilon_1^{\mathcal{A}} \leq \epsilon(v)$, we apply rule (ii) and get the recurrence

$$\begin{aligned} C(v) &\leq 1 + C(|\mathcal{A}_1''||\mathcal{B}|) + \sum_{j=1}^{k}\left(\prod_{y\in(x^j)^\perp} |\widetilde{\mathcal{B}}(y)|\right) C(|\mathcal{A}_1'(x^j)||\mathcal{B}_1'|) \\ &\leq 1 + C(|\mathcal{A}_1''||\mathcal{B}|) + |\widetilde{\mathcal{B}}|^W \sum_{j=1}^{k} C(|\mathcal{A}_1'(x^j)||\mathcal{B}_1'|) \\ &\leq 1 + C((1-\epsilon_1^{\mathcal{A}})v) + |\mathcal{B}|^W C(\epsilon_1^{\mathcal{A}} v) \\ &\leq 1 + C((1-\epsilon_1^{\mathcal{A}})v) + \frac{v^W}{\delta^W} C(\epsilon_1^{\mathcal{A}} v) \\ &\leq C((1-\epsilon)v) + \frac{v^W}{\delta^W - 1} C(\epsilon v), \quad \text{for some } \epsilon \in (0, \epsilon(v)] \end{aligned} \tag{4.25}$$

where the second inequality follows from the fact that $|(x^j)^\perp| \leq W$, the third inequality follows from $\sum_{j=1}^{k} C(|\mathcal{A}_1'(x^j)||\mathcal{B}_1'|) \leq C(\sum_{j=1}^{k} |\mathcal{A}_1'(x^j)||\mathcal{B}_1'|) = C(|\mathcal{A}_1'||\mathcal{B}_1'|)$ since $\{\mathcal{A}_1'(x^j) \mid j = 1, \ldots, k\}$ is a partition of $\mathcal{A}_1'$ and the function $C(\cdot)$ is super-additive, the forth inequality follows from $|\mathcal{B}|^W \leq v(|\mathcal{A}|, |\mathcal{B}|)^W/\delta^W$, and the last inequality follows from the fact that $v \geq \delta^2$ and $\delta \geq 2$.

Step 6. We assume for next steps that $\epsilon_1^{\mathcal{A}} > \epsilon(v)$. Then there exists a point $z \in \mathcal{X}$, such that $|\{a \in \mathcal{A} \mid a_1 \succeq z\} \geq \epsilon_1^{\mathcal{A}}|\mathcal{A}|/|\mathcal{X}| > \epsilon(v)|\mathcal{A}|/W$. Let $\mathcal{R}_1' \leftarrow \mathcal{R}_1 \cap z^+$, $\mathcal{R}_1'' \leftarrow \mathcal{L}_1 \setminus z^+$, and let

$\mathcal{Y}, \mathcal{A}_2', \mathcal{A}_2'', \mathcal{B}_2', \mathcal{B}_2'', \widetilde{\mathcal{A}}$ be as defined in the previous subsection. Let also

$$\epsilon_2^{\mathcal{A}} = \frac{|\mathcal{A}_2'|}{|\mathcal{A}|}, \qquad \epsilon_2^{\mathcal{B}} = \frac{|\mathcal{B}_2''|}{|\mathcal{B}|},$$

and observe that $\epsilon_2^{\mathcal{A}} > \frac{\epsilon(v)}{W}$ by our selection of $z \in \mathcal{L}_1$, and that $\epsilon_2^{\mathcal{B}} > 0$ since $b^o \notin \mathcal{B}_2'$.

Step 7. If $\epsilon_2^{\mathcal{B}} > \epsilon(v)$, then we use decomposition rule (i') which gives

$$\begin{aligned} C(v) &\leq 1 + C(|\mathcal{A}||\mathcal{B}_2'|) + |\mathcal{Y}|C(|\mathcal{A}_2''||\mathcal{B}|) \\ &\leq 1 + C((1-\epsilon_2^{\mathcal{B}})v) + W \cdot C((1-\epsilon_2^{\mathcal{A}})v) \\ &\leq 1 + C((1-\epsilon(v))v) + W \cdot C((1-\frac{\epsilon(v)}{W})v), \end{aligned} \tag{4.26}$$

since $\epsilon_2^{\mathcal{A}} > \frac{\epsilon(v)}{W}$ and $\epsilon_2^{\mathcal{B}} > \epsilon(v)$.

Step 8. Finally if $\epsilon_2^{\mathcal{B}} \leq \epsilon(v)$, use decomposition rule (ii') and get thus the recurrence

$$\begin{aligned} C(v) &\leq 1 + C(|\mathcal{A}||\mathcal{B}_2'|) + \sum_{j=1}^{h} \left(\prod_{x \in (y^j)^\top} |\widetilde{\mathcal{A}}(x)| \right) C(|\mathcal{A}_2''||\mathcal{B}_2''(y^j)|) \\ &\leq C((1-\epsilon)v) + \frac{v^W}{\delta^W - 1} C(\epsilon v), \quad \text{for some } \epsilon \in (0, \epsilon(v)] \end{aligned} \tag{4.27}$$

symmetric to (4.25).

Proof of Theorem 4.2

Following [43], we show by induction on $v = v(\mathcal{A}, \mathcal{B})$, that recurrences (4.24)–(4.27) imply that $C(v) \leq R(v) \stackrel{\text{def}}{=} v^{\chi(v)}$. Since, for $\min\{|\mathcal{A}|, |\mathcal{B}|\} < \delta$, Step 2 of the algorithm implies that $C(v) = 1$, we may assume that $\min\{|\mathcal{A}|, |\mathcal{B}|\} \geq \delta$, i.e., $v \geq \delta^2 = 4$.

Let us consider first recurrence (4.26). Using the induction hypothesis and the monotonicity of $\mathcal{X}(v)$, we obtain

$$\begin{aligned} C(v) &\leq 1 + [(1-\epsilon(v))v]^{\chi(v)} + W[(1-\frac{\epsilon(v)}{W})v]^{\chi(v)} \\ &\leq 1 + \left(e^{-\rho(W)} + We^{-\rho(W)/W}\right) v^{\chi(v)} \leq v^{\chi(v)}, \end{aligned} \tag{4.28}$$

since $1 - e^{-\rho(W)} - We^{-\rho(W)/W} \geq 1/2$ for all $W \geq 1$.

Let us consider next (4.24) and note that the monotonicity of $C(v)$ implies that $C((1 - \epsilon(v))v) \leq C((1-\frac{\epsilon(v)}{W})v)$, concluding by (4.24) and (4.28) that $C(v) \leq R(v)$ for this case too.

Let us now consider (4.25) and apply induction to get

$$C(v) \leq [(1-\epsilon)v]^{\chi(v)} + \frac{v^W}{\delta^W - 1}[\epsilon v]^{\chi(v)} = \psi(\epsilon)v^{\chi(v)},$$

where $\psi(\epsilon) \stackrel{\text{def}}{=} (1-\epsilon)^{\chi(v)} + \frac{v^W}{\delta^W-1}\epsilon^{\chi(v)}$. Since $\psi(\epsilon)$ is convex in ϵ, $\psi(0)=1$, $\epsilon \leq \epsilon(v)$, and

$$\psi(\epsilon(v)) = \left(1 - \frac{\rho(W)}{\chi(v)}\right)^{\chi(v)} + \frac{v^W}{\delta^W - 1}\left(\frac{\rho(W)}{\chi(v)}\right)^{\chi(v)} \leq e^{-\rho(W)} + \frac{v^W}{\delta^W - 1}\left(\frac{\rho(W)}{\chi(v)}\right)^{\chi(v)} = 1,$$

by the definition of $\chi(v)$, it follows that $\psi(\epsilon) \leq 1$ and hence, $C(v) \leq v^{\chi(v)}$.

Finally, by symmetry, recurrence (4.27) will also imply by induction that $C(v) \leq v^{\chi(v)}$.

Note that, for $\delta \geq 2$ and $W \geq 1$, we have $(\chi/\rho(W))^{\chi} < 3\,(v/\delta)^W$, and thus,

$$\chi(v) < \frac{W \log(v/\delta) + \log 3}{\log(\chi/\rho(W))} \sim \frac{W\rho(W)\log v}{\log W + \log\log v}.$$

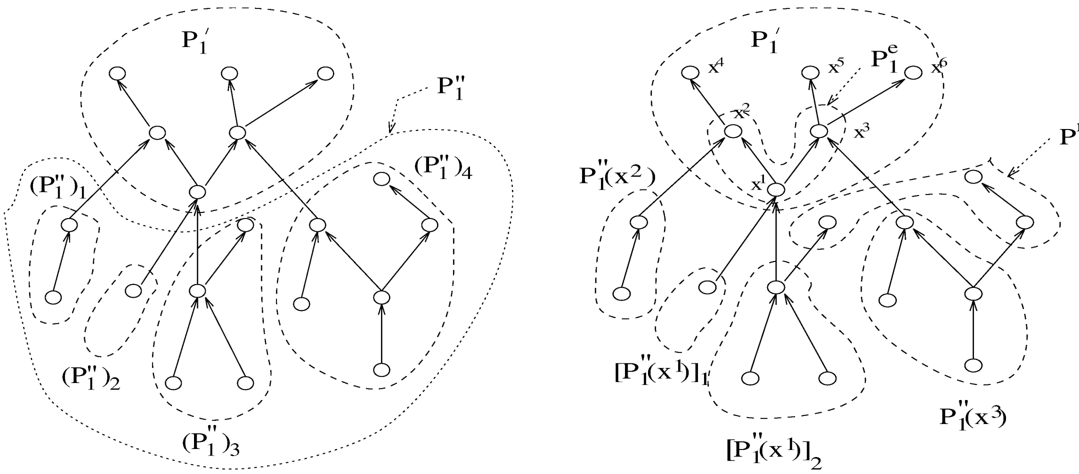
As $v(\mathcal{A},\mathcal{B}) < m^2$, we get $\chi(v) = o(W\rho(W)\log m)$, concluding the proof of the theorem. $\square$

4.5 Dualization in products of forests

4.5.1 Decomposition rules

Let $\mathcal{P} = \mathcal{P}_1 \times \ldots \mathcal{P}_n$, where the precedence graph of each poset $\mathcal{P}_i$ is a forest, and let $\mathcal{A} \subseteq \mathcal{P}$ and $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$. As usual, to decompose $\mathcal{P}$, we pick $a^o \in \mathcal{A}$, $b^o \in \mathcal{B}$ and an $i \in [n]$, such that $a_i^o \not\preceq b_i^o$. Let us assume, without loss of generality, that $i = 1$ and set $\mathcal{P}_1' \leftarrow \mathcal{P}_1 \cap (a_1^o)^+$, $\mathcal{P}_1'' \leftarrow \mathcal{P}_1 \setminus \mathcal{P}_1'$ (it may alternatively necessary to pick $\mathcal{P}_1'' \leftarrow \mathcal{P}_1 \cap (b_1^o)^-$, and $\mathcal{P}_1' \leftarrow \mathcal{P}_1 \setminus \mathcal{P}_1''$, see Step 3 of the algorithm below). Defining $\mathcal{P}' = \mathcal{P}_1' \times \overline{\mathcal{P}}$ and $\mathcal{P}'' = \mathcal{P}_1'' \times \overline{\mathcal{P}}$ to be the two subposets induced by this partitioning (see Figure 4.2–a), and letting $\mathcal{A}'' \stackrel{\text{def}}{=} \mathcal{A}(\mathcal{P}'') = \{a \in \mathcal{A} \mid a_1 \not\succeq a_1^o\}$, $\mathcal{A}' \stackrel{\text{def}}{=} \mathcal{A} \setminus \mathcal{A}''$, $\mathcal{B}' \stackrel{\text{def}}{=} \mathcal{B}(\mathcal{P}') = \{b \in \mathcal{B} \mid b_1 \succeq a_1^o\}$, $\mathcal{B}'' \stackrel{\text{def}}{=} \mathcal{B} \setminus \mathcal{B}'$, we obtain the decomposition (4.4)–(4.5).

As described in Subsection (4.2.4), it is required to maintain the property that each poset $\mathcal{P}_i$ has a connected precedence graph. Clearly, if $\mathcal{P}_1$ has a connected graph, then so does $\mathcal{P}_1'$ by the above definitions (since $a_1^o \in \mathcal{P}_1'$). However, this might not be the case for $\mathcal{P}_1''$, and thus let us denote the connected components of its precedence graph by $(\mathcal{P}_1'')_1, (\mathcal{P}_1'')_2 \ldots, (\mathcal{P}_1'')_h$. Let $\mathcal{A}_j'' \stackrel{\text{def}}{=} \mathcal{A}''\left((\mathcal{P}_1'')_j \times \overline{\mathcal{P}}\right) = \{a \in \mathcal{A}'' \mid a_1^+ \cap (\mathcal{P}_1'')_j \neq \emptyset\}$ and $\mathcal{B}_j'' \stackrel{\text{def}}{=} \mathcal{B}''\left((\mathcal{P}_1'')_j \times \overline{\mathcal{P}}\right) = \{b \in$



a: Rule (i). b: Rules (ii) and (iii).

Figure 4.2: Decomposing the forest $\mathcal{P}_1$.

$\mathcal{B}'' \mid b_1^- \cap (\mathcal{P}_1'')_j \neq \emptyset\}$ for $j = 1, \ldots, h$. Then checking (4.5) becomes equivalent to checking whether

$$(\mathcal{P}_1'')_j \times \overline{\mathcal{P}} \subseteq (\mathcal{A}_j'')^+ \cup (\mathcal{B}' \cup \mathcal{B}_j'')^-, \qquad j = 1, \ldots, h. \tag{4.29}$$

Thus we obtain our first decomposition rule:

Rule (i) Solve subproblem (4.4) together with h subproblems (4.29).

To utilize the dependence between subproblems (4.4) and (4.5) to further decompose (4.5), let us define $\mathcal{P}_1^e = \{x \in \mathcal{P}_1' \mid x^\perp \cap \mathcal{P}_1'' \neq \emptyset\}$ to be the set of elements in $\mathcal{P}_1'$ with immediate predecessors in $\mathcal{P}_1''$ (see Figure 4.2–b). Let, for each $x \in \mathcal{P}_1^e$, the set $\mathcal{P}_1''(x)$ be the subtree of x^-, lying in $\mathcal{P}_1''$ and whose root in $\mathcal{P}_1'$ is x, i.e., $\mathcal{P}_1''(x) = \left(x^\perp \cap \mathcal{P}_1''\right)^- \cap \mathcal{P}_1''$. Observe that $\mathcal{P}_1''(x)$ and $\mathcal{P}_1''(y)$ are independent posets for $x \neq y$, $x, y \in \mathcal{P}_1^e$ since the precedence graph of $\mathcal{P}_1$ is a forest. Let us further define, for each $x \in \mathcal{P}_1^e$, the sets

$$\mathcal{A}''(x) = \mathcal{A}''(\mathcal{P}_1''(x) \times \overline{\mathcal{P}}), \quad \widetilde{\mathcal{A}}(x) = \{a \in \mathcal{A} \mid a_1 \preceq x\},$$
$$\mathcal{B}'(x) = \mathcal{B}'(\mathcal{P}_1''(x) \times \overline{\mathcal{P}}), \quad \mathcal{B}''(x) = \mathcal{B}''(\mathcal{P}_1''(x) \times \overline{\mathcal{P}}).$$

The following simple lemma is implied by the above definitions.

Lemma 4.6 *For every $x \in \mathcal{P}_1^e$ and every $b \in \mathcal{B}'(x)$, we must have $b_1 \succeq x$.*

Proof. Since $b \in \mathcal{B}'$, there must exist a $y \in \mathcal{P}_1'$ such that $b_1 \succeq y$. Also, the definition of $\mathcal{B}'(x)$ and the fact that $b \in \mathcal{B}'(x)$ imply that there exists a $z \in \mathcal{P}_1''(x)$ such that $b_1 \succeq z$. Since $\mathcal{P}_1'$ is assumed to have a connected precedence graph and $x, y \in \mathcal{P}_1'$, there exists an undirected path between y and x in this graph. Also, $b_1 \succeq y$ and $b_1 \succeq z$ imply respectively that there exist paths between b_1, y and b_1, z. On the other hand, $z \in \mathcal{P}_1''(x)$ implies that $z \prec x$ and hence that there is a path from x to z. Thus we conclude that there is a path between b_1 and z going through y and x. Since the precedence graph of $\mathcal{P}_1$ is assumed to be a forest, there exists a unique path from b_1 to z. But then this path must go through x, i.e., $b_1 \succeq x \succeq z$. $\square$

Continuing, let $\mathcal{P}_1^r = \mathcal{P}_1'' \setminus \left(\bigcup_{x \in \mathcal{P}_1^e} \mathcal{P}_1''(x)\right)$, let $(\mathcal{P}_1^r)_1, \ldots, (\mathcal{P}_1^r)_k$ be the connected components of $\mathcal{P}_1^r$, and let $\mathcal{A}_j^r = \mathcal{A}''((\mathcal{P}_1^r)_j \times \overline{\mathcal{P}})$, $\mathcal{B}_j^r = \mathcal{B}''((\mathcal{P}_1^r)^j \times \overline{\mathcal{P}})$, for $j = 1, \ldots, k$. We can now decompose subproblem (4.5) into

$$\mathcal{P}_1''(x) \times \overline{\mathcal{P}} \subseteq \mathcal{A}''(x)^+ \cup (\mathcal{B}''(x) \cup \mathcal{B}'(x))^-, \qquad x \in \mathcal{P}_1^e, \tag{4.30}$$

$$(\mathcal{P}_1^r)_j \times \overline{\mathcal{P}} \subseteq (\mathcal{A}_j^r)^+ \cup (\mathcal{B}_j^r)^-, \qquad j = 1, \ldots, k. \tag{4.31}$$

Given that (4.4) is satisfied, we claim that for each $x \in \mathcal{P}_1^e$, (4.30) is equivalent to

$$\forall a \in \widetilde{\mathcal{A}}(x) : \mathcal{P}_1''(x) \times (\overline{\mathcal{P}} \cap \overline{a}^+) \subseteq \mathcal{A}''(x)^+ \cup \mathcal{B}''(x)^-, \tag{4.32}$$

where $\overline{\mathcal{P}} \cap \overline{a}^+ = (\mathcal{P}_2 \cap a_2^+) \times \ldots \times (\mathcal{P}_n \cap a_n^+)$. To see (4.32), we make use of Lemma 4.1 by taking $z \leftarrow x$ and $\mathcal{P}_1'' \leftarrow \mathcal{P}_1''(x)$. Then $\mathcal{P}_1' \leftarrow x^+ \cap \mathcal{P}_1$, $\mathcal{A}'' \leftarrow \mathcal{A}''(x)$, $\mathcal{A}' \leftarrow \mathcal{A}'(\mathcal{P}_1' \times \overline{\mathcal{P}})$, $\widetilde{\mathcal{A}} \leftarrow \widetilde{\mathcal{A}}(x)$, $\mathcal{B}'' \leftarrow \mathcal{B}''(x)$, and by Lemma 4.6, $\mathcal{B}' \leftarrow \mathcal{B}'(x)$.

Now for each $x \in \mathcal{P}_1^e$, denoting by $[\mathcal{P}_1''(x)]_1, \ldots, [\mathcal{P}_1''(x)]_{k(x)}$ the connected components of $\mathcal{P}_1''(x)$, each problem of the form (4.32) can be further decomposed into the $k(x)$ subproblems:

$$[\mathcal{P}_1''(x)]_j \times (\overline{\mathcal{P}} \cup \overline{a}^+) \subseteq \mathcal{A}_j''(x)^+ \cup \mathcal{B}_j''(x)^-, \qquad j = 1, \ldots, k(x), \quad a \in \widetilde{\mathcal{A}}(x), \tag{4.33}$$

where $\mathcal{A}_j''(x) = \{a \in \mathcal{A}''(x) \mid a_1^+ \cap [\mathcal{P}_1''(x)]_j \neq \emptyset\}$, and $\mathcal{B}_j''(x) = \{b \in \mathcal{B}''(x) \mid b_1^- \cap [\mathcal{P}_1''(x)]_j \neq \emptyset\}$. Thus we arrive at the following decomposition rule:

Rule (ii) Solve subproblem (4.4). If it has a solution then we get an element $q \in \mathcal{P} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. Otherwise, we solve $|\widetilde{\mathcal{A}}(x)|$ subproblems (4.33), for each $x \in \mathcal{P}_1^e$ and $j = 1, \ldots, k(x)$, and finally subproblems (4.31) for $j = 1, \ldots, k$.

Suppose finally that subproblem (4.5), or its equivalent (4.29), has no solution (i.e. there is no $q \in \mathcal{P}'' \setminus [(\mathcal{A}'')^+ \cup \mathcal{B}^-]$). We proceed in this case as follows. For $x \in \mathcal{P}_1$, let $\widetilde{\mathcal{A}}(x) = \{a \in \mathcal{A} \mid a_1 \preceq x\}$, $\widetilde{\mathcal{B}}(x) = \{b \in \mathcal{B} \mid b_1 \succeq x\}$, and $\hat{\mathcal{A}}'(x) = \{a \in \mathcal{A}' \mid a_1 = x\}$. Let us use $x^1, \ldots, x^w$ to denote the elements of $\mathcal{P}_1'$ and assume, without loss of generality, that they are topologically sorted in this order (see Figure 4.2–b). Now we can use the following rule to solve our problem:

a. Solve subproblems (4.29), then

b. (*decompose (4.4):*) for $j = 1$ to w, solve

$$\{x^j\} \times \overline{\mathcal{P}} \subseteq \left[\left(\bigcup_{y \in (x^j)^\perp} \widetilde{\mathcal{A}}(y)\right) \cup \hat{\mathcal{A}}'(x^j)\right]^+ \cup \widetilde{\mathcal{B}}(x^j)^-. \tag{4.34}$$

Similar to Lemma 4.5, we obtain the following decomposition for forests.

Lemma 4.7 *Given $x^j \in \mathcal{P}_1'$, suppose we know that $(y^- \cap \mathcal{P}_1) \times \overline{\mathcal{P}} \subseteq \widetilde{\mathcal{A}}(y)^+ \cup \mathcal{B}^-$ for all $y \in (x^j)^\perp$, then (4.34) is equivalent to*

$$\{x^j\} \times \left[\overline{\mathcal{P}} \cap \left(\bigcap_{y \in (x^j)^\perp} \bar{b}(y)^-\right)\right] \subseteq \hat{\mathcal{A}}'(x^j)^+ \cup \widetilde{\mathcal{B}}(x^j)^-, \tag{4.35}$$

for all collections $\{b(y) \in \widetilde{\mathcal{B}}(y) \mid y \in (x^j)^\perp\}$.

Thus we get

Rule (iii) Solve subproblems (4.29), and if they do not have a solution, then solve subproblems (4.35), for all collections $\{b(y) \in \widetilde{\mathcal{B}}(y) \mid y \in (x^j)^\perp\}$, for $j = 1, \ldots, w$.

Finally it remains to remark that all the decomposition rules described above result, indeed, in posets with connected precedence graphs.

4.5.2 The algorithm

Given subsets $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ that satisfy (4.3), we proceed as follows:

Step 0. If the precedence graph of $\mathcal{P}_i$ is not connected, for some $i \in [n]$ (only can happen initially), use Proposition 4.2 to decompose the problem into a number of subproblems over posets with connected precedence graphs.

Step 1. For each $k \in [n]$:

1. if $a_k^+ \cap \mathcal{P}_k = \emptyset$ for some $a \in \mathcal{A}$ ($b_k^- \cap \mathcal{P}_k = \emptyset$ for some $b \in \mathcal{B}$), then set $\mathcal{A} \leftarrow \mathcal{A} \setminus a$ (respectively, $\mathcal{B} \leftarrow \mathcal{B} \setminus b$);

2. if $a_k \prec \min(a_k^+ \cap \mathcal{P}_k)$ for some $a \in \mathcal{A}$ ($b_k \succ \max(b_k^- \cap \mathcal{P}_k)$ for some $b \in \mathcal{B}$), then set $a_k \leftarrow \min(a_k^+ \cap \mathcal{P}_k)$ (respectively, $b_k \leftarrow \max(b_k^- \cap \mathcal{P}_k)$).

Step 2. If $\min\{|\mathcal{A}|, |\mathcal{B}|\} \leq 3$, then the dualization problem can be solved in polynomial time using Proposition 4.3.

Step 3. Let a^o, b^o be arbitrary elements of $\mathcal{A}, \mathcal{B}$ respectively. Find an $i \in [n]$ such that $a_i^o \not\preceq b_i^o$. Assume, without loss of generality, that $i = 1$. If in-deg$(\mathcal{P}_1) \leq$ out-deg$(\mathcal{P}_1)$, we set $\mathcal{P}_1' \leftarrow \mathcal{P}_1 \cap (a_1^o)^+$, $\mathcal{P}_1'' \leftarrow \mathcal{P}_1 \setminus \mathcal{P}_1'$, otherwise, we set $\mathcal{P}_1'' \leftarrow \mathcal{P}_1 \cap (b_1^o)^-$, and $\mathcal{P}_1' \leftarrow \mathcal{P}_1 \setminus \mathcal{P}_1''$. In the latter case, we should use the symmetric versions of the decomposition rules (i)-(iii), listed above, which we obtain by exchanging the roles of $\mathcal{A}$ and $\mathcal{B}$ in these rules and replacing $\mathcal{P}$ by its dual poset $\mathcal{P}^*$. We assume therefore, without loss of generality, that the former case holds.

Step 4. Let $\mathcal{A}', \mathcal{A}'', \mathcal{B}', \mathcal{B}'', \ldots$ be as defined in the previous section, and define

$$\epsilon^{\mathcal{A}} = \frac{|\mathcal{A}'|}{|\mathcal{A}|}, \qquad \epsilon^{\mathcal{B}} = \frac{|\mathcal{B}''|}{|\mathcal{B}|}.$$

Observe that $\epsilon^{\mathcal{A}} > 0$ and $\epsilon^{\mathcal{B}} > 0$ since $a^o \in \mathcal{A}'$ and $b^o \in \mathcal{B}''$.

Step 5. Define

$$\epsilon(v) = 1/\chi(v), \quad \textit{where } \chi(v)^{\chi(v)} = v^d, \ v = v(\mathcal{A}, \mathcal{B}).$$

If $\min\{\epsilon^{\mathcal{A}}, \epsilon^{\mathcal{B}}\} > \epsilon(v)$, we use decomposition rule (i). This gives rise to the recurrence

$$C(v(\mathcal{A}, \mathcal{B})) \leq 1 + C(|\mathcal{A}||\mathcal{B}'|) + \sum_{j=1}^{h} C(|\mathcal{A}_j''||\mathcal{B}|). \tag{4.36}$$

Step 6. If $\epsilon^{\mathcal{B}} \leq \epsilon(v)$, we apply rule (ii) and get the recurrence

$$C(v) \leq 1 + C(|\mathcal{A}||\mathcal{B}'|) + \sum_{x \in \mathcal{P}_1^e} \sum_{j=1}^{k(x)} |\widetilde{\mathcal{A}}(x)| C(|\mathcal{A}_j''(x)||\mathcal{B}_j''(x)|) + \sum_{j=1}^{k} C(|\mathcal{A}_j^r||\mathcal{B}_j^r|). \tag{4.37}$$

Step 7. Finally, if $\epsilon^{\mathcal{A}} \leq \epsilon(v) < \epsilon^{\mathcal{B}}$, we use rule (iii) which gives

$$C(v(\mathcal{A}, \mathcal{B})) \leq 1 + \sum_{j=1}^{h} C(|\mathcal{A}_j''||\mathcal{B}|) + \sum_{j=1}^{w} \left(\prod_{y \in (x^j)^\perp} |\widetilde{\mathcal{B}}(y)| \right) C(|\hat{\mathcal{A}}'(x^j)||\widetilde{\mathcal{B}}(x^j)|). \tag{4.38}$$

4.5.3 Proof of Theorem 4.3

Again , we show by induction on $v = v(\mathcal{A}, \mathcal{B})$, that recurrences (4.36)–(4.38) imply that $C(v) \leq R(v) \stackrel{\text{def}}{=} v^{\chi(v)}$. Since, for $\min\{|\mathcal{A}|, |\mathcal{B}|\} \leq 3$, Step 2 of the algorithm implies that $C(v) = 1$, we may assume that $\min\{|\mathcal{A}|, |\mathcal{B}|\} \geq 4$, i.e., $v \geq 16$.

Let us consider first recurrence (4.36). Observe that $\mathcal{A}''_1, \ldots, \mathcal{A}''_h$ partition $\mathcal{A}''$ since $\mathcal{P}_1$ is a forest, and therefore, we get by the induction hypothesis and the super-additivity of $R(.)$

$$\begin{aligned} C(v) &\leq 1 + R(|\mathcal{A}||\mathcal{B}'|) + \sum_{j=1}^{h} R(|\mathcal{A}''_j||\mathcal{B}|) \\ &\leq 1 + R(|\mathcal{A}||\mathcal{B}'|) + R(\sum_{j=1}^{h} |\mathcal{A}''_j||\mathcal{B}|) \\ &= 1 + R((1-\epsilon^{\mathcal{B}})v) + R((1-\epsilon^{\mathcal{A}})v), \end{aligned}$$

implying that $C(v) \leq R(v)$ as in the proof of Theorem 4.2.

Let us consider next (4.37) and observe that the sets $\mathcal{A}''_j(x)$, $j = 1, \ldots, k(x)$, $x \in \mathcal{P}_1^e$, are disjoint (again, since $\mathcal{P}_1$ is a forest), that $\mathcal{B}''(x) \subseteq \mathcal{B}''$ for all $x \in \mathcal{P}_1^e$, and that the sets $\mathcal{B}_j^r$, $j = 1, \ldots, k$ are disjoint and each is a subset of $\mathcal{B}''$. Consequently,

$$\begin{aligned} C(v) &\leq 1 + R(|\mathcal{A}||\mathcal{B}'|) + |\mathcal{A}|R(\sum_{x \in \mathcal{P}_1^e} \sum_{j=1}^{k(x)} |\mathcal{A}''_j(x)||\mathcal{B}''_j(x)|) + R(\sum_{j=1}^{k} |\mathcal{A}_j^r||\mathcal{B}_j^r|) \\ &\leq 1 + R((1-\epsilon^{\mathcal{B}})v) + (|\mathcal{A}|+1)R(\epsilon^{\mathcal{B}}v). \end{aligned}$$

Since $\min\{|\mathcal{A}|, |\mathcal{B}|\} \geq 4$ is assumed, we have $|\mathcal{A}| + 1 \leq |\mathcal{A}||\mathcal{B}|/3 = v/3$ and thus

$$\begin{aligned} C(v) &\leq 1 + R((1-\epsilon^{\mathcal{B}})v) + \frac{v}{3}R(\epsilon^{\mathcal{B}}v) \\ &\leq R((1-\epsilon^{\mathcal{B}})v) + \frac{v}{2}R(\epsilon^{\mathcal{B}}v) \leq R(v). \end{aligned} \tag{4.39}$$

Let us consider finally recurrence (4.38) and observe that $|(x^j)^{\perp}| \leq d$ for every $x^j \in \mathcal{P}'_1$, by our assumption that in-deg$(\mathcal{P}_1) \leq$ out-deg$(\mathcal{P}_1)$ (see Step 3 of the algorithm). Thus (4.38) gives

$$C(v(\mathcal{A}, \mathcal{B})) \leq 1 + \sum_{j=1}^{h} C(|\mathcal{A}''_j||\mathcal{B}|) + |\mathcal{B}|^d \sum_{j=1}^{w} C(|\hat{\mathcal{A}}'(x^j)||\widetilde{\mathcal{B}}(x^j)|).$$

Note that this the only place in which the bound d on the degrees appears. Since the sets $\hat{\mathcal{A}}'(x^j)$, $j = 1, \ldots, w$, partition $\mathcal{A}'$, applying induction and noting that $|\mathcal{B}|^d \leq v(|\mathcal{A}|, |\mathcal{B}|)^d/3$ for

$|\mathcal{A}| \geq 3$, we get $C(v) \leq 1 + R((1-\epsilon^{\mathcal{A}})v) + \frac{v^d}{3}R(\epsilon^{\mathcal{A}}v)$. This implies by similarity to (4.39) that $C(v) \leq R(v)$.

Note that $\chi(v) < 2\chi(m) \sim 2d \log m / \log\log m$, and we get the bound stated in Theorem 4.3.

$\square$

Chapter 5

An incremental RNC algorithm for generating all maximal independent sets in hypergraphs of bounded dimension and its generalization in products of lattices

5.1 Introduction

Let $\mathcal{A} \subseteq 2^V$ be a hypergraph on a finite vertex set V. Recall that a vertex set $X \subseteq V$ is called *independent* if X contains no hyperedge of $\mathcal{A}$. Let $\mathcal{I}(\mathcal{A}) \subseteq 2^V$ denote the family of all maximal independent sets of $\mathcal{A}$. In this chapter, we shall assume that $\mathcal{A}$ is given by a list of its hyperedges and consider the problem of incrementally generating $\mathcal{I}(\mathcal{A})$:

$DUAL(\mathcal{A}, \mathcal{B})$: *Given a hypergraph $\mathcal{A}$ and a collection $\mathcal{I} \subseteq \mathcal{I}(\mathcal{A})$ of maximal independent sets for $\mathcal{A}$, either find a new maximal independent set $I \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{I}$, or prove that the given collection is complete: $\mathcal{I} = \mathcal{I}(\mathcal{A})$.*

Our objective in this chapter is to show that for hypergraphs of bounded dimension,

$$\dim(\mathcal{A}) \stackrel{\text{def}}{=} \max\{|A| \ : \ A \in \mathcal{A}\} \leq const,$$

problem DUAL$(\mathcal{A}, \mathcal{B})$ can be efficiently solved in parallel:

Theorem 5.1 *$DUAL(\mathcal{A}, \mathcal{B}) \in NC$ for $dim(\mathcal{A}) \leq 3$, and $DUAL(\mathcal{A}, \mathcal{B}) \in RNC$ for $dim(\mathcal{A}) = 4, 5, \ldots$*

The statements of Theorem 5.1 were previously known [8, 61] only for $\mathcal{I} = \emptyset$, when DUAL$(\mathcal{A}, \mathcal{B})$ turns into the classical problem of computing a single maximal independent set for $\mathcal{A}$ (see [5, 31, 46, 47, 58, 59, 60, 68]). We show that conversely, DUAL$(\mathcal{A}, \mathcal{B})$ can be reduced to the above special case.

Theorem 5.2 *If $dim(\mathcal{A}) \leq const$, then problem $DUAL(\mathcal{A}, \mathcal{B})$ is NC-reducible*[1] *to problem $DUAL(\mathcal{A}', \emptyset)$, where $\mathcal{A}'$ is some induced partial hypergraph of $\mathcal{A}$.*

(Given a hypergraph $\mathcal{A} \subseteq 2^V$, a subfamily $\mathcal{A}' \subseteq \mathcal{A}$ is called a *partial hypergraph* of $\mathcal{A}$, while $\{A \cap U | A \in \mathcal{A}'\}$ for some $U \subseteq V$ is called an *induced partial hypergraph* of $\mathcal{A}$.)

Since the maximal independent sets of a hypergraph $\mathcal{A}$ are complements of minimal transversals $\mathcal{A}^d \stackrel{\text{def}}{=} \{B \subseteq V \mid B \text{ is minimal with the property that } B \cap X \neq \emptyset \text{ for all } X \in \mathcal{A}\}$ (known also as the *dual* hypergraph of $\mathcal{A}$), problem DUAL$(\mathcal{A}, \mathcal{B})$ can be equivalently stated as problem GEN$(\mathcal{A}^d, \mathcal{X})$ of incrementally generating minimal transversals of $\mathcal{A}$. As stated in the introductory chapter of this manuscript, it is an open question whether problem DUAL$(\mathcal{A}, \mathcal{B})$ can be solved in polynomial time for arbitrary hypergraphs. However, as shown in [39, 12], and also recently in [40], for hypergraphs of bounded dimension problem DUAL$(\mathcal{A}, \mathcal{B})$ can be solved in polynomial time. Theorem 5.1 strengthens this result by implying that DUAL$(\mathcal{A}, \mathcal{B}) \in NC$ for $\dim(\mathcal{A}) \leq 3$ and DUAL$(\mathcal{A}, \mathcal{B}) \in RNC$ for $\dim(\mathcal{A}) = 4, 5, \ldots$. As mentioned above, Theorem 5.1 is a corollary of Theorem 5.2 and the results of [8, 61].

A vertex set S is called a *sub-transversal* of $\mathcal{A}$ if $S \subseteq B$ for some minimal transversal $B \in \mathcal{A}^d$. Our proof of Theorem 5.2 makes use of a characterization of sub-transversals suggested in [12]. Even though it is NP-hard in general to test whether a given set $S \subseteq V$ is a sub-transversal of $\mathcal{A}$, for $|S| \leq const$ the sub-transversal criterion of [12] is in NC. This turns out to be sufficient for the proof of Theorem 5.2.

The remainder of this chapter is organized as follows. In Section 5.2 we begin with some applications that motivate the results in the chapter. In Sections 5.3.1 and 5.3.2 we recall the sub-transversal criterion of [12] and prove Theorem 5.2. Then in Section 5.4 we discuss a generalization of the sub-transversal criterion and Theorem 5.2 for the dualization problem on the Cartesian products of n lattices. More precisely, given n lattices $\mathcal{P}_1, \ldots, \mathcal{P}_n$ and a set $\mathcal{A} \subseteq \mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$, we consider the problem DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$ of generating all maximal elements in $\mathcal{P} \setminus \mathcal{A}^+$, where $\mathcal{A}^+$ is the (upper) ideal generated by $\mathcal{A}$. If $\mathcal{P} = \{0, 1\}^n$ is the product of n chains $\{0, 1\}$, then this problem is equivalent to the generation of the transversal hypergraph for

[1] *In fact, our reduction is in AC_0*

$\mathcal{A}$. In general, when $\mathcal{A}$ is a set in $\mathcal{P} = \mathcal{P}_1 \times \cdots \times \mathcal{P}_n$, we define $\dim(\mathcal{A}) = \max\{|\operatorname{Supp}(a)| : a \in \mathcal{A}\}$, where $\operatorname{Supp}(a)$ is the *support* of $a \in \mathcal{P}$, i.e., the set of all non-minimal components of a (the dimension of a (lower) filter is analogously defined). Then we show that for $\dim(\mathcal{A}) \leq const$, the dualization problem on $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ is NC-reducible to the maximal independent set problem for some hypergraphs of dimension at most $\dim(\mathcal{A})$, provided that the number of immediate predecessors of any element in each factor-lattice $\mathcal{P}_i$ is also bounded by a constant.

5.2 Applications

5.2.1 Generating minimal integer solutions of a monotone system of linear inequalities with a fixed number of non-zero coefficients per inequality

Let A be a given real $r \times n$-matrix, b is a given real r-vector, and c is a given non-negative integral n-vector whose components are *bounded* by a constant. Consider a system of r linear inequalities in n integer variables

$$Ax \geq b, \quad x \in \mathcal{C} = \{x \in \mathbb{Z}^n \mid 0 \leq x \leq c\}. \tag{5.1}$$

Assume further that the matrix A has at most k non-zero entries in each row, where k is a given constant. Let $\mathcal{F}$, $\mathcal{I}(\mathcal{F})$ be respectively the set of minimal feasible and maximal infeasible integer vectors of (5.1). In this section we are concerned with the *parallel complexity* of generating $\mathcal{F}$, and $\mathcal{I}(\mathcal{F})$ incrementally when the number of non-zero coefficients per inequality is bounded by a constant k. Observe that the set $\mathcal{I}(\mathcal{F})$, in this case, has bounded dimension $\dim(\mathcal{I}(\mathcal{F})) \leq k$. This follows from the fact if $x \in \mathcal{I}(\mathcal{F})$ then x must violate at least one inequality. But then the monotonicity of (5.1) and the maximality of x imply that x must have $x_i = c_i$ for every coordinate i in which there is a zero-coefficient in the violated inequality.

Thus, it follows that all elements $\mathcal{I}(\mathcal{F})$ can be enumerated efficiently in parallel, since we have only a polynomial number of candidates to check. The same is not directly obvious for the dual set $\mathcal{F}$, but easily follows from the results on products of lattices, to be presented in Section 5.4. Thus we conclude:

Corollary 5.1 *If the number of non-zero coefficients of a monotone system (5.1), in a bounded* *integral box, is bounded by a constant, then all minimal integer solutions can be generated in incremental RNC time.*

5.2.2 Minimal infrequent sets in databases with large dimension

Let us recall the data mining application described in Section 3.5.2. Given a hypergraph (a database) $\mathcal{D} \subseteq 2^V$ and an integer threshold t, a set $X \subseteq V$ is said to be t-frequent if $|S_{\mathcal{D}}(X)| \geq t$, and otherwise t-infrequent, where $S(X) = S_{\mathcal{D}}(X) \stackrel{\text{def}}{=} \{Y \in \mathcal{D} \mid Y \supseteq X\}$, is the family of subsets in $\mathcal{D}$ that *support* $X \subseteq V$. As before, let $\mathcal{F}_{\mathcal{D},t}$ and $\mathcal{I}(\mathcal{F}_{\mathcal{D},t})$ be the sets of minimal t-infrequent and maximal t-frequent elements of $\mathcal{D}$. Clearly, if $\dim(\mathcal{D}) \leq k$, for some constant k, then the number of frequent sets is bounded by n^k, and, therefore, they can be listed efficiently in parallel. On the other hand, it will follow from the results of Section 5.3 that if $\dim(\mathcal{D}) \geq n-k$, and k, t are constants, then both maximal t-frequent and minimal t-infrequent sets can be incrementally generated by an RNC algorithm. Indeed, if $X \in \mathcal{I}(\mathcal{F}_{\mathcal{D},t})$, then x is contained in at least t sets of $\mathcal{D}$, each of which avoids at most k elements of V. Thus, by maximality, X must contain all the non-avoided elements, whose number is at least $n - tk$. The required implication follows then from Theorem 5.1, applied to the complement of $\mathcal{I}(\mathcal{F}_{\mathcal{D},t})$. An analogous result can be obtained for databases whose attributes assume values ranging over lattices with bounded out-degrees.

Corollary 5.2 *Let V be a finite set of binary attributes of a database $\mathcal{D} \subseteq 2^V$, and let t be a constant integer threshold. Then problem GEN($\mathcal{F}_{\mathcal{D},t}, \mathcal{X}$) $\in NC$ for dim($\mathcal{D}$) $\geq n-3$, and GEN($\mathcal{F}_{\mathcal{D},t}, \mathcal{X}$) $\in RNC$ for dim($\mathcal{D}$) $= n-4, n-5, \ldots$*

5.3 Maximal independent sets in hypergraphs

5.3.1 Characterization of sub-transversals to a hypergraph

Given a hypergraph $\mathcal{A} \subseteq 2^V$, a subset $S \subseteq V$, and a vertex $v \in S$, let $\mathcal{A}_v(S) = \{A \in \mathcal{A} \mid A \cap S = \{v\}\}$ denote the family of all hyperedges of $\mathcal{A}$ whose intersection with S is exactly v. Let further $\mathcal{A}_0(S) = \{A \in \mathcal{A} \mid A \cap S = \emptyset\}$ denote the partial hypergraph consisting of the hyperedges of $\mathcal{A}$ disjoint from S. A selection of $|S|$ hyperedges $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ is called *covering* if there exists a hyperedge $A \in \mathcal{A}_0(S)$, such that $A \subseteq \bigcup_{v \in S} A_v$. Proposition 5.1 below, for which we include the proof for completeness, states that a non-empty set S is a sub-transversal of $\mathcal{A}$ if and only if there is a non-covering selection for S.

Proposition 5.1 (cf. [12]) *Let $S \subseteq V$ be a non-empty vertex set in a hypergraph $\mathcal{A} \subseteq 2^V$.*

(i) If S is a sub-transversal for $\mathcal{A}$, then there exists a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S.

(ii) Given a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S, we can extend S to a minimal transversal of $\mathcal{A}$ by solving problem DUAL$(\mathcal{A}', \emptyset)$ for the induced partial hypergraph

$$\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}_0(S)\} \subseteq 2^U, \tag{5.2}$$

where $U = V \setminus \bigcup_{v \in S} A_v$.

Proof. Let us start with the following observations:

(a) If $S \subseteq B \subseteq V$, then $\mathcal{A}_v(B) \subseteq \mathcal{A}_v(S)$ holds for all $v \in S$.

(b) If B is a transversal to $\mathcal{A}$, then B is minimal if and only if $\mathcal{A}_v(B) \neq \emptyset$ for all $v \in B$.

Observation (a) follows directly from the definitions of $\mathcal{A}_v(S)$ and $\mathcal{A}_v(B)$. To see (b), note that if $\mathcal{A}_v(B) = \emptyset$ for some $v \in B$, then $B \setminus \{v\}$ is still a transversal to $\mathcal{A}$.

Proof of (i) Suppose that $\emptyset \neq S \subseteq B$, where $B \in \mathcal{A}^d$ is a minimal transversal. By observations (a) and (b), we have $\emptyset \neq \mathcal{A}_v(B) \subseteq \mathcal{A}_v(S)$ for each $v \in S$. Consider then a selection of the form $\{A_v \in \mathcal{A}_v(B) \mid v \in S\}$. If it covers a hyperedge $A \in \mathcal{A}_0(S)$, then A would be disjoint from B, contradicting the fact that $B \in \mathcal{A}^d$.

Proof of (ii) Suppose we are given a non-covering selection $\{A_v \in \mathcal{A}_v(S) | v \in S\}$. If $\mathcal{A}_0(S) = \emptyset$, then S is obviously a transversal to $\mathcal{A}$. Hence by (b), S itself is a minimal transversal to $\mathcal{A}$. Let us assume now that $\mathcal{A}_0(S) \neq \emptyset$ and consider the hypergraph $\mathcal{A}'$ as defined in (5.2). Since the given selection is non-covering and $\mathcal{A}_0(S) \neq \emptyset$, we conclude that the vertex and edge sets of $\mathcal{A}'$ are not empty, and $\mathcal{A}'$ contains no empty edges. Let T be a minimal transversal to $\mathcal{A}'$. (Such a transversal can be computed by letting $T = U \setminus I$, where $I = \mathit{output}(\mathrm{DUAL}(\mathcal{A}', \emptyset))$.) It is easy to see that $S \cup T$ is a transversal to $\mathcal{A}$. Moreover, $S \cup T$ is minimal, since if we delete a vertex $v \in S$, then $A_v \cap [(S \setminus \{v\}) \cup T] = \emptyset$, while deleting a vertex $v \in T$ results in an empty intersection with some $A \in \mathcal{A}_0(S)$. $\square$

Unfortunately, if the cardinality of S is not bounded, finding a non-covering selection for S (equivalently, testing if S is a sub-transversal) is NP-hard. In fact, this is so even for $\dim(\mathcal{A}) = 2$ (i.e., for graphs and transversals $\equiv$ vertex covers).

Proposition 5.2 *Given an undirected graph $G = (V, E)$ and a vertex set $S \subseteq V$, it is NP-complete to determine whether S can be extended to a minimal vertex cover.*

Proof. We use a polynomial transformation from the satisfiability problem. Let $C = C_1 \wedge \ldots \wedge C_m$ be a conjunctive normal form, and let us consider the graph $G = (V, E)$, where V is the set of all clauses and literals of C, and where E consists of the pairs $(x, \overline{x})$ of mutually negating literals, and the pairs (C_i, u) formed by a clause and one of its literals. Then the set $S = \{C_1, \ldots, C_m\}$ can be extended to a minimal vertex cover of G if and only if C is satisfiable. $\square$

We close this section with the observation that if the size of S is bounded by a constant, then there are only polynomially many selections $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$ for S. All of these selections, including the non-covering ones, can be easily enumerated in parallel.

Corollary 5.3 *For any fixed k, there is an NC algorithm which, given a hypergraph $\mathcal{A} \subseteq 2^V$ and a set S of at most k vertices, determines whether S is a sub-transversal to $\mathcal{A}$ and if so, finds a non-covering selection $\{A_v \in \mathcal{A}_v(S) \mid v \in S\}$.*

Note that Corollary 5.3 holds for hypergraphs $\mathcal{A}$ of arbitrary dimension.

5.3.2 Proof of Theorem 5.2

We prove the theorem for the equivalent problem $\mathrm{GEN}(\mathcal{A}^d, \mathcal{B})$, i.e. show that for $\dim(\mathcal{A}) \leq$ *const*, problem $\mathrm{GEN}(\mathcal{A}^d, \mathcal{B})$ is NC-reducible to $\mathrm{GEN}((\mathcal{A}')^d, \emptyset)$, for some induced partial hypergraph $\mathcal{A}'$ of $\mathcal{A}$. Our reduction consists of several steps.

Step 1. Delete all hyperedges of $\mathcal{A}$ that contain other hyperedges of $\mathcal{A}$. Clearly, this does not change the minimal transversals to $\mathcal{A}$. We assume in the sequel that no hyperedge of $\mathcal{A}$ contains another hyperedge of $\mathcal{A}$, i.e., that

$$\mathcal{A} \text{ is Sperner.} \tag{5.3}$$

Note that the dual hypergraph $\mathcal{A}^d$ is Sperner by definition, and hence $\mathcal{B} \subseteq \mathcal{A}^d$ is Sperner as well.

Step 2 (optional). Delete all vertices in V that are not covered by some $A \in \mathcal{A}$ so that we have $V = \bigcup_{A \in \mathcal{A}} A$. If $\bigcup_{B \in \mathcal{B}} B$ is a proper subset of V, a new minimal transversal in $\mathcal{A}^d \setminus \mathcal{B}$ can be found as follows:

- Pick a vertex $u \in V \setminus \bigcup_{B \in \mathcal{B}} B$.
- The set $S = \{u\}$ is a sub-transversal to A. In view of (5.3), any hyperedge $A_u \in \mathcal{A}$ such that $u \in A_u$ is a non-covering selection for S.
- Let $u \in T \in \mathcal{A}^d$, then $T \notin \mathcal{B}$, because none of the transversals in $\mathcal{B}$ contains u. By Proposition 5.1, the problem of extending $S = \{u\}$ to a minimal transversal T is equivalent to that of computing a maximal independent set for hypergraph (5.2) with $U = V \setminus A_u$.

We can thus assume without loss of generality that $\bigcup_{A \in \mathcal{A}} A = \bigcup_{B \in \mathcal{B}} B = V$.

Step 3. By definition, each set $B \in \mathcal{B}$ is a minimal transversal to $\mathcal{A}$. This implies that each set $A \in \mathcal{A}$ is transversal to $\mathcal{B}$. Check whether each $A \in \mathcal{A}$ is a *minimal* transversal to $\mathcal{B}$. Suppose that some $A^o \in \mathcal{A}$ is not minimal, i.e. there is a vertex $u \in A^o$ such that $A^* = A^o \setminus \{u\}$ is still transversal to $\mathcal{B}$. Then we can proceed as follows.

- Let $\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}\}$, where $U = V \setminus A^*$.
- By (5.3), we have $A \cap U \neq \emptyset$ for each hyperedge $A \in \mathcal{A}$. Hence any minimal transversal T to $\mathcal{A}'$ is also a minimal transversal for $\mathcal{A}$.
- It easy to see that $T \notin \mathcal{B}$. This is because any set $B \in \mathcal{B}$ intersects A^* whereas T is disjoint from A^*. This reduces the computation of a new element in $\mathcal{A}^d \setminus \mathcal{B}$ to problem $\mathrm{GEN}((\mathcal{A}')^d, \emptyset)$.

In the sequel we assume in addition to (5.3) that each set in $\mathcal{A}$ is a minimal transversal to $\mathcal{B}$:

$$\mathcal{A} \subseteq \mathcal{B}^d. \tag{5.4}$$

Before proceeding to the next step of the reduction, we pause to make some observations. Clearly, $(\mathcal{A}^d)^d = \mathcal{A}$ for any Sperner hypergraph $\mathcal{A}$. Therefore, if $B \neq \mathcal{A}^d$ then $\mathcal{A} \neq \mathcal{B}^d$. By (5.4), we then have $\mathcal{B}^d \setminus \mathcal{A} \neq \emptyset$. Hence we arrive at the following duality criterion: $\mathcal{A}^d \setminus \mathcal{B} \neq \emptyset$ if and only if there is a sub-transversal S to $\mathcal{B}$ such that

$$|S| \leq \dim(\mathcal{A}), \quad \text{and} \tag{5.5}$$

$$S \not\subseteq A \quad \text{for all} \quad A \in \mathcal{A}. \tag{5.6}$$

The "if" part is obvious and holds even without assumption (5.4). To show the "only if" part, consider an arbitrary minimal transversal $T \in \mathcal{B}^d \setminus \mathcal{A}$. Clearly, T satisfies (5.6). Let S be a minimal subset of T that still satisfies (5.6) and let v be an arbitrary vertex in S. Since $S \setminus \{v\}$ does not satisfy (5.6) we have $S \setminus \{v\} \subseteq A$ for some $A \in \mathcal{A}$. Assuming $|S| > \dim(\mathcal{A})$, we obtain $A = S \setminus \{v\}$ by (5.6). Hence $A \subset S \subseteq T$. However, both A and T are minimal transversals to $\mathcal{B}$. This contradiction shows (5.5).

So far, we have not relied on the assumption that $\dim(\mathcal{A})$ is bounded. We need this assumption to guarantee that the next step of our reduction is in NC.

Step 4 (Duality test.) For each set S satisfying (5.5), (5.6) and the condition that

$$A \not\subseteq S \text{ for all } A \in \mathcal{A}, \tag{5.7}$$

check whether or not

$$S \text{ is a sub-transversal to } \mathcal{B}. \tag{5.8}$$

Recall that by Proposition 5.1, S satisfies (5.8) if and only if there is a selection

$$\{B_v \in \mathcal{B}_v(S) \mid v \in S\} \tag{5.9}$$

which covers no set $B \in \mathcal{B}_0(S)$. Here as before, $\mathcal{B}_0(S) = \{B \in \mathcal{B} \mid B \cap S = \emptyset\}$ and $\mathcal{B}_v(S) = \{B \in \mathcal{B} \mid B \cap S = \{v\}\}$ for $v \in S$.

If conditions (5.5), (5.6),(5.7) and (5.8) cannot be met, we conclude that $\mathcal{B} = \mathcal{A}^d$ and halt.

Step 5. Suppose we have found a non-covering selection (5.9) for some set S satisfying (5.5), (5.6), (5.7) (and hence (5.8)). We claim that the set

$$Z = S \bigcup \left[V \setminus \bigcup_{v \in S} B_v \right]$$

is independent in $\mathcal{A}$. Suppose to the contrary that $A \subseteq W$ for some $A \in \mathcal{A}$. By (5.6), there is a vertex $u \in S$ such that $u \notin A$. Then $A \cap B_u = \emptyset$, yielding a contradiction. Note also that Z is transversal to $\mathcal{B}$ because selection (5.9) is non-covering.

Let $\mathcal{A}' = \{A \cap U \mid A \in \mathcal{A}\}$, where $U = V \setminus Z$, and let T be a minimal transversal to $\mathcal{A}'$. (As before, we can let $T = U \setminus \mathit{output}(\mathrm{GEN}((\mathcal{A}')^d, \emptyset))$.) Since Z is an independent set of $\mathcal{A}$, we have $T \cap A \neq \emptyset$ for all $A \in \mathcal{A}$, i.e., T is transversal to $\mathcal{A}$. Clearly, T is minimal, i.e. $T \in \mathcal{A}^d$. It remains to argue that T is a *new* minimal transversal to $\mathcal{A}$, i.e., $T \notin \mathcal{B}$. This follows from the fact that Z is transversal to $\mathcal{B}$ and disjoint from T. $\square$

5.4 Maximal Independent elements in products of Lattices

In this section, we recall problem DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$:

DUAL$(\mathcal{P}, \mathcal{A}, \mathcal{B})$**:** *Given a set $\mathcal{A}$ in a poset $\mathcal{P}$ and a collection of maximal independent elements $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$, either find a new maximal independent element $x \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$, or prove that $\mathcal{B} = \mathcal{I}(\mathcal{A})$,*

where $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ is the product of n posets, $\mathcal{A} \subseteq \mathcal{P}$ and $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$. (Recall that we denoted by $\mathcal{A}^+ = \{x \in \mathcal{P} \mid x \succeq a, \text{ for some } a \in \mathcal{A}\}$ and $\mathcal{A}^- = \{x \in \mathcal{P} \mid x \preceq a, \text{ for some } a \in \mathcal{A}\}$ the ideal and the filter generated by $\mathcal{A} \subseteq \mathcal{P}$, and by $\mathcal{I}(\mathcal{A})$ the set of all maximal independent elements for $\mathcal{A}$).

In what follows, we assume that each poset $\mathcal{P}_i$ has a unique minimum element l_i, and let $\mathrm{Supp}(x) = \{i \mid x_i \succ l_i\}$ denote the set of non-minimal components of $x = (x_1, \ldots, x_n) \in \mathcal{P}$. As mentioned in the introduction, we define $\dim(\mathcal{A}) = \max\{|\mathrm{Supp}(a)| : a \in \mathcal{A}\}$. We also denote by $x^\perp$ the set of immediate predecessors of x, i.e., $x^\perp = \{y \in \mathcal{P} \mid y \prec x, \ \nexists z \in \mathcal{P} : y \prec z \prec x\}$, and let $\text{in-deg}(\mathcal{P}) = \max\{|x^\perp| : \ x \in \mathcal{P}\}$. Clearly, $\text{in-deg}(\mathcal{P}) = \sum_{i=1}^n \text{in-deg}(\mathcal{P}_i)$ for $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$. If $\mathcal{P}$ is a lattice, we let $x \vee y$ and $x \wedge y$ denote the maximum and minimum of $x, y \in \mathcal{P}$.

Theorems 5.1 and 5.2 admit the following generalizations.

Theorem 5.3 *Let $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$, where each poset $\mathcal{P}_i$ is a lattice of in-degree $\leq$ const, and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$ be two given sets such that $\mathcal{B} \subseteq \mathcal{I}(\mathcal{A})$. Then $DUAL(\mathcal{P}, \mathcal{A}, \mathcal{B}) \in NC$ for $dim(\mathcal{A}) \leq 3$, and $DUAL(\mathcal{P}, \mathcal{A}, \mathcal{B}) \in RNC$ for $dim(\mathcal{A}) = 4, 5, \ldots$*

Theorem 5.4 *Under the assumptions of Theorem 5.3, $DUAL(\mathcal{P}, \mathcal{A}, \mathcal{B})$ is NC-reducible to $DUAL(\mathcal{P}', \mathcal{A}', \emptyset)$, where $\mathcal{P}' = \{z\}^+$ for some $z \in \mathcal{P}$, $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$, and $dim(\mathcal{A}) \leq$ const.*

Note that for any $z = (z_1, \ldots, z_n) \in \mathcal{P}$, we have $\mathcal{P}' = \{z\}^+ = \{z_1\}^+ \times \cdots \times \{z_n\}^+$, i.e. $\mathcal{P}'$ is still the product of n lattices $\mathcal{P}'_i = \{z_i\}^+$ whose in-degrees are bounded by the in-degrees of the original lattices $\mathcal{P}_i$. Moreover, we have $l'_i = z_i$ is the minimum element in $\mathcal{P}'_i$, and for this reason the dimension of $\mathcal{A}' \subseteq \mathcal{P}'$ does not exceed the dimension of $\mathcal{A}$ in $\mathcal{P}$. In addition, it is easy to see that Theorem 5.4 is indeed a generalization of Theorem 5.2. If $\mathcal{P} = \{0,1\}^n$, then z is the characteristic vector of some set $Z \subseteq V = \{1, \ldots, n\}$ and $\{Z\}^+$ is the family of all supersets of Z. Furthermore, each element $a \in \mathcal{A}$ is then the characteristic vector of some hyperedge $A \subseteq V$. Under this interpretation, $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$ can be regarded as the hypergraph $\{Z \cup A \mid A \in \mathcal{A}\}$. Problem DUAL$(\mathcal{P}', \mathcal{A}', \emptyset)$ calls for computing a set $X \subseteq V$ such that $Z \subseteq X$ and X is a maximal independent set for $\{Z \cup A \mid A \in \mathcal{A}\}$. Letting $U = V \setminus Z$, the latter problem is easily seen to be equivalent to computing a maximal independent set for the induced hypergraph $\{A \cap U \mid A \in \mathcal{A}\}$, as stated in Theorem 5.2.

In addition to Theorems 5.3 and 5.4, we show that if each poset $\mathcal{P}_i$ has a unique minimum element, then problem DUAL$(\mathcal{P}, \mathcal{A}, \emptyset)$ can be reduced to the maximal independent set problem for some hypergraphs.

Theorem 5.5 *For each fixed k, there is an NC-algorithm which, given posets $\mathcal{P}_1, \ldots, \mathcal{P}_n$ with unique minimum elements $l_i \in \mathcal{P}_i$, $i = 1, \ldots, n$, and a set $\mathcal{A} \subseteq \mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ such that $dim(\mathcal{A}) \leq k$, reduces $DUAL(\mathcal{P}, \mathcal{A}, \emptyset)$ to the maximal independent set problem for some hypergraphs of dimension at most k.*

Note that Theorem 5.5 holds for the posets $\mathcal{P}_i$ of arbitrarily large in-degrees and does not require that these posets be lattices. It is also clear that Theorem 5.3 is a corollary of Theorems 5.5 and 5.4.

5.4.1 Characterization of sub-minimal elements of an ideal

Our proof of Theorem 5.4 makes use of an analogue of Proposition 5.1. This analogue, Proposition 5.3 below, assumes that each of the posets $\mathcal{P}_i,\ i \in V = \{1, \dots, n\}$, is a lower semi-lattice, i.e., for any two elements $x, y \in \mathcal{P}_i$ there is a unique minimum element $x \wedge y$. As before, we denote by $x^\perp$ the set of immediate predecessors of x. Note that if $x = (x_1, \dots, x_n) \in \mathcal{P}_1 \times \dots \times \mathcal{P}_n$, then any element $y \in x^\perp$ has the form $y = (x_1, \dots, x_{i-1}, y_i, x_{i+1}, \dots, x_n)$, where $y_i \in x_i^\perp$ is an immediate predecessor of x_i in $\mathcal{P}_i$ and $i \in \mathrm{Supp}(x)$.

Given a set $\mathcal{B} \subseteq \mathcal{P}$ and a vector $s \in \mathcal{P}$, we say that s is *sub-minimal for* $\mathcal{P} \setminus \mathcal{B}^-$ if $s \preceq x$ for some minimal element x of the ideal $\mathcal{P} \setminus \mathcal{B}^-$. We call a subset $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$ a *majorant for* $s^\perp$ if for any $y \in s^\perp$ there is an element $b \in \widetilde{\mathcal{B}}$ such that $b \succeq y$.

Proposition 5.3 *Let $\mathcal{B}$ be a given set in $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$, where each poset $\mathcal{P}_i$ is a lower semi-lattice. A vector $s \in \mathcal{P}$ is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$ if and only if there is a majorant $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$ for $s^\perp$ and a vector $z \in \{s\}^+ \cap (\mathcal{P} \setminus \mathcal{B}^-)$ such that*

(a) *$z[S]$ is minimal in $\mathcal{P}[S] \setminus \widetilde{\mathcal{B}}[S]^-$,*

(b) *$z_i = \wedge\{b_i \mid b = (b_1, \dots, b_n) \in \widetilde{\mathcal{B}}\}$ for all $i \in V \setminus S$, and*

(c) *$|\widetilde{\mathcal{B}}| \leq \sum\{in\text{-}deg(\mathcal{P}_i) \mid i \in S\}$,*

where $S = Supp(s)$ and $z[S], \widetilde{\mathcal{B}}[S]$ are the restrictions respectively, of z and $\widetilde{\mathcal{B}}$ on S.

Let us note that if $|\mathrm{Supp}(s)|$ and all poset in-degrees $\max\{|x^\perp| : x \in \mathcal{P}_i\}$ are bounded, then $|\widetilde{\mathcal{B}}| \leq$ *const* and hence there are only polynomially many sets $\widetilde{\mathcal{B}}$ satisfying condition (c). In addition, (b) and the boundedness of $|\mathrm{Supp}(S)|$ imply that for each $\widetilde{\mathcal{B}}$, there are only polynomially many candidate vectors z that can satisfy (a). It is clear that all such sets $\widetilde{\mathcal{B}} \subseteq \mathcal{B}$ and vectors $z \in \mathcal{P}$ can be generated and tested efficiently in parallel. We shall also make use of the following fact.

Proposition 5.4 *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$ such that $\mathcal{A}^+ \cap \mathcal{B}^- = \emptyset$. Let us assume further that $s \notin \mathcal{A}^-$ is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$, and let $z \in \mathcal{P}$ be the vector proving this, as in Proposition 5.3. Then, $z \notin \mathcal{A}^+$.*

Proof of Proposition 5.3. To show the "only if" part, suppose that s is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$, and let x be a minimal element in $\mathcal{P} \setminus \mathcal{B}^-$ such that $s \preceq x$. Denote by $Y \subseteq \mathcal{P}$ the set of all elements of the form $y = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n)$, where $i \in S = \mathrm{Supp}(s)$ and $y_i \in x_i^\perp$. Clearly, $Y \subseteq x^\perp$ and $|Y| \leq \sum\{\text{in-deg}(\mathcal{P}_i) \mid i \in S\}$. By the minimality of x in $\mathcal{P} \setminus \mathcal{B}^-$, for each $y \in Y$ we can find an element $b = b(y) \in \mathcal{B}$ such that $b \succeq y$. Since $s \preceq x$, it follows that any immediate predecessor of s can be majorized by some $y \in Y$. Hence we conclude that $\widetilde{\mathcal{B}} = \bigcup\{\ b(y) \mid y \in Y\}$ is a majorant for $s^\perp$. By definition, $\widetilde{\mathcal{B}}$ satisfies (c). Now letting

$$z_i = \begin{cases} x_i & \text{if} \quad i \in S \\ \wedge\{b_i \mid b = (b_1, \ldots, b_n) \in \widetilde{\mathcal{B}}\} & \text{if} \quad i \in V \setminus S, \end{cases} \tag{5.10}$$

we readily obtain (b). To prove (a), let us first show that $x[S] \notin \widetilde{\mathcal{B}}[S]^-$. Suppose, to the contrary, that $b[S] \succeq x[S]$ for some $b \in \widetilde{\mathcal{B}}$. By the definition of $\widetilde{\mathcal{B}}$, we have

$$b \succeq y = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n) \tag{5.11}$$

for some $i \in S$ and $y_i \in x_i^\perp$. We cannot have $b_i \succeq x_i$ because this would imply $x \in \mathcal{B}^-$. Hence $b_i \not\succeq x_i$ for some $i \in S$, and consequently, $b[S] \not\succeq x[S]$. We have thus shown that $x[S] \in \mathcal{P}[S] \setminus \widetilde{\mathcal{B}}[S]^-$. Now it is easy to see that $x[S]$ is minimal in $\mathcal{P}[S] \setminus \widetilde{\mathcal{B}}[S]^-$, because any immediate predecessor of $x[S]$ in $\mathcal{P}[S]$ can be majorized by the restriction $b[S]$ of some vector $b \in \widetilde{\mathcal{B}}$, see (5.11). By the first line of (5.10), we have $z[S] = x[S]$ and (a) follows. It remains to show that $z \in \{s\}^+ \cap (\mathcal{P} \setminus \mathcal{B}^-)$. To this end, note that $x \preceq z$, because $x[S] = z[S]$ and for any $b = (b_1, \ldots, b_n) \in \widetilde{\mathcal{B}}$ and $i \in V \setminus S$ we have $x_i \preceq b_i$, see (5.11) and the second line of (5.10). The inequality $x \preceq z$ implies that $z \in \{s\}^+ \cap (\mathcal{P} \setminus \mathcal{B}^-)$, because on the one hand $s \preceq x$, and on the other hand $x \notin \mathcal{B}^-$.

The "if" part of the proof does not require condition (c). Since $z \notin \mathcal{B}^-$, there is a vector $x \notin \mathcal{B}^-$, minimal in $\mathcal{P} \setminus \mathcal{B}^-$, such that $x \preceq z$. We have $x[S] = z[S]$ for this vector, since by (a) and (b), any decrease of z in a coordinate $i \in S$ would yield a vector majorized by some

$b \in \widetilde{\mathcal{B}} \subseteq \mathcal{B}$. From $x[S] = z[S]$ and $z \succeq s$ it follows that $x \succeq s$, thus proving that s is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$. □

Proof of Proposition 5.4. Suppose, to the contrary, that $z \succeq a$ for some $a \in \mathcal{A}$. Let $b \in \widetilde{\mathcal{B}}$. By (b), $b[V \setminus S] \succeq z[V \setminus S]$. This implies $b[V \setminus S] \succeq a[V \setminus S]$. Since $\mathcal{A}^+ \cap \mathcal{B}^- = \emptyset$, it follows that $b[S] \not\succeq a[S]$ for all $b \in \widetilde{\mathcal{B}}$, i.e., $a[S] \notin \widetilde{\mathcal{B}}[S]^-$. On the other hand, $z[S] \succeq a[S]$ by our assumption that $z \succeq a$. Now the minimality of $z[S]$ in $\mathcal{P}[S] \setminus \widetilde{\mathcal{B}}[S]^-$ implies that $z[S] = a[S]$. However, Proposition 5.3 also says that $z \succeq s$ and hence $a[S] \succeq s[S]$. Recalling that $S = \mathrm{Supp}(s)$, we conclude that $a \succeq s$, i.e., $s \in \mathcal{A}^-$. □

5.4.2 Proof of Theorem 5.4

The proof is analogous to that of Theorem 5.2. Without loss of generality we can assume that $\mathcal{A}$ is an antichain in $\mathcal{P}$ (cf. Step 1 in Section 5.3.2). If there exists a vector $a \in \mathcal{A}$ which is not minimal in $\mathcal{P} \setminus \mathcal{B}^-$, then we can find an element $z \in a^\perp$ such that $z \in \mathcal{P} \setminus \mathcal{B}^-$. This can be done fast in parallel. We can then compute a new maximal independent point $b' \in \mathcal{I}(\mathcal{A}) \setminus \mathcal{B}$ by letting $b' = output(\mathrm{DUAL}(\mathcal{P}', \mathcal{A}', \emptyset))$, where $\mathcal{P}' = \{z\}^+$ and $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$ (cf. Step. 3 in the proof of Theorem 5.2). It is clear that b' is indeed a new maximal independent element for $\mathcal{A}$ because $b' \succeq z$ and $z \notin \mathcal{B}^-$. (Note that if $\mathcal{P}$ is not an upper semi-lattice then the set $\mathcal{A}'$ of all minimal elements of $\{z\}^+ \cap \mathcal{A}^+$ may be exponentially large.)

Let us assume now that

$$\text{Each } a \in \mathcal{A} \text{ is minimal in } \mathcal{P} \setminus \mathcal{B}^-. \tag{5.12}$$

If

$$\mathcal{I}(\mathcal{A}) \neq \mathcal{B}, \tag{5.13}$$

then there is a vector $x \in \mathcal{P} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. Without loss of generality, we may assume that x is minimal in $\mathcal{P} \setminus \mathcal{B}^-$. We have $x \notin \mathcal{A}^-$ because otherwise $x \in \mathcal{B}^-$ by (5.12). Let s be a minimal element in $\{x\}^-$ such that $s \notin \mathcal{A}^-$, then $|\mathrm{Supp}(s)|$ does not exceed $\dim(\mathcal{A}) + 1$. Thus, (5.13) implies that

$$\begin{aligned}&\text{There is a vector } s \notin \mathcal{A}^- \text{ such that } |\mathrm{Supp}(s)| \leq \dim(\mathcal{A}) + 1\\&\text{and } s \text{ is sub-minimal for } \mathcal{P} \setminus \mathcal{B}^-.\end{aligned} \tag{5.14}$$

Conversely, (5.14) implies (5.13) even without (5.12) and the assumption that $|\mathrm{Supp}(s)| \leq dim(\mathcal{A}) + 1$. To see this, observe that if $s \notin \mathcal{A}^-$ is sub-minimal for $\mathcal{P} \setminus \mathcal{B}^-$, then by Proposition 5.4 we can find a vector $z \notin \mathcal{A}^+$ which satisfies the conditions of Proposition 5.3. In particular, $z \notin \mathcal{B}^-$, which implies (5.13).

As mentioned in Section 5.4.1, Proposition 5.3 gives an NC test for (5.14) provided that the dimension of $\mathcal{A}$ and the in-degrees of all posets $\mathcal{P}_i$ are bounded (cf. Step 4 in Section 5.3.2). Moreover, if we find an s satisfying (5.14), then, according to Propositions 5.3 and 5.4, we also obtain an element $z \in \mathcal{P} \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. Letting $\mathcal{A}' = \{z \vee a \mid a \in \mathcal{A}\}$, any solution to DUAL($\{z\}^+, \mathcal{A}', \emptyset$) yields a new element in $\mathcal{I}(\mathcal{A})$. □

5.4.3 Proof of Theorem 5.5

Consider the following problem:

DUAL($\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset$): *Given $2n$ non-empty finite posets $\mathcal{R}_i \subseteq \mathcal{P}_i$, $i \in V = \{1, \ldots, n\}$, each of which has a unique minimum element, and a set $\mathcal{A} \subseteq \mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$, find a maximal $\mathcal{A}$-independent element x* in $\mathcal{R} = \mathcal{R}_1 \times \cdots \times \mathcal{R}_n$.

Denoting by l_i and r_i the minimum elements of $\mathcal{P}_i$ and $\mathcal{R}_i$, respectively, we shall assume without loss of generality that

$$r = (r_1, \ldots, r_n) \notin \mathcal{A}^+, \tag{5.15}$$

for otherwise $\mathcal{R}$ contains no $\mathcal{A}$-independent element. As before, we let $\mathrm{Supp}(a) = \{i \in V \mid a_i \succ l_i\}$ and let $\dim(\mathcal{A}) = \max\{|\mathrm{Supp}(a)| : a \in \mathcal{A}\}$ denote the dimension of $\mathcal{A}$. Our goal is to show that for $\dim(\mathcal{A}) \leq k$, problem DUAL($\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset$) is NC-reducible to the maximal independent set problem for some hypergraphs of dimensions $\leq c$. This will prove Theorem 5.5 because DUAL($\mathcal{P}, \mathcal{A}, \emptyset$) is a special case of DUAL($\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset$) for $\mathcal{R} = \mathcal{P}$. Our reduction iteratively decreases $|\mathcal{A}|$ and the maximum cardinality of the posets $\mathcal{R}_i$.

Step 1. If $\max\{|\mathcal{R}_i| : i \in V\} = 1$, return $x = r$ and halt.

Step 2. If $\mathcal{A} = \emptyset$, return any maximal point in $\mathcal{R}$ and halt.

Step 3. Let $\text{Supp}_{\mathcal{R}}(a) = \{i \in V \mid a_i \succ r_i\}$. In view of (5.15), we have $|\text{Supp}_{\mathcal{R}}(a)| \geq 1$ for all $a \in \mathcal{A}$. Remove all points $a \in \mathcal{A}$ with $|\text{Supp}_{\mathcal{R}}(a)| = 1$ and reduce $\mathcal{R}$ accordingly:

$$\mathcal{R}_i \leftarrow \mathcal{R}_i \setminus \bigcup \{ a_i^+ \mid \text{Supp}_{\mathcal{R}}(a) = \{i\},\ a \in \mathcal{A}\}, \quad \mathcal{A} \leftarrow \{a \in \mathcal{A} : \ |\text{Supp}_{\mathcal{R}}(a)| \geq 2\}.$$

Step 4. For each $i \in V = \{1, \ldots, n\}$, topologically sort poset $\mathcal{R}_i$, i.e., find a one-to-one mapping $\phi_i : \mathcal{R}_i \to \{1, \ldots, |\mathcal{R}_i|\}$ such that $\phi_i(x) < \phi_i(y)$ whenever $x \prec y$ in $\mathcal{R}_i$. Let $\mathcal{R}_i^u = \{x \in \mathcal{R}_i \mid \phi_i(x) \geq \lceil |\mathcal{R}_i|/2 \rceil\}$ and let $\mathcal{Q}_i$ denote the antichain consisting of all minimal elements of $\mathcal{R}_i^u$. Note that $\mathcal{R}_i^u$ and hence $\mathcal{Q}_i$ are not empty for all $i \in V$.

Step 5. Let $U = \bigcup_{i=1}^n \mathcal{Q}_i$, and let $\mathcal{H} \subseteq 2^U$ be the hypergraph whose hyperedges are:

1) all pairs of the form $\{x, y\}$, where $x \neq y$ and $x, y \in \mathcal{Q}_i$ for some $i \in V$, and

2) all collections H of at most $k = \dim(\mathcal{A})$ elements of U such that H contains at most one element from each $\mathcal{Q}_i$ and $\pi(H) \succeq a$ for some $a \in \mathcal{A}$, where $\pi(H) = (\pi_1(H), \ldots, \pi_n(H)) \in \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_n$ is the vector with the following components:

$$\pi_i(H) = \begin{cases} H \cap \mathcal{Q}_i & \text{if } H \cap \mathcal{Q}_i \neq \emptyset \\ r_i & \text{otherwise.} \end{cases}$$

Step 6. Compute a maximal independent set I for $\mathcal{H}$. Note that $I \neq \emptyset$ since $\mathcal{H}$ does not contain singletons. Also, by the definition of $\mathcal{H}$, the independent set I contains at most one element from each antichain $\mathcal{Q}_i$ and the vector $\pi(I) \in \mathcal{Q} = \mathcal{Q}_1 \times \ldots \times \mathcal{Q}_n \subseteq \mathcal{R}$ is independent of $\mathcal{A}$.

Step 7. Go to Step 1 and compute $\text{DUAL}(\mathcal{R}' \subseteq \mathcal{P}, \mathcal{A}', \emptyset)$, where $\mathcal{R}' = \mathcal{R}'_1 \times \ldots \times \mathcal{R}'_n$ is defined as follows:

$$\mathcal{R}'_i = \begin{cases} \mathcal{R}_i & \text{if } \ |\mathcal{R}_i| = 1 \\ \mathcal{R}_i \cap \{\pi_i(I)\}^+ & \text{if } \ I \cap \mathcal{Q}_i \neq \emptyset \\ \mathcal{R}_i \setminus \mathcal{R}_i^u & \text{otherwise,} \end{cases}$$

and $\mathcal{A}' = \{a \in \mathcal{A} \mid \{a\}^+ \cap \mathcal{R}' \neq \emptyset\}$.

The correctness of the above iterative procedure can be seen from the following observations:

(a) Each poset $\mathcal{R}'_i$ still has a unique minimum element r'_i;

(b) Since $\pi(I)$ is independent of $\mathcal{A}$, the new minimum element $r' = (r'_1, \ldots, r'_n)$ satisfies (5.15);

(c) Let $x \in \mathcal{R}'$ be a maximal $\mathcal{A}'$-independent element in $\mathcal{R}'$. Then x is a maximal $\mathcal{A}$-independent element of $\mathcal{R} \cap \{\pi(I)\}^+$. Hence x is also a maximal $\mathcal{A}$-independent element of $\mathcal{R}$, i.e., x solves the original problem DUAL($\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset$).

Since each iteration almost halves the maximum size of the posets $\mathcal{R}_i$, our reduction consists of $O\big(\log(\max\{|\mathcal{R}_i| :\ i \in V\})\big)$ iterations and Theorem 5.5 follows. $\square$

Remark. It is essential, in the above result, to assume that each poset $\mathcal{R}_i$ has a unique minimum element, for otherwise problem DUAL($\mathcal{R} \subseteq \mathcal{P}, \mathcal{A}, \emptyset$) becomes NP-hard even for posets $\mathcal{P}_i$ with only 3 elements. To see this, let each poset $\mathcal{P}_i$ be a "$\bigvee$", i.e. composed of 3 elements $\{u, 0, w\}$, where $0 \leq u$ and $0 \leq w$ are the only relations in $\mathcal{P}_i$. Let $R_i = \{u, w\} \subseteq \mathcal{P}_i$. Now given a disjunctive normal form $D = D_1 \vee \ldots \vee D_m$ in n variables $x_1, \ldots, x_n$, let us associate a vector $a^j \in \mathcal{P}$ with every term D_j as follows: a^j_i takes the value u if variable x_i appears in term D_j, the value w if $\overline{x}_i$ appears in D_j, and the value 0 otherwise. Letting $\mathcal{A} = \{a^j \mid j = 1, \ldots, m\} \subseteq \mathcal{P}$, it is then easy to see that $\mathcal{R} \subseteq \mathcal{A}^+$ if and only if D is a tautology.

Chapter 6

Some Open Problems

We conclude in this last chapter by some related open problems.

Tight inequalities for rank functions over a small field . In Chapter 2, we gave a bound on the number of maximal collections of subspaces, from a given list $\mathcal{V}$ in the field $\mathbf{F}^r$, that do not span the entire space, in terms of $|\mathcal{V}|$, r, and the number of minimal collections that do span the space. An example showing that our inequality is reasonably sharp was also given in Section 2.3.2. This example assumes the size of the filed $\mathbf{F}$ is large. It is not known whether a similar sharp example exists, or a stronger inequality holds, if the size of $\mathbf{F}$ is bounded by a constant. This problem is interesting, since it includes, with $|\mathbf{F}| = 2$, the problem of generating minimal collections of subgraphs of a given graph, that interconnect all nodes of the graph.

Dualization in products of partially ordered sets. Given a product $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_n$ of n partially ordered sets and a subset $\mathcal{A} \subseteq \mathcal{P}$ of elements of $\mathcal{P}$, recall that the dualization problem is to incrementally list all those maximal elements in $\mathcal{P}$ that do not succeed any element of $\mathcal{A}$. We have seen that two special cases of posets are known to be quasi-polynomially solvable, namely, when each factor poset $\mathcal{P}_i$ has an acyclic precedence graph of bounded in- or out-degrees, or when $\mathcal{P}_i$ is a lattice with bounded width. This naturally raises the question of the complexity of the dualization problem in general:

- Is the dualization problem on products of general posets is NP-hard?
- Can the known quasi-polynomial bound be extended for forests of arbitrary degrees? Can it be extended for lattices of arbitrary width?
- And, of course, the long-standing open question: Does there exist a polynomial time algorithm for solving the hypergraph dualization problem? Can the currently known

quasi-polynomial bound on the running time be improved?

Parallel complexity of dualization and related problems. We have seen in Chapter 5 that a special case of the dualization problem, namely when the input hypergraph has bounded dimension, can be solved efficiently in parallel.

This already raises a number of questions:

- Given two hypergraphs $\mathcal{A}, \mathcal{B} \subseteq 2^V$ (not necessarily Sperner), with the property that $\mathcal{A}^+ \cap \mathcal{B}^- = \emptyset$, the hypergraph dualization problem can be expressed as of generating *maximal* subsets in $2^V \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$. Thus to develop an efficient parallel algorithm for this problem, one needs first to answer the standing open question about the parallel complexity of finding a single maximal independent set in a hypergraph. A related problem is to aim only at finding a subset in $2^V \setminus (\mathcal{A}^+ \cup \mathcal{B}^-)$, *not necessarily maximal*, and ask about the corresponding parallel complexity (at least for the special cases in which the dualization problem is known to be polynomially solvable [10, 24, 30, 35, 39, 40, 69, 72, 73, 82, 83]). Can this problem be solved efficiently in parallel, given a quasi-polynomial number of processors?

- Given a lattice $\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_n$, where each $\mathcal{L}_i$ has *a bounded in-degree*, we have seen in Chapter 5 that the dualization problem is efficiently solvable in parallel, for antichains of bounded dimension. Can the same result be obtained with the boundedness assumption on the degrees removed?

- The set of minimal integer solutions of a monotone system of linear inequalities can be generated by an RNC algorithm, if the number of non-zero coefficients in each inequality is fixed, and the variables are bounded by a constant. What if the variables are not bounded?

- A similar question arises for the generation of fairly independent sets of a hypergraph of bounded dimension. Let $\mathcal{A} \subseteq 2^V$ be a hypergraph and let $t \in \{0, 1, \ldots, |\mathcal{A}| - 1\}$ be a given threshold. Recall that a vertex set $I \subseteq V$ is called fairly independent or t-independent if I contains at most t hyperedges of $\mathcal{A}$. For $t = const$ and $\dim(\mathcal{A}) \leq const$, we have seen that the generation problem is in RNC. The status of the problem for a bounded-dimension $\mathcal{A}$ but an *arbitrary* t remains open.

Generation problems in directed graphs. In chapter 3, it was shown that the generation of minimal collections of subgraphs of a given *undirected* graph $G(V, E)$, that interconnect all nodes of V, can be done in incremental quasi-polynomial time. Although the other direction, i.e., generating all maximal collections that disconnect the graph is NP-hard in general, the problem becomes easy if each subgraph is a single edge as it reduces to the generation of minimal cuts of an undirected graph. The corresponding problem for directed graphs, when each subgraph is a single arc, is open. Namely, it is not known if the problem of generating all minimal collections of arcs of a given *directed* graph, whose removal will result in a non-strongly connected subgraph, is NP-hard.

References

[1] R. Agrawal, T. Imielinski and A. Swami, Mining associations between sets of items in massive databases, in *Proceedings of the 1993 ACM-SIGMOD International Conference on Management of Data*, pp. 207-216.

[2] R. Agrawal, H. Mannila, R. Srikant, H. Toivonen and A. I. Verkamo, Fast discovery of association rules, in U. M. Fayyad, G. Piatetsky-Shapiro, P. Smyth and R. Uthurusamy eds., *Advances in Knowledge Discovery and Data Mining*, 307-328, AAAI Press, Menlo Park, California, 1996.

[3] R. Agrawal and R. Srikant, Mining sequential patterns, in *Proceedings of the 11th International Conference on Data Engineering, 1995*, pp.3-14.

[4] V.E. Alekseev, On the number of dead-end independent sets in graphs from hereditary classes, *Combinatoric-algebraic Methods in Discrete Optimization* **6** (1991) 5–8 (Nizhni Novgorod, in Russian).

[5] N. Alon, L. Babai and A. Itai, A fast randomized parallel algorithm for the maximal independent set problem, *Journal of Algorithms* **7** (1986) 567–583.

[6] E. Balas and C. S. Yu, On graphs with polynomially solvable maximal-weight clique problem, *Networks* **19** (1989) 247–253.

[7] R.J. Bayardo, Efficiently mining long patterns from databases, in *Proceedings the 1998 ACM-SIGMOD International Conference on Management of Data*, pp. 85–93.

[8] P. Beame and M. Luby, Parallel search for maximal independence given minimal dependence, in *Proceedings of the First Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)* (1990) 212–218.

[9] C. Berge, *Hypergrahs*, North Holland Mathematical Library, Vol. 445, Elsevier-North Holland, Amestrdam, 1989.

[10] P. Bertolazzi and A. Sassano, An $O(mn)$ time algorithm for regular set-covering problems, *Theoretical Computer Science* **54** (1987) 237–247.

[11] J. C. Bioch and T. Ibaraki, Complexity of identification and dualization of positive Boolean functions, *Information and Computation* **123** (1995) 50–63.

[12] E. Boros, V. Gurvich and P.L. Hammer, Dual subimplicants of positive Boolean functions, *Optimization Methods and Software*, **10** (1998) 147–156. (Also available as RUTCOR Research Report 11-93.)

[13] E. Boros, K. Elbassioni, V. Gurvich and L. Khachiyan, An efficient incremental algorithm for generating all maximal independent sets in hypergraphs of bounded dimension, *Parallel Processing Letters*, **10** (2000) 253–266. (DIMACS technical report 2000-21, Rutgers University.)

[14] E. Boros, K. Elbassioni, V. Gurvich and L. Khachiyan, An inequality for polymatroid functions and its applications, to appear in *Discrete Applied Mathematics*, 2002. (DIMACS Technical Report 2001-14, Rutgers University.)

[15] E. Boros, K. Elbassioni, V. Gurvich, L. Khachiyan and K. Makino, On generating all minimal integer solutions to a monotone system of linear inequalities, *Automata, Languages and Programming, 28-th International Colloquium, ICALP 2001* (F. Orejas, P.G. Spirakis and Jan van Leeuwen, eds.), *Lecture Notes in Computer Science* 2076, pp. 92–103, 2001. (DIMACS Technical Report 2001-12, Rutgers University.)

[16] E. Boros, K. Elbassioni, V. Gurvich, L. Khachiyan and K. Makino, Dual-bounded generating problems: All minimal integer solutions for a monotone system of linear inequalities, to appear in *SIAM Journal on Computing*

[17] E. Boros, K. Elbassioni, V. Gurvich, L. Khachiyan, and K. Makino, Dual-bounded hypergraphs: A survey, in *Proceedings of the SIAM Workshop on Discrete Mathematics and Data Mining (DM & DM)*, Arlington, VA, April 2002, pp. 87–98.

[18] E. Boros, K. Elbassioni, V. Gurvich and L. Khachiyan, Matroid intersections, polymatroid inequalities, and related problems, to appear in *Proceedings of the 27th International Symposium on Mathematical Foundations of Computer Science (MFCS)*, Warszawa–Otwock, Poland, August 26-30, 2002.

[19] E. Boros, V. Gurvich, L. Khachiyan and K.Makino, Generating partial and multiple transversals of a hypergraph, in *Automata, Languanges and Programming, 27th International Colloquium, ICALP 2000* (Montanari, J.D.P. Rolim and E. Welzl, eds.), *Lecture Notes in Computer Science* 1853, pp. 588–599 (Springer Verlag, Berlin, Heidelberg, New York, July 2000).

[20] E. Boros, V. Gurvich, L. Khachiyan and K. Makino, Dual-bounded generating problems: partial and multiple transversals of a hypergraph, *SIAM Journal on Computing*, **30** (2001) 2036–2050.

[21] E. Boros, V. Gurvich, L. Khachiyan and K. Makino, On the complexity of generating maximal frequent and minimal infrequent sets, in *Proceedings of the 19th International Symposium on Theoretical Aspects of Computer Science, (STACS)*, (H. Alt and A. Ferreira, eds.), Mrach 2002, *Lecture Notes in Computer Science* 2285, pp. 133–141.

[22] E. Boros, V. Gurvich, L. Khachiyan and K. Makino, Generating weighted transversals of a hypergraph, in *Proceedings of the 2nd Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications*, pp. 13–22. Hungarian Academy of Sciences, Budapest, Hungary, April 20-23, 2001.

[23] E. Boros, P.L. Hammer and J.N. Hooker, Predicting cause-effect relationships from incomplete discrete observations, *SIAM Journal on Discrete Mathematics*, **7** (1994), 481–491.

[24] E. Boros, P. L. Hammer, T. Ibaraki and K. Kawakami, Polynomial time recognition of 2-monotonic positive Boolean functions given by an oracle, *SIAM Journal on Computing*, **26** (1997) 93–109.

[25] D. D. Bremner, On the complexity of vetrex and facet enumeration for convex polytopes, Ph.D. thesis, School of Computer Science, McGill University, 1997.

[26] S. Brin, R. Motwani and C. Silverstein, Beyond market basket: Generalizing association rules to correlations, in *Proceedings of the 1997 ACM-SIGMOD Conference on Management of Data*, pp. 265–276.

[27] S. Brin, R. Motwani, J. Ullman and S. Tsur, Dynamic itemset counting and implication rules for market basket data, in *Proceedings of the 1997 ACM-SIGMOD Conference on Management of Data*, pp. 255–264.

[28] L. Budach, B. Graw, C. Meinel and S. Waack , *Algebraic and topological properties of finite partially ordered sets*, Teubner-texte zur Mathematik, Band 109, 1988.

[29] C. J. Colbourn, *The combinatorics of network reliability*, Oxford University Press, 1987.

[30] Y. Crama, Dualization of regular Boolean functions, *Discrete Applied Mathematics*, **16** (1987) 79–85.

[31] E. Dahlhaus and M. Karpinski, An efficient algorithm for 3MIS problem, Technical Report TR-89-052, September 1989, International Computer Science Institute, Berkeley, CA.

[32] D. Dentcheva, A. Prékopa and A. Ruszczynski, Concavity and efficient points of discrete distributions in Probabilistic Programming, *Mathematical Programming* **89** (2000) 55–77.

[33] G. Ding, Bounding the number of bases of a matroid, *Combinatorica*, **15** (2) (1996) 159–165.

[34] G. Ding, Bounding the number of circuits of a graph, *Combinatorica*, **16** (3) (1996) 331–341.

[35] C. Domingo, N. Mishra and L. Pitt, Efficient read-k monotone CNF/DNF dualization using a learning with membership queries approach, *Machine Learning*, **37** (1999) 89–110.

[36] G. Dong and J. Li, Efficient mining of emerging patterns, in *Proceedings the 1999 ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pp. 43–52.

[37] W. Duckworth, D. Manlove and M. Zito, On the approximability of the maximum induced matching problem, to appear in *Journal of Combinatorial Optimization*, 2000.

[38] A. Durand, M. Hermann and P. G. Kolaitis, Subtractive reductions and complete problems for counting complexity classes, in *Proceedings of the 25th International Symposium on Mathematical Foundations of Computer Science (MFCS)*, M. Nielsen and B. Rovan, eds., Bratislava, Slovakia, Lecture Notes in Computer Science, vol. 1893, Springer-Verlag, August 2000, pp. 323–332.

[39] T. Eiter and G. Gottlob, Identifying the minimal transversals of a hypergraph and related problems, *SIAM Journal on Computing*, **24** (1995) 1278–1304.

[40] T. Eiter, G. Gottlob and K. Makino, New results on monotone dualization and generating hypergraph transversals, to appear in *the 34th Annual ACM Symposium on Theory of Computing (STOC)*, 2002.

[41] K. Elbassioni, On dualization in products of forests, in *Proceedings of the 19th International Symposium on Theoretical Aspects of Computer Science (STACS)*, (H. Alt and A. Ferreira, eds.), March 14-16, 2002, Antibes, Juan les pins, France, *Lecture Notes in Computer Science* 2285, pp. 142–153.

[42] K. Elbassioni, An algorithm for dualization in products of lattices and its applications, Manuscript.

[43] M. L. Fredman and L. Khachiyan, On the complexity of dualization of monotone disjunctive normal forms, *Journal of Algorithms*, **21** (1996) 618–628.

[44] M. Galań, I. Garciá-Vargas, F.V. Fernańdez and A. Rodriǵuez-Vaźquez, A new matroid intersection algorithm for symbolic large circuit analysis, in *Proceedings of the 4th International Workshop on Symbolic Methods and Applications to Circuit Design*, Leuven (Belgium), October 1996.

[45] D. R. Gaur, Satisfiability and self-duality of monotone Boolean functions, Ph.D. thesis, School of Computing Science, Simon Fraser University, January 1999.

[46] M. Goldberg and T. Spencer, Constructing a maximal independent set in parallel, SIAM Journal on Discrete Mathematics **2** (1989) 322–328.

[47] M. Goldberg and T. Spencer, A new parallel algorithm for the maximal independent set problem, SIAM Journal on Computing **18** (1989) 419–427.

[48] D. Gunopulos, R. Khardon, H. Mannila and H. Toivonen, Data mining, hypergraph transversals and machine learning, in *Proceedings of the 16th ACM-SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems*, (1997) pp. 12–15.

[49] V. Gurvich and L. Khachiyan, On generating the irredundant conjunctive and disjunctive normal forms of monotone Boolean functions, *Discrete Applied Mathematics*, **96-97** (1999) 363–373. (See also Rutcor Research Report, RRR-35-95, Rutgers University.)

[50] V. Gurvich and L. Khachiyan, On the frequency of the most frequently occurring variable in dual DNFs, *Discrete Mathematics* **169** (1997) 245–248.

[51] P. L. Hammer, U.N. Peled and M.A. Pollatschek, An algorithm to dualize a regular switching function, *IEEE Transactions on Computers* **28** (1979), 238–243.

[52] J. Han, Y. Cai and N. Cercone, Data driven discovery of quantitative rules in relational databases, *IEEE Transactions on Knowledge and Data Engineering*, **5** (1) (1993) 29–40.

[53] J. Han and Y. Fu, Discovery of multiple-level association rules from large databases, in *Proceedings of the 21st International Conference on Very Large Data Bases*, pp. 420–431, 1995.

[54] J. Han, J. Pei and Y. Yin, Mining frequent patterns without candidate generation, in *Proceedings of the 2000 ACM-SIGMOD Conference on Management of Data*, pp. 1–12.

[55] T. Helgason, Aspects of the theory of hypermatroids, in Hypergraph Seminar, eds C. Berge and D.K. Ray-Chaudhuri), Lecture Notes in Mathematics, **411** (1975) 191–214.

[56] D. S. Johnson, M. Yannakakis and C. H. Papadimitriou, On generating all maximal independent sets, *Information Processing Letters*, **27** (1988) 119–123.

[57] A. Kanevsky, Finding all minimum size separating vertex sets in a graph, *Networks* **23** (1993) 533–541.

[58] R. Karp and V. Ramachandran, Parallel algorithms for shared memory machines, in Handbook of Theoretical Computer Science, J. van Leeuwen, ed., North Holland (1990) 869–941.

[59] R. Karp and A. Wigderson, A fast parallel algorithm for the maximal independent set problem, Journal of the ACM **32** (1985) 762–773.

[60] P. Kelsen, An efficient parallel algorithm for finding an mis in hypergraphs of dimension 3, Manuscript, Dept. of Computer Sciences, Univ. of Texas, Austin, TX, Jan. 1990.

[61] P. Kelsen, On the parallel complexity of computing a maximal independent set in a hypergraph, in *Proceedings of the 24th Anual ACM Symposium on Theory of Computing (STOC)*, 1992.

[62] L. Khachiyan, Transversal hypergraphs and families of polyhedral cones, in *Advances in Convex Analysis and Global Optimization, Proceedings of the International conference honoring the memory of K. Carathéodory, Samos, Greece, June 2000*, P. Pardalos ed. (Kluwer, 2001), pp. 105–118.

[63] L. Khachiyan, Private communication.

[64] E. L. Lawler, *Combinatorial Optimization: Networks and Matroids*, Holt, Rinehart and Winston, New York, 1976.

[65] E. L. Lawler, J. K. Lenstra and A. H. G. Rinnooy Kan, Generating all maximal independent sets: NP-hardness and polynomial-time algorithms, *SIAM Journal on Computing*, **9** (1980) 558–565.

[66] D. Lin and Z.M. Kedem and Pincer-search: a new algorithm for discovering the maximum frequent set, in *Proceedings of the 6th European Conference on Extending Database Technology (EDBT)*, to appear.

[67] L. Lovász, Submodular functions and convexity, in *Mathematical Programming: The State of the Art*, Bonn 1982, pp. 235–257, (Springer Verlag, 1983).

[68] M. Luby, Removing randomness in parallel computation without a processor penalty, *Proceedings of the 29th IEEE Symposium on Foundations of Computer Science (FOCS)*, (1988) 162–173.

[69] K. Makino, Efficient dualization of $O(\log n)$-term monotone disjunctive normal forms, to appear in *Discrete Applied Mathematics.* (Technical Report 00-07, Discrete Mathematics and Systems Science, Osaka University, 2000.)

[70] K. Makino and T. Ibaraki, Interior and exterior functions of Boolean functions, *Discrete Applied Mathematics*, **69** (1996) 209–231. (Rutcor Research Report, RRR-44-94, Rutgers University.)

[71] K. Makino and T. Ibaraki, Inner-core and outer-core functions of partially defined Boolean functions, *Discrete Applied Mathematics*, **96-97** (1999) 307–326.

[72] K. Makino and T. Ibaraki. The maximum latency and identification of positive Boolean functions. *SIAM Journal on Computing*, **26** (1997) 1363–1383.

[73] K. Makino and T. Ibaraki, A fast and simple algorithm for identifying 2-monotonic positive Boolean functions, *Journal of Algorithms*, **26** (1998) 291–305.

[74] H. Mannila and H. Toivonen, Multiple uses of frequent sets and condensed representations, in *Proceedings of the 2nd International Conference on Knowledge Discovery and Data Mining*, (1996) pp. 189–194.

[75] H. Mannila and H. Toivonen, Levelwise search and borders of theories in knowledge discovery, Series of Publications C-1997-8, University of Helsinki, Department of Computer Science (1997).

[76] H. Mannila, H. Toivonen and A. I. Verkamo. Discovery of frequent episodes in event sequences. *Data Mining and Knowledge Discovery*, **1** (1997) 259–289.

[77] C.G.H. McDiarmid, Rado's theorem for polymatroids, *Mathematical Proceedings of the Cambridge Philosophical Society* **78** (1975) 263–281.

[78] N. Pasquier, Y. Bastide, R. Taouil and L. Lakhal, Discovering frequent closed itemsets for association rules, in *Proceedings of the 7th International Conefernce on Database Theory (ICDT)*, January 10-12, 1999, *Lecture Notes in Computer Science*, 1540, pp. 398–416, Springer Verlag, 1999.

[79] N. Pasquier, Y. Bastide, R. Taouil and L. Lakhal, Closed set based discovery of small covers for association rules, in *Proceedings of the 15emes Journees Bases de Donnees Avancees, BDA*, pp. 361–381, 1999.

[80] M. C. Paull and S. H. Unger, Minimizing the number of states in incompletely specified sequential switching functions, *IRE Trans. Electr. Comput.*, **EC-8** (1959) 356–367.

[81] E. Prisner, Graphs with few cliques, *Graph theory, combinatorics and algorithms*, **1,2** (Kalamazoo, MI, 1992) 945–956, Wisley-Intersi. Publ., Wisley, New York, 1995.

[82] U. N. Peled and B. Simeone, Polynomial-time algorithm for regular set-covering and threshold synthesis, *Discrete Applied Mathematics*, **12** (1985) 57–69.

[83] U. N. Peled and B. Simeone, An $O(nm)$-time algorithm for computing the dual of a regular Boolean function, *Discrete Applied Mathematics*, **49** (1994) 309–323.

[84] A. Prékopa, *Stochastic Programming*, (Kluwer, Dordrecht, 1995).

[85] J. S. Provan, Efficient enumeration of the vertices of polyhedra associated with network LP's, *Math Programming*, **63** (1994), pp. 47–64.

[86] J. S. Provan and D. R. Shier, A paradigm for listing (s, t) cuts in graphs, *Algorithmica* **15**, (1996), pp. 351–372.

[87] K. G. Ramamurthy, *Coherent Structures and Simple Games*, Kluwer Academic Publishers, 1990.

[88] R. C. Read and R. E. Tarjan, Bounds on backtrack algorithms for listing cycles, paths, and spanning trees, *Networks*, **5** (1975) 237–252.

[89] R. H. Sloan, K. Takata and G. Turan, On frequent sets of Boolean matrices, *Annals of Mathematics and Artificial Intelligence*, **24** (1998) 1-4.

[90] R. Srikant and R. Agrawal, Mining generalized association rules, in *Proceedings of the 21st International Conference on Very Large Data Bases*, pp. 407–419, 1995.

[91] R. Srikant and R. Agrawal, Mining quantitative association rules in large relational tables, in *Proceedings of the ACM-SIGMOD 1996 Conference on Management of Data*, pp. 1–12, 1996.

[92] Larry J. Stockmeyer and Vijay V. Vazirani. NP-completeness of some generalizations of the maximum matching problem, *Information Processing Letters*, **15** (1) 14–19, 19 August 1982.

[93] H. Tamaki, Space-efficient enumeration of minimal transversals of a hypergraph, IPSJ-AL **75** (2000) 29–36.

[94] S. Tsukiyama, M. Ide, H. Ariyoshi and I. Shirakawa, A new algorithm for generating all maximal independent sets, *SIAM Journal on Computing*, **6** (1977) 505–517.

[95] D.J.A. Welsh, Matroid Theory (Academic Press, London, New York, San Francisco 1976).

[96] M.J. Zaki and M. Ogihara, Theoretical foundations of association rules, in *Proceedings od the 3rd SIGMOD Workshop on Research Issues in Data Mining and Knowledge Discovery*, June 1998.

Vita

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