PROBABILISTIC ROBOTICS: SIMULTANEOUS LOCALIZATION AND MAPPING

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1

Let d the dimension of state space in EKF SLAM, and N the number of landmarks in the map. We have $d = 3N + 3$. The complexity of motion update in EKF SLAM is asymptotically dominated by order d matrix multiplications, so the complexity is $O(d^\omega)$ or equivalently $O(N^\omega)$, where $\omega < 2.3728639$(cf. [1]). On the other hand, the complexity of EKF algorithm for localization is dominated by the loop over all sensings, which is executed $O(N)$ times and contains a constant number of constant size matrix multiplications / inversions; hence an asymptotic complexity of $O(N)$.

2

A solution to initialize landmarks during bearing-only SLAM can be found in paper [2]; It uses the framework described in [3], [4] to manage relationships between spatial uncertainties. I will give only a summary of the main ideas here and refer the reader to the papers for the details.

In [4], the concept of *approximate transformation* (AT) is defined: it expresses the uncertain relative location of one frame $\mathbf{B} = (X_2, Y_2, \theta_2)$ with respect to another frame $\mathbf{A} = (X_1, Y_1, \theta_1)$. The transformation has a mean and a covariance matrix. Think of it as the pose of the robot or the location of a landmark.

(1) The ATs can be *compounded*: knowing the AT $\mathbf{A}$ with respect to $\mathbf{W}$ and the AT $\mathbf{B}$ with respect to $\mathbf{A}$, compute the AT $\mathbf{B}$ with respect to $\mathbf{W}$, in which case the uncertainty is the sum of the original ones (think of 2 successive motion update in Kalman filter without sensing).
(2) The ATs can be *merged*; knowing a first AT $\mathbf{A_1}$ with respect to $\mathbf{W}$ and a second AT of the same frame $\mathbf{A_2}$ with respect to $\mathbf{W}$, compute the AT $\mathbf{A}$ with respect to $\mathbf{W}$ by combining the 2 spatial relationships, in which case the resulting uncertainty is less than both of original uncertainties (think of motion update followed by sensing a known location landmark).

An insightful analogy can be made with series / parallel electric resistances.

Back to the landmark initialization problem: Obviously, a single bearing measurement of a landmark does not allow to determine a location for it (cf. 1). The idea is to accumulate over time evidences of the presence of a new landmark before integrating it in the state variable. We see in figure 2 that potential landmarks location arise at the intersections of measurements from different positions. For each of the intersection, we *compound* the covariance Σ_t of robot uncertain pose at time t with covariance of the measurement Q to get an approximate landmark location $\hookrightarrow \mathcal{N}(\mu, C_1)$. We do it again from location at time $t + 1$ to get another estimate $\hookrightarrow \mathcal{N}(\mu, C_2)$ and then *merge* both estimates (cf. figure 3) to get a clearer position $\mathcal{N}(\mu, C_3)$.

Note that every measurements intersection is not necessarily a landmark, that is why we need to introduce the notion of *persistance*: as the persistence count increases, the certainty that the gaussian corresponds to an actual landmark increases. In figure 4 again estimate is build from measurement intersection $\mathcal{N}(\mu_4, C_4)$. This estimate is close enough (we need to define a distance between Gaussian distributions) from the previous one, so the estimates are merged and persistance count is incremented. When the count reaches a certain threshold, the landmark is appended to state variable and initialized with current estimate's mean and covariance.

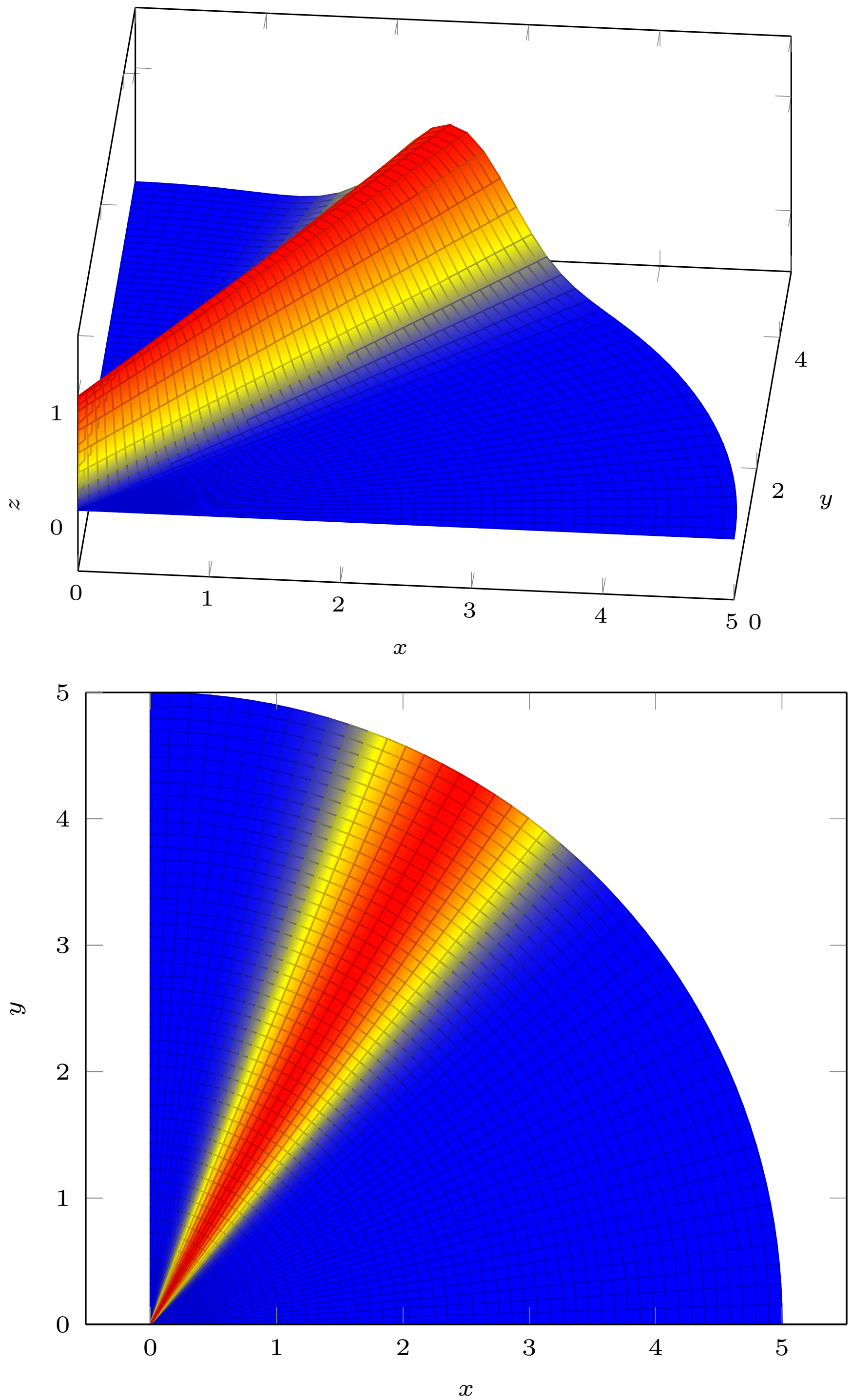

FIGURE 1. scaled probability density of location in robot local frame of a landmark, detected at $\phi = \frac{\pi}{3}$ with noise variance $\sigma^2 = 0.01$

We have seen previously that for a given known landmark k, the r.v. $\pi_k = (z - h(y_t, k))\Psi_k^{-1}(z - h(y_t, k))$ is a chi-squared r.v. if landmark k is the actual target of measurement z;so we can do a Chi-squared test and decide a threshold αabove which the landmark is associated to the measurement (we need to choose a policy to handle incompatibilities); if the measurement cannot be associated to any already known landmark, it is used in above initialization algorithm.

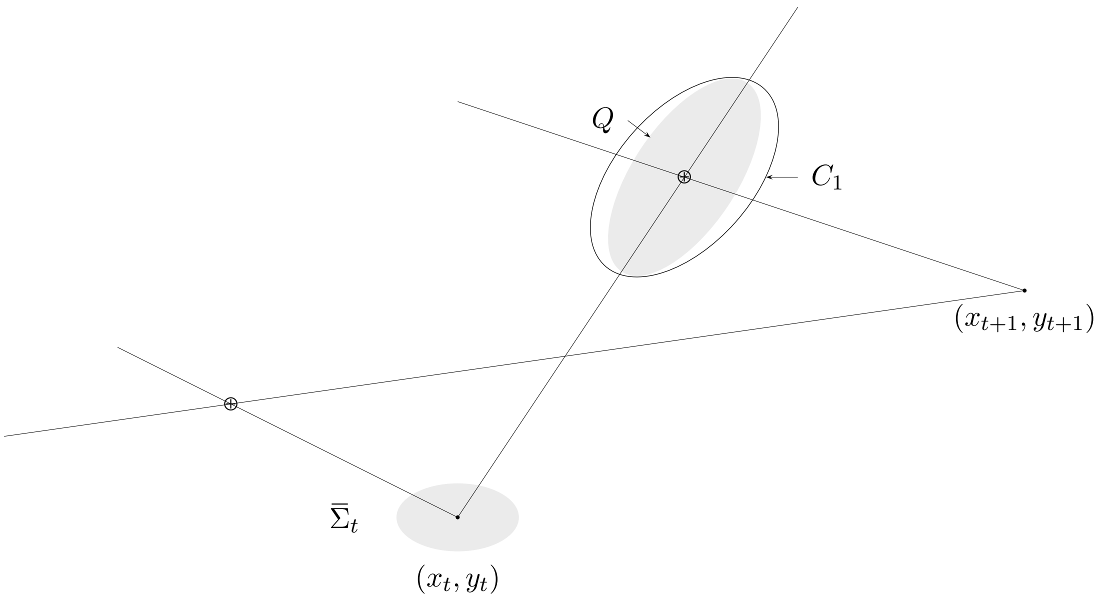

FIGURE 2. Uncertainty of the robot pose is compounded with noise uncertainty to estimate landmark position

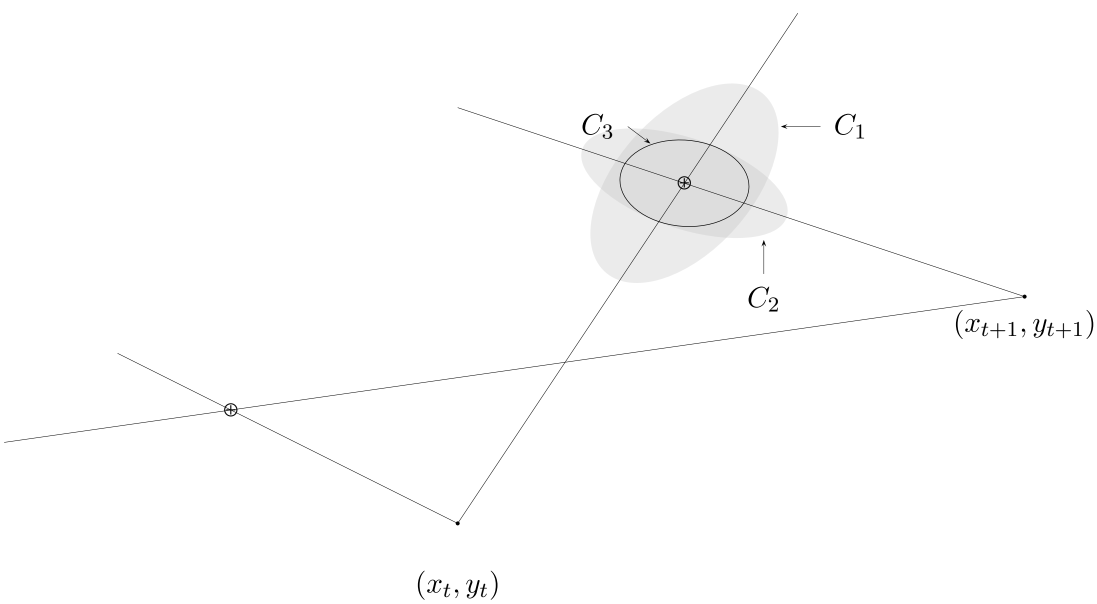

FIGURE 3. Merging of estimates from 2 different positions

3

First we review the matrix

$$F_{x,j} = \begin{bmatrix} 1 & 0 & 0 & 0\dots0 & 0 & 0 & 0 & 0\dots0 \\ 0 & 1 & 0 & 0\dots0 & 0 & 0 & 0 & 0\dots0 \\ 0 & 0 & 1 & 0\dots0 & 0 & 0 & 0 & 0\dots0 \\ 0 & 0 & 0 & 0\dots0 & 1 & 0 & 0 & 0\dots0 \\ 0 & 0 & 0 & 0\dots0 & 0 & 1 & 0 & 0\dots0 \\ 0 & 0 & 0 & \underbrace{0\dots0}_{3j-3} & 0 & 0 & 1 & \underbrace{0\dots0}_{3N-3j} \end{bmatrix}$$

We observe that multiplying on the right any 6 columns matrix by $F_{x,j}$ has the effect to insert $3j-3$ null columns between columns 3 and 4 and $3N-3j$ null columns on the end so that we end

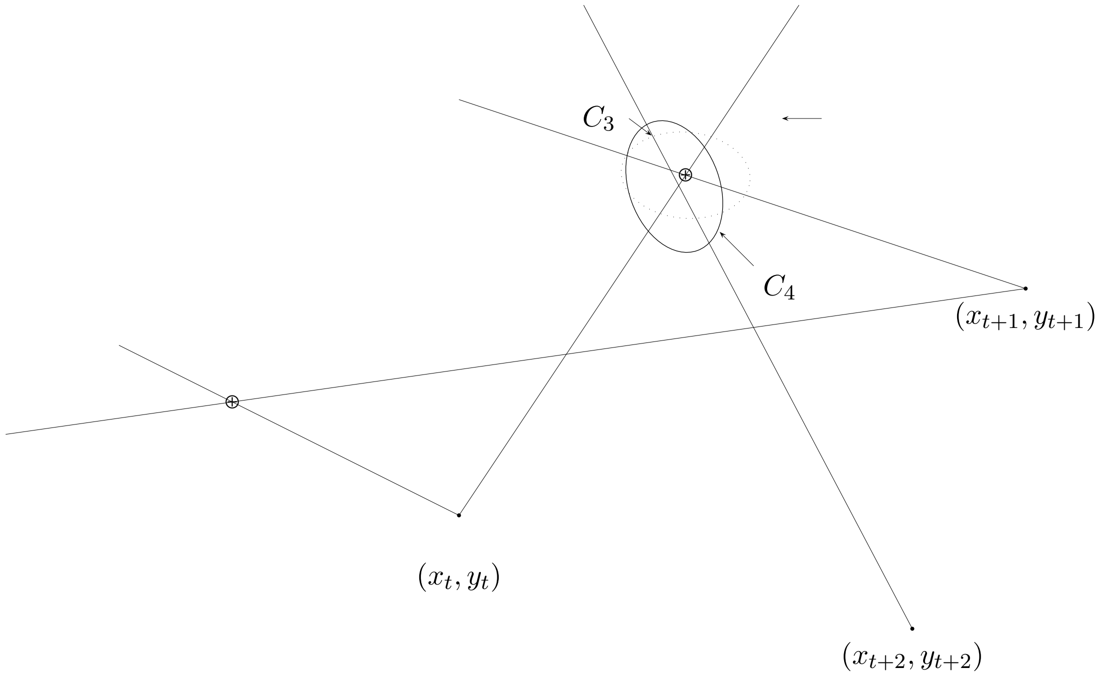

FIGURE 4. Persistance count of landmark is increased when seen again at a later stage

with a matrix with $3N+3$ columns. Similarly, multiplying on the left any 6 rows matrix by $F_{x,j}^T$ has the effect to insert $3j-3$ null rows between rows 3 and 4 and $3N-3j$ null rows on the end so that we end with a matrix with $3N+3$ rows. Remember we have initialized the covariance matrix of the state with equation (10.9):

$$\Sigma_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \infty & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \infty \end{bmatrix}$$

Note that in this pratical context ∞ actually means huge number so that $0 \times \infty = 0$.We write the covariance matrix Σ_t of by block:

$$\Sigma_t = \begin{bmatrix} \Sigma_t^X & \dots & [\Sigma_t^{X,k}]^T & \dots \\ \vdots & \ddots & \vdots & \dots \\ \Sigma_t^{X,k} & \dots & \Sigma_t^k & \dots \\ \vdots & \dots & \vdots & \ddots \end{bmatrix}$$

where $\Sigma_t^{X,k}$ is the cross-covariance between the landmark number k and the robot pose (index X).We now restate measurement update line 23 of EKF SLAM:

$$\begin{aligned} \mu_t &= \bar{\mu}_t + K_t^i(z_t^i - \hat{z}_t^{j(i)}) \\ \Leftrightarrow \mu_t &= \bar{\mu}_t + \bar{\Sigma}_t [H_t^{j(i)}]^T \Psi_{j(i)}^{-1}(z_t^i - \hat{z}_t^{j(i)}) \\ \Leftrightarrow \mu_t &= \bar{\mu}_t + \bar{\Sigma}_t F_{x,j(i)}^T h_t^{j(i)} \Psi_{j(i)}^{-1}(z_t^i - \hat{z}_t^{j(i)}) \end{aligned}$$

By the previous remarks, a little thought show that if the lines / columns related to landmark k have not changed since initialization ($\forall j \neq k, \Sigma_t^{j,k} = 0$ and $\Sigma_t^k = \begin{bmatrix} \infty & 0 & 0 \\ 0 & \infty & 0 \\ 0 & 0 & \infty \end{bmatrix}$) and if $j(i) \neq k$, the position estimate of landmark k stay untouched by the update. If we also consider covariance update:

$$
\begin{aligned}
&\Sigma_t = (I - K_t^i H_t^{j(i)})\bar{\Sigma}_t \\
\Leftrightarrow &\Sigma_t = (I - \bar{\Sigma}_t [H_t^{j(i)}]^T \Psi_{j(i)}^{-1} H_t^{j(i)})\bar{\Sigma}_t \\
\Leftrightarrow &\Sigma_t = \bar{\Sigma}_t - \bar{\Sigma}_t [H_t^{j(i)}]^T \Psi_{j(i)}^{-1} H_t^{j(i)} \bar{\Sigma}_t \\
\Leftrightarrow &\Sigma_t = \bar{\Sigma}_t - \bar{\Sigma}_t F_{x,j(i)}^T h_t^{j(i)} \Psi_{j(i)}^{-1} h_t^{j(i)} F_{x,j(i)} \bar{\Sigma}_t
\end{aligned}
$$

we can also see that if $j(i) \neq k$, the lines / columns in covariance matrix related to landmark k stay untouched. By applying same kind of reasoning on motion update, we can assert that as long as a given landmark has not been observed, the corresponding lines / columns in both μ and Σ keep their initial values during the execution of the algorithm; this was expected since the cross covariances being set to 0 initially, updating robot and others landmarks locations should not affect the landmark.

Turning now to what happens when the landmark $k = j(i)$ is observed for the first time, the mean is updated line 9 of the EKF SLAM algorithm:

$$
\begin{aligned}
\begin{bmatrix} \bar{\mu}_{k,x} \\ \bar{\mu}_{k,y} \\ \bar{\mu}_{k,s} \end{bmatrix} &= \begin{bmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \\ s_t^i \end{bmatrix} + r_t^i \begin{bmatrix} \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \\ 0 \end{bmatrix} \\
&= [h_m^t]^{-1}(z_t^i)
\end{aligned}
$$

where

$$
h_m^t = h(y_t, .) : \begin{bmatrix} m_x \\ m_y \\ m_s \end{bmatrix} \mapsto h(y_t, \begin{bmatrix} m_x \\ m_y \\ m_s \end{bmatrix}) = \begin{bmatrix} \sqrt{(m_x - x_t)^2 + (m_y - y_t)} \\ \text{atan2}(m_y - y_t, m_x - x_t) - \theta_t \\ m_s \end{bmatrix}
$$

this is indeed the best guess of the location we can make from the measurement z_t^i alone. The position is subsequently updated line 23 by:

$$
\mu_t = \bar{\mu}_t + K_t^i (z_t^i - \hat{z}_t^k) \qquad ①
$$

where

$$
\begin{aligned}
K_t^i &= \bar{\Sigma}_t [H_t^k]^T \Psi_k^{-1} \\
&= \bar{\Sigma}_t F_{x,k}^T [h_t^k]^T \Psi_k^{-1}
\end{aligned}
$$

We have

$$
\begin{aligned}
F_{x,k}^T [h_t^k]^T &= F_{x,k}^T \begin{bmatrix} J^X_{(\bar\mu_X,\bar\mu_k)} & J^k_{(\bar\mu_X,\bar\mu_k)} \end{bmatrix}^T \\
&= F_{x,k}^T \begin{bmatrix} [J^X_{(\bar\mu_X,\bar\mu_k)}]^T \\ [J^k_{(\bar\mu_X,\bar\mu_k)}]^T \end{bmatrix} \\
&= \begin{array}{l} \left[\begin{array}{ccc} & A^T & \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ & B^T & \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{array}\right] \end{array}
\begin{array}{l} \}\, 3 \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} 3k-3 \\ \}\, 3 \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} 3N-3k \end{array}
\end{aligned}
$$

where $A = J^X_{(\bar\mu_X,\bar\mu_k)}$ and $B = J^k_{(\bar\mu_X,\bar\mu_k)}$ are the Jacobians at current position of measurement function h, respectively with respects to robot's and landmark's positions. Then,

$$
\begin{aligned}
\bar\Sigma_t F_{x,k}^T [h_t^k]^T &= \begin{bmatrix} \overbrace{\bar\Sigma_t^X}^{3} & \overbrace{\cdots}^{3k-3} & \overbrace{0}^{3} & \overbrace{\cdots}^{3N-3k} \\ \vdots & \ddots & \vdots & \cdots \\ 0 & \cdots & \Sigma_0^k & \cdots \\ \vdots & \cdots & \vdots & \ddots \end{bmatrix}
\begin{bmatrix} & A^T & \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ & B^T & \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \bar\Sigma_t^X A^T \\ \bar\Sigma_t^{X,1} A^T \\ \vdots \\ \bar\Sigma_t^{X,k-1} A^T \\ \Sigma_0^k B^T \\ \bar\Sigma_t^{X,k+1} A^T \\ \vdots \\ \bar\Sigma_t^{X,N} A^T \end{bmatrix} \qquad ②
\end{aligned}
$$

Also,

$$\begin{aligned}
H_t^k &= h_t^k F_{x,k} \\
&= \begin{bmatrix} & & 0 & \dots & 0 & & & 0 & \dots & 0 \\ & A & 0 & \dots & 0 & & B & 0 & \dots & 0 \\ & & 0 & \dots & 0 & & & 0 & \dots & 0 \end{bmatrix}
\end{aligned}$$

so that

$$\begin{aligned}
\Psi_k &= H_t^k \bar{\Sigma}_t [H_t^k]^T + Q_t \\
&= A\bar{\Sigma}_t^X A^T + B\Sigma_0^k B^T + Q_t
\end{aligned}$$

Now we can assume $\Sigma_0^k = C \times I_3$, where C is a huge positive constant. Assuming also BB^T is (symmetric positive) definite,

$$\begin{aligned}
\Psi_k &= A\bar{\Sigma}_t^X A^T + CBB^T + Q_t \\
&= [I_3 + \frac{1}{C}\underbrace{(A\bar{\Sigma}_t^X A^T + Q_t)(BB^T)^{-1}}_{N}]CBB^T
\end{aligned}$$

We have given a proof in chapter 7 for the existence and formula of the inverse of $I_3 - M$ when $\|M\| < 1$:

$$(I_3 + \frac{1}{C}N)^{-1} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{C^n} N^n$$

with a norm such that $\left\| \frac{(-1)^n}{C^n} N^n \right\| \leqslant \left\| \frac{1}{C} N \right\|^n$, we write to first order:

$$\begin{aligned}
(I_3 + \frac{1}{C}N)^{-1} &\underset{C \to +\infty}{=} I_3 - \frac{1}{C}N + o(\left\| \frac{1}{C} \right\|) \\
&\approx I_3 - \frac{1}{C}N
\end{aligned}$$

so

$$\begin{aligned}
\Psi_k^{-1} &= \frac{1}{C}(BB^T)^{-1}(I_3 - \frac{1}{C}N) \\
&= \frac{1}{C}(BB^T)^{-1}(I_3 - \frac{1}{C}N) \\
\Psi_k^{-1} &= (BB^T)^{-1}(\frac{1}{C}I_3 - \frac{1}{C^2}N) \qquad ③
\end{aligned}$$

Multiplying ② and ③, and restricting to order 0 in $\frac{1}{C}$(we discard terms in $\frac{1}{C}$ and smaller),

$$\bar{\Sigma}_t[H_t^k]^T\Psi_k^{-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B^T(BB^T)^{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ B^{-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This shows that measurement update ① updates only landmark k position with:

$$\begin{aligned} \mu_k &= \bar{\mu}_k + B^{-1}(z_t^i - \hat{z}_t^k) \\ &= \bar{\mu}_k + B^{-1}(\underbrace{h(y_t, \bar{\mu}_k) - h(y_t, \bar{\mu}_k)}_{=0}) \\ \mu_k &= \begin{bmatrix} \bar{\mu}_{t,x} \\ \bar{\mu}_{t,y} \\ s_t^i \end{bmatrix} + r_t^i \begin{bmatrix} \cos(\phi_t^i + \bar{\mu}_{t,\theta}) \\ \sin(\phi_t^i + \bar{\mu}_{t,\theta}) \\ 0 \end{bmatrix} \end{aligned} \quad ④$$

Line 24 of the algorithm, covariance is updated with

$$\begin{aligned} \Sigma_t &= (I - K_t^i H_t^i)\bar{\Sigma}_t \\ &= \bar{\Sigma}_t - K_t^i H_t^k \bar{\Sigma}_t \end{aligned}$$

$$
\begin{aligned}
&K_t^i H_t^k \bar{\Sigma}_t \\
&\quad = \quad \bar{\Sigma}_t [H_t^k]^T \Psi_k^{-1} H_t^k \bar{\Sigma}_t \\
&\quad = \quad \begin{bmatrix} \bar{\Sigma}_t^X A^T \\ \bar{\Sigma}_t^{X,1} A^T \\ \vdots \\ \bar{\Sigma}_t^{X,k-1} A^T \\ CB^T \\ \bar{\Sigma}_t^{X,k+1} A^T \\ \vdots \\ \bar{\Sigma}_t^{X,N} A^T \end{bmatrix} (BB^T)^{-1}\left(\frac{1}{C} I_3 - \frac{1}{C^2} N\right) \begin{bmatrix} A\bar{\Sigma}_t^X & A\bar{\Sigma}_t^{X,1} & \dots & A\bar{\Sigma}_t^{X,k-1} & CB & \dots & A\bar{\Sigma}_t^{X,N} \end{bmatrix} \\
&\underset{C \to +\infty}{\approx} \begin{bmatrix} 0 & \dots & 0 & \bar{\Sigma}_t^X A^T [B^T]^{-1} & 0 & \dots \\ \vdots & \ddots & & \bar{\Sigma}_t^{X,1} A^T [B^T]^{-1} & \vdots & \dots \\ \vdots & & \ddots & \vdots & \vdots & \dots \\ B^{-1} A \bar{\Sigma}_t^X & \dots & B^{-1} A \bar{\Sigma}_t^{X,k-1} & C I_3 - B^{-1} N B & B^{-1} A \bar{\Sigma}_t^{X,k+1} & \dots \\ 0 & \dots & 0 & \bar{\Sigma}_t^{X,k+1} A^T [B^T]^{-1} & 0 & \dots \\ \vdots & & \vdots & \vdots & \vdots & \end{bmatrix}
\end{aligned}
$$

to order 0 in $\frac{1}{C}$, with

$$
\begin{aligned}
B^{-1} N B &= B^{-1}(A\bar{\Sigma}_t^X A^T + Q_t)(BB^T)^{-1} B \\
&= B^{-1}(A\bar{\Sigma}_t^X A^T + Q_t)[B^T]^{-1} \qquad \textcircled{5}
\end{aligned}
$$

The latter expression should be familiar: it is the compound of independant uncertainties of robot position and measurement, transformed back to landmark position frame. Besides,

$$
\begin{aligned}
z_t^i &= A(\overline{X}_t - \mu_t) + B(X^k - \mu_k) + \underbrace{\xi^k}_{\hookrightarrow \mathcal{N}(0, Q_t)} \\
\Rightarrow z_t^i (X^j - \mu_j)^T &= A(\overline{X}_t - \mu_t)(X^j - \mu_j)^T + B(X^k - \mu_k)(X^j - \mu_j)^T + \xi^k (X^j - \mu_j)^T
\end{aligned}
$$

Taking the expectancy of this relation and since we can assume if $k \neq j$ that ξ^k and X_j are independant, we get

$$
\begin{aligned}
\Rightarrow \quad & 0 = A\overline{\Sigma}_{tt}^{X,j} + B\Sigma_{k,j} + 0 \\
\Rightarrow & B\Sigma^{k,j} = -A\bar{\Sigma}_t^{X,j} \\
\Leftrightarrow \quad & \Sigma^{k,j} = -B^{-1} A \bar{\Sigma}_t^{X,j} \qquad \textcircled{6}
\end{aligned}
$$

Thus we have proven that the EKF SLAM presented in the book is equivalent to:

(1) start with a state including only robot pose.
(2) each time a new landmark is found, augment the state variable using formulas ④, ⑤, ⑥ to initialize mean and covariances matrices related to the landmark.

References

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