PROBABILISTIC ROBOTICS:
MOBILE ROBOT LOCALIZATION: MARKOV AND GAUSSIAN

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1.1. If the measurement is not noisy, the possible robot poses (x, y, θ) for a given range / bearing sensing (r, ϕ) are located on an helix in $\mathbb{R}^3$, which projects on a circle in $\mathbb{R}^2$(cf. figure 1). This situation cannot be captured by a Gaussian properly. When two measurements are done, the helices intersect in at one pose (x, y, θ)(cf. figure 2). Thus the probability distribution for noisy measurements has only one mode and can be modelized by a Gaussian.

1.2. The range / bearing measurement model is:

$$\forall i \in [\![1, k]\!], \quad \begin{cases} r_i = \sqrt{(m_{j,x} - x)^2 + (m_{j,y} - y)^2} + \epsilon_{\sigma_r^2} \\ \\ \phi_i = \operatorname{atan2}(m_{j,y} - y, m_{j,x} - x) - \theta + \epsilon_{\sigma_\phi^2} \end{cases}$$

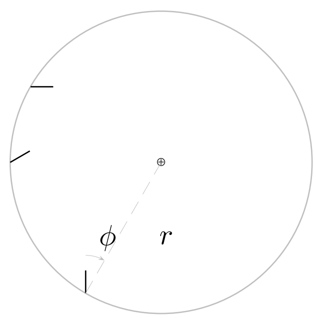

FIGURE 1. estimating pose from sensing a single landmark, $r = 2$, $\phi = -\frac{\pi}{6}$

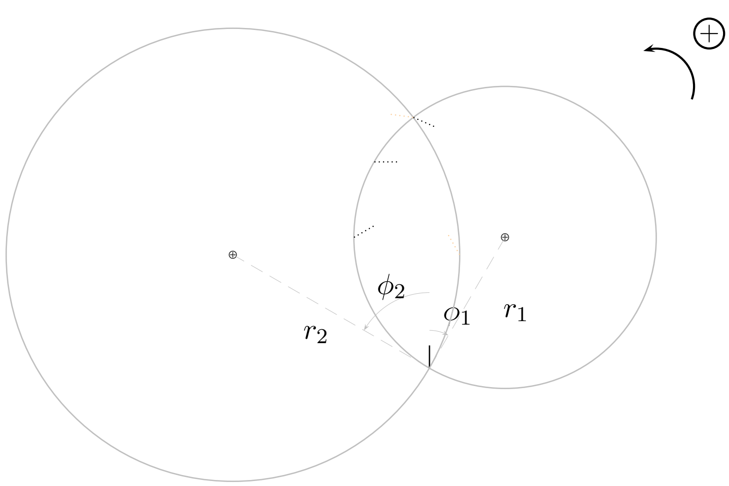

FIGURE 2. estimating pose from sensing two landmarks, $r_1 = 2$, $\phi_1 = -\frac{\pi}{6}$,$r_2 = 3$, $\phi_2 = \frac{\pi}{3}$

where $c(i) = j$, $\epsilon_{\sigma^2} \hookrightarrow \mathcal{N}(0, \sigma^2)$.

$$p(x, y, \theta \mid z_{1:k}) = \eta p(z_{1:k} \mid x, y, \theta) p(x, y, \theta)$$

Under the conditional independance assumption,

$$\begin{aligned} p(x, y, \theta \mid z_{1:k}) &= \eta (\prod_{i=1}^{k} p(z_i \mid x, y, \theta)) p(x, y, \theta) \\ &= \eta p(x, y, \theta) \prod_{i=1}^{k} p(r_i \mid x, y) p(\phi_i \mid x, y, \theta) \end{aligned}$$

Since the prior is uniform,

$$\begin{aligned} p(x, y, \theta \mid z_{1:k}) &= \eta \prod_{i=1}^{k} p(r_i \mid x, y) p(\phi_i \mid x, y, \theta) \\ &= \eta e^{-\frac{1}{2\sigma_r^2} \sum_{i=1}^{k} (r_i - \sqrt{(m_{c(i),x} - x)^2 + (m_{c(i),y} - y)^2})^2 - \frac{1}{2\sigma_\phi^2} \sum_{i=1}^{k} (\phi_i - \operatorname{atan2}(m_{c(i),y} - y, m_{c(i),x} - x) + \theta)^2} \\ &= \eta e^{-F(x, y, \theta)} \end{aligned}$$

We will approximate F by a quadratic function. Supposing it exists and it is unique, let μ the solution of

$$\begin{aligned} &\vec{\nabla} F(x, y, \theta) = \vec{0} \\ \Leftrightarrow &\begin{cases} \frac{\partial F}{\partial x}(x, y, \theta) = 0 \\ \frac{\partial F}{\partial y}(x, y, \theta) = 0 \\ \frac{\partial F}{\partial \theta}(x, y, \theta) = 0 \end{cases} \end{aligned}$$

$$\Leftrightarrow \begin{cases} \frac{1}{\sigma_r^2} \sum_{i=1}^{k} \frac{m_{c(i),x} - x}{\sqrt{(m_{c(i),x} - x)^2 + (m_{c(i),y} - y)^2}} (r_i - \sqrt{(m_{c(i),x} - x)^2 + (m_{c(i),y} - y)^2}) \\ - \frac{1}{\sigma_\phi^2} \sum_{i=1}^{k} \frac{m_{c(i),y} - y}{(m_{c(i),x} - x)^2 + (m_{c(i),y} - y)^2} (\phi_i - \operatorname{atan2}(m_{c(i),y} - y, m_{c(i),x} - x) + \theta) = 0 \\ \frac{1}{\sigma_r^2} \sum_{i=1}^{k} \frac{m_{c(i),y} - y}{\sqrt{(m_{c(i),x} - x)^2 + (m_{c(i),y} - y)^2}} (r_i - \sqrt{(m_{c(i),x} - x)^2 + (m_{c(i),y} - y)^2}) \\ + \frac{1}{\sigma_\phi^2} \sum_{i=1}^{k} \frac{m_{c(i),x} - x}{(m_{c(i),x} - x)^2 + (m_{c(i),y} - y)^2} (\phi_i - \operatorname{atan2}(m_{c(i),y} - y, m_{c(i),x} - x) + \theta) = 0 \\ \frac{1}{\sigma_\phi^2} \sum_{i=1}^{k} (\phi_i - \operatorname{atan2}(m_{c(i),y} - y, m_{c(i),x} - x) + \theta) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{1}{\sigma_r^2}\sum_{i=1}^{k}\frac{m_{c(i),x}-x}{\sqrt{(m_{c(i),x}-x)^2+(m_{c(i),y}-y)^2}}(r_i-\sqrt{(m_{c(i),x}-x)^2+(m_{c(i),y}-y)^2}) \\ -\frac{1}{\sigma_\phi^2}\sum_{i=1}^{k}\frac{m_{c(i),y}-y}{(m_{c(i),x}-x)^2+(m_{c(i),y}-y)^2}(\phi_i-\operatorname{atan2}(m_{c(i),y}-y,m_{c(i),x}-x)+\theta)=0 \\ \frac{1}{\sigma_r^2}\sum_{i=1}^{k}\frac{m_{c(i),y}-y}{\sqrt{(m_{c(i),x}-x)^2+(m_{c(i),y}-y)^2}}(r_i-\sqrt{(m_{c(i),x}-x)^2+(m_{c(i),y}-y)^2}) \\ +\frac{1}{\sigma_\phi^2}\sum_{i=1}^{k}\frac{m_{c(i),x}-x}{(m_{c(i),x}-x)^2+(m_{c(i),y}-y)^2}(\phi_i-\operatorname{atan2}(m_{c(i),y}-y,m_{c(i),x}-x)+\theta)=0 \\ \theta=\frac{1}{k}\sum_{i=1}^{k}(\operatorname{atan2}(m_{c(i),y}-y,m_{c(i),x}-x)-\phi_i) \end{cases}$$

The system can be solved using numerical procedure like Newton's method (cf. A.1). The curvature is given by the Hessian of F evaluated at $\mu=(\mu_x,\mu_y,\mu_\theta)$

$$\begin{aligned} H_F(\mu) &= \begin{bmatrix} \frac{\partial^2 F}{\partial^2 x} & \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial \theta \partial x} \\ \frac{\partial^2 F}{\partial x \partial y} & \frac{\partial^2 F}{\partial^2 y} & \frac{\partial^2 F}{\partial \theta \partial y} \\ \frac{\partial^2 F}{\partial x \partial \theta} & \frac{\partial^2 F}{\partial y \partial \theta} & \frac{\partial^2 F}{\partial^2 \theta} \end{bmatrix} \\ &= \Sigma^{-1} \end{aligned}$$

Second order Taylor expansion gives

$$\begin{aligned} F(\underbrace{x,y,\theta}_{X}) &\underset{X\to\mu}{=} F(\mu_x,\mu_y,\mu_\theta)+\underbrace{\vec{\nabla}F(\mu)^T}_{\vec{0}}(X-\mu)+\frac{1}{2}(X-\mu)^T H_F(\mu)(X-\mu)+o(\|X-\mu\|^2) \\ &\underset{X\to\mu}{=} F(\mu_x,\mu_y,\mu_\theta)+\frac{1}{2}(X-\mu)^T H_F(\mu)(X-\mu)+o(\|X-\mu\|^2) \\ &\approx \underbrace{F(\mu_x,\mu_y,\mu_\theta)}_{\text{does not depend on} X}+\frac{1}{2}(X-\mu)^T H_F(\mu)(X-\mu) \end{aligned}$$

hence,

$$p(x,y,\theta \mid z_{1:k}) \approx \eta e^{-\frac{1}{2}(X-\mu)^T H_F(\mu)(X-\mu)}$$

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The setting of this problem is unclear to me.

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3.1. We will use the following first order motion model:

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{bmatrix} \underset{\Delta t \to 0}{=} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} + \begin{bmatrix} \dot{x}_t \\ \dot{y}_t \\ \dot{z}_t \end{bmatrix} \times \Delta t + o((\Delta t)^2)$$

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ z_{t+1} \end{bmatrix} \underset{\Delta t \to 0}{=} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_t} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} + \underbrace{\begin{bmatrix} \dot{x}_t \\ \dot{y}_t \\ \dot{z}_t \end{bmatrix}}_{u_t} \times \Delta t + o((\Delta t)^2)$$

$$X_{t+1} \underset{\Delta t \to 0}{\approx} X_t + \dot{X}_t \Delta t$$

We add noise $\epsilon_t \hookrightarrow \mathcal{N}(0, \sigma^2 I_3)$:

$$X_{t+1} = X_t + \dot{X}_t \Delta t + \epsilon_t$$

The measurement model is:

$$z_t = \begin{bmatrix} r_t^1 & r_t^2 & \dots & r_t^k \end{bmatrix}$$

where k is the number of beacons reachable at date t.

$$r_t^i = \underbrace{\sqrt{(m_{j,x} - x_t)^2 + (m_{j,y} - y_t)^2 + (m_{j,z} - z_t)^2}}_{h(X_t, i)}$$

where the correspondance $c(i) = j$ is known.We have to linearize about $X = \bar{\mu}$:

$$r_t^i \underset{X \to \mu}{=} h(\mu, i) + \vec{\nabla}_X h(\mu, i)^T (X - \mu) + o(\|X - \mu\|)$$

$$\vec{\nabla}_X h(\mu, i)^T = \begin{bmatrix} -\frac{m_{j,x} - \mu_x}{h(\bar{\mu}, i)} & -\frac{m_{j,y} - \mu_y}{h(\bar{\mu}, i)} & -\frac{m_{j,z} - \mu_z}{h(\bar{\mu}, i)} \end{bmatrix}$$

$$\Rightarrow \quad r_t^i \underset{X \to \bar{\mu}}{\approx} h(\bar{\mu}, i) + \underbrace{\begin{bmatrix} -\frac{m_{j,x} - \mu_x}{h(\bar{\mu}, i)} & -\frac{m_{j,y} - \mu_y}{h(\bar{\mu}, i)} & -\frac{m_{j,z} - \mu_z}{h(\bar{\mu}, i)} \end{bmatrix}}_{H_{t,i}} (X - \bar{\mu})$$

3.2. The code for simulation of environment and EKF algorithm is in `simulation.m` and `ekf.m`. I plotted some results in figure 3 and figure 4.

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In this exercise and the following, I will adopt a strictly equivalent and more compact representation; the available actions will be :

Action U: Go up
Action D: Go down
Action L: Go left
Action R: Go right

All the motions stop when hitting a wall.

4.1. Shortest open loops are of length 3. For instance, `U L U`.

4.2. Shortest open loops are of length 6. For instance, `L U R U L U`.

4.3. Shortest open loops are of length 7. For instance, `D R U L D L U`.

4.4. Shortest open loops are of length 4. For instance, `D R U R`.

4.5. Shortest open loops are of length 6. For instance, `R D L U R U`.

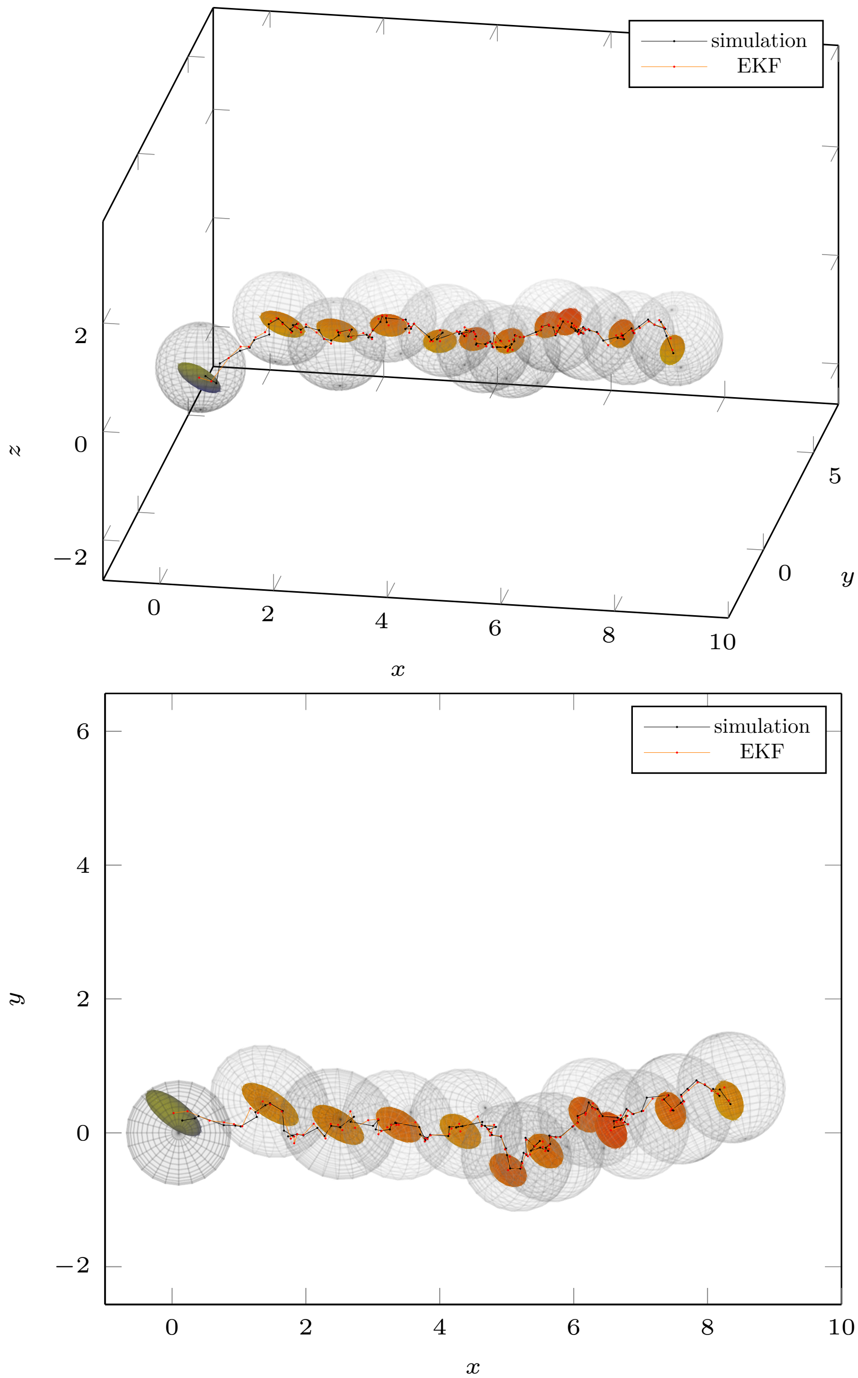

Figure 3. EKF algorithm for commands $v_t = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

4.6. We observe that the robot can never escape corridors at the top and bottom unless it is initially in front of the "door". Moreover, there is no way for the robot to ever determine in which area it was initially. Hence there is no open loop which ends at a predictable location in this case.

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The robot still can't escape the corridors, so there will be different ending positions according to whether the robot was initially. For the ending position to be determined, the robot must deduce from the number of steps of its motions where it was at first. The only way to discriminate between

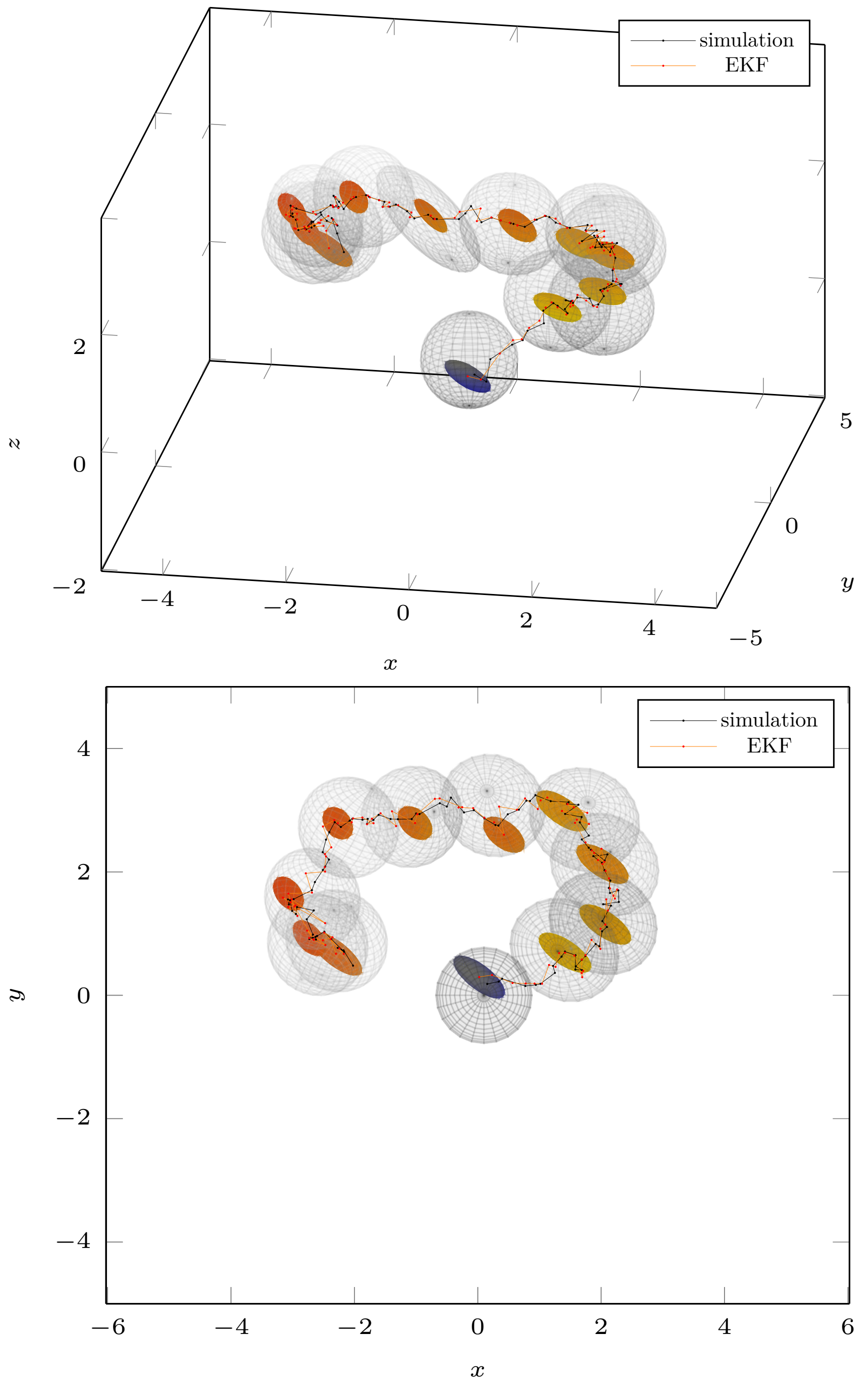

FIGURE 4. EKF algorithm for commands $v_t = \begin{bmatrix}\cos(0.6t) & \sin(0.6t) & 0.1\end{bmatrix}$

the two corridors is to make a full lateral swing; hence the fact that the sequence must include at least one R and one L. One such sequence of length 2 being obviously not sufficient, we can discard one by one the possible sequence of length 3 by bringing up ambiguous initial locations(cf. figure 6). Thus the shortest sequence is of length 4, for instance `U L D R`. See figure 7 for the final positions according to initial position.

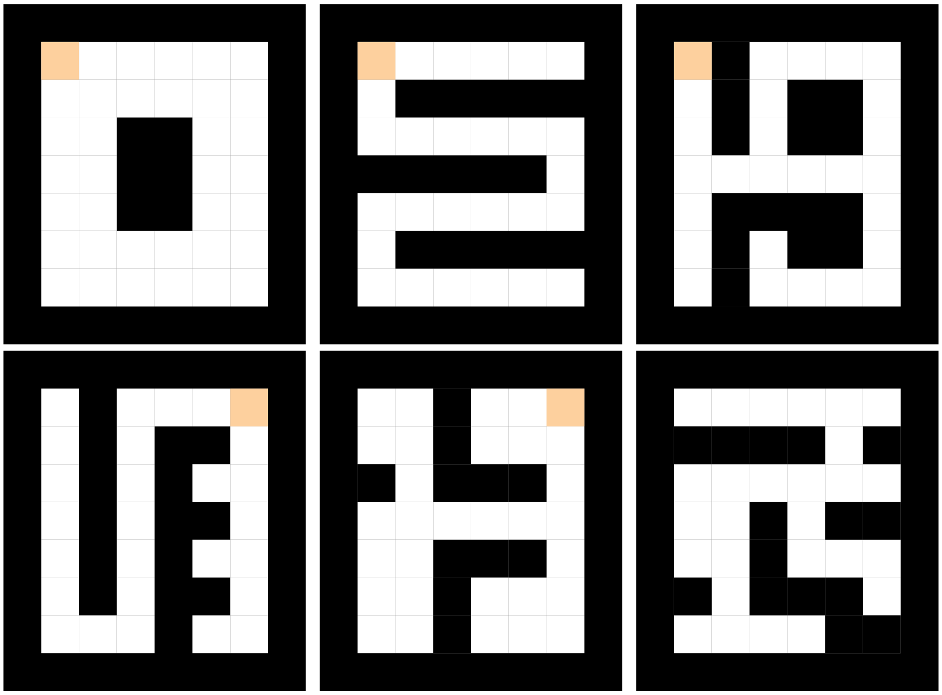

FIGURE 5. Position of the robot at the end of the open-loop

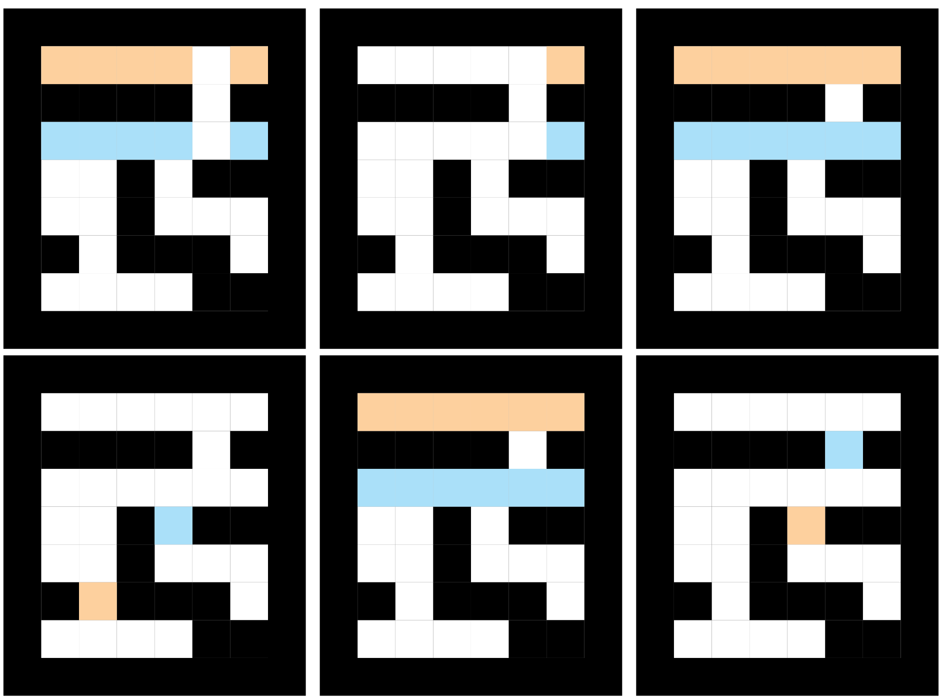

FIGURE 6. Ambiguous initial locations for the respective sequences `U _ _`, `D _ _`, `_ U _`, `L D R`, `R D L`, `_ _ U?D`, left to right, top to bottom

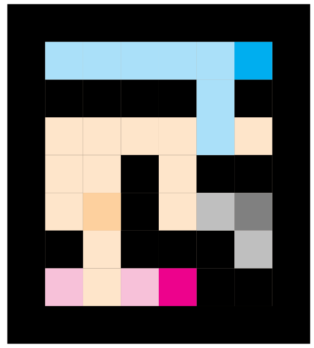

FIGURE 7. 4 Final positions according to initial position `U L D R`

Appendix A

A.1.

Lemma. Let E a Banach space and $u \in \mathcal{L}_c(E)$ (the space of continuous maps of E), such that $\|u\| < 1$. Then $\mathrm{id} - u$ is non singular and

$$(\mathrm{id} - u)^{-1} = \sum_{n=0}^{+\infty} u^n$$

and

$$\left\|(\mathrm{id} - u)^{-1}\right\| \leqslant \frac{1}{1 - \|u\|}$$

Proof.

$$\forall (n,p) \in \mathbb{N}^2, \quad n \geqslant p, \quad \left\|\sum_{k=p}^{n} u^n\right\| \leqslant \sum_{k=p}^{n} \|u\|^n$$

shows that $(u^n)_{n\in\mathbb{N}}$ is a Cauchy sequence, thus converges since $\mathcal{L}_c(E)$ is complete. Then

$$\begin{aligned}(\mathrm{id} - u)\sum_{n=0}^{+\infty} u^n &= \sum_{n=0}^{+\infty} u^n - \sum_{n=1}^{+\infty} u^n \\ &= Id\end{aligned}$$

and

$$\begin{aligned}\left\|(\mathrm{id} - u)^{-1}\right\| &= \lim_{N\to+\infty} \left\|\sum_{n=0}^{N} u^n\right\| \\ &\leqslant \sum_{n=0}^{+\infty} \|u\|^n \\ &= \frac{1}{1 - \|u\|}\end{aligned}$$

□

Lemma. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in the open convex set $D \subset \mathbb{R}^n$, $x \in D$, and let J be γ Lipschitz continuous at x in the neighborhood D. Then,

$$\forall y \in D, \quad \|F(y) - F(x) - J(x)(y - x)\| \leqslant \frac{\gamma}{2}\|y - x\|^2$$

Proof. We have, by chain derivation rule,

$$\begin{aligned}F(y) - F(x) - J(x)(y - x) &= \int_0^1 J(x + t(y - x))(y - x)\,\mathrm{d}t - J(x)(y - x) \\ &= \int_0^1 [J(x + t(y - x)) - J(x)](y - x)\,\mathrm{d}t\end{aligned}$$

so

$$\begin{aligned}
\|F(y)-F(x)-J(x)(y-x)\| &\leqslant \int_0^1 \|J(x+t(y-x))-J(x)\|\,\|(y-x)\|\,\mathrm{d}t \\
&\leqslant \int_0^1 \gamma\,\|t(y-x)\|\,\|(y-x)\|\,\mathrm{d}t \\
&= \gamma\,\|y-x\|^2 \int_0^1 t\,\mathrm{d}t \\
&= \frac{\gamma}{2}\,\|y-x\|^2
\end{aligned}$$

□

Theorem. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable in an open convex set $D \subset \mathbb{R}^n$. Assume that there exists $x_* \in \mathbb{R}^n$ and r, $\beta > 0$, such that $N(x_*, r) = \{x \in \mathbb{R}^n, \|x - x_*\| < r\} \subset D$, $F(x_*) = 0$, $J(x_*)^{-1}$ exists with $\left\|J(x_*)^{-1}\right\| \leqslant \beta$, and $J \in \mathrm{Lip}_\gamma(N(x_*, r))$. Then there exists $\epsilon > 0$ such that for all $x_0 \in N(x_*, \epsilon)$ the sequence $(x_n)_{n\in\mathbb{N}}$ defined by

$$\forall k \in \mathbb{N}, \quad x_{k+1} = x_k - J(x_k)^{-1} F(x_k)$$

is well defined, converges to x_*, and obeys

$$\forall k \in \mathbb{N}, \quad \|x_{k+1} - x_*\| \leqslant \beta\gamma\,\|x_k - x_*\|^2$$

Proof. Proof from [1]. Let $\epsilon = \min\{r, \frac{1}{2\beta\gamma}\}$. We first show that $J(x_0)$ is non singular. From $\|x_0 - x_*\| \leqslant \epsilon$, the Lipschitz continuity of J at x_*, it follows that

$$\begin{aligned}
\left\|J(x_*)^{-1}(J(x_0)-J(x_*))\right\| &\leqslant \left\|J(x_*)^{-1}\right\|\,\|J(x_0)-J(x_*)\| \\
&\leqslant \beta\gamma\,\|x_0 - x_*\| \\
&\leqslant \beta\gamma\epsilon \\
&\leqslant \frac{1}{2}
\end{aligned}$$

Thus, by first lemma above, $J(x_0)$ is non singular and

$$\begin{aligned}
\left\|J(x_0)^{-1}\right\| &\leqslant \frac{\left\|J(x_*)^{-1}\right\|}{1 - \|J(x_*)^{-1}(J(x_0)-J(x_*))\|} \\
&\leqslant 2\left\|J(x_*)^{-1}\right\| \\
&\leqslant 2\beta
\end{aligned}$$

Therefore x_1 is well defined and

$$\begin{aligned}
x_1 - x_* &= x_0 - x_* - J(x_0)^{-1}F(x_0) \\
&= x_0 - x_* - J(x_0)^{-1}(F(x_0) - F(x_*)) \\
&= J(x_0)^{-1}[F(x_*) - F(x_0) - J(x_0)(x_* - x_0)]
\end{aligned}$$

[1] Robert B. Schnabel and J. E. Dennis : *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*, Society for Industrial and Applied Mathematics (1996)

Using the second lemma above,

$$\begin{aligned}\|x_1 - x_*\| &\leqslant \left\|J(x_0)^{-1}\right\| \|F(x_*) - F(x_0) - J(x_0)(x_* - x_0)\| \\ &\leqslant 2\beta\frac{\gamma}{2}\|x_* - x_0\|^2 \\ &= \beta\gamma\|x_* - x_0\|^2\end{aligned}$$

Moreover, since $\|x_0 - x_*\| \leqslant \frac{1}{2\beta\gamma}$,

$$\|x_1 - x_*\| \leqslant \frac{1}{2}\|x_* - x_0\|$$

so that $x_1 \in \mathcal{N}(x_*, \epsilon)$ and the proof of the theorem can be completed with induction. □