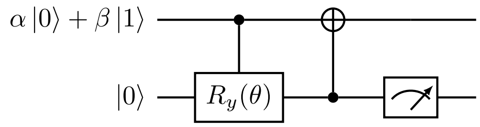
QUANTUM COMPUTATION AND QUANTUM INFORMATION:
QUANTUM NOISE AND QUANTUM OPERATIONS

Pierre-Paul TACHER

20. Circuit model for amplitude damping

We want to prove that the following circuit models the amplitude damping operation



Recall that

$$\begin{aligned} R_y(\theta) &= e^{-i\frac{\theta}{2}Y} \\ &= \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \end{aligned}$$

Initially the two-qubit state is

$$(\alpha |0\rangle + \beta |1\rangle) |0\rangle = \alpha |00\rangle + \beta |10\rangle$$

After the controlled R_y gate it becomes

$$\begin{aligned} \alpha |00\rangle + \beta |1\rangle R_y(\theta) |0\rangle &= \alpha |00\rangle + \beta |1\rangle (\cos(\frac{\theta}{2}) |0\rangle + \sin(\frac{\theta}{2}) |1\rangle) \\ &= \alpha |00\rangle + \beta(\cos(\frac{\theta}{2}) |10\rangle + \sin(\frac{\theta}{2}) |11\rangle) \end{aligned}$$

After the controlled not gate,

$$\alpha |00\rangle + \beta(\cos(\frac{\theta}{2}) |10\rangle + \sin(\frac{\theta}{2}) |01\rangle)$$

This is the effect of amplitude damping, with probability of 1 be switched to 0, or one photon being lost to environment, being $\gamma = \sin^2(\frac{\theta}{2})$.

21. Amplitude damping of a harmonic oscillator

The principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian:

$$H = \chi(a^\dagger b + b^\dagger a)$$

where $a^\dagger, a$ and $b^\dagger, b$ are the creation, annihilation operators for the principal and environment oscillators, respectively.

The time evolution of the coupled system is governed by the unitary operator:

$$U = e^{-iH\Delta t}$$

21.1. Operation elements. We recall some results for the harmonic oscillator:

$$\forall n \in \mathbb{N}, \quad a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

and similarly in the environment space

$$\forall n \in \mathbb{N}, \quad b^\dagger |n\rangle_b = \sqrt{n+1}\,|n+1\rangle_b$$

Here we use the subscript b to differentiate the eigenvectors of the Hermitian operator $bb^\dagger$ which live in the environment space from the eigenvectors of $aa^\dagger$ in the principal space:

$$\begin{aligned}\forall n \in \mathbb{N}, \quad bb^\dagger |n\rangle_b &= (n+1)\,|n\rangle_b \\ \forall n \in \mathbb{N}, \quad aa^\dagger |n\rangle &= (n+1)\,|n\rangle\end{aligned}$$

Each set of vectors constitute an orthonormal basis:

$$\begin{aligned}\forall (n,m) \in \mathbb{N}^2, \quad \langle n|m\rangle &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases} \\ &= \delta_{nm}\end{aligned}$$

We also have

$$\begin{aligned} aa^\dagger - a^\dagger a &= [a, a^\dagger] \\ &= 1 \\ bb^\dagger - b^\dagger b &= [b, b^\dagger] \\ &= 1 \end{aligned}$$

where 1 stands for the identity operator.

Each of the operators $a, a^\dagger$ commutes with each of the operators $b, b^\dagger$ since they act on different spaces

$$\begin{aligned} 0 &= [a^\dagger, b^\dagger] \\ &= [a, b^\dagger] \\ &= [a^\dagger, b] \\ &= [a, b] \end{aligned}$$

The Baker-Campbell-Hausdorff formula states that, for any operators A, G such that e^G exists,

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} C_n$$

where the operators C_n are defined recursively by

$$\begin{aligned} C_0 &= A \\ C_1 &= [G, A] \\ \forall n \in \mathbb{N}, \quad C_{n+1} &= [G, C_n] \end{aligned}$$

Lets compute a simplified expression for the operator $Ua^\dagger U^\dagger$ acting on the product space:

$$\begin{aligned} Ua^\dagger U^\dagger &= e^{-iH\Delta t} a^\dagger e^{iH\Delta t} \\ &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \end{aligned} \tag{1}$$

The first commutators C_n are

$$\begin{aligned}
C_0 &= a^\dagger \\
C_1 &= [H, a^\dagger] \\
&= [\chi b^\dagger a, a^\dagger] \\
&= \chi b^\dagger [a, a^\dagger] \\
&= \chi b^\dagger \\
C_2 &= [H, C_1] \\
&= [\chi a^\dagger b, \chi b^\dagger] \\
&= \chi^2 a^\dagger [b, b^\dagger] \\
&= \chi^2 a^\dagger
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\forall n \in \mathbb{N}, \quad C_{2n} &= \chi^{2n} a^\dagger \\
C_{2n+1} &= \chi^{2n+1} b^\dagger
\end{aligned}$$

We now rewrite equation 1

$$\begin{aligned}
U a^\dagger U^\dagger &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \\
&= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n}}{(2n)!} C_{2n} + \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n+1}}{(2n+1)!} C_{2n+1} \\
&= a^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n}}{(2n)!} + b^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n+1}}{(2n+1)!} \\
&= a^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n}}{(2n)!} - i b^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n+1}}{(2n+1)!} \\
&= \cos(\chi\Delta t) a^\dagger - i \sin(\chi\Delta t) b^\dagger
\end{aligned}$$

Let us now compute the effect of U on $|0\rangle |0\rangle_b = |00\rangle$:

$$\begin{aligned}
U |00\rangle &= e^{-iH\Delta t} |00\rangle \\
&= \sum_{n=0}^{+\infty} \frac{(-iH\Delta t)^n}{n!} |00\rangle
\end{aligned}$$

Since $a |0\rangle = 0$ and $b |0\rangle_b = 0$, we have

$$H |00\rangle = 0$$

and

$$\forall n \in \mathbb{N}^*, \quad , H^n |00\rangle = 0$$

from which it follows there is only one non nul term in the previous sum and

$$U |00\rangle = |00\rangle$$

Let us compute the effect of U on $|1\rangle |0\rangle_b = |10\rangle$:

$$\begin{aligned} U|10\rangle &= Ua^\dagger |00\rangle \\ &= Ua^\dagger \underbrace{U^\dagger U}_{=1} |00\rangle \\ &= Ua^\dagger U^\dagger |00\rangle \\ &= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)|00\rangle \\ &= \cos(\chi\Delta t)|10\rangle - i\sin(\chi\Delta t)|01\rangle \\ &= \cos(\chi\Delta t)|1\rangle|0\rangle_b - i\sin(\chi\Delta t)|0\rangle|1\rangle_b \end{aligned}$$

Similarly,

$$\begin{aligned} \sqrt{n!}U|n\rangle|0\rangle_b &= \sqrt{n!}U|n0\rangle \\ &= U(a^\dagger)^n|00\rangle \\ &= U(a^\dagger)^n U^\dagger U|00\rangle \\ &= (Ua^\dagger U^\dagger)^n|00\rangle \\ &= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)^n|00\rangle \end{aligned}$$

Since $[a^\dagger, b^\dagger] = 0$,

$$\begin{aligned} \sqrt{n!}U|n\rangle|0\rangle_b &= \left(\sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)(a^\dagger)^{n-k}(b^\dagger)^k\right)|00\rangle \\ &= \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)\sqrt{(n-k)!}\sqrt{k!}\,|n-k\rangle|k\rangle_b \end{aligned}$$

so that

$$\begin{aligned} U|n0\rangle &= \sum_{k=0}^{n}\binom{n}{k}\sqrt{\frac{(n-k)!k!}{n!}}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)|n-k\rangle|k\rangle_b \\ &= \sum_{k=0}^{n}\sqrt{\binom{n}{k}}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)|n-k\rangle|k\rangle_b \end{aligned}$$

We can think of the number

$$\binom{n}{k}\cos^{2(n-k)}(\chi\Delta t)\sin^{2k}(\chi\Delta t)$$

as the probability of losing k quanta of energy to the environment.

Let $E_m = \langle m|_b U|0\rangle_b, \quad m \in \mathbb{N}$ the operation elements of U. They are operators acting on the principal space. We can compute the action of E_m on $|n\rangle$ (i.e. compute the nth column of the matrix of E_m) from the previous formula:

$$\begin{aligned} E_m|n\rangle &= (\langle m|_b U|0\rangle_b)|n\rangle \\ &= \langle m|_b(U|n\rangle|0\rangle_b) \\ &= \langle m|_b U|n0\rangle \end{aligned}$$

First it is clear that if $n < m$, $E_m |n\rangle = 0$. Then if $n \geqslant m$,

$$\begin{aligned}
E_m |n\rangle &= \langle m|_b \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t) |n-k\rangle |k\rangle_b \\
&= \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t) |n-k\rangle \underbrace{\langle m|k\rangle_b}_{=\delta_{mk}} \\
&= (-i)^m \sin^m(\chi\Delta t) \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle
\end{aligned}$$

This shows that the matrix of E_m has non nul elements only on the mth superior diagonal. E_m corresponds to the physical process of losing m quanta of energy to the environment.

We can also reconstruct the full formula for E_m using braket calculus:

$$\begin{aligned}
E_m &= E_m \underbrace{\sum_{n=0}^{+\infty} |n\rangle\langle n|}_{=1} \\
&= \sum_{n=0}^{+\infty} E_m |n\rangle\langle n| \\
&= \sum_{n=m}^{+\infty} E_m |n\rangle\langle n| \\
&= (-i)^m \sin^m(\chi\Delta t) \sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle\langle n|
\end{aligned}$$

Note that the sole effect of factor $(-i)^m$ is to add a global phase so it may as well be omitted.

$$E_m = \sin^m(\chi\Delta t) \begin{array}{cc}
\begin{array}{ccccc} \quad m & & & & n \quad \end{array} & \\
\begin{bmatrix}
1 & & & & \\
& \sqrt{\binom{m+1}{m}} \cos(\chi\Delta t) & & & \mathbf{0} \\
& & \sqrt{\binom{m+2}{m}} \cos^2(\chi\Delta t) & & \\
& & & \ddots & \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) \\
& \mathbf{0} & & & \ddots \\
& & & & \\
& & & &
\end{bmatrix} & \begin{array}{c} \\ \\ \\ n-m \\ \\ \\ \\ \end{array}
\end{array}$$

21.2. Trace-preserving property. Matrix calculus or braket calculus show that the matrices $E_m^\dagger E_m$ are diagonals, with the first m elements are 0:

$$\begin{aligned}
E_m^\dagger E_m &= \sin^{2m}(\chi\Delta t)\left(\sum_{n=m}^{+\infty}\sqrt{\binom{n}{m}}\cos^{n-m}(\chi\Delta t)\left|n\right\rangle\left\langle n-m\right|\right)\left(\sum_{l=m}^{+\infty}\sqrt{\binom{l}{m}}\cos^{l-m}(\chi\Delta t)\left|l-m\right\rangle\left\langle l\right|\right)\\
&= \sin^{2m}(\chi\Delta t)\sum_{n=m}^{+\infty}\sum_{l=m}^{+\infty}\sqrt{\binom{n}{m}}\sqrt{\binom{l}{m}}\cos^{n-m}(\chi\Delta t)\cos^{l-m}(\chi\Delta t)\left|n\right\rangle\underbrace{\left\langle n-m|l-m\right\rangle}_{=\delta_{nl}}\left\langle l\right|\\
&= \sin^{2m}(\chi\Delta t)\sum_{n=m}^{+\infty}\binom{n}{m}\cos^{2(n-m)}(\chi\Delta t)\left|n\right\rangle\left\langle n\right|
\end{aligned}$$

$$E_m^\dagger E_m = \sin^{2m}(\chi\Delta t)
\begin{array}{cc}
\begin{array}{cccccc} \phantom{\mathbf{0}} & m & & & n & \end{array} & \\
\left[\begin{array}{cccccc}
\mathbf{0} & & & & \mathbf{0} & \\
 & 1 & & & & \\
 & & \binom{m+1}{m}\cos^2(\chi\Delta t) & & & \\
 & & & \binom{m+1}{m}\cos^4(\chi\Delta t) & & \\
 & & & & \ddots & \\
 & & & & \binom{n}{m}\cos^{2(n-m)}(\chi\Delta t) & \\
\mathbf{0} & & & & & \ddots
\end{array}\right] & \begin{array}{c} \\ m \\ \\ \\ \\ n \\ \\ \end{array}
\end{array}$$

It follows that the operator $\sum_{m=0}^{+\infty} E_m^\dagger E_m$ is also diagonal, and diagonal elements are

$$\begin{aligned}
\left\langle n\right|\sum_{m=0}^{+\infty}E_m^\dagger E_m\left|n\right\rangle &= \sum_{m=0}^{+\infty}\left\langle n\right|E_m^\dagger E_m\left|n\right\rangle\\
&= \sum_{m=0}^{n}\left\langle n\right|E_m^\dagger E_m\left|n\right\rangle\\
&= \sum_{m=0}^{n}\binom{n}{m}\sin^{2m}(\chi\Delta t)\cos^{2(n-m)}(\chi\Delta t)\\
&= (\sin^2(\chi\Delta t)+\cos^2(\chi\Delta t))^n\\
&= 1
\end{aligned}$$

i.e. $\sum_{m=0}^{+\infty} E_m^\dagger E_m = 1$ and the quantum operation is trace-preserving.

22. Amplitude damping of a single qubit density matrix

Let

$$\rho = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

The amplitude damping operation is defined by

$$\varepsilon_{AD}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$$

where

$$
\begin{aligned}
E_0 &= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \\
E_1 &= \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}
\end{aligned} \tag{2}
$$

Straightforward matrix calculus show that

$$
E_0 \rho E_0^\dagger = \begin{bmatrix} a & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}
$$

and

$$
\begin{aligned}
E_1 \rho E_1^\dagger &= \begin{bmatrix} c\gamma & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} (1-a)\gamma & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

because $1 = \operatorname{Tr} \rho = a + c$.

Thus we have

$$
\begin{aligned}
\varepsilon_{AD}(\rho) &= \begin{bmatrix} a + (1-a)\gamma & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} \\
&= \begin{bmatrix} 1 - (1-a)(1-\gamma) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}
\end{aligned} \tag{3}
$$

23. Amplitude damping of dual-rail qubits

Let

$$
|\psi\rangle = a\,|01\rangle + b\,|10\rangle
$$

Applying $\varepsilon_{AD} \otimes \varepsilon_{AD}$ to $\rho = |\psi\rangle\langle\psi|$ is equivalent to applying unitary $B \otimes B$ to $|\psi\rangle$, where $B = e^{\theta(a^\dagger b - a b^\dagger)}$. Let's do this by making explicit the 2 environment qubits initially set to 0, dnoted by subscript b:

$$
|\psi\rangle = a\,|01\rangle\,|00\rangle_b + b\,|10\rangle\,|00\rangle_b
$$

$$
\begin{aligned}
B \otimes B\,|\psi\rangle &= a\,|0\rangle\,|0\rangle_b\,(B\,|1\rangle\,|0\rangle_b) + b(B\,|1\rangle\,|0\rangle_b)\,|0\rangle\,|0\rangle_b \\
&= a\,|0\rangle\,|0\rangle_b\,(\cos(\theta)\,|1\rangle\,|0\rangle_b + \sin(\theta)\,|0\rangle\,|1\rangle_b) + b(\cos(\theta)\,|1\rangle\,|0\rangle_b + \sin(\theta)\,|0\rangle\,|1\rangle_b)\,|0\rangle\,|0\rangle_b \\
&= a\cos(\theta)\,|0\rangle\,|0\rangle_b\,|1\rangle\,|0\rangle_b + a\sin(\theta)\,|0\rangle\,|0\rangle_b\,|0\rangle\,|1\rangle_b + b\cos(\theta)\,|1\rangle\,|0\rangle_b\,|0\rangle\,|0\rangle_b + b\sin(\theta)\,|0\rangle\,|1\rangle_b\,|0\rangle\,|0\rangle_b
\end{aligned}
$$

We reorder the qubits to put the environments qubits at the end since we will trace them out:

$$
\begin{aligned}
B \otimes B\,|\psi\rangle &= a\cos(\theta)\,|01\rangle\,|00\rangle_b + a\sin(\theta)\,|00\rangle\,|01\rangle_b + b\cos(\theta)\,|10\rangle\,|00\rangle_b + b\sin(\theta)\,|00\rangle\,|10\rangle_b \\
&= |\varphi\rangle
\end{aligned} \tag{4}
$$

Now we have to find the dual vector $\langle\varphi|$ of this state. We can recall the not so trivial following facts related to product space: Let $\{|a_i\rangle\}$,$\{|b_j\rangle\}$ be basis of two Hilbert spaces A and B.

The dual of $|a_i b_j\rangle = |a_i\rangle \otimes |b_j\rangle$ is

$$
\langle a_i| \otimes \langle b_j| = \langle a_i b_j|
$$

so that

$$
\langle\varphi| = a^*\cos(\theta)\,\langle 01|\,\langle 00|_b + a^*\sin(\theta)\,\langle 00|\,\langle 01|_b + b^*\cos(\theta)\,\langle 10|\,\langle 00|_b + b^*\sin(\theta)\,\langle 00|\,\langle 10|_b
$$

We have also

$$|a_k b_l\rangle \langle a_i b_j| = |a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|$$

We could then use equation 4 to compute the density $|\varphi\rangle \langle\varphi|$, but this would be a messy sum with 16 terms.

Since we will trace out the environment, we recall the partial trace formula:

$$\begin{aligned}
\mathrm{Tr}_B(|a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|) &= |a_k\rangle \langle a_i| \,\mathrm{Tr}(|b_l\rangle \langle b_j|) \\
&= |a_k\rangle \langle a_i| \langle b_l|b_j\rangle
\end{aligned}$$

Since $\{|00\rangle_b, |01\rangle_b, |10\rangle_b, |11\rangle_b\}$ is an orthonormal basis, there are only 6 out of 16 terms left after the partial trace operation:

$$\begin{aligned}
\mathrm{Tr}_b(|\varphi\rangle \langle\varphi|) &= |a|^2 \cos^2(\theta) |01\rangle \langle 01| + ab^* \cos^2(\theta) |01\rangle \langle 10| + |a|^2 \sin^2(\theta) |00\rangle \langle 00| \\
&\quad + |b|^2 \cos^2(\theta) |10\rangle \langle 10| + ba^* \cos^2(\theta) |10\rangle \langle 01| + |b|^2 \sin^2(\theta) |00\rangle \langle 00| \\
&= |a|^2(1-\gamma) |01\rangle \langle 01| + ab^*(1-\gamma) |01\rangle \langle 10| + |a|^2\gamma |00\rangle \langle 00| \\
&\quad + |b|^2(1-\gamma) |10\rangle \langle 10| + ba^*(1-\gamma) |10\rangle \langle 01| + |b|^2\gamma |00\rangle \langle 00| \\
&= (\underbrace{|a|^2 + |b|^2}_{=1})\gamma |00\rangle \langle 00| + (1-\gamma) \big(|a|^2 |01\rangle \langle 01| + ab^* |01\rangle \langle 10| + |b|^2 |10\rangle \langle 10| + ba^* |10\rangle \langle 01|\big) \\
&= \gamma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + (1-\gamma) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & |a|^2 & ab^* & 0 \\ 0 & a^*b & |b|^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \gamma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + (1-\gamma)\rho
\end{aligned}$$

It is a mixed state:

- with probability γ, the state is projected to $|00\rangle$, orthogonal to $|\psi\rangle$.
- with probability $1-\gamma$, state is unchanged.

Since $|00\rangle$ is orthogonal to $|\psi\rangle$, one can detect amplitude damping errors with measurement operators:

$$\begin{aligned}
M_0 &= |00\rangle \langle 00| \qquad \text{orthogonal projector on } \mathrm{span}\{|00\rangle\} \\
M_1 &= |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11| \qquad \text{orthogonal projector on } \mathrm{span}\{|01\rangle, |10\rangle\ |11\rangle\}
\end{aligned}$$

- If the state decayed to $|00\rangle$, then with probability 1 the result of the measurement will be $|00\rangle$.
- Otherwise, with probability 1 the result of the measurement will be the original $|\psi\rangle$.

It can be easily checked that the quantum operation can be described with 3 operators:

$$\begin{aligned}
E_0^{dr} &= \sqrt{1-\gamma} I \\
E_1^{dr} &= \sqrt{\gamma} |00\rangle \langle 01| \\
&= \sqrt{\gamma} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
E_2^{dr} &= \sqrt{\gamma} |00\rangle \langle 10| \\
&= \sqrt{\gamma} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

It is interesting to see that these operators are the restriction to $\text{span}\{|01\rangle, |10\rangle\}$ of the operators

$$\begin{gathered} E_0 \otimes E_0 \\ E_0 \otimes E_1 \\ E_1 \otimes E_0 \\ E_1 \otimes E_1 \end{gathered}$$

where E_0, E_1 are the operators of amplitude damping for single qubit, defined in 2.

24. Spontaneous emission is amplitude damping

From equation (7.77) in the book, the time evolution of the single atom interacting with single photon is governed by unitary

$$\begin{aligned} U = e^{-i\delta t} |00\rangle \langle 00| &+ (\cos(\Omega t) + i\frac{\delta}{\Omega}\sin(\Omega t)) |01\rangle \langle 01| \\ &+ (\cos(\Omega t) - i\frac{\delta}{\Omega}\sin(\Omega t)) |10\rangle \langle 10| - i\frac{g}{\Omega}\sin(\Omega t)(|01\rangle \langle 10| + |10\rangle \langle 01|) \end{aligned}$$

the left label corresponds to the electric field, the right label corresponds to the atom. The derivation of this formula from the Hamiltonian can be found in appendix A.

The *Rabi frequency* is

$$\Omega = \sqrt{g^2 + \delta^2}$$

If we set $\delta = 0$ and if $g > 0$, then $\Omega = g$ and

$$\begin{aligned} U = |00\rangle \langle 00| &+ \cos(\Omega t)(|01\rangle \langle 01| + |10\rangle \langle 10|) \\ &- i\sin(\Omega t)(|01\rangle \langle 10| + |10\rangle \langle 01|) \end{aligned}$$

Let us apply U to

$$\begin{aligned} |\psi\rangle &= |0\rangle (a|0\rangle + b|1\rangle) \\ &= a|00\rangle + b|01\rangle) \end{aligned}$$

We find

$$\begin{aligned} U|\psi\rangle &= a|00\rangle + b(\cos(\Omega t)|01\rangle - i\sin(\Omega t)|10\rangle) \\ &= |\varphi\rangle \end{aligned} \tag{5}$$

Now we have to find the dual vector $\langle\varphi|$ of this state. We can recall the not so trivial following facts related to product space: Let $\{|a_i\rangle\}$,$\{|b_j\rangle\}$ be basis of two Hilbert spaces A and B.

The dual of $|a_i b_j\rangle = |a_i\rangle \otimes |b_j\rangle$ is

$$\langle a_i| \otimes \langle b_j| = \langle a_i b_j|$$

so that

$$\langle\varphi| = a\langle 00| + b^*(\cos(\Omega t)\langle 01| + i\sin(\Omega t)\langle 10|)$$

We have also

$$|a_k b_l\rangle \langle a_i b_j| = |a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|$$

We could then use equation 5 to compute the density $|\varphi\rangle\langle\varphi|$, but this would be a ugly sum with 9 terms.

Since we will trace out the photon space, we recall the partial trace formula:

$$\begin{aligned}\mathrm{Tr}_B(|a_k\rangle\langle a_i|\otimes|b_l\rangle\langle b_j|) &= |a_k\rangle\langle a_i|\,\mathrm{Tr}(|b_l\rangle\langle b_j|)\\ &= |a_k\rangle\langle a_i|\langle b_l|b_j\rangle\end{aligned}$$

Since $\{|0\rangle, |1\rangle\}$ is an orthonormal basis of the state space A of the photon, there are only 5 out of 9 terms left after the partial trace operation over the photon (those where the bit for the photon is the same in the ket and in the bra):

$$\begin{aligned}\mathrm{Tr}_A(|\varphi\rangle\langle\varphi|) &= (|a|^2+|b|^2\sin^2(\Omega t))\,|0\rangle\langle 0| + ab^*\cos(\Omega t)\,|0\rangle\langle 1|\\ &\quad + a^*b\cos(\Omega t)\,|1\rangle\langle 0| + |b|^2\cos^2(\Omega t)\\ &= \begin{bmatrix}|a|^2+(1-|a|^2)\gamma & ab^*\sqrt{1-\gamma}\\ a^*b\sqrt{1-\gamma} & |b|^2(1-\gamma)\end{bmatrix}\end{aligned}$$

with $\gamma = \sin^2(\Omega t)$. Now compare with equation 3 and recall that

$$\rho = \begin{bmatrix}|a|^2 & ab^*\\ a^*b & |b|^2\end{bmatrix}$$

to see that this is indeed the amplitude damping operation.

25.

We consider the density operator

$$\rho = \begin{bmatrix}p & 0\\ 0 & 1-p\end{bmatrix}$$

The qubit is in state $|0\rangle$ with probability $p_0 = p$ and in state $|1\rangle$ with probability $p_1 = 1-p$.

Let us compute T as a function of E_0, E_1 and p:

$$\begin{aligned}\mathcal{Z} = \frac{e^{-\frac{E_0}{k_BT}}}{p} &= e^{-\frac{E_0}{k_BT}} + e^{-\frac{E_1}{k_BT}}\\ \Leftrightarrow\qquad \frac{1}{p} &= 1 + e^{-\frac{E_1-E_0}{k_BT}}\\ \Leftrightarrow\qquad \frac{1}{p} - 1 &= e^{-\frac{E_1-E_0}{k_BT}}\\ \Leftrightarrow\qquad \frac{1}{p} - 1 &= e^{-\frac{E_1-E_0}{k_BT}}\\ \Leftrightarrow\qquad -\frac{E_1-E_0}{k_BT} &= \ln(\frac{1-p}{p})\\ \Leftrightarrow\qquad T &= -\frac{1}{k_B}\frac{E_1-E_0}{\ln(\frac{1-p}{p})}\end{aligned}$$

Assuming $E_1 > E_0$,

- the regular amplitude damping case corresponds to $T \to 0^+$, $p = 1$.
- When $T \to +\infty$, $p \to \frac{1}{2}$.

Appendix A. Derivation of the formula of unitary evolution for atom photon interaction

We consider a system formed by a two-level atom and a cavity confined electric field. The Jaynes-Cummings Hamiltonian is

$$H = \delta Z + g(a\sigma_- + a^\dagger \sigma_+)$$

where g is some constant which describes the strength of the interaction, $\delta = \frac{\omega - \omega_0}{2}$ is the *detuning*, $a^\dagger, a$ are respectively the creation,annihilation operators [1] on the single mode field, and $\sigma_\pm$ are operators acting on the two-level atom, namely:

$$\begin{aligned} \sigma_+ &= \frac{1}{2}(X + iY) \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \sigma_- &= \frac{1}{2}(X - iY) \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

We recall

$$\begin{aligned} \forall n \in \mathbb{N}, \quad a^\dagger |n\rangle &= \sqrt{n+1}\,|n+1\rangle \\ a\,|n+1\rangle &= \sqrt{n+1}\,|n\rangle \\ a\,|0\rangle &= 0 \end{aligned}$$

The first label corresponding to electric field, the second to atom, we have:

$$\begin{aligned} Z\,|00\rangle &= |00\rangle \\ Z\,|01\rangle &= -\,|01\rangle \\ Z\,|10\rangle &= |10\rangle \\ a\sigma_-\,|00\rangle &= 0 \\ a\sigma_-\,|01\rangle &= 0 \\ a\sigma_-\,|10\rangle &= |01\rangle \\ a^\dagger\sigma_+\,|00\rangle &= 0 \\ a^\dagger\sigma_+\,|01\rangle &= |10\rangle \\ a^\dagger\sigma_+\,|10\rangle &= 0 \end{aligned}$$

This shows that $F = \text{span}\{|00\rangle, |01\rangle, |10\rangle\}$ is an invariant subspace for H, i.e. $H(F) \subset F$. The same is true for $H^n, n \in \mathbb{N}$ and $U = e^{-iH\Delta t} = \sum \frac{(-i\Delta t)^n}{n!} H^n$.

Let's find the representation of H in the basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$.

[1] It seems to me the book mixes up $a^\dagger$ with a in several places.

The representation of $Z = I \otimes Z$ is

$$\boldsymbol{Z} \qquad I \otimes Z = \begin{bmatrix} 0\boldsymbol{Z} & 1\boldsymbol{Z} \\ & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{Z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The representation of annihilation operator in the $(|n\rangle)_{n\in\mathbb{N}}$ basis of the electric field state space is

$$\boldsymbol{0} \qquad a = \begin{array}{c c} & n \\ 1\boldsymbol{Z} \; 0\boldsymbol{\sigma}_- & \begin{bmatrix} 0 & 1 & & & & & \\ \vdots & & \sqrt{2} & & & & \\ & & & \sqrt{3} & & & \\ & & & & \sqrt{4} & & \\ & & & & & \ddots & \\ & & & & & \sqrt{n} & \\ \boldsymbol{0} & & & & 0\boldsymbol{Z} & & \ddots \\ \vdots & & & & 1\boldsymbol{\sigma}_- & & \end{bmatrix} & n-1 \end{array}$$

The representation of $a \otimes \sigma_-$ is then

$$\boldsymbol{0} \qquad a \otimes \sigma_- = \begin{bmatrix} 0\boldsymbol{\sigma}_- & 0\boldsymbol{\sigma}_- \\ & \boldsymbol{\sigma}_- \end{bmatrix} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly,

$$
\begin{aligned}
a^\dagger \otimes \sigma_+ &= \begin{bmatrix} \mathbf{0} & \\ 1\boldsymbol{\sigma}_+ & 0\boldsymbol{\sigma}_+ \end{bmatrix} \quad \mathbf{0} \\
&= \begin{bmatrix} & \\ \boldsymbol{\sigma}_+ & \mathbf{0} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Thus the representation of H in the basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$ is

$$
H = \begin{bmatrix} \delta & 0 & 0 & 0 \\ 0 & -\delta & g & 0 \\ 0 & g & \delta & 0 \\ 0 & 0 & 0 & -\delta \end{bmatrix}
$$

The representation of the restriction of H in the basis $(|00\rangle, |01\rangle, |10\rangle)$ is

$$
\begin{aligned}
H &= \begin{bmatrix} \delta & 0 & 0 \\ 0 & -\delta & g \\ 0 & g & \delta \end{bmatrix} \quad 0\boldsymbol{\sigma}_+ \quad 0\boldsymbol{\sigma}_- \\
&= \begin{bmatrix} \delta & 0 & 0 \\ 0 & \boldsymbol{H_1} & \\ 0 & & \end{bmatrix}
\end{aligned}
$$

Block calculus shows that

$$
e^{-iH\Delta t} = \begin{bmatrix} e^{-i\delta\Delta t} & 0 & 0 \\ 0 & \boldsymbol{e^{-iH_1\Delta t}} & \\ 0 & & \end{bmatrix}
$$

Let $\Omega = \sqrt{g^2 + \delta^2}$, the *Rabi frequency.*

$$
\begin{aligned}
H_1^2 &= \begin{bmatrix} \Omega^2 & 0 \\ 0 & \Omega^2 \end{bmatrix} \\
&= \Omega^2 I_2
\end{aligned}
$$

This shows that

$$
\begin{aligned}
\forall n \in \mathbb{N}, \quad H_1^{2n} &= \Omega^{2n} I_2 \\
H_1^{2n+1} &= \Omega^{2n} H_1
\end{aligned}
$$

Then,

$$\begin{aligned} e^{-iH_1\Delta t} &= \sum_{n=0}^{+\infty} \frac{(-i\Omega\Delta t)^{2n}}{(2n)!} I_2 + \frac{1}{\Omega} \sum_{n=0}^{+\infty} \frac{(-i\Omega\Delta t)^{2n+1}}{(2n+1)!} H_1 \\ &= \sum_{n=0}^{+\infty} (-1)^n \frac{(\Omega\Delta t)^{2n}}{(2n)!} I_2 - i\frac{1}{\Omega} \sum_{n=0}^{+\infty} (-1)^n \frac{(\Omega\Delta t)^{2n+1}}{(2n+1)!} H_1 \\ &= \cos(\Omega t) I_2 - i\frac{1}{\Omega} \sin(\Omega t) H_1 \end{aligned}$$

Finally the matrix U is

$$e^{-iH\Delta t} = \begin{array}{cc} \begin{array}{ccc} U\,|00\rangle & U\,|01\rangle & U\,|10\rangle \end{array} & \\ \begin{bmatrix} e^{-i\delta\Delta t} & 0 & 0 \\ 0 & \cos(\Omega t) + i\frac{\delta}{\Omega}\sin(\Omega t) & -i\frac{g}{\Omega}\sin(\Omega t) \\ 0 & -i\frac{g}{\Omega}\sin(\Omega t) & \cos(\Omega t) - i\frac{\delta}{\Omega}\sin(\Omega t) \end{bmatrix} & \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \end{array} \end{array}$$