

# QUANTUM COMPUTATION AND QUANTUM INFORMATION: THE QUANTUM FOURIER TRANSFORM

## 1

We consider the linear map in  $\mathbb{C}^N$  which acts on the computational basis as

$$|j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2i\pi jk}{N}} |k\rangle$$

Let  $A$  be the matrix of the transformation in the computational basis.

$$\forall (k, l) \in \llbracket 0, N-1 \rrbracket^2, \quad a_{kl} = \frac{1}{\sqrt{N}} e^{\frac{2i\pi kl}{N}}$$

The adjoint matrix  $A^\dagger$  is then

$$\begin{aligned} \forall (k, l) \in \llbracket 0, N-1 \rrbracket^2, \quad b_{kl} &= a_{lk}^* \\ &= \frac{1}{\sqrt{N}} e^{-\frac{2i\pi kl}{N}} \end{aligned}$$

We compute the coefficient  $k, l$  of the product  $AA^\dagger$ :

$$\begin{aligned} \forall (k, l) \in \llbracket 0, N-1 \rrbracket^2, \quad c_{kl} &= \sum_{j=0}^{N-1} a_{kj} b_{jl} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} e^{\frac{2i\pi j}{N} (k-l)} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} (e^{\frac{2i\pi}{N} (k-l)})^j \\ &= \delta_{kl} \end{aligned}$$

which shows that  $AA^\dagger = I$  i.e.  $A$  is unitary.

## 2

Here the dimension  $N = 2^n$ . The Fourier transform of the  $n$  qubit state  $|00\dots 0\rangle$  is

$$A|0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle$$

or in product representation,

$$= \frac{1}{2^{n/2}} \underbrace{(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \dots (|0\rangle + |1\rangle)}_{n \text{ qubits}}$$