QUANTUM COMPUTATION AND QUANTUM INFORMATION: QUANTUM COMPUTERS: PHYSICAL REALIZATION

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17. Eigenstates of the Jaynes-Cummings Hamiltonian

We consider a system formed by a two-level atom and a cavity confined electric field. The Hamiltonian is

$$H = g(a\sigma_- + a^\dagger\sigma_+)$$

where g is some constant which describes the strength of the interaction, $a^\dagger, a$ are respectively the creation,annihilation operators [1] on the single mode field, and $\sigma_\pm$ are operators acting on the two-level atom, namely:

$$\begin{aligned}
\sigma_+ &= \frac{1}{2}(X + iY) \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
\sigma_- &= \frac{1}{2}(X - iY) \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

We recall

$$\begin{aligned}
\forall n \in \mathbb{N}, \quad a^\dagger |n\rangle &= \sqrt{n+1}\, |n+1\rangle \\
a\, |n+1\rangle &= \sqrt{n+1}\, |n\rangle \\
a\, |0\rangle &= 0
\end{aligned}$$

We define the vectors

$$\begin{aligned}
|\chi_n\rangle &= \frac{1}{\sqrt{2}}(|n,1\rangle + |n+1,0\rangle) \\
|\overline{\chi}_n\rangle &= \frac{1}{\sqrt{2}}(|n,1\rangle - |n+1,0\rangle)
\end{aligned}$$

Let's apply the Hamiltonian

$$\begin{aligned}
H\,|\chi_n\rangle &= \frac{g}{\sqrt{2}}(a\sigma_- + a^\dagger\sigma_+)(|n,1\rangle + |n+1,0\rangle) \\
&= \frac{g}{\sqrt{2}}(a\,|n\rangle\, \sigma_-\, |1\rangle + a^\dagger\, |n\rangle\, \sigma_+\, |1\rangle + a\, |n+1\rangle\, \sigma_-\, |0\rangle + a^\dagger\, |n+1\rangle\, \sigma_+\, |0\rangle) \\
&= \frac{g}{\sqrt{2}}\sqrt{n+1}(|n+1\rangle\, |0\rangle + |n\rangle\, |1\rangle) \\
&= \frac{g}{\sqrt{2}}\sqrt{n+1}(|n+1,0\rangle + |n,1\rangle) \\
H\,|\chi_n\rangle &= g\sqrt{n+1}\, |\chi_n\rangle
\end{aligned}$$

and

[1] It seems to me the book mixes up $a^\dagger$ and a in several places.

$$
\begin{aligned}
H\left|\overline{\chi}_n\right\rangle &= \frac{g}{\sqrt{2}}(a\sigma_- + a^\dagger\sigma_+)(|n,1\rangle - |n+1,0\rangle) \\
&= \frac{g}{\sqrt{2}}(a\left|n\right\rangle\sigma_-\left|1\right\rangle + a^\dagger\left|n\right\rangle\sigma_+\left|1\right\rangle - a\left|n+1\right\rangle\sigma_-\left|0\right\rangle - a^\dagger\left|n+1\right\rangle\sigma_+\left|0\right\rangle) \\
&= \frac{g}{\sqrt{2}}\sqrt{n+1}(\left|n+1\right\rangle\left|0\right\rangle - \left|n\right\rangle\left|1\right\rangle) \\
&= \frac{g}{\sqrt{2}}\sqrt{n+1}(|n+1,0\rangle - |n,1\rangle) \\
H\left|\overline{\chi}_n\right\rangle &= -g\sqrt{n+1}\left|\overline{\chi}_n\right\rangle
\end{aligned}
$$

It proves that $|\chi_n\rangle$ and $|\overline{\chi}_n\rangle$ are eigenvectors of the Hamiltonian, with eigenvalues $g\sqrt{n+1}$ and $-g\sqrt{n+1}$, respectively.