

QUANTUM COMPUTATION AND QUANTUM INFORMATION: QUANTUM NOISE AND QUANTUM OPERATIONS

21. Amplitude damping of a harmonic oscillator.

The principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian:

$$H = \chi(a^\dagger b + b^\dagger a)$$

where a^\dagger, a and b^\dagger, b are the creation, annihilation operators for the principal and environment oscillators, respectively.

The time evolution of the coupled system is governed by the unitary operator:

$$U = e^{-iH\Delta t}$$

21.1. Operation elements. We recall some results for the harmonic oscillator:

$$\forall n \in \mathbb{N}, \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

and similarly in the environment space

$$\forall n \in \mathbb{N}, \quad b^\dagger |n\rangle_b = \sqrt{n+1} |n+1\rangle_b$$

Here we use the subscript b to differentiate the eigenvectors of the Hermitian operator bb^\dagger which live in the environment space from the eigenvectors of aa^\dagger in the principal space:

$$\begin{aligned} \forall n \in \mathbb{N}, \quad bb^\dagger |n\rangle_b &= (n+1) |n\rangle_b \\ \forall n \in \mathbb{N}, \quad aa^\dagger |n\rangle &= (n+1) |n\rangle \end{aligned}$$

Each set of vectors constitute an orthonormal basis:

$$\begin{aligned} \forall (n, m) \in \mathbb{N}^2, \quad \langle n|m \rangle &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases} \\ &= \delta_{nm} \end{aligned}$$

We also have

$$\begin{aligned} aa^\dagger - a^\dagger a &= [a, a^\dagger] \\ &= 1 \\ bb^\dagger - b^\dagger b &= [b, b^\dagger] \\ &= 1 \end{aligned}$$

where 1 stands for the identity operator.

Each of the operators a, a^\dagger commutes with each of the operators b, b^\dagger since they act on different spaces

$$\begin{aligned} 0 &= [a, b] \\ &= [a^\dagger, b] \\ &= [a^\dagger, b^\dagger] \\ &= [a, b^\dagger] \end{aligned}$$

The Baker-Campbell-Hausdorff formula states that, for any operators A, G such that e^G exists,

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} C_n$$

where the operators C_n are defined recursively by

$$\begin{aligned} C_0 &= A \\ C_1 &= GA - AG \\ &= [G, A] \\ \forall n \in \mathbb{N}^*, \quad C_{n+1} &= [G, C_n] \end{aligned}$$

Let's compute a simplified expression for the operator $U a^\dagger U^\dagger$ acting on the product space:

$$\begin{aligned} U a^\dagger U^\dagger &= e^{-iH\Delta t} a^\dagger e^{iH\Delta t} \\ &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \end{aligned}$$

The first commutators C_n are

$$\begin{aligned} C_0 &= a^\dagger \\ C_1 &= [H, a^\dagger] \\ &= [\chi b^\dagger a, a^\dagger] \\ &= \chi b^\dagger [a, a^\dagger] \\ &= \chi b^\dagger \\ C_2 &= [H, C_1] \\ &= [\chi a^\dagger b, \chi b^\dagger] \\ &= \chi^2 a^\dagger [b, b^\dagger] \\ &= \chi^2 a^\dagger \end{aligned}$$

from which it follows that

$$\begin{aligned} \forall n \in \mathbb{N}, \quad C_{2n} &= \chi^{2n} a^\dagger \\ C_{2n+1} &= \chi^{2n+1} b^\dagger \end{aligned}$$

Thus we have

$$\begin{aligned} U a^\dagger U^\dagger &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \\ &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n}}{2n!} C_{2n} + \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n+1}}{2n!} C_{2n+1} \\ &= a^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n}}{2n!} + b^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n+1}}{2n!} \\ &= a^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n}}{2n!} - i b^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n+1}}{2n!} \\ &= \cos(\chi\Delta t) a^\dagger - i \sin(\chi\Delta t) b^\dagger \end{aligned}$$

Let us compute the effect of U on $|0\rangle|0\rangle_b = |00\rangle$:

$$\begin{aligned} U|00\rangle &= e^{-iH\Delta t} |00\rangle \\ &= \sum_{k=0}^{+\infty} \frac{(-iH\Delta t)^k}{k!} |00\rangle \end{aligned}$$

Since $a|0\rangle = 0$ and $b|0\rangle_b = 0$, we have

$$H|00\rangle = 0$$

and

$$\forall n \in \mathbb{N}^*, \quad H^n|00\rangle = 0$$

from which it follows

$$U|00\rangle = |00\rangle$$

Let us compute the effect of U on $|1\rangle|0\rangle_b = |10\rangle$:

$$\begin{aligned} U|10\rangle &= Ua^\dagger|00\rangle \\ &= Ua^\dagger \underbrace{U^\dagger U}_{=1}|00\rangle \\ &= Ua^\dagger U^\dagger|00\rangle \\ &= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)|00\rangle \\ &= \cos(\chi\Delta t)|10\rangle - i\sin(\chi\Delta t)b^\dagger|01\rangle \\ &= \cos(\chi\Delta t)|1\rangle|0\rangle_b - i\sin(\chi\Delta t)|0\rangle|1\rangle_b \end{aligned}$$

Similarly,

$$\begin{aligned} \sqrt{n!}U|n0\rangle &= U(a^\dagger)^n|00\rangle \\ &= U(a^\dagger)^n U^\dagger U|00\rangle \\ &= (Ua^\dagger U^\dagger)^n|00\rangle \\ &= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)^n|00\rangle \end{aligned}$$

Since $[a^\dagger, b^\dagger] = 0$,

$$\begin{aligned} \sqrt{n!}U|n0\rangle &= \left(\sum_{k=0}^n \binom{n}{k} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) (a^\dagger)^{n-k} (b^\dagger)^k \right) |00\rangle \\ &= \sum_{k=0}^n \binom{n}{k} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) \sqrt{(n-k)!k!} |n-k\rangle|k\rangle_b \end{aligned}$$

so that

$$\begin{aligned} U|n0\rangle &= \sum_{k=0}^n \binom{n}{k} \sqrt{\frac{(n-k)!k!}{n!}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle|k\rangle_b \\ &= \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle|k\rangle_b \end{aligned}$$

Let $E_m = \langle m|_b U|0\rangle_b$, $m \in \mathbb{N}$ the operations elements of U . They are operators acting on the principal space. We can compute the action of E_p on $|n\rangle$ (i.e. compute the n th column of the matrix of E_p) from the previous formula:

$$\begin{aligned} E_m|n\rangle &= (\langle m|_b U|0\rangle_b)|n\rangle \\ &= \langle m|_b (U|n\rangle|0\rangle_b) \\ &= \langle m|_b U|n0\rangle \end{aligned}$$

First it is clear that if $n < m$, $E_m |n\rangle = 0$. Then if $n \geq m$,

$$\begin{aligned} E_m |n\rangle &= \langle m |_b \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle |k\rangle_b \\ &= (-i)^m \sin^m(\chi\Delta t) \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle \end{aligned}$$

We can obtain again the full formula for E_m using bracket calculus:

$$\begin{aligned} E_m &= E_m \underbrace{\sum_{n=0}^{+\infty} |n\rangle \langle n|}_{=1} \\ &= \sum_{n=0}^{+\infty} E_m |n\rangle \langle n| \\ &= \sum_{n=m}^{+\infty} E_m |n\rangle \langle n| \\ &= (-i)^m \sin^m(\chi\Delta t) \sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle \langle n| \end{aligned}$$

Note that the factor $(-i)^m$ only has the effect of adding a global phase so it may as well be omitted.