

# QUANTUM COMPUTATION AND QUANTUM INFORMATION: THE QUANTUM FOURIER TRANSFORM

1.

We consider the linear map in  $\mathbb{C}^N$  which acts on the computational basis as

$$|j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2i\pi jk}{N}} |k\rangle$$

Let  $A$  be the matrix of the transformation in the computational basis.

$$\forall (k, l) \in \llbracket 0, N-1 \rrbracket^2, \quad a_{kl} = \frac{1}{\sqrt{N}} e^{\frac{2i\pi kl}{N}}$$

The adjoint matrix  $A^\dagger$  is then

$$\begin{aligned} \forall (k, l) \in \llbracket 0, N-1 \rrbracket^2, \quad b_{kl} &= a_{lk}^* \\ &= \frac{1}{\sqrt{N}} e^{-\frac{2i\pi kl}{N}} \end{aligned}$$

We compute the coefficient  $k, l$  of the product  $AA^\dagger$ :

$$\begin{aligned} \forall (k, l) \in \llbracket 0, N-1 \rrbracket^2, \quad c_{kl} &= \sum_{j=0}^{N-1} a_{kj} b_{jl} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} e^{\frac{2i\pi j}{N} (k-l)} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} (e^{\frac{2i\pi}{N} (k-l)})^j \\ &= \begin{cases} \frac{1}{N} \frac{1 - (e^{\frac{2i\pi}{N} (k-l)})^N}{1 - e^{\frac{2i\pi}{N} (k-l)}} = 0 & \text{if } e^{\frac{2i\pi}{N} (k-l)} \neq 1, \\ 1 & \text{if } e^{\frac{2i\pi}{N} (k-l)} = 1. \end{cases} \\ &= \begin{cases} 0 & \text{if } k \neq l, \\ 1 & \text{if } k = l. \end{cases} \\ &= \delta_{kl} \end{aligned}$$

which shows that  $AA^\dagger = A^\dagger A = I$  i.e.  $A$  is unitary.

2.

Here the dimension of the state space is  $N = 2^n$ . The Fourier transform of the  $n$  qubit state  $|00 \dots 0\rangle$  is

$$A|0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle$$

we can write  $k$  in binary  $k_{n-1} \dots k_1 k_0$

$$A|0\rangle = \frac{1}{2^{n/2}} \sum_{k_0, k_1, \dots, k_{n-1}=0}^1 |k_{n-1} \dots k_1 k_0\rangle$$

or in product representation,

$$= \frac{1}{2^{n/2}} \underbrace{(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \dots (|0\rangle + |1\rangle)}_{n \text{ qubits}}$$

## 3.

Let  $N = 2^n$  and  $Y = (y_k)_{k \in \llbracket 0, N-1 \rrbracket}$  be the classical fourier transform of  $X = (x_k)_{k \in \llbracket 0, N-1 \rrbracket}$ .

$$\forall k \in \llbracket 0, N-1 \rrbracket, \quad y_k = \sum_{j=0}^{N-1} e^{\frac{2i\pi k j}{2^n}} x_j$$

The factor  $\frac{1}{\sqrt{N}}$  is omitted for clarity. We can write  $j$  in binary  $j_{n-1} \dots j_1 j_0$

$$\begin{aligned} y_k &= \sum_{j_0, j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-1}j_{n-1} + \dots + 2j_1 + j_0)}{2^n}} x_j \\ &= \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-1}j_{n-1} + \dots + 2j_1)}{2^n}} x_{j_{n-1} \dots j_1 0} + \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-1}j_{n-1} + \dots + 2j_1 + 1)}{2^n}} x_{j_{n-1} \dots j_1 1} \\ &= \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-1}j_{n-1} + \dots + 2j_1)}{2^n}} x_{j_{n-1} \dots j_1 0} + e^{\frac{2i\pi k}{2^n}} \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-1}j_{n-1} + \dots + 2j_1)}{2^n}} x_{j_{n-1} \dots j_1 1} \\ &= \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-2}j_{n-1} + \dots + j_1)}{2^{n-1}}} x_{j_{n-1} \dots j_1 0} + e^{\frac{2i\pi k}{2^n}} \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-2}j_{n-1} + \dots + j_1)}{2^{n-1}}} x_{j_{n-1} \dots j_1 1} \end{aligned}$$

We see the first sum is the  $k^{th}$  coefficient of the FT of the sequence  $(x_{2k})_{k \in \llbracket 0, N/2-1 \rrbracket}$  and the second is the  $k^{th}$  coefficient of the FT of  $(x_{2k+1})_{k \in \llbracket 0, N/2-1 \rrbracket}$ . This shows that to compute FT of sequence of length  $N$ , we have to compute 2 FT of sequence of length  $\frac{N}{2}$  and do  $2N$  complex additions/multiplications. The complexity of the operation  $T(N)$  follows the recurrence:

$$T(N) = 2T\left(\frac{N}{2}\right) + 2N$$

We can use the Master theorem [1]:

**Theorem.** Let  $a \geq 1$  and  $b > 1$  be constants, let  $f(n)$  be a function, and let  $T(n)$  be defined on the non negative integers by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where we interpret  $\frac{n}{b}$  to mean either  $\lfloor \frac{n}{b} \rfloor$  or  $\lceil \frac{n}{b} \rceil$ . Then  $T(n)$  has the following asymptotic bounds:

- (1) If  $f(n) = O(n^{\log_b a - \epsilon})$  for some constant  $\epsilon > 0$ , then  $T(n) = \Theta(n^{\log_b a})$ .
- (2) If  $f(n) = \Theta(n^{\log_b a})$ , then  $T(n) = \Theta(n^{\log_b a} \log n)$ .
- (3) If  $f(n) = \Omega(n^{\log_b a + \epsilon})$  for some constant  $\epsilon > 0$ , and if  $af(\frac{n}{b}) \leq cf(n)$  for some constant  $c < 1$  and  $n$  sufficiently large, then  $T(n) = \Theta(f(n))$ .

Here we are in the second case of the theorem, so  $T(N) = \Theta(N \log(N)) = \Theta(n 2^n)$ .

Instead of  $\mathbb{C}$ , the Fourier transform may be used in any ring as soon as we are given a  $N$ th root of unity. The book *The design and analysis of computer algorithms* [2] provides an overview of the FFT, an algorithm using bits operations and application to fast integer multiplication.

## 5.

The inverse Fourier Transform

$$|j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-\frac{2i\pi j k}{N}} |k\rangle$$

is the adjoint of the Fourier Transform. The quantum circuit of figure 1 is obtained from the FT's circuit, replacing each  $R_k$  gate by its adjoint

$$R_k^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{2i\pi}{2^k}} \end{bmatrix}$$



FIGURE 1. Quantum circuit for IFT.

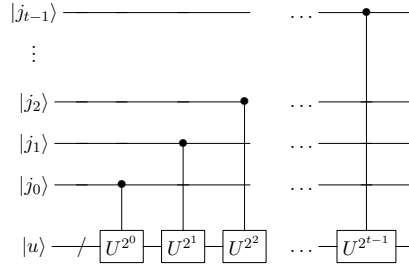


FIGURE 2. Sequence of controlled U.

7.

In figure 2, the  $t$  qubits of the first register are prepared with  $|j\rangle = |j_{t-1} \dots j_1 j_0\rangle$ , the second register is prepared with some state  $|u\rangle$ . After the first controlled-U operation, the state is  $|j\rangle |U^{j_0 2^0} u\rangle$ . After the second controlled-U, the state is  $|j\rangle |U^{j_1 2^1} U^{j_0 2^0} u\rangle = |j\rangle |U^{j_0 2^0 + j_1 2^1} u\rangle$  and so on. The final state is  $|j\rangle |U^{j_0 2^0 + j_1 2^1 + \dots + j_{t-1} 2^{t-1}} u\rangle = |j\rangle |U^j u\rangle$ .

8.

By linearity, the phase estimation algorithm takes input  $|0\rangle |\Sigma_{u \in A} c_u |u\rangle\rangle$ , where  $A$  is some orthonormal basis of eigenstates of  $U$ , to output  $\sum_{u \in A} c_u |\widetilde{\varphi}_u\rangle |u\rangle$ , where  $\widetilde{\varphi}_u$  is an estimation of the phase of the eigenvalue associated with eigenstate  $u$ . If we fix  $u_0 \in A$  beforehand, the probability to measure  $\widetilde{\varphi}_{u_0}$  when measuring the first register in the computational basis is

$$\begin{aligned}
 & \left( \sum_{u \in A} c_u^* \langle \widetilde{\varphi}_u | \langle u | \right) P_{\widetilde{\varphi}_{u_0}} \otimes I \left( \sum_{u \in A} c_u |\widetilde{\varphi}_u\rangle |u\rangle \right) = \left( \sum_{u \in A} c_u^* \langle \widetilde{\varphi}_u | \langle u | \right) \left( \sum_{\substack{u \in A \\ \widetilde{\varphi}_u = \widetilde{\varphi}_{u_0}}} c_u |\widetilde{\varphi}_u\rangle |u\rangle \right) \\
 & = \left( \sum_{u \in A} c_u^* \langle \widetilde{\varphi}_u | \langle u | \right) \left( \sum_{\substack{u \in A \\ \widetilde{\varphi}_u = \widetilde{\varphi}_{u_0}}} c_u |\widetilde{\varphi}_{u_0}\rangle |u\rangle \right) \\
 & = \sum_{\substack{v \in A \\ u \in A \\ \widetilde{\varphi}_u = \widetilde{\varphi}_{u_0}}} c_v^* c_u \langle \widetilde{\varphi}_v | \widetilde{\varphi}_u \rangle \langle v | u \rangle \\
 & = \sum_{\substack{v \in A \\ u \in A \\ \widetilde{\varphi}_u = \widetilde{\varphi}_{u_0}}} c_v^* c_u \langle \widetilde{\varphi}_v | \widetilde{\varphi}_u \rangle \delta_{vu} \\
 & = \sum_{\substack{u \in A \\ \widetilde{\varphi}_u = \widetilde{\varphi}_{u_0}}} |c_u|^2 \\
 & \geq |c_{u_0}|^2
 \end{aligned}$$

FIGURE 3. Phase estimation circuit with  $t = 1$ .

$I$  is the identity operator of whatever state space  $U$  operates on, while  $P_{\widetilde{\varphi_{u_0}}}$  is the orthonormal projector onto the space generated by the vector  $|\widetilde{\varphi_{u_0}}\rangle$  of the computational basis. Besides, following the analysis of the book,  $\widetilde{\varphi_{u_0}}$  is an approximation to  $\varphi_{u_0}$  to an accuracy  $2^{-n}$  with probability at least  $1 - \epsilon$  if we make use of  $t = n + \lceil \log(2 + \frac{1}{2\epsilon}) \rceil$  bits in the first register. We conclude we get the desired approximation of  $\varphi_{u_0}$  at the end of the phase estimation algorithm with probability at least  $|c_{u_0}|^2(1 - \epsilon)$ .

## 9.

$U$  being unitary with eigenvalues  $-1$  and  $+1$ , the state space is the direct sum of the two orthogonal eigenspaces  $E_{-1} \oplus E_1$ . Thus we can uniquely decompose any  $|\psi\rangle = |\psi_{-1}\rangle + |\psi_{+1}\rangle$ , with  $|\psi_{-1}\rangle \in E_{-1}$  and  $|\psi_{+1}\rangle \in E_{+1}$ . Then  $-1 = e^{i\pi} = e^{2i\pi 0.1}$  and  $1 = e^0 = e^{2i\pi 0.0}$  shows that it is sufficient to make use of  $t = 1$  wire in the first register in the phase estimation procedure to read directly the phase of any eigenvector. If we use  $|0\rangle |\psi\rangle$  as input in the circuit of figure 3, the output before the final measurement will be  $|0\rangle |\psi_{+1}\rangle + |1\rangle |\psi_{-1}\rangle$ .

When we measure the first register, we obtain 0 with probability

$$\begin{aligned} (\langle 0| \langle \psi_{+1}| + \langle 1| \langle \psi_{-1}|) P_0 \otimes I(|0\rangle |\psi_{+1}\rangle + |1\rangle |\psi_{-1}\rangle) &= (\langle 0| \langle \psi_{+1}| + \langle 1| \langle \psi_{-1}|)(|0\rangle |\psi_{+1}\rangle) \\ &= \langle 0|0\rangle \langle \psi_{+1}|\psi_{+1}\rangle \\ &= \langle \psi_{+1}|\psi_{+1}\rangle \end{aligned}$$

or 1 with probability

$$\begin{aligned} (\langle 0| \langle \psi_{+1}| + \langle 1| \langle \psi_{-1}|) P_1 \otimes I(|0\rangle |\psi_{+1}\rangle + |1\rangle |\psi_{-1}\rangle) &= (\langle 0| \langle \psi_{+1}| + \langle 1| \langle \psi_{-1}|)(|1\rangle |\psi_{-1}\rangle) \\ &= \langle 1|1\rangle \langle \psi_{-1}|\psi_{-1}\rangle \\ &= \langle \psi_{-1}|\psi_{-1}\rangle \end{aligned}$$

The state will collapse respectively into  $\frac{1}{\sqrt{\langle \psi_{+1}|\psi_{+1}\rangle}} |0\rangle |\psi_{+1}\rangle$  or  $\frac{1}{\sqrt{\langle \psi_{-1}|\psi_{-1}\rangle}} |1\rangle |\psi_{-1}\rangle$ . Thus if we read 0 in the first register, that means that we have an eigenvector associated to eigenvalue  $+1$  in the second register, and if we read 1 in the first register, that means that we have an eigenvector associated to eigenvalue  $-1$  in the second register.

Once we have noticed that the FT in dimension  $N = 2^1$  is just the Hadamard operator, we conclude the phase estimation circuit in this particular case is the just the same as the circuit of exercise 4.34.

## 10.

$$\begin{aligned} x^2 &= 25 = 4 \\ x^3 &= 20 = -1 \\ x^4 &= 4^2 = 16 \\ x^5 &= 16 \times 5 = 80 \\ &= 17 \\ x^6 &= (-1)^2 = 1 \end{aligned}$$

## 11.

**Theorem (Euler).** For  $N \in \mathbb{N}^*$ , let

$$\varphi(N) = \#\{m \in \llbracket 1, N \rrbracket, m \wedge N = 1\}$$

We have

$$\forall x \in \mathbb{N}^*, \quad x \wedge N = 1 \Rightarrow x^{\varphi(N)} = 1 \pmod{N}$$

Then by definition of the order  $r$ ,  $r \leq \varphi(N) \leq N$ .

**12.**

Since  $x \wedge N = 1$ , from Bezout's Theorem  $\exists(u, v) \in \mathbb{Z}^2$  such that  $ux + vN = 1$  that is  $\exists u$  such that  $ux = 1 \pmod N$  which shows that  $x$  has a multiplicative inverse  $x^{-1} = u$  in the ring  $(\frac{\mathbb{Z}}{N\mathbb{Z}}, +, \times)$ . We define the linear map  $U'$  on  $(\mathbb{C}^2)^{\otimes L} \cong \mathbb{C}^{2^L}$  that acts on the computational basis as

$$\forall y \in \{0, 1\}^L, \quad U' |y\rangle = \begin{cases} |x^{-1}y \pmod N\rangle & \text{if } y < N, \\ y & \text{if } y \in \llbracket N, 2^L - 1 \rrbracket. \end{cases}$$

We have

$$\begin{aligned} \forall y_1, y_2 \in \{0, 1\}^L, \quad \langle y_1 | U(y_2) \rangle = 1 &\Leftrightarrow y_1 = y_2 \in \llbracket N, 2^L - 1 \rrbracket \text{ or } (y_1, y_2 < N \text{ and } xy_2 = y_1 \pmod N) \\ &\Leftrightarrow y_1 = y_2 \in \llbracket N, 2^L - 1 \rrbracket \text{ or } (y_1, y_2 < N \text{ and } \exists k \in \mathbb{Z}, xy_2 = y_1 + kN) \\ &\Leftrightarrow y_1 = y_2 \in \llbracket N, 2^L - 1 \rrbracket \text{ or } (y_1, y_2 < N \text{ and } \exists k \in \mathbb{Z}, y_2 = x^{-1}y_1 + x^{-1}kN) \\ &\Leftrightarrow y_1 = y_2 \in \llbracket N, 2^L - 1 \rrbracket \text{ or } (y_1, y_2 < N \text{ and } \exists k' \in \mathbb{Z}, y_2 = x^{-1}y_1 + k'N) \\ &\Leftrightarrow y_1 = y_2 \in \llbracket N, 2^L - 1 \rrbracket \text{ or } (y_1, y_2 < N \text{ and } x^{-1}y_1 = y_2 \pmod N) \\ &\Leftrightarrow \langle U'(y_1) | y_2 \rangle = 1 \end{aligned}$$

so, since  $\langle U'(y_1) | y_2 \rangle, \langle y_1 | U(y_2) \rangle \in \{0, 1\}$ ,

$$\forall y_1, y_2 \in \{0, 1\}^L, \quad \langle y_1 | U(y_2) \rangle = \langle U'(y_1) | y_2 \rangle$$

This shows that  $U' = U^\dagger$ . since it is obvious that  $U$  is invertible and  $U^\dagger = U^{-1}$ , we have shown that  $U$  is unitary.

**13.**

$(|u_s\rangle)_{s \in \llbracket 0, r-1 \rrbracket}$  is defined to be the IFT of the sequence  $(|x^k \pmod N\rangle)_{k \in \llbracket 0, r-1 \rrbracket}$ :

$$\forall s \in \llbracket 0, r-1 \rrbracket, \quad |u_s\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-\frac{2i\pi sk}{r}} |x^k \pmod N\rangle$$

Thus the equalities

$$\forall k \in \llbracket 0, r-1 \rrbracket, \quad |x^k \pmod N\rangle = \frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} e^{\frac{2i\pi sk}{r}} |u_s\rangle$$

just state the fact that  $(|x^k \pmod N\rangle)_{k \in \llbracket 0, r-1 \rrbracket}$  is the FT of the sequence  $(|u_s\rangle)_{s \in \llbracket 0, r-1 \rrbracket}$ . Let's check this. Let  $k \in \llbracket 0, r-1 \rrbracket$ ,

$$\begin{aligned} \frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} e^{\frac{2i\pi sk}{r}} |u_s\rangle &= \frac{1}{r} \sum_{s=0}^{r-1} e^{\frac{2i\pi sk}{r}} \sum_{j=0}^{r-1} e^{-\frac{2i\pi sj}{r}} |x^j \pmod N\rangle \\ &= \frac{1}{r} \sum_{j=0}^{r-1} \left( \sum_{s=0}^{r-1} (e^{\frac{2i\pi(k-j)}{r}})^s \right) |x^j \pmod N\rangle \\ &= \frac{1}{r} \sum_{j=0}^{r-1} r \delta_{jk} |x^j \pmod N\rangle \\ &= |x^k \pmod N\rangle \end{aligned}$$

For  $k = 0$  we obtain

$$\frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} |u_s\rangle = |1\rangle$$

**15.**

The easiest way is to think with the prime decomposition of the integers  $x$  and  $y$ . Let  $d = x \wedge y$  and  $m = x \vee y$ . Let  $p_0, p_1, \dots, p_n$  be the prime numbers which appear in either prime decomposition. We can write

$$\begin{aligned} x &= p_0^{\alpha_0} p_1^{\alpha_1} \dots p_n^{\alpha_n} \\ y &= p_0^{\beta_0} p_1^{\beta_1} \dots p_n^{\beta_n} \end{aligned}$$

where  $\alpha_i, \beta_i \in \mathbb{N}$ . Then it is clear that

$$\begin{aligned} d &= p_0^{\gamma_0} p_1^{\gamma_1} \dots p_n^{\gamma_n} \\ m &= p_0^{\delta_0} p_1^{\delta_1} \dots p_n^{\delta_n} \end{aligned}$$

where  $\gamma_i = \min(\alpha_i, \beta_i)$  and  $\delta_i = \max(\alpha_i, \beta_i)$ . We have  $\alpha_i + \beta_i = \gamma_i + \delta_i$ . Then,

$$\begin{aligned} md &= p_0^{\gamma_0} p_1^{\gamma_1} \dots p_n^{\gamma_n} p_0^{\delta_0} p_1^{\delta_1} \dots p_n^{\delta_n} \\ &= p_0^{\gamma_0 + \delta_0} p_1^{\gamma_1 + \delta_1} \dots p_n^{\gamma_n + \delta_n} \\ &= p_0^{\alpha_0 + \beta_0} p_1^{\alpha_1 + \beta_1} \dots p_n^{\alpha_n + \beta_n} \\ &= xy \end{aligned}$$

**16.**

Let  $x \geq 2$ .

$$\begin{aligned} \int_x^{x+1} \frac{1}{y^2} dy &= \frac{1}{x} - \frac{1}{x+1} \\ &= \frac{1}{x(x+1)} \end{aligned}$$

since

$$x+1 \leq \frac{3}{2}x \Leftrightarrow 2 \leq x$$

$$\int_x^{x+1} \frac{1}{y^2} dy = \frac{1}{x(x+1)} \geq \frac{2}{3x^2}$$

If we sum these inequalities

$$\sum_{q=2}^{+\infty} \frac{1}{q^2} \leq \frac{3}{2} \sum_{q=2}^{+\infty} \int_q^{q+1} \frac{1}{y^2} dy = \frac{3}{2} \int_2^{+\infty} \frac{1}{y^2} dy = \frac{3}{4}$$

and finally

$$\sum_{\substack{q \in \mathbb{N}^* \\ q \text{ is prime}}} \frac{1}{q^2} \leq \sum_{q=2}^{+\infty} \frac{1}{q^2} \leq \frac{3}{4}$$

**17.**

**17.1.** The assertion  $N = a^b \Rightarrow b \leq L$  is obviously wrong if  $N = a = 1$ . Since we aim to prove an asymptotical result, we can assume that  $N \geq 2$ .

$$\begin{aligned} N = a^b &\Leftrightarrow \log N = b \log a \\ &\Leftrightarrow \frac{\log N}{\log a} = b & (N \geq 2 \Rightarrow a \geq 2 \Rightarrow \log a \geq 1 > 0) \\ &\Rightarrow b \leq \log N \\ &\Leftrightarrow b \leq \lfloor \log N \rfloor = L - 1 < L \quad (b \in \mathbb{N}) \end{aligned}$$

**17.2.** Let  $N = 2^l + a_{l-1}2^{l-1} + \dots + a_12 + a_0$  with  $l + 1 \leq L$  and  $a_i \in \{0, 1\}$ .

$$\begin{aligned} N &= 2^l(1 + a_{l-1}2^{-1} + \dots + a_12^{-l+1} + a_02^{-l}) \\ &= 2^l(1 + f) \end{aligned}$$

with  $f \in [0, 1[$ .

$$\begin{aligned} \log N &= l + \log(1 + a_{l-1}2^{-1} + \dots + a_12^{-l+1} + a_02^{-l}) \\ &= l + \log(1 + f) \end{aligned}$$

where  $\log$  is  $\log_2$ . This shows that to compute an approximation to  $\log N$ , we just need an approximation of  $\log$  in range  $[1, 2[$  or any interval of the form  $[t, 2t[$  for instance  $[\frac{3}{4}, \frac{1}{2}[$ . Besides,

$$\begin{aligned} \forall x \in ]-1, 1], \quad \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n+1} \frac{x^n}{n} + \dots \\ &= \sum_{k=1}^{+\infty} (-1)^{k+1} \frac{x^k}{k} \end{aligned}$$

Let's write it until order  $L - 1$ :

$$\forall x \in ]-1, +\infty[, \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{L+1} \frac{x^{L-1}}{L-1} + \sum_{k=L}^{+\infty} (-1)^{k+1} \frac{x^k}{k}$$

For  $x \in [0, \frac{1}{2}[$ , this is an alternating series and we can bound the rest by

$$\begin{aligned} \left| \sum_{k=L}^{+\infty} (-1)^{k+1} \frac{x^k}{k} \right| &\leq \left| (-1)^L \frac{x^L}{L} \right| \\ &\leq \frac{1}{2^L L} \end{aligned}$$

For  $x \in [-\frac{1}{4}, 0[$ , we can use Lagrange formula to bound the rest by

$$\begin{aligned} \exists \xi \in [-\frac{1}{4}, 0[, \quad \left| \sum_{k=L}^{+\infty} (-1)^{k+1} \frac{x^k}{k} \right| &= \left| \frac{\log^{(L)}(\xi)}{(L)!} x^L \right| \\ &= \frac{(L-1)!}{(1+\xi)^{L-1} L!} |x^L| \\ &= \frac{1}{(1+\xi)^{L-1} L} |x^L| \\ &\leq \frac{1}{(1+\xi)^{L-1} L} \frac{1}{4^L} \\ &\leq \frac{1}{(\frac{3}{4})^{L-1} L} \frac{1}{4^L} \\ &= \frac{1}{3^{L-1} L} \end{aligned}$$

This shows that we can use the Taylor series up to order  $L - 1$  to approximate  $\ln(x)$  with precision  $2^{-L}$  on the range  $[\frac{3}{4}, \frac{1}{2}[$ . This is to simplify the complexity analysis. In actual implementation though better and faster approximation are used: See the book by Cheney [4] for mathematical foundations of the approximation of functions by polynomials including the Remez algorithm. See also this insightful post [3] which discusses tradeoffs between accuracy and speed in approximating this log function, taking into account error induced by floating-point representation of real numbers. Here [5] can be found an actual implementation of the C standard library.

In addition to the Taylor error, there is an error occuring when computing the polynomial using floating-point arithmetic. If we store the significand of the floating-point variables in binary on  $L+1$  bits, and use

$O(L)$  bits to do arithmetic operations, each operation will incur a relative error of at most  $\epsilon = 2^{-L-1}$ , i.e.

$$\begin{aligned} x \oplus y &= (x + y)(1 + \xi) \\ x \ominus y &= (x - y)(1 + \xi) \\ x \otimes y &= (x \times y)(1 + \xi) \\ x \oslash y &= (x \div y)(1 + \xi) \end{aligned}$$

where  $|\xi| \leq \epsilon$  and the values on the left are the value computed exactly and then rounded on  $L + 1$  digits. For the details on floating point arithmetic see [6]. The previous polynomial can be rewritten as:

$$\begin{aligned} P(x) &= \sum_{k=1}^n (-1)^{k+1} \frac{1}{k} x^k \\ &= x \left( 1 + x \left( -\frac{1}{2} + x \left( \frac{1}{3} + \dots + ((-1)^{n-1} \frac{1}{n-1} + (-1)^n \frac{1}{n} x) \dots \right) \right) \right) \end{aligned}$$

This shows that the evaluation costs  $n$  fused multiply-add operations. If one rounding error occurs for each of the multiply-add, we have the following bound on the error due to floating-point arithmetic (see [7] for a detailed analysis):

$$\begin{aligned} |\bar{P}(x) - \tilde{P}(x)| &= \left| \sum_{j=1}^{n-1} (\xi_j \sum_{i=j}^n (-1)^{i+1} \frac{1}{i} x^i) \right| \\ &= \left| \sum_{i=1}^{n-1} \left( \sum_{j=1}^i \xi_j \right) (-1)^{i+1} \frac{1}{i} x^i + \left( \sum_{j=1}^{n-1} \xi_j \right) (-1)^n \frac{1}{n} x^n \right| \\ &\leq \sum_{i=1}^{n-1} \left( \sum_{j=1}^i \epsilon \right) \frac{1}{i} |x^i| + \left( \sum_{j=1}^{n-1} \epsilon \right) \frac{1}{n} |x^n| \\ &= \epsilon \sum_{i=1}^{n-1} |x^i| + \epsilon \frac{n-1}{n} |x^n| \\ &\leq \epsilon \sum_{i=1}^n \frac{1}{2^i} \\ &= \epsilon \left( 1 - \left( \frac{1}{2} \right)^n \right) \\ &\leq \epsilon \end{aligned}$$

and we add the error due to just storing the coefficients of the polynomial on  $L$  bits: for instance  $\frac{1}{3} = 0.010101\dots$  is rounded when storing in binary. If  $\tilde{a}_i$  is the rounded value of  $a_i = (-1)^{i+1} \frac{1}{i}$ , the error will be:

$$\begin{aligned} |P(x) - \tilde{P}(x)| &\leq \sum_{i=1}^n |a_i - \tilde{a}_i| |x^i| \\ &\leq \sum_{i=1}^n \epsilon |a_i| |x^i| \\ &= \epsilon \sum_{i=1}^n \frac{1}{i} |x^i| \\ &\leq \epsilon \sum_{i=1}^n |x^i| \\ &\leq \epsilon \end{aligned}$$



Taking into consideration the three types of error, we see that the Taylor series of order  $L$  is a approximation to  $\ln(x)$  on range  $[\frac{3}{4}, \frac{1}{2}[$  with precision  $2^{-L}$  since :

$$\begin{aligned} & \frac{1}{2^{L+1}(L+1)} + 2\epsilon \leq 2^{-L} \\ \Leftrightarrow & \frac{2^{-L-1}}{L} + 2^{-L} \leq 2^{-L} \\ \Leftrightarrow & 2^{-L-1}(\frac{1}{L} + 1) \leq 2^{-L} \\ \Leftrightarrow & L \geq 1 \end{aligned}$$

This analysis shows that the procedure LOG2 computes an approximation of  $\log(N)$  to precision  $2^{-L}$ . Binary addition-substraction costs  $\Theta(L)$  operations, grade-school multiplication-division costs  $\Theta(L^2)$ . Multiplication complexity can be improved to:

- $O(L^{\log_2(3)})$  using Karatsuba algorithm [2].
- $O(L \log(L) \log \log(L))$  using Schönhage-Strassen algorithm [2].
- $O(L \log L \log^* L)$  using Furer algorithm [8].

Faster division  $x \div y$  consists in computing  $\frac{1}{y}$  in  $O(\log(L))$  multiplications, then doing  $x \div y = x \times \frac{1}{y}$  (cf. [9]).

In the end computing  $\log_2 N$  has an  $O(L^3)$  time complexity. If we are given an approximating polynomial and are assured it gives the desired precision for any input size considered, the complexity is  $O(L^2)$ . The complexity of finding  $\lfloor \log_2(N) \rfloor$  given the binary representation of  $N$  is  $O(L)$ .

LOG2( $N, L$ )

```
//  $L \geq l + 1$  where  $l = \lfloor \log_2(N) \rfloor$ , i.e.  $2^l \leq N < 2^{l+1}$ .
for  $j = 1$  to  $L$ 
     $A[j] = (-1)^{j+1} \frac{1}{j}$ 
 $m = \lfloor \log_2(N) \rfloor$ 
 $f = \frac{N}{2^m} - 1$ 
if  $f \geq \frac{1}{2}$ 
     $f = \frac{1/2 - f}{2}$ 
     $m = m + 1$ 
 $q = 0$ 
for  $j = L$  downto  $0$ 
     $q = q \times f + A[j]$ 
 $q = q \div \ln(2)$ 
return  $q + m$ 
```

18.

$$\begin{aligned} x^2 &= 16 \\ x^3 &= 64 = -27 \\ x^4 &= 108 = 17 \\ x^5 &= 68 = -23 \\ x^6 &= 92 = 1 \end{aligned}$$

shows that the order of  $x$  is 6.

19.

1 is not composite and all the odd integers less than 15 are prime except  $9 = 3^2$ .

**20.**

Let  $l \in \llbracket 0, N-1 \rrbracket$ .

$$\begin{aligned}
 \sum_{x=0}^{N-1} e^{-\frac{2i\pi lx}{N}} f(x) &= \sum_{k=0}^{\frac{N}{r}-1} \sum_{x=0}^{r-1} e^{-\frac{2i\pi l(kr+x)}{N}} f(kr+x) \\
 &= \sum_{k=0}^{\frac{N}{r}-1} \sum_{x=0}^{r-1} e^{-\frac{2i\pi lkr}{N}} e^{-\frac{2i\pi lx}{N}} f(x) \\
 &= \sum_{x=0}^{r-1} \left( \sum_{k=0}^{\frac{N}{r}-1} e^{-\frac{2i\pi lkr}{N}} \right) e^{-\frac{2i\pi lx}{N}} f(x) \\
 &= \sum_{x=0}^{r-1} \left( \sum_{k=0}^{\frac{N}{r}-1} e^{-\frac{2i\pi lkr}{N}} \right) e^{-\frac{2i\pi lx}{N}} f(x)
 \end{aligned}$$

we have

$$\sum_{k=0}^{\frac{N}{r}-1} e^{-\frac{2i\pi lkr}{N}} = \begin{cases} \frac{N}{r} & \text{if } \frac{lr}{N} \in \mathbb{N}, \\ \frac{1 - (e^{-\frac{2i\pi lr}{N}})^{\frac{N}{r}}}{1 - e^{-\frac{2i\pi lr}{N}}} = 0 & \text{otherwise.} \end{cases}$$

so if  $l = l' \frac{N}{r}$ , with  $l' \in \llbracket 0, r-1 \rrbracket$ ,

$$\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{-\frac{2i\pi lx}{N}} f(x) = \sqrt{\frac{N}{r}} \frac{1}{\sqrt{r}} \sum_{x=0}^{r-1} e^{-\frac{2i\pi l'x}{r}} f(x)$$

otherwise

$$\frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} e^{-\frac{2i\pi lx}{N}} f(x) = 0$$

**22.**

First we observe that because of the periodicity, we have

$$\begin{aligned}
 f(x_1, x_2) &= f(x_1 - 1, x_2 + s) \\
 &= \dots \\
 &= f(1, x_2 + (x_1 - 1)s) \\
 &= f(0, x_2 + x_1 s)
 \end{aligned}$$

Second, for a fixed  $x_1$ , the application

$$\begin{aligned}
 \varphi_{x_1} : \llbracket 0, r-1 \rrbracket &\rightarrow \llbracket 0, r-1 \rrbracket \\
 x_2 &\mapsto x_2 + x_1 s \pmod{r}
 \end{aligned}$$

is a permutation of  $\llbracket 0, r-1 \rrbracket$ , i.e. for each  $j \in \llbracket 0, r-1 \rrbracket$ , there is exactly one  $x_2$  such that  $x_2 + x_1 s = j \pmod{r}$ . Finally, if  $x_2 + x_1 s = j \pmod{r}$ ,

$$\begin{aligned}
 e^{-\frac{2i\pi(l_1 x_1 + l_2 x_2)}{r}} &= e^{-\frac{2i\pi(l_1 x_1 + l_2(j - x_1 s + kr))}{r}} \\
 &= e^{-\frac{2i\pi((l_1 - l_2 s)x_1 + l_2 j)}{r}}
 \end{aligned}$$

FIGURE 4. Periodicity of  $f$  with  $r = 15$  and  $s = 4$ .

Let's now rewrite the sum. Let  $l_1, l_2 \in \llbracket 0, r-1 \rrbracket$ :

$$\begin{aligned}
 |\hat{f}(l_1, l_2)\rangle &= \sum_{x_1, x_2=0}^{r-1} e^{-\frac{2i\pi(l_1 x_1 + l_2 x_2)}{r}} |f(x_1, x_2)\rangle \\
 &= \sum_{x_1=0}^{r-1} \sum_{x_2=0}^{r-1} e^{-\frac{2i\pi(l_1 x_1 + l_2 x_2)}{r}} |f(x_1, x_2)\rangle \\
 &= \sum_{x_1=0}^{r-1} e^{-\frac{2i\pi l_1 x_1}{r}} \sum_{x_2=0}^{r-1} e^{-\frac{2i\pi l_2 x_2}{r}} |f(0, x_2 + x_1 s)\rangle \\
 &= \sum_{x_1=0}^{r-1} e^{-\frac{2i\pi l_1 x_1}{r}} \sum_{j=0}^{r-1} e^{-\frac{2i\pi l_2 (-x_1 s + j)}{r}} |f(0, j)\rangle \\
 &= \sum_{x_1=0}^{r-1} e^{-\frac{2i\pi(l_1 - l_2 s)x_1}{r}} \sum_{j=0}^{r-1} e^{-\frac{2i\pi l_2 j}{r}} |f(0, j)\rangle \\
 &= \sum_{x_1=0}^{r-1} (e^{-\frac{2i\pi(l_1 - l_2 s)}{r}})^{x_1} \sum_{j=0}^{r-1} e^{-\frac{2i\pi l_2 j}{r}} |f(0, j)\rangle
 \end{aligned}$$

Using the usual argument, the first factor which is a geometric sum is 0 unless  $l_1 - l_2 s = kr$  with  $k \in \mathbb{Z}$  and in that case

$$\sum_{x_1, x_2=0}^{r-1} e^{-\frac{2i\pi(l_1 x_1 + l_2 x_2)}{r}} |f(x_1, x_2)\rangle = r \sum_{j=0}^{r-1} e^{-\frac{2i\pi l_2 j}{r}} |f(0, j)\rangle$$

### 23.

Let  $x_1, x_2 \in \llbracket 0, r-1 \rrbracket$ .

$$\sum_{l_1, l_2=0}^{r-1} e^{\frac{2i\pi(x_1 l_1 + x_2 l_2)}{r}} |\hat{f}(l_1, l_2)\rangle = \sum_{l_2=0}^{r-1} \sum_{l_1=0}^{r-1} e^{\frac{2i\pi(x_1 l_1 + x_2 l_2)}{r}} |\hat{f}(l_1, l_2)\rangle$$

for a given value of  $l_2$ , there is exactly one  $l_1 \in \llbracket 0, r-1 \rrbracket$  such that  $l_1 = l_2 s \pmod r$ , so there is exactly one term in each inner sum which is non zero and

$$\begin{aligned} \sum_{l_2=0}^{r-1} \sum_{l_1=0}^{r-1} e^{\frac{2i\pi(x_1 l_1 + x_2 l_2)}{r}} |\hat{f}(l_1, l_2)\rangle &= r \sum_{l_2=0}^{r-1} e^{\frac{2i\pi(x_1 l_2 s + x_2 l_2)}{r}} \sum_{j=0}^{r-1} e^{-\frac{2i\pi l_2 j}{r}} |f(0, j)\rangle \\ &= r \sum_{j=0}^{r-1} \left( \sum_{l_2=0}^{r-1} e^{\frac{2i\pi(x_1 l_2 s + x_2 l_2 - j l_2)}{r}} \right) |f(0, j)\rangle \\ &= r \sum_{j=0}^{r-1} \left( \sum_{l_2=0}^{r-1} (e^{\frac{2i\pi(x_1 s + x_2 - j)}{r}})^{l_2} \right) |f(0, j)\rangle \end{aligned}$$

Again a geometric sum

$$\sum_{l_2=0}^{r-1} (e^{\frac{2i\pi(x_1 s + x_2 - j)}{r}})^{l_2} = \begin{cases} r & \text{if } j = x_2 + x_1 s \pmod r, \\ 0 & \text{otherwise.} \end{cases}$$

So finally

$$\begin{aligned} \sum_{l_2=0}^{r-1} \sum_{l_1=0}^{r-1} e^{\frac{2i\pi(x_1 l_1 + x_2 l_2)}{r}} |\hat{f}(l_1, l_2)\rangle &= r^2 |f(0, x_2 + x_1 s + kr)\rangle \\ &= r^2 |f(0, x_2 + x_1 s)\rangle \\ &= r^2 |f(x_1, x_2)\rangle \end{aligned}$$

## 26.

Any finite abelian group is a direct sum of cyclic groups of prime power order (cf. [10] or these shorter notes [11]).

$$G \cong \mathbb{Z}/p_1^{\beta_1} \mathbb{Z} \times \mathbb{Z}/p_2^{\beta_2} \mathbb{Z} \times \cdots \times \mathbb{Z}/p_n^{\beta_n} \mathbb{Z}$$

Such a decomposition of the group  $G$  does not necessarily imply a similar decomposition of subgroup  $K$  as a product of subgroup. Let's take the example of  $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$  and  $K = \langle (1, 2) \rangle$ : not only  $K$  is not a product but there exists no isomorphism of  $G$  which maps  $K$  into the desired form (cf. [12]).

**As a basic case, we start by making the additional assumption that  $K$  is decomposed as a product of subgroups of the direct factors of  $G$ .**

$$K = p_1^{\alpha_1} \mathbb{Z}/p_1^{\beta_1} \mathbb{Z} \times p_2^{\alpha_2} \mathbb{Z}/p_2^{\beta_2} \mathbb{Z} \times \cdots \times p_n^{\alpha_n} \mathbb{Z}/p_n^{\beta_n} \mathbb{Z} \quad (1)$$

where  $\alpha_i \leq \beta_i$ . Note that

$$|K| = p_1^{\beta_1 - \alpha_1} \times p_2^{\beta_2 - \alpha_2} \times \cdots \times p_n^{\beta_n - \alpha_n}$$

and that we then have the following decomposition for the quotient of group  $G$  by its normal subgroup  $K$ :

$$G/K = \mathbb{Z}/p_1^{\alpha_1} \mathbb{Z} \times \mathbb{Z}/p_2^{\alpha_2} \mathbb{Z} \times \cdots \times \mathbb{Z}/p_n^{\alpha_n} \mathbb{Z}$$

Let  $l = (l_1, l_2, \dots, l_n) \in \mathbb{Z}/p_1^{\beta_1}\mathbb{Z} \times \mathbb{Z}/p_2^{\beta_2}\mathbb{Z} \times \dots \times \mathbb{Z}/p_n^{\beta_n}\mathbb{Z}$ . We define  $|\hat{f}(l)\rangle$  by:

$$\begin{aligned}
\sqrt{|G||K|}|\hat{f}(l)\rangle &= \sum_{g \in G} e^{-2i\pi(l_1 \frac{g_1}{p_1^{\beta_1}} + l_2 \frac{g_2}{p_2^{\beta_2}} + \dots + l_n \frac{g_n}{p_n^{\beta_n}})} |f(g_1, g_2, \dots, g_n)\rangle \\
&= \sum_{k \in K} \sum_{x \in G/K} e^{-2i\pi(l_1 \frac{k_1+x_1}{p_1^{\beta_1}} + l_2 \frac{k_2+x_2}{p_2^{\beta_2}} + \dots + l_n \frac{k_n+x_n}{p_n^{\beta_n}})} |f(k_1+x_1, k_2+x_2, \dots, k_n+x_n)\rangle \\
&= \sum_{k \in K} e^{-2i\pi(l_1 \frac{k_1}{p_1^{\beta_1}} + l_2 \frac{k_2}{p_2^{\beta_2}} + \dots + l_n \frac{k_n}{p_n^{\beta_n}})} \sum_{x \in G/K} e^{-2i\pi(l_1 \frac{x_1}{p_1^{\beta_1}} + l_2 \frac{x_2}{p_2^{\beta_2}} + \dots + l_n \frac{x_n}{p_n^{\beta_n}})} |f(k_1+x_1, k_2+x_2, \dots, k_n+x_n)\rangle \\
&= \sum_{k \in K} e^{-2i\pi l_1 \frac{k_1}{p_1^{\beta_1}}} e^{-2i\pi l_2 \frac{k_2}{p_2^{\beta_2}}} \times \dots \times e^{-2i\pi l_n \frac{k_n}{p_n^{\beta_n}}} \sum_{x \in G/K} e^{-2i\pi(l_1 \frac{x_1}{p_1^{\beta_1}} + l_2 \frac{x_2}{p_2^{\beta_2}} + \dots + l_n \frac{x_n}{p_n^{\beta_n}})} |f(x_1, x_2, \dots, x_n)\rangle \quad (2) \\
&= \left( \sum_{k_1=0}^{p_1^{\beta_1}-\alpha_1-1} e^{-2i\pi l_1 \frac{k_1 p_1^{\alpha_1}}{p_1^{\beta_1}}} \right) \left( \sum_{k_2=0}^{p_2^{\beta_2}-\alpha_2-1} e^{-2i\pi l_2 \frac{k_2 p_2^{\alpha_2}}{p_2^{\beta_2}}} \right) \times \dots \times \left( \sum_{k_n=0}^{p_n^{\beta_n}-\alpha_n-1} e^{-2i\pi l_n \frac{k_n p_n^{\alpha_n}}{p_n^{\beta_n}}} \right) \\
&\quad \times \sum_{x \in G/K} e^{-2i\pi(l_1 \frac{x_1}{p_1^{\beta_1}} + l_2 \frac{x_2}{p_2^{\beta_2}} + \dots + l_n \frac{x_n}{p_n^{\beta_n}})} |f(x_1, x_2, \dots, x_n)\rangle \\
&= \left( \sum_{k_1=0}^{p_1^{\beta_1}-\alpha_1-1} (e^{-2i\pi l_1 \frac{p_1^{\alpha_1}}{p_1^{\beta_1}}})^{k_1} \right) \left( \sum_{k_2=0}^{p_2^{\beta_2}-\alpha_2-1} (e^{-2i\pi l_2 \frac{p_2^{\alpha_2}}{p_2^{\beta_2}}})^{k_2} \right) \times \dots \times \left( \sum_{k_n=0}^{p_n^{\beta_n}-\alpha_n-1} (e^{-2i\pi l_n \frac{p_n^{\alpha_n}}{p_n^{\beta_n}}})^{k_n} \right) \\
&\quad \times \sum_{x \in G/K} e^{-2i\pi(l_1 \frac{x_1}{p_1^{\beta_1}} + l_2 \frac{x_2}{p_2^{\beta_2}} + \dots + l_n \frac{x_n}{p_n^{\beta_n}})} |f(x_1, x_2, \dots, x_n)\rangle
\end{aligned}$$

where we have used the fact that  $f$  is constant on the cosets of  $K$ . The  $n$  first factors are geometric sums, everyone of them is non zero if and only if

$$(l_1, l_2, \dots, l_n) = (l'_1 p_1^{\beta_1-\alpha_1}, l'_2 p_2^{\beta_2-\alpha_2}, \dots, l'_n p_n^{\beta_n-\alpha_n})$$

with  $(l'_1, l'_2, \dots, l'_n) \in G/K$ . In that case the last sum becomes:

$$\sum_{x \in G/K} e^{-2i\pi(l'_1 \frac{x_1}{p_1^{\alpha_1}} + l'_2 \frac{x_2}{p_2^{\alpha_2}} + \dots + l'_n \frac{x_n}{p_n^{\alpha_n}})} |f(x_1, x_2, \dots, x_n)\rangle$$

Since the values of  $f$  on 2 different cosets are distincts, the family of vectors  $|\hat{f}\rangle$  is an orthonormal basis of the  $\frac{|G|}{|K|}$ -dimensional subspace spanned by the vectors  $|f(x_1, x_2, \dots, x_n)\rangle$ . Inverting these equalities allows us to write

$$\forall (x_1, x_2, \dots, x_n) \in G, \quad (3)$$

$$|f(x_1, x_2, \dots, x_n)\rangle = \sqrt{\frac{|K|}{|G|}} \sum_{l' \in G/K} e^{2i\pi(l'_1 \frac{x_1}{p_1^{\alpha_1}} + l'_2 \frac{x_2}{p_2^{\alpha_2}} + \dots + l'_n \frac{x_n}{p_n^{\alpha_n}})} |\hat{f}(l'_1 p_1^{\beta_1-\alpha_1}, l'_2 p_2^{\beta_2-\alpha_2}, \dots, l'_n p_n^{\beta_n-\alpha_n})\rangle$$

Applying the phase estimation algorithm will thus allow to determine

$$\left( \frac{l'_1}{p_1^{\alpha_1}}, \frac{l'_2}{p_2^{\alpha_2}}, \dots, \frac{l'_n}{p_n^{\alpha_n}} \right)$$

and the continued fraction algorithm will provide  $(p_1^{\alpha_1}, p_2^{\alpha_2}, \dots, p_n^{\alpha_n})$  which is all we need to characterize the hidden subgroup  $K$ .

**We now remove the assumption 1 that  $K$  is a product of subgroups.** We have to find another way to write the first sum in equation 2; let  $(u_1, u_2, \dots, u_m)$  be a minimal set of generators of  $K$ . Every element of  $K$  can be decomposed uniquely using this generators:

$$\forall k \in K, \quad \exists! (\lambda_1, \lambda_2, \dots, \lambda_m) \in (\llbracket 0, |u_1| - 1 \rrbracket, \llbracket 0, |u_2| - 1 \rrbracket, \dots, \llbracket 0, |u_m| - 1 \rrbracket), \quad k = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_m u_m$$

where  $|u_i| = |\langle u_i \rangle|$  is the order of group element  $u_i$ . If we note the components of  $u_i \in G$  as

$$u_i = (u_{i1}, u_{i2}, \dots, u_{in})$$

we rewrite the sum in equation 2 as

$$\begin{aligned}
\sum_{k \in K} e^{-2i\pi l_1 \frac{k_1}{p_1^{\beta_1}}} e^{-2i\pi l_2 \frac{k_2}{p_2^{\beta_2}}} \times \dots \times e^{-2i\pi l_n \frac{k_n}{p_n^{\beta_n}}} &= \sum_{\lambda_1, \lambda_2, \dots, \lambda_m} e^{-2i\pi l_1 \frac{\lambda_1 u_{11} + \lambda_2 u_{21} + \dots + \lambda_m u_{m1}}{p_1^{\beta_1}}} e^{-2i\pi l_2 \frac{\lambda_1 u_{12} + \lambda_2 u_{22} + \dots + \lambda_m u_{m2}}{p_2^{\beta_2}}} \times \dots \\
&\times e^{-2i\pi l_n \frac{\lambda_1 u_{1n} + \lambda_2 u_{2n} + \dots + \lambda_m u_{mn}}{p_n^{\beta_n}}} \\
&= \sum_{\lambda_1, \lambda_2, \dots, \lambda_m} e^{-2i\pi (l_1 \frac{u_{11}}{p_1^{\beta_1}} + l_2 \frac{u_{12}}{p_2^{\beta_2}} + \dots + l_n \frac{u_{1n}}{p_n^{\beta_n}}) \lambda_1} e^{-2i\pi (l_1 \frac{u_{21}}{p_1^{\beta_1}} + l_2 \frac{u_{22}}{p_2^{\beta_2}} + \dots + l_n \frac{u_{2n}}{p_n^{\beta_n}}) \lambda_2} \times \dots \\
&\times e^{-2i\pi (l_1 \frac{u_{m1}}{p_1^{\beta_1}} + l_2 \frac{u_{m2}}{p_2^{\beta_2}} + \dots + l_n \frac{u_{mn}}{p_n^{\beta_n}}) \lambda_m} \\
&= \left( \sum_{\lambda_1=0}^{|u_1|-1} e^{-2i\pi (l_1 \frac{u_{11}}{p_1^{\beta_1}} + l_2 \frac{u_{12}}{p_2^{\beta_2}} + \dots + l_n \frac{u_{1n}}{p_n^{\beta_n}}) \lambda_1} \right) \left( \sum_{\lambda_2=0}^{|u_2|-1} e^{-2i\pi (l_1 \frac{u_{21}}{p_1^{\beta_1}} + l_2 \frac{u_{22}}{p_2^{\beta_2}} + \dots + l_n \frac{u_{2n}}{p_n^{\beta_n}}) \lambda_2} \right) \times \dots \\
&\times \left( \sum_{\lambda_m=0}^{|u_m|-1} e^{-2i\pi (l_1 \frac{u_{m1}}{p_1^{\beta_1}} + l_2 \frac{u_{m2}}{p_2^{\beta_2}} + \dots + l_n \frac{u_{mn}}{p_n^{\beta_n}}) \lambda_m} \right)
\end{aligned}$$

The following notation will be useful: for  $l = (l_1, l_2, \dots, l_n) \in G$ , we define the group homomorphism  $\chi_l$ :

$$\begin{aligned}
\chi_l : (G, +) &\rightarrow (\mathbb{C}^*, \times) \\
u &\mapsto e^{-2i\pi (l_1 \frac{u_1}{p_1^{\beta_1}} + l_2 \frac{u_2}{p_2^{\beta_2}} + \dots + l_n \frac{u_n}{p_n^{\beta_n}})}
\end{aligned}$$

$$\begin{aligned}
\sum_{k \in K} e^{-2i\pi l_1 \frac{k_1}{p_1^{\beta_1}}} e^{-2i\pi l_2 \frac{k_2}{p_2^{\beta_2}}} \times \dots \times e^{-2i\pi l_n \frac{k_n}{p_n^{\beta_n}}} &= \left( \sum_{\lambda_1=0}^{|u_1|-1} \chi_l(u_1)^{\lambda_1} \right) \left( \sum_{\lambda_2=0}^{|u_2|-1} \chi_l(u_2)^{\lambda_2} \right) \times \dots \\
&\times \left( \sum_{\lambda_m=0}^{|u_m|-1} \chi_l(u_m)^{\lambda_m} \right)
\end{aligned}$$

Since  $\chi_l(u_i)^{|u_i|} = 1$ , the previous product is not zero if and only if

$$\begin{aligned}
&\forall i \in \llbracket 1, m \rrbracket, \quad \chi_l(u_i) = 1 \\
&\Leftrightarrow \quad \forall k \in K, \quad \chi_l(k) = 1
\end{aligned}$$

We define

$$K^\perp = \{g \in G \mid \forall k \in K, \quad \chi_g(k) = 1\}$$

It is easy to see that  $K^\perp$  is a subgroup of  $G$  and it can be shown that  $K^\perp \cong G/K$  (cf. A). Let's write explicitly the equations characterizing  $K^\perp$ :

$$\begin{aligned}
l_1 \frac{u_{11}}{p_1^{\beta_1}} + l_2 \frac{u_{12}}{p_2^{\beta_2}} + \dots + l_n \frac{u_{1n}}{p_n^{\beta_n}} &\in \mathbb{N} \\
l_1 \frac{u_{21}}{p_1^{\beta_1}} + l_2 \frac{u_{22}}{p_2^{\beta_2}} + \dots + l_n \frac{u_{2n}}{p_n^{\beta_n}} &\in \mathbb{N} \\
&\vdots \\
l_1 \frac{u_{m1}}{p_1^{\beta_1}} + l_2 \frac{u_{m2}}{p_2^{\beta_2}} + \dots + l_n \frac{u_{mn}}{p_n^{\beta_n}} &\in \mathbb{N}
\end{aligned}$$

if we let  $N = p_1^{\beta_1} \vee p_2^{\beta_2} \vee \dots \vee p_n^{\beta_n}$ ,

$$\begin{aligned}
l_1 u_{11} \frac{N}{p_1^{\beta_1}} + l_2 u_{12} \frac{N}{p_2^{\beta_2}} + \dots + l_n u_{1n} \frac{N}{p_n^{\beta_n}} &= 0 \pmod{N} \\
l_1 u_{21} \frac{N}{p_1^{\beta_1}} + l_2 u_{22} \frac{N}{p_2^{\beta_2}} + \dots + l_n u_{2n} \frac{N}{p_n^{\beta_n}} &= 0 \pmod{N} \\
&\vdots \\
l_1 u_{m1} \frac{N}{p_1^{\beta_1}} + l_2 u_{m2} \frac{N}{p_2^{\beta_2}} + \dots + l_n u_{mn} \frac{N}{p_n^{\beta_n}} &= 0 \pmod{N}
\end{aligned} \tag{4}$$

We can reorder the direct factors of  $G$  to group together those related to the same prime; since we can choose the generators  $u_i$  so that their orders are prime powers, let  $|u_i| = p^\alpha$ : we notice that the components of  $u_i$  in the factors of the form  $\mathbb{Z}/q^\beta\mathbb{Z}$  with  $q \neq p$  are zero and the previous system is rectangular with integer coefficients.

To sum up, for  $l = (l_1, l_2, \dots, l_n) \in K^\perp$  we have

$$\begin{aligned} \sqrt{|G||K|} |\hat{f}(l)\rangle &= |K| \sum_{x \in G/K} e^{-2i\pi(l_1 \frac{x_1}{p_1^{\beta_1}} + l_2 \frac{x_2}{p_2^{\beta_2}} + \dots + l_n \frac{x_n}{p_n^{\beta_n}})} |f(x_1, x_2, \dots, x_n)\rangle \\ \Leftrightarrow |\hat{f}(l)\rangle &= \sqrt{\frac{|K|}{|G|}} \sum_{x \in G/K} e^{-2i\pi(l_1 \frac{x_1}{p_1^{\beta_1}} + l_2 \frac{x_2}{p_2^{\beta_2}} + \dots + l_n \frac{x_n}{p_n^{\beta_n}})} |f(x_1, x_2, \dots, x_n)\rangle \end{aligned}$$

where  $G/K$  is an abuse of notation to refer to a set of representatives of each equivalence class.

For  $l \in G \setminus K^\perp$ ,

$$|\hat{f}(l)\rangle = 0$$

We can invert these relations to obtain the equivalent of equations 3

$$\begin{aligned} \forall (x_1, x_2, \dots, x_n) \in G/K, \\ |f(x_1, x_2, \dots, x_n)\rangle &= \sqrt{\frac{|K|}{|G|}} \sum_{l \in K^\perp} e^{2i\pi(l_1 \frac{x_1}{p_1^{\beta_1}} + l_2 \frac{x_2}{p_2^{\beta_2}} + \dots + l_n \frac{x_n}{p_n^{\beta_n}})} |\hat{f}(l_1, l_2, \dots, l_n)\rangle \end{aligned} \quad (5)$$

Actually, we can write these equalities for any  $x \in G$ , as it could be expected. If  $x'$  is the chosen representative of the class of  $x$ , we have  $x = \underbrace{x - x'}_{\in K} + \underbrace{x'}_{\in G/K}$  and

$$\begin{aligned} \sqrt{\frac{|K|}{|G|}} \sum_{l \in K^\perp} e^{2i\pi(l_1 \frac{x_1}{p_1^{\beta_1}} + l_2 \frac{x_2}{p_2^{\beta_2}} + \dots + l_n \frac{x_n}{p_n^{\beta_n}})} |\hat{f}(l_1, l_2, \dots, l_n)\rangle &= \sqrt{\frac{|K|}{|G|}} \sum_{l \in K^\perp} \chi_l(x) |\hat{f}(l_1, l_2, \dots, l_n)\rangle \\ &= \sqrt{\frac{|K|}{|G|}} \sum_{l \in K^\perp} \chi_l(x - x' + x') |\hat{f}(l_1, l_2, \dots, l_n)\rangle \\ &= \sqrt{\frac{|K|}{|G|}} \sum_{l \in K^\perp} \underbrace{\chi_l(x - x')}_{=1} \chi_l(x') |\hat{f}(l_1, l_2, \dots, l_n)\rangle \\ &= \sqrt{\frac{|K|}{|G|}} \sum_{l \in K^\perp} \chi_l(x') |\hat{f}(l_1, l_2, \dots, l_n)\rangle \\ &= |f(x'_1, x'_2, \dots, x'_n)\rangle \\ &= |f(x_1, x_2, \dots, x_n)\rangle \end{aligned}$$

28.

We start with the initial state

$$\underbrace{|0\rangle |0\rangle \dots |0\rangle}_{t=t_1+t_2+\dots+t_n \text{ bits}} \quad \overbrace{|0\rangle}^{m \text{ bits}}$$

We create the superposition

$$\frac{1}{\sqrt{2^t}} \sum_{x_1=0}^{2^{t_1}-1} \sum_{x_2=0}^{2^{t_2}-1} \dots \sum_{x_n=0}^{2^{t_n}-1} |x_1\rangle |x_2\rangle \dots |x_n\rangle |0\rangle$$

$f$  is a function from  $G$  to  $\llbracket 0, 2^m - 1 \rrbracket$  which is constant on the cosets of  $K$ , such that the values on 2 different cosets are distinct:

$$f : \mathbb{Z}/p_1^{\beta_1}\mathbb{Z} \times \mathbb{Z}/p_2^{\beta_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_n^{\beta_n}\mathbb{Z} \rightarrow \llbracket 0, 2^{m-1} \rrbracket$$

$$(g_1, g_2, \dots, g_n) \mapsto f(g_1, g_2, \dots, g_n)$$

We define  $\tilde{f}$  to be an extension of the function  $f$ :

$$\tilde{f} : \llbracket 0, 2^{t_1} - 1 \rrbracket \times \llbracket 0, 2^{t_2} - 1 \rrbracket \times \cdots \times \llbracket 0, 2^{t_n} - 1 \rrbracket \rightarrow \llbracket 0, 2^{m-1} \rrbracket$$

$$(x_1, x_2, \dots, x_n) \mapsto f(x_1 \bmod p_1^{\beta_1}, x_2 \bmod p_2^{\beta_2}, \dots, x_n \bmod p_n^{\beta_n})$$

We apply the unitary operator

$$U :$$

$$|x_1\rangle |x_2\rangle \cdots |x_n\rangle |y\rangle \mapsto |x_1\rangle |x_2\rangle \cdots |x_n\rangle |y \oplus \tilde{f}(x_1, x_2, \dots, x_n)\rangle$$

to get the state

$$\frac{1}{\sqrt{2^t}} \sum_{x_1=0}^{2^{t_1}-1} \sum_{x_2=0}^{2^{t_2}-1} \cdots \sum_{x_n=0}^{2^{t_n}-1} |x_1\rangle |x_2\rangle \cdots |x_n\rangle |\tilde{f}(x_1, x_2, \dots, x_n)\rangle$$

**From there we are making the additional assumption that  $K$  is decomposed as a product of subgroups of the direct factors of  $G$ .** We can express the  $|f\rangle$  in the  $|\hat{f}\rangle$  basis:

$$\frac{1}{\sqrt{2^t}} \sqrt{\frac{|K|}{|G|}} \sum_{x_1=0}^{2^{t_1}-1} \sum_{x_2=0}^{2^{t_2}-1} \cdots \sum_{x_n=0}^{2^{t_n}-1} |x_1\rangle |x_2\rangle \cdots |x_n\rangle \sum_{l' \in G/K} e^{2i\pi(l'_1 \frac{x_1}{p_1^{\alpha_1}} + l'_2 \frac{x_2}{p_2^{\alpha_2}} + \cdots + l'_n \frac{x_n}{p_n^{\alpha_n}})} |\hat{f}(l'_1 p_1^{\beta_1 - \alpha_1}, l'_2 p_2^{\beta_2 - \alpha_2}, \dots, l'_n p_n^{\beta_n - \alpha_n})\rangle$$

$$= \sqrt{\frac{|K|}{|G|}} \sum_{l' \in G/K} \left( \frac{1}{\sqrt{2^t}} \sum_{x_1=0}^{2^{t_1}-1} \sum_{x_2=0}^{2^{t_2}-1} \cdots \sum_{x_n=0}^{2^{t_n}-1} e^{2i\pi(l'_1 \frac{x_1}{p_1^{\alpha_1}} + l'_2 \frac{x_2}{p_2^{\alpha_2}} + \cdots + l'_n \frac{x_n}{p_n^{\alpha_n}})} |x_1\rangle |x_2\rangle \cdots |x_n\rangle \right) |\hat{f}(l'_1 p_1^{\beta_1 - \alpha_1}, l'_2 p_2^{\beta_2 - \alpha_2}, \dots, l'_n p_n^{\beta_n - \alpha_n})\rangle$$

$$= \sqrt{\frac{|K|}{|G|}} \sum_{l' \in G/K} \left( \frac{1}{\sqrt{2^{t_1}}} \sum_{x_1=0}^{2^{t_1}-1} e^{2i\pi l'_1 \frac{x_1}{p_1^{\alpha_1}}} |x_1\rangle \right) \left( \frac{1}{\sqrt{2^{t_2}}} \sum_{x_2=0}^{2^{t_2}-1} e^{2i\pi l'_2 \frac{x_2}{p_2^{\alpha_2}}} |x_2\rangle \right) \cdots \left( \frac{1}{\sqrt{2^{t_n}}} \sum_{x_n=0}^{2^{t_n}-1} e^{2i\pi l'_n \frac{x_n}{p_n^{\alpha_n}}} |x_n\rangle \right) |\hat{f}(l'_1 p_1^{\beta_1 - \alpha_1}, l'_2 p_2^{\beta_2 - \alpha_2}, \dots, l'_n p_n^{\beta_n - \alpha_n})\rangle$$

We apply inverse fourier transform on each of the first  $n$  registers:

$$\sqrt{\frac{|K|}{|G|}} \sum_{l' \in G/K} \left( \widetilde{|\frac{l'_1}{p_1^{\alpha_1}}\rangle} \widetilde{|\frac{l'_2}{p_2^{\alpha_2}}\rangle} \cdots \widetilde{|\frac{l'_n}{p_n^{\alpha_n}}\rangle} \right) |\hat{f}(l'_1 p_1^{\beta_1 - \alpha_1}, l'_2 p_2^{\beta_2 - \alpha_2}, \dots, l'_n p_n^{\beta_n - \alpha_n})\rangle$$

We measure the first  $n$  registers to get

$$\left( \widetilde{|\frac{l'_1}{p_1^{\alpha_1}}\rangle}, \widetilde{|\frac{l'_2}{p_2^{\alpha_2}}\rangle}, \dots, \widetilde{|\frac{l'_n}{p_n^{\alpha_n}}\rangle} \right)$$

The continued fraction algorithm provides

$$(p_1^{\alpha_1}, p_2^{\alpha_2}, \dots, p_n^{\alpha_n})$$

In register  $i$ , if we use  $t_i = 2\lceil \log p_i^{\beta_i} \rceil + 1 + \lceil \log(2 + \frac{1}{2\epsilon}) \rceil$ , with probability at least  $1 - \epsilon$ , we will have some an approximation of  $\frac{l'_i}{p_i^{\alpha_i}}$  accurate to  $2\lceil \log p_i^{\beta_i} \rceil + 1$  bits, for some  $l'_i \in \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$ . The continued fraction algorithm will then succeed in providing  $p_i^{\alpha_i}$  unless  $l'_i \in p_i\mathbb{Z}$ , which happens with probability  $\frac{p_i^{\alpha_i-1}}{p_i^{\alpha_i}} = \frac{1}{p_i}$ . A lower bound on the probability that the algorithm succeeds in finding  $K$  is thus:

$$(1 - \epsilon)^n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_n}\right)$$

We can improve this noticing that when  $p_i = 2$ , we are sure that  $\frac{l'_i}{2^{\alpha_i}}$  can be recovered exactly by the phase estimation algorithm if using  $t_i = \beta_i \geq \alpha_i$  bits. If we let  $n_2$  be  $|\{i \in \llbracket 1, n \rrbracket \mid p_i = 2\}|$ , the bound becomes

$$(1 - \epsilon)^{n-n_2} \prod_{p_i \neq 2} \left(1 - \frac{1}{p_i}\right)$$

If we repeat the algorithm  $N$  times, the probability to determine  $K$  is at least

$$\prod_{p_i=2} \left(1 - \frac{1}{2^N}\right) \prod_{p_i \neq 2} \left(1 - (1 - (1 - \epsilon)\left(1 - \frac{1}{p_i}\right))^N\right) = \left(1 - \frac{1}{2^N}\right)^{n_2} \prod_{p_i \neq 2} \left(1 - (\epsilon + (1 - \epsilon)\frac{1}{p_i})^N\right)$$



**We now turn to the general case.** We can express the  $|f\rangle$  in the  $|\hat{f}\rangle$  basis:

$$\begin{aligned}
& \frac{1}{\sqrt{2^t}} \sum_{x_1=0}^{2^{t_1}-1} \sum_{x_2=0}^{2^{t_2}-1} \cdots \sum_{x_n=0}^{2^{t_n}-1} |x_1\rangle |x_2\rangle \cdots |x_n\rangle |\tilde{f}(x_1, x_2, \dots, x_n)\rangle \\
&= \frac{1}{\sqrt{2^t}} \sqrt{\frac{|K|}{|G|}} \sum_{x_1=0}^{2^{t_1}-1} \sum_{x_2=0}^{2^{t_2}-1} \cdots \sum_{x_n=0}^{2^{t_n}-1} |x_1\rangle |x_2\rangle \cdots |x_n\rangle \sum_{l \in K^\perp} e^{2i\pi(l_1 \frac{x_1}{p_1^{\beta_1}} + l_2 \frac{x_2}{p_2^{\beta_2}} + \cdots + l_n \frac{x_n}{p_n^{\beta_n}})} |\hat{f}(l_1, l_2, \dots, l_n)\rangle \\
&= \sqrt{\frac{|K|}{|G|}} \sum_{l \in K^\perp} \left( \frac{1}{\sqrt{2^t}} \sum_{x_1=0}^{2^{t_1}-1} \sum_{x_2=0}^{2^{t_2}-1} \cdots \sum_{x_n=0}^{2^{t_n}-1} e^{2i\pi(l_1 \frac{x_1}{p_1^{\beta_1}} + l_2 \frac{x_2}{p_2^{\beta_2}} + \cdots + l_n \frac{x_n}{p_n^{\beta_n}})} |x_1\rangle |x_2\rangle \cdots |x_n\rangle \right) |\hat{f}(l_1, l_2, \dots, l_n)\rangle \\
&= \sqrt{\frac{|K|}{|G|}} \sum_{l \in K^\perp} \left( \frac{1}{\sqrt{2^{t_1}}} \sum_{x_1=0}^{2^{t_1}-1} e^{2i\pi l_1 \frac{x_1}{p_1^{\beta_1}}} |x_1\rangle \right) \left( \frac{1}{\sqrt{2^{t_2}}} \sum_{x_2=0}^{2^{t_2}-1} e^{2i\pi l_2 \frac{x_2}{p_2^{\beta_2}}} |x_2\rangle \right) \cdots \left( \frac{1}{\sqrt{2^{t_n}}} \sum_{x_n=0}^{2^{t_n}-1} e^{2i\pi l_n \frac{x_n}{p_n^{\beta_n}}} |x_n\rangle \right) |\hat{f}(l_1, l_2, \dots, l_n)\rangle
\end{aligned}$$

We apply inverse fourier transform on each of the first  $n$  registers:

$$\sqrt{\frac{|K|}{|G|}} \sum_{l \in K^\perp} |\widetilde{\frac{l_1}{p_1^{\beta_1}}}\rangle |\widetilde{\frac{l_2}{p_2^{\beta_2}}}\rangle \cdots |\widetilde{\frac{l_n}{p_n^{\beta_n}}}\rangle |\hat{f}(l_1, l_2, \dots, l_n)\rangle$$

We measure the first  $n$  registers to get

$$\left( \widetilde{\frac{l_1}{p_1^{\beta_1}}}, \widetilde{\frac{l_2}{p_2^{\beta_2}}}, \dots, \widetilde{\frac{l_n}{p_n^{\beta_n}}} \right)$$

The continued fraction algorithm provides

$$(l_1, l_2, \dots, l_n) \in K^\perp$$

In register  $i$ , if we use  $t_i = 2\lceil \log p_i^{\beta_i} \rceil + 1 + \lceil \log(2 + \frac{1}{2\epsilon}) \rceil$ , with probability at least  $1 - \epsilon$ , we will have some an approximation of  $\frac{l_i}{p_i^{\beta_i}}$  accurate to  $2\lceil \log p_i^{\beta_i} \rceil + 1$  bits, for some  $l_i \in \mathbb{Z}/p_i^{\beta_i}\mathbb{Z}$ . The continued fraction algorithm

will then succeed in providing  $l_i$  unless  $l_i \in p_i\mathbb{Z} \setminus \{0\}$ , which happens with probability  $\frac{p_i^{\beta_i-2}}{p_i^{\beta_i}} = \frac{1}{p_i^2}$ . A lower bound on the probability that the algorithm succeeds in finding all the coordinates of some element  $l \in K^\perp$  is thus:

$$(1 - \epsilon)^{n-n_2} \prod (1 - \frac{1}{p_i^2})$$

The idea is then to repeat the algorithm until we find a set of generators of  $K^\perp$ ; Then we can solve a system similar to 4 to fully determine  $(K^\perp)^\perp = K$  (cf. A for the proof of the latter fact).

**Appendix A.****Theorem.***Proof.*

□

**Theorem.***Proof.*

□

## References

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