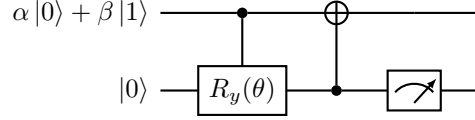


QUANTUM COMPUTATION AND QUANTUM INFORMATION: QUANTUM NOISE AND QUANTUM OPERATIONS

20. Circuit model for amplitude damping

We want to prove that the following circuit models the amplitude damping operation



Recall that

$$R_y(\theta) = e^{-i\frac{\theta}{2}Y} = \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}$$

Initially the two-qubit state is

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle = \alpha|00\rangle + \beta|10\rangle$$

After the controlled R_y gate it becomes

$$\begin{aligned} \alpha|00\rangle + \beta|1\rangle R_y(\theta)|0\rangle &= \alpha|00\rangle + \beta|1\rangle (\cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})|1\rangle) \\ &= \alpha|00\rangle + \beta(\cos(\frac{\theta}{2})|10\rangle + \sin(\frac{\theta}{2})|11\rangle) \end{aligned}$$

After the controlled not gate,

$$\alpha|00\rangle + \beta(\cos(\frac{\theta}{2})|10\rangle + \sin(\frac{\theta}{2})|01\rangle)$$

This is the effect of amplitude damping, with probability of 1 be switched to 0, or one photon being lost to environment, being $\gamma = \sin^2(\frac{\theta}{2})$.

21. Amplitude damping of a harmonic oscillator

The principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian:

$$H = \chi(a^\dagger b + b^\dagger a)$$

where a^\dagger, a and b^\dagger, b are the creation, annihilation operators for the principal and environment oscillators, respectively.

The time evolution of the coupled system is governed by the unitary operator:

$$U = e^{-iH\Delta t}$$

21.1. Operation elements. We recall some results for the harmonic oscillator:

$$\forall n \in \mathbb{N}, \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

and similarly in the environment space

$$\forall n \in \mathbb{N}, \quad b^\dagger |n\rangle_b = \sqrt{n+1} |n+1\rangle_b$$

Here we use the subscript b to differentiate the eigenvectors of the Hermitian operator bb^\dagger which live in the environment space from the eigenvectors of aa^\dagger in the principal space:

$$\begin{aligned} \forall n \in \mathbb{N}, \quad bb^\dagger |n\rangle_b &= (n+1) |n\rangle_b \\ \forall n \in \mathbb{N}, \quad aa^\dagger |n\rangle &= (n+1) |n\rangle \end{aligned}$$

Each set of vectors constitute an orthonormal basis:

$$\begin{aligned} \forall (n, m) \in \mathbb{N}^2, \quad \langle n|m \rangle &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases} \\ &= \delta_{nm} \end{aligned}$$

We also have

$$\begin{aligned} aa^\dagger - a^\dagger a &= [a, a^\dagger] \\ &= 1 \\ bb^\dagger - b^\dagger b &= [b, b^\dagger] \\ &= 1 \end{aligned}$$

where 1 stands for the identity operator.

Each of the operators a, a^\dagger commutes with each of the operators b, b^\dagger since they act on different spaces

$$\begin{aligned} 0 &= [a^\dagger, b^\dagger] \\ &= [a, b^\dagger] \\ &= [a^\dagger, b] \\ &= [a, b] \end{aligned}$$

The Baker-Campbell-Hausdorff formula states that, for any operators A, G such that e^G exists,

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} C_n$$

where the operators C_n are defined recursively by

$$\begin{aligned} C_0 &= A \\ C_1 &= [G, A] \\ \forall n \in \mathbb{N}, \quad C_{n+1} &= [G, C_n] \end{aligned}$$

Lets compute a simplified expression for the operator $U a^\dagger U^\dagger$ acting on the product space:

$$\begin{aligned} U a^\dagger U^\dagger &= e^{-iH\Delta t} a^\dagger e^{iH\Delta t} \\ &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \end{aligned} \tag{1}$$

The first commutators C_n are

$$\begin{aligned}
C_0 &= a^\dagger \\
C_1 &= [H, a^\dagger] \\
&= [\chi b^\dagger a, a^\dagger] \\
&= \chi b^\dagger [a, a^\dagger] \\
&= \chi b^\dagger \\
C_2 &= [H, C_1] \\
&= [\chi a^\dagger b, \chi b^\dagger] \\
&= \chi^2 a^\dagger [b, b^\dagger] \\
&= \chi^2 a^\dagger
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\forall n \in \mathbb{N}, \quad C_{2n} &= \chi^{2n} a^\dagger \\
C_{2n+1} &= \chi^{2n+1} b^\dagger
\end{aligned}$$

We now rewrite equation 1

$$\begin{aligned}
U a^\dagger U^\dagger &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \\
&= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n}}{(2n)!} C_{2n} + \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n+1}}{(2n+1)!} C_{2n+1} \\
&= a^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n}}{(2n)!} + b^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n+1}}{(2n+1)!} \\
&= a^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n}}{(2n)!} - i b^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n+1}}{(2n+1)!} \\
&= \cos(\chi\Delta t) a^\dagger - i \sin(\chi\Delta t) b^\dagger
\end{aligned}$$

Let us now compute the effect of U on $|0\rangle|0\rangle_b = |00\rangle$:

$$\begin{aligned}
U|00\rangle &= e^{-iH\Delta t}|00\rangle \\
&= \sum_{n=0}^{+\infty} \frac{(-iH\Delta t)^n}{n!} |00\rangle
\end{aligned}$$

Since $a|0\rangle = 0$ and $b|0\rangle_b = 0$, we have

$$H|00\rangle = 0$$

and

$$\forall n \in \mathbb{N}^*, \quad H^n |00\rangle = 0$$

from which it follows there is only one non nul term in the previous sum and

$$U|00\rangle = |00\rangle$$

Let us compute the effect of U on $|1\rangle|0\rangle_b = |10\rangle$:

$$\begin{aligned}
U|10\rangle &= Ua^\dagger|00\rangle \\
&= Ua^\dagger \underbrace{U^\dagger U}_{=1}|00\rangle \\
&= Ua^\dagger U^\dagger|00\rangle \\
&= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)|00\rangle \\
&= \cos(\chi\Delta t)|10\rangle - i\sin(\chi\Delta t)|01\rangle \\
&= \cos(\chi\Delta t)|1\rangle|0\rangle_b - i\sin(\chi\Delta t)|0\rangle|1\rangle_b
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sqrt{n!}U|n\rangle|0\rangle_b &= \sqrt{n!}U|n0\rangle \\
&= U(a^\dagger)^n|00\rangle \\
&= U(a^\dagger)^n U^\dagger U|00\rangle \\
&= (Ua^\dagger U^\dagger)^n|00\rangle \\
&= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)^n|00\rangle
\end{aligned}$$

Since $[a^\dagger, b^\dagger] = 0$,

$$\begin{aligned}
\sqrt{n!}U|n\rangle|0\rangle_b &= \left(\sum_{k=0}^n \binom{n}{k} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) (a^\dagger)^{n-k} (b^\dagger)^k \right) |00\rangle \\
&= \sum_{k=0}^n \binom{n}{k} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) \sqrt{(n-k)!k!} |n-k\rangle|k\rangle_b
\end{aligned}$$

so that

$$\begin{aligned}
U|n0\rangle &= \sum_{k=0}^n \binom{n}{k} \sqrt{\frac{(n-k)!k!}{n!}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle|k\rangle_b \\
&= \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle|k\rangle_b
\end{aligned}$$

Let $E_m = \langle m|_b U|0\rangle_b$, $m \in \mathbb{N}$ the operation elements of U . They are operators acting on the principal space. We can compute the action of E_m on $|n\rangle$ (i.e. compute the n th column of the matrix of E_m) from the previous formula:

$$\begin{aligned}
E_m|n\rangle &= (\langle m|_b U|0\rangle_b)|n\rangle \\
&= \langle m|_b (U|n\rangle|0\rangle_b) \\
&= \langle m|_b U|n0\rangle
\end{aligned}$$

First it is clear that if $n < m$, $E_m|n\rangle = 0$. Then if $n \geq m$,

$$\begin{aligned}
E_m|n\rangle &= \langle m|_b \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle|k\rangle_b \\
&= \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle \underbrace{\langle m|k\rangle_b}_{=\delta_{mk}} \\
&= (-i)^m \sin^m(\chi\Delta t) \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle
\end{aligned}$$

This shows that the matrix of E_m has non nul elements only on the m th superior diagonal.

We can also reconstruct the full formula for E_m using braket calculus:

$$\begin{aligned}
E_m &= E_m \underbrace{\sum_{n=0}^{+\infty} |n\rangle \langle n|}_{=1} \\
&= \sum_{n=0}^{+\infty} E_m |n\rangle \langle n| \\
&= \sum_{n=m}^{+\infty} E_m |n\rangle \langle n| \\
&= (-i)^m \sin^m(\chi\Delta t) \sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle \langle n|
\end{aligned}$$

Note that the sole effect of factor $(-i)^m$ is to add a global phase so it may as well be omitted.

$$E_m = \sin^m(\chi\Delta t) \left[\begin{array}{ccc} & \begin{matrix} m \\ 1 \end{matrix} & \begin{matrix} n \\ \end{matrix} \\ & \sqrt{\binom{m+1}{m}} \cos(\chi\Delta t) & \mathbf{0} \\ & \sqrt{\binom{m+2}{m}} \cos^2(\chi\Delta t) & \\ & \vdots & \vdots \\ & \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) & \\ & \vdots & \vdots \\ \mathbf{0} & & \end{array} \right] \begin{matrix} n-m \\ \end{matrix}$$

21.2. Trace-preserving property. Matrix calculus or braket calculus show that the matrices $E_m^\dagger E_m$ are diagonals, with the first m elements are 0:

$$\begin{aligned}
E_m^\dagger E_m &= \sin^{2m}(\chi\Delta t) \left(\sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n\rangle \langle n-m| \right) \left(\sum_{l=m}^{+\infty} \sqrt{\binom{l}{m}} \cos^{l-m}(\chi\Delta t) |l-m\rangle \langle l| \right) \\
&= \sin^{2m}(\chi\Delta t) \sum_{n=m}^{+\infty} \sum_{l=m}^{+\infty} \sqrt{\binom{n}{m}} \sqrt{\binom{l}{m}} \cos^{n-m}(\chi\Delta t) \cos^{l-m}(\chi\Delta t) |n\rangle \underbrace{\langle n-m|l-m\rangle}_{=\delta_{nl}} \langle l| \\
&= \sin^{2m}(\chi\Delta t) \sum_{n=m}^{+\infty} \binom{n}{m} \cos^{2(n-m)}(\chi\Delta t) |n\rangle \langle n|
\end{aligned}$$

$$E_m^\dagger E_m = \sin^{2m}(\chi \Delta t)$$

It follows that the operator $\sum_{m=0}^{+\infty} E_m^\dagger E_m$ is also diagonal, and diagonal elements are

$$\begin{aligned}
 \langle n | \sum_{m=0}^{+\infty} E_m^\dagger E_m | n \rangle &= \sum_{m=0}^{+\infty} \langle n | E_m^\dagger E_m | n \rangle \\
 &= \sum_{m=0}^n \langle n | E_m^\dagger E_m | n \rangle \\
 &= \sum_{m=0}^n \binom{n}{m} \sin^{2m}(\chi \Delta t) \cos^{2(n-m)}(\chi \Delta t) \\
 &= (\sin^2(\chi \Delta t) + \cos^2(\chi \Delta t))^n \\
 &= 1
 \end{aligned}$$

i.e. $\sum_{m=0}^{+\infty} E_m^\dagger E_m = 1$ and the quantum operation is trace-preserving.