

QUANTUM COMPUTATION AND QUANTUM INFORMATION: QUANTUM COMPUTERS: PHYSICAL REALIZATION

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17. Eigenstates of the Jaynes-Cummings Hamiltonian

We consider a system formed by a two-level atom and a cavity confined electric field. The Hamiltonian is

$$H = g(a\sigma_- + a^\dagger\sigma_+)$$

where g is some constant which describes the strength of the interaction, a^\dagger, a are respectively the creation, annihilation operators¹ on the single mode field, and σ_\pm are operators acting on the two-level atom, namely:

$$\begin{aligned}\sigma_+ &= \frac{1}{2}(X + iY) \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \sigma_- &= \frac{1}{2}(X - iY) \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\end{aligned}$$

We recall

$$\begin{aligned}\forall n \in \mathbb{N}, \quad a^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \\ a |n+1\rangle &= \sqrt{n+1} |n\rangle \\ a |0\rangle &= 0\end{aligned}$$

We define the vectors

$$\begin{aligned}|\chi_n\rangle &= \frac{1}{\sqrt{2}}(|n, 1\rangle + |n+1, 0\rangle) \\ |\bar{\chi}_n\rangle &= \frac{1}{\sqrt{2}}(|n, 1\rangle - |n+1, 0\rangle)\end{aligned}$$

Let's apply the Hamiltonian

$$\begin{aligned}H |\chi_n\rangle &= \frac{g}{\sqrt{2}}(a\sigma_- + a^\dagger\sigma_+)(|n, 1\rangle + |n+1, 0\rangle) \\ &= \frac{g}{\sqrt{2}}(a |n\rangle \sigma_- |1\rangle + a^\dagger |n\rangle \sigma_+ |1\rangle + a |n+1\rangle \sigma_- |0\rangle + a^\dagger |n+1\rangle \sigma_+ |0\rangle) \\ &= \frac{g}{\sqrt{2}}\sqrt{n+1}(|n+1\rangle |0\rangle + |n\rangle |1\rangle) \\ &= \frac{g}{\sqrt{2}}\sqrt{n+1}(|n+1, 0\rangle + |n, 1\rangle) \\ H |\chi_n\rangle &= g\sqrt{n+1} |\chi_n\rangle\end{aligned}$$

and

¹It seems to me the book mixes up a^\dagger with a in several places.

$$\begin{aligned}
H |\bar{\chi}_n\rangle &= \frac{g}{\sqrt{2}} (a\sigma_- + a^\dagger\sigma_+) (|n, 1\rangle - |n+1, 0\rangle) \\
&= \frac{g}{\sqrt{2}} (a |n\rangle \sigma_- |1\rangle + a^\dagger |n\rangle \sigma_+ |1\rangle - a |n+1\rangle \sigma_- |0\rangle - a^\dagger |n+1\rangle \sigma_+ |0\rangle) \\
&= \frac{g}{\sqrt{2}} \sqrt{n+1} (|n+1\rangle |0\rangle - |n\rangle |1\rangle) \\
&= \frac{g}{\sqrt{2}} \sqrt{n+1} (|n+1, 0\rangle - |n, 1\rangle) \\
H |\bar{\chi}_n\rangle &= -g\sqrt{n+1} |\bar{\chi}_n\rangle
\end{aligned}$$

It proves that $|\chi_n\rangle$ and $|\bar{\chi}_n\rangle$ are eigenvectors of the Hamiltonian, with eigenvalues $g\sqrt{n+1}$ and $-g\sqrt{n+1}$, respectively.