

# QUANTUM COMPUTATION AND QUANTUM INFORMATION: THE QUANTUM FOURIER TRANSFORM

## 1.

We consider the linear map in  $\mathbb{C}^N$  which acts on the computational basis as

$$|j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2i\pi jk}{N}} |k\rangle$$

Let  $A$  be the matrix of the transformation in the computational basis.

$$\forall (k, l) \in \llbracket 0, N-1 \rrbracket^2, \quad a_{kl} = \frac{1}{\sqrt{N}} e^{\frac{2i\pi kl}{N}}$$

The adjoint matrix  $A^\dagger$  is then

$$\begin{aligned} \forall (k, l) \in \llbracket 0, N-1 \rrbracket^2, \quad b_{kl} &= a_{lk}^* \\ &= \frac{1}{\sqrt{N}} e^{-\frac{2i\pi kl}{N}} \end{aligned}$$

We compute the coefficient  $k, l$  of the product  $AA^\dagger$ :

$$\begin{aligned} \forall (k, l) \in \llbracket 0, N-1 \rrbracket^2, \quad c_{kl} &= \sum_{j=0}^{N-1} a_{kj} b_{jl} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} e^{\frac{2i\pi j}{N} (k-l)} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} (e^{\frac{2i\pi}{N} (k-l)})^j \\ &= \begin{cases} \frac{1}{N} \frac{1 - (e^{\frac{2i\pi}{N} (k-l)})^N}{1 - e^{\frac{2i\pi}{N} (k-l)}} = 0 & \text{if } e^{\frac{2i\pi}{N} (k-l)} \neq 1, \\ 1 & \text{if } e^{\frac{2i\pi}{N} (k-l)} = 1. \end{cases} \\ &= \begin{cases} 0 & \text{if } k \neq l, \\ 1 & \text{if } k = l. \end{cases} \\ &= \delta_{kl} \end{aligned}$$

which shows that  $AA^\dagger = A^\dagger A = I$  i.e.  $A$  is unitary.

## 2.

Here the dimension of the state space is  $N = 2^n$ . The Fourier transform of the  $n$  qubit state  $|00 \dots 0\rangle$  is

$$A|0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle$$

we can write  $k$  in binary  $k_{n-1} \dots k_1 k_0$

$$A|0\rangle = \frac{1}{2^{n/2}} \sum_{k_0, k_1, \dots, k_{n-1}=0}^1 |k_{n-1} \dots k_1 k_0\rangle$$

or in product representation,

$$= \frac{1}{2^{n/2}} \underbrace{(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \dots (|0\rangle + |1\rangle)}_{n \text{ qubits}}$$

## 3.

Let  $N = 2^n$  and  $Y = (y_k)_{k \in \llbracket 0, N-1 \rrbracket}$  be the classical fourier transform of  $X = (x_k)_{k \in \llbracket 0, N-1 \rrbracket}$ .

$$\forall k \in \llbracket 0, N-1 \rrbracket, \quad y_k = \sum_{j=0}^{N-1} e^{\frac{2i\pi k j}{2^n}} x_j$$

The factor  $\frac{1}{\sqrt{N}}$  is omitted for clarity. We can write  $j$  in binary  $j_{n-1} \dots j_1 j_0$

$$\begin{aligned} y_k &= \sum_{j_0, j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-1}j_{n-1} + \dots + 2j_1 + j_0)}{2^n}} x_j \\ &= \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-1}j_{n-1} + \dots + 2j_1)}{2^n}} x_{j_{n-1} \dots j_1 0} + \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-1}j_{n-1} + \dots + 2j_1 + 1)}{2^n}} x_{j_{n-1} \dots j_1 1} \\ &= \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-1}j_{n-1} + \dots + 2j_1)}{2^n}} x_{j_{n-1} \dots j_1 0} + e^{\frac{2i\pi k}{2^n}} \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-1}j_{n-1} + \dots + 2j_1)}{2^n}} x_{j_{n-1} \dots j_1 1} \\ &= \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-2}j_{n-1} + \dots + j_1)}{2^{n-1}}} x_{j_{n-1} \dots j_1 0} + e^{\frac{2i\pi k}{2^n}} \sum_{j_1, \dots, j_{n-1}=0}^1 e^{\frac{2i\pi k(2^{n-2}j_{n-1} + \dots + j_1)}{2^{n-1}}} x_{j_{n-1} \dots j_1 1} \end{aligned}$$

We see the first sum is the  $k^{th}$  coefficient of the FT of the sequence  $(x_{2k})_{k \in \llbracket 0, N/2-1 \rrbracket}$  and the second is the  $k^{th}$  coefficient of the FT of  $(x_{2k+1})_{k \in \llbracket 0, N/2-1 \rrbracket}$ . This shows that to compute FT of sequence of length  $N$ , we have to compute 2 FT of sequence of length  $\frac{N}{2}$  and do  $2N$  complex additions/multiplications. The complexity of the operation  $T(N)$  follows the recurrence:

$$T(N) = 2T\left(\frac{N}{2}\right) + 2N$$

We can use the Master theorem [1]:

**Theorem.** Let  $a \geq 1$  and  $b > 1$  be constants, let  $f(n)$  be a function, and let  $T(n)$  be defined on the non negative integers by the recurrence

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$

where we interpret  $\frac{n}{b}$  to mean either  $\lfloor \frac{n}{b} \rfloor$  or  $\lceil \frac{n}{b} \rceil$ . Then  $T(n)$  has the following asymptotic bounds:

- (1) If  $f(n) = O(n^{\log_b a - \epsilon})$  for some constant  $\epsilon > 0$ , then  $T(n) = \Theta(n^{\log_b a})$ .
- (2) If  $f(n) = \Theta(n^{\log_b a})$ , then  $T(n) = \Theta(n^{\log_b a} \log n)$ .
- (3) If  $f(n) = \Omega(n^{\log_b a + \epsilon})$  for some constant  $\epsilon > 0$ , and if  $af(\frac{n}{b}) \leq cf(n)$  for some constant  $c < 1$  and  $n$  sufficiently large, then  $T(n) = \Theta(f(n))$ .

Here we are in the second case of the theorem, so  $T(N) = \Theta(N \log(N)) = \Theta(n 2^n)$ .

Instead of  $\mathbb{C}$ , the Fourier transform may be used in any ring as soon as we are given a  $N$ th root of unity. The book *The design and analysis of computer algorithms* [2] provides an overview of the FFT, an algorithm using bits operations and application to fast integer multiplication.

## 5.

The inverse Fourier Transform

$$|j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-\frac{2i\pi j k}{N}} |k\rangle$$

is the adjoint of the Fourier Transform. The quantum circuit of figure 1 is obtained from the FT's circuit, replacing each  $R_k$  gate by its adjoint

$$R_k^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{2i\pi}{2^k}} \end{bmatrix}$$



FIGURE 1. Quantum circuit for IFT.



FIGURE 2. Sequence of controlled U.

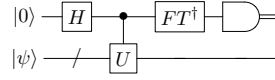
## 7.

In figure 2, the  $t$  qubits of the first register are prepared with  $|j\rangle = |j_{t-1} \dots j_1 j_0\rangle$ , the second register is prepared with some state  $|u\rangle$ . After the first controlled-U operation, the state is  $|j\rangle |U^{j_0 2^0} u\rangle$ . After the second controlled-U, the state is  $|j\rangle |U^{j_1 2^1} U^{j_0 2^0} u\rangle = |j\rangle |U^{j_0 2^0 + j_1 2^1} u\rangle$  and so on. The final state is  $|j\rangle |U^{j_0 2^0 + j_1 2^1 + \dots + j_{t-1} 2^{t-1}} u\rangle = |j\rangle |U^j u\rangle$ .

## 8.

By linearity, the phase estimation algorithm takes input  $|0\rangle |\sum_{u \in A} c_u |u\rangle\rangle$ , where  $A$  is some orthonormal basis of eigenstates of  $U$ , to output  $\sum_{u \in A} c_u |\widetilde{\varphi}_u\rangle |u\rangle$ , where  $\widetilde{\varphi}_u$  is an estimation of the phase of the eigenvalue associated with eigenstate  $u$ . If we fix  $u_0 \in A$  beforehand, the probability to measure  $\widetilde{\varphi}_{u_0}$  when measuring the first register in the computational basis is

$$\begin{aligned}
 & \left( \sum_{u \in A} c_u^* \langle \widetilde{\varphi}_u | \langle u | \right) P_{\widetilde{\varphi}_{u_0}} \otimes I \left( \sum_{u \in A} c_u |\widetilde{\varphi}_u\rangle |u\rangle \right) = \left( \sum_{u \in A} c_u^* \langle \widetilde{\varphi}_u | \langle u | \right) \left( \sum_{\substack{u \in A \\ \widetilde{\varphi}_u = \widetilde{\varphi}_{u_0}}} c_u |\widetilde{\varphi}_u\rangle |u\rangle \right) \\
 & = \left( \sum_{u \in A} c_u^* \langle \widetilde{\varphi}_u | \langle u | \right) \left( \sum_{\substack{u \in A \\ \widetilde{\varphi}_u = \widetilde{\varphi}_{u_0}}} c_u |\widetilde{\varphi}_{u_0}\rangle |u\rangle \right) \\
 & = \sum_{\substack{v \in A \\ u \in A \\ \widetilde{\varphi}_u = \widetilde{\varphi}_{u_0}}} c_v^* c_u \langle \widetilde{\varphi}_v | \widetilde{\varphi}_u \rangle \langle v | u \rangle \\
 & = \sum_{\substack{v \in A \\ u \in A \\ \widetilde{\varphi}_u = \widetilde{\varphi}_{u_0}}} c_v^* c_u \langle \widetilde{\varphi}_v | \widetilde{\varphi}_u \rangle \delta_{vu} \\
 & = \sum_{\substack{u \in A \\ \widetilde{\varphi}_u = \widetilde{\varphi}_{u_0}}} |c_u|^2 \\
 & \geq |c_{u_0}|^2
 \end{aligned}$$

FIGURE 3. Phase estimation circuit with  $t = 1$ .

$I$  is the identity operator of whatever state space  $U$  operates on, while  $P_{\widetilde{\varphi_{u_0}}}$  is the orthonormal projector onto the space generated by the vector  $|\widetilde{\varphi_{u_0}}\rangle$  of the computational basis. Besides, following the analysis of the book,  $\widetilde{\varphi_{u_0}}$  is an approximation to  $\varphi_{u_0}$  to an accuracy  $2^{-n}$  with probability at least  $1 - \epsilon$  if we make use of  $t = n + \lceil \log(2 + \frac{1}{2\epsilon}) \rceil$  bits in the first register. We conclude we get the desired approximation of  $\varphi_{u_0}$  at the end of the phase estimation algorithm with probability at least  $|c_{u_0}|^2(1 - \epsilon)$ .

## 9.

$U$  being unitary with eigenvalues  $-1$  and  $+1$ , the state space is the direct sum of the two orthogonal eigenspaces  $E_{-1} \oplus E_1$ . Thus we can uniquely decompose any  $|\psi\rangle = |\psi_{-1}\rangle + |\psi_{+1}\rangle$ , with  $|\psi_{-1}\rangle \in E_{-1}$  and  $|\psi_{+1}\rangle \in E_1$ . Then  $-1 = e^{i\pi} = e^{2i\pi 0.1}$  and  $1 = e^0 = e^{2i\pi 0.0}$  shows that it is sufficient to make use of  $t = 1$  wire in the first register in the phase estimation procedure to read directly the phase of any eigenvector. If we use  $|0\rangle |\psi\rangle$  as input in the circuit of figure 3, the output before the final measurement will be  $|0\rangle |\psi_{+1}\rangle + |1\rangle |\psi_{-1}\rangle$ .

When we measure the first register, we obtain 0 with probability

$$\begin{aligned} (\langle 0| \langle \psi_{+1}| + \langle 1| \langle \psi_{-1}|) P_0 \otimes I(|0\rangle |\psi_{+1}\rangle + |1\rangle |\psi_{-1}\rangle) &= (\langle 0| \langle \psi_{+1}| + \langle 1| \langle \psi_{-1}|)(|0\rangle |\psi_{+1}\rangle) \\ &= \langle 0|0\rangle \langle \psi_{+1}|\psi_{+1}\rangle \\ &= \langle \psi_{+1}|\psi_{+1}\rangle \end{aligned}$$

or 1 with probability

$$\begin{aligned} (\langle 0| \langle \psi_{+1}| + \langle 1| \langle \psi_{-1}|) P_1 \otimes I(|0\rangle |\psi_{+1}\rangle + |1\rangle |\psi_{-1}\rangle) &= (\langle 0| \langle \psi_{+1}| + \langle 1| \langle \psi_{-1}|)(|1\rangle |\psi_{-1}\rangle) \\ &= \langle 1|1\rangle \langle \psi_{-1}|\psi_{-1}\rangle \\ &= \langle \psi_{-1}|\psi_{-1}\rangle \end{aligned}$$

The state will collapse respectively into  $\frac{1}{\sqrt{\langle \psi_{+1}|\psi_{+1}\rangle}} |0\rangle |\psi_{+1}\rangle$  or  $\frac{1}{\sqrt{\langle \psi_{-1}|\psi_{-1}\rangle}} |1\rangle |\psi_{-1}\rangle$ . Thus if we read 0 in the first register, that means that we have an eigenvector associated to eigenvalue  $+1$  in the second register, and if we read 1 in the first register, that means that we have an eigenvector associated to eigenvalue  $-1$  in the second register.

Once we have noticed that the FT in dimension  $N = 2^1$  is just the Hadamard operator, we conclude the phase estimation circuit in this particular case is the just the same as the circuit of exercise 4.34.

## 10.

$$\begin{aligned} x^2 &= 25 = 4 \\ x^3 &= 20 = -1 \\ x^4 &= 4^2 = 16 \\ x^5 &= 16 \times 5 = 80 \\ &= 17 \\ x^6 &= (-1)^2 = 1 \end{aligned}$$

## 11.

**Theorem (Euler).** For  $N \in \mathbb{N}^*$ , let

$$\varphi(N) = \#\{m \in \llbracket 1, N \rrbracket, m \wedge N = 1\}$$

We have

$$\forall x \in \mathbb{N}^*, \quad x \wedge N = 1 \Rightarrow x^{\varphi(N)} = 1 \pmod{N}$$

Then by definition of the order  $r$ ,  $r \leq \varphi(N) \leq N$ .

## 12.

Since  $x \wedge N = 1$ , from Bezout's Theorem  $\exists(u, v) \in \mathbb{Z}^2$  such that  $ux + vN = 1$  that is  $\exists u$  such that  $ux = 1 \pmod N$  which shows that  $x$  has a multiplicative inverse  $x^{-1} = u$  in the ring  $(\frac{\mathbb{Z}}{N\mathbb{Z}}, +, \times)$ . We define the linear map  $U'$  on  $(\mathbb{C}^2)^{\otimes L} \cong \mathbb{C}^{2^L}$  that acts on the computational basis as

$$\forall y \in \{0, 1\}^L, \quad U' |y\rangle = \begin{cases} |x^{-1}y \pmod N\rangle & \text{if } y < N, \\ y & \text{if } y \in \llbracket N, 2^L - 1 \rrbracket. \end{cases}$$

We have

$$\begin{aligned} \forall y_1, y_2 \in \{0, 1\}^L, \quad \langle y_1 | U(y_2) \rangle = 1 &\Leftrightarrow y_1 = y_2 \in \llbracket N, 2^L - 1 \rrbracket \text{ or } (y_1, y_2 < N \text{ and } xy_2 = y_1 \pmod N) \\ &\Leftrightarrow y_1 = y_2 \in \llbracket N, 2^L - 1 \rrbracket \text{ or } (y_1, y_2 < N \text{ and } \exists k \in \mathbb{Z}, xy_2 = y_1 + kN) \\ &\Leftrightarrow y_1 = y_2 \in \llbracket N, 2^L - 1 \rrbracket \text{ or } (y_1, y_2 < N \text{ and } \exists k \in \mathbb{Z}, y_2 = x^{-1}y_1 + x^{-1}kN) \\ &\Leftrightarrow y_1 = y_2 \in \llbracket N, 2^L - 1 \rrbracket \text{ or } (y_1, y_2 < N \text{ and } \exists k' \in \mathbb{Z}, y_2 = x^{-1}y_1 + k'N) \\ &\Leftrightarrow y_1 = y_2 \in \llbracket N, 2^L - 1 \rrbracket \text{ or } (y_1, y_2 < N \text{ and } x^{-1}y_1 = y_2 \pmod N) \\ &\Leftrightarrow \langle U'(y_1) | y_2 \rangle = 1 \end{aligned}$$

so, since  $\langle U'(y_1) | y_2 \rangle, \langle y_1 | U(y_2) \rangle \in \{0, 1\}$ ,

$$\forall y_1, y_2 \in \{0, 1\}^L, \quad \langle y_1 | U(y_2) \rangle = \langle U'(y_1) | y_2 \rangle$$

This shows that  $U' = U^\dagger$ . since it is obvious that  $U$  is invertible and  $U^\dagger = U^{-1}$ , we have shown that  $U$  is unitary.

## 13.

$(|u_s\rangle)_{s \in \llbracket 0, r-1 \rrbracket}$  is defined to be the IFT of the sequence  $(|x^k \pmod N\rangle)_{k \in \llbracket 0, r-1 \rrbracket}$ :

$$\forall s \in \llbracket 0, r-1 \rrbracket, \quad |u_s\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-\frac{2i\pi sk}{r}} |x^k \pmod N\rangle$$

Thus the equalities

$$\forall k \in \llbracket 0, r-1 \rrbracket, \quad |x^k \pmod N\rangle = \frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} e^{\frac{2i\pi sk}{r}} |u_s\rangle$$

just state the fact that  $(|x^k \pmod N\rangle)_{k \in \llbracket 0, r-1 \rrbracket}$  is the FT of the sequence  $(|u_s\rangle)_{s \in \llbracket 0, r-1 \rrbracket}$ . Let's check this. Let  $k \in \llbracket 0, r-1 \rrbracket$ ,

$$\begin{aligned} \frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} e^{\frac{2i\pi sk}{r}} |u_s\rangle &= \frac{1}{r} \sum_{s=0}^{r-1} e^{\frac{2i\pi sk}{r}} \sum_{j=0}^{r-1} e^{-\frac{2i\pi sj}{r}} |x^j \pmod N\rangle \\ &= \frac{1}{r} \sum_{j=0}^{r-1} \left( \sum_{s=0}^{r-1} (e^{\frac{2i\pi(k-j)}{r}})^s \right) |x^j \pmod N\rangle \\ &= \frac{1}{r} \sum_{j=0}^{r-1} r \delta_{jk} |x^j \pmod N\rangle \\ &= |x^k \pmod N\rangle \end{aligned}$$

For  $k = 0$  we obtain

$$\frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} |u_s\rangle = |1\rangle$$

## 15.

The easiest way is to think with the prime decomposition of the integers  $x$  and  $y$ . Let  $d = x \wedge y$  and  $m = x \vee y$ . Let  $p_0, p_1, \dots, p_n$  be the prime numbers which appear in either prime decomposition. We can write

$$\begin{aligned} x &= p_0^{\alpha_0} p_1^{\alpha_1} \dots p_n^{\alpha_n} \\ y &= p_0^{\beta_0} p_1^{\beta_1} \dots p_n^{\beta_n} \end{aligned}$$

where  $\alpha_i, \beta_i \in \mathbb{N}$ . Then it is clear that

$$\begin{aligned} d &= p_0^{\gamma_0} p_1^{\gamma_1} \dots p_n^{\gamma_n} \\ m &= p_0^{\delta_0} p_1^{\delta_1} \dots p_n^{\delta_n} \end{aligned}$$

where  $\gamma_i = \min(\alpha_i, \beta_i)$  and  $\delta_i = \max(\alpha_i, \beta_i)$ . We have  $\alpha_i + \beta_i = \gamma_i + \delta_i$ . Then,

$$\begin{aligned} md &= p_0^{\gamma_0} p_1^{\gamma_1} \dots p_n^{\gamma_n} p_0^{\delta_0} p_1^{\delta_1} \dots p_n^{\delta_n} \\ &= p_0^{\gamma_0 + \delta_0} p_1^{\gamma_1 + \delta_1} \dots p_n^{\gamma_n + \delta_n} \\ &= p_0^{\alpha_0 + \beta_0} p_1^{\alpha_1 + \beta_1} \dots p_n^{\alpha_n + \beta_n} \\ &= xy \end{aligned}$$

## 16.

Let  $x \geq 2$ .

$$\begin{aligned} \int_x^{x+1} \frac{1}{y^2} dy &= \frac{1}{x} - \frac{1}{x+1} \\ &= \frac{1}{x(x+1)} \end{aligned}$$

since

$$x+1 \leq \frac{3}{2}x \Leftrightarrow 2 \leq x$$

$$\int_x^{x+1} \frac{1}{y^2} dy = \frac{1}{x(x+1)} \geq \frac{2}{3x^2}$$

If we sum these inequalities

$$\sum_{q=2}^{+\infty} \frac{1}{q^2} \leq \frac{3}{2} \sum_{q=2}^{+\infty} \int_q^{q+1} \frac{1}{y^2} dy = \frac{3}{2} \int_2^{+\infty} \frac{1}{y^2} dy = \frac{3}{4}$$

and finally

$$\sum_{\substack{q \in \mathbb{N}^* \\ q \text{ is prime}}} \frac{1}{q^2} \leq \sum_{q=2}^{+\infty} \frac{1}{q^2} \leq \frac{3}{4}$$

## 17.

**17.1.** The assertion  $N = a^b \Rightarrow b \leq L$  is obviously wrong if  $N = a = 1$ . Since we aim to prove an asymptotical result, we can assume that  $N \geq 2$ .

$$\begin{aligned} N = a^b &\Leftrightarrow \log N = b \log a \\ &\Leftrightarrow \frac{\log N}{\log a} = b & (N \geq 2 \Rightarrow a \geq 2 \Rightarrow \log a \geq 1 > 0) \\ &\Rightarrow b \leq \log N \\ &\Leftrightarrow b \leq \lfloor \log N \rfloor = L - 1 < L \quad (b \in \mathbb{N}) \end{aligned}$$

**17.2.** Let  $N = 2^l + a_{l-1}2^{l-1} + \dots + a_12 + a_0$  with  $l + 1 \leq L$  and  $a_i \in \{0, 1\}$ .

$$\begin{aligned} N &= 2^l(1 + a_{l-1}2^{-1} + \dots + a_12^{-l+1} + a_02^{-l}) \\ &= 2^l(1 + f) \end{aligned}$$

with  $f \in [0, 1[$ .

$$\begin{aligned} \log N &= l + \log(1 + a_{l-1}2^{-1} + \dots + a_12^{-l+1} + a_02^{-l}) \\ &= l + \log(1 + f) \end{aligned}$$

where  $\log$  is  $\log_2$ . This shows that to compute an approximation to  $\log N$ , we just need an approximation of  $\log$  in range  $[1, 2[$  or any interval of the form  $[t, 2t[$  for instance  $[\frac{3}{4}, \frac{1}{2}[$ . Besides,

$$\begin{aligned} \forall x \in ]-1, 1], \quad \log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n+1} \frac{x^n}{n} + \dots \\ &= \sum_{k=1}^{+\infty} (-1)^{k+1} \frac{x^k}{k} \end{aligned}$$

Let's write it until order  $L - 1$ :

$$\forall x \in ]-1, +\infty[, \quad \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{L+1} \frac{x^{L-1}}{L-1} + \sum_{k=L}^{+\infty} (-1)^{k+1} \frac{x^k}{k}$$

For  $x \in [0, \frac{1}{2}[$ , this is an alternating series and we can bound the rest by

$$\begin{aligned} \left| \sum_{k=L}^{+\infty} (-1)^{k+1} \frac{x^k}{k} \right| &\leq \left| (-1)^L \frac{x^L}{L} \right| \\ &\leq \frac{1}{2^L L} \end{aligned}$$

For  $x \in [-\frac{1}{4}, 0[$ , we can use Lagrange formula to bound the rest by

$$\begin{aligned} \exists \xi \in [-\frac{1}{4}, 0[, \quad \left| \sum_{k=L}^{+\infty} (-1)^{k+1} \frac{x^k}{k} \right| &= \left| \frac{\log^{(L)}(\xi)}{(L)!} x^L \right| \\ &= \frac{(L-1)!}{(1+\xi)^{L-1}} |x^L| \\ &= \frac{1}{(1+\xi)^{L-1}} |x^L| \\ &\leq \frac{1}{(1+\xi)^{L-1}} \frac{1}{4^L} \\ &\leq \frac{1}{(\frac{3}{4})^{L-1}} \frac{1}{4^L} \\ &= \frac{1}{3^{L-1} 4^L} \end{aligned}$$

This shows that we can use the Taylor series up to order  $L - 1$  to approximate  $\log(x)$  with precision  $2^{-L}$  on the range  $[\frac{3}{4}, \frac{1}{2}[$ . This is to simplify the complexity analysis. In actual implementation though better and faster approximation are used: See the book by Cheney [4] for mathematical foundations of the approximation of functions by polynomials including the Remez algorithm. See also this insightful post [3] which discusses tradeoffs between accuracy and speed in approximating this log function, taking into account error induced by floating-point representation of real numbers. Here [5] can be found an actual implementation of the C standard library.

In addition to the Taylor error, there is an error occuring when computing the polynomial using floating-point arithmetic. If we store the significand of the floating-point variables in binary on  $L+1$  bits, and use

$O(L)$  bits to do arithmetic operations, each operation will incur a relative error of at most  $\epsilon = 2^{-L-1}$ , i.e.

$$\begin{aligned} x \oplus y &= (x + y)(1 + \xi) \\ x \ominus y &= (x - y)(1 + \xi) \\ x \otimes y &= (x \times y)(1 + \xi) \\ x \oslash y &= (x \div y)(1 + \xi) \end{aligned}$$

where  $|\xi| \leq \epsilon$  and the values on the left are the value computed exactly and then rounded on  $L + 1$  digits. For the details on floating point arithmetic see [6]. The previous polynomial can be rewritten as:

$$\begin{aligned} \log(1 + x) &= \sum_{k=1}^n (-1)^{k+1} \frac{1}{k} x^k \\ &= x(1 + x(-\frac{1}{2} + x(\frac{1}{3} + \dots + ((-1)^{n-1} \frac{1}{n-1} + (-1)^n \frac{1}{n} x) \dots))) \end{aligned}$$

This shows that the evaluation costs  $n$  fused multiply-add operations. If one rounding error occurs for each of the multiply-add, we have the following bound on the error due to floating-point arithmetic (see [7] for a detailed analysis):

$$\begin{aligned} |\bar{P}(x) - \tilde{P}(x)| &= \left| \sum_{j=1}^{n-1} (\xi_j \sum_{i=j}^n (-1)^{i+1} \frac{1}{i} x^i) \right| \\ &= \left| \sum_{i=1}^{n-1} \left( \sum_{j=1}^i \xi_j \right) (-1)^{i+1} \frac{1}{i} x^i + \left( \sum_{j=1}^{n-1} \xi_j \right) (-1)^n \frac{1}{n} x^n \right| \\ &\leq \sum_{i=1}^{n-1} \left( \sum_{j=1}^i \epsilon \right) \frac{1}{i} |x^i| + \left( \sum_{j=1}^{n-1} \epsilon \right) \frac{1}{n} |x^n| \\ &= \epsilon \sum_{i=1}^{n-1} |x^i| + \epsilon \frac{n-1}{n} |x^n| \\ &\leq \epsilon \sum_{i=1}^n \frac{1}{2^i} \\ &= \epsilon (1 - (\frac{1}{2})^n) \\ &\leq \epsilon \end{aligned}$$

and we add the error due to just storing the coefficients of the polynomial on  $L$  bits: for instance  $\frac{1}{3} = 0.010101\dots$  is rounded when storing in binary. If  $\tilde{a}_i$  is the rounded value of  $a_i = (-1)^{i+1} \frac{1}{i}$ , the error will be:

$$\begin{aligned} |P(x) - \tilde{P}(x)| &\leq \sum_{i=1}^n |a_i - \tilde{a}_i| |x^i| \\ &\leq \sum_{i=1}^n \epsilon |a_i| |x^i| \\ &= \epsilon \sum_{i=1}^n \frac{1}{i} |x^i| \\ &\leq \epsilon \sum_{i=1}^n |x^i| \\ &\leq \epsilon \end{aligned}$$



Taking into consideration the three types of error, we see that the taylor series of order  $L$  is a approximation to  $\log(x)$  on range  $[\frac{3}{4}, \frac{1}{2}[$  with precision  $2^{-L}$  since :

$$\frac{1}{2^{L+1}(L+1)} + 2\epsilon \leq 2^{-L} \quad (1)$$

$$\Leftrightarrow \frac{2^{-L-1}}{L} + 2^{-L} \leq 2^{-L} \quad (2)$$

$$\Leftrightarrow 2^{-L-1}(\frac{1}{L} + 1) \leq 2^{-L} \quad (3)$$

$$\Leftrightarrow L \geq 1 \quad (4)$$

$$(5)$$

This analysis shows that the procedure LOG2 computes an approximation of  $\log(N)$  to precision  $2^{-L}$ . Binary addition-substraction costs  $\Theta(L)$  operations, grade-school multiplication-division costs  $\Theta(L^2)$ . Multiplication complexity can be improved to:

- $O(L^{\log_2(3)})$  using Karatsuba algorithm [2].
- $O(L \log(L) \log \log(L))$  using Schönhage-Strassen algorithm [2].
- $O(L \log L \log^* L)$  using Furer algorithm [8].

In the end computing  $\log_2 N$  has an  $O(L^3)$  time complexity. If we are given an approximating polynomial and are assured it gives the desired precision for any input size considered, the complexity is  $O(L^2)$ . The complexity of finding  $\lfloor \log_2(N) \rfloor$  given the binary representation of  $N$  is  $O(L)$ .

LOG2( $N, L$ )

```

1  //  $L \geq l + 1$  where  $l = \lfloor \log_2(N) \rfloor$ , i.e.  $2^l \leq N < 2^{l+1}$ .
2  for  $j = 1$  to  $L$ 
3       $A[j] = (-1)^{j+1} \frac{1}{j}$ 
4  //  $N = 2^m(1 + f)$ 
5   $m = \lfloor \log_2(N) \rfloor$ 
6   $f = \frac{N}{2^m} - 1$  // no rounding error in  $f$ .
7  if  $f \geq \frac{1}{2}$ 
8       $f = \frac{1/2 - f}{2}$ 
9       $m = m + 1$ 
10  $q = 0$ 
11 for  $j = L$  downto 0
12      $q = q \times f + A[j]$ 
13  $q = q \div \log_e(2)$ 
14 return  $q + m$ 
```

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