QUANTUM COMPUTATION AND QUANTUM INFORMATION: THE QUANTUM FOURIER TRANSFORM

1

We consider the linear map in $\mathbb{C}^N$ which acts on the computational basis as

$$|j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2i\pi jk}{N}} |k\rangle$$

Let A be the matrix of the transformation in the computational basis.

$$\forall (k,l) \in [\![0, N-1]\!]^2, \quad a_{kl} = \frac{1}{\sqrt{N}} e^{\frac{2i\pi kl}{N}}$$

The adjoint matrix $A^\dagger$ is then

$$\forall (k,l) \in [\![0, N-1]\!]^2, \quad b_{kl} = a_{lk}^*$$
$$= \frac{1}{\sqrt{N}} e^{-\frac{2i\pi kl}{N}}$$

We compute the coefficient k, l of the product $AA^\dagger$:

$$\forall (k,l) \in [\![0, N-1]\!]^2, \quad c_{kl} = \sum_{j=0}^{N-1} a_{kj} b_{jl}$$
$$= \frac{1}{N} \sum_{j=0}^{N-1} e^{\frac{2i\pi j}{N}(k-l)}$$
$$= \frac{1}{N} \sum_{j=0}^{N-1} (e^{\frac{2i\pi}{N}(k-l)})^j$$
$$= \begin{cases} \dfrac{1}{N} \dfrac{1 - (e^{\frac{2i\pi}{N}(k-l)})^N}{1 - e^{\frac{2i\pi}{N}(k-l)}} = 0 & \text{if } e^{\frac{2i\pi}{N}(k-l)} \neq 1, \\ 1 & \text{if } e^{\frac{2i\pi}{N}(k-l)} = 1. \end{cases}$$
$$= \begin{cases} 0 & \text{if } k \neq l, \\ 1 & \text{if } k = l. \end{cases}$$
$$= \delta_{kl}$$

which shows that $AA^\dagger = A^\dagger A = I$ i.e. A is unitary.

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Here the dimension of the state space is $N = 2^n$. The Fourier transform of the n qubit state $|00\ldots0\rangle$ is

$$A|0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle$$

we can write k in binary $k_{n-1}\ldots k_1 k_0$

$$A|0\rangle = \frac{1}{2^{n/2}} \sum_{k_0, k_1, \ldots, k_{n-1}=0}^{1} |k_{n-1}\ldots k_1 k_0\rangle$$

or in product representation,

$$= \frac{1}{2^{n/2}} \underbrace{(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)\ldots(|0\rangle + |1\rangle)}_{n \text{ qubits}}$$

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Let $N = 2^n$ and $Y = (y_k)_{k \in [\![0, N-1]\!]}$ be the classical fourier transform of $X = (x_k)_{k \in [\![0, N-1]\!]}$.

$$\forall k \in [\![0, N-1]\!], \quad y_k = \sum_{j=0}^{N-1} e^{\frac{2i\pi kj}{2^n}} x_j$$

The factor $\frac{1}{\sqrt{N}}$ is omitted for clarity. We can write j in binary $j_{n-1} \ldots j_1 j_0$

$$
\begin{aligned}
y_k &= \sum_{j_0, j_1, \ldots, j_{n-1}=0}^{1} e^{\frac{2i\pi k(2^{n-1}j_{n-1}+\cdots+2j_1+j_0)}{2^n}} x_j \\
&= \sum_{j_1, \ldots, j_{n-1}=0}^{1} e^{\frac{2i\pi k(2^{n-1}j_{n-1}+\cdots+2j_1)}{2^n}} x_{j_{n-1}\ldots j_1 0} + \sum_{j_1, \ldots, j_{n-1}=0}^{1} e^{\frac{2i\pi k(2^{n-1}j_{n-1}+\cdots+2j_1+1)}{2^n}} x_{j_{n-1}\ldots j_1 1} \\
&= \sum_{j_1, \ldots, j_{n-1}=0}^{1} e^{\frac{2i\pi k(2^{n-1}j_{n-1}+\cdots+2j_1)}{2^n}} x_{j_{n-1}\ldots j_1 0} + e^{\frac{2i\pi k}{2^n}}\sum_{j_1, \ldots, j_{n-1}=0}^{1} e^{\frac{2i\pi k(2^{n-1}j_{n-1}+\cdots+2j_1)}{2^n}} x_{j_{n-1}\ldots j_1 1} \\
&= \sum_{j_1, \ldots, j_{n-1}=0}^{1} e^{\frac{2i\pi k(2^{n-2}j_{n-1}+\cdots+j_1)}{2^{n-1}}} x_{j_{n-1}\ldots j_1 0} + e^{\frac{2i\pi k}{2^n}}\sum_{j_1, \ldots, j_{n-1}=0}^{1} e^{\frac{2i\pi k(2^{n-2}j_{n-1}+\cdots+j_1)}{2^{n-1}}} x_{j_{n-1}\ldots j_1 1}
\end{aligned}
$$

We see the first sum is the kth coefficient of the FT of the sequence $(x_{2k})_{k \in [\![0, N/2-1]\!]}$ and the second is the kth coefficient of the FT of $(x_{2k+1})_{k \in [\![0, N/2-1]\!]}$. This shows that to compute FT of sequence of length N, we have to compute 2 FT of sequence of length $\frac{N}{2}$ and do $2N$ complex additions/multiplications. The complexity of the operation $T(N)$ follows the recurrence:

$$T(N) = 2T(\frac{N}{2}) + 2N$$

We can use the Master theorem [1]:

Theorem. Let $a \geqslant 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the non negative integers by the recurrence

$$T(n) = aT(\frac{n}{b}) + f(n)$$

where we interpret $\frac{n}{b}$ to mean either $\lfloor \frac{n}{b} \rfloor$ or $\lceil \frac{n}{b} \rceil$. Then $T(n)$ has the following asymptotic bounds:

  (1) If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
  (2) If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \log n)$.
  (3) If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(\frac{n}{b}) \leqslant cf(n)$ for some constant $c < 1$ and n sufficiently large, then $T(n) = \Theta(f(n))$.

Here we are in the second case of the theorem, so $T(N) = \Theta(N \log(N)) = \Theta(n2^n)$.

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The inverse Fourier Transform

$$|j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{-\frac{2i\pi jk}{N}} |k\rangle$$

is the adjoint of the Fourier Transform. The quantum circuit of figure 1 is obtained from the FT's circuit, replacing each R_k gate by its adjoint

$$R_k^{\dagger} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{2i\pi}{2^k}} \end{bmatrix}$$

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[1] Thomas H. Cormen and Charles E. Leiserson : *Introduction to algorithms*, MIT Press (2009)

FIGURE 1. Quantum circuit for IFT.
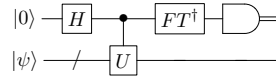

FIGURE 2. Sequence of controlled U.

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In figure 2, the t qubits of the first register are prepared with $|j\rangle = |j_{t-1} \dots j_1 j_0\rangle$, the second register is prepared with some state $|u\rangle$. After the first controlled-U operation, the state is $|j\rangle |U^{j_0 2^0} u\rangle$. After the second controlled-U, the state is $|j\rangle |U^{j_1 2^1} U^{j_0 2^0} u\rangle = |j\rangle |U^{j_0 2^0 + j_1 2^1} u\rangle$ and so on. The final state is $|j\rangle |U^{j_0 2^0 + j_1 2^1 + \dots + j_{t-1} 2^{t-1}} u\rangle = |j\rangle |U^j u\rangle$.

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By linearity, the phase estimation algorithm takes input $|0\rangle |\Sigma_{u \in A} c_u |u\rangle\rangle$, where A is some orthonormal basis of eigenstates of U, to output $\sum_{u \in A} c_u |\widetilde{\varphi_u}\rangle |u\rangle$, where $\widetilde{\varphi_u}$ is an estimation of the phase of the eigenvalue associated with eigenstate u. If we fix $u_0 \in A$ beforehand, the probability to measure $\widetilde{\varphi_{u_0}}$ when measuring the first register in the computational basis is

$$
\begin{aligned}
(\sum_{u \in A} c_u^* \langle \widetilde{\varphi_u}| \langle u|) P_{\widetilde{\varphi_{u_0}}} \otimes I(\sum_{u \in A} c_u |\widetilde{\varphi_u}\rangle |u\rangle) &= (\sum_{u \in A} c_u^* \langle \widetilde{\varphi_u}| \langle u|)(\sum_{\substack{u \in A \\ \widetilde{\varphi_u} = \widetilde{\varphi_{u_0}}}} c_u |\widetilde{\varphi_u}\rangle |u\rangle) \\
&= (\sum_{u \in A} c_u^* \langle \widetilde{\varphi_u}| \langle u|)(\sum_{\substack{u \in A \\ \widetilde{\varphi_u} = \widetilde{\varphi_{u_0}}}} c_u |\widetilde{\varphi_{u_0}}\rangle |u\rangle) \\
&= \sum_{\substack{v \in A \\ u \in A \\ \widetilde{\varphi_u} = \widetilde{\varphi_{u_0}}}} c_v^* c_u \langle \widetilde{\varphi_v}|\widetilde{\varphi_u}\rangle \langle v|u\rangle \\
&= \sum_{\substack{v \in A \\ u \in A \\ \widetilde{\varphi_u} = \widetilde{\varphi_{u_0}}}} c_v^* c_u \langle \widetilde{\varphi_v}|\widetilde{\varphi_u}\rangle \delta_{vu} \\
&= \sum_{\substack{u \in A \\ \widetilde{\varphi_u} = \widetilde{\varphi_{u_0}}}} |c_u|^2 \\
&\geqslant |c_{u_0}|^2
\end{aligned}
$$

FIGURE 3. Phase estimation circuit with $t = 1$.

I is the identity operator of whatever state space U operates on, while $P_{\widetilde{\varphi_{u_0}}}$ is the orthonormal projector onto the space generated by the vector $|\widetilde{\varphi_{u_0}}\rangle$ of the computational basis. Besides, following the analysis of the book, $\widetilde{\varphi_{u_0}}$ is an approximation to φ_{u_0} to an accuracy 2^{-n} with probability at least $1 - \epsilon$ if we make use of $t = n + \lceil \log(2 + \frac{1}{2\epsilon}) \rceil$ bits in the first register. We conclude we get the desired approximation of φ_{u_0} at the end of the phase estimation algorithm with probability at least $|c_{u_0}|^2(1 - \epsilon)$.

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U being unitary with eigenvalues -1 and +1, the state space is the direct sum of the two orthogonal eigenspaces $E_{-1} \oplus E_1$. Thus we can uniquely decompose any $|\psi\rangle = |\psi_{-1}\rangle + |\psi_{+1}\rangle$, with $|\psi_{-1}\rangle \in E_{-1}$ ans $|\psi_{+1}\rangle \in E_{+1}$. Then $-1 = e^{i\pi} = e^{2i\pi 0.1}$ and $1 = e^0 = e^{2i\pi 0.0}$ shows that is is sufficient to make use of $t = 1$ wire in the first register in the phase estimation procedure to read directly the phase of any eigenvector. If we use $|0\rangle |\psi\rangle$ as input in the circuit of figure 3, the output before the final measurement will be $|0\rangle |\psi_{+1}\rangle + |1\rangle |\psi_{-1}\rangle$.

When we measure the first register, we obtain 0 with probability

$$(\langle 0| \langle\psi_{+1}| + \langle 1| \langle\psi_{-1}|)P_0 \otimes I(|0\rangle |\psi_{+1}\rangle + |1\rangle |\psi_{-1}\rangle) = (\langle 0| \langle\psi_{+1}| + \langle 1| \langle\psi_{-1}|)(|0\rangle |\psi_{+1}\rangle)$$
$$= \langle 0|0\rangle \langle\psi_{+1}|\psi_{+1}\rangle$$
$$= \langle\psi_{+1}|\psi_{+1}\rangle$$

or 1 with probability

$$(\langle 0| \langle\psi_{+1}| + \langle 1| \langle\psi_{-1}|)P_1 \otimes I(|0\rangle |\psi_{+1}\rangle + |1\rangle |\psi_{-1}\rangle) = (\langle 0| \langle\psi_{+1}| + \langle 1| \langle\psi_{-1}|)(|1\rangle |\psi_{-1}\rangle)$$
$$= \langle 1|1\rangle \langle\psi_{-1}|\psi_{-1}\rangle$$
$$= \langle\psi_{-1}|\psi_{-1}\rangle$$

The state will collapse respectively into $\frac{1}{\sqrt{\langle\psi_{+1}|\psi_{+1}\rangle}} |0\rangle |\psi_{+1}\rangle$ or $\frac{1}{\sqrt{\langle\psi_{-1}|\psi_{-1}\rangle}} |1\rangle |\psi_{-1}\rangle$. Thus if we read 0 in the first register, that means that we have an eigenvector associated to eigenvalue +1 in the second register, and if we read 1 in the first register, that means that we have an eigenvector associated to eigenvalue -1 in the second register.

Once we have noticed that the FT in dimension $N = 2^1$ is just the Hadamard operator, we conclude the phase estimation circuit in this particular case is the just the same as the circuit of exercice 4.34.

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$$
\begin{aligned}
x^2 &= & 25 & = 4 \\
x^3 &= & 20 & = -1 \\
x^4 &= & 4^2 & = 16 \\
x^5 &= 16 \times 5 & = 80 \\
& & & = 17 \\
x^6 &= (-1)^2 & = 1
\end{aligned}
$$

11

Theorem (*Euler*). For $N \in \mathbb{N}^*$, let

$$\varphi(N) = \#\{m \in [\![1, N]\!], m \wedge N = 1\}$$

We have

$$\forall x \in \mathbb{N}^*, \quad x \wedge N = 1 \Rightarrow x^{\varphi(N)} = 1 \mod N$$

Then by definition of the order r, $r \leqslant \varphi(N) \leqslant N$.

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Since $x \wedge N = 1$, from Bezout's Theorem $\exists (u, v) \in \mathbb{Z}^2$ such that $ux + vN = 1$ that is $\exists u$ such that $ux = 1$ mod N which shows that x has a multiplicative inverse $x^{-1} = u$ in the ring $(\frac{\mathbb{Z}}{N\mathbb{Z}}, +, \times)$. We define the linear map U' on $(\mathbb{C}^2)^{\otimes L} \cong \mathbb{C}^{2^L}$ that acts on the computational basis as

$$\forall y \in \{0, 1\}^L, \quad U' |y\rangle = \begin{cases} |x^{-1}y \mod N\rangle & \text{if } y < N, \\ y & \text{if } y \in [\![N, 2^L - 1]\!]. \end{cases}$$

We have

$$\begin{aligned}
\forall y_1, y_2 \in \{0, 1\}^L, \quad \langle y_1 | U(y_2) \rangle = 1 &\Leftrightarrow y_1 = y_2 \in [\![N, 2^L - 1]\!] \text{ or } (y_1, y_2 < N \text{ and } xy_2 = y_1 \mod N) \\
&\Leftrightarrow y_1 = y_2 \in [\![N, 2^L - 1]\!] \text{ or } (y_1, y_2 < N \text{ and } \exists k \in \mathbb{Z}, xy_2 = y_1 + kN) \\
&\Leftrightarrow y_1 = y_2 \in [\![N, 2^L - 1]\!] \text{ or } (y_1, y_2 < N \text{ and } \exists k \in \mathbb{Z}, y_2 = x^{-1}y_1 + x^{-1}kN) \\
&\Leftrightarrow y_1 = y_2 \in [\![N, 2^L - 1]\!] \text{ or } (y_1, y_2 < N \text{ and } \exists k' \in \mathbb{Z}, y_2 = x^{-1}y_1 + k'N) \\
&\Leftrightarrow y_1 = y_2 \in [\![N, 2^L - 1]\!] \text{ or } (y_1, y_2 < N \text{ and } x^{-1}y_1 = y_2 \mod N) \\
&\Leftrightarrow \langle U'(y_1) | y_2 \rangle = 1
\end{aligned}$$

so, since $\langle U'(y_1) | y_2 \rangle, \langle y_1 | U(y_2) \rangle \in \{0, 1\}$,

$$\forall y_1, y_2 \in \{0, 1\}^L, \quad \langle y_1 | U(y_2) \rangle = \langle U'(y_1) | y_2 \rangle$$

This shows that $U' = U^\dagger$. since it is obvious that U is invertible and $U^\dagger = U^{-1}$, we have shown that U is unitary.

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$(|u_s\rangle)_{s \in [\![0, r-1]\!]}$ is defined to be the IFT of the sequence $(|x^k \mod N\rangle)_{k \in [\![0, r-1]\!]}$:

$$\forall s \in [\![0, r-1]\!], \quad |u_s\rangle = \frac{1}{\sqrt{r}} \sum_{k=0}^{r-1} e^{-\frac{2i\pi sk}{r}} |x^k \mod N\rangle$$

Thus the equalities

$$\forall k \in [\![0, r-1]\!], \quad |x^k \mod N\rangle = \frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} e^{\frac{2i\pi sk}{r}} |u_s\rangle$$

just express the fact that $(|x^k \mod N\rangle)_{k \in [\![0, r-1]\!]}$ is the FT of the sequence $(|u_s\rangle)_{s \in [\![0, r-1]\!]}$. Let's check this. Let $k \in [\![0, r-1]\!]$,

$$\begin{aligned}
\frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} e^{\frac{2i\pi sk}{r}} |u_s\rangle &= \frac{1}{r} \sum_{s=0}^{r-1} e^{\frac{2i\pi sk}{r}} \sum_{j=0}^{r-1} e^{-\frac{2i\pi sj}{r}} |x^j \mod N\rangle \\
&= \frac{1}{r} \sum_{j=0}^{r-1} (\sum_{s=0}^{r-1} (e^{\frac{2i\pi(k-j)}{r}})^s) |x^j \mod N\rangle \\
&= \frac{1}{r} \sum_{j=0}^{r-1} r\delta_{jk} |x^j \mod N\rangle \\
&= |x^k \mod N\rangle
\end{aligned}$$

For $k = 0$ we obtain

$$\frac{1}{\sqrt{r}} \sum_{s=0}^{r-1} |u_s\rangle = |1\rangle$$

15

The easiest way is to think with the prime decomposition of the integers x and y. Let $d = x \wedge y$ and $m = x \vee y$. Let $p_0, p_1, \ldots, p_n$ be the prime numbers which appear in either prime decomposition. We can write

$$x = p_0^{\alpha_0} p_1^{\alpha_1} \ldots p_n^{\alpha_n}$$
$$y = p_0^{\beta_0} p_1^{\beta_1} \ldots p_n^{\beta_n}$$

where $\alpha_i, \beta_i \in \mathbb{N}$. Then it is clear that

$$d = p_0^{\gamma_0} p_1^{\gamma_1} \ldots p_n^{\gamma_n}$$
$$m = p_0^{\delta_0} p_1^{\delta_1} \ldots p_n^{\delta_n}$$

where $\gamma_i = \min(\alpha_i, \beta_i)$ and $\delta_i = \max(\alpha_i, \beta_i)$. We have $\alpha_i + \beta_i = \gamma_i + \delta_i$. Then,

$$\begin{aligned} md &= p_0^{\gamma_0} p_1^{\gamma_1} \ldots p_n^{\gamma_n} p_0^{\delta_0} p_1^{\delta_1} \ldots p_n^{\delta_n} \\ &= p_0^{\gamma_0 + \delta_0} p_1^{\gamma_1 + \delta_1} \ldots p_n^{\gamma_n + \delta_n} \\ &= p_0^{\alpha_0 + \beta_0} p_1^{\alpha_1 + \beta_1} \ldots p_n^{\alpha_n + \beta_n} \\ &= xy \end{aligned}$$

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Let $x \geqslant 2$.

$$\begin{aligned} \int_x^{x+1} \frac{1}{y^2} \, dy &= \frac{1}{x} - \frac{1}{x+1} \\ &= \frac{1}{x(x+1)} \end{aligned}$$

since

$$x + 1 \leqslant \frac{3}{2} x \Leftrightarrow 2 \leqslant x$$

$$\int_x^{x+1} \frac{1}{y^2} \, dy = \frac{1}{x(x+1)} \geqslant \frac{2}{3x^2}$$

If we sum these inequalities

$$\sum_{q=2}^{+\infty} \frac{1}{q^2} \leqslant \frac{3}{2} \sum_{q=2}^{+\infty} \int_q^{q+1} \frac{1}{y^2} \, dy = \frac{3}{2} \int_2^{+\infty} \frac{1}{y^2} \, dy = \frac{3}{4}$$

and finally

$$\sum_{\substack{q \in \mathbb{N}^* \\ q \text{ is prime}}} \frac{1}{q^2} \leqslant \sum_{q=2}^{+\infty} \frac{1}{q^2} \leqslant \frac{3}{4}$$

17

17.1. The assertion $N = a^b \Rightarrow b \leqslant L$ is obviously wrong if $N = a = 1$. We now assume that $N \geqslant 2$.

$$\begin{aligned} N = a^b &\Leftrightarrow \log N = b \log a \\ &\Leftrightarrow \frac{\log N}{\log a} = b \\ &\Rightarrow b \leqslant \log N \\ &\Leftrightarrow b \leqslant \lfloor \log N \rfloor = L - 1 < L \quad (b \in \mathbb{N}) \end{aligned}$$