

QUANTUM COMPUTATION AND QUANTUM INFORMATION: QUANTUM NOISE AND QUANTUM OPERATIONS

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20. Circuit model for amplitude damping

We want to prove that the following circuit models the amplitude damping operation



Recall that

$$\begin{aligned} R_y(\theta) &= e^{-i\frac{\theta}{2}Y} \\ &= \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \end{aligned}$$

Initially the two-qubit state is

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle = \alpha|00\rangle + \beta|10\rangle$$

After the controlled R_y gate it becomes

$$\begin{aligned} \alpha|00\rangle + \beta|1\rangle R_y(\theta)|0\rangle &= \alpha|00\rangle + \beta|1\rangle (\cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})|1\rangle) \\ &= \alpha|00\rangle + \beta(\cos(\frac{\theta}{2})|10\rangle + \sin(\frac{\theta}{2})|11\rangle) \end{aligned}$$

After the controlled not gate,

$$\alpha|00\rangle + \beta(\cos(\frac{\theta}{2})|10\rangle + \sin(\frac{\theta}{2})|01\rangle)$$

This is the effect of amplitude damping, with probability of 1 be switched to 0, or one photon being lost to environment, being $\gamma = \sin^2(\frac{\theta}{2})$.

21. Amplitude damping of a harmonic oscillator

The principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian:

$$H = \chi(a^\dagger b + b^\dagger a)$$

where a^\dagger, a and b^\dagger, b are the creation, annihilation operators for the principal and environment oscillators, respectively.

The time evolution of the coupled system is governed by the unitary operator:

$$U = e^{-iH\Delta t}$$

21.1. Operation elements. We recall some results for the harmonic oscillator:

$$\forall n \in \mathbb{N}, \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

and similarly in the environment space

$$\forall n \in \mathbb{N}, \quad b^\dagger |n\rangle_b = \sqrt{n+1} |n+1\rangle_b$$

Here we use the subscript b to differentiate the eigenvectors of the Hermitian operator bb^\dagger which live in the environment space from the eigenvectors of aa^\dagger in the principal space:

$$\begin{aligned} \forall n \in \mathbb{N}, \quad bb^\dagger |n\rangle_b &= (n+1) |n\rangle_b \\ \forall n \in \mathbb{N}, \quad aa^\dagger |n\rangle &= (n+1) |n\rangle \end{aligned}$$

Each set of vectors constitute an orthonormal basis:

$$\begin{aligned} \forall (n, m) \in \mathbb{N}^2, \quad \langle n|m \rangle &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases} \\ &= \delta_{nm} \end{aligned}$$

We also have

$$\begin{aligned} aa^\dagger - a^\dagger a &= [a, a^\dagger] \\ &= 1 \\ bb^\dagger - b^\dagger b &= [b, b^\dagger] \\ &= 1 \end{aligned}$$

where 1 stands for the identity operator.

Each of the operators a, a^\dagger commutes with each of the operators b, b^\dagger since they act on different spaces

$$\begin{aligned} 0 &= [a^\dagger, b^\dagger] \\ &= [a, b^\dagger] \\ &= [a^\dagger, b] \\ &= [a, b] \end{aligned}$$

The Baker-Campbell-Hausdorff formula states that, for any operators A, G such that e^G exists,

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} C_n$$

where the operators C_n are defined recursively by

$$\begin{aligned} C_0 &= A \\ C_1 &= [G, A] \\ \forall n \in \mathbb{N}, \quad C_{n+1} &= [G, C_n] \end{aligned}$$

Lets compute a simplified expression for the operator $U a^\dagger U^\dagger$ acting on the product space:

$$\begin{aligned} U a^\dagger U^\dagger &= e^{-iH\Delta t} a^\dagger e^{iH\Delta t} \\ &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \end{aligned} \tag{1}$$

The first commutators C_n are

$$\begin{aligned}
 C_0 &= a^\dagger \\
 C_1 &= [H, a^\dagger] \\
 &= [\chi b^\dagger a, a^\dagger] \\
 &= \chi b^\dagger [a, a^\dagger] \\
 &= \chi b^\dagger \\
 C_2 &= [H, C_1] \\
 &= [\chi a^\dagger b, \chi b^\dagger] \\
 &= \chi^2 a^\dagger [b, b^\dagger] \\
 &= \chi^2 a^\dagger
 \end{aligned}$$

from which it follows that

$$\begin{aligned}
 \forall n \in \mathbb{N}, \quad C_{2n} &= \chi^{2n} a^\dagger \\
 C_{2n+1} &= \chi^{2n+1} b^\dagger
 \end{aligned}$$

We now rewrite equation 1

$$\begin{aligned}
 U a^\dagger U^\dagger &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \\
 &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n}}{(2n)!} C_{2n} + \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n+1}}{(2n+1)!} C_{2n+1} \\
 &= a^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n}}{(2n)!} + b^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n+1}}{(2n+1)!} \\
 &= a^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n}}{(2n)!} - i b^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n+1}}{(2n+1)!} \\
 &= \cos(\chi\Delta t) a^\dagger - i \sin(\chi\Delta t) b^\dagger
 \end{aligned}$$

Let us now compute the effect of U on $|0\rangle|0\rangle_b = |00\rangle$:

$$\begin{aligned}
 U |00\rangle &= e^{-iH\Delta t} |00\rangle \\
 &= \sum_{n=0}^{+\infty} \frac{(-iH\Delta t)^n}{n!} |00\rangle
 \end{aligned}$$

Since $a|0\rangle = 0$ and $b|0\rangle_b = 0$, we have

$$H |00\rangle = 0$$

and

$$\forall n \in \mathbb{N}^*, \quad H^n |00\rangle = 0$$

from which it follows there is only one non nul term in the previous sum and

$$U |00\rangle = |00\rangle$$

Let us compute the effect of U on $|1\rangle|0\rangle_b = |10\rangle$:

$$\begin{aligned}
U|10\rangle &= Ua^\dagger|00\rangle \\
&= Ua^\dagger \underbrace{U^\dagger U}_{=1}|00\rangle \\
&= Ua^\dagger U^\dagger|00\rangle \\
&= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)|00\rangle \\
&= \cos(\chi\Delta t)|10\rangle - i\sin(\chi\Delta t)|01\rangle \\
&= \cos(\chi\Delta t)|1\rangle|0\rangle_b - i\sin(\chi\Delta t)|0\rangle|1\rangle_b
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sqrt{n!}U|n\rangle|0\rangle_b &= \sqrt{n!}U|n0\rangle \\
&= U(a^\dagger)^n|00\rangle \\
&= U(a^\dagger)^n U^\dagger U|00\rangle \\
&= (Ua^\dagger U^\dagger)^n|00\rangle \\
&= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)^n|00\rangle
\end{aligned}$$

Since $[a^\dagger, b^\dagger] = 0$,

$$\begin{aligned}
\sqrt{n!}U|n\rangle|0\rangle_b &= \left(\sum_{k=0}^n \binom{n}{k} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) (a^\dagger)^{n-k} (b^\dagger)^k \right) |00\rangle \\
&= \sum_{k=0}^n \binom{n}{k} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) \sqrt{(n-k)!k!} |n-k\rangle|k\rangle_b
\end{aligned}$$

so that

$$\begin{aligned}
U|n0\rangle &= \sum_{k=0}^n \binom{n}{k} \sqrt{\frac{(n-k)!k!}{n!}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle|k\rangle_b \\
&= \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle|k\rangle_b
\end{aligned}$$

We can think of the number

$$\binom{n}{k} \cos^{2(n-k)}(\chi\Delta t) \sin^{2k}(\chi\Delta t)$$

as the probability of losing k quanta of energy to the environment.

Let $E_m = \langle m|_b U|0\rangle_b$, $m \in \mathbb{N}$ the operation elements of U . They are operators acting on the principal space. We can compute the action of E_m on $|n\rangle$ (i.e. compute the n th column of the matrix of E_m) from the previous formula:

$$\begin{aligned}
E_m|n\rangle &= (\langle m|_b U|0\rangle_b)|n\rangle \\
&= \langle m|_b (U|n\rangle|0\rangle_b) \\
&= \langle m|_b U|n0\rangle
\end{aligned}$$

$$\begin{aligned} E_m |n\rangle &= \langle m|_b \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle |k\rangle_b \\ &= \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle \underbrace{\langle m|k\rangle_b}_{=\delta_{mk}} \\ &= (-i)^m \sin^m(\chi\Delta t) \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle \end{aligned}$$

We can also reconstruct the full formula for E_m using bracket calculus:

$$\begin{aligned}
 E_m &= E_m \underbrace{\sum_{n=0}^{+\infty} |n\rangle \langle n|}_{=1} \\
 &= \sum_{n=0}^{+\infty} E_m |n\rangle \langle n| \\
 &= \sum_{n=m}^{+\infty} E_m |n\rangle \langle n| \\
 &= (-i)^m \sin^m(\chi \Delta t) \sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi \Delta t) |n-m\rangle \langle n|
 \end{aligned}$$

Note that the sole effect of factor $(-i)^m$ is to add a global phase so it may as well be omitted.

Diagram illustrating a quantum circuit for a binomial distribution. The circuit starts with an input state $E_m = \sin^m(\chi \Delta t)$ on the left. The circuit consists of a series of gates, each labeled with a binomial coefficient and a cosine term: $\sqrt{\binom{m+1}{m}} \cos(\chi \Delta t)$, $\sqrt{\binom{m+2}{m}} \cos^2(\chi \Delta t)$, ..., $\sqrt{\binom{n}{m}} \cos^{n-m}(\chi \Delta t)$. The final output state on the right is labeled $n-m$. The diagram is enclosed in a large rectangle with a vertical line on the left and a vertical line on the right.

21.2. Trace-preserving property. Matrix calculus or bracket calculus show that the matrices $E_m^\dagger E_m$ are diagonals, with the first m elements are 0:

$$\begin{aligned}
E_m^\dagger E_m &= \sin^{2m}(\chi\Delta t) \left(\sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n\rangle \langle n-m| \right) \left(\sum_{l=m}^{+\infty} \sqrt{\binom{l}{m}} \cos^{l-m}(\chi\Delta t) |l-m\rangle \langle l| \right) \\
&= \sin^{2m}(\chi\Delta t) \sum_{n=m}^{+\infty} \sum_{l=m}^{+\infty} \sqrt{\binom{n}{m}} \sqrt{\binom{l}{m}} \cos^{n-m}(\chi\Delta t) \cos^{l-m}(\chi\Delta t) |n\rangle \underbrace{\langle n-m|l-m\rangle}_{=\delta_{nl}} \langle l| \\
&= \sin^{2m}(\chi\Delta t) \sum_{n=m}^{+\infty} \binom{n}{m} \cos^{2(n-m)}(\chi\Delta t) |n\rangle \langle n|
\end{aligned}$$

It follows that the operator $\sum_{m=0}^{+\infty} E_m^\dagger E_m$ is also diagonal, and diagonal elements are

$$\begin{aligned}
\langle n | \sum_{m=0}^{+\infty} E_m^\dagger E_m | n \rangle &= \sum_{m=0}^{+\infty} \langle n | E_m^\dagger E_m | n \rangle \\
&= \sum_{m=0}^n \langle n | E_m^\dagger E_m | n \rangle \\
&= \sum_{m=0}^n \binom{n}{m} \sin^{2m}(\chi\Delta t) \cos^{2(n-m)}(\chi\Delta t) \\
&= (\sin^2(\chi\Delta t) + \cos^2(\chi\Delta t))^n \\
&= 1
\end{aligned}$$

i.e. $\sum_{m=0}^{+\infty} E_m^\dagger E_m = 1$ and the quantum operation is trace-preserving.

22. Amplitude damping of a single qubit density matrix

Let

$$\rho = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

The amplitude damping operation is defined by

$$\varepsilon_{AD}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

where

$$\begin{aligned} E_0 &= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \\ E_1 &= \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (2)$$

Straightforward matrix calculus show that

$$E_0 \rho E_0^\dagger = \begin{bmatrix} a & b\sqrt{1-\gamma} \\ b^* \sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}$$

and

$$\begin{aligned} E_1 \rho E_1^\dagger &= \begin{bmatrix} c\gamma & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (1-a)\gamma & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

because $1 = \text{Tr } \rho = a + c$.

Thus we have

$$\begin{aligned} \varepsilon_{AD}(\rho) &= \begin{bmatrix} a + (1-a)\gamma & b\sqrt{1-\gamma} \\ b^* \sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} \\ &= \begin{bmatrix} 1 - (1-a)(1-\gamma) & b\sqrt{1-\gamma} \\ b^* \sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} \end{aligned} \quad (3)$$

23. Amplitude damping of dual-rail qubits

Let

$$|\psi\rangle = a|01\rangle + b|10\rangle$$

Applying $\varepsilon_{AD} \otimes \varepsilon_{AD}$ to $\rho = |\psi\rangle\langle\psi|$ is equivalent to applying unitary $B \otimes B$ to $|\psi\rangle$, where $B = e^{\theta(a^\dagger b - ab^\dagger)}$. Let's do this by making explicit the 2 environment qubits initially set to 0, denoted by subscript b :

$$|\psi\rangle = a|01\rangle|00\rangle_b + b|10\rangle|00\rangle_b$$

$$\begin{aligned} B \otimes B |\psi\rangle &= a|0\rangle|0\rangle_b (B|1\rangle|0\rangle_b) + b(B|1\rangle|0\rangle_b)|0\rangle|0\rangle_b \\ &= a|0\rangle|0\rangle_b (\cos(\theta)|1\rangle|0\rangle_b + \sin(\theta)|0\rangle|1\rangle_b) + b(\cos(\theta)|1\rangle|0\rangle_b + \sin(\theta)|0\rangle|1\rangle_b)|0\rangle|0\rangle_b \\ &= a\cos(\theta)|0\rangle|0\rangle_b|1\rangle|0\rangle_b + a\sin(\theta)|0\rangle|0\rangle_b|0\rangle|1\rangle_b + b\cos(\theta)|1\rangle|0\rangle_b|0\rangle|0\rangle_b + b\sin(\theta)|0\rangle|1\rangle_b|0\rangle|0\rangle_b \end{aligned}$$

We reorder the qubits to put the environments qubits at the end since we will trace them out:

$$\begin{aligned} B \otimes B |\psi\rangle &= a\cos(\theta)|01\rangle|00\rangle_b + a\sin(\theta)|00\rangle|01\rangle_b + b\cos(\theta)|10\rangle|00\rangle_b + b\sin(\theta)|00\rangle|10\rangle_b \\ &= |\varphi\rangle \end{aligned} \quad (4)$$

Now we have to find the dual vector $\langle\varphi|$ of this state. We can recall the not so trivial following facts related to product space: Let $\{|a_i\rangle\}, \{|b_j\rangle\}$ be basis of two Hilbert spaces A and B .

The dual of $|a_i b_j\rangle = |a_i\rangle \otimes |b_j\rangle$ is

$$\langle a_i | \otimes \langle b_j | = \langle a_i b_j |$$

so that

$$\langle\varphi| = a^* \cos(\theta) \langle 01| \langle 00|_b + a^* \sin(\theta) \langle 00| \langle 01|_b + b^* \cos(\theta) \langle 10| \langle 00|_b + b^* \sin(\theta) \langle 00| \langle 10|_b$$

We have also

$$|a_k b_l\rangle \langle a_i b_j| = |a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|$$

We could then use equation 4 to compute the density $|\varphi\rangle \langle \varphi|$, but this would be a messy sum with 16 terms.

Since we will trace out the environment, we recall the partial trace formula:

$$\begin{aligned} \text{Tr}_B(|a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|) &= |a_k\rangle \langle a_i| \text{Tr}(|b_l\rangle \langle b_j|) \\ &= |a_k\rangle \langle a_i| \langle b_l| b_j \rangle \end{aligned}$$

Since $\{|00\rangle_b, |01\rangle_b, |10\rangle_b, |11\rangle_b\}$ is an orthonormal basis, there are only 6 out of 16 terms left after the partial trace operation:

$$\begin{aligned} \text{Tr}_b(|\varphi\rangle \langle \varphi|) &= |a|^2 \cos^2(\theta) |01\rangle \langle 01| + ab^* \cos^2(\theta) |01\rangle \langle 10| + |a|^2 \sin^2(\theta) |00\rangle \langle 00| \\ &\quad + |b|^2 \cos^2(\theta) |10\rangle \langle 10| + ba^* \cos^2(\theta) |10\rangle \langle 01| + |b|^2 \sin^2(\theta) |00\rangle \langle 00| \\ &= |a|^2(1-\gamma) |01\rangle \langle 01| + ab^*(1-\gamma) |01\rangle \langle 10| + |a|^2 \gamma |00\rangle \langle 00| \\ &\quad + |b|^2(1-\gamma) |10\rangle \langle 10| + ba^*(1-\gamma) |10\rangle \langle 01| + |b|^2 \gamma |00\rangle \langle 00| \\ &= \underbrace{(|a|^2 + |b|^2)\gamma |00\rangle \langle 00|}_{=1} + (1-\gamma) (|a|^2 |01\rangle \langle 01| + ab^* |01\rangle \langle 10| + |b|^2 |10\rangle \langle 10| + ba^* |10\rangle \langle 01|) \\ &= \gamma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + (1-\gamma) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & |a|^2 & ab^* & 0 \\ 0 & a^*b & |b|^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \gamma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + (1-\gamma)\rho \end{aligned}$$

It is a mixed state:

- with probability γ , the state is projected to $|00\rangle$, orthogonal to $|\psi\rangle$.
- with probability $1-\gamma$, state is unchanged.

Since $|00\rangle$ is orthogonal to $|\psi\rangle$, one can detect amplitude damping errors with measurement operators:

$$\begin{aligned} M_0 &= |00\rangle \langle 00| \quad \text{orthogonal projector on } \text{span}\{|00\rangle\} \\ M_1 &= |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11| \quad \text{orthogonal projector on } \text{span}\{|01\rangle, |10\rangle, |11\rangle\} \end{aligned}$$

- If the state decayed to $|00\rangle$, then with probability 1 the result of the measurement will be $|00\rangle$.
- Otherwise, with probability 1 the result of the measurement will be the original $|\psi\rangle$.

It can be easily checked that the quantum operation can be described with 3 operators:

$$\begin{aligned} E_0^{dr} &= \sqrt{1-\gamma} I \\ E_1^{dr} &= \sqrt{\gamma} |00\rangle \langle 01| \\ &= \sqrt{\gamma} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ E_2^{dr} &= \sqrt{\gamma} |00\rangle \langle 10| \\ &= \sqrt{\gamma} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

It is interesting to see that these operators are the restriction to $\text{span}\{|01\rangle, |10\rangle\}$ of the operators

$$\begin{aligned} E_0 \otimes E_0 \\ E_0 \otimes E_1 \\ E_1 \otimes E_0 \\ E_1 \otimes E_1 \end{aligned}$$

where E_0, E_1 are the operators of amplitude damping for single qubit, defined in 2.

24. Spontaneous emission is amplitude damping

From equation (7.77) in the book, the time evolution of the single atom interacting with single photon is governed by unitary

$$\begin{aligned} U = e^{-i\delta t} |00\rangle \langle 00| + (\cos(\Omega t) + i \frac{\delta}{\Omega} \sin(\Omega t)) |01\rangle \langle 01| \\ + (\cos(\Omega t) - i \frac{\delta}{\Omega} \sin(\Omega t)) |10\rangle \langle 10| - i \frac{\delta}{\Omega} \sin(\Omega t) (|01\rangle \langle 10| + |10\rangle \langle 01|) \end{aligned}$$

the left label corresponds to the electric field, the right label corresponds to the atom.
The *Rabi frequency* is

$$\Omega = \sqrt{g^2 + \delta^2}$$

If we set $\delta = 0$ and if $g > 0$, then $\Omega = g$ and

$$\begin{aligned} U = |00\rangle \langle 00| + \cos(\Omega t) (|01\rangle \langle 01| + |10\rangle \langle 10|) \\ - i \sin(\Omega t) (|01\rangle \langle 10| + |10\rangle \langle 01|) \end{aligned}$$

Let us apply U to

$$\begin{aligned} |\psi\rangle &= |0\rangle (a|0\rangle + b|1\rangle) \\ &= a|00\rangle + b|01\rangle \end{aligned}$$

We find

$$\begin{aligned} U|\psi\rangle &= a|00\rangle + b(\cos(\Omega t)|01\rangle - i \sin(\Omega t)|10\rangle) \\ &= |\varphi\rangle \end{aligned} \tag{5}$$

Now we have to find the dual vector $\langle\varphi|$ of this state. We can recall the not so trivial following facts related to product space: Let $\{|a_i\rangle\}, \{|b_j\rangle\}$ be basis of two Hilbert spaces A and B .

The dual of $|a_i b_j\rangle = |a_i\rangle \otimes |b_j\rangle$ is

$$\langle a_i| \otimes \langle b_j| = \langle a_i b_j|$$

so that

$$\langle\varphi| = a^* \cos(\theta) \langle 01| \langle 00|_b + a^* \sin(\theta) \langle 00| \langle 01|_b + b^* \cos(\theta) \langle 10| \langle 00|_b + b^* \sin(\theta) \langle 00| \langle 10|_b$$

We have also

$$|a_k b_l\rangle \langle a_i b_j| = |a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|$$

We could then use equation 5 to compute the density $|\varphi\rangle \langle\varphi|$, but this would be a messy sum with 9 terms.

Since we will trace out the photon space, we recall the partial trace formula:

$$\begin{aligned}\mathrm{Tr}_B(|a_k\rangle\langle a_i| \otimes |b_l\rangle\langle b_j|) &= |a_k\rangle\langle a_i| \mathrm{Tr}(|b_l\rangle\langle b_j|) \\ &= |a_k\rangle\langle a_i| \langle b_l|b_j\rangle\end{aligned}$$

Since $\{|0\rangle, |1\rangle\}$ is an orthonormal basis of the state space A of the photon, there are only 5 out of 9 terms left after the partial trace operation over the photon (those where the bit for the photon is the same in the ket and in the bra):

$$\begin{aligned}\mathrm{Tr}_A(|\varphi\rangle\langle\varphi|) &= (|a|^2 + |b|^2 \sin^2(\Omega t)) |0\rangle\langle 0| + ab^* \cos(\Omega t) |0\rangle\langle 1| \\ &\quad + a^*b \cos(\Omega t) |1\rangle\langle 0| + |b|^2 \cos^2(\Omega t) |1\rangle\langle 1| \\ &= \begin{bmatrix} |a|^2 + (1 - |a|^2)\gamma & ab^*\sqrt{1 - \gamma} \\ a^*b\sqrt{1 - \gamma} & |b|^2(1 - \gamma) \end{bmatrix}\end{aligned}$$

with $\gamma = \sin^2(\Omega t)$. Now compare with equation 3 and recall that

$$\rho = \begin{bmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{bmatrix}$$

to see that this is indeed the amplitude damping operation.