QUANTUM COMPUTATION AND QUANTUM INFORMATION: QUANTUM NOISE AND QUANTUM OPERATIONS

21. Amplitude dampling of a harmonic oscillator

The principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian:

$$H = \chi(a^\dagger b + b^\dagger a)$$

where $a^\dagger, a$ and $b^\dagger, b$ are the creation, annihilation operators for the principal and environment oscillators, respectively.

The time evolution of the coupled system is governed by the unitary operator:

$$U = e^{-iH\Delta t}$$

21.1. Operation elements. We recall some results for the harmonic oscillator:

$$\forall n \in \mathbb{N}, \quad a^\dagger \ket{n} = \sqrt{n+1} \ket{n+1}$$

and similarly in the environment space

$$\forall n \in \mathbb{N}, \quad b^\dagger \ket{n}_b = \sqrt{n+1} \ket{n+1}_b$$

Here we use the subscript b to differentiate the eigenvectors of the Hermitian operator $bb^\dagger$ which live in the environment space from the eigenvectors of $aa^\dagger$ in the principal space:

$$\forall n \in \mathbb{N}, \quad bb^\dagger \ket{n}_b = (n+1) \ket{n}_b$$
$$\forall n \in \mathbb{N}, \quad aa^\dagger \ket{n} = (n+1) \ket{n}$$

Each set of vectors constitute an orthonormal basis:

$$\forall (n,m) \in \mathbb{N}^2, \quad \braket{n|m} = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases}$$
$$= \delta_{nm}$$

We also have

$$aa^\dagger - a^\dagger a = [a, a^\dagger]$$
$$= 1$$
$$bb^\dagger - b^\dagger b = [b, b^\dagger]$$
$$= 1$$

where 1 stands for the identity operator.

Each of the operators $a, a^\dagger$ commutes with each of the operators $b, b^\dagger$ since they act on different spaces

$$0 = [a^\dagger, b^\dagger]$$
$$= [a, b^\dagger]$$
$$= [a^\dagger, b]$$
$$= [a, b]$$

The Baker-Campbell-Hausdorff formula states that, for any operators A, G such that e^G exists,

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} C_n$$

where the operators C_n are defined recursively by

$$C_0 = A$$
$$C_1 = [G, A]$$
$$\forall n \in \mathbb{N}, \quad C_{n+1} = [G, C_n]$$

Lets compute a simplified expression for the operator $U a^\dagger U^\dagger$ acting on the product space:

$$U a^\dagger U^\dagger = e^{-iH\Delta t} a^\dagger e^{iH\Delta t}$$
$$= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \qquad (1)$$

The first commutators C_n are

$$C_0 = a^\dagger$$
$$C_1 = [H, a^\dagger]$$
$$= [\chi b^\dagger a, a^\dagger]$$
$$= \chi b^\dagger [a, a^\dagger]$$
$$= \chi b^\dagger$$
$$C_2 = [H, C_1]$$
$$= [\chi a^\dagger b, \chi b^\dagger]$$
$$= \chi^2 a^\dagger [b, b^\dagger]$$
$$= \chi^2 a^\dagger$$

from which it follows that

$$\forall n \in \mathbb{N}, \quad C_{2n} = \chi^{2n} a^\dagger$$
$$C_{2n+1} = \chi^{2n+1} b^\dagger$$

We now rewrite equation 1

$$U a^\dagger U^\dagger = \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n$$
$$= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n}}{(2n)!} C_{2n} + \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n+1}}{(2n+1)!} C_{2n+1}$$
$$= a^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n}}{(2n)!} + b^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n+1}}{(2n+1)!}$$
$$= a^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n}}{(2n)!} - ib^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n+1}}{(2n+1)!}$$
$$= \cos(\chi\Delta t) a^\dagger - i \sin(\chi\Delta t) b^\dagger$$

Let us now compute the effect of U on $|0\rangle |0\rangle_b = |00\rangle$:

$$U |00\rangle = e^{-iH\Delta t} |00\rangle$$
$$= \sum_{n=0}^{+\infty} \frac{(-iH\Delta t)^n}{n!} |00\rangle$$

Since $a\,|0\rangle = 0$ and $b\,|0\rangle_b = 0$, we have

$$H\,|00\rangle = 0$$

and

$$\forall n \in \mathbb{N}^*, \quad , H^n\,|00\rangle = 0$$

from which it follows there is only one non nul term in the previous sum and

$$U\,|00\rangle = |00\rangle$$

Let us compute the effect of U on $|1\rangle\,|0\rangle_b = |10\rangle$:

$$
\begin{aligned}
U\,|10\rangle &= U a^\dagger\,|00\rangle \\
&= U a^\dagger \underbrace{U^\dagger U}_{=1}\,|00\rangle \\
&= U a^\dagger U^\dagger\,|00\rangle \\
&= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)\,|00\rangle \\
&= \cos(\chi\Delta t)\,|10\rangle - i\sin(\chi\Delta t)\,|01\rangle \\
&= \cos(\chi\Delta t)\,|1\rangle\,|0\rangle_b - i\sin(\chi\Delta t)\,|0\rangle\,|1\rangle_b
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\sqrt{n!}U\,|n\rangle\,|0\rangle_b &= \sqrt{n!}U\,|n0\rangle \\
&= U(a^\dagger)^n\,|00\rangle \\
&= U(a^\dagger)^n U^\dagger U\,|00\rangle \\
&= (U a^\dagger U^\dagger)^n\,|00\rangle \\
&= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)^n\,|00\rangle
\end{aligned}
$$

Since $[a^\dagger, b^\dagger] = 0$,

$$
\begin{aligned}
\sqrt{n!}U\,|n\rangle\,|0\rangle_b &= \left(\sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)(a^\dagger)^{n-k}(b^\dagger)^k\right)\,|00\rangle \\
&= \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)\sqrt{(n-k)!}\sqrt{k!}\,|n-k\rangle\,|k\rangle_b
\end{aligned}
$$

so that

$$
\begin{aligned}
U\,|n0\rangle &= \sum_{k=0}^{n}\binom{n}{k}\sqrt{\frac{(n-k)!k!}{n!}}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)\,|n-k\rangle\,|k\rangle_b \\
&= \sum_{k=0}^{n}\sqrt{\binom{n}{k}}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)\,|n-k\rangle\,|k\rangle_b
\end{aligned}
$$

Let $E_m = \langle m|_b\,U\,|0\rangle_b$, $\quad m \in \mathbb{N}$ the operation elements of U. They are operators acting on the principal space. We can compute the action of E_m on $|n\rangle$ (i.e. compute the nth column of the matrix of E_m) from the previous formula:

$$
\begin{aligned}
E_m\,|n\rangle &= (\langle m|_b\,U\,|0\rangle_b)\,|n\rangle \\
&= \langle m|_b\,(U\,|n\rangle\,|0\rangle_b) \\
&= \langle m|_b\,U\,|n0\rangle
\end{aligned}
$$

First it is clear that if $n < m$, $E_m \left| n \right\rangle = 0$. Then if $n \geqslant m$,

$$
\begin{aligned}
E_m \left| n \right\rangle &= \left\langle m \right|_b \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi \Delta t)(-i)^k \sin^k(\chi \Delta t) \left| n-k \right\rangle \left| k \right\rangle_b \\
&= \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi \Delta t)(-i)^k \sin^k(\chi \Delta t) \left| n-k \right\rangle \underbrace{\left\langle m | k \right\rangle_b}_{=\delta_{mk}} \\
&= (-i)^m \sin^m(\chi \Delta t) \sqrt{\binom{n}{m}} \cos^{n-m}(\chi \Delta t) \left| n-m \right\rangle
\end{aligned}
$$

This shows that the matrix of E_m has non nul elements only on the mth superior diagonal.
    We can also reconstruct the full formula for E_m using braket calculus:

$$
\begin{aligned}
E_m &= E_m \underbrace{\sum_{n=0}^{+\infty} \left| n \right\rangle \left\langle n \right|}_{=1} \\
&= \sum_{n=0}^{+\infty} E_m \left| n \right\rangle \left\langle n \right| \\
&= \sum_{n=m}^{+\infty} E_m \left| n \right\rangle \left\langle n \right| \\
&= (-i)^m \sin^m(\chi \Delta t) \sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi \Delta t) \left| n-m \right\rangle \left\langle n \right|
\end{aligned}
$$

Note that the sole effect of factor $(-i)^m$ is to add a global phase so it may as well be omitted.

$$
E_m = \sin^m(\chi \Delta t)
\begin{bmatrix}
& & \overset{\displaystyle m}{\underset{1}{}} & & & \overset{\displaystyle n}{} & \\
& & \sqrt{\binom{m+1}{m}} \cos(\chi \Delta t) & & & \mathbf{0} & \\
& & & \sqrt{\binom{m+2}{m}} \cos^2(\chi \Delta t) & & & \\
& & & & \ddots & & \\
& & & & & \sqrt{\binom{n}{m}} \cos^{n-m}(\chi \Delta t) & \\
& & \mathbf{0} & & & & \ddots \\
& & & & & &
\end{bmatrix}
\begin{matrix} \\ \\ \\ \\ \\ n-m \\ \\ \end{matrix}
$$

21.2. Trace-preserving property. Matrix calculus or braket calculus show that the matrices $E_m^\dagger E_m$ are diagonals, with the first m elements are 0:

$$E_m^\dagger E_m = \sin^{2m}(\chi\Delta t) \left(\sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) \, |n\rangle \langle n-m| \right) \left(\sum_{l=m}^{+\infty} \sqrt{\binom{l}{m}} \cos^{l-m}(\chi\Delta t) \, |l-m\rangle \langle l| \right)$$

$$= \sin^{2m}(\chi\Delta t) \sum_{n=m}^{+\infty} \sum_{l=m}^{+\infty} \sqrt{\binom{n}{m}} \sqrt{\binom{l}{m}} \cos^{n-m}(\chi\Delta t) \cos^{l-m}(\chi\Delta t) \, |n\rangle \underbrace{\langle n-m|l-m\rangle}_{=\delta_{nl}} \langle l|$$

$$= \sin^{2m}(\chi\Delta t) \sum_{n=m}^{+\infty} \binom{n}{m} \cos^{2(n-m)}(\chi\Delta t) \, |n\rangle \langle n|$$

$$E_m^\dagger E_m = \sin^{2m}(\chi\Delta t) \begin{bmatrix} & \overset{\displaystyle m}{} & & & & \overset{\displaystyle n}{} & \\ \mathbf{0} & & & & & \mathbf{0} & \\ & 1 & & & & & \\ & & \binom{m+1}{m}\cos^2(\chi\Delta t) & & & & \\ & & & \binom{m+1}{m}\cos^4(\chi\Delta t) & & & \\ & & & & \ddots & & \\ & & & & & \binom{n}{m}\cos^{2(n-m)}(\chi\Delta t) & \\ \mathbf{0} & & & & & & \ddots \end{bmatrix} \begin{matrix} m \\ \\ \\ \\ n \end{matrix}$$

It follows that the operator $\sum_{m=0}^{+\infty} E_m^\dagger E_m$ is also diagonal, and diagonal elements are

$$\langle n| \sum_{m=0}^{+\infty} E_m^\dagger E_m |n\rangle = \sum_{m=0}^{+\infty} \langle n|E_m^\dagger E_m|n\rangle$$

$$= \sum_{m=0}^{n} \langle n|E_m^\dagger E_m|n\rangle$$

$$= \sum_{m=0}^{n} \binom{n}{m} \sin^{2m}(\chi\Delta t) \cos^{2(n-m)}(\chi\Delta t)$$

$$= (\sin^2(\chi\Delta t) + \cos^2(\chi\Delta t))^n$$

$$= 1$$

i.e. $\sum_{m=0}^{+\infty} E_m^\dagger E_m = 1$ and the quantum operation is trace-preserving.