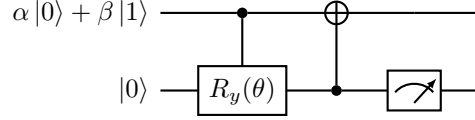# QUANTUM COMPUTATION AND QUANTUM INFORMATION: QUANTUM NOISE AND QUANTUM OPERATIONS

20. Circuit model for amplitude dampling

We want to prove that the following circuit models the amplitude dampling operation

Recall that

$$R_y(\theta) = e^{-i\frac{\theta}{2}Y}$$
$$= \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}$$

Initially the two-qubit state is

$$(\alpha |0\rangle + \beta |1\rangle) |0\rangle = \alpha |00\rangle + \beta |10\rangle$$

After the controlled R_y gate it becomes

$$\alpha |00\rangle + \beta |1\rangle R_y(\theta) |0\rangle = \alpha |00\rangle + \beta |1\rangle (\cos(\frac{\theta}{2}) |0\rangle + \sin(\frac{\theta}{2}) |1\rangle)$$
$$= \alpha |00\rangle + \beta(\cos(\frac{\theta}{2}) |10\rangle + \sin(\frac{\theta}{2}) |11\rangle)$$

After the controlled not gate,

$$\alpha |00\rangle + \beta(\cos(\frac{\theta}{2}) |10\rangle + \sin(\frac{\theta}{2}) |01\rangle)$$

This is the effect of amplitude dampling, with probability of 1 be switched to 0, or one photon being lost to environment, being $\gamma = \sin^2(\frac{\theta}{2})$.

21. Amplitude dampling of a harmonic oscillator

The principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian:

$$H = \chi(a^\dagger b + b^\dagger a)$$

where $a^\dagger, a$ and $b^\dagger, b$ are the creation, annihilation operators for the principal and environment oscillators, respectively.

The time evolution of the coupled system is governed by the unitary operator:

$$U = e^{-iH\Delta t}$$

21.1. Operation elements. We recall some results for the harmonic oscillator:

$$\forall n \in \mathbb{N}, \quad a^\dagger \left| n \right\rangle = \sqrt{n+1} \left| n+1 \right\rangle$$

and similarly in the environment space

$$\forall n \in \mathbb{N}, \quad b^\dagger \left| n \right\rangle_b = \sqrt{n+1} \left| n+1 \right\rangle_b$$

Here we use the subscript b to differentiate the eigenvectors of the Hermitian operator $bb^\dagger$ which live in the environment space from the eigenvectors of $aa^\dagger$ in the principal space:

$$\forall n \in \mathbb{N}, \quad bb^\dagger \left| n \right\rangle_b = (n+1) \left| n \right\rangle_b$$
$$\forall n \in \mathbb{N}, \quad aa^\dagger \left| n \right\rangle = (n+1) \left| n \right\rangle$$

Each set of vectors constitute an orthonormal basis:

$$\forall (n,m) \in \mathbb{N}^2, \quad \left\langle n | m \right\rangle = \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases}$$
$$= \delta_{nm}$$

We also have

$$aa^\dagger - a^\dagger a = [a, a^\dagger]$$
$$= 1$$
$$bb^\dagger - b^\dagger b = [b, b^\dagger]$$
$$= 1$$

where 1 stands for the identity operator.

Each of the operators $a, a^\dagger$ commutes with each of the operators $b, b^\dagger$ since they act on different spaces

$$0 = [a^\dagger, b^\dagger]$$
$$= [a, b^\dagger]$$
$$= [a^\dagger, b]$$
$$= [a, b]$$

The Baker-Campbell-Hausdorff formula states that, for any operators A, G such that e^G exists,

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} C_n$$

where the operators C_n are defined recursively by

$$C_0 = A$$
$$C_1 = [G, A]$$
$$\forall n \in \mathbb{N}, \quad C_{n+1} = [G, C_n]$$

Lets compute a simplified expression for the operator $U a^\dagger U^\dagger$ acting on the product space:

$$U a^\dagger U^\dagger = e^{-iH\Delta t} a^\dagger e^{iH\Delta t}$$
$$= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \tag{1}$$

The first commutators C_n are

$$
\begin{aligned}
C_0 &= a^\dagger \\
C_1 &= [H, a^\dagger] \\
&= [\chi b^\dagger a, a^\dagger] \\
&= \chi b^\dagger [a, a^\dagger] \\
&= \chi b^\dagger \\
C_2 &= [H, C_1] \\
&= [\chi a^\dagger b, \chi b^\dagger] \\
&= \chi^2 a^\dagger [b, b^\dagger] \\
&= \chi^2 a^\dagger
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
\forall n \in \mathbb{N}, \quad C_{2n} &= \chi^{2n} a^\dagger \\
C_{2n+1} &= \chi^{2n+1} b^\dagger
\end{aligned}
$$

We now rewrite equation 1

$$
\begin{aligned}
U a^\dagger U^\dagger &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \\
&= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n}}{(2n)!} C_{2n} + \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n+1}}{(2n+1)!} C_{2n+1} \\
&= a^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n}}{(2n)!} + b^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n+1}}{(2n+1)!} \\
&= a^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n}}{(2n)!} - i b^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n+1}}{(2n+1)!} \\
&= \cos(\chi\Delta t) a^\dagger - i\sin(\chi\Delta t) b^\dagger
\end{aligned}
$$

Let us now compute the effect of U on $|0\rangle \, |0\rangle_b = |00\rangle$:

$$
\begin{aligned}
U |00\rangle &= e^{-iH\Delta t} |00\rangle \\
&= \sum_{n=0}^{+\infty} \frac{(-iH\Delta t)^n}{n!} |00\rangle
\end{aligned}
$$

Since $a |0\rangle = 0$ and $b |0\rangle_b = 0$, we have

$$
H |00\rangle = 0
$$

and

$$
\forall n \in \mathbb{N}^*, \quad , H^n |00\rangle = 0
$$

from which it follows there is only one non nul term in the previous sum and

$$
U |00\rangle = |00\rangle
$$

Let us compute the effect of U on $|1\rangle |0\rangle_b = |10\rangle$:

$$
\begin{aligned}
U |10\rangle &= U a^\dagger |00\rangle \\
&= U a^\dagger \underbrace{U^\dagger U}_{=1} |00\rangle \\
&= U a^\dagger U^\dagger |00\rangle \\
&= (\cos(\chi\Delta t) a^\dagger - i\sin(\chi\Delta t) b^\dagger) |00\rangle \\
&= \cos(\chi\Delta t) |10\rangle - i\sin(\chi\Delta t) |01\rangle \\
&= \cos(\chi\Delta t) |1\rangle |0\rangle_b - i\sin(\chi\Delta t) |0\rangle |1\rangle_b
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\sqrt{n!} U |n\rangle |0\rangle_b &= \sqrt{n!} U |n0\rangle \\
&= U (a^\dagger)^n |00\rangle \\
&= U (a^\dagger)^n U^\dagger U |00\rangle \\
&= (U a^\dagger U^\dagger)^n |00\rangle \\
&= (\cos(\chi\Delta t) a^\dagger - i\sin(\chi\Delta t) b^\dagger)^n |00\rangle
\end{aligned}
$$

Since $[a^\dagger, b^\dagger] = 0$,

$$
\begin{aligned}
\sqrt{n!} U |n\rangle |0\rangle_b &= \left(\sum_{k=0}^n \binom{n}{k} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t)(a^\dagger)^{n-k}(b^\dagger)^k \right) |00\rangle \\
&= \sum_{k=0}^n \binom{n}{k} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t)\sqrt{(n-k)!}\sqrt{k!} |n-k\rangle |k\rangle_b
\end{aligned}
$$

so that

$$
\begin{aligned}
U |n0\rangle &= \sum_{k=0}^n \binom{n}{k} \sqrt{\frac{(n-k)!k!}{n!}} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t) |n-k\rangle |k\rangle_b \\
&= \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t) |n-k\rangle |k\rangle_b
\end{aligned}
$$

We can think of the number

$$
\binom{n}{k} \cos^{2(n-k)}(\chi\Delta t) \sin^{2k}(\chi\Delta t)
$$

as the probability of losing k quanta of energy to the environment.

Let $E_m = \langle m|_b U |0\rangle_b, \quad m \in \mathbb{N}$ the operation elements of U. They are operators acting on the principal space. We can compute the action of E_m on $|n\rangle$ (i.e. compute the nth column of the matrix of E_m) from the previous formula:

$$
\begin{aligned}
E_m |n\rangle &= (\langle m|_b U |0\rangle_b) |n\rangle \\
&= \langle m|_b (U |n\rangle |0\rangle_b) \\
&= \langle m|_b U |n0\rangle
\end{aligned}
$$

First it is clear that if $n < m$, $E_m \left| n \right> = 0$. Then if $n \geqslant m$,

$$
\begin{aligned}
E_m \left| n \right> &= \left< m \right|_b \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi \Delta t)(-i)^k \sin^k(\chi \Delta t) \left| n - k \right> \left| k \right>_b \\
&= \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi \Delta t)(-i)^k \sin^k(\chi \Delta t) \left| n - k \right> \underbrace{\left< m | k \right>_b}_{=\delta_{mk}} \\
&= (-i)^m \sin^m(\chi \Delta t) \sqrt{\binom{n}{m}} \cos^{n-m}(\chi \Delta t) \left| n - m \right>
\end{aligned}
$$

This shows that the matrix of E_m has non nul elements only on the mth superior diagonal. E_m corresponds to the physical process of losing m quanta of energy to the environment.

We can also reconstruct the full formula for E_m using braket calculus:

$$
\begin{aligned}
E_m &= E_m \underbrace{\sum_{n=0}^{+\infty} \left| n \right> \left< n \right|}_{=1} \\
&= \sum_{n=0}^{+\infty} E_m \left| n \right> \left< n \right| \\
&= \sum_{n=m}^{+\infty} E_m \left| n \right> \left< n \right| \\
&= (-i)^m \sin^m(\chi \Delta t) \sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi \Delta t) \left| n - m \right> \left< n \right|
\end{aligned}
$$

Note that the sole effect of factor $(-i)^m$ is to add a global phase so it may as well be omitted.

$$
E_m = \sin^m(\chi \Delta t) \begin{bmatrix}
\overset{m}{\underset{1}{}} & & & & & \overset{n}{} \\
& \sqrt{\binom{m+1}{m}} \cos(\chi \Delta t) & & & \mathbf{0} & \\
& & \sqrt{\binom{m+2}{m}} \cos^2(\chi \Delta t) & & & \\
& & & \ddots & & \\
& & & & \sqrt{\binom{n}{m}} \cos^{n-m}(\chi \Delta t) & \\
& & \mathbf{0} & & & \ddots \\
& & & & &
\end{bmatrix} \begin{matrix} \\ \\ \\ \\ \\ n-m \\ \\ \end{matrix}
$$

21.2. Trace-preserving property. Matrix calculus or braket calculus show that the matrices $E_m^\dagger E_m$ are diagonals, with the first m elements are 0:

$$E_m^\dagger E_m = \sin^{2m}(\chi\Delta t)\left(\sum_{n=m}^{+\infty}\sqrt{\binom{n}{m}}\cos^{n-m}(\chi\Delta t)\,|n\rangle\langle n-m|\right)\left(\sum_{l=m}^{+\infty}\sqrt{\binom{l}{m}}\cos^{l-m}(\chi\Delta t)\,|l-m\rangle\langle l|\right)$$

$$= \sin^{2m}(\chi\Delta t)\sum_{n=m}^{+\infty}\sum_{l=m}^{+\infty}\sqrt{\binom{n}{m}}\sqrt{\binom{l}{m}}\cos^{n-m}(\chi\Delta t)\cos^{l-m}(\chi\Delta t)\,|n\rangle\underbrace{\langle n-m|l-m\rangle}_{=\delta_{nl}}\langle l|$$

$$= \sin^{2m}(\chi\Delta t)\sum_{n=m}^{+\infty}\binom{n}{m}\cos^{2(n-m)}(\chi\Delta t)\,|n\rangle\langle n|$$

It follows that the operator $\sum_{m=0}^{+\infty}E_m^\dagger E_m$ is also diagonal, and diagonal elements are

$$\langle n|\sum_{m=0}^{+\infty}E_m^\dagger E_m|n\rangle = \sum_{m=0}^{+\infty}\langle n|E_m^\dagger E_m|n\rangle$$

$$= \sum_{m=0}^{n}\langle n|E_m^\dagger E_m|n\rangle$$

$$= \sum_{m=0}^{n}\binom{n}{m}\sin^{2m}(\chi\Delta t)\cos^{2(n-m)}(\chi\Delta t)$$

$$= (\sin^2(\chi\Delta t)+\cos^2(\chi\Delta t))^n$$

$$= 1$$

i.e. $\sum_{m=0}^{+\infty}E_m^\dagger E_m = 1$ and the quantum operation is trace-preserving.

22. Amplitude dampling of a single qubit density matrix

Let

$$\rho = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

The amplitude dampling operation is defined by

$$\varepsilon_{AD}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$$

where

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} \tag{2}$$

Straightforward matrix calculus show that

$$E_0 \rho E_0^\dagger = \begin{bmatrix} a & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}$$

and

$$E_1 \rho E_1^\dagger = \begin{bmatrix} c\gamma & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} (1-a)\gamma & 0 \\ 0 & 0 \end{bmatrix}$$

because $1 = \operatorname{Tr} \rho = a + c$.

Thus we have

$$\varepsilon_{AD}(\rho) = \begin{bmatrix} a + (1-a)\gamma & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}$$
$$= \begin{bmatrix} 1 - (1-a)(1-\gamma) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}$$

23. Amplitude dampling of dual-rail qubits

Let

$$|\psi\rangle = a\,|01\rangle + b\,|10\rangle$$

Applying $\varepsilon_{AD} \otimes \varepsilon_{AD}$ to $\rho = |\psi\rangle\langle\psi|$ is equivalent to applying unitary $B \otimes B$ to $|\psi\rangle$, where $B = e^{\theta(a^\dagger b - a b^\dagger)}$. Let's do this by making explicit the 2 environment qubits initially set to 0, dnoted by subscript b:

$$|\psi\rangle = a\,|01\rangle\,|00\rangle_b + b\,|10\rangle\,|00\rangle_b$$

$$B \otimes B\,|\psi\rangle = a\,|0\rangle\,|0\rangle_b\,(B\,|1\rangle\,|0\rangle_b) + b(B\,|1\rangle\,|0\rangle_b)\,|0\rangle\,|0\rangle_b$$
$$= a\,|0\rangle\,|0\rangle_b\,(\cos(\theta)\,|1\rangle\,|0\rangle_b + \sin(\theta)\,|0\rangle\,|1\rangle_b) + b(\cos(\theta)\,|1\rangle\,|0\rangle_b + \sin(\theta)\,|0\rangle\,|1\rangle_b)\,|0\rangle\,|0\rangle_b$$
$$= a\cos(\theta)\,|0\rangle\,|0\rangle_b\,|1\rangle\,|0\rangle_b + a\sin(\theta)\,|0\rangle\,|0\rangle_b\,|0\rangle\,|1\rangle_b + b\cos(\theta)\,|1\rangle\,|0\rangle_b\,|0\rangle\,|0\rangle_b + b\sin(\theta)\,|0\rangle\,|1\rangle_b\,|0\rangle\,|0\rangle_b$$

We reorder the qubits to put the environments qubits at the end since we will trace them out:

$$B \otimes B\,|\psi\rangle = a\cos(\theta)\,|01\rangle\,|00\rangle_b + a\sin(\theta)\,|00\rangle\,|01\rangle_b + b\cos(\theta)\,|10\rangle\,|00\rangle_b + b\sin(\theta)\,|00\rangle\,|10\rangle_b \tag{3}$$
$$= |\varphi\rangle$$

Now we have to find the dual vector $\langle\varphi|$ of this state. We can recall the not so trivial following facts related to product space: Let $\{|a_i\rangle\},\{|b_j\rangle\}$ be basis of two Hilbert spaces A and B.

The dual of $|a_i b_j\rangle = |a_i\rangle \otimes |b_j\rangle$ is

$$\langle a_i| \otimes \langle b_j| = \langle a_i b_j|$$

so that

$$\langle\varphi| = a^*\cos(\theta)\,\langle 01|\,\langle 00|_b + a^*\sin(\theta)\,\langle 00|\,\langle 01|_b + b^*\cos(\theta)\,\langle 10|\,\langle 00|_b + b^*\sin(\theta)\,\langle 00|\,\langle 10|_b$$

We have also

$$|a_k b_l\rangle \langle a_i b_j| = |a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|$$

We could then use equation 3 to compute the density $|\varphi\rangle \langle\varphi|$, but this would be a messy sum with 16 terms.

Since we will trace out the environment, we recall the partial trace formula:

$$\mathrm{Tr}_B(|a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|) = |a_k\rangle \langle a_i| \mathrm{Tr}(|b_l\rangle \langle b_j|)$$
$$= |a_k\rangle \langle a_i| \langle b_l|b_j\rangle$$

Since $\{|00\rangle_b, |01\rangle_b, |10\rangle_b, |11\rangle_b\}$ is an orthonormal basis, there are only 6 out of 16 terms left after the partial trace operation:

$$
\begin{aligned}
\mathrm{Tr}_b(|\varphi\rangle \langle\varphi|) &= |a|^2 \cos^2(\theta) |01\rangle \langle 01| + ab^* \cos^2(\theta) |01\rangle \langle 10| + |a|^2 \sin^2(\theta) |00\rangle \langle 00| \\
&\quad + |b|^2 \cos^2(\theta) |10\rangle \langle 10| + ba^* \cos^2(\theta) |10\rangle \langle 01| + |b|^2 \sin^2(\theta) |00\rangle \langle 00| \\
&= |a|^2(1-\gamma) |01\rangle \langle 01| + ab^*(1-\gamma) |01\rangle \langle 10| + |a|^2 \gamma |00\rangle \langle 00| \\
&\quad + |b|^2(1-\gamma) |10\rangle \langle 10| + ba^*(1-\gamma) |10\rangle \langle 01| + |b|^2 \gamma |00\rangle \langle 00| \\
&= \underbrace{(|a|^2 + |b|^2)}_{=1}\gamma |00\rangle \langle 00| + (1-\gamma)\left(|a|^2 |01\rangle \langle 01| + ab^* |01\rangle \langle 10| + |b|^2 |10\rangle \langle 10| + ba^* |10\rangle \langle 01|\right)
\end{aligned}
$$

$$
= \gamma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + (1-\gamma) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & |a|^2 & ab^* & 0 \\ 0 & a^*b & |b|^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
= \gamma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + (1-\gamma)\rho
$$

It is a mixed state:

 – with probability γ, the state is projected to $|00\rangle$, orthogonal to $|\psi\rangle$.
 – with probability $1-\gamma$, state is unchanged.

Since $|00\rangle$ is orthogonal to $|\psi\rangle$, you can detect amplitude dampling errors with measurement operators:

$$M_0 = |00\rangle \langle 00| \qquad \text{orthogonal projector on } \mathrm{span}\{|00\rangle\}$$
$$M_1 = |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11| \qquad \text{orthogonal projector on } \mathrm{span}\{|01\rangle, |10\rangle \, |11\rangle\}$$

 – If the state decayed to $|00\rangle$, then with probability 1 the result of the measurement will be $|00\rangle$.
 – Otherwise, with probability 1 the result of the measurement will be the original $|\psi\rangle$.

It can be easily checked that the quantum operation can be described with 3 operators:

$$E_0^{dr} = \sqrt{1-\gamma}I$$
$$E_1^{dr} = \sqrt{\gamma} |00\rangle \langle 01|$$
$$= \sqrt{\gamma} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$E_2^{dr} = \sqrt{\gamma} |00\rangle \langle 10|$$
$$= \sqrt{\gamma} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is interesting to see that these operators are the restriction to $\text{span}\{|01\rangle, |10\rangle\}$ of the operators

$$E_0 \otimes E_0$$
$$E_0 \otimes E_1$$
$$E_1 \otimes E_0$$
$$E_1 \otimes E_1$$

where E_0, E_1 are the operators of amplitude dampling for single qubit, defined in 2.