

# QUANTUM COMPUTATION AND QUANTUM INFORMATION: QUANTUM NOISE AND QUANTUM OPERATIONS

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## 20. Circuit model for amplitude damping

We want to prove that the following circuit models the amplitude damping operation



Recall that

$$R_y(\theta) = e^{-i\frac{\theta}{2}Y} = \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}$$

Initially the two-qubit state is

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle = \alpha|00\rangle + \beta|10\rangle$$

After the controlled  $R_y$  gate it becomes

$$\begin{aligned} \alpha|00\rangle + \beta|1\rangle R_y(\theta)|0\rangle &= \alpha|00\rangle + \beta|1\rangle (\cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})|1\rangle) \\ &= \alpha|00\rangle + \beta(\cos(\frac{\theta}{2})|10\rangle + \sin(\frac{\theta}{2})|11\rangle) \end{aligned}$$

After the controlled not gate,

$$\alpha|00\rangle + \beta(\cos(\frac{\theta}{2})|10\rangle + \sin(\frac{\theta}{2})|01\rangle)$$

This is the effect of amplitude damping, with probability of 1 be switched to 0, or one photon being lost to environment, being  $\gamma = \sin^2(\frac{\theta}{2})$ .

## 21. Amplitude damping of a harmonic oscillator

The principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian:

$$H = \chi(a^\dagger b + b^\dagger a)$$

where  $a^\dagger, a$  and  $b^\dagger, b$  are the creation, annihilation operators for the principal and environment oscillators, respectively.

The time evolution of the coupled system is governed by the unitary operator:

$$U = e^{-iH\Delta t}$$

**21.1. Operation elements.** We recall some results for the harmonic oscillator:

$$\forall n \in \mathbb{N}, \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

and similarly in the environment space

$$\forall n \in \mathbb{N}, \quad b^\dagger |n\rangle_b = \sqrt{n+1} |n+1\rangle_b$$

Here we use the subscript  $b$  to differentiate the eigenvectors of the Hermitian operator  $bb^\dagger$  which live in the environment space from the eigenvectors of  $aa^\dagger$  in the principal space:

$$\begin{aligned} \forall n \in \mathbb{N}, \quad bb^\dagger |n\rangle_b &= (n+1) |n\rangle_b \\ \forall n \in \mathbb{N}, \quad aa^\dagger |n\rangle &= (n+1) |n\rangle \end{aligned}$$

Each set of vectors constitute an orthonormal basis:

$$\begin{aligned} \forall (n, m) \in \mathbb{N}^2, \quad \langle n|m \rangle &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases} \\ &= \delta_{nm} \end{aligned}$$

We also have

$$\begin{aligned} aa^\dagger - a^\dagger a &= [a, a^\dagger] \\ &= 1 \\ bb^\dagger - b^\dagger b &= [b, b^\dagger] \\ &= 1 \end{aligned}$$

where 1 stands for the identity operator.

Each of the operators  $a, a^\dagger$  commutes with each of the operators  $b, b^\dagger$  since they act on different spaces

$$\begin{aligned} 0 &= [a^\dagger, b^\dagger] \\ &= [a, b^\dagger] \\ &= [a^\dagger, b] \\ &= [a, b] \end{aligned}$$

The Baker-Campbell-Hausdorff formula states that, for any operators  $A, G$  such that  $e^G$  exists,

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} C_n$$

where the operators  $C_n$  are defined recursively by

$$\begin{aligned} C_0 &= A \\ C_1 &= [G, A] \\ \forall n \in \mathbb{N}, \quad C_{n+1} &= [G, C_n] \end{aligned}$$

Lets compute a simplified expression for the operator  $U a^\dagger U^\dagger$  acting on the product space:

$$\begin{aligned} U a^\dagger U^\dagger &= e^{-iH\Delta t} a^\dagger e^{iH\Delta t} \\ &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \end{aligned} \tag{1}$$

The first commutators  $C_n$  are

$$\begin{aligned}
 C_0 &= a^\dagger \\
 C_1 &= [H, a^\dagger] \\
 &= [\chi b^\dagger a, a^\dagger] \\
 &= \chi b^\dagger [a, a^\dagger] \\
 &= \chi b^\dagger \\
 C_2 &= [H, C_1] \\
 &= [\chi a^\dagger b, \chi b^\dagger] \\
 &= \chi^2 a^\dagger [b, b^\dagger] \\
 &= \chi^2 a^\dagger
 \end{aligned}$$

from which it follows that

$$\begin{aligned}
 \forall n \in \mathbb{N}, \quad C_{2n} &= \chi^{2n} a^\dagger \\
 C_{2n+1} &= \chi^{2n+1} b^\dagger
 \end{aligned}$$

We now rewrite equation 1

$$\begin{aligned}
 U a^\dagger U^\dagger &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \\
 &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n}}{(2n)!} C_{2n} + \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n+1}}{(2n+1)!} C_{2n+1} \\
 &= a^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n}}{(2n)!} + b^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n+1}}{(2n+1)!} \\
 &= a^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n}}{(2n)!} - i b^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n+1}}{(2n+1)!} \\
 &= \cos(\chi\Delta t) a^\dagger - i \sin(\chi\Delta t) b^\dagger
 \end{aligned}$$

Let us now compute the effect of  $U$  on  $|0\rangle|0\rangle_b = |00\rangle$ :

$$\begin{aligned}
 U |00\rangle &= e^{-iH\Delta t} |00\rangle \\
 &= \sum_{n=0}^{+\infty} \frac{(-iH\Delta t)^n}{n!} |00\rangle
 \end{aligned}$$

Since  $a|0\rangle = 0$  and  $b|0\rangle_b = 0$ , we have

$$H |00\rangle = 0$$

and

$$\forall n \in \mathbb{N}^*, \quad H^n |00\rangle = 0$$

from which it follows there is only one non nul term in the previous sum and

$$U |00\rangle = |00\rangle$$

Let us compute the effect of  $U$  on  $|1\rangle|0\rangle_b = |10\rangle$ :

$$\begin{aligned}
U|10\rangle &= Ua^\dagger|00\rangle \\
&= Ua^\dagger \underbrace{U^\dagger U}_{=1}|00\rangle \\
&= Ua^\dagger U^\dagger|00\rangle \\
&= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)|00\rangle \\
&= \cos(\chi\Delta t)|10\rangle - i\sin(\chi\Delta t)|01\rangle \\
&= \cos(\chi\Delta t)|1\rangle|0\rangle_b - i\sin(\chi\Delta t)|0\rangle|1\rangle_b
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sqrt{n!}U|n\rangle|0\rangle_b &= \sqrt{n!}U|n0\rangle \\
&= U(a^\dagger)^n|00\rangle \\
&= U(a^\dagger)^n U^\dagger U|00\rangle \\
&= (Ua^\dagger U^\dagger)^n|00\rangle \\
&= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)^n|00\rangle
\end{aligned}$$

Since  $[a^\dagger, b^\dagger] = 0$ ,

$$\begin{aligned}
\sqrt{n!}U|n\rangle|0\rangle_b &= \left( \sum_{k=0}^n \binom{n}{k} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) (a^\dagger)^{n-k} (b^\dagger)^k \right) |00\rangle \\
&= \sum_{k=0}^n \binom{n}{k} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) \sqrt{(n-k)!k!} |n-k\rangle|k\rangle_b
\end{aligned}$$

so that

$$\begin{aligned}
U|n0\rangle &= \sum_{k=0}^n \binom{n}{k} \sqrt{\frac{(n-k)!k!}{n!}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle|k\rangle_b \\
&= \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle|k\rangle_b
\end{aligned}$$

We can think of the number

$$\binom{n}{k} \cos^{2(n-k)}(\chi\Delta t) \sin^{2k}(\chi\Delta t)$$

as the probability of losing  $k$  quanta of energy to the environment.

Let  $E_m = \langle m|_b U|0\rangle_b$ ,  $m \in \mathbb{N}$  the operation elements of  $U$ . They are operators acting on the principal space. We can compute the action of  $E_m$  on  $|n\rangle$  (i.e. compute the  $n$ th column of the matrix of  $E_m$ ) from the previous formula:

$$\begin{aligned}
E_m|n\rangle &= (\langle m|_b U|0\rangle_b)|n\rangle \\
&= \langle m|_b (U|n\rangle|0\rangle_b) \\
&= \langle m|_b U|n0\rangle
\end{aligned}$$

First it is clear that if  $n < m$ ,  $E_m |n\rangle = 0$ . Then if  $n \geq m$ ,

$$\begin{aligned}
 E_m |n\rangle &= \langle m |_b \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle |k\rangle_b \\
 &= \sum_{k=0}^n \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t) (-i)^k \sin^k(\chi\Delta t) |n-k\rangle \underbrace{\langle m|k\rangle_b}_{=\delta_{mk}} \\
 &= (-i)^m \sin^m(\chi\Delta t) \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle
 \end{aligned}$$

This shows that the matrix of  $E_m$  has non nul elements only on the  $m$ th superior diagonal.  $E_m$  corresponds to the physical process of losing  $m$  quanta of energy to the environment.

We can also reconstruct the full formula for  $E_m$  using braket calculus:

$$\begin{aligned}
 E_m &= E_m \underbrace{\sum_{n=0}^{+\infty} |n\rangle \langle n|}_{=1} \\
 &= \sum_{n=0}^{+\infty} E_m |n\rangle \langle n| \\
 &= \sum_{n=m}^{+\infty} E_m |n\rangle \langle n| \\
 &= (-i)^m \sin^m(\chi\Delta t) \sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle \langle n|
 \end{aligned}$$

Note that the sole effect of factor  $(-i)^m$  is to add a global phase so it may as well be omitted.

$$E_m = \sin^m(\chi\Delta t) \begin{bmatrix} & m & & & n \\ & 1 & & & 0 \\ & \sqrt{\binom{m+1}{m}} \cos(\chi\Delta t) & & & \\ & & \sqrt{\binom{m+2}{m}} \cos^2(\chi\Delta t) & & \\ & & \dots & \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) & \\ & & & \dots & \\ 0 & & & & \\ & & & & \dots & \\ & & & & & n-m \end{bmatrix}$$

**21.2. Trace-preserving property.** Matrix calculus or bracket calculus show that the matrices  $E_m^\dagger E_m$  are diagonals, with the first  $m$  elements are 0:

$$\begin{aligned}
E_m^\dagger E_m &= \sin^{2m}(\chi\Delta t) \left( \sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n\rangle \langle n-m| \right) \left( \sum_{l=m}^{+\infty} \sqrt{\binom{l}{m}} \cos^{l-m}(\chi\Delta t) |l-m\rangle \langle l| \right) \\
&= \sin^{2m}(\chi\Delta t) \sum_{n=m}^{+\infty} \sum_{l=m}^{+\infty} \sqrt{\binom{n}{m}} \sqrt{\binom{l}{m}} \cos^{n-m}(\chi\Delta t) \cos^{l-m}(\chi\Delta t) |n\rangle \underbrace{\langle n-m|l-m\rangle}_{=\delta_{nl}} \langle l| \\
&= \sin^{2m}(\chi\Delta t) \sum_{n=m}^{+\infty} \binom{n}{m} \cos^{2(n-m)}(\chi\Delta t) |n\rangle \langle n|
\end{aligned}$$

It follows that the operator  $\sum_{m=0}^{+\infty} E_m^\dagger E_m$  is also diagonal, and diagonal elements are

$$\begin{aligned}
\langle n | \sum_{m=0}^{+\infty} E_m^\dagger E_m | n \rangle &= \sum_{m=0}^{+\infty} \langle n | E_m^\dagger E_m | n \rangle \\
&= \sum_{m=0}^n \langle n | E_m^\dagger E_m | n \rangle \\
&= \sum_{m=0}^n \binom{n}{m} \sin^{2m}(\chi\Delta t) \cos^{2(n-m)}(\chi\Delta t) \\
&= (\sin^2(\chi\Delta t) + \cos^2(\chi\Delta t))^n \\
&= 1
\end{aligned}$$

i.e.  $\sum_{m=0}^{+\infty} E_m^\dagger E_m = 1$  and the quantum operation is trace-preserving.

## 22. Amplitude damping of a single qubit density matrix

Let

$$\rho = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

The amplitude damping operation is defined by

$$\varepsilon_{AD}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

where

$$\begin{aligned} E_0 &= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \\ E_1 &= \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} \end{aligned} \quad (2)$$

Straightforward matrix calculus show that

$$E_0 \rho E_0^\dagger = \begin{bmatrix} a & b\sqrt{1-\gamma} \\ b^* \sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}$$

and

$$\begin{aligned} E_1 \rho E_1^\dagger &= \begin{bmatrix} c\gamma & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (1-a)\gamma & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

because  $1 = \text{Tr } \rho = a + c$ .

Thus we have

$$\begin{aligned} \varepsilon_{AD}(\rho) &= \begin{bmatrix} a + (1-a)\gamma & b\sqrt{1-\gamma} \\ b^* \sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} \\ &= \begin{bmatrix} 1 - (1-a)(1-\gamma) & b\sqrt{1-\gamma} \\ b^* \sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} \end{aligned}$$

### 23. Amplitude damping of dual-rail qubits

Let

$$|\psi\rangle = a|01\rangle + b|10\rangle$$

Applying  $\varepsilon_{AD} \otimes \varepsilon_{AD}$  to  $\rho = |\psi\rangle\langle\psi|$  is equivalent to applying unitary  $B \otimes B$  to  $|\psi\rangle$ , where  $B = e^{\theta(a^\dagger b - ab^\dagger)}$ . Let's do this by making explicit the 2 environment qubits initially set to 0, denoted by subscript  $b$ :

$$|\psi\rangle = a|01\rangle|00\rangle_b + b|10\rangle|00\rangle_b$$

$$\begin{aligned} B \otimes B |\psi\rangle &= a|0\rangle|0\rangle_b (B|1\rangle|0\rangle_b) + b(B|1\rangle|0\rangle_b)|0\rangle|0\rangle_b \\ &= a|0\rangle|0\rangle_b (\cos(\theta)|1\rangle|0\rangle_b + \sin(\theta)|0\rangle|1\rangle_b) + b(\cos(\theta)|1\rangle|0\rangle_b + \sin(\theta)|0\rangle|1\rangle_b)|0\rangle|0\rangle_b \\ &= a\cos(\theta)|0\rangle|0\rangle_b|1\rangle|0\rangle_b + a\sin(\theta)|0\rangle|0\rangle_b|0\rangle|1\rangle_b + b\cos(\theta)|1\rangle|0\rangle_b|0\rangle|0\rangle_b + b\sin(\theta)|0\rangle|1\rangle_b|0\rangle|0\rangle_b \end{aligned}$$

We reorder the qubits to put the environments qubits at the end since we will trace them out:

$$\begin{aligned} B \otimes B |\psi\rangle &= a\cos(\theta)|01\rangle|00\rangle_b + a\sin(\theta)|00\rangle|01\rangle_b + b\cos(\theta)|10\rangle|00\rangle_b + b\sin(\theta)|00\rangle|10\rangle_b \\ &= |\varphi\rangle \end{aligned} \quad (3)$$

Now we have to find the dual vector  $\langle\varphi|$  of this state. We can recall the not so trivial following facts related to product space: Let  $\{|a_i\rangle\}, \{|b_j\rangle\}$  be basis of two Hilbert spaces  $A$  and  $B$ .

The dual of  $|a_i b_j\rangle = |a_i\rangle \otimes |b_j\rangle$  is

$$\langle a_i | \otimes \langle b_j | = \langle a_i b_j |$$

so that

$$\langle\varphi| = a^* \cos(\theta) \langle 01| \langle 00|_b + a^* \sin(\theta) \langle 00| \langle 01|_b + b^* \cos(\theta) \langle 10| \langle 00|_b + b^* \sin(\theta) \langle 00| \langle 10|_b$$

We have also

$$|a_k b_l\rangle \langle a_i b_j| = |a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|$$

We could then use equation 3 to compute the density  $|\varphi\rangle \langle \varphi|$ , but this would be a messy sum with 16 terms.

Since we will trace out the environment, we recall the partial trace formula:

$$\begin{aligned} \text{Tr}_B(|a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|) &= |a_k\rangle \langle a_i| \text{Tr}(|b_l\rangle \langle b_j|) \\ &= |a_k\rangle \langle a_i| \langle b_l| b_j \rangle \end{aligned}$$

Since  $\{|00\rangle_b, |01\rangle_b, |10\rangle_b, |11\rangle_b\}$  is an orthonormal basis, there are only 6 out of 16 terms left after the partial trace operation:

$$\begin{aligned} \text{Tr}_b(|\varphi\rangle \langle \varphi|) &= |a|^2 \cos^2(\theta) |01\rangle \langle 01| + ab^* \cos^2(\theta) |01\rangle \langle 10| + |a|^2 \sin^2(\theta) |00\rangle \langle 00| \\ &\quad + |b|^2 \cos^2(\theta) |10\rangle \langle 10| + ba^* \cos^2(\theta) |10\rangle \langle 01| + |b|^2 \sin^2(\theta) |00\rangle \langle 00| \\ &= |a|^2(1-\gamma) |01\rangle \langle 01| + ab^*(1-\gamma) |01\rangle \langle 10| + |a|^2 \gamma |00\rangle \langle 00| \\ &\quad + |b|^2(1-\gamma) |10\rangle \langle 10| + ba^*(1-\gamma) |10\rangle \langle 01| + |b|^2 \gamma |00\rangle \langle 00| \\ &= \underbrace{(|a|^2 + |b|^2)\gamma |00\rangle \langle 00|}_{=1} + (1-\gamma) (|a|^2 |01\rangle \langle 01| + ab^* |01\rangle \langle 10| + |b|^2 |10\rangle \langle 10| + ba^* |10\rangle \langle 01|) \\ &= \gamma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + (1-\gamma) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & |a|^2 & ab^* & 0 \\ 0 & a^*b & |b|^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \gamma \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + (1-\gamma)\rho \end{aligned}$$

It is a mixed state:

- with probability  $\gamma$ , the state is projected to  $|00\rangle$ , orthogonal to  $|\psi\rangle$ .
- with probability  $1-\gamma$ , state is unchanged.

Since  $|00\rangle$  is orthogonal to  $|\psi\rangle$ , one can detect amplitude damping errors with measurement operators:

$$\begin{aligned} M_0 &= |00\rangle \langle 00| \quad \text{orthogonal projector on } \text{span}\{|00\rangle\} \\ M_1 &= |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11| \quad \text{orthogonal projector on } \text{span}\{|01\rangle, |10\rangle, |11\rangle\} \end{aligned}$$

- If the state decayed to  $|00\rangle$ , then with probability 1 the result of the measurement will be  $|00\rangle$ .
- Otherwise, with probability 1 the result of the measurement will be the original  $|\psi\rangle$ .

It can be easily checked that the quantum operation can be described with 3 operators:

$$\begin{aligned} E_0^{dr} &= \sqrt{1-\gamma} I \\ E_1^{dr} &= \sqrt{\gamma} |00\rangle \langle 01| \\ &= \sqrt{\gamma} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ E_2^{dr} &= \sqrt{\gamma} |00\rangle \langle 10| \\ &= \sqrt{\gamma} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$



It is interesting to see that these operators are the restriction to  $\text{span}\{|01\rangle, |10\rangle\}$  of the operators

$$E_0 \otimes E_0$$

$$E_0 \otimes E_1$$

$$E_1 \otimes E_0$$

$$E_1 \otimes E_1$$

where  $E_0, E_1$  are the operators of amplitude damping for single qubit, defined in 2.