QUANTUM COMPUTATION AND QUANTUM INFORMATION: THE QUANTUM FOURIER TRANSFORM

1

We consider the linear map in $\mathbb{C}^N$ which acts on the computational basis as

$$|j\rangle \mapsto \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{\frac{2i\pi jk}{N}} |k\rangle$$

Let A be the matrix of the transformation in the computational basis.

$$\forall (k,l) \in [\![0, N-1]\!]^2, \quad a_{kl} = \frac{1}{\sqrt{N}} e^{\frac{2i\pi kl}{N}}$$

The adjoint matrix $A^\dagger$ is then

$$\forall (k,l) \in [\![0, N-1]\!]^2, \quad b_{kl} = a_{lk}^*$$
$$= \frac{1}{\sqrt{N}} e^{-\frac{2i\pi kl}{N}}$$

We compute the coefficient k, l of the product $AA^\dagger$:

$$\forall (k,l) \in [\![0, N-1]\!]^2, \quad c_{kl} = \sum_{j=0}^{N-1} a_{kj} b_{jl}$$
$$= \frac{1}{N} \sum_{j=0}^{N-1} e^{\frac{2i\pi j}{N}(k-l)}$$
$$= \frac{1}{N} \sum_{j=0}^{N-1} (e^{\frac{2i\pi}{N}(k-l)})^j$$
$$= \begin{cases} \frac{1}{N} \frac{1-(e^{\frac{2i\pi}{N}(k-l)})^N}{1-e^{\frac{2i\pi}{N}(k-l)}} = 0 & \text{if } e^{\frac{2i\pi}{N}(k-l)} \neq 1, \\ 1 & \text{if } e^{\frac{2i\pi}{N}(k-l)} = 1. \end{cases}$$
$$= \begin{cases} 0 & \text{if } k \neq l, \\ 1 & \text{if } k = l. \end{cases}$$
$$= \delta_{kl}$$

which shows that $AA^\dagger = I$ i.e. A is unitary.

2

Here the dimension $N = 2^n$. The Fourier transform of the n qubit state $|00\ldots 0\rangle$ is

$$A|0\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} |k\rangle$$

we can write k in binary $k_{n-1}\ldots k_1 k_0$

$$A|0\rangle = \frac{1}{2^{n/2}} \sum_{k_0, k1, \ldots, k_{n-1}=0}^{1} |k_{n-1}\ldots k_1 k_0\rangle$$

or in product representation,

$$= \frac{1}{2^{n/2}} \underbrace{(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)\ldots(|0\rangle + |1\rangle)}_{n \text{ qubits}}$$