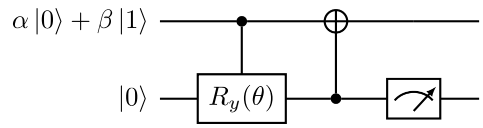
QUANTUM COMPUTATION AND QUANTUM INFORMATION: QUANTUM NOISE AND QUANTUM OPERATIONS

20. Circuit model for amplitude dampling

We want to prove that the following circuit models the amplitude dampling operation



Recall that

$$\begin{aligned} R_y(\theta) &= e^{-i\frac{\theta}{2}Y} \\ &= \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \end{aligned}$$

Initially the two-qubit state is

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle = \alpha|00\rangle + \beta|10\rangle$$

After the controlled R_y gate it becomes

$$\begin{aligned} \alpha|00\rangle + \beta|1\rangle R_y(\theta)|0\rangle &= \alpha|00\rangle + \beta|1\rangle(\cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})|1\rangle) \\ &= \alpha|00\rangle + \beta(\cos(\frac{\theta}{2})|10\rangle + \sin(\frac{\theta}{2})|11\rangle) \end{aligned}$$

After the controlled not gate,

$$\alpha|00\rangle + \beta(\cos(\frac{\theta}{2})|10\rangle + \sin(\frac{\theta}{2})|01\rangle)$$

This is the effect of amplitude dampling, with probability of 1 be switched to 0, or one photon being lost to environment, being $\gamma = \sin^2(\frac{\theta}{2})$.

21. Amplitude dampling of a harmonic oscillator

The principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian:

$$H = \chi(a^\dagger b + b^\dagger a)$$

where $a^\dagger, a$ and $b^\dagger, b$ are the creation, annihilation operators for the principal and environment oscillators, respectively.

The time evolution of the coupled system is governed by the unitary operator:

$$U = e^{-iH\Delta t}$$

21.1. Operation elements. We recall some results for the harmonic oscillator:

$$\forall n \in \mathbb{N}, \quad a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

and similarly in the environment space

$$\forall n \in \mathbb{N}, \quad b^\dagger |n\rangle_b = \sqrt{n+1}\,|n+1\rangle_b$$

Here we use the subscript b to differentiate the eigenvectors of the Hermitian operator $bb^\dagger$ which live in the environment space from the eigenvectors of $aa^\dagger$ in the principal space:

$$\begin{aligned} \forall n \in \mathbb{N}, \quad bb^\dagger |n\rangle_b &= (n+1)\,|n\rangle_b \\ \forall n \in \mathbb{N}, \quad aa^\dagger |n\rangle &= (n+1)\,|n\rangle \end{aligned}$$

Each set of vectors constitute an orthonormal basis:

$$\begin{aligned} \forall (n,m) \in \mathbb{N}^2, \quad \langle n|m\rangle &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases} \\ &= \delta_{nm} \end{aligned}$$

We also have

$$\begin{aligned} aa^\dagger - a^\dagger a &= [a, a^\dagger] \\ &= 1 \\ bb^\dagger - b^\dagger b &= [b, b^\dagger] \\ &= 1 \end{aligned}$$

where 1 stands for the identity operator.

Each of the operators $a, a^\dagger$ commutes with each of the operators $b, b^\dagger$ since they act on different spaces

$$\begin{aligned} 0 &= [a^\dagger, b^\dagger] \\ &= [a, b^\dagger] \\ &= [a^\dagger, b] \\ &= [a, b] \end{aligned}$$

The Baker-Campbell-Hausdorff formula states that, for any operators A, G such that e^G exists,

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} C_n$$

where the operators C_n are defined recursively by

$$\begin{aligned} C_0 &= A \\ C_1 &= [G, A] \\ \forall n \in \mathbb{N}, \quad C_{n+1} &= [G, C_n] \end{aligned}$$

Lets compute a simplified expression for the operator $Ua^\dagger U^\dagger$ acting on the product space:

$$\begin{aligned} Ua^\dagger U^\dagger &= e^{-iH\Delta t} a^\dagger e^{iH\Delta t} \\ &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \end{aligned} \tag{1}$$

The first commutators C_n are

$$\begin{aligned}
C_0 &= a^\dagger \\
C_1 &= [H, a^\dagger] \\
&= [\chi b^\dagger a, a^\dagger] \\
&= \chi b^\dagger [a, a^\dagger] \\
&= \chi b^\dagger \\
C_2 &= [H, C_1] \\
&= [\chi a^\dagger b, \chi b^\dagger] \\
&= \chi^2 a^\dagger [b, b^\dagger] \\
&= \chi^2 a^\dagger
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\forall n \in \mathbb{N}, \quad C_{2n} &= \chi^{2n} a^\dagger \\
C_{2n+1} &= \chi^{2n+1} b^\dagger
\end{aligned}$$

We now rewrite equation 1

$$\begin{aligned}
U a^\dagger U^\dagger &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \\
&= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n}}{(2n)!} C_{2n} + \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n+1}}{(2n+1)!} C_{2n+1} \\
&= a^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n}}{(2n)!} + b^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n+1}}{(2n+1)!} \\
&= a^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n}}{(2n)!} - i b^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n+1}}{(2n+1)!} \\
&= \cos(\chi\Delta t) a^\dagger - i \sin(\chi\Delta t) b^\dagger
\end{aligned}$$

Let us now compute the effect of U on $|0\rangle |0\rangle_b = |00\rangle$:

$$\begin{aligned}
U|00\rangle &= e^{-iH\Delta t} |00\rangle \\
&= \sum_{n=0}^{+\infty} \frac{(-iH\Delta t)^n}{n!} |00\rangle
\end{aligned}$$

Since $a|0\rangle = 0$ and $b|0\rangle_b = 0$, we have

$$H|00\rangle = 0$$

and

$$\forall n \in \mathbb{N}^*, \quad , H^n |00\rangle = 0$$

from which it follows there is only one non nul term in the previous sum and

$$U|00\rangle = |00\rangle$$

Let us compute the effect of U on $|1\rangle |0\rangle_b = |10\rangle$:

$$\begin{aligned} U|10\rangle &= Ua^\dagger |00\rangle \\ &= Ua^\dagger \underbrace{U^\dagger U}_{=1} |00\rangle \\ &= Ua^\dagger U^\dagger |00\rangle \\ &= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)|00\rangle \\ &= \cos(\chi\Delta t)|10\rangle - i\sin(\chi\Delta t)|01\rangle \\ &= \cos(\chi\Delta t)|1\rangle|0\rangle_b - i\sin(\chi\Delta t)|0\rangle|1\rangle_b \end{aligned}$$

Similarly,

$$\begin{aligned} \sqrt{n!}U|n\rangle|0\rangle_b &= \sqrt{n!}U|n0\rangle \\ &= U(a^\dagger)^n|00\rangle \\ &= U(a^\dagger)^n U^\dagger U|00\rangle \\ &= (Ua^\dagger U^\dagger)^n|00\rangle \\ &= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)^n|00\rangle \end{aligned}$$

Since $[a^\dagger, b^\dagger] = 0$,

$$\begin{aligned} \sqrt{n!}U|n\rangle|0\rangle_b &= \left(\sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)(a^\dagger)^{n-k}(b^\dagger)^k\right)|00\rangle \\ &= \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)\sqrt{(n-k)!}\sqrt{k!}\,|n-k\rangle|k\rangle_b \end{aligned}$$

so that

$$\begin{aligned} U|n0\rangle &= \sum_{k=0}^{n}\binom{n}{k}\sqrt{\frac{(n-k)!k!}{n!}}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)|n-k\rangle|k\rangle_b \\ &= \sum_{k=0}^{n}\sqrt{\binom{n}{k}}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)|n-k\rangle|k\rangle_b \end{aligned}$$

We can think of the number

$$\binom{n}{k}\cos^{2(n-k)}(\chi\Delta t)\sin^{2k}(\chi\Delta t)$$

as the probability of losing k quanta of energy to the environment.

Let $E_m = \langle m|_b U|0\rangle_b\,, \quad m\in\mathbb{N}$ the operation elements of U. They are operators acting on the principal space. We can compute the action of E_m on $|n\rangle$ (i.e. compute the nth column of the matrix of E_m) from the previous formula:

$$\begin{aligned} E_m|n\rangle &= (\langle m|_b U|0\rangle_b)|n\rangle \\ &= \langle m|_b (U|n\rangle|0\rangle_b) \\ &= \langle m|_b U|n0\rangle \end{aligned}$$

First it is clear that if $n < m$, $E_m |n\rangle = 0$. Then if $n \geqslant m$,

$$\begin{aligned} E_m |n\rangle &= \langle m|_b \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t) |n-k\rangle |k\rangle_b \\ &= \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t) |n-k\rangle \underbrace{\langle m|k\rangle_b}_{=\delta_{mk}} \\ &= (-i)^m \sin^m(\chi\Delta t) \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle \end{aligned}$$

This shows that the matrix of E_m has non nul elements only on the mth superior diagonal. E_m corresponds to the physical process of losing m quanta of energy to the environment.

We can also reconstruct the full formula for E_m using braket calculus:

$$\begin{aligned} E_m &= E_m \underbrace{\sum_{n=0}^{+\infty} |n\rangle \langle n|}_{=1} \\ &= \sum_{n=0}^{+\infty} E_m |n\rangle \langle n| \\ &= \sum_{n=m}^{+\infty} E_m |n\rangle \langle n| \\ &= (-i)^m \sin^m(\chi\Delta t) \sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle \langle n| \end{aligned}$$

Note that the sole effect of factor $(-i)^m$ is to add a global phase so it may as well be omitted.

$$E_m = \sin^m(\chi\Delta t) \begin{array}{cc} \begin{array}{cccc} m & & & n \end{array} & \\ \begin{bmatrix} 1 & & & & \\ & \sqrt{\binom{m+1}{m}} \cos(\chi\Delta t) & & \mathbf{0} & \\ & & \sqrt{\binom{m+2}{m}} \cos^2(\chi\Delta t) & & \\ & & & \ddots & \\ & & & \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) & \\ & & & & \ddots \\ & \mathbf{0} & & & \end{bmatrix} & \begin{array}{c} \\ \\ \\ \\ n-m \\ \\ \\ \end{array} \end{array}$$

21.2. Trace-preserving property. Matrix calculus or braket calculus show that the matrices $E_m^\dagger E_m$ are diagonals, with the first m elements are 0:

$$\begin{aligned}
E_m^\dagger E_m &= \sin^{2m}(\chi\Delta t)\left(\sum_{n=m}^{+\infty}\sqrt{\binom{n}{m}}\cos^{n-m}(\chi\Delta t)\left|n\right\rangle\left\langle n-m\right|\right)\left(\sum_{l=m}^{+\infty}\sqrt{\binom{l}{m}}\cos^{l-m}(\chi\Delta t)\left|l-m\right\rangle\left\langle l\right|\right)\\
&= \sin^{2m}(\chi\Delta t)\sum_{n=m}^{+\infty}\sum_{l=m}^{+\infty}\sqrt{\binom{n}{m}}\sqrt{\binom{l}{m}}\cos^{n-m}(\chi\Delta t)\cos^{l-m}(\chi\Delta t)\left|n\right\rangle\underbrace{\left\langle n-m|l-m\right\rangle}_{=\delta_{nl}}\left\langle l\right|\\
&= \sin^{2m}(\chi\Delta t)\sum_{n=m}^{+\infty}\binom{n}{m}\cos^{2(n-m)}(\chi\Delta t)\left|n\right\rangle\left\langle n\right|
\end{aligned}$$

$$E_m^\dagger E_m = \sin^{2m}(\chi\Delta t)\begin{array}{cc}
\begin{array}{ccccccc} \qquad\qquad & m & & & & n & \qquad\qquad \end{array} & \\
\begin{bmatrix}
\mathbf{0} & & & & & \mathbf{0} & \\
 & 1 & & & & & \\
 & & \binom{m+1}{m}\cos^2(\chi\Delta t) & & & & \\
 & & & \binom{m+1}{m}\cos^4(\chi\Delta t) & & & \\
 & & & & \ddots & & \\
 & & & & & \binom{n}{m}\cos^{2(n-m)}(\chi\Delta t) & \\
\mathbf{0} & & & & & & \ddots
\end{bmatrix} & \begin{array}{c} \\ m \\ \\ \\ \\ n \\ \\ \end{array}
\end{array}$$

It follows that the operator $\sum_{m=0}^{+\infty} E_m^\dagger E_m$ is also diagonal, and diagonal elements are

$$\begin{aligned}
\left\langle n\right|\sum_{m=0}^{+\infty}E_m^\dagger E_m\left|n\right\rangle &= \sum_{m=0}^{+\infty}\left\langle n|E_m^\dagger E_m|n\right\rangle\\
&= \sum_{m=0}^{n}\left\langle n|E_m^\dagger E_m|n\right\rangle\\
&= \sum_{m=0}^{n}\binom{n}{m}\sin^{2m}(\chi\Delta t)\cos^{2(n-m)}(\chi\Delta t)\\
&= (\sin^2(\chi\Delta t)+\cos^2(\chi\Delta t))^n\\
&= 1
\end{aligned}$$

i.e. $\sum_{m=0}^{+\infty} E_m^\dagger E_m = 1$ and the quantum operation is trace-preserving.

21.3. Amplitude dampling of a single qubit density matrix. Let

$$\rho = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

The amplitude dampling operation is defined by

$$\varepsilon_{AD}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$$

where

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}$$
$$E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

Straightforward matrix calculus show that

$$E_0 \rho E_0^\dagger = \begin{bmatrix} a & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}$$

and

$$\begin{aligned} E_1 \rho E_1^\dagger &= \begin{bmatrix} c\gamma & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (1-a)\gamma & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

because $1 = \operatorname{Tr}\rho = a + c$.

Thus we have

$$\begin{aligned} \varepsilon_{AD}(\rho) &= \begin{bmatrix} a + (1-a)\gamma & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} \\ &= \begin{bmatrix} 1 - (1-a)(1-\gamma) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} \end{aligned}$$