QUANTUM COMPUTATION AND QUANTUM INFORMATION:
QUANTUM NOISE AND QUANTUM OPERATIONS

21. Amplitude dampling of a harmonic oscillator

The principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian:

$$H = \chi(a^\dagger b + b^\dagger a)$$

where $a^\dagger, a$ and $b^\dagger, b$ are the creation, annihilation operators for the principal and environment oscillators, respectively.

The time evolution of the coupled system is governed by the unitary operator:

$$U = e^{-iH\Delta t}$$

21.1. Operation elements.

We recall some results for the harmonic oscillator:

$$\forall n \in \mathbb{N}, \quad a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$$

and similarly in the environment space

$$\forall n \in \mathbb{N}, \quad b^\dagger |n\rangle_b = \sqrt{n+1}\, |n+1\rangle_b$$

Here we use the subscript b to differentiate the eigenvectors of the Hermitian operator $bb^\dagger$ which live in the environment space from the eigenvectors of $aa^\dagger$ in the principal space:

$$\begin{aligned} \forall n \in \mathbb{N}, \quad bb^\dagger |n\rangle_b &= (n+1)\, |n\rangle_b \\ \forall n \in \mathbb{N}, \quad aa^\dagger |n\rangle &= (n+1)\, |n\rangle \end{aligned}$$

Each set of vectors constitute an orthonormal basis:

$$\begin{aligned} \forall (n,m) \in \mathbb{N}^2, \quad \langle n|m\rangle &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases} \\ &= \delta_{nm} \end{aligned}$$

We also have

$$\begin{aligned} aa^\dagger - a^\dagger a &= [a, a^\dagger] \\ &= 1 \\ bb^\dagger - b^\dagger b &= [b, b^\dagger] \\ &= 1 \end{aligned}$$

where 1 stands for the identity operator.

Each of the operators $a, a^\dagger$ commutes with each of the operators $b, b^\dagger$ since they act on different spaces

$$\begin{aligned} 0 &= [a^\dagger, b^\dagger] \\ &= [a, b^\dagger] \\ &= [a^\dagger, b] \\ &= [a, b] \end{aligned}$$

The Baker-Campbell-Hausdorff formula states that, for any operators A, G such that e^G exists,

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} C_n$$

where the operators C_n are defined recursively by

$$\begin{aligned} C_0 &= A \\ C_1 &= [G, A] \\ \forall n \in \mathbb{N}, \quad C_{n+1} &= [G, C_n] \end{aligned}$$

Lets compute a simplified expression for the operator $U a^\dagger U^\dagger$ acting on the product space:

$$\begin{aligned} U a^\dagger U^\dagger &= e^{-iH\Delta t} a^\dagger e^{iH\Delta t} \\ &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \end{aligned} \tag{1}$$

The first commutators C_n are

$$\begin{aligned} C_0 &= a^\dagger \\ C_1 &= [H, a^\dagger] \\ &= [\chi b^\dagger a, a^\dagger] \\ &= \chi b^\dagger [a, a^\dagger] \\ &= \chi b^\dagger \\ C_2 &= [H, C_1] \\ &= [\chi a^\dagger b, \chi b^\dagger] \\ &= \chi^2 a^\dagger [b, b^\dagger] \\ &= \chi^2 a^\dagger \end{aligned}$$

from which it follows that

$$\begin{aligned} \forall n \in \mathbb{N}, \quad C_{2n} &= \chi^{2n} a^\dagger \\ C_{2n+1} &= \chi^{2n+1} b^\dagger \end{aligned}$$

We now rewrite equation 1

$$\begin{aligned} U a^\dagger U^\dagger &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \\ &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n}}{(2n)!} C_{2n} + \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n+1}}{(2n+1)!} C_{2n+1} \\ &= a^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n}}{(2n)!} + b^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n+1}}{(2n+1)!} \\ &= a^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n}}{(2n)!} - i b^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n+1}}{(2n+1)!} \\ &= \cos(\chi\Delta t) a^\dagger - i \sin(\chi\Delta t) b^\dagger \end{aligned}$$

Let us now compute the effect of U on $|0\rangle |0\rangle_b = |00\rangle$:

$$\begin{aligned} U|00\rangle &= e^{-iH\Delta t} |00\rangle \\ &= \sum_{n=0}^{+\infty} \frac{(-iH\Delta t)^n}{n!} |00\rangle \end{aligned}$$

Since $a\left|0\right\rangle = 0$ and $b\left|0\right\rangle_b = 0$, we have

$$H\left|00\right\rangle = 0$$

and

$$\forall n \in \mathbb{N}^*, \quad , H^n\left|00\right\rangle = 0$$

from which it follows there is only one non nul term in the previous sum and

$$U\left|00\right\rangle = \left|00\right\rangle$$

Let us compute the effect of U on $\left|1\right\rangle\left|0\right\rangle_b = \left|10\right\rangle$:

$$\begin{aligned}
U\left|10\right\rangle &= Ua^\dagger\left|00\right\rangle \\
&= Ua^\dagger \underbrace{U^\dagger U}_{=1}\left|00\right\rangle \\
&= Ua^\dagger U^\dagger\left|00\right\rangle \\
&= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)\left|00\right\rangle \\
&= \cos(\chi\Delta t)\left|10\right\rangle - i\sin(\chi\Delta t)\left|01\right\rangle \\
&= \cos(\chi\Delta t)\left|1\right\rangle\left|0\right\rangle_b - i\sin(\chi\Delta t)\left|0\right\rangle\left|1\right\rangle_b
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sqrt{n!}U\left|n\right\rangle\left|0\right\rangle_b &= \sqrt{n!}U\left|n0\right\rangle \\
&= U(a^\dagger)^n\left|00\right\rangle \\
&= U(a^\dagger)^n U^\dagger U\left|00\right\rangle \\
&= (Ua^\dagger U^\dagger)^n\left|00\right\rangle \\
&= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)^n\left|00\right\rangle
\end{aligned}$$

Since $[a^\dagger, b^\dagger] = 0$,

$$\begin{aligned}
\sqrt{n!}U\left|n\right\rangle\left|0\right\rangle_b &= \left(\sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)(a^\dagger)^{n-k}(b^\dagger)^k\right)\left|00\right\rangle \\
&= \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)\sqrt{(n-k)!}\sqrt{k!}\left|n-k\right\rangle\left|k\right\rangle_b
\end{aligned}$$

so that

$$\begin{aligned}
U\left|n0\right\rangle &= \sum_{k=0}^{n}\binom{n}{k}\sqrt{\frac{(n-k)!k!}{n!}}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)\left|n-k\right\rangle\left|k\right\rangle_b \\
&= \sum_{k=0}^{n}\sqrt{\binom{n}{k}}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)\left|n-k\right\rangle\left|k\right\rangle_b
\end{aligned}$$

Let $E_m = \left\langle m\right|_b U\left|0\right\rangle_b, \quad m \in \mathbb{N}$ the operation elements of U. They are operators acting on the principal space. We can compute the action of E_m on $\left|n\right\rangle$ (i.e. compute the nth column of the matrix of E_m) from the previous formula:

$$\begin{aligned}
E_m\left|n\right\rangle &= (\left\langle m\right|_b U\left|0\right\rangle_b)\left|n\right\rangle \\
&= \left\langle m\right|_b (U\left|n\right\rangle\left|0\right\rangle_b) \\
&= \left\langle m\right|_b U\left|n0\right\rangle
\end{aligned}$$

First it is clear that if $n < m$, $E_m \left|n\right\rangle = 0$. Then if $n \geqslant m$,

$$\begin{aligned}
E_m \left|n\right\rangle &= \left\langle m\right|_b \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t) \left|n-k\right\rangle \left|k\right\rangle_b \\
&= \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t) \left|n-k\right\rangle \underbrace{\left\langle m|k\right\rangle_b}_{=\delta_{mk}} \\
&= (-i)^m \sin^m(\chi\Delta t) \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) \left|n-m\right\rangle
\end{aligned}$$

This shows that the matrix of E_m has non nul elements only on the mth superior diagonal.

We can also reconstruct the full formula for E_m using braket calculus:

$$\begin{aligned}
E_m &= E_m \underbrace{\sum_{n=0}^{+\infty} \left|n\right\rangle \left\langle n\right|}_{=1} \\
&= \sum_{n=0}^{+\infty} E_m \left|n\right\rangle \left\langle n\right| \\
&= \sum_{n=m}^{+\infty} E_m \left|n\right\rangle \left\langle n\right| \\
&= (-i)^m \sin^m(\chi\Delta t) \sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) \left|n-m\right\rangle \left\langle n\right|
\end{aligned}$$

Note that the sole effect of factor $(-i)^m$ is to add a global phase so it may as well be omitted.

$$E_m = \sin^m(\chi\Delta t) \begin{array}{cc}
\begin{array}{cccccc} \quad & m & \qquad\qquad & \qquad\qquad & n & \quad \end{array} & \\
\begin{bmatrix}
 & 1 & & & \mathbf{0} & \\
 & & \sqrt{\binom{m+1}{m}}\cos(\chi\Delta t) & & & \\
 & & & \sqrt{\binom{m+2}{m}}\cos^2(\chi\Delta t) & & \\
 & & & \ddots & & \\
 & & & & \sqrt{\binom{n}{m}}\cos^{n-m}(\chi\Delta t) & \\
 & & & & & \ddots \\
 & \mathbf{0} & & & & \\
 & & & & &
\end{bmatrix} & \begin{array}{c} \\ \\ \\ \\ n-m \\ \\ \\ \\ \end{array}
\end{array}$$