QUANTUM COMPUTATION AND QUANTUM INFORMATION:
QUANTUM NOISE AND QUANTUM OPERATIONS

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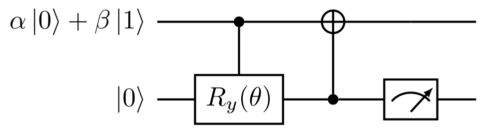
20. Circuit model for amplitude damping

We want to prove that the following circuit models the amplitude damping operation



Recall that

$$\begin{aligned} R_y(\theta) &= e^{-i\frac{\theta}{2}Y} \\ &= \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \end{aligned}$$

Initially the two-qubit state is

$$(\alpha |0\rangle + \beta |1\rangle) |0\rangle = \alpha |00\rangle + \beta |10\rangle$$

After the controlled R_y gate it becomes

$$\begin{aligned} \alpha |00\rangle + \beta |1\rangle R_y(\theta) |0\rangle &= \alpha |00\rangle + \beta |1\rangle (\cos(\frac{\theta}{2}) |0\rangle + \sin(\frac{\theta}{2}) |1\rangle) \\ &= \alpha |00\rangle + \beta(\cos(\frac{\theta}{2}) |10\rangle + \sin(\frac{\theta}{2}) |11\rangle) \end{aligned}$$

After the controlled not gate,

$$\alpha |00\rangle + \beta(\cos(\frac{\theta}{2}) |10\rangle + \sin(\frac{\theta}{2}) |01\rangle)$$

This is the effect of amplitude damping, with probability of 1 be switched to 0, or one photon being lost to environment, being $\gamma = \sin^2(\frac{\theta}{2})$.

21. Amplitude damping of a harmonic oscillator

The principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian:

$$H = \chi(a^\dagger b + b^\dagger a)$$

where $a^\dagger, a$ and $b^\dagger, b$ are the creation, annihilation operators for the principal and environment oscillators, respectively.

The time evolution of the coupled system is governed by the unitary operator:

$$U = e^{-iH\Delta t}$$

21.1. Operation elements. We recall some results for the harmonic oscillator:

$$\forall n \in \mathbb{N}, \quad a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$$

and similarly in the environment space

$$\forall n \in \mathbb{N}, \quad b^\dagger |n\rangle_b = \sqrt{n+1}\, |n+1\rangle_b$$

Here we use the subscript b to differentiate the eigenvectors of the Hermitian operator $bb^\dagger$ which live in the environment space from the eigenvectors of $aa^\dagger$ in the principal space:

$$\begin{aligned}\forall n \in \mathbb{N}, \quad bb^\dagger |n\rangle_b &= (n+1)\, |n\rangle_b \\ \forall n \in \mathbb{N}, \quad aa^\dagger |n\rangle &= (n+1)\, |n\rangle\end{aligned}$$

Each set of vectors constitute an orthonormal basis:

$$\begin{aligned}\forall (n,m) \in \mathbb{N}^2, \quad \langle n|m\rangle &= \begin{cases} 0 & \text{if } n \neq m, \\ 1 & \text{if } n = m. \end{cases} \\ &= \delta_{nm}\end{aligned}$$

We also have

$$\begin{aligned}aa^\dagger - a^\dagger a &= [a, a^\dagger] \\ &= 1 \\ bb^\dagger - b^\dagger b &= [b, b^\dagger] \\ &= 1\end{aligned}$$

where 1 stands for the identity operator.

Each of the operators $a, a^\dagger$ commutes with each of the operators $b, b^\dagger$ since they act on different spaces

$$\begin{aligned}0 &= [a^\dagger, b^\dagger] \\ &= [a, b^\dagger] \\ &= [a^\dagger, b] \\ &= [a, b]\end{aligned}$$

The Baker-Campbell-Hausdorff formula states that, for any operators A, G such that e^G exists,

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} C_n$$

where the operators C_n are defined recursively by

$$\begin{aligned}C_0 &= A \\ C_1 &= [G, A] \\ \forall n \in \mathbb{N}, \quad C_{n+1} &= [G, C_n]\end{aligned}$$

Lets compute a simplified expression for the operator $Ua^\dagger U^\dagger$ acting on the product space:

$$\begin{aligned}Ua^\dagger U^\dagger &= e^{-iH\Delta t} a^\dagger e^{iH\Delta t} \\ &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n\end{aligned} \tag{1}$$

The first commutators C_n are

$$\begin{aligned}
C_0 &= a^\dagger \\
C_1 &= [H, a^\dagger] \\
&= [\chi b^\dagger a, a^\dagger] \\
&= \chi b^\dagger [a, a^\dagger] \\
&= \chi b^\dagger \\
C_2 &= [H, C_1] \\
&= [\chi a^\dagger b, \chi b^\dagger] \\
&= \chi^2 a^\dagger [b, b^\dagger] \\
&= \chi^2 a^\dagger
\end{aligned}$$

from which it follows that

$$\begin{aligned}
\forall n \in \mathbb{N}, \quad C_{2n} &= \chi^{2n} a^\dagger \\
C_{2n+1} &= \chi^{2n+1} b^\dagger
\end{aligned}$$

We now rewrite equation 1

$$\begin{aligned}
U a^\dagger U^\dagger &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \\
&= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n}}{(2n)!} C_{2n} + \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^{2n+1}}{(2n+1)!} C_{2n+1} \\
&= a^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n}}{(2n)!} + b^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^{2n+1}}{(2n+1)!} \\
&= a^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n}}{(2n)!} - i b^\dagger \sum_{n=0}^{+\infty} (-1)^n \frac{(\chi\Delta t)^{2n+1}}{(2n+1)!} \\
&= \cos(\chi\Delta t) a^\dagger - i \sin(\chi\Delta t) b^\dagger
\end{aligned}$$

Let us now compute the effect of U on $|0\rangle |0\rangle_b = |00\rangle$:

$$\begin{aligned}
U |00\rangle &= e^{-iH\Delta t} |00\rangle \\
&= \sum_{n=0}^{+\infty} \frac{(-iH\Delta t)^n}{n!} |00\rangle
\end{aligned}$$

Since $a|0\rangle = 0$ and $b|0\rangle_b = 0$, we have

$$H |00\rangle = 0$$

and

$$\forall n \in \mathbb{N}^*, \quad , H^n |00\rangle = 0$$

from which it follows there is only one non nul term in the previous sum and

$$U |00\rangle = |00\rangle$$

Let us compute the effect of U on $|1\rangle |0\rangle_b = |10\rangle$:

$$\begin{aligned} U|10\rangle &= Ua^\dagger |00\rangle \\ &= Ua^\dagger \underbrace{U^\dagger U}_{=1} |00\rangle \\ &= Ua^\dagger U^\dagger |00\rangle \\ &= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)|00\rangle \\ &= \cos(\chi\Delta t)|10\rangle - i\sin(\chi\Delta t)|01\rangle \\ &= \cos(\chi\Delta t)|1\rangle|0\rangle_b - i\sin(\chi\Delta t)|0\rangle|1\rangle_b \end{aligned}$$

Similarly,

$$\begin{aligned} \sqrt{n!}U|n\rangle|0\rangle_b &= \sqrt{n!}U|n0\rangle \\ &= U(a^\dagger)^n|00\rangle \\ &= U(a^\dagger)^n U^\dagger U|00\rangle \\ &= (Ua^\dagger U^\dagger)^n|00\rangle \\ &= (\cos(\chi\Delta t)a^\dagger - i\sin(\chi\Delta t)b^\dagger)^n|00\rangle \end{aligned}$$

Since $[a^\dagger, b^\dagger] = 0$,

$$\begin{aligned} \sqrt{n!}U|n\rangle|0\rangle_b &= \left(\sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)(a^\dagger)^{n-k}(b^\dagger)^k\right)|00\rangle \\ &= \sum_{k=0}^{n}\binom{n}{k}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)\sqrt{(n-k)!}\sqrt{k!}\,|n-k\rangle|k\rangle_b \end{aligned}$$

so that

$$\begin{aligned} U|n0\rangle &= \sum_{k=0}^{n}\binom{n}{k}\sqrt{\frac{(n-k)!k!}{n!}}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)|n-k\rangle|k\rangle_b \\ &= \sum_{k=0}^{n}\sqrt{\binom{n}{k}}\cos^{n-k}(\chi\Delta t)(-i)^k\sin^k(\chi\Delta t)|n-k\rangle|k\rangle_b \end{aligned}$$

We can think of the number

$$\binom{n}{k}\cos^{2(n-k)}(\chi\Delta t)\sin^{2k}(\chi\Delta t)$$

as the probability of losing k quanta of energy to the environment.

Let $E_m = \langle m|_b U|0\rangle_b, \quad m \in \mathbb{N}$ the operation elements of U. They are operators acting on the principal space. We can compute the action of E_m on $|n\rangle$ (i.e. compute the nth column of the matrix of E_m) from the previous formula:

$$\begin{aligned} E_m|n\rangle &= (\langle m|_b U|0\rangle_b)|n\rangle \\ &= \langle m|_b(U|n\rangle|0\rangle_b) \\ &= \langle m|_b U|n0\rangle \end{aligned}$$

First it is clear that if $n < m$, $E_m |n\rangle = 0$. Then if $n \geqslant m$,

$$\begin{aligned}
E_m |n\rangle &= \langle m|_b \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t) |n-k\rangle |k\rangle_b \\
&= \sum_{k=0}^{n} \sqrt{\binom{n}{k}} \cos^{n-k}(\chi\Delta t)(-i)^k \sin^k(\chi\Delta t) |n-k\rangle \underbrace{\langle m|k\rangle_b}_{=\delta_{mk}} \\
&= (-i)^m \sin^m(\chi\Delta t)\sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle
\end{aligned}$$

This shows that the matrix of E_m has non nul elements only on the mth superior diagonal. E_m corresponds to the physical process of losing m quanta of energy to the environment.

We can also reconstruct the full formula for E_m using braket calculus:

$$\begin{aligned}
E_m &= E_m \underbrace{\sum_{n=0}^{+\infty} |n\rangle \langle n|}_{=1} \\
&= \sum_{n=0}^{+\infty} E_m |n\rangle \langle n| \\
&= \sum_{n=m}^{+\infty} E_m |n\rangle \langle n| \\
&= (-i)^m \sin^m(\chi\Delta t) \sum_{n=m}^{+\infty} \sqrt{\binom{n}{m}} \cos^{n-m}(\chi\Delta t) |n-m\rangle \langle n|
\end{aligned}$$

Note that the sole effect of factor $(-i)^m$ is to add a global phase so it may as well be omitted.

$$E_m = \sin^m(\chi\Delta t) \begin{array}{cc}
\begin{array}{cccccc} & m & & & n & \end{array} & \\
\begin{bmatrix}
 & 1 & & & & \\
 & & \sqrt{\binom{m+1}{m}}\cos(\chi\Delta t) & & \mathbf{0} & \\
 & & & \sqrt{\binom{m+2}{m}}\cos^2(\chi\Delta t) & & \\
 & & & \ddots & & \\
 & & & & \sqrt{\binom{n}{m}}\cos^{n-m}(\chi\Delta t) & \\
 & & & & & \ddots \\
 & \mathbf{0} & & & & \\
 & & & & &
\end{bmatrix} & \begin{array}{c} \\ \\ \\ \\ n-m \\ \\ \\ \\ \end{array}
\end{array}$$

21.2. Trace-preserving property. Matrix calculus or braket calculus show that the matrices $E_m^\dagger E_m$ are diagonals, with the first m elements are 0:

$$\begin{aligned}
E_m^\dagger E_m &= \sin^{2m}(\chi\Delta t)\left(\sum_{n=m}^{+\infty}\sqrt{\binom{n}{m}}\cos^{n-m}(\chi\Delta t)\left|n\right\rangle\left\langle n-m\right|\right)\left(\sum_{l=m}^{+\infty}\sqrt{\binom{l}{m}}\cos^{l-m}(\chi\Delta t)\left|l-m\right\rangle\left\langle l\right|\right)\\
&= \sin^{2m}(\chi\Delta t)\sum_{n=m}^{+\infty}\sum_{l=m}^{+\infty}\sqrt{\binom{n}{m}}\sqrt{\binom{l}{m}}\cos^{n-m}(\chi\Delta t)\cos^{l-m}(\chi\Delta t)\left|n\right\rangle\underbrace{\left\langle n-m|l-m\right\rangle}_{=\delta_{nl}}\left\langle l\right|\\
&= \sin^{2m}(\chi\Delta t)\sum_{n=m}^{+\infty}\binom{n}{m}\cos^{2(n-m)}(\chi\Delta t)\left|n\right\rangle\left\langle n\right|
\end{aligned}$$

$$E_m^\dagger E_m = \sin^{2m}(\chi\Delta t)
\begin{array}{cc}
\begin{array}{ccccc} & m & & n & \end{array} & \\
\begin{bmatrix}
\mathbf{0} & & & & \mathbf{0} \\
& 1 & & & \\
& & \binom{m+1}{m}\cos^2(\chi\Delta t) & & \\
& & & \binom{m+1}{m}\cos^4(\chi\Delta t) & \\
& & & & \ddots \\
& & & & \binom{n}{m}\cos^{2(n-m)}(\chi\Delta t) \\
\mathbf{0} & & & & \ddots
\end{bmatrix} & \begin{array}{c} \\ m \\ \\ \\ \\ n \\ \\ \end{array}
\end{array}$$

It follows that the operator $\sum_{m=0}^{+\infty} E_m^\dagger E_m$ is also diagonal, and diagonal elements are

$$\begin{aligned}
\left\langle n\right|\sum_{m=0}^{+\infty}E_m^\dagger E_m\left|n\right\rangle &= \sum_{m=0}^{+\infty}\left\langle n\right|E_m^\dagger E_m\left|n\right\rangle\\
&= \sum_{m=0}^{n}\left\langle n\right|E_m^\dagger E_m\left|n\right\rangle\\
&= \sum_{m=0}^{n}\binom{n}{m}\sin^{2m}(\chi\Delta t)\cos^{2(n-m)}(\chi\Delta t)\\
&= (\sin^2(\chi\Delta t)+\cos^2(\chi\Delta t))^n\\
&= 1
\end{aligned}$$

i.e. $\sum_{m=0}^{+\infty} E_m^\dagger E_m = 1$ and the quantum operation is trace-preserving.

22. Amplitude damping of a single qubit density matrix

Let

$$\rho = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}$$

The amplitude damping operation is defined by

$$\varepsilon_{AD}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$$

where

$$\begin{aligned} E_0 &= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \\ E_1 &= \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{2}$$

Straightforward matrix calculus show that

$$E_0 \rho E_0^\dagger = \begin{bmatrix} a & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}$$

and

$$\begin{aligned} E_1 \rho E_1^\dagger &= \begin{bmatrix} c\gamma & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (1-a)\gamma & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

because $1 = \operatorname{Tr} \rho = a + c$.

Thus we have

$$\begin{aligned} \varepsilon_{AD}(\rho) &= \begin{bmatrix} a + (1-a)\gamma & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} \\ &= \begin{bmatrix} 1 - (1-a)(1-\gamma) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} \end{aligned} \tag{3}$$

23. Amplitude damping of dual-rail qubits

Let

$$|\psi\rangle = a\,|01\rangle + b\,|10\rangle$$

Applying $\varepsilon_{AD} \otimes \varepsilon_{AD}$ to $\rho = |\psi\rangle\langle\psi|$ is equivalent to applying unitary $B \otimes B$ to $|\psi\rangle$, where $B = e^{\theta(a^\dagger b - a b^\dagger)}$. Let's do this by making explicit the 2 environment qubits initially set to 0, dnoted by subscript b:

$$|\psi\rangle = a\,|01\rangle\,|00\rangle_b + b\,|10\rangle\,|00\rangle_b$$

$$\begin{aligned} B \otimes B\,|\psi\rangle &= a\,|0\rangle\,|0\rangle_b\,(B\,|1\rangle\,|0\rangle_b) + b(B\,|1\rangle\,|0\rangle_b)\,|0\rangle\,|0\rangle_b \\ &= a\,|0\rangle\,|0\rangle_b\,(\cos(\theta)\,|1\rangle\,|0\rangle_b + \sin(\theta)\,|0\rangle\,|1\rangle_b) + b(\cos(\theta)\,|1\rangle\,|0\rangle_b + \sin(\theta)\,|0\rangle\,|1\rangle_b)\,|0\rangle\,|0\rangle_b \\ &= a\cos(\theta)\,|0\rangle\,|0\rangle_b\,|1\rangle\,|0\rangle_b + a\sin(\theta)\,|0\rangle\,|0\rangle_b\,|0\rangle\,|1\rangle_b + b\cos(\theta)\,|1\rangle\,|0\rangle_b\,|0\rangle\,|0\rangle_b + b\sin(\theta)\,|0\rangle\,|1\rangle_b\,|0\rangle\,|0\rangle_b \end{aligned}$$

We reorder the qubits to put the environments qubits at the end since we will trace them out:

$$\begin{aligned} B \otimes B\,|\psi\rangle &= a\cos(\theta)\,|01\rangle\,|00\rangle_b + a\sin(\theta)\,|00\rangle\,|01\rangle_b + b\cos(\theta)\,|10\rangle\,|00\rangle_b + b\sin(\theta)\,|00\rangle\,|10\rangle_b \\ &= |\varphi\rangle \end{aligned} \tag{4}$$

Now we have to find the dual vector $\langle\varphi|$ of this state. We can recall the not so trivial following facts related to product space: Let $\{|a_i\rangle\}$,$\{|b_j\rangle\}$ be basis of two Hilbert spaces A and B.

The dual of $|a_i b_j\rangle = |a_i\rangle \otimes |b_j\rangle$ is

$$\langle a_i| \otimes \langle b_j| = \langle a_i b_j|$$

so that

$$\langle\varphi| = a^* \cos(\theta)\,\langle 01|\,\langle 00|_b + a^* \sin(\theta)\,\langle 00|\,\langle 01|_b + b^* \cos(\theta)\,\langle 10|\,\langle 00|_b + b^* \sin(\theta)\,\langle 00|\,\langle 10|_b$$

We have also

$$|a_k b_l\rangle \langle a_i b_j| = |a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|$$

We could then use equation 4 to compute the density $|\varphi\rangle \langle\varphi|$, but this would be a messy sum with 16 terms.

Since we will trace out the environment, we recall the partial trace formula:

$$\begin{aligned}\mathrm{Tr}_B(|a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|) &= |a_k\rangle \langle a_i| \,\mathrm{Tr}(|b_l\rangle \langle b_j|)\\ &= |a_k\rangle \langle a_i| \langle b_l|b_j\rangle\end{aligned}$$

Since $\{|00\rangle_b, |01\rangle_b, |10\rangle_b, |11\rangle_b\}$ is an orthonormal basis, there are only 6 out of 16 terms left after the partial trace operation:

$$\begin{aligned}\mathrm{Tr}_b(|\varphi\rangle \langle\varphi|) &= |a|^2 \cos^2(\theta) |01\rangle \langle 01| + ab^* \cos^2(\theta) |01\rangle \langle 10| + |a|^2 \sin^2(\theta) |00\rangle \langle 00|\\ &\quad + |b|^2 \cos^2(\theta) |10\rangle \langle 10| + ba^* \cos^2(\theta) |10\rangle \langle 01| + |b|^2 \sin^2(\theta) |00\rangle \langle 00|\\ &= |a|^2(1-\gamma) |01\rangle \langle 01| + ab^*(1-\gamma) |01\rangle \langle 10| + |a|^2\gamma |00\rangle \langle 00|\\ &\quad + |b|^2(1-\gamma) |10\rangle \langle 10| + ba^*(1-\gamma) |10\rangle \langle 01| + |b|^2\gamma |00\rangle \langle 00|\\ &= (\underbrace{|a|^2 + |b|^2}_{=1})\gamma |00\rangle \langle 00| + (1-\gamma) \big(|a|^2 |01\rangle \langle 01| + ab^* |01\rangle \langle 10| + |b|^2 |10\rangle \langle 10| + ba^* |10\rangle \langle 01|\big)\\ &= \gamma \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} + (1-\gamma) \begin{bmatrix}0 & 0 & 0 & 0\\ 0 & |a|^2 & ab^* & 0\\ 0 & a^*b & |b|^2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\\ &= \gamma \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} + (1-\gamma)\rho\end{aligned}$$

It is a mixed state:

- with probability γ, the state is projected to $|00\rangle$, orthogonal to $|\psi\rangle$.
- with probability $1-\gamma$, state is unchanged.

Since $|00\rangle$ is orthogonal to $|\psi\rangle$, one can detect amplitude damping errors with measurement operators:

$$\begin{aligned}M_0 &= |00\rangle \langle 00| \qquad \text{orthogonal projector on } \mathrm{span}\{|00\rangle\}\\ M_1 &= |01\rangle \langle 01| + |10\rangle \langle 10| + |11\rangle \langle 11| \qquad \text{orthogonal projector on } \mathrm{span}\{|01\rangle, |10\rangle\ |11\rangle\}\end{aligned}$$

- If the state decayed to $|00\rangle$, then with probability 1 the result of the measurement will be $|00\rangle$.
- Otherwise, with probability 1 the result of the measurement will be the original $|\psi\rangle$.

It can be easily checked that the quantum operation can be described with 3 operators:

$$\begin{aligned}E_0^{dr} &= \sqrt{1-\gamma} I\\ E_1^{dr} &= \sqrt{\gamma} |00\rangle \langle 01|\\ &= \sqrt{\gamma} \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\\ E_2^{dr} &= \sqrt{\gamma} |00\rangle \langle 10|\\ &= \sqrt{\gamma} \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\end{aligned}$$

It is interesting to see that these operators are the restriction to $\mathrm{span}\{|01\rangle, |10\rangle\}$ of the operators

$$\begin{aligned} &E_0 \otimes E_0 \\ &E_0 \otimes E_1 \\ &E_1 \otimes E_0 \\ &E_1 \otimes E_1 \end{aligned}$$

where E_0, E_1 are the operators of amplitude damping for single qubit, defined in 2.

24. Spontaneous emission is amplitude damping

From equation (7.77) in the book, the time evolution of the single atom interacting with single photon is governed by unitary

$$\begin{aligned} U = e^{-i\delta t} |00\rangle \langle 00| + (\cos(\Omega t) + i\frac{\delta}{\Omega}\sin(\Omega t)) |01\rangle \langle 01| \\ + (\cos(\Omega t) - i\frac{\delta}{\Omega}\sin(\Omega t)) |10\rangle \langle 10| - i\frac{g}{\Omega}\sin(\Omega t)(|01\rangle \langle 10| + |10\rangle \langle 01|) \end{aligned}$$

the left label corresponds to the electric field, the right label corresponds to the atom. The derivation of this formula from the Hamiltonian can be found in appendix A.

The *Rabi frequency* is

$$\Omega = \sqrt{g^2 + \delta^2}$$

If we set $\delta = 0$ and if $g > 0$, then $\Omega = g$ and

$$\begin{aligned} U = |00\rangle \langle 00| + \cos(\Omega t)(|01\rangle \langle 01| + |10\rangle \langle 10|) \\ - i\sin(\Omega t)(|01\rangle \langle 10| + |10\rangle \langle 01|) \end{aligned}$$

Let us apply U to

$$\begin{aligned} |\psi\rangle &= |0\rangle (a|0\rangle + b|1\rangle) \\ &= a|00\rangle + b|01\rangle) \end{aligned}$$

We find

$$\begin{aligned} U|\psi\rangle &= a|00\rangle + b(\cos(\Omega t)|01\rangle - i\sin(\Omega t)|10\rangle) \\ &= |\varphi\rangle \end{aligned} \tag{5}$$

Now we have to find the dual vector $\langle\varphi|$ of this state. We can recall the not so trivial following facts related to product space: Let $\{|a_i\rangle\}$,$\{|b_j\rangle\}$ be basis of two Hilbert spaces A and B.

The dual of $|a_i b_j\rangle = |a_i\rangle \otimes |b_j\rangle$ is

$$\langle a_i| \otimes \langle b_j| = \langle a_i b_j|$$

so that

$$\langle\varphi| = a\langle 00| + b^*(\cos(\Omega t)\langle 01| + i\sin(\Omega t)\langle 10|)$$

We have also

$$|a_k b_l\rangle \langle a_i b_j| = |a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|$$

We could then use equation 5 to compute the density $|\varphi\rangle\langle\varphi|$, but this would be a ugly sum with 9 terms.

Since we will trace out the photon space, we recall the partial trace formula:

$$\begin{aligned}\mathrm{Tr}_B(|a_k\rangle\langle a_i| \otimes |b_l\rangle\langle b_j|) &= |a_k\rangle\langle a_i| \,\mathrm{Tr}(|b_l\rangle\langle b_j|)\\ &= |a_k\rangle\langle a_i| \langle b_l|b_j\rangle\end{aligned}$$

Since $\{|0\rangle, |1\rangle\}$ is an orthonormal basis of the state space A of the photon, there are only 5 out of 9 terms left after the partial trace operation over the photon (those where the bit for the photon is the same in the ket and in the bra):

$$\begin{aligned}\mathrm{Tr}_A(|\varphi\rangle\langle\varphi|) &= (|a|^2 + |b|^2\sin^2(\Omega t))\,|0\rangle\langle 0| + ab^*\cos(\Omega t)\,|0\rangle\langle 1|\\ &\quad + a^* b\cos(\Omega t)\,|1\rangle\langle 0| + |b|^2\cos^2(\Omega t)\\ &= \begin{bmatrix} |a|^2 + (1-|a|^2)\gamma & ab^*\sqrt{1-\gamma} \\ a^* b\sqrt{1-\gamma} & |b|^2(1-\gamma)\end{bmatrix}\end{aligned}$$

with $\gamma = \sin^2(\Omega t)$. Now compare with equation 3 and recall that

$$\rho = \begin{bmatrix} |a|^2 & ab^* \\ a^* b & |b|^2 \end{bmatrix}$$

to see that this is indeed the amplitude damping operation.

25.

We consider the density operator

$$\rho = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}$$

The qubit is in state $|0\rangle$ with probability $p_0 = p$ and in state $|1\rangle$ with probability $p_1 = 1-p$.

Let us compute T as a function of E_0, E_1 and p:

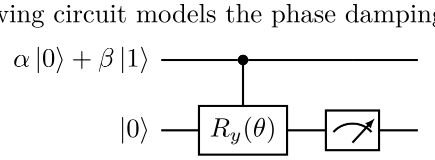
$$\begin{aligned} & & \mathcal{Z} = \frac{e^{-\frac{E_0}{k_B T}}}{p} &= e^{-\frac{E_0}{k_B T}} + e^{-\frac{E_1}{k_B T}}\\ &\Leftrightarrow & \frac{1}{p} &= 1 + e^{-\frac{E_1 - E_0}{k_B T}}\\ &\Leftrightarrow & \frac{1}{p} - 1 &= e^{-\frac{E_1 - E_0}{k_B T}}\\ &\Leftrightarrow & \frac{1}{p} - 1 &= e^{-\frac{E_1 - E_0}{k_B T}}\\ &\Leftrightarrow & -\frac{E_1 - E_0}{k_B T} &= \ln\left(\frac{1-p}{p}\right)\\ &\Leftrightarrow & T &= -\frac{1}{k_B}\frac{E_1 - E_0}{\ln\left(\frac{1-p}{p}\right)}\end{aligned}$$

Assuming $E_1 > E_0$,

- the regular amplitude damping case corresponds to $T \to 0^+$, $p = 1$.
- When $T \to +\infty$, $p \to \frac{1}{2}$.

26. Circuit model for phase damping

We want to prove that the following circuit models the phase damping operation



Recall that

$$\begin{aligned} R_y(\theta) &= e^{-i\frac{\theta}{2}Y} \\ &= \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \end{aligned}$$

Initially the two-qubit state is

$$(\alpha |0\rangle + \beta |1\rangle) |0\rangle = \alpha |00\rangle + \beta |10\rangle$$

After the controlled R_y gate it becomes

$$\begin{aligned} \alpha |00\rangle + \beta |1\rangle R_y(\theta) |0\rangle &= \alpha |00\rangle + \beta |1\rangle (\cos(\frac{\theta}{2}) |0\rangle + \sin(\frac{\theta}{2}) |1\rangle) \\ &= \alpha |00\rangle + \beta(\cos(\frac{\theta}{2}) |10\rangle + \sin(\frac{\theta}{2}) |11\rangle) \\ &= |\varphi\rangle \end{aligned}$$

Now we have to find the dual vector $\langle\varphi|$ of this state. We can recall the not so trivial following facts related to product space: Let $\{|a_i\rangle\}$,$\{|b_j\rangle\}$ be basis of two Hilbert spaces A and B.

The dual of $|a_i b_j\rangle = |a_i\rangle \otimes |b_j\rangle$ is

$$\langle a_i| \otimes \langle b_j| = \langle a_i b_j|$$

so that

$$\langle\varphi| = \alpha^* \langle 00| + \beta^*(\cos(\frac{\theta}{2}) \langle 10| + \sin(\frac{\theta}{2}) \langle 11|)$$

We have also

$$|a_k b_l\rangle \langle a_i b_j| = |a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|$$

We could then use equation 5 to compute the density $|\varphi\rangle \langle\varphi|$, but this would be a ugly sum with 9 terms. Since we will trace out the environment, we recall the partial trace formula:

$$\begin{aligned} \mathrm{Tr}_B(|a_k\rangle \langle a_i| \otimes |b_l\rangle \langle b_j|) &= |a_k\rangle \langle a_i| \,\mathrm{Tr}(|b_l\rangle \langle b_j|) \\ &= |a_k\rangle \langle a_i| \langle b_l|b_j\rangle \end{aligned}$$

Since $\{|0\rangle, |1\rangle\}$ is an orthonormal basis of the state space B of the environment, there are only 5 out of 9 terms left after the partial trace operation over the photon (those where the bit for the environment is the same in the ket and in the bra):

$$\begin{aligned} \mathrm{Tr}_B(|\varphi\rangle \langle\varphi|) &= |\alpha|^2 |0\rangle \langle 0| + \alpha\beta^* \cos(\frac{\theta}{2}) |0\rangle \langle 1| + |\beta|^2 \cos^2(\frac{\theta}{2}) |1\rangle \langle 1| \\ &\quad + \alpha^*\beta \cos(\frac{\theta}{2}) |1\rangle \langle 0| + |\beta|^2 \sin^2(\frac{\theta}{2}) |1\rangle \langle 1| \\ &= |\alpha|^2 |0\rangle \langle 0| + \alpha\beta^* \cos(\frac{\theta}{2}) |0\rangle \langle 1| + \alpha^*\beta \cos(\frac{\theta}{2}) |1\rangle \langle 0| + \\ &\quad + |\beta|^2 |1\rangle \langle 1| \\ &= \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \cos(\frac{\theta}{2}) \\ \alpha^*\beta \cos(\frac{\theta}{2}) & |\beta|^2 \end{bmatrix} \\ &= \begin{bmatrix} |\alpha|^2 & \alpha\beta^* e^{-\lambda} \\ \alpha^*\beta e^{-\lambda} & |\beta|^2 \end{bmatrix} \end{aligned}$$

with $e^{-\lambda} = \cos(\frac{\theta}{2}) \Leftarrow \theta = 2\arccos e^{-\lambda}$ (indeed $\lambda \geqslant 0$ because it is a variance so that $0 < e^{-\lambda} \leqslant 1$).

Now compare with equation 8.125 of the book and recall that the density corresponding to pure state φ is

$$\begin{aligned}\rho &= |\varphi\rangle\langle\varphi| \\ &= \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2 \end{bmatrix}\end{aligned}$$

to see that this is indeed the phase damping operation. Thus the result is true for a pure state.

Since every quantum operations is linear with respects to the density operator and every density operator is a convex linear combination of pure state $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$, the conclusion holds for all density operators.

27. Phase damping is phase flip

Let

$$\begin{aligned}E_0 &= \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\lambda} \end{bmatrix} \\ E_1 &= \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\lambda} \end{bmatrix}\end{aligned}$$

the operations elements of the phase damping channel.

Let

$$\begin{aligned}\tilde{E}_0 &= \sqrt{\alpha}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \tilde{E}_1 &= \sqrt{1-\alpha}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\end{aligned}$$

the operations elements of the phase flip channel, where $\alpha = \frac{1+\sqrt{1-\lambda}}{2}$.

We want to find a unitary matrix U such that:

$$\tilde{E}_0 = u_{00}E_0 + u_{01}E_1$$

and

$$\tilde{E}_1 = u_{10}E_0 + u_{11}E_1$$

$$\begin{aligned}\tilde{E}_0 = u_{00}E_0 + u_{01}E_1 &\Leftrightarrow \begin{cases} \sqrt{\alpha} = & u_{00} \\ \sqrt{\alpha} = & \sqrt{1-\lambda}u_{00} + \sqrt{\lambda}u_{01} \end{cases} \\ &\Leftrightarrow \begin{cases} \sqrt{\alpha} = & u_{00} \\ \sqrt{\alpha} = & \sqrt{1-\lambda}\sqrt{\alpha} + \sqrt{\lambda}u_{01} \end{cases} \\ &\Leftrightarrow \begin{cases} \sqrt{\alpha} & = u_{00} \\ \sqrt{\alpha}\dfrac{1-\sqrt{1-\lambda}}{\sqrt{\lambda}} & = u_{01} \end{cases} \\ &\Leftrightarrow \begin{cases} \sqrt{\alpha} & = u_{00} \\ \sqrt{1-\alpha} & = u_{01} \end{cases}\end{aligned}$$

Similarly,

$$\tilde{E}_1 = u_{10}E_0 + u_{11}E_1 \Leftrightarrow \begin{cases} \sqrt{1-\alpha} & = u_{10} \\ -\sqrt{1-\alpha} & = \sqrt{1-\lambda}u_{00} + \sqrt{\lambda}u_{11} \end{cases}$$
$$\Leftrightarrow \begin{cases} \sqrt{1-\alpha} & = u_{10} \\ -\sqrt{1-\alpha} & = \sqrt{1-\lambda}\sqrt{1-\alpha} + \sqrt{\lambda}u_{11} \end{cases}$$
$$\Leftrightarrow \begin{cases} \sqrt{1-\alpha} & = u_{10} \\ -\sqrt{1-\alpha}\dfrac{1+\sqrt{1-\lambda}}{\sqrt{\lambda}} & = u_{11} \end{cases}$$
$$\Leftrightarrow \begin{cases} \sqrt{1-\alpha} & = u_{10} \\ -\sqrt{\alpha} & = u_{11} \end{cases}$$

It is easy to check that the matrix

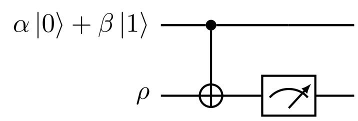
$$U = \begin{bmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{bmatrix} = \begin{bmatrix} \sqrt{\alpha} & \sqrt{1-\alpha} \\ \sqrt{1-\alpha} & -\sqrt{\alpha} \end{bmatrix}$$

is unitary, which shows that the phase damping and the phase flip correspond to the same physical quantum operation.

28. One *CNOT* phase damping model circuit

We have proved in the previous exercise that phase flip is the same as phase damping. We will show now that phase flip can be modeled with one *CNOT*, provided the environment is initially in mixed state.

Thus we consider the folloing circuit:



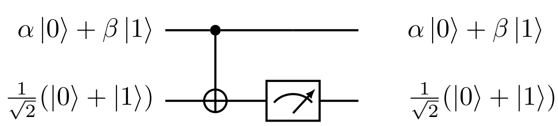
Let us assume the environment is initially in the mixed state:

$$\varphi = \begin{cases} |+\rangle = \dfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle) & \text{with probability } p \\ |-\rangle = \dfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle) & \text{with probability } 1-p \end{cases}$$

or equivalently,

$$\rho = p\,|+\rangle\langle+| + (1-p)\,|-\rangle\langle-|$$

Thus, with probability p, the circuit acts as



Indeed,

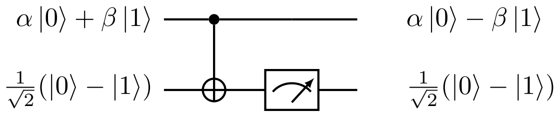
$$(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(\alpha|00\rangle + \alpha|01\rangle + \beta|10\rangle + \beta|11\rangle)$$

After the *CNOT*, the 2 qubit state becomes

$$\begin{aligned}\frac{1}{\sqrt{2}}(\alpha|00\rangle + \alpha|01\rangle + \beta|11\rangle + \beta|10\rangle) &= \frac{1}{\sqrt{2}}(\alpha|00\rangle + \alpha|01\rangle + \beta|10\rangle + \beta|11\rangle) \\ &= (\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\end{aligned}$$

In this case the principal qubit is left untouched.

With probability $1-p$, the circuit acts as



Indeed,

$$(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(\alpha|00\rangle - \alpha|01\rangle + \beta|10\rangle - \beta|11\rangle)$$

After the *CNOT*, the 2 qubit state becomes

$$\begin{aligned}\frac{1}{\sqrt{2}}(\alpha|00\rangle - \alpha|01\rangle + \beta|11\rangle - \beta|10\rangle) &= \frac{1}{\sqrt{2}}(\alpha|00\rangle - \alpha|01\rangle - \beta|10\rangle + \beta|11\rangle) \\ &= (\alpha|0\rangle - \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\end{aligned}$$

In this case the phase of principal qubit is flipped.

We have proved the circuit models the phase flip operation, and thus also the phase damping.

29. Unitality

For phase damping,

$$E_0 = \begin{bmatrix}1 & 0 \\ 0 & \sqrt{1-\lambda}\end{bmatrix}$$
$$E_1 = \begin{bmatrix}0 & 0 \\ 0 & \sqrt{\lambda}\end{bmatrix}$$

Thus,

$$\begin{aligned}\varepsilon(I) &= E_0E_0^\dagger + E_1E_1^\dagger \\ &= \begin{bmatrix}1 & 0 \\ 0 & 1-\lambda\end{bmatrix} + \begin{bmatrix}0 & 0 \\ 0 & \lambda\end{bmatrix} \\ &= I\end{aligned}$$

For depolarizing channel, according to equation 8.103[1]

$$\varepsilon(\rho) = (1-p)\rho + \frac{1}{3}p(X\rho X + Y\rho Y + Z\rho Z)$$

so:

$$\begin{aligned}\varepsilon(I) &= (1-p)I + \frac{1}{3}p(X^2 + Y^2 + Z^2) \\ &= (1-p)I + \frac{1}{3}p(3I) \\ &= I\end{aligned}$$

[1]Note that equation 8.100 is equivalent to 8.103 only for valid density operators, which I is not ($\frac{1}{2}I$ is).

For amplitude damping,

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}$$
$$E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

Thus,

$$\begin{aligned} \varepsilon(I) &= E_0 E_0^\dagger + E_1 E_1^\dagger \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1-\gamma \end{bmatrix} + \begin{bmatrix} \gamma & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1+\gamma & 0 \\ 0 & 1-\gamma \end{bmatrix} \\ &\neq I \qquad \text{if } \gamma > 0 \end{aligned}$$

This proves amplitude damping channel is not unital.
It is interesting to note that

$$E_1^\dagger E_1 = \begin{bmatrix} 0 & 0 \\ 0 & \gamma \end{bmatrix}$$

so that

$$E_0^\dagger E_0 + E_1^\dagger E_1 = I$$

which is the trace preserving property, that can be easily confused with the unitality property.

30. exponential time decay of matrix elements

We will use equation 3 of exercise 22 for the amplitude damping:

$$\begin{aligned} \varepsilon_{AD}(\rho) &= \begin{bmatrix} a+(1-a)\gamma & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} \\ &= \begin{bmatrix} \gamma + a(1-\gamma) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} \end{aligned}$$

The time constant T_1 of the decay of diagonal elements is defined by equation:

$$\begin{aligned} & e^{-\frac{t}{T_1}} = 1-\gamma \\ \Leftrightarrow \quad & T_1 = -\frac{1}{\ln 1-\gamma} t \end{aligned}$$

The time constant T_2 of the decay of off-diagonal elements is defined by equation:

$$\begin{aligned} & e^{-\frac{t}{T_2}} = \sqrt{1-\gamma} \\ \Leftrightarrow \quad & T_2 = -\frac{2}{\ln 1-\gamma} t \end{aligned}$$

We thus have

$$\frac{T_2}{T_1} = 2$$

If we apply amplitude damping followed by phase damping, the off-diagonal elements decay faster: using equation 8.125:

$$\varepsilon(\rho) = \begin{bmatrix} \gamma + a(1-\gamma) & b\sqrt{1-\gamma}e^{-\lambda} \\ b^*\sqrt{1-\gamma}e^{-\lambda} & c(1-\gamma) \end{bmatrix}$$

so that the time constant T_2 is now

$$\begin{aligned} \tilde{T_2} &= -\frac{2}{\ln(1-\gamma)e^{-\lambda}}t \\ &= -\frac{2}{\ln(1-\gamma)-\lambda}t \\ &\leqslant T_2 = 2T_1 \end{aligned}$$

T_2 and T_1 have the dimension of time. The quantities $\frac{1}{T_1}$ and $\frac{1}{T_2}$ are decay rates in s^{-1} and are related by equation:

$$\frac{1}{T_2} = \frac{1}{2}\frac{1}{T_1}$$

31. Exponential sensitivity to phase damping

We will use the same line of reasoning as in exercise 21. Please refer there for a summary of the formulas of the quantum harmonic oscillator.

The principal system, a harmonic oscillator, interacts with an environment, modeled as another harmonic oscillator, through the Hamiltonian:

$$H = \chi a^\dagger a(b + b^\dagger)$$

where $a^\dagger, a$ and $b^\dagger, b$ are the creation, annihilation operators for the principal and environment oscillators, respectively.

The time evolution of the coupled system is governed by the unitary operator:

$$U = e^{-iH\Delta t}$$

The Baker-Campbell-Hausdorff formula states that, for any operators A, G such that e^G exists,

$$e^{\lambda G}Ae^{-\lambda G} = \sum_{n=0}^{+\infty}\frac{\lambda^n}{n!}C_n$$

where the operators C_n are defined recursively by

$$\begin{aligned} C_0 &= A \\ C_1 &= [G, A] \\ \forall n \in \mathbb{N}, \quad C_{n+1} &= [G, C_n] \end{aligned}$$

Lets compute a simplified expression for the operator $Ua^\dagger U^\dagger$ acting on the product space:

$$\begin{aligned} Ua^\dagger U^\dagger &= e^{-iH\Delta t}a^\dagger e^{iH\Delta t} \\ &= \sum_{n=0}^{+\infty}\frac{(-i\Delta t)^n}{n!}C_n \end{aligned} \tag{6}$$

The first commutators C_n are

$$\begin{aligned}
C_0 &= a^\dagger \\
C_1 &= [H, a^\dagger] \\
&= [\chi a^\dagger a(b + b^\dagger), a^\dagger] \\
&= \chi(b + b^\dagger)[a^\dagger a, a^\dagger] \\
&= \chi(b + b^\dagger)a^\dagger[a, a^\dagger] \\
&= \chi(b + b^\dagger)a^\dagger \\
C_2 &= [H, C_1] \\
&= [\chi a^\dagger a(b + b^\dagger), \chi(b + b^\dagger)a^\dagger] \\
&= \chi^2(b + b^\dagger)^2[a^\dagger a, a^\dagger] \\
&= \chi^2(b + b^\dagger)^2 a^\dagger[a, a^\dagger] \\
&= \chi^2(b + b^\dagger)^2 a^\dagger
\end{aligned}$$

from which it follows easily that

$$\forall n \in \mathbb{N}, \quad C_n = \chi^n(b + b^\dagger)^n a^\dagger$$

We now rewrite equation 6

$$\begin{aligned}
Ua^\dagger U^\dagger &= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{n!} C_n \\
&= \sum_{n=0}^{+\infty} \frac{(-i\Delta t)^n}{(n)!} \chi^n(b + b^\dagger)^n a^\dagger \\
&= a^\dagger \sum_{n=0}^{+\infty} \frac{(-i\chi\Delta t)^n}{(n)!} (b + b^\dagger)^n \\
&= a^\dagger e^{-i\chi\Delta t(b+b^\dagger)}
\end{aligned}$$

Let us now compute the effect of U on $|0\rangle |0\rangle_b = |00\rangle$ (the subscript b is used to refer to vector in the environment state space):

$$\begin{aligned}
U|00\rangle &= e^{-iH\Delta t}|00\rangle \\
&= \sum_{n=0}^{+\infty} \frac{(-iH\Delta t)^n}{n!} |00\rangle
\end{aligned}$$

Since $a|0\rangle = 0$, we have

$$\begin{aligned}
H|00\rangle &= \chi a^\dagger a|0\rangle \otimes (b + b^\dagger)|0\rangle_b \\
&= \chi 0 \otimes (b + b^\dagger)|0\rangle_b \\
&= 0
\end{aligned}$$

and

$$\forall n \in \mathbb{N}^*, \quad , H^n|00\rangle = 0$$

from which it follows there is only one non nul term in the previous sum (the first) and

$$U|00\rangle = |00\rangle$$

Let us compute the effect of U on $|1\rangle|0\rangle_b = |10\rangle$:

$$\begin{aligned}
U|10\rangle &= Ua^\dagger|00\rangle \\
&= Ua^\dagger \underbrace{U^\dagger U}_{=1}|00\rangle \\
&= Ua^\dagger U^\dagger|00\rangle \\
&= a^\dagger e^{-i\chi\Delta t(b+b^\dagger)}|00\rangle \\
&= a^\dagger|0\rangle \otimes e^{-i\chi\Delta t(b+b^\dagger)}|0\rangle_b \\
&= |1\rangle \otimes e^{-i\chi\Delta t(b+b^\dagger)}|0\rangle_b
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sqrt{n!}U|n\rangle|0\rangle_b &= \sqrt{n!}U|n0\rangle \\
&= U(a^\dagger)^n|00\rangle \\
&= U(a^\dagger)^n U^\dagger U|00\rangle \\
&= (Ua^\dagger U^\dagger)^n|00\rangle \\
&= (a^\dagger)^n|0\rangle \otimes e^{-in\chi\Delta t(b+b^\dagger)}|0\rangle_b \\
&= \sqrt{n!}\,|n\rangle \otimes e^{-in\chi\Delta t(b+b^\dagger)}|0\rangle_b
\end{aligned}$$

it follows that

$$U|n\rangle|0\rangle_b = |n\rangle \otimes e^{-in\chi\Delta t(b+b^\dagger)}|0\rangle_b$$

If we call $|\psi\rangle$ the initial state of the system, letting the initial state of the environment be $|0\rangle_b$,

$$|\psi\rangle = \sum_{n=0}^{+\infty}\alpha_n|n0\rangle$$

After a time t, the state of the system becomes:

$$\begin{aligned}
U|\psi\rangle &= U\sum_{n=0}^{+\infty}\alpha_n|n0\rangle \\
&= \sum_{n=0}^{+\infty}\alpha_n U|n0\rangle \\
&= \sum_{n=0}^{+\infty}\alpha_n|n\rangle \otimes e^{-in\chi\Delta t(b+b^\dagger)}|0\rangle_b
\end{aligned}$$

The only term of the joint density operator $U|\psi\rangle\langle\psi|U^\dagger$ that will contribute to coefficient (n,m) $\langle n|U|\psi\rangle\langle\psi|U\dagger|m\rangle$ is

$$\begin{aligned}
&= \alpha_n\alpha_m^*|n\rangle \otimes e^{-in\chi\Delta t(b+b^\dagger)}|0\rangle_b\langle m| \otimes e^{im\chi\Delta t(b+b^\dagger)}\langle 0|_b \\
&= \alpha_n\alpha_m^*|n\rangle\langle m| \otimes e^{-in\chi\Delta t(b+b^\dagger)}|0\rangle_b e^{im\chi\Delta t(b+b^\dagger)}\langle 0|_b
\end{aligned}$$

We recall the partial trace formula:

$$\begin{aligned}
\mathrm{Tr}_B(|a_k\rangle\langle a_i| \otimes |b_l\rangle\langle b_j|) &= |a_k\rangle\langle a_i|\,\mathrm{Tr}(|b_l\rangle\langle b_j|) \\
&= |a_k\rangle\langle a_i|\langle b_l|b_j\rangle
\end{aligned}$$

Thus taking the partial trace of the previous expression gives:

$$\begin{aligned}\alpha_n\alpha_m^*\,\mathrm{Tr}_B(|n\rangle\langle m|\otimes e^{-in\chi\Delta t(b+b^\dagger)}|0\rangle_b\, e^{im\chi\Delta t(b+b^\dagger)}\langle 0|_b) &= \alpha_n\alpha_m^*\,|n\rangle\langle m|\times\langle 0|e^{in\chi\Delta t(b+b^\dagger)}e^{-im\chi\Delta t(b+b^\dagger)}|0\rangle_b\\ &= \alpha_n\alpha_m^*\,|n\rangle\langle m|\times\langle 0|e^{i(n-m)\chi\Delta t(b+b^\dagger)}|0\rangle_b \qquad (7)\end{aligned}$$

The operators b and $b^\dagger$ do not commute, but since their commutator is

$$\begin{aligned}[b,b^\dagger] &= bb^\dagger - b^\dagger b\\ &= 1\end{aligned}$$

we have

$$\begin{aligned}[b,[b,b^\dagger]] &= [b,1]\\ &= 0\end{aligned}$$

and

$$\begin{aligned}[b^\dagger,[b,b^\dagger]] &= [b^\dagger,1]\\ &= 0\end{aligned}$$

We can use the Glauber formula:

$$e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]}$$

to obtain:

$$\begin{aligned}e^{i(n-m)\chi\Delta t(b+b^\dagger)} &= e^{i(n-m)\chi\Delta t b^\dagger}e^{i(n-m)\chi\Delta t b}e^{-\frac{1}{2}(i(n-m)\chi\Delta t)^2[b^\dagger,b]}\\ &= e^{i(n-m)\chi\Delta t b^\dagger}e^{i(n-m)\chi\Delta t b}e^{-\frac{1}{2}(n-m)^2\chi^2(\Delta t)^2 1}\end{aligned}$$

Let us apply this operator to $|0\rangle_b$:

$$\begin{aligned}e^{i(n-m)\chi\Delta t(b+b^\dagger)}|0\rangle_b &= e^{i(n-m)\chi\Delta t b^\dagger}e^{i(n-m)\chi\Delta t b}e^{-\frac{1}{2}(n-m)^2\chi^2(\Delta t)^2 1}|0\rangle_b\\ &= \underbrace{e^{-\frac{1}{2}(n-m)^2\chi^2(\Delta t)^2}}_{\in\mathbb{R}^+}e^{i(n-m)\chi\Delta t b^\dagger}e^{i(n-m)\chi\Delta t b}|0\rangle_b\\ &= e^{-\frac{1}{2}(n-m)^2\chi^2(\Delta t)^2}e^{i(n-m)\chi\Delta t b^\dagger}(1\,|0\rangle_b)\\ &= e^{-\frac{1}{2}(n-m)^2\chi^2(\Delta t)^2}e^{i(n-m)\chi\Delta t b^\dagger}|0\rangle_b\\ &= e^{-\frac{1}{2}(n-m)^2\chi^2(\Delta t)^2}\sum_{k=0}^{+\infty}\frac{(i(n-m)\chi\Delta t)^k}{k!}\sqrt{k!}\,|k\rangle_b\end{aligned}$$

In equation 7, we take the hermitian inner product of the latter quantity with $|0\rangle_b$, there is only one term left in the sum ($k=0$):

$$\alpha_n\alpha_m^*\,|n\rangle\langle m|\times\langle 0|e^{i(n-m)\chi\Delta t(b+b^\dagger)}|0\rangle_b = \alpha_n\alpha_m^*\,|n\rangle\langle m|\times e^{-\frac{1}{2}(n-m)^2\chi^2(\Delta t)^2}$$

We can conclude the coefficient ρ_{nm} of the density operator of the principal system after a time t is

$$\begin{aligned}\rho_{nm} &= \langle n|\rho|m\rangle\\ &= \langle n|\alpha_n\alpha_m^*\,|n\rangle\langle m|\times e^{-\frac{1}{2}(n-m)^2\chi^2(\Delta t)^2}|m\rangle\\ &= \alpha_n\alpha_m^* e^{-\frac{1}{2}(n-m)^2\chi^2(\Delta t)^2}\end{aligned}$$

The number $\alpha_n\alpha_m^*$ is the element ρ_{nm} at $t=0$. It decays exponentially as $e^{-\lambda(n-m)^2}$ with $\lambda = \frac{1}{2}\chi^2(\Delta t)^2$.

32.

In the case of non trace-preserving operations, we have to solve for the full set of d^4 parameters $\chi_{mn}, \quad (m,n) \in [\![1, d^2]\!]^2$, because the relation

$$\sum_i E_i^\dagger E_i = I$$

is not satisfied.

33. Specifying a quantum process

A quantum operation on a single qubit is a linear map from the complex vector space of 2 by 2 complex matrices to itself (endomorphism)

$$\begin{aligned} \varepsilon : M_2(\mathbb{C}) &\to M_2(\mathbb{C}) \\ \rho &\mapsto \varepsilon(\rho) \end{aligned}$$

Since $\dim_{\mathbb{C}} M_2(\mathbb{C}) = 4$,the images of at least 4 points are needed to completely specify ε.

If there is less than 4 points, it cannot form a basis of the vector space, and if one completes the set of points to form a basis, the images of the added points can be arbitrarily set, so that there is an infinity of map which will fit the initial description.

34. Process tomography for two qubits

34.1. One qubit operations process tomography. There is a lot to do in this exercise, but one must first understand the one qubit case; indeed, formulas are given in Box 8.5 in the book without any explanations, and moreover the conventions used does not match the ones presented in the main text.

We use the Pauli matrices for the operators $\tilde{E_i}$:

$$\begin{aligned} \tilde{E_0} &= I \\ \tilde{E_1} &= X \\ \tilde{E_2} &= -iY \\ \tilde{E_3} &= Z \end{aligned}$$

We use the canonical basis for $M_2(\mathbb{C})$:

$$\rho_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \rho_1 &= \rho_0 X \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \rho_2 &= X \rho_0 \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \rho_3 &= X \rho_0 X \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

The coefficients β_{jk}^{mn} are defined in equation (8.156) in the book:

$$\forall (j,k,m,n) \in [\![0,3]\!]^4, \quad \tilde{E_m} \rho_j \tilde{E_n}^\dagger = \sum_{k=0}^{3} \beta_{jk}^{mn} \rho_k$$

i.e. the $(\beta_{jk}^{mn})_{k\in[\![0,3]\!]}$ are the coordinates of $\tilde{E_m}\rho_j\tilde{E_n}^\dagger$ in the canonical basis of $M_2(\mathbb{C})$.
Let's find these:

$$
\begin{aligned}
\forall j \in [\![0,3]\!]^2, \quad \tilde{E_0}\rho_j\tilde{E_0}^\dagger &= \rho_j \\
\tilde{E_0}\rho_0\tilde{E_1}^\dagger &= \rho_0 X \\
&= \rho_1 \\
\tilde{E_0}\rho_1\tilde{E_1}^\dagger &= \rho_1 X \\
&= \rho_0 X^2 \\
&= \rho_0 \\
\tilde{E_0}\rho_2\tilde{E_1}^\dagger &= X\rho_0 X \\
&= \rho_3 \\
\tilde{E_0}\rho_3\tilde{E_1}^\dagger &= X\rho_0 X^2 \\
&= X\rho_0 \\
&= \rho_2 \\
\tilde{E_0}\rho_0\tilde{E_2}^\dagger &= \rho_0 iY \\
&= \rho_1 \\
\tilde{E_0}\rho_1\tilde{E_2}^\dagger &= \rho_0 iXY \\
&= -\rho_0 Z \\
&= -\rho_0 \\
\tilde{E_0}\rho_2\tilde{E_2}^\dagger &= X\rho_0 iY \\
&= \rho_3 \\
\tilde{E_0}\rho_3\tilde{E_2}^\dagger &= X\rho_0 iXY \\
&= -X\rho_0 Z \\
&= -\rho_2 \\
\tilde{E_0}\rho_0\tilde{E_3}^\dagger &= \rho_0 Z \\
&= \rho_0 \\
\tilde{E_0}\rho_1\tilde{E_3}^\dagger &= \rho_0 XZ \\
&= -i\rho_0 Y \\
&= -\rho_1 \\
\tilde{E_0}\rho_2\tilde{E_3}^\dagger &= X\rho_0 Z \\
&= \rho_2 \\
\tilde{E_0}\rho_3\tilde{E_3}^\dagger &= X\rho_0 XZ \\
&= -X\rho_1 \\
&= -\rho_4
\end{aligned}
$$

this gives us the first 4 columns of the matrix β, for $m = 0$ and $n \in [\![0,3]\!]$. I won't go through all the computations which do not present any difficulties using the basic properties of the Pauli matrices; here is the final β obtained with this method (the rows are labelled by j, k and the columns are labelled by m, n):

$$
\beta = \begin{array}{c}
\begin{array}{cccccccccccccccc} {\scriptstyle 00} & {\scriptstyle 01} & {\scriptstyle 02} & {\scriptstyle 03} & {\scriptstyle 10} & {\scriptstyle 11} & {\scriptstyle 12} & {\scriptstyle 13} & {\scriptstyle 20} & {\scriptstyle 21} & {\scriptstyle 22} & {\scriptstyle 23} & {\scriptstyle 30} & {\scriptstyle 31} & {\scriptstyle 32} & {\scriptstyle 33} \end{array} \\
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{bmatrix}
\end{array}
\begin{array}{c} \\ {\scriptstyle 00} \\ {\scriptstyle 01} \\ {\scriptstyle 02} \\ {\scriptstyle 03} \\ {\scriptstyle 10} \\ {\scriptstyle 11} \\ {\scriptstyle 12} \\ {\scriptstyle 13} \\ {\scriptstyle 20} \\ {\scriptstyle 21} \\ {\scriptstyle 22} \\ {\scriptstyle 23} \\ {\scriptstyle 30} \\ {\scriptstyle 31} \\ {\scriptstyle 32} \\ {\scriptstyle 33} \end{array}
\tag{8}
$$

The coefficients λ_{jk} are defined in equation (8.155) in the book:

$$\forall j \in [\![0,3]\!], \quad \varepsilon(\rho_j) = \sum_{k=0}^{3} \lambda_{jk}\rho_k$$

i.e. the $(\lambda_{jk})_{k\in[\![0,3]\!]}$ are the coordinates of $\varepsilon(\rho_j)$ in the canonical basis of $M_2(\mathbb{C})$.

However, if one looks at the following expression in box 8.5 for χ (corrected for reasons explained later on):

$$\chi = \frac{1}{4}\Lambda \begin{bmatrix} \boldsymbol{\rho'_0} & \boldsymbol{\rho'_1} \\ \boldsymbol{\rho'_2} & \boldsymbol{\rho'_3} \end{bmatrix} \Lambda \tag{9}$$

one can guess the numbering is different: the coordinates of $\rho'_0 = \varepsilon(\rho_0)$ are not $\lambda_{00}, \lambda_{01}, \lambda_{02}, \lambda_{03}$, but $\lambda_{00}, \lambda_{01}, \lambda_{10}, \lambda_{11}$.

the coordinates of $\rho'_1 = \varepsilon(\rho_1)$ are not $\lambda_{10}, \lambda_{11}, \lambda_{12}, \lambda_{13}$, but $\lambda_{02}, \lambda_{03}, \lambda_{12}, \lambda_{13}$.

the coordinates of $\rho'_2 = \varepsilon(\rho_2)$ are not $\lambda_{20}, \lambda_{21}, \lambda_{22}, \lambda_{23}$, but $\lambda_{20}, \lambda_{21}, \lambda_{30}, \lambda_{31}$.

the coordinates of $\rho'_3 = \varepsilon(\rho_3)$ are not $\lambda_{30}, \lambda_{31}, \lambda_{32}, \lambda_{33}$, but $\lambda_{22}, \lambda_{23}, \lambda_{32}, \lambda_{33}$.

That means that it we want the equation (8.158) to still hold in this new convention

$$\sum_{m,n\in[\![0,3]\!]^2} \beta_{jk}^{mn}\chi_{mn} = \lambda_{jk}$$

we have to

- swap rows $(j,k) = (0,2), (0,3)$ of matrix β with $(j,k) = (1,0), (1,1)$, respectively.
- swap rows $(j,k) = (2,2), (2,3)$ with $(j,k) = (3,0), (3,1)$, respectively.

The matrix β thus becomes:

$$\beta = \begin{array}{cc}
\begin{array}{cccccccccccccccc} 00 & 01 & 02 & 03 & 10 & 11 & 12 & 13 & 20 & 21 & 22 & 23 & 30 & 31 & 32 & 33 \end{array} & \\
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{bmatrix} &
\begin{array}{c} 00 \\ 01 \\ 02 \\ 03 \\ 10 \\ 11 \\ 12 \\ 13 \\ 20 \\ 21 \\ 22 \\ 23 \\ 30 \\ 31 \\ 32 \\ 33 \end{array}
\end{array}$$

If we define[2]:

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} \boldsymbol{I_2} & \boldsymbol{X} \\ \boldsymbol{X} & \boldsymbol{-I_2} \end{bmatrix}$$

We can write

$$\beta = \Lambda \otimes \Lambda$$

Next, in the book main text, χ is seen as a column vector, indexed by m, n, here of size 16. Multiplying on the left by the 16 by 16 matrix $\beta = \Lambda \otimes \Lambda$ is equivalent to first "fold" the vector χ into a square matrix of size 4 by 4, multiply on the left by Λ and on the right by $\Lambda^T = \Lambda$, that is applyind the map:

$$M_4\mathbb{C} \to M_4\mathbb{C}$$
$$M \mapsto \Lambda M \Lambda$$

Refer to appendix B for the proof of that fact.

[2] I think the factor $\frac{1}{2}$ in the book is a mistake; it appears in the expression of χ though, because $\Lambda^{-1} = \frac{1}{2}\Lambda$

Thus, if we use the notation $\overrightarrow{\chi}$ to refer to the column vector, and χ to refer to the corresponding square matrix,

$$\beta\overrightarrow{\chi} = \overrightarrow{\lambda} \Leftrightarrow \Lambda\chi\Lambda = \lambda = \begin{bmatrix} \boldsymbol{\rho'_0} & \boldsymbol{\rho'_1} \\ \boldsymbol{\rho'_2} & \boldsymbol{\rho'_3} \end{bmatrix}$$

It is easy to check that

$$\Lambda^{-}1 = \frac{1}{2}\Lambda$$

It follows that the matrix defined in equation 9 is indeed the solution we are looking for:

$$\begin{aligned} \Lambda\chi\Lambda &= \frac{1}{2}\Lambda^2 \begin{bmatrix} \boldsymbol{\rho'_0} & \boldsymbol{\rho'_1} \\ \boldsymbol{\rho'_2} & \boldsymbol{\rho'_3} \end{bmatrix} \frac{1}{2}\Lambda^2 \\ &= \begin{bmatrix} \boldsymbol{\rho'_0} & \boldsymbol{\rho'_1} \\ \boldsymbol{\rho'_2} & \boldsymbol{\rho'_3} \end{bmatrix} \end{aligned}$$

34.2. Two qubits operations process tomography. Conceptually, the process is exactly the same as in the first part, but all the indices are "doubled", causing the formulas to be very messy to write down.

It is easy to check the matrix ρ_{ij} are just the cannonical basis for $M_4(\mathbb{C})$, the set of square matrices of size 4.

We use the sixteen operators:

$$\tilde{E}_m \otimes \tilde{E}_n \quad (m,n) \in [\![0,3]\!]^2$$

Let us calculate the β matrix:

$$\begin{aligned} \tilde{E}_m \otimes \tilde{E}_{m'}\rho_{jj'}\tilde{E}_n^\dagger \otimes \tilde{E}_{n'}^\dagger &= \tilde{E}_m \otimes \tilde{E}_{m'}(\rho_j \otimes \rho_{j'})\tilde{E}n^\dagger \otimes \tilde{E}_{n'}^\dagger \\ &= (\tilde{E}_m\rho_j\tilde{E}_n^\dagger) \otimes (\tilde{E}_{m'}\rho_{j'} \otimes \tilde{E}_{n'}^\dagger) \\ &= (\sum_{k=0}^{3} \beta_{jk}^{mn}\rho_k) \otimes (\sum_{k'=0}^{3} \beta_{j'k'}^{m'n'}\rho_{k'}) \\ &= \sum_{(k,k')\in[\![0,3]\!]^2} \beta_{jk}^{mn}\beta_{j'k'}^{m'n'}\rho_k \otimes \rho_{k'} \\ &= \sum_{(k,k')\in[\![0,3]\!]^2} \beta_{jk}^{mn}\beta_{j'k'}^{m'n'}\rho_{kk'} \end{aligned}$$

We see that $\beta_{jk}^{mn}\beta_{j'k'}^{m'n'}$ is just the coefficient row $(j,k)(j',k')$ and column $(m,n)(m',n')$ of matrix $\beta_2 = \beta \otimes \beta$, where β is the matrix defined in 8 (before the rows swapping).

We are looking for 16×16 complex numbers $\chi_{mn,m'n'}$ such that

$$\forall (j,j') \in [\![0,3]\!]^2, \qquad \sum_{(k,k')\in[\![0,3]\!]^2} \sum_{\substack{(m,n)\in[\![0,3]\!]^2 \\ (m',n')\in[\![0,3]\!]^2}} \beta_{jk}^{mn}\beta_{j'k'}^{m'n'}\chi_{mn,m'n'}\rho_{kk'} = \varepsilon(\rho_{jj'}) = \rho'_{jj'}$$

A little thought shows that the expression $\sum_{\substack{(m,n)\in[\![0,3]\!]^2 \\ (m',n')\in[\![0,3]\!]^2}} \beta_{jk}^{mn}\beta_{j'k'}^{m'n'}\chi_{mn,m'n'}$ is the coefficient on row indexed by $(j,k)(j',k')$ of the product $\beta_2\overrightarrow{\chi}$.

So all we have to do is put the values of $\rho_{jj'}$ in 16 rows indexed by $(j,k)(j',k'),\quad (k,k') \in [\![0,3]\!]^2$ of a column vector $\overrightarrow{\lambda}$, and solve

$$\begin{aligned} &\beta_2 \overrightarrow{\chi} = \lambda \\ \Leftrightarrow \quad &\chi = \beta_2^{-1}\lambda \\ \Leftrightarrow \quad &\chi = \beta^{-1} \otimes \beta^{-1} \overrightarrow{\lambda} \end{aligned}$$

Or equivalently in the square matrix representation:

$$\chi = \beta^{-1}\lambda(\beta^{-1})^T$$

Note that the coordinates of a given $\rho_{jj'}$ are placed in indices $(j,k)(j',k'),\quad (k,k') \in [\![0,3]\!]^2$, which correspond to a 4 by 4 block in the matrix λ, which is precisely what we want:

$$\lambda = \begin{array}{cc}
\begin{array}{cccc} \mathbf{j'}\ 0 & 1 & 2 & 3 \\ \mathbf{k'}\ 0\ 1\ 2\ 3 & 0\ 1\ 2\ 3 & 0\ 1\ 2\ 3 & 0\ 1\ 2\ 3 \end{array} & \mathbf{j\ k} \\
\begin{bmatrix} \boldsymbol{\rho'_{00}} & \boldsymbol{\rho'_{01}} & \boldsymbol{\rho'_{02}} & \boldsymbol{\rho'_{03}} \\ \boldsymbol{\rho'_{10}} & \boldsymbol{\rho'_{11}} & \boldsymbol{\rho'_{12}} & \boldsymbol{\rho'_{13}} \\ \boldsymbol{\rho'_{20}} & \boldsymbol{\rho'_{21}} & \boldsymbol{\rho'_{22}} & \boldsymbol{\rho'_{23}} \\ \boldsymbol{\rho'_{30}} & \boldsymbol{\rho'_{31}} & \boldsymbol{\rho'_{32}} & \boldsymbol{\rho'_{33}} \end{bmatrix} & \begin{array}{c} 0\ 0 \\ 1 \\ 2 \\ 3 \\ 1\ 0 \\ 1 \\ 2 \\ 3 \\ 2\ 0 \\ 1 \\ 2 \\ 3 \\ 3\ 0 \\ 1 \\ 2 \\ 3 \end{array}
\end{array}$$

The last thing to do is to relate the matrix β to $\Lambda_2 = \Lambda \otimes \Lambda$. Recall from the first part that Λ_2 was obtained from β by swapping rows; this corresponds to left multiplication by matrix:

$$\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1&0&0&0 \\ 0&0&1&0 \\ 0&1&0&0 \\ 0&0&0&1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we call the latter matrix P, we have $\beta = P\Lambda_2$. The solution can thus be written:

$$\begin{aligned}
\chi &= (P\Lambda_2)^{-1}\lambda((P\Lambda_2)^{-1})^T \\
&= (\Lambda_2)^{-1}P^{-1}\lambda(P^{-1})^T((\Lambda_2)^{-1})^T \\
&= (\Lambda \otimes \Lambda)^{-1}P\lambda P((\Lambda \otimes \Lambda)^{-1})^T \\
&= (\Lambda^{-1} \otimes \Lambda^{-1})P\lambda P(\Lambda^{-1} \otimes \Lambda^{-1})^T \\
&= \frac{1}{16}(\Lambda \otimes \Lambda)P\lambda P(\Lambda \otimes \Lambda)^T \\
\chi &= \frac{1}{16}\Lambda_2 P\lambda P\Lambda_2
\end{aligned}$$

Appendix A. Derivation of the formula of unitary evolution for atom photon interaction

We consider a system formed by a two-level atom and a cavity confined electric field. The Jaynes-Cummings Hamiltonian is

$$H = \delta Z + g(a\sigma_- + a^\dagger \sigma_+)$$

where g is some constant which describes the strength of the interaction, $\delta = \frac{\omega - \omega_0}{2}$ is the *detuning*, $a^\dagger, a$ are respectively the creation,annihilation operators [3] on the single mode field, and $\sigma_\pm$ are operators acting on the two-level atom, namely:

$$\begin{aligned}
\sigma_+ &= \frac{1}{2}(X + iY) \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
\sigma_- &= \frac{1}{2}(X - iY) \\
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}
\end{aligned}$$

We recall

$$\begin{aligned}
\forall n \in \mathbb{N}, \quad a^\dagger |n\rangle &= \sqrt{n+1}\,|n+1\rangle \\
a\,|n+1\rangle &= \sqrt{n+1}\,|n\rangle \\
a\,|0\rangle &= 0
\end{aligned}$$

The first label corresponding to electric field, the second to atom, we have:

$$\begin{aligned}
Z\,|00\rangle &= |00\rangle \\
Z\,|01\rangle &= -\,|01\rangle \\
Z\,|10\rangle &= |10\rangle \\
a\sigma_-\,|00\rangle &= 0 \\
a\sigma_-\,|01\rangle &= 0 \\
a\sigma_-\,|10\rangle &= |01\rangle \\
a^\dagger\sigma_+\,|00\rangle &= 0 \\
a^\dagger\sigma_+\,|01\rangle &= |10\rangle \\
a^\dagger\sigma_+\,|10\rangle &= 0
\end{aligned}$$

This shows that $F = \text{span}\{|00\rangle, |01\rangle, |10\rangle\}$ is an invariant subspace for H, i.e. $H(F) \subset F$. The same is true for $H^n, n \in \mathbb{N}$ and $U = e^{-iH\Delta t} = \sum \frac{(-i\Delta t)^n}{n!} H^n$.

Let's find the representation of H in the basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$.

[3] It seems to me the book mixes up $a^\dagger$ with a in several places.

The representation of $Z = I \otimes Z$ is

$$
\begin{aligned}
I \otimes Z &= \begin{bmatrix} 1\boldsymbol{Z} & 0\boldsymbol{Z} \\ 0\boldsymbol{Z} & 1\boldsymbol{Z} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{Z} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Z} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}
\end{aligned}
$$

The representation of annihilation operator in the $(|n\rangle)_{n\in\mathbb{N}}$ basis of the electric field state space is

$$
a = \begin{array}{cc}
\begin{matrix} & & & & & n & & \end{matrix} & \\
\begin{bmatrix}
0 & 1 & & & & & \\
 & \ddots & \sqrt{2} & & & & \\
 & & \ddots & \sqrt{3} & & \mathbf{0} & \\
 & & & \ddots & \sqrt{4} & & \\
 & & & & \ddots & \ddots & \\
 & & & & & \ddots & \sqrt{n} \\
 & \mathbf{0} & & & & & \ddots \\
 & & & & & & \ddots
\end{bmatrix} & n-1
\end{array}
$$

The representation of $a \otimes \sigma_-$ is then

$$
\begin{aligned}
a \otimes \sigma_- &= \begin{bmatrix} 0\boldsymbol{\sigma}_- & 1\boldsymbol{\sigma}_- \\ 0\boldsymbol{\sigma}_- & 0\boldsymbol{\sigma}_- \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0} & \boldsymbol{\sigma}_- \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
a^\dagger \otimes \sigma_+ &= \begin{bmatrix} 0\boldsymbol{\sigma}_+ & 0\boldsymbol{\sigma}_- \\ 1\boldsymbol{\sigma}_+ & 0\boldsymbol{\sigma}_+ \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{\sigma}_+ & \mathbf{0} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

Thus the representation of H in the basis $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$ is

$$
H = \begin{bmatrix} \delta & 0 & 0 & 0 \\ 0 & -\delta & g & 0 \\ 0 & g & \delta & 0 \\ 0 & 0 & 0 & -\delta \end{bmatrix}
$$

The representation of the restriction of H in the basis $(|00\rangle, |01\rangle, |10\rangle)$ is

$$
\begin{aligned}
H &= \begin{bmatrix} \delta & 0 & 0 \\ 0 & -\delta & g \\ 0 & g & \delta \end{bmatrix} \\
&= \begin{bmatrix} \delta & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\boldsymbol{H_1}$}} \\ 0 & & \end{bmatrix}
\end{aligned}
$$

Block calculus shows that

$$
e^{-iH\Delta t} = \begin{bmatrix} e^{-i\delta\Delta t} & 0 & 0 \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\boldsymbol{e^{-iH_1\Delta t}}$}} \\ 0 & & \end{bmatrix}
$$

Let $\Omega = \sqrt{g^2 + \delta^2}$, the *Rabi frequency.*

$$
\begin{aligned}
H_1^2 &= \begin{bmatrix} \Omega^2 & 0 \\ 0 & \Omega^2 \end{bmatrix} \\
&= \Omega^2 I_2
\end{aligned}
$$

This shows that

$$
\begin{aligned}
\forall n \in \mathbb{N}, \quad H_1^{2n} &= \Omega^{2n} I_2 \\
H_1^{2n+1} &= \Omega^{2n} H_1
\end{aligned}
$$

Then,

$$\begin{aligned} e^{-iH_1\Delta t} &= \sum_{n=0}^{+\infty} \frac{(-i\Omega\Delta t)^{2n}}{(2n)!} I_2 + \frac{1}{\Omega} \sum_{n=0}^{+\infty} \frac{(-i\Omega\Delta t)^{2n+1}}{(2n+1)!} H_1 \\ &= \sum_{n=0}^{+\infty} (-1)^n \frac{(\Omega\Delta t)^{2n}}{(2n)!} I_2 - i\frac{1}{\Omega} \sum_{n=0}^{+\infty} (-1)^n \frac{(\Omega\Delta t)^{2n+1}}{(2n+1)!} H_1 \\ &= \cos(\Omega t) I_2 - i\frac{1}{\Omega} \sin(\Omega t) H_1 \end{aligned}$$

Finally the matrix U is

$$e^{-iH\Delta t} = \begin{array}{c c} \begin{array}{ccc} \scriptstyle U|00\rangle & \scriptstyle U|01\rangle & \scriptstyle U|10\rangle \end{array} & \\ \begin{bmatrix} e^{-i\delta\Delta t} & 0 & 0 \\ 0 & \cos(\Omega t) + i\frac{\delta}{\Omega}\sin(\Omega t) & -i\frac{g}{\Omega}\sin(\Omega t) \\ 0 & -i\frac{g}{\Omega}\sin(\Omega t) & \cos(\Omega t) - i\frac{\delta}{\Omega}\sin(\Omega t) \end{bmatrix} & \begin{array}{c} \scriptstyle |00\rangle \\ \scriptstyle |01\rangle \\ \scriptstyle |10\rangle \end{array} \end{array}$$

Appendix B. equivalence between column vector and matrix representation

Lemma. Let $A \in M_{d_0,d_1}(\mathbb{C})$, $C \in M_{d_2,d_3}(\mathbb{C})$.

Let $B = A \otimes C \in M_{d_0 d_2, d_1 d_3}(\mathbb{C})$, and $\overrightarrow{X} \in M_{d_1 d_3,1}(\mathbb{C})$ a column vector.

Suppose we fold $\overrightarrow{X}$ into a square matrix $X = (x_{ij})_{(i,j)\in[\![0,d_1-1]\!]\times[\![0,d_3-1]\!]}$ of size d_1 by d_3, and we index the rows of β by $(i,j) \in [\![0, d_0-1]\!] \times [\![0, d_2-1]\!]$, then the coefficient (i,j) of $\beta\overrightarrow{X}$ is equal to the coefficient (i,j) of AXC^T:

$$\forall (i,j) \in [\![0, d_0-1]\!] \times [\![0, d_2-1]\!], \quad (\beta\overrightarrow{x})_{i,j} = (AXC^T)_{i,j}$$

Proof. The coefficient (i,j) of AXC^T is:

$$\begin{aligned} (AXC^T)_{i,j} &= \sum_{k=0}^{d_1} a_{ik} \left(\sum_{l=0}^{d_3} x_{kl} c_{jl} \right) \\ &= \sum_{k=0}^{d_1} \sum_{l=0}^{d_3} a_{ik} x_{kl} c_{jl} \\ &= \left(\sum_{k=0}^{d_1} \sum_{l=0}^{d_3} \underbrace{a_{ik} c_{jl}}_{\substack{\text{coeff. row } (i,j) \\ \text{and column } (k,l) \text{ of } A\otimes C}} \right) x_{kl} \\ &= (\beta\overrightarrow{x})_{i,j} \end{aligned}$$

□