

Highlights

Sparsification Lower Bound for Linear Spanners in Directed Graphs

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- DIRECTED LINEAR SPANNER parameterized by the number of vertices n admits no polynomial compression of size $\mathcal{O}(n^{2-\epsilon})$ for any $\epsilon > 0$ unless $\text{NP} \subseteq \text{coNP}/poly$.
- Similar results hold for DIRECTED ADDITIVE SPANNER and DIRECTED MULTIPLICATIVE SPANNER problems.
- The sparsification lower bound holds even when input is a directed acyclic graph and the error terms α, β are *any* computable functions of the distance being approximated.

Sparsification Lower Bound for Linear Spanners in Directed Graphs

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Abstract

For $\alpha \geq 1$, $\beta \geq 0$, and a graph G , a spanning subgraph H of G is said to be an (α, β) -spanner if $\text{dist}(u, v, H) \leq \alpha \cdot \text{dist}(u, v, G) + \beta$ holds for any pair of vertices u and v . These type of spanners, called *linear spanners*, generalizes *additive spanners* and *multiplicative spanners*. Recently, Fomin, Golovach, Lochet, Misra, Saurabh, and Sharma initiated the study of additive and multiplicative spanners for directed graphs (IPEC 2020). In this article, we continue this line of research and prove that DIRECTED LINEAR SPANNER parameterized by the number of vertices n admits no polynomial compression of size $\mathcal{O}(n^{2-\epsilon})$ for any $\epsilon > 0$ unless $\text{NP} \subseteq \text{coNP}/\text{poly}$. We show that similar results hold for DIRECTED ADDITIVE SPANNER and DIRECTED MULTIPLICATIVE SPANNER problems. This sparsification lower bound holds even when the input is a directed acyclic graph and α, β are *any* computable functions of the distance being approximated.

Keywords: Additive Spanners, Multiplicative Spanners, Sparsification, Directed Graphs

1. Introduction

For a graph or a digraph, a spanner is its subgraph that preserves lengths of shortest paths between any two pair of vertices in it up to some additive and/or multiplicative error. Spanners can be classified as additive spanners,

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multiplicative spanners, or linear or mixed spanners depending on the type of error is allowed. We refer readers to the recent survey by Ahmed et al. [1] for motivations, applications, and literature regarding this topic.

A subgraph H of G is its *spanning subgraph* if $V(H) = V(G)$. We use $\text{dist}(u, v, G)$ to denote the shortest distance between u and v in G . For $\alpha \geq 1$, $\beta \geq 0$, a spanning subgraph H of G is said to be (α, β) -*spanner* if $\text{dist}(u, v, H) \leq \alpha \cdot \text{dist}(u, v, G) + \beta$ holds for any pair of vertices u and v . These type of spanners are called *linear spanners*. *Additive spanners* and *multiplicative spanners* are $(1, \beta)$ -spanners and $(\alpha, 0)$ -spanners, respectively, for some $\beta \geq 1$ and $\alpha > 1$. Thorup and Zwick [2] considered spanners with additive error terms that are sub-linear in the distance being approximated. They constructed a spanning subgraph H such that $\text{dist}(u, v, H) \leq \text{dist}(u, v, G) + f_\beta(\text{dist}(u, v, G))$ where f_β is a sub-linear function of the form $f_\beta(d) = c \cdot d^{1/(1-q)}$ for some constants c and $q \geq 2$.

As any graph is a spanner for itself, a non-trivial question is to find a spanner with as few edges as possible. Liestman and Shermer [3] proved that for every fixed $\beta \geq 1$, given a graph G and a positive integer k , it is **NP-Complete** to decide whether G admits a $(1, \beta)$ -spanner (also called *additive β -spanner*) with at most $|E(G)| - k$ edges. It is also known that deciding whether G has an $(\alpha, 0)$ -spanner (also called *multiplicative α -spanner*) with at most $|E(G)| - k$ edges is **NP-Complete** for every fixed $\alpha \geq 2$ ([4], [5]).

Recently, the problem of finding optimum additive, multiplicative, and linear spanners for undirected graph have been studied from the Parameterized Complexity framework. In this framework, we measure the computational complexity as a function of the input size of a problem and a secondary measure. We define relevant notations in Section 2. Kobayashi proved that **MULTIPLICATIVE SPANNER** admits a polynomial kernel of size $\mathcal{O}(k^2\alpha^2)$ [6], and **ADDITIVE SPANNER**, **LINEAR SPANNER** problems are fixed parameter tractable [7]. Fomin et al. [8] started the study of additive and multiplicative spanners for directed graphs. They proved that **DIRECTED MULTIPLICATIVE SPANNER** admits a kernel of size $\mathcal{O}(k^4\alpha^5)$ whereas **DIRECTED ADDITIVE SPANNER** is not fixed parameter tractable under a widely believed conjecture.

We continue this line of research and present a sparsification lower bound for these problems. We remark that the problem of finding a spanner (not necessarily an optimum spanner) becomes ‘easier’ as the error factor increases (see Related Work below). To present a lower bound for a general case, we consider the spanners that allow the error as *any* computable function of the

distance being approximated. Note that such a formulation is considered by Thorup and Zwick [2] for undirected graphs. We highlight that unlike in their case, where only sub-linear functions are allowed, the following formulation allows any computable function.

DIRECTED LINEAR SPANNER

Input: Digraph D , two monotonically non-decreasing computable functions $f_\alpha, f_\beta : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$, and a positive integer k .

Question: Does there exist a spanning subgraph H of D with at most $|A(D)| - k$ arcs such that $\text{dist}(u, v, H) \leq f_\alpha(\text{dist}(u, v, D)) \cdot \text{dist}(u, v, D) + f_\beta(\text{dist}(u, v, D))$ holds for any pair of vertices u and v ?

In this article, we investigate the problem from the perspective of polynomial-time *sparsification*: the method of reducing an input instance to an equivalent object that needs fewer bits to encode. For example, if input instance comprises a graph or a CNF-formula then the aim is to find an equivalent graph or CNF-formula in which ratio of edges to vertices or clauses to variable is smaller than the original instance. Consider an instance $(D, f_\alpha, f_\beta, k)$ of DIRECTED LINEAR SPANNER problem. We assume, throughout the article, that any function given as a part of input can be encoded using the constant number of bits. Hence, this instance can be encoded with $\mathcal{O}(|V(D)|^2)$ bits as for any non-trivial instance $k \leq |A(D)|$. The goal of sparsification is to examine existence of a polynomial time algorithm that given an instance $(D, f_\alpha, f_\beta, k)$, maps it to an equivalent instance of *any* problem that uses $\mathcal{O}(|V(D)|^{2-\epsilon})$ bits for some $\epsilon > 0$. We answer this question in the negative.

Theorem 1. *Consider two monotonically non-decreasing computable functions $f_\alpha, f_\beta : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ such that $1 \leq f_\alpha(1)$ and $2 \leq f_\alpha(1) + f_\beta(1)$. Unless $\text{NP} \subseteq \text{coNP}/\text{poly}$, an arbitrary instance $(D, f_\alpha, f_\beta, k)$ of DIRECTED LINEAR SPANNER does not admit a polynomial compression of size $\mathcal{O}(|V(D)|^{2-\epsilon})$ for any $\epsilon > 0$, even when D a directed acyclic graph.*

We justify the condition on the sum of the values of two functions at 1 in Section 3. Consider special case when $f_\alpha(d) = \alpha$ and $f_\beta(d) = \beta$ for all $d \in \mathbb{N}$ for some non-negative constants α, β such that $\alpha + \beta \geq 2$. For $\alpha = 1$ and $\beta = 0$, Theorem 1 implies, respectively, that DIRECTED ADDITIVE SPANNER and DIRECTED MULTIPLICATIVE SPANNER do not admit polynomial compressions of size $\mathcal{O}(|V(D)|^{2-\epsilon})$ for any $\epsilon > 0$ unless $\text{NP} \subseteq \text{coNP}/\text{poly}$.

We remark that Theorem 1 *does not* imply that there are digraphs which do not have the spanners that satisfy the properties mentioned in the def-

inition of the problem. Rather, it implies that to find an optimum linear spanner, one needs information about almost all the arcs in the digraph. And hence, it is not possible to compress the instance in polynomial time and with non-trivial number of bits without solving it. Considering theoretical and practical applications of spanners, we believe Theorem 1 provides a non-trivial lower bound.

Related Work. It is known that all undirected graphs have additive 2-spanners with $\mathcal{O}(n^{1.5})$ edges ([9], [10], [11]), additive 4-spanners with $\mathcal{O}(n^{1.4})$ edges ([12], [13]), and additive 6-spanners with $\mathcal{O}(n^{1.33})$ edges ([14], [15]). Abboud and Bodwin [16] proved that for $0 < \epsilon < 1/3$, one cannot compress an input graph into $\mathcal{O}(n^{1+\epsilon})$ bits, so that one can recover distance information for each pair of vertices within $n^{o(1)}$ additive error. A well-known trade-off between the sparsity and the multiplicative factor is: for any positive integer α and any graph G , there is a multiplicative $(2\alpha - 1)$ -spanner with $\mathcal{O}(n^{1+1/\alpha})$ edges [17]. This bound is conjectured to be tight based on the popular Girth Conjecture of Erdős [18].

The general quest for sparsification algorithms is motivated by the fact that they allow instances to be stored, manipulated, and solved more efficiently. As sparsification preserves the exact answer to the problem, it suffices to solve the sparsified instance. The notion is fruitful in theoretical [19] and practical [20] settings. The growing list of problems for which the existence of non-trivial sparsification algorithms has been ruled out under the same assumption includes VERTEX COVER [21], DOMINATING SET [22], FEEDBACK ARC SET [22], TREewidth [23], LIST H -COLORING [24], and BOOLEAN CONSTRAINT SATISFACTION problems [25].

Organization of the article. We organize the remaining article as follows. In Section 2, we present some preliminaries. In Section 3, we present a parameter preserving reduction from DOMINATING SET to DIRECTED LINEAR SPANNER. We use this reduction to present a proof of Theorem 1. We conclude this article in Section 4.

2. Preliminaries

We denote the set of positive integers and the set of non-negative real numbers by \mathbb{N} and $\mathbb{R}_{\geq 0}$, respectively. For a positive integer q , we denote set $\{1, 2, \dots, q\}$ by $[q]$.

We consider graphs and directed graphs with a finite number of vertices that do not have loops or multiple edges/arcs as they are irrelevant for distances. For an undirected graph G , by $V(G)$ and $E(G)$ we denote the set of vertices and edges of G respectively. Two vertices u, v are said to be *adjacent* in G if there is an edge $(u, v) \in E(G)$. The *neighborhood* of a vertex v , denoted by $N_G(v)$, is the set of vertices adjacent to v . The *closed neighborhood* of a vertex is $N_G[v] = N_G(v) \cup \{v\}$. We say $u \in V(G)$ *dominates* $v \in V(G)$ if v is in $N[u]$. A set $X \subseteq V(G)$ is a *dominating set* of G if $V(G) = N_G[X]$. For a *directed graph* (or *digraph*) D , by $V(D)$ and $A(D)$ we denote the sets of vertices and directed arcs in D , respectively. For $F \subseteq A(D)$, $D - F$ is the graph obtained by deleting arcs in F from D .

A *directed path* P in D is an ordered sequence of vertices $\langle v_1, v_2, \dots, v_q \rangle$ such that $(v_i, v_{i+1}) \in A(D)$ for all $i \in [q - 1]$. We denote the set of arcs $\{(v_i, v_{i+1}) \mid i \in [q - 1]\}$ by $A(P)$. The length of directed path P is $|A(P)|$. For two vertices $u, v \in V(D)$, $\text{dist}(u, v, D)$ denotes the length of a shortest directed path from u to v . If there is no directed path from u to v , then we assign $\text{dist}(u, v, D) = +\infty$. Note that $\text{dist}(u, v, D)$ may not be equal to $\text{dist}(v, u, D)$. Consider two directed paths $P_1 = \langle v_1, v_2, \dots, v_q \rangle$ and $P_2 = \langle u_1, u_2, \dots, u_p \rangle$. If $v_q = u_1$ then we denote directed path $P = \langle v_1, \dots, v_q = u_1, \dots, u_p \rangle$ by $P_1 \circ P_2$. A digraph D is said to be *acyclic* if the following statement is true for any two distinct vertices: if there is a directed path from u to v then there is no directed path from v to u .

A *subdivision* of an arc $(u, v) \in A(D)$ is an operation that deletes arc (u, v) , adds a vertex w to $V(D)$, and adds arcs (u, w) and (w, v) . We say arc (u, v) is subdivided q times if we delete arc (u, v) and add a directed path of length $(q + 1)$ from u to v . If digraph D' is obtained from D by subdividing q times arc (u, v) then $\text{dist}(u, v, D') = q + 1$. A *contraction* of an arc $(u, v) \in A(D)$ is an operation that results in a digraph D' on the vertex set $V(D') = (V(D) \setminus \{u, v\}) \cup \{w\}$ with $A(D') = \{(x, y) \mid (x, y) \in A(D) \text{ and } x, y \in V(D')\} \cup \{(x, w) \mid (x, u) \in A(D)\} \cup \{(w, y) \mid (u, y) \in A(D)\} \cup \{(x, w) \mid (x, v) \in A(D)\} \cup \{(w, y) \mid (v, y) \in A(D)\}$.

We refer the readers to the recent books [26], [27] for the detailed introduction of Parameterized Complexity theory. A parameterized language \mathcal{Q} is a subset of $\Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet. The second component of a tuple $(x, k) \in \Sigma^* \times \mathbb{N}$ is called the *parameter*. A parameterized language is said to be fixed-parameter tractable (or **FPT**) if there exists an algorithm that given a tuple $(x, k) \in \Sigma^* \times \mathbb{N}$, runs in $f(k) \cdot |x|^{\mathcal{O}(1)}$, for some computable function $f(\cdot)$, and correctly determines whether $(x, k) \in \mathcal{Q}$. The notion of

kernelization is used to capture various forms of efficient preprocessing. We define it in its general form.

Definition 1 (Definition 1.5 in [27]). *A polynomial compression of a parameterized language $\mathcal{Q} \subseteq \Sigma^* \times \mathbb{N}$ into a language $\mathcal{R} \subseteq \Sigma^*$ is an algorithm that takes as input an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, runs in time polynomial in $|x| + k$, and returns a string y such that: (i) $|y| \leq p(k)$ for some polynomial $p(\cdot)$, and (ii) $y \in \mathcal{R}$ if and only if $(x, k) \in \mathcal{Q}$.*

We need the following result regarding sparsification.

Proposition 1 (Theorem 4 in [22]). *Unless $\text{NP} \subseteq \text{coNP}/\text{poly}$, DOMINATING SET parameterized by the number of vertices n does not admit a polynomial compression of size $\mathcal{O}(n^{2-\epsilon})$ for any $\epsilon > 0$.*

3. Proof of Theorem 1

To prove the theorem, we present a reduction from DOMINATING SET to DIRECTED LINEAR SPANNER. In the DOMINATING SET problem, an input is an undirected graph G and an integer l . The objective is to decide whether there is a dominating set of size at most l in G .

Consider an instance $(D, f_\alpha, f_\beta, k)$ of DIRECTED LINEAR SPANNER such that $f_\alpha(1) + f_\beta(1) < 2$. Recall that k is a positive integer. By the definition of the problem, it is safe to consider that D does not have parallel arcs. Assume that there exist a set of arcs $F \subseteq A(D)$ of size at least k such that $D - F$ satisfies the properties mention in the problem statement. For the endpoints of an arc $(u, v) \in F$, we have $\text{dist}(u, v, D - F) \leq 2 \leq f_\alpha(\text{dist}(u, v, D)) + f_\beta(\text{dist}(u, v, D)) \leq f_\alpha(1) + f_\beta(1) < 2$, which is a contradiction. Hence, our assumption is wrong, and no such set of arcs exists. In this case the input is a NO-instance. This fact can be encoded in the constant bit-size. To avoid this trivial case, we consider the functions for which $2 \leq f_\alpha(1) + f_\beta(1)$. As a technical requirement, we need $1 \leq f_\alpha(1)$ and f_α, f_β are monotonically non-decreasing functions.

Reduction. The reduction takes as input an instance (G, l) of DOMINATING SET and two monotonically non-decreasing computable functions $f_\alpha, f_\beta : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$. It outputs an instance $(D, f_\alpha, f_\beta, k)$ of DIRECTED LINEAR SPANNER. Let $V(G) = \{v_1, v_2, \dots, v_{|V(G)|}\}$. The reduction creates digraph D as follows:

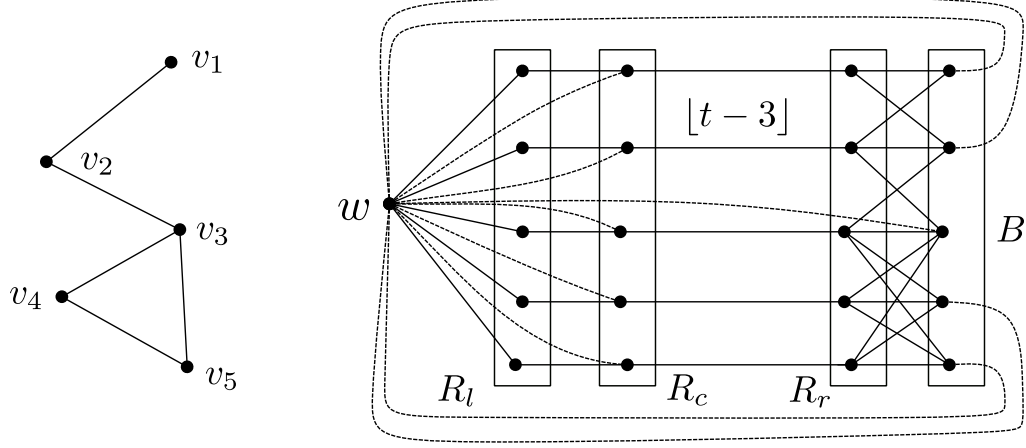


Figure 1: Overview of the reduction. Left side is an input graph G and right side shows the corresponding digraph D constructed by the reduction. All arcs in D are directed from their left end-point to right end-point. Every arc starting in R_c and ending in R_r is sub-divided $\lfloor t - 3 \rfloor$ times. For every vertex in $V(G)$, four vertices on the same horizontal line are the vertices corresponding to it in $V(D)$. Only the dotted arcs can be omitted from a spanner of D .

- It creates the following four copies of $V(G)$: R_l, R_c, R_r , and B . For every vertex v_i in $V(G)$, let $R_l[i], R_c[i], R_r[i], B[i]$ denote these four copies. It adds a new vertex w .
- For every vertex v_i in $V(G)$, it adds the following six arcs $(w, R_l[i]), (w, R_c[i]), (w, B[i]), (R_l[i], R_c[i]), (R_c[i], R_r[i]),$ and $(R_r[i], B[i])$.
- For every edge (v_i, v_j) in $E(G)$, it adds the following arcs: $(R_r[i], B[j]), (R_r[j], B[i])$.
- Let $t = f_\alpha(1) + f_\beta(1)$. Recall that, by our assumption on the functions, $t \geq 2$. If $\lfloor t \rfloor = 2$, then the reduction contracts arc $(R_c[i], R_r[i])$ for every $i \in [|V(G)|]$. If $\lfloor t \rfloor = 3$, it does not modify the graph. If $\lfloor t \rfloor \geq 4$, it subdivides $\lfloor t - 3 \rfloor$ -times arc $(R_c[i], R_r[i])$ for every $i \in [|V(G)|]$.

This completes the construction of D . The reduction sets $k = 2 \cdot |V(G)| - l$ and returns the instance $(D, f_\alpha, f_\beta, k)$. See Figure 1 for an illustration.

Lemma 1. *If (G, l) is a YES instance of DOMINATING SET then $(D, f_\alpha, f_\beta, k)$ is a YES instance of DIRECTED LINEAR SPANNER.*

Proof. Let X be a dominating set of size at most l in G . Consider the subset of arcs $F \subseteq A(D)$ that contains all the arcs of the form $(w, B[i])$ and arcs $(w, R_c[i])$ corresponding to every vertex *not* in X . Formally,

$$F := \{(w, B[i]) \mid \forall i \in [|V(G)|]\} \cup \{(w[i], R_c[i]) \mid \forall i \in [|V(G)|] \text{ s.t. } v_i \notin X\}.$$

Note that $|F| \geq |V(G)| + |V(G)| - l \geq k$. We argue that $D - F$ satisfies the properties mentioned in the definition of DIRECTED LINEAR SPANNER.

It is easy to see that for any vertex $u \in V(D) \setminus \{w\}$ and vertex $v \in V(D) \setminus B$, if there is a directed path from u to v in D then the same path is also present in $D - F$. Hence, for all pairs of such vertices, we have $\text{dist}(u, v, D - F) = \text{dist}(u, v, D)$.

Consider the case when $u = w$ and $v \in V(D) \setminus B$. It is easy to verify that if $\text{dist}(u, v, D) = d$, then $\text{dist}(u, v, D - F) \leq d + 1$. Hence, it is sufficient to argue that $d + 1 \leq f_\alpha(d) \cdot d + f_\beta(d)$ for every $d \geq 1$. Recall that $1 \leq f_\alpha(1)$ and $2 \leq f_\alpha(1) + f_\beta(1)$. For $d \geq 1$, this implies that $(d - 1) + 2 \leq (d - 1) \cdot f_\alpha(1) + f_\alpha(1) + f_\beta(1) \leq d \cdot f_\alpha(1) + f_\beta(1)$. As f_α, f_β are monotonically non-decreasing functions and $d \geq 1$, we have $d + 1 \leq d \cdot f_\alpha(d) + f_\beta(d)$. This implies that $\text{dist}(u, v, D - F) \leq f_\alpha(\text{dist}(u, v, D)) \cdot \text{dist}(u, v, D) + f_\beta(\text{dist}(u, v, D))$.

For any $i \in [|V(G)|]$ there are following three directed paths that starts at w and ends at $B[i]$: (i) $\langle w, B[i] \rangle$, (ii) $\langle w, R_c[j] \rangle \circ \langle R_c[j], \dots, R_r[j] \rangle \circ \langle R_r[j], B[i] \rangle$, and (iii) $\langle w, R_l[j], R_c[j] \rangle \circ \langle R_c[j], \dots, R_r[j] \rangle \circ \langle R_r[j], B[i] \rangle$. For the last two types of directed paths, $j \in [|V(G)|]$ such that either $i = j$ or $(v_i, v_j) \in E(G)$. The lengths of these three types of directed paths are 1, $2 + \lfloor t - 2 \rfloor$, and $3 + \lfloor t - 2 \rfloor$, respectively. As X is a dominating set, for any $i \in [|V(G)|]$, at least one directed path of the second type is present in $D - F$. This implies $\text{dist}(w, B[i], D - F) = 2 + \lfloor t - 2 \rfloor \leq t = f_\alpha(1) + f_\beta(1) = f_\alpha(\text{dist}(w, B[i], D)) \cdot \text{dist}(w, B[i], D) + f_\beta(\text{dist}(w, B[i], D))$.

This implies $D - F$ satisfies the properties mentioned in the problem definition of DIRECTED LINEAR SPANNER. Hence, if (G, ℓ) is a YES instance of DOMINATING SET then $(D, f_\alpha, f_\beta, k)$ is a YES instance of DIRECTED LINEAR SPANNER. \square

Lemma 2. *If $(D, f_\alpha, f_\beta, k)$ is a YES instance of DIRECTED LINEAR SPANNER then (G, l) is a YES instance of DOMINATING SET.*

Proof. Let $F \subseteq A(D)$ be the set of arcs in $A(D)$ such that $D - F$ satisfies the properties mentioned in the definition of DIRECTED LINEAR SPANNER. We argue that arcs in F are of the form $(w, R_c[i])$ or $(w, B[i])$ for some $i \in [|V(D)|]$. For any arc (u, v) in $A(D)$ that is not of the above forms, there is no directed path from u to v in digraph $D - (u, v)$. Hence, if (u, v) is in F , then $\text{dist}(u, v, D - F) = \infty$ which is not upper bounded by $f_\alpha(\text{dist}(u, v, D)) \cdot \text{dist}(u, v, D) + f_\beta(\text{dist}(u, v, D))$. This contradicts the fact that $G - F$ satisfies the properties mentioned in the definition of the problem.

We say arc $(w, R_c[i])$ corresponds to vertex v_i in $V(G)$ for every $i \in [|V(G)|]$. Consider the subset X of vertices in $V(G)$ whose corresponding to arcs are *not* in F . Formally, $X := \{v_i \mid i \in [|V(G)|] \text{ and } (w, R_c[i]) \notin F\}$. We argue that set $X \cup (V(G) \setminus N_G[X])$ is a dominating set of size at most l in G .

We claim that for a vertex $v_i \in V(G) \setminus N_G[X]$, set F does not contain arc $(w, B[i])$. Assume, for the sake contradiction, that $(w, B[i])$ is in F . By the construction of X , arc $(w, R_c[i])$ is present in F . Any directed path from w to $B[i]$ in $D - F$ is either of the form $\langle w, R_c[j] \rangle \circ \langle R_c[j], \dots, R_r[j] \rangle \circ \langle R_r[j], B[i] \rangle$ or $\langle w, R_l[j], R_c[j] \rangle \circ \langle R_c[j], \dots, R_r[j] \rangle \circ \langle R_r[j], B[i] \rangle$ for some $j \in [|V(G)|]$ such that $(v_i, v_j) \in E(G)$. As $v_i \in V(G) \setminus N_G[X]$, for every $j \in [|V(G)|]$ such that $(v_i, v_j) \in E(G)$, we have $v_j \notin X$. Hence, arc $(w, R_c[j])$ is in F and not present in $D - F$. This implies any directed path from w to $B[i]$ is of the second form. Recall that the length of such directed path is $3 + \lfloor t - 2 \rfloor$. This implies $3 + \lfloor t - 2 \rfloor \leq f_\alpha(\text{dist}(w, B[i], D)) \cdot \text{dist}(w, B[i], D) + f_\beta(\text{dist}(w, B[i], D)) \leq f_\alpha(1) \cdot 1 + f_\beta(1) = t$, which is a contradiction. Hence, our assumption was wrong and for any $v_i \in V(G) \setminus N_G[X]$, set F does not contain arc $(w, B[i])$.

Define set $Y = V(G) \setminus N_G[X]$. Note that $X \cup Y$ dominates every vertex in $V(G)$. It remains to argue the bound on the size of $X \cup Y$. By the definition of X , set F contains $|V(G)| - |X|$ many arcs of the form $(w, R_c[i])$ for some $i \in [|V(G)|]$. By the definition of Y , set F contains $|V(G)| - |Y|$ many arcs of the form $(w, B[i])$ for some $i \in [|V(G)|]$. As $|F| \geq 2|V(G)| - l$, we have $|X| + |Y| \leq l$. This implies there is a dominating set of size at most l in G , and hence (G, l) is a YES instance of DOMINATING SET. This concludes the proof of the lemma. \square

Proof. (of Theorem 1) Assume, for the sake of contradiction, that there is an algorithm \mathcal{A} that given a constant $\epsilon > 0$ and an instance $(D_1, f_\alpha, f_\beta, k)$ of DIRECTED LINEAR SPANNER, where D_1 is a directed acyclic graph, runs in polynomial time and computes its polynomial compression of size $\mathcal{O}(|V(D_1)|^{2-\epsilon})$.

Consider the following algorithm \mathcal{B} that takes as input an instance (G, l) of DOMINATING SET and returns its polynomial compression. Algorithm \mathcal{B} runs the reduction mentioned above as a subroutine with input (G, l) and functions f_α, f_β . Let $(D, f_\alpha, f_\beta, k)$ be the equivalent instance returned by the reduction. It is easy to verify that D is a directed acyclic graph. Algorithm \mathcal{B} uses Algorithm \mathcal{A} as a subroutine to obtain a polynomial compression of size $\mathcal{O}(|V(D)|^{2-\epsilon})$. It then returns this as polynomial compression for (D, l) . This completes the description of Algorithm \mathcal{B} .

Algorithm \mathcal{B} runs in polynomial time and its correctness follows from Lemma 1, Lemma 2, and the correctness of Algorithm \mathcal{A} . By the description of Algorithm \mathcal{B} , the number of vertices in D is at most $4 \cdot |V(G)| + |V(G)| \cdot (f_\alpha(1) + f_\beta(1)) + 1 \in \mathcal{O}(|V(G)|)$. This implies Algorithm \mathcal{B} computes a polynomial compression of DOMINATING SET of size $\mathcal{O}(|V(G)|^{2-\epsilon})$. But, this contradicts Proposition 1. Hence, our assumption was wrong and no such algorithm exists. This concludes the proof of the theorem. \square

4. Conclusion

In this article, we proved that unless $\text{NP} \subseteq \text{coNP}/\text{poly}$, DIRECTED LINEAR SPANNER parameterized by the number of vertices n admits no generalized kernel of size $\mathcal{O}(n^{2-\epsilon})$ for any $\epsilon > 0$. This lower bound holds even when input is a directed acyclic graph and α, β are *any* computable functions of the distance being approximated. Abboud and Bodwin [16] proved that unconditional sparsification lower bound for undirected graphs with weaker constants in the exponent. It will be interesting to investigate whether their lower bound can be strengthened in case of directed graphs.

We can extend our result to more generalized problem at the cost stronger condition on the error function. Consider the generalization of the problem, called DIRECTED SPANNER, in which the error function $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ is not restricted to be linear in terms of the distance. Using the identical arguments, it is not hard to see that if $d + 1 \leq f(d)$ for every $d \geq 1$, then the DIRECTED SPANNER problem does not admit a polynomial compression of size $\mathcal{O}(|V(D)|^{2-\epsilon})$ for any $\epsilon > 0$, even when D a directed acyclic graph. Note that this condition, i.e. $d + 1 \leq f(d)$ for every $d \geq 1$, is stronger than the conditions, i.e. $1 \leq f_\alpha(1)$, $2 \leq f_\alpha(1) + f_\beta(1)$, and both f_α, f_β are monotonically non-decreasing, we used to prove the similar result for DIRECTED LINEAR SPANNER.

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