Languages and Algorithms for Artificial Intelligence (Third Module)

A Glimpse into Computational Learning Theory

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Learning Problems as Computational Problems

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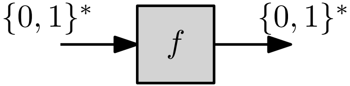

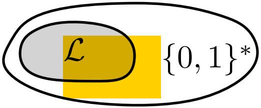

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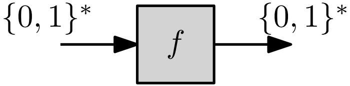

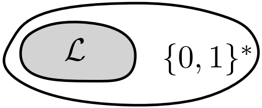

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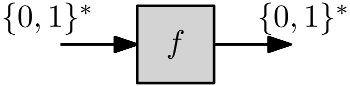

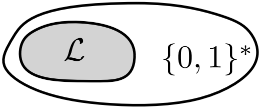

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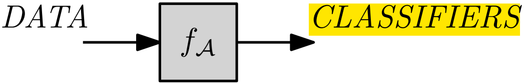

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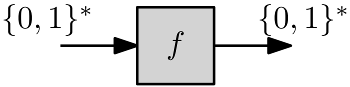

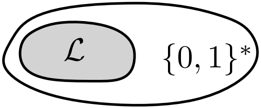

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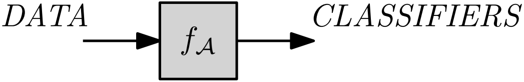
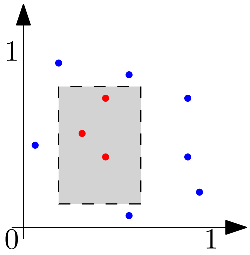


  - Could data and classifiers be encoded as strings, thus turning $f_{\mathcal{A}}$ as a function of the kind we know?
  - How could we formalize the fact that $\mathcal{A}$ correctly solves a given learning task?

An Example Problem

Suppose that the data the algorithm $\mathcal{A}$ takes in input are points $(x, y) \in \mathbb{R}_{[0,1]} \times \mathbb{R}_{[0,1]}$ (where $\mathbb{R}_{[0,1]}$ is the set of real numbers between 0 and 1). These are labelled as positive or negative depending on they being inside a rectangle



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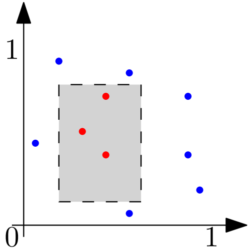

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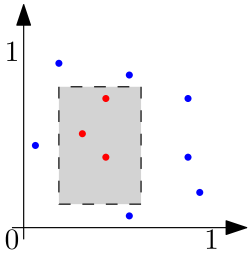

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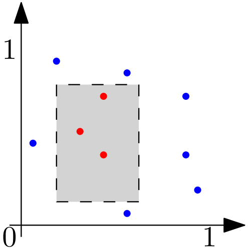

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- The algorithm $\mathcal{A}$ cannot guess the rectangle R with perfect accuracy if the data it receives in input are too few. As the data in D grow in number, we would expect the rectangle $f_{\mathcal{A}}(D)$ to converge to R, wouldn't we?

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- $\mathcal{A}$ is an ordinary algorithm.
  - Ultimately, it can be seen as a TM.
  - We thus assume that real numbers can be appropriately approximated as binary strings.
  - In some cases, it is useful to assume $\mathcal{A}$ to have the possibility to "flip a coin", i.e., to be a randomized algorithm.

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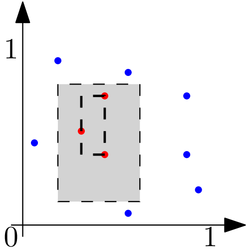

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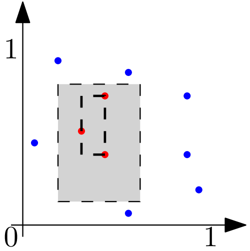

- The output classifier is a rectangle, which can be easily represented as a pair of coordinates.

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- As the number of samples in D grows, the result $\mathcal{A}_{\textsf{BFP}}(D)$ does *not* necessarily approach the target rectangle, but its probability of error approaches zero.

Theorem

For every distribution $\mathbf{D}$, *for every* $0 < \varepsilon < \frac{1}{2}$ *and for every* $0 < \delta < \frac{1}{2}$, *if* $m \geq \frac{4}{\varepsilon} \ln\left(\frac{4}{\delta}\right)$, *then*

$$\Pr_{D\sim\mathbf{D}^m}[error_{\mathbf{D},T}(\mathcal{A}_{\textsf{BFP}}(T(D)) < \varepsilon] > 1 - \delta$$

The General Model — Terminology

- We assume to work within an **instance space** X.
  - X is the set of (encodings) of instances of objects the learner wants to classify.
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- **Concepts** are subsets of X, i.e. collections of objects. These should be thought of as properties of objects.
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  - In the example, concepts are arbitrary subsets of $X = \mathbb{R}^2_{[0,1]}$, i.e. arbitrary regions within $\mathbb{R}^2_{[0,1]}$.
- A **concept class** $\mathcal{C}$ is a collection of concepts, namely a subset of $\mathcal{P}(X)$. These are the concepts which are sufficiently simple to describe, and that algorithms can handle.
  - The concept class $\mathcal{C}$ we work with in the example is the one of rectangles whose sides are parallel to the axes.
  - The **target concept** $c \in \mathcal{C}$ is the concept the learner wants to build a classifier for.

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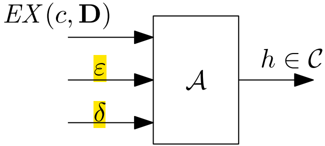

where:

  - ε is **error parameter**, while δ is the **confidence parameter**;
  - $EX(c, \mathbf{D})$ should be though as an *oracle*, a procedure that $\mathcal{A}$ can call as many times she wants, and which returns an element $x \sim \mathbf{D}$ from X, labelled according to whether it is in c or not.

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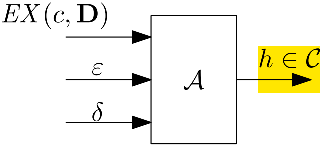

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- The **error** of any $h \in \mathcal{C}$ is defined as $error_{\mathbf{D},c} = \Pr_{x \sim \mathbf{D}}[h(x) \neq c(x)]$.

The General Model — PAC Concept Classes

- Let $\mathcal{C}$ be a concept class over the instance space X. We say that $\mathcal{C}$ is **PAC learnable** iff there is an algorithm $\mathcal{A}$ such that for every $c \in \mathcal{C}$, for every distribution $\mathbf{D}$, for every $0 < \varepsilon < \frac{1}{2}$ and for every $0 < \delta < \frac{1}{2}$, then

$$\Pr[error_{\mathbf{D},c}(\mathcal{A}(EX(c,\mathbf{D}),\varepsilon,\delta)) < \varepsilon] > 1 - \delta$$

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Corollary

The concept-class of axis-aligned rectangles over $\mathbb{R}^2_{[0,1]}$ is efficiently PAC-learnable.

Thank You!

Questions?