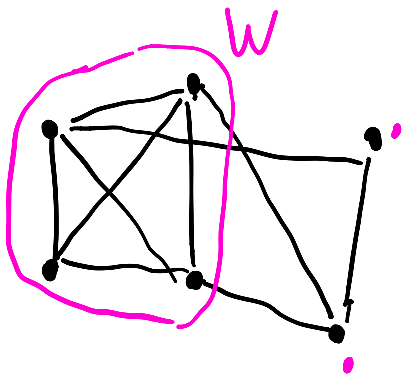
EXERCISE

Suppose CLIQUE is the following set:

$$\text{CLIQUE} = \{(G, k) \mid G \text{ is an undirected graph containing a clique of size at least } k\}$$

a clique in an (undirected) graph is a set of vertices $W \subseteq V$ such that for every $v, w \in W$ $\{v, w\} \in E$. As an example:

$G =$ 

$(G, 4) \in \text{CLIQUE}$

$(G, 3) \in \text{CLIQUE}$

$(G, 5) \notin \text{CLIQUE}$

Prove that CLIQUE is NP-complete by showing that $3SAT \leq_p \text{CLIQUE}$

Hint: consider any 3CNF F as a graph whose vertices are the occurrences of literals in F.

SOLUTION

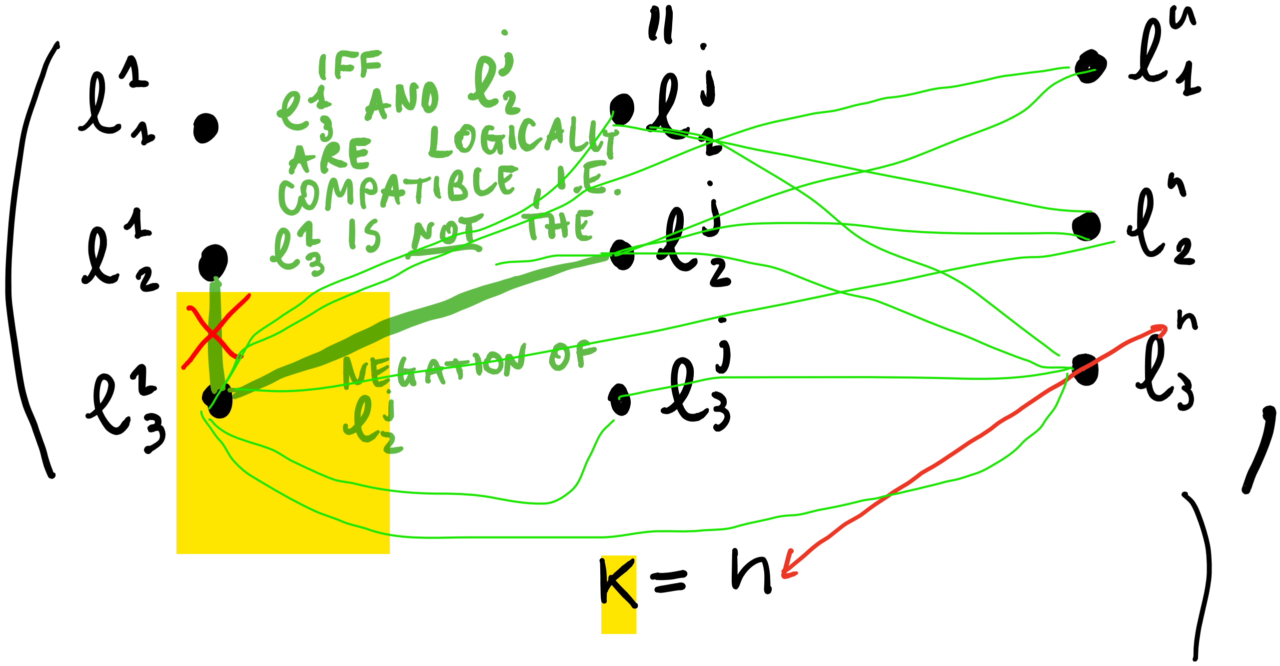
- The fact that CLIQUE is in NP is easy to prove: the certificate is nothing more that $W \subseteq V$ because:

  → Its size (i.e. its cardinality) is smaller that the one of V

→ Checking that W is indeed a clique of size $\geq$ to k can be done in polynomial time: it suffices to check that any pair of distinct vertices $v, w \in W$ are connected by an edge.

- About completeness, we define a function f such that $f(F)$ is a pair (G, k) such that G has a clique of cardinality at least k iff F is satisfiable. f is defined as follows:

$$f\left((l_1^1 \vee l_2^1 \vee l_3^1) \wedge \dots \wedge (l_1^n \vee l_2^n \vee l_3^n) \right) =$$



- IN other words there is an edge between l_j^i and l_q^s iff $i \neq s$ and l_j^i and l_q^s are logically compatible.
- Let's prove that $f(F) \in$ CLIQUE iff F is satisfiable

$\Rightarrow$ Suppose that $f(F) \in$ CLIQUE, then G has a clique of size at least n. By the way we have built G, it's clear that this implies that n literals $l_{i_1}^1, \dots l_{i_n}^n$ are logically compatible with each other. To each variable x, then, we can assign a truth value in such a way that $l_{i_1}^1, \dots, l_{i_n}^n$ are all true in this assignment. As a consequence F itself is true in this assignmen (i.e. $[\![F]\!]_\rho = 1$, where ρ is the assignment)

$\Leftarrow$ Suppose that $[\![F]\!]_\rho = 1$, namely that ρ satisfies F. It means that all the clauses in F are true in ρ and that, as a consequence, one literal $l^j_{i_j}$ is true for each clause C_j in F. The fact that the $\{l^j_{i_j}\}_j$ are true in the same assignment implies that these literals $l^1_{i_1}, \dots, l^n_{i_n}$ are all logically compatible. As a consequence they form a clique of size n in G. In other words $f(F) \in$ CLIQUE.

EXERCISE 2

We know that SAT and 3SAT are both NP-complete problems. How about 2SAT, namely

$$2SAT = \{ \llcorner F \lrcorner \mid F \text{ is a satisfiable 2CNF} \} ?$$

Hint: prove that 2SAT is in P by seeing any clause $C = l_1 \vee l_2$ in a 2CNF F as an implication, namely $\neg l_1 \Rightarrow l_2$ or $\neg l_2 \Rightarrow l_1$.

SOLUTION.

- Of course $l_1 \vee l_2$ is logically equivalent to $\neg l_1 \Rightarrow l_2$ or $\neg l_2 \Rightarrow l_1$ because the connective $\Rightarrow$ can be expressed in terms of $\vee$

$$(\neg l_1 \Rightarrow l_2) \sim (\neg\neg l_1 \vee l_2) \sim l_1 \vee l_2$$

$$(\neg l_2 \Rightarrow l_1) \sim (\neg\neg l_2 \vee l_1) \sim l_1 \vee l_2$$

- This way, you can build a graph whose nodes are the literals involving the variables which occur in any 2CNF F. The edges are exactly those we can derive from F following the interpretation of clauses as implications.

EXAMPLES

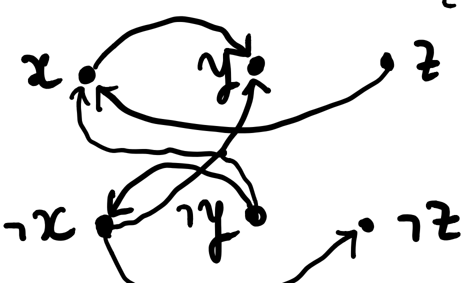
$$F_1 = (\neg x \vee y) \wedge (x \vee \neg z) \wedge (x \vee y) \qquad x \mapsto 1,\ y \mapsto 1,\ z \mapsto 1$$

$$F_2 = \underbrace{(x \vee \neg y)}_{y \text{ CANNOT HOLD}} \wedge \underbrace{(x \vee y)}_{y \text{ MUST HOLD}} \wedge \underbrace{(\neg x \vee \neg x)}_{x \text{ MUST BE FALSE}}$$

$\neg x \vee y$: $x \Rightarrow y$, $\neg y \Rightarrow \neg x$

$x \vee y$: $\neg x \Rightarrow y$, $\neg y \Rightarrow x$

$x \vee \neg z$: $\neg x \Rightarrow \neg z$, $z \Rightarrow x$

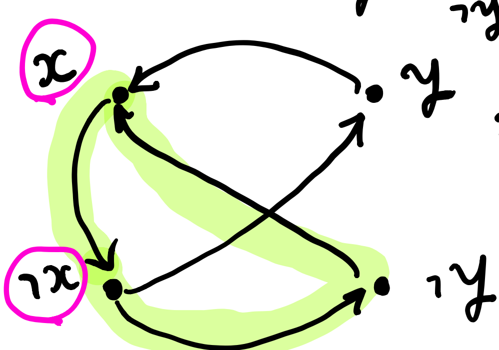


$\mathbb{G}_1$

$x \vee \neg y$: $\neg x \Rightarrow \neg y$, $y \Rightarrow x$

$x \vee y$: $\neg x \Rightarrow y$, $\neg y \Rightarrow x$

$\neg x \vee \neg x$: $x \Rightarrow \neg x$



$\mathbb{G}_2$

- We can check, then, whether a 2CNF F is satisfiable by constructing the corresponding graph, and checking whether the graph includes a loop crossing BOTH literals in which a variable occurs.
- We don't have time to complete the exercise, but we should prove that:
  - → Looking for loops in graph can be done in polynomial time!
  - → We also need to be sure that the graph for F does not contain a loop iff F is satisfiable.

EXERCISE 3

Suppose that $\mathcal{L}_1, \mathcal{L}_2 \in NP$. What can we say about $\mathcal{L}_1 \cap \mathcal{L}_2$? Is it that $\mathcal{L}_1 \cap \mathcal{L}_2 \in NP$? Please prove your claim. Can we say the same for $\mathcal{L}_1 \cup \mathcal{L}_2$?