Optimization

A gentle introduction

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Linear programming

Linear Programming

When both objective function and constraints are linear function, optimization is named linear programming.

Essentially a tool for optimal allocation of scarce resources, among a number of competing alternatives.

Powerful model: generalizes many classic problems (shortest path, max flow, MST, matching, …)

Ranked among most important scientific advances of 20^{th} century.

Helps finding "good" solutions to NP-hard optimization problems.

- optimal solutions (branch-and-bound/cut/prize/…)
- provably good and heuristic solutions

Example: Brewery Problem*

A small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit (€)
Ale	5	4	35	13
Beer	15	4	20	23
Available quantity	480	160	1190	

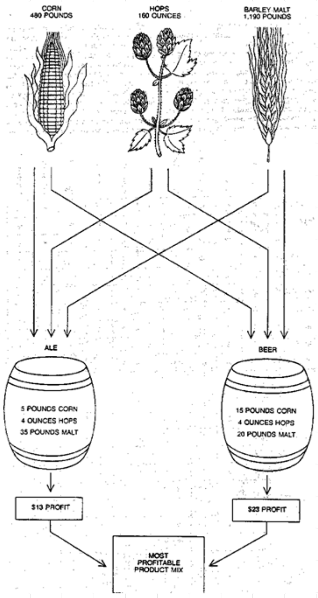
How can brewer maximize profits?

- Devote all resources to beer: 32 barrels of beer ⇒ € 736.
- Devote all resources to ale: 34 barrels of ale ⇒ € 442.
- 7½ barrels of ale, 29½ barrels of beer ⇒ € 776.
- 12 barrels of ale, 28 barrels of beer ⇒ € 800.

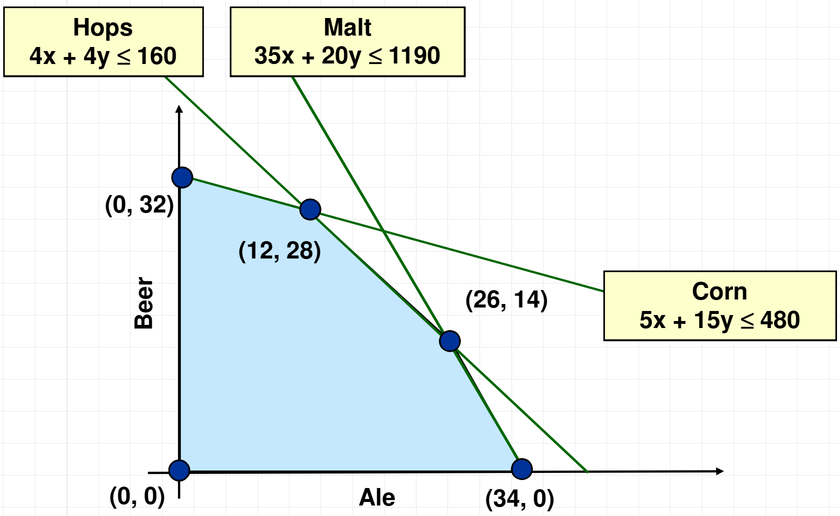
** Standard toy problem, cannot trace the origin (must be before 2002)*

Brewery Problem

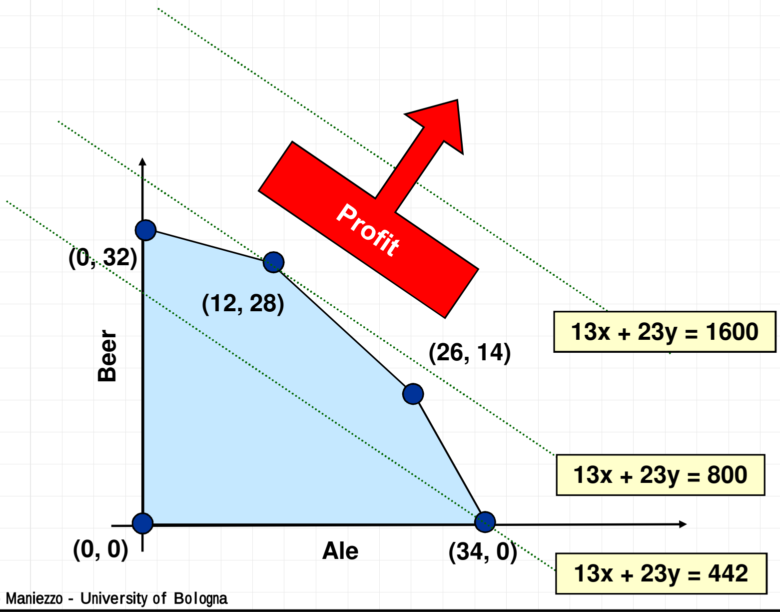
$$
\begin{array}{llrrcr}
\text{max} & 13\,x & +23\,y & & \\
\text{s.t.} & 5\,x & +15\,y & \leq & 480 \\
 & 4\,x & +4\,y & \leq & 160 \\
 & 35\,x & +20\,y & \leq & 1190 \\
 & x, y & & \geq & 0
\end{array}
$$



Brewery Problem: Feasible Region

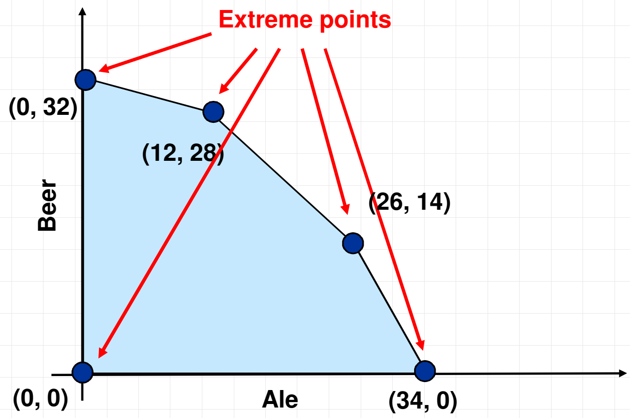


Brewery Problem: Objective Function



Brewery Problem: geometry

Brewery problem observation. Regardless of coefficients of linear objective function, there exists an optimal solution that is an extreme point.



Linear Programming

LP “canonical" form.

- Input data: rational numbers c_j, b_i, a_{ij} .
- Max/min-imize linear objective function.
- Subject to linear inequalities.

$$
\begin{array}{llrcll}
\textbf{(P)} & \max & \sum_{j=1}^{n} c_j x_j & & & \\
 & \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & \le & b_i & 1 \le i \le m \\
 & & x_j & \ge & 0 & 1 \le j \le n
\end{array}
$$

$$
\begin{array}{llrcl}
\textbf{(P)} & \max & c \bullet x & & \\
 & \text{s.t.} & Ax & \le & b \\
 & & x & \ge & 0
\end{array}
$$

Formulations

The two most common formulations. Inequalities can be turned into equalities by means of slack variables.

<u>Canonical form</u>

$$
\begin{array}{ll}
\textit{min} & c^T x \\
\textit{s.t.} & Ax \ge b \\
 & x \ge 0
\end{array}
$$

slack variables →

<u>Standard form</u>

$$
\begin{array}{ll}
\textit{min} & c^T x \\
\textit{s.t.} & Ax = b \\
 & x \ge 0
\end{array}
$$

e.g.

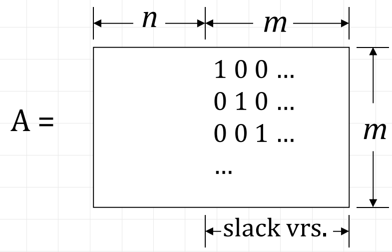
$$
\begin{array}{l}
7x_1 + 5x_2 \ge 7 \\
x_1, x_2 \ge 0
\end{array}
\xrightarrow{y_1}
\begin{array}{l}
7x_1 + 5x_2 - y_1 = 7 \\
x_1, x_2, y_1 \ge 0
\end{array}
$$

Equivalent, but deals only with equalities

Slack as matrices

If before adding the slack variables A has size m x n then after it has size m x $(n + m)$

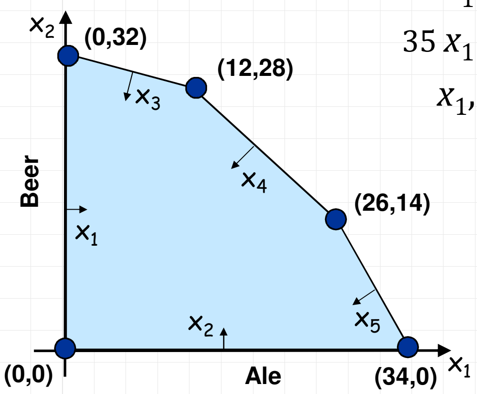
m can be larger or smaller than n



Assuming rows to be independent, the solution space of Ax = b is a n dimensional subspace.

Example, with slacks

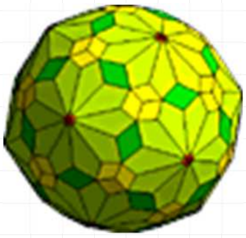
$$\begin{aligned} &min\ z = -13\, x_1 - 23\, x_2 \\ &s.t. \quad 5\, x_1 + 15\, x_2 + x_3 = 480 \\ &\quad\quad 4\, x_1 + 4\, x_2 + x_4 = 160 \\ &\quad\quad 35\, x_1 + 20\, x_2 + x_5 = 1190 \\ &\quad\quad x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$



Geometric View

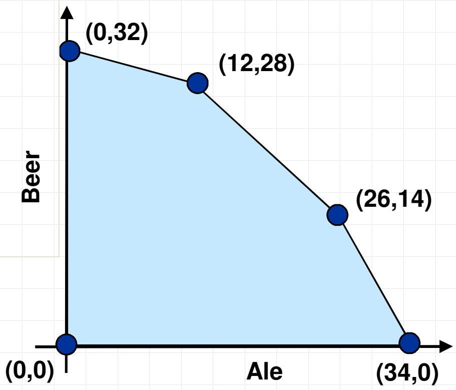
A polyhedron in n-dimensional space

- Each inequality corresponds to a half-space.
- The "feasible set" is the intersection of the half-spaces.
- This corresponds to a polyhedron (a "simplex", it is the *convex hull of its vertices*)
- An optimal solution is at a vertex.



Geometric View

$$
\begin{array}{llll}
min\ z = & -13x_1 & -23x_2 & \\
\text{s.t.} & 5x_1 & + 15x_2 & \leq 480 \\
 & 4x_1 & + 4x_2 & \leq 160 \\
 & 35x_1 & + 20x_2 & \leq 1190 \\
 & & x_1, x_2 \geq 0 &
\end{array}
$$



The feasible region is an intersection of 5 constraints (halfspaces)

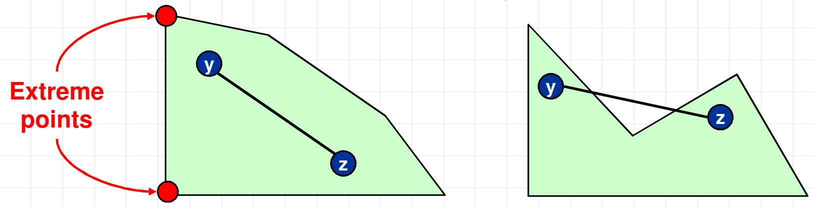
LP: Geometry

Geometry.

- Forms an n-dimensional polyhedron.

$$
\begin{array}{llrcll}
(P) & \max & \sum_{j=1}^{n} c_j x_j & & & \\
& \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & \le & b_i & 1 \le i \le m \\
& & x_j & \ge & 0 & 1 \le j \le n
\end{array}
$$

- Convex: if y and z are feasible solutions, then so is any $\alpha y + \beta z$ ($\alpha, \beta > 0$).
- Extreme point: a feasible solution x that can't be written as $\alpha y + \beta z$ for any two distinct feasible solutions y and z.



Convex   Not convex

LP: Geometry

Extreme Point Theorem. If there exists an optimal solution to standard form LP (P), then there exists one that is an extreme point.

- Only need to consider finitely many possible solutions.

Greedy. Local optima are global optima.

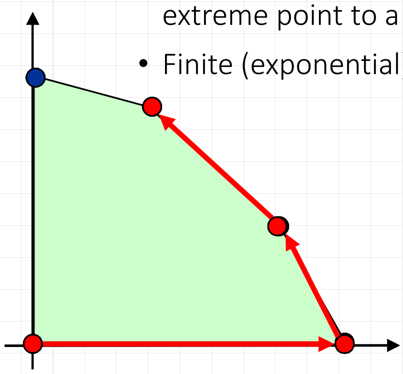
Two classes of solution methods:

- Simplex algorithms move on the surface of the polyhedron
- Interior-point algorithms move within the polyhedron

LP: Algorithms

Simplex. *(Dantzig 1947)*

- Developed shortly after WW2 in response to logistical problems: used for 1948 Berlin airlift.
- Practical solution method that moves from one extreme point to a neighboring extreme point.
- Finite (exponential) complexity (even though LP is in P).



LP: Polynomial Algorithms

Ellipsoid. (Khachian 1979, 1980)

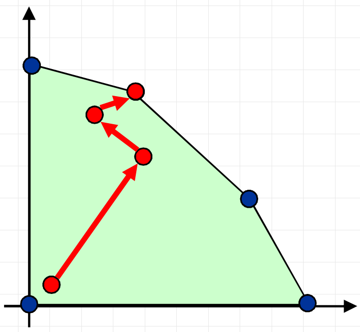
- Solvable in polynomial time: $O(n^4 L)$ bit operations.
  - n = # variables
  - L = # bits in input
- Theoretically important, not practical.

Karmarkar's algorithm. (Karmarkar 1984)

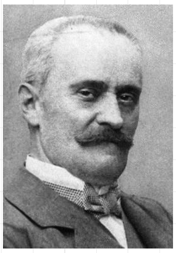
- $O(n^{3.5} L)$.
- Polynomial and reasonably efficient implementations possible.

Interior point algorithms.

- $O(n^3 L)$.
- Often dominates simplex.
- Extends to even more general problems.



Farkas' lemma (1894)



https://aardvarks.bandcamp.com/track/farkas-lemma

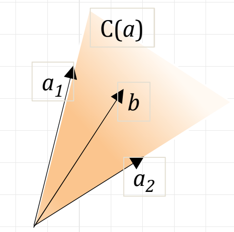
Consider a linear equation system $\mathrm{A}\mathrm{x}=\mathrm{b}$, where A is a general mxn matrix and x and b are respectively an n-dimensional and an m-dimensional vectors.

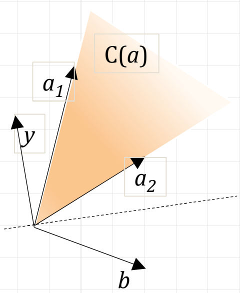
Farkas' lemma states that one but not both of the following two statements is true:

1. there is a vector $\mathrm{x} \in \mathbb{R}^n$ solving the equation with $\mathrm{x} \geq 0$;
2. there is a vector $\mathrm{y} \in \mathbb{R}^m$ satisfying $\mathrm{A'y} > 0$ and $\mathrm{b'y} < 0$ (where 0 is the zero vector).

(intuitions follow …)

Farkas' lemma, geometry





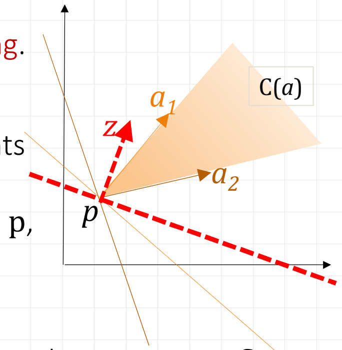
Geometrically, this is equivalent to saying that:
1) b is in the cone $\mathrm{C(A)}$, spanned by A, or
2) not.

1. Point 1 requires directly that b is in $\mathrm{C(A)}$, if we require $\mathrm{x} \geq 0$.
2. b is not in $\mathrm{C(A)}$: there is a separating hyperplane between $\mathrm{C(A)}$ and b. We can thus find a vector y such that $\mathrm{y}'a_i \geq 0$, $i = 1, \dots, m$, and $\mathrm{y'b} < 0$ such that the hyperplane $\mathrm{y'x} = 0$ separates b from $a_1, \dots, a_m$. i.e., b has an angle with y greater than $\pi/2$.

Farkas' lemma, linear programming

Immediate application to linear programming.

If we are interested in minimizing the linear objective $z(x)$ subject to the linear constraints $a_i x \geq b_i$ for $i \in I$,
suppose that we have found a feasible point p,
and wish to know if there is a direction y
for which we can improve the objective.



Then if A gives the indices of those constraints that are active, C gives us the set of o.f. augmenting directions, and Farkas' lemma tells us that improvement will be possible if and only if z is not in C.

Farkas' lemma, algebra

Consider the equation $a_1x_1 + a_2x_2 + \ ... + a_nx_n = b$.
If x≥0 then $\boldsymbol{a}$ and b cannot be general. If we have all the a_i>0: then b>0.
If b<0 and a>0 there exist a number $y = b/\sum a_i$ solves the equation, there cannot be b<0. it is impossible to choose all the x_i positive and have them satisfy the equation.

This generalizes to linear combination of a system of equations

$$\begin{matrix} a_{11}x_1 & \cdots & a_{1n}x_n \\ \vdots & \ddots & \vdots \\ a_{m1}x_1 & \cdots & a_{mn}x_n \end{matrix} \quad = \quad \begin{matrix} b_1 \\ \vdots \\ b_m \end{matrix}$$

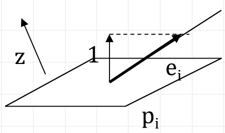
No sol of $(\sum_i y_i a_{i1})x_1 + (\sum_i y_i a_{i2})x_2 + \cdots + (\sum_i y_i a_{in})x_n = \sum_i y_i b_i$ asking x_i≥0 with positive coefficients y_i and a negative number on the right-hand side.

Now, what Farkas' Lemma says is that this is in fact the only bad thing that can happen. If the system Ax=b doesn't have a solution x≥0, then this must be because there exists a nasty linear combination of some equations (yTA)x=yTb
that obstructs solvability because it satisfies yTA≥0 but yTb≤0
In a sense, Farkas' Lemma shows that the non-solvability of the system can be "certified"; the vector y is a "certificate" for the fact that the system cannot be solved.
The system Ax=b is infeasible iff using positive linear combinations of the inequalities there existλ1,λ2,...λm≥0 such that

Optimality and Reduced Cost

The optimal solution must include a vertex.

The reduced cost for a hyperplane at a vertex is the cost of moving one unit away from the plane along its corresponding edge.

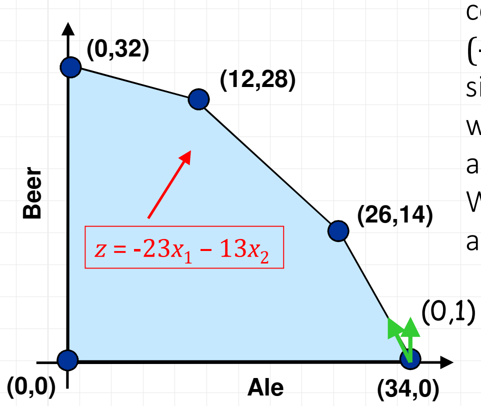


$r_i = z \cdot e_i$

For minimization, if all reduced cost are non-negative, then we are at an optimal solution.

Finding the most negative reduced cost is a heuristic for choosing an edge to leave on.

Reduced cost example

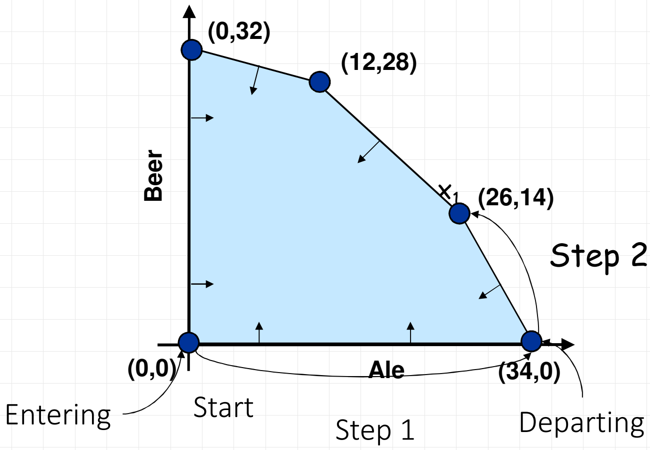


In the example the reduced cost of leaving the plane x_1 is (-23, -13) * (-1.57, 1) = 23.11 since moving one unit off of x_1 will move us (-1.57, 1) units along the edge.
We take the dot product of this and the cost function.

Simplex Algorithm

1. Find a corner of the feasible region
2. Repeat
   2.1 For each of the n hyperplanes intersecting at the corner, calculate its reduced cost
   2.2 If they are all non-negative,
   then STOP
   else, pick the most negative reduced cost
   2.3 Move along corresponding edge until the next corner is reached

Example



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Simplex, rough psudocode

1. Start with the initial basis associated with identity matrix.
2. Calculate the relative profits.

   For MAX problems
   If all the relative profits are less than or equal to 0, then the current basis is the optimal one. STOP.
   else continue to 3.

   For MIN problems
   If all the relative profits are greater than or equal to 0, then the current basis is the optimal one. STOP.
   else continue to 3.
3. Find the column corresponding to max relative profit.
   Let column j have the max rel. profit: xj will enter the basis.
4. Perform a min ratio test ONLY on positive elements to determine which variable will leave the basis.
   min ratio test: b_r/a_{rj} = min_i{ b_i/a_{ij} }
   The index of the min element, r, is the leaving row.
   The variable at index r of the basic variables vector will leave the basis.
5. Make the identity matrix again.
   The element at index (r,j) will be the pivot element and row r will be the pivot row.
   Divide the r-th row by pivot to make it 1. And subtract c*(rth row) from other rows to make them 0, where c is the coefficient required to make that row 0.

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Simplex, the OR way

First reformulate the problem in standard form and express the o.f. by a dedicated variable.

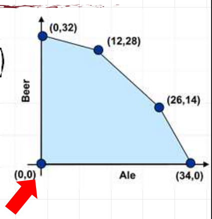
$$
\begin{array}{lrrcr}
\max & 13\,x & +23\,y & & \\
\text{s.t.} & 5\,x & +15\,y & \leq & 480 \\
 & 4\,x & +4\,y & \leq & 160 \\
 & 35\,x & +20\,y & \leq & 1190 \\
 & x, y & & \geq & 0
\end{array}
$$

Becomes (with inverted sign)

$$
\begin{array}{llllllll}
\min z & & & & & & & \\
s.t. & z & +13\,x_1 & +23\,x_2 & & & & = 0 \\
 & & 5\,x_1 & +15\,x_2 & +x_3 & & & = 480 \\
 & & 4\,x_1 & +4\,x_2 & & +x_4 & & = 160 \\
 & & 35\,x_1 & +20\,x_2 & & & +x_5 & = 1190 \\
 & & \multicolumn{6}{l}{x_1, x_2, x_3, x_4, x_5 \geq 0}
\end{array}
$$

Simplex, the OR way

Next, represent the problem by a table (its *tableau*)



$$
\begin{array}{llllllll}
s.t. & z & +13\,x_1 & +23\,x_2 & & & & = 0 \\
 & & 5\,x_1 & +15\,x_2 & +x_3 & & & = 480 \\
 & & 4\,x_1 & +4\,x_2 & & +x_4 & & = 160 \\
 & & 35\,x_1 & +20\,x_2 & & & +x_5 & = 1190
\end{array}
$$

	z	x_1	x_2	x_3	x_4	x_5	
r_0^0	1	13	23	0	0	0	0
r_1^0	0	5	15	1	0	0	480
r_2^0	0	4	4	0	1	0	160
r_3^0	0	35	20	0	0	1	1190

Will hold the relative profits

Solution cost (inverted sign)

Solution values (initially RHS)
Nonbasic variables are at 0

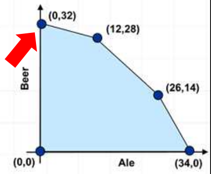
Highest value (will enter the basis)

Min ratio (will pivot here)
480/15=32, 160/4-40, 1190=20=59.5

Basis (solution) is in the identity matrix

Simplex, the OR way

Intermediate non optimal basis



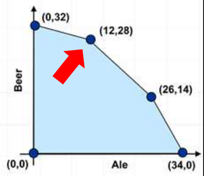
	z	x_1	x_2	x_3	x_4	x_5		
r_0^1	1	16/3	0	-23/15	0	0	-736	$r_0^0 - 23r_1^1$
r_1^1	0	1/3	1	1/15	0	0	32	$r_1^0/15$
r_2^1	0	8/3	0	-4/15	1	0	32	$r_2^0 - 4r_1^1$
r_3^1	0	85/3	0	-20/15	0	1	550	$r_3^0 - 20r_1^1$

Highest value (will enter the basis)

Min ratio (will pivot here)
32/(1/3)=96, **32/(8/3)=12**,
550/(85/3)=19.4

Simplex, the OR way

The final tableau



No positive values. Optimal!

	z	x_1	x_2	x_3	x_4	x_5		
r_0^2	1	0	0	-1	-2	0	-800	$r_0^1 - 16/3r_2^2$
r_1^2	0	0	1	-1/10	-1/8	0	28	$r_1^1 - 1/3r_2^2$
r_2^2	0	1	0	-1/10	3/8	0	12	$r_2^1/(8/3)$
r_3^2	0	0	35	15/10	-85/8	1	210	$r_3^1 - 85/3r_2^2$

Duality

Every LP problem has another special LP problem associated with it. The original problem is named the primal, the second one the dual. Actually, they are duals of one another. Duality is important:

- to determine the optimality of a solution
- to determine the sensitivity of a solution to (small) changes of the problem parameters. The dual variable on a constraint represents the incremental change in the optimal solution value per unit increase in the RHS of the constraint.
- Identifying near-optimal solutions: a good dual solution can be used to bound the values of primal solutions, and it can be used to identify when a primal solution is nearoptimal.
- Karush-Kuhn-Tucker conditions: the optimal solution to the dual problem is a vector of KKT multipliers.
- Convergence of improvement algorithms: the dual problem can be used in the convergence analysis of algorithms.
- Good structure: often, the dual problem has some good mathematical, geometric, or computational structure that can exploited in computing solutions to both the primal and the dual problem.
- ...

Duality

Duality arises in nonlinear (and linear) optimization models in a wide variety of settings. Some immediate examples of duality are in:

- Models of electrical networks The current flows are “primal variables" and the voltage differences are the “dual variables" that arise in consideration of optimization (and equilibrium) in electrical networks.
- Models of economic markets In these models, the “primal" variables are production levels and consumption levels, and the “dual" variables are prices of goods and services.
- Models in structural design. In these models, the tensions on the beams are “primal" variables, and the nodal displacements are the “dual" variables.
- ...

Duality

Given a linear program in standard form

min $c'x$
subject to $Ax = b$
$x \geq 0$

we can define a function - the Lagrangian - which has the form $\mathcal{L}(x,y,z) = c'x + y'(Ax - b) - z'x$

where $y \in \mathbb{R}^m$, $z \in \mathbb{R}^n$ are called dual variables for the constraints $Ax = b$ and $x \geq 0$ respectively

Duality

Note that

$$\max_{y,z\geq 0} \mathcal{L}(x,y,z) = \begin{cases} c'x & if\ Ax = b, x \geq 0 \\ \infty & otherwise \end{cases}$$

We can write the original optimization problem (called the primal problem) as

$$\min_{x} \max_{y,z\geq 0} \mathcal{L}(x,y,z)$$

The constraints of the standard form problem (the primal) are expressed using the min/max sequence.

We can also flip the order of the min/max and obtain what is called the dual problem

$$\max_{y,z\geq 0} \min_{x} \mathcal{L}(x,y,z)$$

Weak and strong duality

Let p^* be the optimal solution to the primal and d^* that of the dual. We have the following (weak duality)

$$d^* = \max_{y,z\geq 0} \min_{x} \mathcal{L}(x,y,z) \leq \min_{x} \max_{y,z\geq 0} \mathcal{L}(x,y,z) = p^*$$

(if we maximize over y, z first and then minimize over x we get a lower value than if we minimize over x first and then maximize over y, z)

I.e., any feasible solution to the dual problem provides a lower bound to the cost of the optimal primal solution

Note: for linear programs, we have $p^* = d^*$ (strong duality)

The simplex algorithm, gives us the solutions of both primal and dual problems

Weak and strong duality

Assuming primal and dual to have both feasible solutions:

- Weak duality (min problem): the cost of any feasible solution of the dual is always less or equal to the cost of any feasible solution of the primal (duality gap).
- Strong duality: the values of the optimal solutions to the primal and dual are always equal.

Any primal solution gives an upper bound to the dual.

Any dual solution gives a lower bound to the primal.

Feasible primal cost equal to feasible dual cost is a certificate of optimality.

Obtaining the dual

Given primal LP problem $min\, cx\ s.t.\{A\boldsymbol{x}\ \geq \boldsymbol{b}, \boldsymbol{x}\ \geq 0\}$ we want the tightest lower bound: the highest value guaranteed to be less or equal to the primal optimal cost.

1) Associate a variable $y_i\ \geq 0$ with each constraint i=1, … m and multiply each constraint $A^i\boldsymbol{x}\ \geq b_i$ by the corresponding coefficient, to obtain $y_i A^i x\ \geq y_i b_i$, valid provided **y**≥0.
2) Sum over i: $\sum_i y_i A^i\, x\ \geq \sum_i y_i b_i$ => **y**A**x** ≥ **yb**
3) Look for **y** such that **yb** is as stringent as possible, but still a lower bound **yb ≤ cx**
4) Since **yb ≤ cx** and **yb ≤ y**A**x**, set **y**A=**c** and maximize **yb**
5) We have the LP dual: *max* **yb** *s.t.* {**y**A=**c**, **y**≥0}.

Obtaining the dual (2nd way)

Obtaining the dual via KKT conditions.

	Primal	Lagrangian multipliers
min	cx	
s.t.	Ax ≥ b	y
	x ≥ 0	μ

Lagrangian $\mathcal{L}$(x, y, μ) = cx + μ(0-x) + y(b - Ax)

Lagrangian opt (saddle): $\max_{y,\mu} \min_{x} (c-\mu-yA)x+yb$

KKT for x optimality:

a) $\frac{\partial L}{\partial x} = c-\mu-yA=0$ x optimality

b) μx=0, y(b-Ax)=0 complementary slackness

c) μ,y ≥ 0 coming from inequality constraints

Obtaining the dual (2nd way)

Lagrangian opt subject to a) and c)

$$\max_{y,\mu} \min_{x} (c - \mu - yA)x + yb$$

$s.t.\ c - \mu - yA = 0$, μ,y ≥ 0

Becomes

$$\max_{y,\mu} yb$$

s.t. $yA + \mu = c$, μ,y ≥ 0

Interpreting μ as slacks:

max **yb** *s.t.* {**y**A≤**c**, **y**≥0}, the LP dual.

Primal-dual correspondences

Primal	Dual
variables $x_1, \ldots, x_n$	n constraints
m constraints	variables $y_1, \ldots, y_m$
objective function $\mathbf{c}$	right-hand side $\mathbf{c}$
right-hand side $\mathbf{b}$	objective function $\mathbf{b}$
$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{b}^T\mathbf{y}$
constraint matrix $\mathbf{A}$	constraint matrix $\mathbf{A}^T$
ith constraint is "$\leq$"	$y_i \geq 0$
ith constraint is "$\geq$"	$y_i \leq 0$
ith constraint is "$=$"	$y_i \in \mathbb{R}$
$x_j \geq 0$	jth constraint is "$\geq$"
$x_j \leq 0$	jth constraint is "$\leq$"
$x_j \in \mathbb{R}$	jth constraint is "$=$"

LP Duality

Primal problem.

$$\begin{array}{llllllllll} \max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\ \text{s.t.} & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \le & 1 \\ & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \le & 55 \\ & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \le & 3 \\ & x_1 & , & x_2 & , & x_3 & , & x_4 & \ge & 0 \end{array}$$

Dual problem.

$$\begin{array}{llllllll} \min & y_1 & + & 55y_2 & + & 3y_3 & & \\ \text{s.t.} & y_1 & + & 5y_2 & - & y_3 & \ge & 4 \\ & -y_1 & + & y_2 & + & 2y_3 & \ge & 1 \\ & -y_1 & + & 3y_2 & + & 3y_3 & \ge & 5 \\ & 3y_1 & + & 8y_2 & - & 5y_3 & \ge & 3 \\ & y_1 & , & y_2 & , & y_3 & \ge & 0 \end{array}$$

LP Duality

Primal and dual linear programs: given rational numbers a_{ij}, b_i, c_j, find values x_i, α_j that optimize (P) and (D).

$$\begin{array}{lllllll} \text{(P)} & \max & \sum_{j=1}^{n} c_j x_j & & & \\ & \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j & \le & b_i & 1 \le i \le m \\ & & x_j & \ge & 0 & 1 \le j \le n \end{array}$$

$$\begin{array}{lllllll} \text{(D)} & \min & \sum_{i=1}^{m} b_i \alpha_i & & & \\ & \text{s.t.} & \sum_{i=1}^{m} a_{ij} \alpha_i & \ge & c_j & 1 \le j \le n \\ & & \alpha_i & \ge & 0 & 1 \le i \le m \end{array}$$

Duality Theorem (Dantzig-von Neumann 1947).

If (P) and (D) are nonempty then max = min.

- Dual solution provides certificate of optimality.
- Sensitivity analysis.

LP Duality: Economic Interpretation

Brewer's problem: find optimal mix of beer and ale to maximize profits.

$$\begin{array}{llrcrcr} \text{(P)} & \max & 13A & + & 23B & & \\ & \text{s. t.} & 5A & + & 15B & \leq & 480 \\ & & 4A & + & 4B & \leq & 160 \\ & & 35A & + & 20B & \leq & 1190 \\ & & A & , & B & \geq & 0 \end{array}$$

A* = 12
B* = 28
OPT = 800

Entrepreneur's problem: buy individual resources from brewer (assume he’s the only one who can sell them) at minimum cost.

- C, H, M = unit price for corn, hops, malt.
- Brewer won't agree to sell resources if 5C + 4H + 35M < 13

$$\begin{array}{llrcrcrcr} \text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\ & \text{s. t.} & 5C & + & 4H & + & 35M & \geq & 13 \\ & & 15C & + & 4H & + & 20M & \geq & 23 \\ & & C & , & H & , & M & \geq & 0 \end{array}$$

C* = 1
H* = 2
M* = 0
OPT = 800

LP Duality: Economic Interpretation

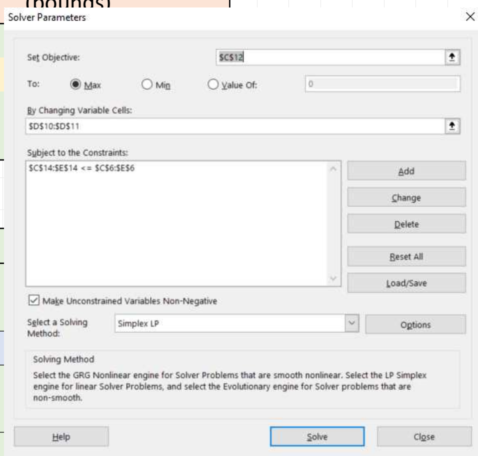
Sensitivity analysis.

- How much should brewer be willing to pay (marginal price) for additional supplies of scarce resources?
  - corn € 1, hops € 2, malt € 0.
- Suppose a new product "light beer" is proposed. It requires 2 corn, 5 hops, 24 malt. How much profit must be obtained from light beer to justify diverting resources from production of beer and ale?
  - At least 2 (€ 1) + 5 (€ 2) + 24 (0€) = € 12 / barrel.

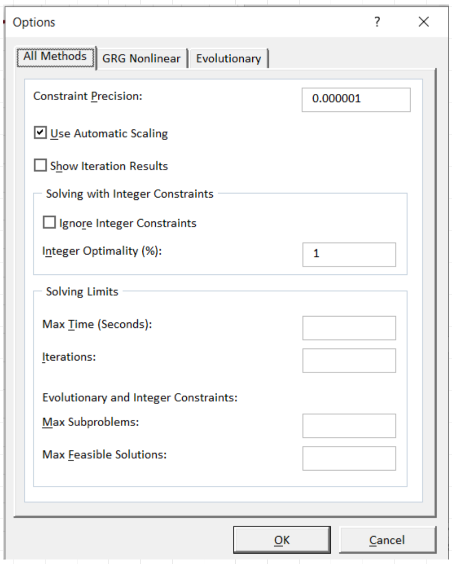
Excel Formulation (brewer probl.)

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit (€)
Ale	5	4		
Beer	15	4		
Available quantity	480	160		

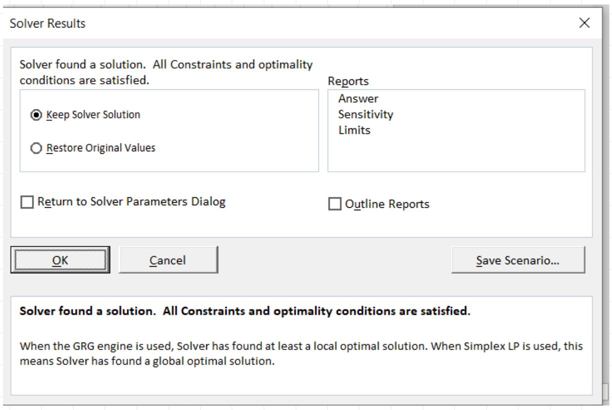
Production		
	Ale	12
	Beer	28
Revenue	800	
Resource Needs	Corn	Hops
	480	160



Excel Solver



Excel Solver



Press the Ctrl key and use the mouse to select both the Answer and Sensitivity Reports

Python

Several possibilities for solving LP in python:

- scipy.optimize.linprog (https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.linprog.html)
- PuLP (https://pythonhosted.org/PuLP/)
- CPLEX (https://pypi.org/project/cplex/)
- … and more

We will be using PuLP

Brewery, PuLP

```python
import pulp
# prob contains the problem data, problem defined as min
prob = pulp.LpProblem("The Brewery Problem",pulp.LpMinimize)
# The 2 variables Beer and Alre are created with a lower limit of zero
x1=pulp.LpVariable("Beer",0)
x2=pulp.LpVariable("Ale",0)
# The objective function is min, profits are negative
prob += -23*x1 -13*x2, "Total profit per unit of product"
# Availability constraints
prob += 15*x1 +  5*x2 <=  480, "Corn availability"
prob +=  4*x1 +  4*x2 <=  160, "Hops availability"
prob += 20*x1 + 35*x2 <= 1190, "Malt availability"
# The problem data is written to an .lp file
prob.writeLP("Brewery.lp")
# The problem is solved using PuLP's choice of Solver or
#prob.solve()  # let pulp decide the solver
#prob.solve(CPLEX())
prob.solve(pulp.PULP_CBC_CMD(fracGap = 0.00001, maxSeconds = 500, threads = None))
# The status of the solution
print("Status:", pulp.LpStatus[prob.status])
# The optimal objective function value
print("Total revenue = ", -1*pulp.value(prob.objective))
# Primal and dual variables optimal value
for v in prob.variables():
    print(v.name, "=", v.varValue)
for name, c in list(prob.constraints.items()):
    print(name, ":", c, "\t dual", c.pi, "\tslack", c.slack)
```


PuLP, .lp file

```
\* The Brewery Problem *\
Minimize
Total_profit_per_unit_of_product: - 13 Ale - 23 Beer
Subject To
Corn_availability: 5 Ale + 15 Beer <= 480
Hops_availability: 4 Ale + 4 Beer <= 160
Malt_availability: 35 Ale + 20 Beer <= 1190
End
```

Excel Solver Answer Report

Microsoft Excel 16.0 Answer Report
Worksheet: [Brewery.xlsx]Sheet1
Report Created: 2/5/2020 3:44:49 PM
Result: Solver found a solution. All Constraints and optimality conditions are satisfied.
Solver Engine
Engine: Simplex LP
Solution Time: 0.015 Seconds.
Iterations: 2 Subproblems: 0
Solver Options
Max Time Unlimited, Iterations Unlimited, Precision 0.000001, Use Automatic Scaling
Max Subproblems Unlimited, Max Integer Sols Unlimited, Integer Tolerance 1%, Assume NonNegative

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$C$12	Revenue Beer	0	800

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$D$10	Ale Hops (ounces)	0	12	Contin
$D$11	Beer Hops (ounces)	0	28	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$C$14	Corn	480	$C$14<=$C$6	Binding	0
$D$14	Hops	160	$D$14<=$D$6	Binding	0
$E$14	Malt	980	$E$14<=$E$6	Not Binding	210

Excel Solver Sensitivity Report

Microsoft Excel 16.0 Sensitivity Report
Worksheet: [Brewery.xlsx]Sheet1
Report Created: 2/5/2020 3:44:49 PM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$D$10	Ale Hops (ounces)	12	0	13	10	5.333333333
$D$11	Beer Hops (ounces)	28	0	23	16	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$C$14	Corn	480	1	480	120	140
$D$14	Hops	160	2	160	19.76470588	32
$E$14	Malt	980	0	1190	1E+30	210

Sensitivity Report: changing cells

Reduced cost

Measures change in objective function (target cell per unit increase of that decision variable (changing cell)

Objective Coefficient

Decision variable's coefficient in objective function

Allowable Increase and Allowable Decrease

Delimit range within which a change of the objective function coefficient would not change any decision variable's value.

Interpreting LP Solutions

Objective ranging

For each objective function coefficient, there is an upper and lower boundary range of values within which the optimal solution does not change.

When an objective coefficient's value changes:

- the optimal objective function value, the shadow prices, and the reduced costs change.
- The values of the optimal basic (used in solution) variables do not change.

Objective ranging provides a sensitivity analysis of how the solution changes as we move *past the bounds* of the original solution. In this case, to obtain the exact solution, the model must be re-solved.

Excel solver sensitivity report

Reduced costs and objective ranging

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$3	Ale Quant.	12	0	13	10	5.333333333
$B$4	Beer Quant.	28	0	23	16	10

A management consultant offers to improve efficiency in the production of ale.

This would increase the contribution by € 5 to €18.

QUEST. What is the new mix?

ANS. No change in the product mix.

Sensitivity report: constraints

Final Value (amount of resource used)

Shadow price (of a resource)

- Change in optimal objective function value per unit increase in right-hand-side
- Additional unit of resource is attractive at

  price = (regular price + premium)

  IF premium <= shadow price

Right-hand-side ranging

- Allowable increase and decrease in RHS such that shadow prices remain valid within this range.
- Multiple RHS changes are possible.

Excel Solver Sensitivity Report

Shadow prices and right-hand-side ranging

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$8	Corn Quant.	480	1	480	120	140
$B$9	Hops Quant.	160	2	160	19.76470588	32
$B$10	Malt Quant.	980	0	1190	1E+30	210

How much would you pay for one additional pound of malt?

Nothing: malt is not constraining the solution!

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Excel Solver Sensitivity Report

Shadow prices and right-hand-side ranging

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$B$8	Corn Quant.	480	1	480	120	140
$B$9	Hops Quant.	160	2	160	19.76470588	32
$B$10	Malt Quant.	980	0	1190	1E+30	210

A fault is found on 30 pounds of corn reducing corn from 480 to 450 pounds. How does this affect total contribution?

total contribution - (shadow price x pounds)

€ 800 - € 1/pound x 30 pounds = € 770

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Limits of the sensitivity report

We need to re-solve the LP model when

- We want to know the new distribution of the decision variables that would provide the optimal solution resulting from a change in the constraints.
- We want to know the effect of changing a constraint beyond the constraint's allowable increase/decrease.
- We want to know the effect of changing several constraints.
- We want to know the decision variables values after the objective coefficients change more than the allowable increase/decrease.