Propositional Logic

## Credits

- Part of the content of the following slides is from:
  - van Dalen's book "Logic and structure"
  - Slides by E. Clarke, CMU

## Summary

- Syntax
- Semantics
- Normal forms
- Deduction and refutation
- Natural Deduction

## Syntax

Recursive definition of **well-formed formulas**

1. An atom is a formula
2. If $S$ is a formula, $\neg S$ is a formula (**negation**)
3. If $S_1$ and $S_2$ are formulas, $S_1 \wedge S_2$ is a formula (**conjunction**)
4. If $S_1$ and $S_2$ are formulas, $S_1 \vee S_2$ is a formula (**disjunction**)
5. All well-formed formulas are generated by applying above rules

Shortcuts:

- $S_1 \rightarrow S_2$ can be written as $\neg S1 \vee S2$

## Using BNF

*Backus-Naur Form (BNF)*:

$$
\begin{array}{lcl}
<\text{formula}> & ::= & \text{Atomic Proposition} \\
 & | & \neg <\text{formula}> \\
 & | & <\text{formula}> \wedge <\text{formula}> \\
 & | & <\text{formula}> \vee <\text{formula}> \\
 & | & <\text{formula}> \rightarrow <\text{formula}> \\
 & | & <\text{formula}> \Leftrightarrow <\text{formula}> \\
 & | & (<\text{formula}>)
\end{array}
$$



## Semantics

**Relationships between truth values of atoms and truth values of formulas**

| | | | | | |
|---|---|---|---|---|---|
| $\neg S$ | is true iff | $S$ | is false | | |
| $S_1 \wedge S_2$ | is true iff | $S_1$ | is true **and** | $S_2$ | is true |
| $S_1 \vee S_2$ | is true iff | $S_1$ | is true **or** | $S_2$ | is true |
| $S_1 \rightarrow S_2$ | is true iff | $S_1$ | is false **or** | $S_2$ | is true |
| i.e., | is false iff | $S_1$ | is true **and** | $S_2$ | is false |
| $S_1 \leftrightarrow S_2$ | is true iff | $S_1 \rightarrow S_2$ | is true **and** | $S_2 \rightarrow S_1$ | is true |

## Semantics

**Example (Truth Tables for main logical connectives)**

| $P_1$ | $P_2$ | $\neg P_1$ | $P_1 \wedge P_2$ | $P_1 \vee P_2$ | $P_1 \rightarrow P_2$ | $P_1 \leftrightarrow P_2$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $T$ | $T$ |

## Evaluation of a formula

**Recursive Evaluation**

Consider the formula $G \triangleq \neg P_1 \wedge (P_2 \vee P_3)$
Suppose we know that $P_1 = F,\ P_2 = F,\ P_3 = T$
Then we have

$$\neg P_1 \wedge (P_2 \vee P_3) = \mathit{true} \wedge (\mathit{false} \vee \mathit{true}) = \mathit{true} \wedge \mathit{true} = \mathit{true}$$

**Note**

We evaluate $\neg P_1$ before $P_1 \wedge P_2$, this is because the following decreasing rank for connectives operator holds:

$\leftrightarrow \quad \rightarrow \quad \vee \quad \wedge \quad \neg$

## Exercise

**Example (XOR)**

Write the truth table for the formula:

$$G \triangleq (P \vee Q) \wedge \neg(P \wedge Q)$$


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**Sol.**

| $P$ | $Q$ | $P \vee Q$ | $P \wedge Q$ | $\neg(P \wedge Q)$ | $G$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $F$ |

## Interpretation

**Definition**

Interpretation: Given a propositional formula $G$, let $\{A_1, \cdots, A_n\}$ be the set of atoms which occur in the formula, an **Interpretation** $I$ of $G$ is an assignment of truth values to $\{A_1, \cdots, A_n\}$.

**Example**

Consider the formula: $G \triangleq (P \vee Q) \wedge \neg(P \wedge Q)$
Set of atoms: $\{P, Q\}$
Interpretation for $G$: $I = \{P = \mathbf{T}, Q = \mathbf{F}\}$

## Interpretation (ctnd)

- Each atom $A_i$ can be assigned either **True** or **False** but never both.
- Given an interpretation $I$ a formula $G$ is said to be true in $I$ iff $G$ is evaluated to **True** in the interpretation
- Given a formula $G$ with $n$ distinct atoms there will be $2^n$ distinct interpretations for the atoms in $G$.
- Convention: $\{P, \neg Q, \neg R, S\}$ represents an interpretation $I : \{P = T, Q = F, R = F, S = T\}$.
- Given a formula $G$ and an interpretation $I$, if $G$ is true under $I$ we say that $I$ is a model for $G$.and we can write $I \models G$

## Validity

**Definition**

Valid Formula: A formula $F$ is **valid** iff it is true in all its interpretation

- A valid formula can be also called a **Tautology**
- A formula which is not valid is **invalid**
- If $F$ is valid we can write $\models F$

**Example (de Morgan's Law)**

$(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$ is a valid formula

| $P$ | $Q$ | $\neg(P \wedge Q)$ | $\neg P \vee \neg Q$ | $(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |

## Inconsistency

**Definition**

Inconsistent Formula: A formula $F$ is **inconsistent** iff it is false in all its interpretation

- An inconsistent formula is said to be **unsatisfiable**
- A formula which is not inconsistent is **consistent** or **satisfiable**
- Invalid is different from Inconsistent

**Example**

$\neg((\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)))$ is inconsistent

| $P$ | $Q$ | $(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$ | $\neg(\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q))$ |
|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ |

## Inconsistency vs validity

- A formula is valid iff its negation is inconsistent (and vice versa)
- A formula is invalid (consistent) iff there is at least an interpretation in which the formula is false (true)
- An inconsistent formula is invalid but the opposite does not hold
- A valid formula is consistent but the opposite does not hold

**Example**

The formula $G \triangleq P \vee Q$ is invalid (e.g., it is false when P and Q are false) but is not inconsistent because it is true in all other cases. Moreover, $G$ is consistent (e.g., it is true whenever P or Q are false) but is not valid because it is false when both P and Q are false.

## Decidability

**Property**

*Propositional Logic is decidable: there is a terminating method to decide whether a formula is* **valid**.

- To decide whether a formula is valid:
  1. we can enumerate all possible interpretations
  2. for each interpretation evaluate the formula
- Number of interpretations for a formula are finite ($2^n$)
- Decidability is a very strong and desirable property for a Logical System
- Trade off between representational power and decidability

## Decidability

The fastest known algorithms for deciding propositional satisfiability are based on the Davis-Putnam Algorithm.

A *unit clause* is a clause that consists of a single literal.

```
function Satisfiable (clause list S) returns boolean;
    /* unit propagation */
    repeat
        for each unit clause L ∈ S do
            delete from S every clause containing L
            delete ¬L from every clause of S in which it occurs
        end for
        if S is empty then return TRUE
        else if null clause is in S then return FALSE end if
    until no further changes result end repeat
    /* splitting */
    choose a literal L occurring in S
    if Satisfiable (S ∪ {L}) then return TRUE
    else if Satisfiable (S ∪ {¬L} then return TRUE
    else return FALSE end if
end function
```

## Logical equivalence

**Definition**

Logical Equivalence: Two formulas $F$ and $G$ are logically equivalent $F \equiv G$ iff the truth values of $F$ and $G$ are the same under every interpretation of $F$ and $G$.

Useful equivalence rules

$$
\begin{array}{rcll}
(P \wedge Q) & \equiv & (Q \wedge P) & \text{commutativity of } \wedge \\
(P \vee Q) & \equiv & (Q \vee P) & \text{commutativity of } \vee \\
((P \wedge Q) \wedge R) & \equiv & (P \wedge (Q \wedge R)) & \text{associativity of } \wedge \\
((P \vee Q) \vee R) & \equiv & (P \vee (Q \vee R)) & \text{associativity of } \vee \\
\neg(\neg P) & \equiv & P & \text{double-negation elimination} \\
(P \rightarrow Q) & \equiv & (\neg Q \rightarrow \neg P) & \text{contraposition} \\
(P \rightarrow Q) & \equiv & (\neg P \vee Q) & \text{implication elimination} \\
(P \leftrightarrow Q) & \equiv & ((P \rightarrow Q) \wedge (Q \rightarrow P)) & \text{biconditional elimination} \\
\neg(P \wedge Q) & \equiv & (\neg P \vee \neg Q) & \text{de Morgan} \\
\neg(P \vee Q) & \equiv & (\neg P \wedge \neg Q) & \text{de Morgan} \\
(P \wedge (Q \vee R)) & \equiv & ((P \wedge Q) \vee (P \wedge R)) & \text{distributivity of } \wedge \text{ over } \vee \\
(P \vee (Q \wedge R)) & \equiv & ((P \vee Q) \wedge (P \vee R)) & \text{distributivity of } \vee \text{ over } \wedge
\end{array}
$$

## Normal forms

Standard ways of writing formulas
Two main normal forms:

- Conjunctive Normal Form (CNF)
- Disjunctive Normal Form (DNF)

**Definition**

Literal: a literal is an atom or the negation of an atom

**Definition**

Negation Normal Form: A formula is in Negation Normal Form (NNF) iff negations appears only in front of atoms

## CNF

**Definition**

Conjunctive Normal Form: A formula $F$ is in Conjunctive Normal Form (CNF) iff it is in Negation Normal Form and it has the form $F \triangleq F_1 \wedge F_2 \wedge \cdots \wedge F_n$, where each $F_i$ is a disjunction of literals.

- If $F$ is in CNF Each $F_i$ is called a **clause**
- CNF is also refered to as Clausal Form

**Example**

The formula $G \triangleq (\neg P \vee Q) \wedge (\neg P \vee R)$ is in CNF. We can write $G$ as a set of clauses $\{C_1, C_2\}$ where $C_1 = \neg P \vee Q$ and $C_2 = \neg P \vee R$.
The formula $G \triangleq \neg(P \vee Q) \wedge (\neg P \vee R)$ is not in CNF because negation appears in front of a formula and not only in front of atoms.

## DNF

**Definition**

Disjunctive Normal Form: A formula $F$ is in Disjunctive Normal Form (DNF) iff it is in Negation Normal Form and it has the form $F \triangleq F_1 \vee F_2 \vee \cdots \vee F_n$, where each $F_i$ is a conjunction of literals.

**Example**

The formula $G \triangleq (\neg P \wedge R) \vee (Q \wedge \neg P) \vee (Q \wedge P)$ is in DNF.

Any formula can be transformed into a normal form by using the equivalence rules given above.

## Example of trasnformation

**Example (Formula transformations)**

Prove that the following logical equivalences hold by transforming formulas:
$P \vee Q \wedge \neg(P \wedge Q) \leftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q) \leftrightarrow (\neg P \wedge Q) \vee (P \wedge \neg Q)$

**Sol.**

Given $P \vee Q \wedge \neg(P \wedge Q)$ apply de Morgan's law on the second part and directly obtain $(P \vee Q) \wedge (\neg P \vee \neg Q)$
For more examples see Examples 2.8, 2.9 [Chang and Lee Ch. 2]
Try to prove the other equivalence

## Logical consequence

**Definition**

Given a set of formulas $\{F_1, \cdots, F_n\}$ and a formula $G$, $G$ is said to be a logical consequence of $F_1, \cdots, F_n$ iff for any interpretation $I$ in which $F_1 \wedge \cdots \wedge F_n$ is true $G$ is also true.

- If $G$ is a logical consequence of $\{F_1, \cdots, F_n\}$ we write $F_1 \wedge \cdots \wedge F_n \models G$.
- $F_1, \cdots, F_n$ are called axioms or premises for $G$.
- $F \equiv Q$ iff $F \models Q$ and $Q \models F$

**Example**

$S \rightarrow C, C \rightarrow F, S$ are premises for $F$

## Deduction theorem

**Theorem**

*Given a set of formulas* $\{F_1, \cdots, F_n\}$ *and a formula* $G$, $(F_1 \wedge \cdots \wedge F_n) \models G$ *if and only if* $\models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.

**Sketch of proof.**

- $\Rightarrow$ For each interpretation $I$ in which $F_1 \wedge \cdots \wedge F_n$ is true $G$ is true, $I \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$, however for every interpretation $I'$ in which $F_1 \wedge \cdots \wedge F_n$ is false then $(F_1 \wedge \cdots \wedge F_n \rightarrow G)$ is true, thus $I' \models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$. Therefore, $\models (F_1 \wedge \cdots \wedge F_n) \rightarrow G$.
- $\Leftarrow$ for every interpretation we have that when $F_1 \wedge \cdots \wedge F_n$ is true $G$ is true therefore $(F_1 \wedge \cdots \wedge F_n) \models G$.

## Proof by refutation

**Theorem**

*Given a set of formulas* $\{F_1, \cdots, F_n\}$ *and a formula* $G$, $(F_1 \wedge \cdots \wedge F_n) \models G$ *if and only if* $F_1 \wedge \cdots \wedge F_n \wedge \neg G$ *is inconsistent.*

**Sketch of proof.**

$(F_1 \wedge \cdots \wedge F_n) \models G$ holds iff for every interpretation under which $F_1 \wedge \cdots \wedge F_n$ is true also $G$ is true. This holds iff there is no interpretation for which $F_1 \wedge \cdots \wedge F_n$ is true and $G$ is false, but this happens precisely when $F_1 \wedge \cdots \wedge F_n \wedge \neg G$ is false for every interpretation, i.e. when $F_1 \wedge \cdots \wedge F_n \wedge \neg G$ is inconsistent. □

## Discussion

Previous theorems show that:

- We can prove logical consequence by proving validity of a formula
- We can prove logical consequence by refuting a given formula, i.e. by proving a given formula is inconsistent

Notice that we did not use any specific properties of propositional logic

## Example

**Example**

We want to show that $(P \rightarrow Q) \wedge P \models Q$

**Using definition**

We show that for each interpretation in which $(P \rightarrow Q) \wedge P$ is true, also $Q$ is true. We can do that by writing the truth table of the formulas.

# Example

**Using deduction theorem**

We know from the deduction theorem that $(P \to Q) \land P \models Q$ iff $\models ((P \to Q) \land P) \to Q$. Therefore we need to show that $((P \to Q) \land P) \to Q$ is valid, we can do that by writing the truth table of the formula and verifying that the formula is evaluated true for all its possible interpretation.

**Using Refutation**

We know that $(P \to Q) \land P \models Q$ iff $(P \to Q) \land P \land \neg Q$ is inconsistent. Therefore we need to show that $(P \to Q) \land P \land \neg Q$ is inconsistent, we can do that by writing the truth table of the formula and verifying that the formula is evaluated false for all its possible interpretation.

# Exercise

**Exercise**

- Consider the following formulas: $F_1 \triangleq (P \to Q)$, $F_2 \triangleq \neg Q$, $G \triangleq \neg P$. Show that $F_1 \land F_2 \models G$ using all three approaches [Chang-Lee example 2.11]
- Given that if the congress refuses to enact new laws, then the strike will not be over unless it lasts for more than a year or the president of the firm resigns, will the strike be over if the congress refuses to act and the strike just started ? [Chang-Lee example 2.12]

# Exercise

Which of the following arguments are valid?

1. If I am wealthy, then I am happy. I am happy. Therefore, I am wealthy.
2. If John drinks beer, he is at least 18 years old. John does not drink beer. Therefore, John is not yet 18 years old.
3. If girls are blonde, they are popular with boys. Ugly girls are unpopular with boys. Intellectual girls are ugly. Therefore, blonde girls are not intellectual.
4. If I study, then I will not fail basket weaving 101. If I do not play cards to often, then I will study. I failed basket weaving 101. Therefore, I played cards too often.

# Exercise

The following example is due to Lewis Carroll. Prove that it is a valid argument.

1. All the dated letters in this room are written on blue paper.
2. None of them are in black ink, except those that are written in the third person.
3. I have not filed any of those that I can read.
4. None of those that are written on one sheet are undated.
5. All of those that are not crossed out are in black ink.
6. All of those that are written by Brown begin with "Dear Sir."
7. All of those that are written on blue paper are filed.
8. None of those that are written on more than one sheet are crossed out.
9. None of those that begin with "Dear sir" are written in the third person.

Therefore, I cannot read any of Brown's letters.

## Exercise (ctnd)

Let

- $p$ be "the letter is dated,"
- $q$ be "the letter is written on blue paper,"
- $r$ be "the letter is written in black ink,"
- $s$ be "the letter is written in the third person,"
- $t$ be "the letter is filed,"
- $u$ be "I can read the letter,"
- $v$ be "the letter is written on one sheet,"
- $w$ be "the letter is crossed out,"
- $x$ be "the letter is written by Brown,"
- $y$ be "the letter begins with 'Dear Sir' "

## Exercise (ctnd)

Now, we can write the argument in propositional logic.

1. $p \to q$
2. $\neg s \to \neg r$
3. $u \to \neg t$
4. $v \to p$
5. $\neg w \to r$
6. $x \to y$
7. $q \to t$
8. $\neg v \to \neg w$
9. $y \to \neg s$

Therefore $x \to \neg u$

## Natural deduction

- We consider only the connectives $\wedge, \to$ and $\perp$.
- The reason is that V has a quite different meaning in constructive and non costrutive approaches.
- This is not a limitation, since the above set of connectives is functionally complete.

## Natural deduction

- We present a set of rules which allow to deduce, or derive, conclusions from premises.
- Rules can introduce or eliminate connectives
- Rules cam be composed to form derivations
- Of course we would like that our rules are correct w.r.t. the semantics based on truth tables … more on this later

## Natural deduction

| INTRODUCTION RULES | ELIMINATION RULES |
| --- | --- |
| $(\wedge I)\quad \dfrac{\varphi \quad \psi}{\varphi \wedge \psi}\wedge I$ | $(\wedge E)\quad \dfrac{\varphi \wedge \psi}{\varphi}\wedge E \qquad \dfrac{\varphi \wedge \psi}{\psi}\wedge E$ |
| $(\to I)\quad \begin{array}{c}[\varphi]\\ \vdots \\ \psi \\ \hline \varphi \to \psi\end{array} \to I$ | $(\to E)\quad \dfrac{\varphi \quad \varphi \to \psi}{\psi} \to E$ |

Note the discard of the hypothesis indicated with [ ]

## Natural deduction

Ex falso sequitur quodlibet

$$(\bot)\ \frac{\bot}{\varphi}\bot$$

Reduction Ad Absurdum

$$(\mathrm{RAA})\quad \begin{array}{c}[\neg\varphi]\\ \vdots \\ \bot \\ \hline \varphi\end{array}\ \mathrm{RAA}$$

## Some examples of derivations (proofs)

**I**

$$\dfrac{\dfrac{\dfrac{[\varphi \wedge \psi]^1}{\psi}\wedge E \qquad \dfrac{[\varphi \wedge \psi]^1}{\varphi}\wedge E}{\psi \wedge \varphi}\wedge I}{\varphi \wedge \psi \to \psi \wedge \varphi}\to I_1$$

## Some examples

**II**

$$\dfrac{\dfrac{\dfrac{[\varphi]^2 \qquad [\varphi \to \bot]^1}{\bot}\to E}{(\varphi \to \bot) \to \bot}\to I_1}{\varphi \to ((\varphi \to \bot) \to \bot)}\to I_2$$

## Some examples

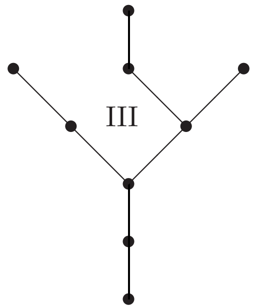


**III**

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi\wedge\psi]^1}{\psi}\wedge E \qquad \dfrac{\dfrac{[\varphi\wedge\psi]^1}{\varphi}\wedge E \qquad [\varphi\to(\psi\to\sigma)]^2}{\psi\to\sigma}\to E}{\sigma}\to E}{\varphi\wedge\psi\to\sigma}\to I_1}{(\varphi\to(\psi\to\sigma))\to(\varphi\wedge\psi\to\sigma)}\to I_2$$

## Derivations are trees

Each derivation is a tree: the leafs contain the premises while the root is the conclusion.




Previous example III

## Derivations

**Definition 1.4.1** *The set of derivations is the smallest set $X$ such that*

*(1) The one element tree $\varphi$ belongs to $X$ for all $\varphi \in PROP$.*

*(2∧) If* $\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}, \begin{matrix}\mathcal{D}'\\ \varphi'\end{matrix} \in X$, *then* $\dfrac{\begin{matrix}\mathcal{D}\\ \varphi\end{matrix} \quad \begin{matrix}\mathcal{D}'\\ \varphi'\end{matrix}}{\varphi\wedge\varphi'} \in X$.

*If* $\begin{matrix}\mathcal{D}\\ \varphi\wedge\psi\end{matrix} \in X$, *then* $\dfrac{\begin{matrix}\mathcal{D}\\ \varphi\wedge\psi\end{matrix}}{\varphi}, \dfrac{\begin{matrix}\mathcal{D}\\ \varphi\wedge\psi\end{matrix}}{\psi} \in X$.

## Definition of Derivations (ctnd)

*2→) If* $\begin{matrix}\varphi\\ \mathcal{D}\\ \psi\end{matrix} \in X$, *then* $\dfrac{\begin{matrix}[\varphi]\\ \mathcal{D}\\ \psi\end{matrix}}{\varphi\to\psi} \in X$.

*If* $\begin{matrix}\mathcal{D}\\ \varphi\end{matrix}, \begin{matrix}\mathcal{D}'\\ \varphi\to\psi\end{matrix} \in X$, *then* $\dfrac{\begin{matrix}\mathcal{D}\\ \varphi\end{matrix} \qquad \begin{matrix}\mathcal{D}'\\ \varphi\to\psi\end{matrix}}{\psi} \in X$.

## Definition of Derivations (ctnd)

$$(2\bot)\ \textit{If}\ \begin{array}{c}\mathcal{D}\\ \bot\end{array} \in X,\ \textit{then}\ \begin{array}{c}\mathcal{D}\\ \bot\\ \hline \varphi\end{array} \in X.$$

$$\textit{If}\ \begin{array}{c}\neg\varphi\\ \mathcal{D}\\ \bot\end{array} \in X,\ \textit{then}\ \begin{array}{c}[\neg\varphi]\\ \mathcal{D}\\ \bot\\ \hline \varphi\end{array} \in X.$$

## Derivation and theorems

We write

$$\Gamma \vdash \varphi$$

If there exists a derivation with (uncancelled) hypothesis (premises) $\Gamma$ and conclusion $\varphi$

When $\Gamma = \emptyset$ we say that $\varphi$ is a theorem

## Completeness theorem

One can problem that the notion of derivability and the notion of truth coincide, that is the following holds:

$$\Gamma \vdash \varphi \Leftrightarrow \Gamma \models \varphi.$$

Corollary: the set of theorems coincide with the set of tautologies

## Soundness

$$\textbf{(Soundness)}\quad \Gamma \vdash \varphi \Rightarrow \Gamma \models \varphi.$$

Proof: Induction on the derivations (see book)

## Completeness

$$\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$$

Proof: One proves that $\Gamma \nvdash \varphi \Rightarrow \Gamma \not\models \varphi$ by showing that:

$\Gamma \nvdash \varphi$ iff

$\Gamma \cup \{\neg\varphi\}$ is consistent iff

there is a valuation v such that $[\![\psi]\!] = 1$

for all $\psi \in \Gamma \cup \{\neg\varphi\}$

## Completeness

$$\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$$

Proof: One proves that $\Gamma \nvdash \varphi \Rightarrow \Gamma \not\models \varphi$ by showing that:

$\Gamma \nvdash \varphi$ iff

$\Gamma \cup \{\neg\varphi\}$ is consistent iff

there is a valuation v such that $[\![\psi]\!] = 1$

for all $\psi \in \Gamma \cup \{\neg\varphi\}$

## A useful Lemma

Definition .

$\Gamma$ is consistent if $\Gamma \nvdash \perp$

$\Gamma$ is inconsistent if $\Gamma \vdash \perp$

**Lemma 1.5.5** *(a)* $\Gamma \cup \{\neg\varphi\}$ *is inconsistent* $\Rightarrow \Gamma \vdash \varphi$,
*(b)* $\Gamma \cup \{\varphi\}$ *is inconsistent* $\Rightarrow \Gamma \vdash \neg\varphi$.

Or, if $\Gamma \nvdash \varphi$ then $\Gamma \cup \{\neg\varphi\}$ is consistent