logic-summary

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0.1 Propositional logic

- **Interpretation**
  - given a propositional formula G, let {A1, ..., An} be the set of atoms which occurs in the formula, an **interpretation** I in G is an assignment of truth values to {A1, ..., An}
  - given an interpretation I, a formula G is said to be true in I iff G is evaluated to True in the interpretation
- **Valid / Invalid formula**
  - a formula F is **valid** iff it is True in all its interpretation
    - a valid formula can be also called a **Tautology**
  - a formula which is not valid is **invalid**. So a formula is invalid if there is at least an interpretation in which the formula is False
- **Inconsistent / consistent formula (satisfiable)**
  - a formula F is **inconsistent** iff it is False in all its interpretation
    - an inconsistent formula is said to be **unsatisfiable**
  - a formula which is not inconsistent is **consistent** or **satisfiable**
- **Decidability**
  - PL is decidable: there is a termining method to decide whether a formula is valid
  - to decide whether a formula is valid:
    - we can enumerate all possible interpretations and for each interpretation evaluate the formula
    - the number of interpretations for a formula are finite (2^n)
  - decidability is a very strong and desiderable property for a Logical System
  - trade off between representional power and decidability
- **Deduction Theorem**
  - given a set of formuls {F1, ..., Fn} and a formula G, (F1 $\wedge$ ... $\wedge$ Fn) $\models$ G if and only if $\models$ (F1 $\wedge$ ... $\wedge$ Fn) $\rightarrow$ G
- **Proof by refutation**
  - given a set of formuls {F1, ..., Fn} and a formula G, (F1 $\wedge$ ... $\wedge$ Fn) $\models$ G if and only if F1 $\wedge$ ... $\wedge$ Fn $\wedge$ $\neg$ G is inconsistent
- **Natural deduction**
  - natural deduction is a kind of proof calculus (using just syntax) in which logical reasoning is expressed by inference rules closely related to the "natural" way of reasoning
  - we consider only connectives $\wedge$, $\rightarrow$ and $\perp$
  - we present a set of **rules** (that can introduce or eliminate) which allow to deduce, or derive, **conclusions from premises**
- **Axiom** (postulate or assumption)
  - an axiom it is a sentence taken to be True, to serve as a premise for reasoning / inference

- **Theorem**
  - it is a sentence which has been proved: a logical consequence of axiom or of other theorems, proved by Natural Deduction
  - more precisely: $\vdash$ F indicates that F is a theorem which only premises are axioms
  - in Natural Deduction, a theorem is something which all premises are discardable
- **Soundness**
  - an algorithm / theory is sound if, whenever if gives you an answer, that is correct
  - all model checking algorithms are sound in PL
  - for proof theory is useful the **Soundness Theorem** which says:
    * $P \vdash Q \implies P \models Q$ which means that if I obtain the logical consequence between P and Q syntactically, the id is also an entilment (semantic consequence) which is what we mean with "correct"
  - in general any answer must be an entilment
- **Correctness**
  - an algorithm / theory is complete if it is able to obtain any possible entilment
  - it is more difficult to obtain. Sometimes completeness misses to obtain a faster algorithm
  - for proof theory is useful the **Completeness Theorem** which says:
    * $P \vdash Q \iff P \models Q$ which means that the algorithm can derive syntactically any entilment (= semantic consequence)
- **Formalization**
  - given propositions A and B:
    * if A then B: $A \rightarrow B$
    * A only if B: $A \rightarrow B$
    * B if A : $A \rightarrow B$
    * A if and only if B : $A \iff B$

0.2 First Order Logic

Everything said until now for PL is true also in FOL. Here we just add things. In FOL there are **variables** and they can be quantified ("some of them", "all of them", "none of them", "at least one", ...) thanks to quantifiers $\forall$ and $\exists$.

- In PL we **only** could treat sentences with a verb! Instead in FOL we can treat:
  - facts
  - objects
  - relations.
- **Terms**
  - refers to objects
  - they **do not contain verbs**
  - example: mother(X), if this is a term (function) this object indicates **the mother of X**
  - objects are any things, they come from a set (finite or infinite) called **Universe** (or domain of discourse)
- **Sentence (Formula)**
  - sentence **has predicates / verbs**
  - can have functions
  - example: mother(X, Y), if this is a sentence it means that X is mother of Y
  - note that predicates can have 0, ..., n inputs

- if a predicate has inputs, the output can be only True of False

- **Herbrand**
  - Herbrand Interpretation
    * The formula A is valid in I, I $\models$ A, if I , $\models$ A for every valuation . This requires to fix a universe U as both I and use U.
    * Jacques Herbrand discovered that there is a universal domain together with a universal interpretation, s.t. that any universally valid formula is valid in any interpretation. Therefore, only interpretations in the Herbrand universe need to be checked, provided the Herbrand universe is infinite.
  - Herbrand Theorem
    * Let P be a set of universal sentences. The following are equivalent:
      · P has an Herbrand model
      · P has a model
      · ground(P) is satisfiable
- **Skolemization**
  - Skolemization is that process that allow us to remove $\exists$ quantifiers
  - what we obtain is called **skolemized form of F**
  - F is satisfiable iff F skolemized is satisfiable
  - F $\not\equiv$ F skolemized
- **Unification**
  - is that procedure which finds substitutions that makes two literal look identical **syntactically**
  - **Most General Unifier (MGU)**
    * there could be many substitutions which makes unification possible, which one I am looking for? The MGU!
    * considering the terms w1 and w2, g is their MGU iff for any other unifier s, does exist s' s.t. (w1)s = ((w1)g)s' and (w2)s = ((w2)g)s'

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