Languages and Algorithms for Artificial Intelligence

Third Module

Exercise Book

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1 Mathematical Preliminaries

Exercise 1.1. What is the cardinality of the set S^n, where S is any finite alphabet? Prove your claim.

Exercise 1.2. What is the cardinality of the subset of $\{0,1\}^{2n}$ consisting of all and only the *palindrome* words? Prove your claim.

Exercise 1.3. What is the cardinality of the subset of $\{0,1\}^n$ consisting of all and only the words which have even *parity* (and the parity of a binary string is the number of occurrences of the symbol 1 inside it)? Prove your claim.

Exercise 1.4. Relate the following pair of functions (f_i, g_i) by way of $O(\cdot)$, $\Omega(\cdot)$ or $\Theta(\cdot)$ notation:

$$
\begin{aligned}
f_1(n) &= n^2 & g_1(n) &= 4n^1 + 100\log(n) \\
f_2(n) &= n\log(n) & g_2(n) &= 10n\log(\log(n)) \\
f_3(n) &= 2^n n^2 & g_3(n) &= 3^n \\
f_4(n) &= 100n & g_4(n) &= \frac{1}{100}n\log(n) \\
f_5(n) &= \log^3(n) & g_5(n) &= n^{\frac{2}{3}}
\end{aligned}
$$

Exercise 1.5. Define appropriate encodings of the following countable sets into the set of $\{0,1\}^*$ binary strings:

- The set $\mathbb{Q}$ of all rational numbers.
- The disjoint union $\mathbb{N} \uplus \mathbb{Z}$ of the set $\mathbb{N}$ of the natural numbers and of the set $\mathbb{Z}$ of the integer numbers.
- The class of all finite, directed graphs, namely pairs of the form (V, E) where V is a finite set, and E is a subset of $V \times V$.

2 The Computational Model

Exercise 2.1. Define an efficient 1-tape Turing Machine computing the function $\mathit{inverse} : \{0,1\}^* \to \{0,1\}^*$ such that $\mathit{inverse}(x)$ is the binary string obtained by flipping all bits in x, e.g. $\mathit{inverse}(01011)$ is 10100. Give the Turing Machine explicitly as a triple in the form (Γ, Q, δ).

Exercise 2.2. Define an efficient 1-tape Turing Machine computing the successor function on the natural numbers, when natural numbers are encoded in in pure binary. In order to make your task simpler, you can safely suppose that the binary string encoding a natural number has least signifcant bits on the left and most significant bits on the right, e.g. 12 is encoded as 0011 rather than as 1100. Give the Turing Machine explicitly as a triple in the form (Γ, Q, δ).

Exercise 2.3. Do Exercise 2.2, but assume, now, that natural numbers are encoded as usual, e.g., 12 is encoded as 1100.

Exercise 2.4. A pair (x, y) of binary strings *of equal length* can be easily encoded into a single binary string in many ways. Pick one, and write $\llcorner(x, y)\lrcorner$ for the encoding of the pair. Define an efficient 3-tape Turing Machine computing the following function $xor : \{0, 1\}^* \to \{0, 1\}^*$ such that $xor(\llcorner(x, y)\lrcorner) = x \oplus y$, where $\oplus$ is the bitwise exclusive or operator, i.e. $100 \oplus 101$ is 001.