# Languages and Algorithms for Artificial Intelligence (Third Module)

## Introduction

Ugo Dal Lago



University of Bologna, Academic Year 2019/2020

Section 1

# The Logistics

# Organisation

- **Webpage**
  - At IOL: `http://iol.unibo.it/`
- **Email**
  - `ugo.dallago@unibo.it`
- **Office Hours**
  - *Where*: see `http://www.cs.unibo.it/~dallago`
  - *When*: there is no fixed office hours, just write me an email and we will fix an appointment.
- **Teaching Assistant**
  - Francesco Gavazzo, `francesco.gavazzo2@unibo.it`
  - Please contact both of us if you need help, this way your request will be taken into account ASAP.

# Structure and Schedule

- The *schedule* of this module's lectures will vary along the coming weeks.
  - Please keep an eye to to the online schedule.
- **Content**: the module is divided into two parts:
  1. The very basics of the theory of *computational complexity*, with implications for AI.
  2. Some rudiments of *computational learning theory*.
- **Requirements**:
  - We assume all students know what a *program* is and what an *algorithm* is.
  - We also assume familiarity with the basics of *probability theory*, *discrete mathematics* (natural numbers, induction, graphs, and the like) and *mathematical logic* (propositional logic and its semantics).
- All lectures will be given by Ugo Dal Lago.
- Some of the exercise sessions will be given by Francesco Gavazzo

# Textbooks and Exams

- **Textbooks**

  The module will be as self contained as possible, but there are plenty of nice books from which you can learn more than what we will do, if you are interested.

  - Sanjeev Arora and Boaz Barak. *Computational Complexity: A Modern Approach*. Cambridge University Press. 2007.
  - Christos Papadimitriou. *Computational Complexity*. Addison-Wesley. 1994.
  - Shai Shalev-Shwartz and Shai Ben-David. *Understanding Machine Learning: from Theory to Algorithms* Cambridge University Press. 2014.
  - Michael Kearns and Umesh Vazirani. *An Introduction to Computational Learning Theory* The MIT Press. 1994.

- **Exams**

  - You need to pass a *written exam*.
  - As you know, you will get just *one* mark for the whole of this course, and the three modules have the same weight.

# Section 2

## What? Why?

# Is There any Limit on what AI can do?

- **AI and ML are Everywhere**.
  - The great impact to our society of technologies derived from Artificial Intellingence and Machine Learning is in front of us.
  - Learning the way tools from AI and ML can be employed in a variety of application scenarios (vision, social-media, recommendation systems, etc.) is of course important for you and your career.
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  - A number of courses in the Master Degree you are enrolled in are specifically about the above.
- **This is *not* What this Module is About**!
  - The question we will try to ask you is rather the following: *is there any intrinsic limit to what we can do with AI and ML, or to the accurancy or efficiency of algorithms solving a give problem?*

# A Paradigm Shift

**"Algorithm $X$ solves Problem $Y$ Efficiently and with Great Accuracy"**

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**"Problem $Y$ Cannot be Solved by any Algorithm $X$".**

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**"Problem $Y$ Cannot be Solved by any Algorithm $X$".**

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**"Algorithm $X$ solves Problem $Y$ Efficiently and with Great Accuracy"**

$\Downarrow$

**"Any Algorithm Solving $Y$ is bound to be Inefficient, or not Accurate".**

# The Value of Negative Results

## A Quote from Thomas A. Edison

*"Negative results are just what I want. They're just as valuable to me as positive results. I can never find the thing that does the job best until I find the ones that don't."*

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- If we know that the task we want to accomplish is intrinsically *hard*, and we know *in what* its hardness resides, we can:
  1. Direct our work towards methodologies *specifically designed* for hard tasks.
  2. *Avoid* spending our time trying to design solutions that cannot exist.
  3. Inform the stakeholders *in advance* that the solution we are going to design will likely be inefficient.
  4. Decide that the task is *too difficult*, and that it cannot be accomplished the way we want.
  5. Decide that it is better to switch to a *relaxed* version of the task, which instead admits reasonable solutions.

# Example Problems - I

## Natural Number Multiplication

Suppose you want to write a `Python` program which multiplies two positive integer numbers, which can both be expressed as $n$-digits numbers, but which are represented as lists rather than scalars. Suppose that the operations at your disposal are just the basic arithmetic operations *on digits*, so that you cannot just multiply the two numbers. How should you write your program?

# Example Problems - II

## Maximizing the Number of Invitees

Suppose you are going to marry your boyfriend (or girlfriend) soon, and you want to decide the list of invitees. Since your soon-to-be father-in-law will pay all bill, you have all the interest in keeping the list of invitees as large as possible. But actually, it would be impossible to invite all the people in the set $L = \{a_1, \ldots, a_m\}$ of all candidate invitees, since some of them are kind of incompatible, and cannot be invited together, i.e. there is another set $I = \{(b_1, c_1), \ldots, (b_n, c_n)\} \subseteq L \times L$ whose elements are those candidate invitees pairs which are incompatible. Would it be possible to maximize the length of the list of invitees?

# Example Problems - III

## Learning Programs with Finite Inputs and Outputs

Suppose your ML teacher asks you to write a learning algorithm capable of reconstructing a program by querying it as a black box, without looking at the code. You can assume that the program takes in input a list of booleans of a fixed (rather big) length $n$, and produces in output one boolean value. How would you proceed? On which input would you query the program? Would it be possible to query the program on significantly less inputs than the $2^n$ possible ones?

# The Module's Structure

1. **Introduction to Computational Complexity**
   - Historical, Conceptual, and Mathematical Preliminaries
   - The Computational Model, and Uncomputability
   - The Class P
   - The Class NP and NP-Completeness

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   - The Computational Model, and Uncomputability
   - The Class P
   - The Class NP and NP-Completeness
2. **Some Rudiments of Computational Learning Theory**
   - From Computation to Learning
   - Positive and Negative Results about the Learning Process
   - The VC Dimension

# Thank You!

Questions?

# Languages and Algorithms for Artificial Intelligence (Third Module)

## Historical, Conceptual, and Mathematical Preliminaries

Ugo Dal Lago



University of Bologna, Academic Year 2019/2020

Section 1

# A Bit of History

# Computation and Computers

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  - Computation was seen as a **form of art**, rather than a concept to be studied scientifically.
- The modern **theory of computation** (ToC) is one of the many gifts humanity has received from science during the first half of the twentieth century.
  - A *precise* definition of the computable has been given, together with many results about it.
  - Actually, many alternative definitions of the computable have been given, but have been proved to be essentially *equivalent*.

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  - A *precise* definition of the computable has been given, together with many results about it.
  - Actually, many alternative definitions of the computable have been given, but have been proved to be essentially *equivalent*.
- In the second half of the twentieth century, ToC has evolved into a **fully fledged** scientific field.

# Computation, Science and Technology

- The outfalls of modern theory of computation, since the 1940s, have been twofold:
  1. **From ToC to the other Sciences**
     - Examples: Genomics, Physics, ...
     - Results from the other sciences have had a very strong impact to the theory of computation, as well.
  2. **From ToC to ICT**
     - The theoretical notion of computation was already there when the first modern, electronic, computers appeared.
     - Concepts from the theory of computation (e.g. universality) informed the design of computer from the very beginning.
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- One can easily observe a trend, in the last thirty years:
  - **Concepts** from ToC have more and more influence on the other sciences.
  - **Problems** from ToC are considered worth being solved by, e.g., researchers in mathematics and physics.

# Computability and Complexity

- Until the Late 60s, ToC was mainly concerned with understanding **computability**.
  - Main Question: is a certain task computable?
  - If the answer is negative, the task is said to be *uncomputable* (and there can be many ways in which this can happen).
  - This is the so-called **computability theory**.

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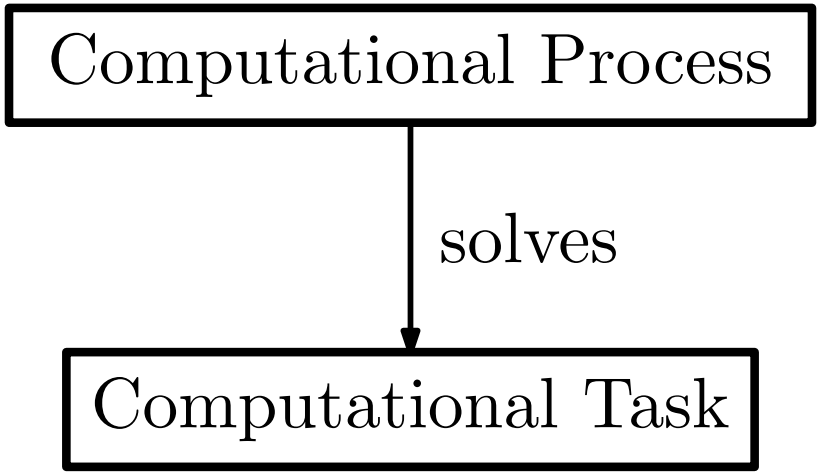
  - This is the so-called **computability theory**.
- Since the pioneering works by Hartmanis, Stearns, and Cobham (all from the late 60s), a new branch of ToC has emerged, which deals with **efficiency**.
  - Main Question: is a certain task solvable *in a reasonable amount* of time, or space (i.e. working memory)?
  - If the answer is negative, the task is maybe computable, but requires so much time or space, that it pecomes *practically* uncomputable.
  - This new branch of ToC has been named **computational complexity theory**.
- This module will mainly deal with computational complexity theory.

# Section 2

# Modelling Computation

# The Standard Paradigm



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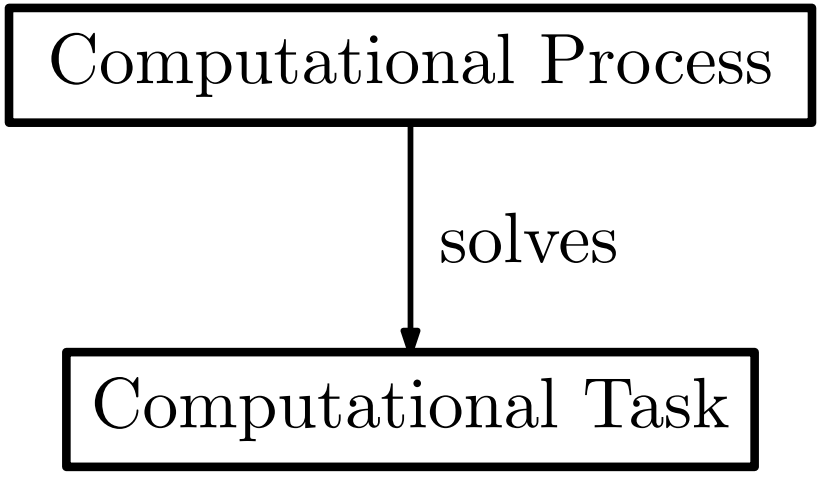


- Processes and tasks **are different**.
- In some cases, there could be **many** distinct processes solving the same tasks.
- In some other cases, the task is so hard that **no** process can solve it.

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$$a \cdot b = \underbrace{a + a + \ldots + a}_{b \text{ times}}$$

Since addition of two $n$-digit numbers is a relatively easy process can easily be carried out in a linear amount of steps, the total number of steps is proportional to $b \cdot n$. Call this process `RepeatedAddition`.

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  Since addition of two $n$-digit numbers is a relatively easy process can easily be carried out in a linear amount of steps, the total number of steps is proportional to $b \cdot n$. Call this process `RepeatedAddition`.
- Another **process** solving the task above consists in rather multipling $a$ and $b$ with the elementary school direct algorithm, which takes an amount of steps which is proportional to $n \cdot n$. Call this process `GridMethod`

# The Multiplication Problem

- There is an exponential difference between the performances of `RepeatedAddition` and `GridMethod`.
  - Indeed, $b$ can be exponential in $n$, i.e. at most $10^n - 1$.
  - When, e.g., $n$ is 100, there is a huge different between $100 \cdot 100$ and $(10^{100} - 1) \cdot 100$.
  - If each basic instruction takes a millisecond, `GridMethod` would take one second, while `RepeatedAddition` would take more than $10^{80}$ years.
- Distinct processes can solve the same task in very different ways, and not all of them are acceptable.
- `GridMethod` witnesses the fact that the task is in a class of tasks called **P**

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- The problem is thus known to be in the class **NP**.
- The real question is **can we do better**?
  - If we manage to do significantly better, we would have solved the question of whether **NP** is equal to **P**.

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- We are **very rarely** able to prove the nonexistence of efficient algorithm.
  - Finding mathematical techniques which allow us to do so is *the crux* of computational complexity theory.
- Rather than proving the non-existence of certain algorithms, complexity theory **interrelates** different tasks.

# Some Research Questions in Complexity Theory

- The **P** vs. **NP** question.
- Can the use of randomness help in speeding up computation?
- Can hard tasks become easier if we allow algorithms to err on a relatively small subset of the inputs?
- Can we exploit the hardness of certain tasks?
- Can we make use of some counterintuitive properties of quantum mechanics to build faster computing devices?

# Section 3

# Some Mathematical Preliminaries

# Sets and Numbers

- The cardinality of any set $X$ is indicated as $|X|$.
  - It can be finite or infinite.
- We will mainly work with discrete numbers.
  - $\mathbb{N}$ is the set of natural numbers, while $\mathbb{Z}$ is the set of integers.
- We say that a condition $P(n)$ depending on $n \in \mathbb{N}$ holds for **sufficiently large** $n$ if there is $N \in \mathbb{N}$ such that $P(n)$ holds for every $n > N$.
- Some common (and useful notation):
  - For a given real number $x$, $\lceil x \rceil$ is the the smallest element of $\mathbb{Z}$ such that $\lceil x \rceil \geq x$. Whenever a real number $x$ is use in place of anatural number, we implicitly read it as $\lceil x \rceil$
  - For a natural number $n$, $[n]$ is the set $\{1, \ldots, n\}$.
  - Contrarily to the common mathematical notation, $\log x$ is **base** 2 logarithm.

# Strings

- If $S$ is a finite set, then a **string** over the alphabet $S$ is a finite, ordered, possibly empty, tuple of elements from $S$.
  - Most often, the alphabet $S$ will be the set $\{0, 1\}$.
- The set of all strings over $S$ of length exactly $n \in \mathbb{N}$ is indicated as $S^n$ (where $S^0$ is the set containing only the empty string $\varepsilon$).
- The set of *all strings* over $S$ is $\bigcup_{n=0}^{\infty} S^n$ and is indicated as $S^*$.
- The concatenation of two strings $x$ and $y$ over $S$ is indicated as $xy$. The string over $S$ obtained by concatenating $x$ with itself $k \in \mathbb{N}$ times is indicated as $x^k$.
  - Strings form a monoid!
- The **length** of a string $x$ is indicated as $|x|$.
- Examples

$$\{0, 1\}^2 = \{\varepsilon, 0, 1, 00, 01, 10, 11\} \qquad |000101| = 6$$

# Tasks as Functions

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  - If the input and/or the output are not strings, but they are taken from a discrete set, they can be **represented** as strings, following some encoding, which however should be as simple as possible.
- Summing up, we always assume that the task we want to solve **is given** as a function $f : A \to B$ where both the domain $A$ and $B$ are discrete (i.e. countable) sets, sometime leaving the encoding of $A$ and $B$ into $\{0,1\}^*$ implicit.
- The encoding of any element $x$ of $A$ as a string is often indicated as $\llcorner x \lrcorner$ or simply as $x$.

# Representing Objects as Strings: Examples

- Strings in $S^*$, where $S = \{a, b, c\}$.
  - $a$ can be encoded as 00, $b$ becomes 01 and $c$ becomes 10.
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- Pairs of binary strings, i.e. $\{0, 1\}^* \times \{0, 1\}^*$.
  - One can encode the pair $(x, y)$ as the string $x\#y$ over the alphabet $\{0, 1, \#\}$, and the proceed as before.

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# Languages and Decision Problems

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- An important class of functions from $\{0,1\}^*$ to $\{0,1\}^*$ are those whose range are strings of length exactly one, called **boolean functions**.
  - They are characteristic functions.
- We identify such a function $f$ with the subset $\mathcal{L}_f$ of $\{0,1\}^*$ defined as follows:

$$\mathcal{L}_f = \{x \in \{0,1\}^* \mid f(x) = 1\}.$$

- This way, a **decision problem** for a given language $\mathcal{M}$ (i.e. does $x \in \{0,1\}^*$ is in $\mathcal{M}$?) can be seen as the task of computing $f$ such that $\mathcal{M} = \mathcal{L}_f$.

# Asymptotic Notation

- A function $f : \mathbb{N} \to \mathbb{N}$ is $O(g)$ if there is a positive real constants $c$ such that $f(n) \leq c \cdot g(n)$ for sufficiently large $n$.
  - Example: the function $n \mapsto 3 \cdot n^2 + 4 \cdot n$ is $O(n^2)$, but also $O(n^3)$, and certainly $O(2^n)$. It is not, however, $O(n)$.
- A function $f : \mathbb{N} \to \mathbb{N}$ is $\Omega(g)$ if there if a positive real constants $c$ such that $f(n) \geq c \cdot g(n)$ for sufficiently large $n$
  - Example: the function $n \mapsto 3 \cdot n^2 + 4 \cdot n$ is $\Omega(n^2)$, but also $\Omega(n)$, but not $\Omega(n^3)$.
- A function $f$ is $\Theta(g)$ if $f$ is both $O(g)$ and $\Omega(g)$.
  - Example: the function $n \mapsto 3 \cdot n^2 + 4 \cdot n + 7$ is $\Theta(n^2)$.

# Thank You!

Questions?

# Languages and Algorithms for Artificial Intelligence (Third Module)

## The Computational Model

Ugo Dal Lago



University of Bologna, Academic Year 2019/2020

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  - If we choose one specific, concrete machine without relating to the other ones, then our negative results would be *vacuous*.
- The only way out is to define a **model of computation** in the form of an abstract machine, keeping in mind that:
  - It should be as simple as possible, to *facilitate proofs*.
  - It must be able to simulate with reasonable overhead *all physically realistic* machines.

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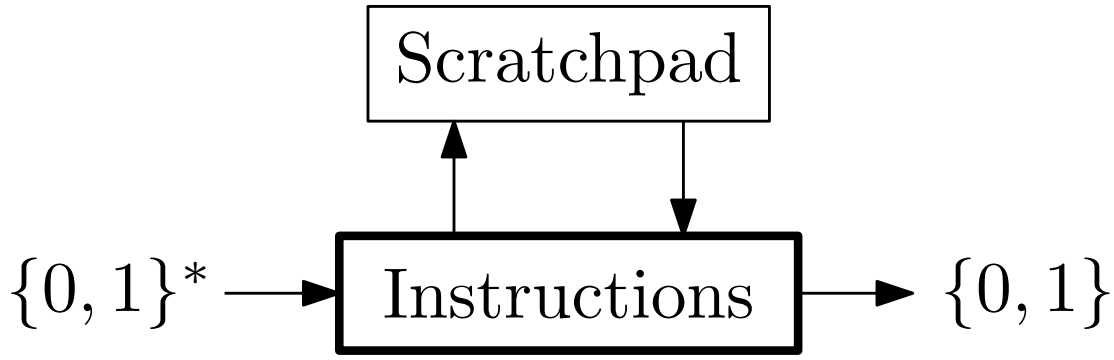
  - If we choose one specific, concrete machine without relating to the other ones, then our negative results would be *vacuous*.
- The only way out is to define a **model of computation** in the form of an abstract machine, keeping in mind that:
  - It should be as simple as possible, to *facilitate proofs*.
  - It must be able to simulate with reasonable overhead *all physically realistic* machines.
- There is a universally accepted model of computation, that we will take as our reference model, namely the **Turing Machine**.
  - This part of the module is specifically about it.

# Part I

# The Model, Informally

# How to Compute $f : \{0,1\}^* \to \{0,1\}$



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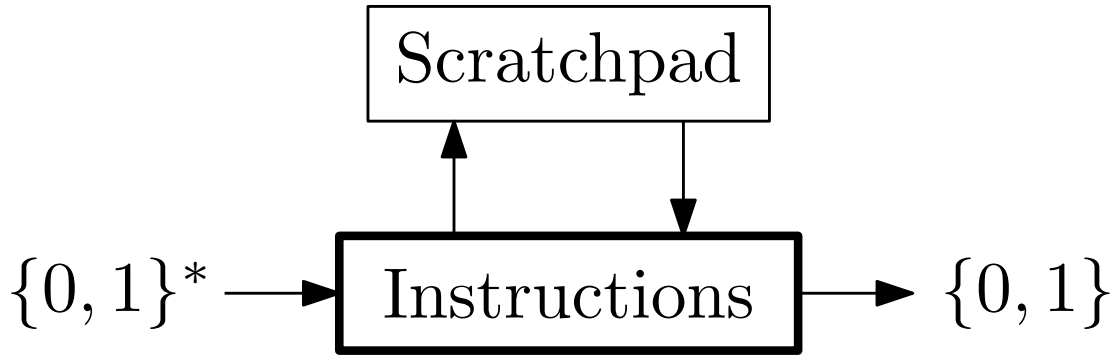


- The set of instructions to be followed is fixed (and should work for every input $x$), and finite.
- The same instruction can be used potentially many times.
- Every instruction proceeds by:
  - Reading a bit of the input;
  - Reading a symbol from the scratchpad;

  and based on that decide what to do next, namely:
  - either write symbol to the scratchpad, and proceed to another instruction;
  - or declare the computation finished, by stopping it and outputting either 0 or 1.

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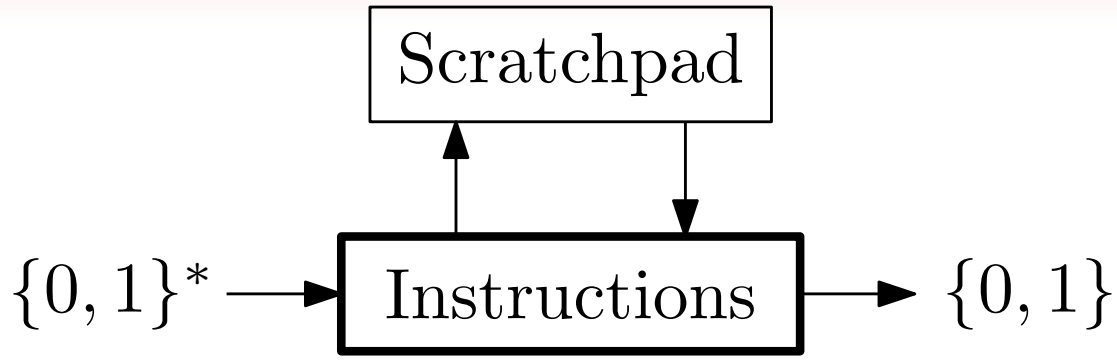


- The **running time** of this machine/process/algorithm on $x$ is simply the number of these basic instructions which are executed on a certain input $x$.
- We say that the machine **runs in time** $T(n)$ if it performs *at most* $T(n)$ instructions on input strings *of length* $n$.

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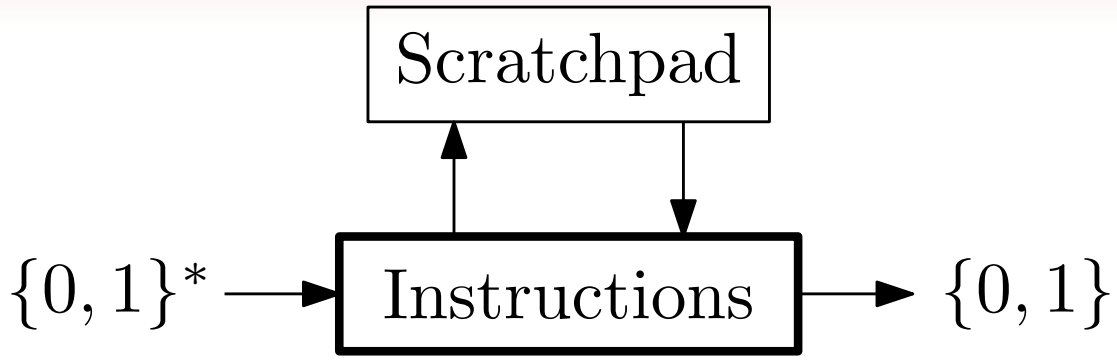


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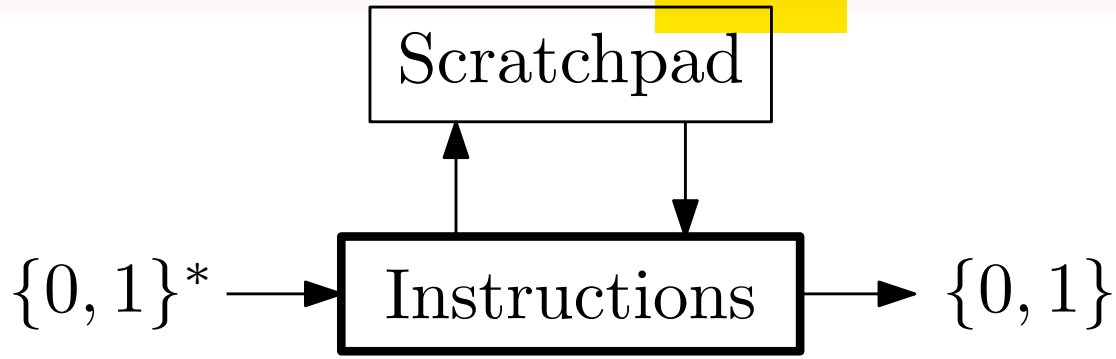


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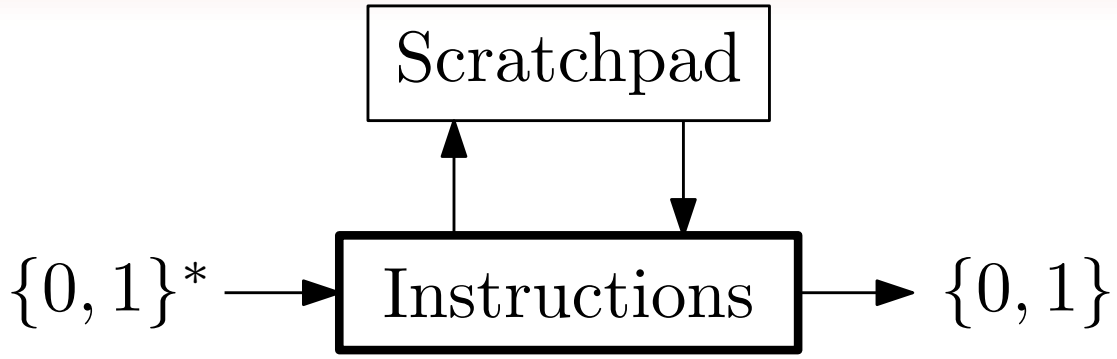
- The model is **robust** to many tweaks in the definition, (e.g. changing the alphabet, allowing multiple scratchpads rather than one): the simplest model can simulate the more complicated with *a polynomial overhead* in time.
- Since there are finitely many instructions, machine descriptions can be **encoded as binary strings** themselves.

# How to Compute $f : \{0,1\}^* \to \{0,1\}$

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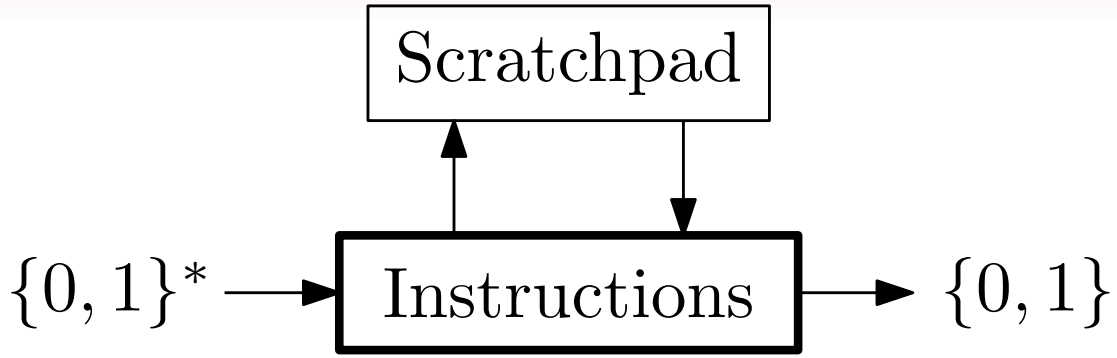
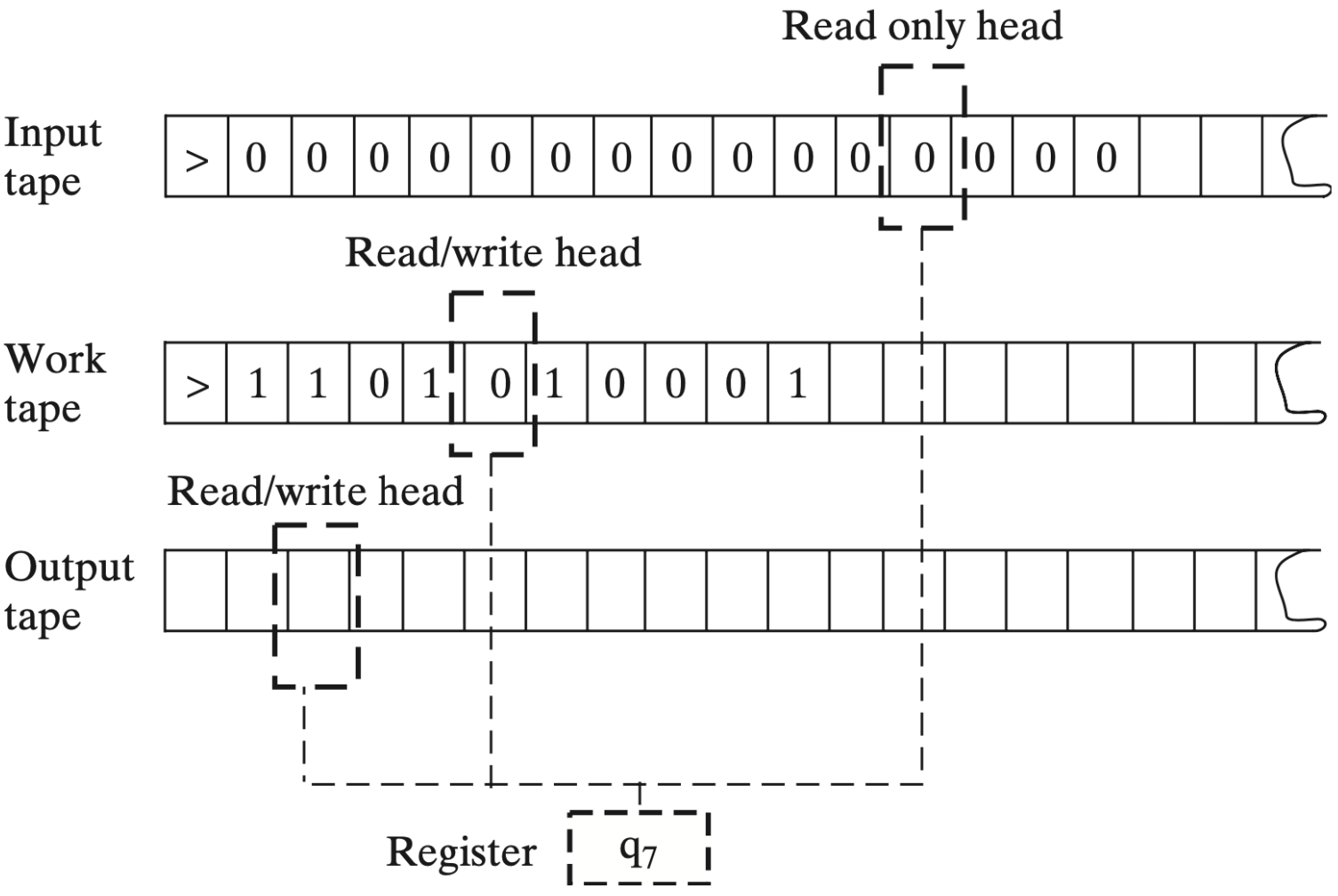



- Given a string $\alpha$, we indicate as $\mathcal{M}_\alpha$ the Turing Machine $\alpha$ encodes.
- There is a so-called **Universal** Turing Machine $\mathcal{U}$, which simulates any other Turing Machine given its string representation: from a pair of strings $(x, \alpha)$, the machine $\mathcal{U}$ simulates the behaviour of $\mathcal{M}_\alpha$ on $x$.
  - The simulation is very efficient: if the running time of $\mathcal{M}_\alpha$ were $T(|x|)$, then $\mathcal{U}$ would take time $O(T(|x|) \log T(|x|))$.
- There are functions $f : \{0,1\}^* \to \{0,1\}$ which are **intrinsically uncomputable** by Turing Machines, and this can be proved formally.
  - This has intimate connections to Gödel's famous **incompleteness theorem**.

# Part II

# The Model, Formally

# A More Detailed View



# The Scratchpad(s)

- It consists of $k$ **tapes**, where a tape is an infinite one-directional line of cells, each of which can hold a simbol from a finite alphabel $\Gamma$, the *alphabet* of the machine.
- Each tape is equipped with **tape head**, which can read or write symbols *from* or *to* the tape. Each head can move left or right.
- The first tape is designated as **input tape**, and is read-only.
- The last tape is the **output tape**, and contains the result of the computation.
  - This slightly deviates from what we have said in our informal account, and is needed to compute arbitrary functions on $\{0,1\}^*$.

# The Instructions

- The machine has a finite set of *states*, called $Q$, which determine the action to be taken at the next step.
- At *each step*, the machine:
  1. **Read** the symbols under the $k$ tape heads.
  2. For the $k-1$ read-write tapes, **replace** the symbol with a new one, or leave it unchanged.
  3. **Change** its state to a new one.
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  4. **Move** each of the $k$ tape heads to the left or to the right (or stay in place).
- These instructions are of course very basic, and far from being close to the kind of instructions programming languages offer.
  - The point here is to have a *simple*, but *expressive* model.

# The Formal Definition

A **Turing Machine** (TM for short) working on $k$ tapes is described as a triple $(\Gamma, Q, \delta)$ containing

- A finite set $\Gamma$ of **tape symbols**, which we assume contains the *blank symbol* $\square$, the *start symbol* $\triangleright$, and the binary digits 0 and 1.

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- A **transition function**

$$\delta : Q \times \Gamma^k \to Q \times \Gamma^{k-1} \times \{\texttt{L}, \texttt{S}, \texttt{R}\}^k$$

describing the instructions regulating the functioning of the machine at each step.

  - When the first parameter is $q_{\texttt{halt}}$, then $\delta$ cannot touch the tapes nor the heads:

$$\delta(q_{\texttt{halt}}, (\sigma_1, \ldots, \sigma_k)) = (q_{\texttt{halt}}, (\sigma_2 \ldots, \sigma_k), (\texttt{S}, \ldots, \texttt{S}))$$

and the machine is *stuck*.

# Machine Configurations and Computations

Given a TM $\mathcal{M} = (\Gamma, Q, \delta)$ working on $k$ tapes:

- A **configuration** consists of
  - The current state $q$.
  - The contents of the $k$ tapes.
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- A **final configuration** for the output $y \in \{0,1\}^*$ is any configuration whose state is $q_{\texttt{halt}}$ and in which the content of the output tape is $\triangleright y$, followed by blank symbols.

# Machine Computations

Given a TM $\mathcal{M} = (\Gamma, Q, \delta)$ working on $k$ tapes:

- Given any configuration $C$, the transition function $\delta$ determines in a natural way the next configuration $D$, and we write $C \xrightarrow{\delta} D$ if this is the case.
- We say that $\mathcal{M}$ **returns** $y \in \{0,1\}^*$ **on input** $x \in \{0,1\}^*$ **in** $t$ **steps** if

$$\mathcal{I}_x \xrightarrow{\delta} C_1 \xrightarrow{\delta} C_2 \xrightarrow{\delta} \ldots \xrightarrow{\delta} C_t$$

  where $C_t$ is a final configuration for $y$. We write $\mathcal{M}(x)$ for $y$ if this holds.
- Finally, we say that $\mathcal{M}$ **computes** a function $f : \{0,1\}^* \to \{0,1\}^*$ iff $\mathcal{M}(x) = f(x)$ for every $x \in \{0,1\}^*$. In this case, $f$ is said to be **computable.**
  - Beware: for the moment, we do not put any constraint *on the number of steps* $\mathcal{M}$ needs to compute $f(x)$ from $x$.

# The Expressive Power of TMs

- Please take a loot at

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- Explicitly constructing TMs is **very tedious**.
  - One needs to give the set of states, the set of instructions, etc.
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  - This is fine, *provided* program or algorithm instructions can be **simulated** by TMs.
- There are many other formalisms which are perfectly equivalent to TMs as for the class of computable functions they induce.
  - Examples: Random Access Machines, the $\lambda$-calculus, URMs, Partial Recursive Functions, ...

# Efficiency and Runtime

- A TM $\mathcal{M}$ computes a function $f : \{0,1\}^* \to \{0,1\}^*$ **in time** $T : \mathbb{N} \to \mathbb{N}$ iff $\mathcal{M}$ returns $f(x)$ on input $x$ in a number of steps smaller or equal to $T(|x|)$ for every $x \in \{0,1\}^*$. In this case, $f$ is said to be **computable** in time $T$.

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  - Basic operations like addition and multiplication are computable in polynomial time.
- A function $T : \mathbb{N} \to \mathbb{N}$ is **time-constructible** iff the function on $\{0,1\}^*$ defined as $x \mapsto \llcorner T(|x|) \lrcorner$ is computable.

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  - Just **one tape**, rather than many.
  - Tapes can be infinite **in both directions**.
- In all the cases above (and in many others), one can prove that the more restrictive notion of machine **simulates** the more general one **with polynomial overhead**.
  - We do not have time to see all that, but you are encouraged to take a look at [AroraBarak2009], Section 1.3.1.

# Machines as Strings

- One of the very nice consequences of keeping our definition of a Turing Machine very simple is that any machine $\mathcal{M} = (\Gamma, Q, \delta)$ is in fact copmletely determined from the graph of $\delta$, seen as a subset of

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- Any parismomious encoding of $\delta$ as a binary string in $\{0,1\}$ thus constitutes an acceptable encoding $\llcorner \mathcal{M} \lrcorner$ of $\mathcal{M}$, provided the following two conditions are satisfied:
  1. **Every string** in $\{0,1\}^*$ represents a TM, i.e. for every $x \in \{0,1\}^*$ there is $\mathcal{M}$ such that $x = \llcorner \mathcal{M} \lrcorner$.
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  2. Every TM $\mathcal{M}$ is represented by an **infinitely many strings** (although exactly one is the "canonical" representation $\llcorner \mathcal{M} \lrcorner$ of $\mathcal{M}$.
- The two conditions above are not essential in any other contexts (i.e. when the encoded data is not a program), but are technically crucial here.

# The Universal Turing Machine

**Theorem (UTM, Efficiently)**

*There exists a TM $\mathcal{U}$ such that for every $x, \alpha \in \{0,1\}^*$, it holds that $\mathcal{U}(x, \alpha) = \mathcal{M}_\alpha(x)$, where $\mathcal{M}_\alpha$ denotes the TM represented by $\alpha$. Moreover, if $\mathcal{M}_\alpha$ halts on input $x$ within $T$ steps then $\mathcal{U}(x, \alpha)$ halts within $CT\log(T)$ steps, where $C$ is independent of $|x|$ and depending only on $\mathcal{M}_\alpha$.*

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- A proof of the Theorem above in its full generality requires quite a bit of work.
- We will see the proof of a version of it in which the $O(T\log(T))$ bound is replaced by a polynomial one.

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- This would be a contraddiction, because by definition

$$uc(\llcorner \mathcal{M} \lrcorner) = 1 \Leftrightarrow \mathcal{M}(\llcorner \mathcal{M} \lrcorner) \neq 1$$

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**Theorem (Uncomputability of *halt*)**

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- This result could be seen as a way to reinterpret Gödel's first incompleteness theorem "computationally".

# Thank You!

Questions?

# Languages and Algorithms for Artificial Intelligence (Third Module)

## Polynomial Time Computable Problems

Ugo Dal Lago



University of Bologna, Academic Year 2019/2020

# Complexity Classes

- A **complexity class** is a set of *tasks* which can be computed within some prescribed resource bounds.
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# Complexity Classes

- A **complexity class** is a set of *tasks* which can be computed within some prescribed resource bounds.
  - It is *not* a set of TMs, although it is defined based on TMs.
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- The letter "D" in $\mathbf{DTIME}(\cdot)$ refers to *determinism*: the machines on which the class is based work deterministically.
- Should we study efficiently solvable tasks by way of classes in the form $\mathbf{DTIME}(T(n))$?
  - The answer is bound to be negative, because these classes are not **robust**, they depends too much on
  - We need a larger class.

# The Class **P**

- The class **P** is defined as follows:

$$\mathbf{P} = \bigcup_{c \geq 1} \mathbf{DTIME}(n^c).$$

- In other words, the class **P** includes all those languages $\mathcal{L}$:
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- **P** is generally considered as *the* class of efficiently decidable languages.

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  - Every physically realizable computer can be simulated by a TM with a (possibly *very large*) overhead in time.
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  - Most scientists believe in it.
- **The Strong Church-Turing Thesis**
  - Every physically realizable computer can be simulated by a TM with a *polynomial* overhead in time. ($n$ steps on the computer requires $n^c$ on TMs, where $c$ only depends on the computer), and viceversa.
  - The class **P** would be *the same* if defined based on other realistic models of computation.
  - This is more controversial (due to, e.g., quantum computation).

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  - For many problems of interest and in **P**, there are TMs working within quadratic or cubit bounds.
- **Nice Closure Properties**
  - The class is closed various operations on programs, e.g. composition and bounded loops (with some restrictions!).
  - As a consequence, it is relatively easy to prove that a given problem/task is *in* the class: it suffices to give an algorithm solving the problem and working in polynomial time, without constructing the TM explicitly.

# Some Criticisms on **P**

- **Worst-Case is Not Realistic**
  - The definition of **P** is intrinsically based on worst-case complexity: there must be *a* polynomial and *a* TM such that *for every input*...
  - It is good enough is our problem takes little time *on the types of inputs* which arise in practice, and not on *all* of them.
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  - Solutions: Average-case Complexity, Approximation Algorithms
- **Alternative Computational Models**
  - Feasibility can be also be defined for classes dealine with arbitrary precision computation, with randomized computation, or with quantum computation.
  - Solutions: the class **P** can be spelled out with other computational models in mind, giving rise to other classes (e.g. **BPP** or **BQP**).
- **Why Just Decision Problems?**
  - As alreday pointed out, not all tasks can be modeled this way.

# Thank You!

Questions?

# The Complexity Class **FP**

- Sometime, one would like to classify *functions* rather than *languages*. This can be done by slightly generalizing a couple of concepts we have previously introduced:
  - Let $T : \mathbb{N} \to \mathbb{N}$. A function $f$ is in the class $\mathbf{FDTIME}(T(n))$ iff *there is* a TM computing $f$ and running in time $n \mapsto c \cdot T(n)$ for some constant $c$.
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- For every $\mathcal{L} \in \mathbf{P}$, the characteristic function $f$ of $\mathcal{L}$ is trivially in **FP**.
- For certain classes of functions (e.g. those corresponding to optimization problems), there are canonical ways to turn a function $f$ into a language $\mathcal{L}_f$
  - In general, however, it is not true that $f \in \mathbf{FP}$ implies $\mathcal{L}_f \in \mathbf{P}$.

# Example Problems in **P** or **FP**

- **Lists**: inverting a list, sorting a list, finding the maximum or minimum element in a list, etc.

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- **Optimization Problems**: linear programming, maximum cost flow, etc.

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- In theory, one should give a TM working within some polynomial bounds, and prove that the machines decides the language (or computes the function).
- This is however too cumbersome, and instead of going through TMs, one often goes informal and uses the so called pseudocode.
- Example.
  - Suppose you want to show the following problem to be computable in polynomial time: given two strings $x, y \in \{0,1\}^*$. determine if the $x$ contains an instance of $y$.
  - A pseudocode solving the problem above is the following:

```
i ← 1;
while i ≤ |x| − |y| + 1 do
    if x[i : i + |y| − 1] = y then
        i ← i + 1
    else
        return True
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  - Indeed, $i$ cannot be greater than $O(|x|)$, thus its length is $O(\lg |x|)$.
- Each instruction takes polynomial time to be simulated.
  - Comparing two strings of length $|y|$ can be done in polynomial time in $|y|$, thus polynomial in $|\llcorner(x, y)\lrcorner|$.

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**Theorem**

*The two inclusions above are strict.*

# Thank You!

Questions?

# Languages and Algorithms for Artificial Intelligence (Third Module)

## Between the Feasible and the Unfeasible

Ugo Dal Lago



University of Bologna, Academic Year 2019/2020

# The Border Between the Tractable and the Intractable

- Up to now, we have encountered (essentially speaking) two complexity classes, namely:
  - **P**, which contains those problems which can be solved in polynomial time, so the **tractable** ones.
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- It's now time to study the "border" between tractability and intractability:
  - Between **P** and **EXP**, one can define *many other* classes. i.e. there are many ways of defining a class **A** such that

$$\mathbf{P} \subseteq \mathbf{A} \subseteq \mathbf{EXP}$$

  - This is a formidable tool to classify those problems in **EXP** for which *we do not know* whether they are in **P** (and there are *so many* of them).

# Creating vs. Verifying

- Very often, the language we would like to classify can be written as follows:

$$\mathcal{L} = \{x \in \{0,1\}^* \mid \exists y \in \{0,1\}^{p(|x|)}.(x,y) \in \mathcal{A}\}$$

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  - Of course this does *not* rule out other strategies to decide $\mathcal{L}$.
- The *take-away message* is thus the following: **crafting** a solution for the problem $x$ (i.e., finding $y$) can potentially be more difficult than just **checking** $y$ to be a solution to $x$.

# The Complexity Class **NP**

- A language $\mathcal{L} \subseteq \{0,1\}^*$ is in the class **NP** iff there exist a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial time TM $\mathcal{M}$ such that

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**Theorem**

**P $\subseteq$ NP $\subseteq$ EXP**

# Examples Problems in **NP**

1. **Maximum Independent Set**
   - In its decision form, it asks whether a pair $(\mathbb{G}, k)$ of an undirected graph and a natural number $k \in \mathbb{N}$ is such that $\mathbb{G} = (V, E)$ admits an independent set $W \subseteq V$ of cardinality at least $k$.

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2. **Subset Sum**
   - It asks whether, given a sequence of natural numbers $n_1, \ldots, n_m$ and a number $k$, there exists a subset $I$ of $\{1, \ldots, m\}$ such that $\sum_{i \in I} n_i = k$.

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   - It asks whether, given a sequence of natural numbers $n_1, \ldots, n_m$ and a number $k$, there exists a subset $I$ of $\{1, \ldots, m\}$ such that $\sum_{i \in I} n_i = k$.
   - Certificate: again, the certificate here is just $I$: checking whether $\sum_{i \in I} n_i = k$ just amounts to some additions and comparisons.

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4. **Factoring**
   - Given three numbers $n, m, l$, it asks whether $n$ has a prime factor in the interval $[m, l]$.
   - Certificate: it can be taken to be the prime number $p$: checking that $p \in [m, l]$ and that $p$ divides $n$ is easy. Instead, checking that $p$ is prime requires a lot of work, but can indeed be done in polynomial time.

# Examples Problems in **NP**

3. **Decisional Linear Programming**
   - Given a sequence of $m$ linear inequalities with rational coefficients over $n$ variables (i.e. inequalities of the form $\sum_{i=1}^{n} a_i x_i \leq b$, where the coefficients $a_1, \ldots, a_n, b$ are in $\mathbb{Q}$), decide whether there is a rational assignment to the variables $x_1, \ldots, x_n$ which makes all the inequalities true.

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# Examples Problems in **NP**

- Some of the aforementioned problems are also in **P**:
  - *Decisional linear programming* can be proved to be in **P** thanks to, e.g., the Ellipsoid algorithm.
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  - The *composite numbers* problem can be proved itself to be in **P**, thanks to a breakthrough recent result, namely the so-called AKS algorithm.
- All the other problems are currently **not known** to be in **P**.
  - Are they all equivalent in terms of their inherent computational difficulty?
  - Is there any way isolate those problems in **NP** whose difficult is maximal, i.e. they are at least as hard as all other problems in **NP**?

# Nondeterministic Turing Machines

- The class **NP** can also be defined using a variant of Turing machines, called the *nondeterministic* Turing machines (NDTM for short).
  - This is the original definition by Hartmanis and Stearns, the founding fathers of computational complexity.
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- The only differences between a NDTM and an ordinary TM is that the former has:
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- We say that a NDTM $M$:
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# Time-Bounded NDTMs

- We say that a NDTM $M$ *runs in time* $T : \mathbb{N} \to \mathbb{N}$ iff for every $x \in \{0,1\}^*$ *and for every possible nondeterministic evolution*, $M$ reaches either the halting state or $q_{\texttt{accept}}$ within $c \cdot T(|x|)$ steps, where $c > 0$.

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- For every function $T : \mathbb{N} \to \mathbb{N}$ and $\mathcal{L} \subseteq \{0,1\}^*$, we say that $\mathcal{L} \in \mathbf{NDTIME}(T(n))$ iff there is a NDTM $M$ working in time $T$ and such that $M(x) = 1$ iff $x \in \mathcal{L}$.

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- All this being said, NDTMs, contrarily to TMs, are *not* meant to model any form of physically realisable machine.

# Thank You!

Questions?

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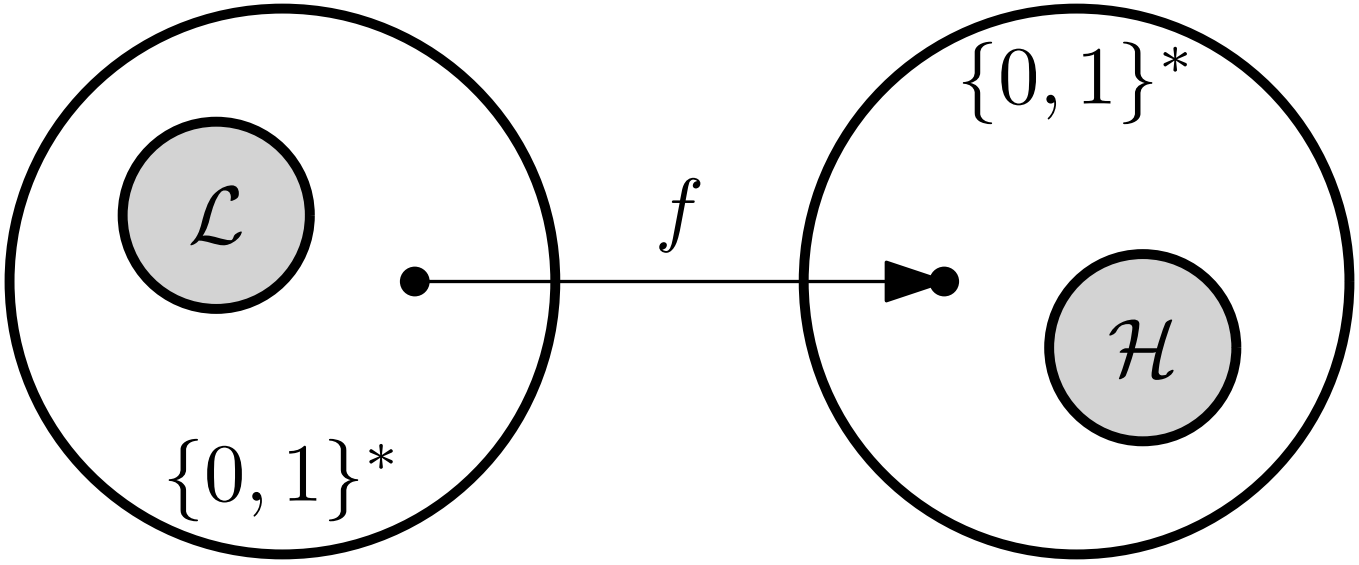
  - Not much, actually! We can only conclude that it is not *too* complicated to solve it.
- We need something else, namely a (pre-order) relation between languages such that two languages being in relation tells us something precise about the **relative** difficulty of deciding them.

# Reductions



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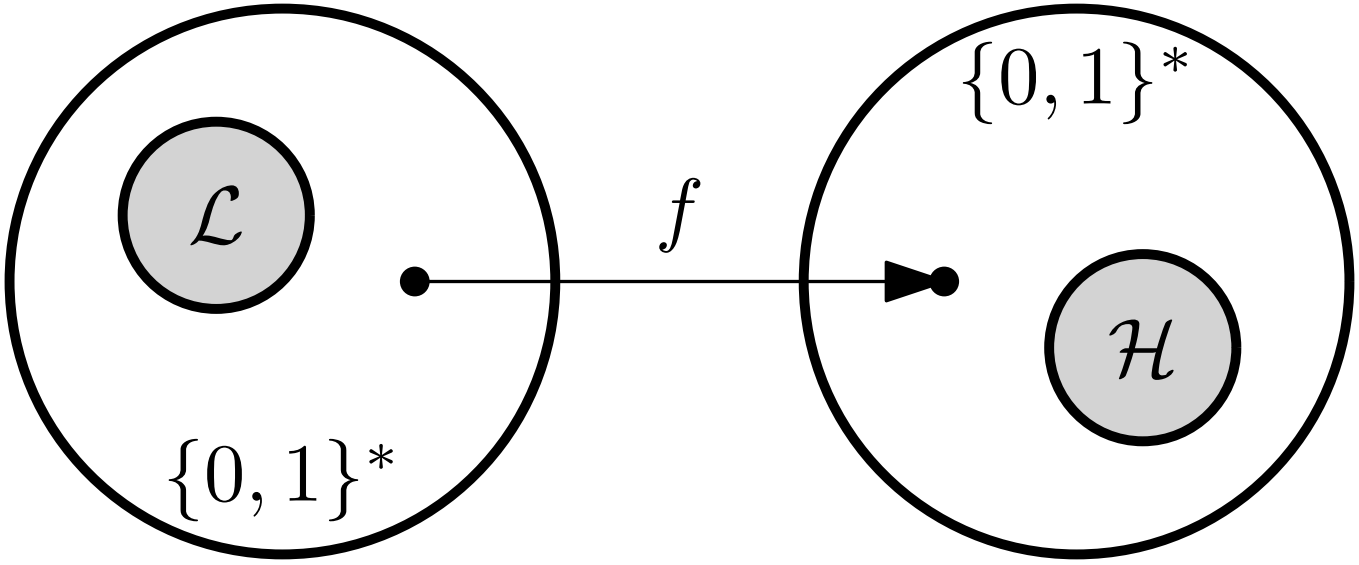


- The language $\mathcal{L}$ is said to be **polynomial-time reducible** to another language $\mathcal{H}$ iff there is a polytime computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that $x \in \mathcal{L}$ iff $f(x) \in \mathcal{H}$.
- In this case, we write $\mathcal{L} \leq_p \mathcal{H}$.

# Reductions and Complexity

- If $\mathcal{L} \leq_p \mathcal{H}$, then $\mathcal{H}$ is at least as difficult as $\mathcal{L}$, at least as far as classes like $\mathbf{P}$ (or above it) are concerned.
  - If, e.g., $\mathcal{L} \leq_p \mathcal{H}$ and $\mathcal{H} \in \mathbf{P}$, then also $\mathcal{L} \in \mathbf{P}$: a way to decide if $x \in \mathcal{L}$ consists in traslating it into $f(x)$ (which can be done in polynomial time), then checking whether $f(x) \in \mathcal{H}$.

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- A language $\mathcal{H} \subseteq \{0,1\}^*$ is said to be:
  - **NP**-hard if $\mathcal{L} \leq_p \mathcal{H}$ for every $\mathcal{L} \in \mathbf{NP}$.
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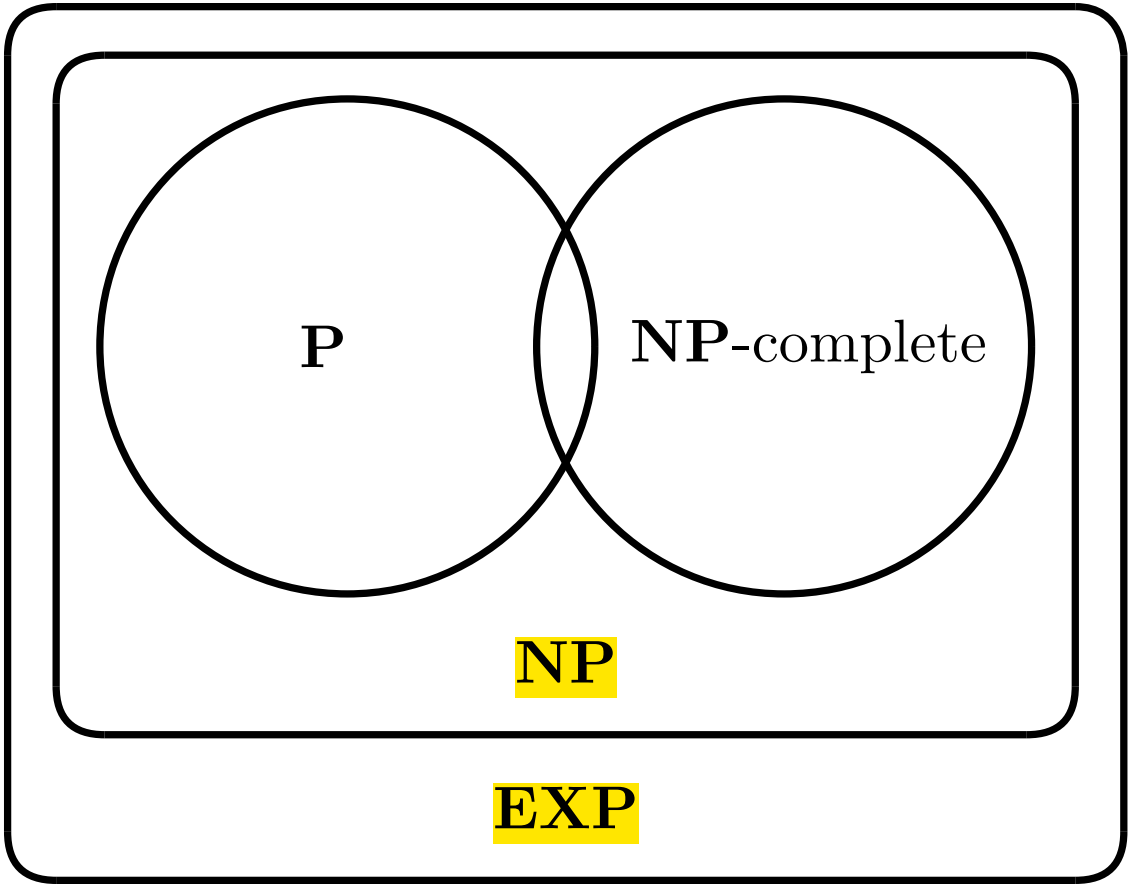
**Theorem**

1. *The relation $\leq_p$ is a pre-order (i.e. it is reflexive and transitive).*
2. *If $\mathcal{L}$ is $\mathbf{NP}$-hard and $\mathcal{L} \in \mathbf{P}$, then $\mathbf{P} = \mathbf{NP}$.*
3. *If $\mathcal{L}$ is $\mathbf{NP}$-complete, then $\mathcal{L} \in \mathbf{P}$ iff $\mathbf{P} = \mathbf{NP}$.*


P
NP-complete
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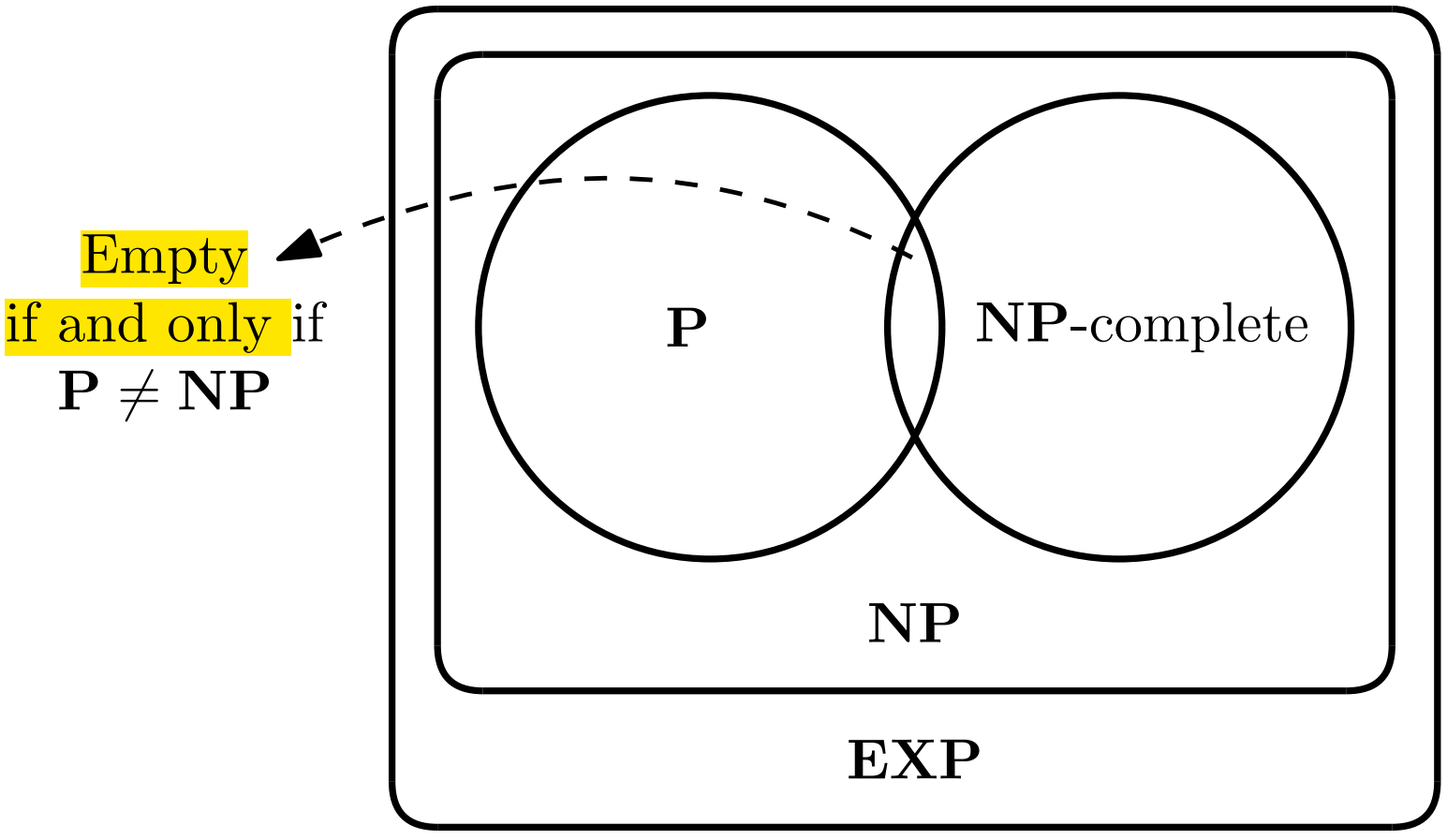
Empty
if and only if
$\mathbf{P} \neq \mathbf{NP}$
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$$\texttt{TMSAT} = \{(\alpha, x, 1^n, 1^t) \mid \exists u \in \{0,1\}^*. \mathcal{M}_\alpha \text{ outputs 1 on input } (x, u) \text{ within } t \text{ steps}\}$$

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**Theorem**

`TMSAT` *is* **NP***-complete.*

- Although interesting from a purely theoretical perspective, the language `TMSAT` is very specifically tied to Turing Machines, and thus of no practical importance.

# A Quick Recap on Propositional Logic

- Formulas of **propositional logic** are either:
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# The Cook-Levin Theorem

- A propositional formula $F$ is said to be in **conjunctive normal form** (or a **CNF**) when it is a conjuction of disjunctions of *literals* (a literal being a variable or its negation).
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**Theorem (Cook-Levin)**

*The following two languages are* **NP***-complete:*

$$\texttt{SAT} = \{\llcorner F \lrcorner \mid F \text{ is a satisfiable CNF}\}$$

$$\texttt{3SAT} = \{\llcorner F \lrcorner \mid F \text{ is a satisfiable 3CNF}\}$$

# Thank You!

Questions?

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  - To do that, we consider any possible polynomial $p$ and any polytime deterministic TM $\mathcal{M}$.
    - Intuitively, these exist because $\mathcal{L} \in \mathbf{NP}$
  - We define a polynomial-time transformation $x \mapsto \varphi_x$ from strings to CNFs such that

$$\varphi_x \in \texttt{SAT} \Leftrightarrow \exists y \in \{0,1\}^{p(|x|)}.\mathcal{M}(x,y) = 1$$

    - This is the crucial step, and requires quite some work. It amounts to showing that computation in TM is *inherently local*.

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  - We consider any language $\mathcal{L} \in \mathbf{NP}$, and we show that $\mathcal{L} \leq_p \texttt{SAT}$.
  - To do that, we consider any possible polynomial $p$ and any polytime deterministic TM $\mathcal{M}$.
    - Intuitively, these exist because $\mathcal{L} \in \mathbf{NP}$
  - We define a polynomial-time transformation $x \mapsto \varphi_x$ from strings to CNFs such that

$$\varphi_x \in \texttt{SAT} \Leftrightarrow \exists y \in \{0,1\}^{p(|x|)}.\mathcal{M}(x,y) = 1$$

    - This is the crucial step, and requires quite some work. It amounts to showing that computation in TM is *inherently local*.
  - Finally, we need to show that $\texttt{SAT} \leq_p \texttt{3SAT}$.

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- Suppose you are studying the complexity of a language $\mathcal{L}$, and you are convinced that the underlying problem is **hard**. How should you back your claim?
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- By proving that $\mathcal{L} \in \mathbf{NP}$?
  - Again, this **does not** mean much.
- By proving that $\mathcal{L}$ is **NP**-complete?
  - **Yes**, this way you prove that the problem is not so hard (being in **NP**), but not so easy either (unless $\mathbf{P} = \mathbf{NP}$).

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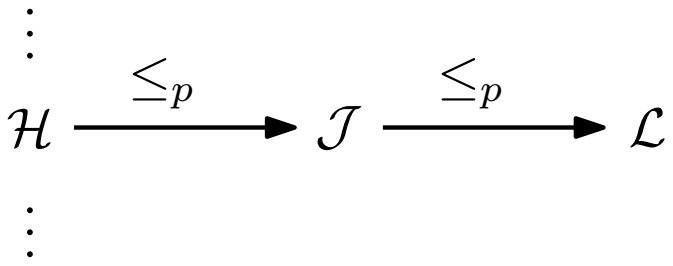
     - This is typically rather easy.
  2. That any other language $\mathcal{H} \in \mathbf{NP}$ is such that $\mathcal{H} \leq_p \mathcal{L}$.
     - We can of course prove the statement directly.
     - More often (e.g. when showing `3SAT` **NP**-complete), one rather proves that $\mathcal{J} \leq_p \mathcal{L}$ for a language $\mathcal{J}$ which is already known to be **NP**-complete.
     - This is correct, simply because $\leq_p$ is transitive:



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**Theorem**

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- There is an easy polytime reduction from `SAT` to `ILP`.

# The Graph of **NP**-complete Problems

- For any pair $\mathcal{L}, \mathcal{H}$ of **NP**-complete problems, we have that

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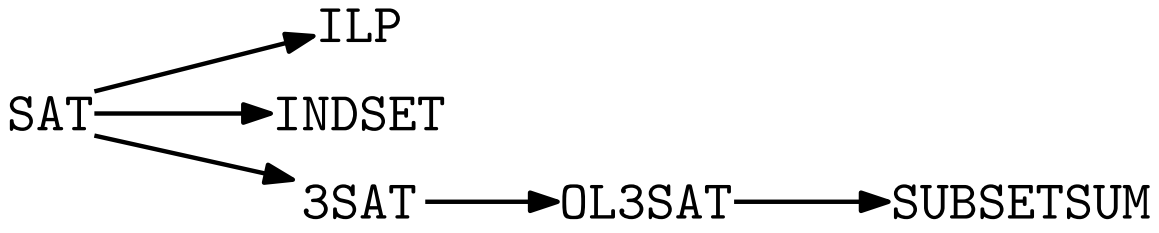

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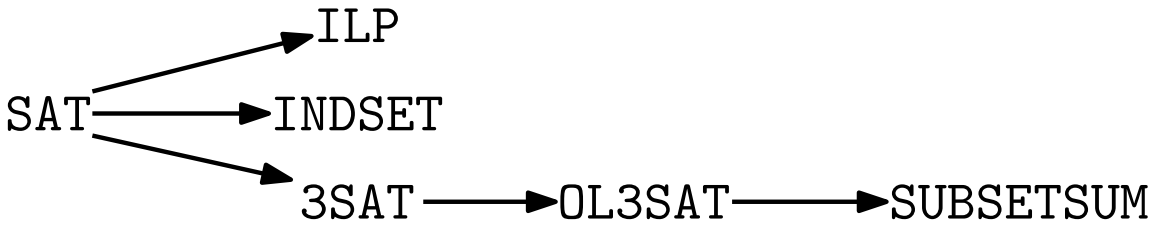


  - In fact, this graph is **huge**: thousands of different problems are known to be **NP**-complete

# Mitigating the Computational Difficulty of **NP**-complete Problems

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- This is often **very useful**, because specialised tools for `SAT`, called SAT-solvers do exist.
  - They do not work in polynomial time.
  - Concretely, they work extremely well on a relatively large class of formulas.

# Thank You!

Questions?

# Languages and Algorithms for Artificial Intelligence (Third Module)

## A Glimpse into Computational Learning Theory

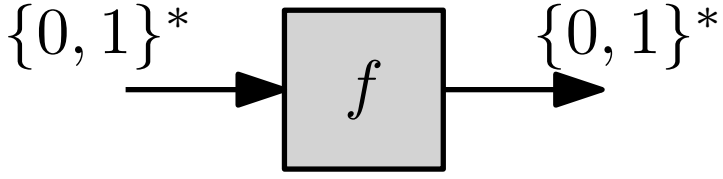
Ugo Dal Lago

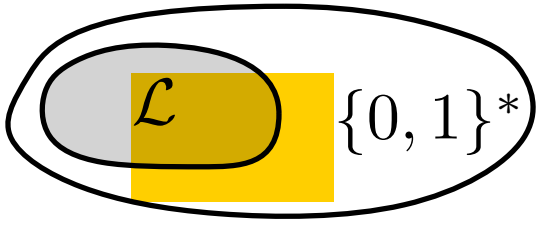


University of Bologna, Academic Year 2019/2020

# Learning Problems as Computational Problems

- We have so far taken functions and languages as our notions of **computational tasks**:



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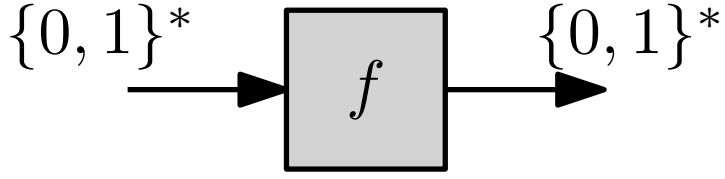


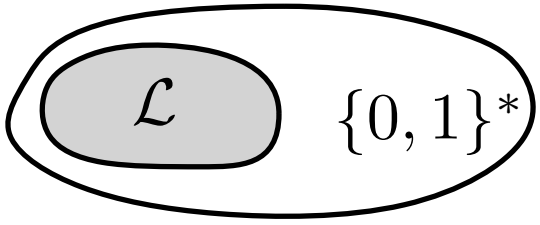


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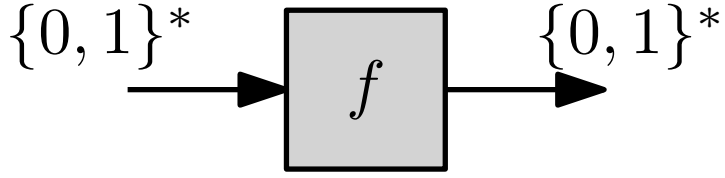
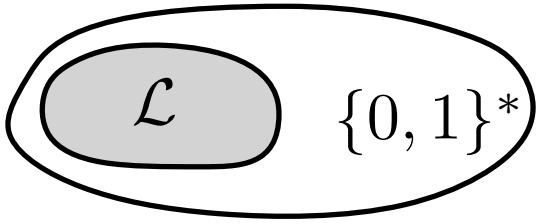
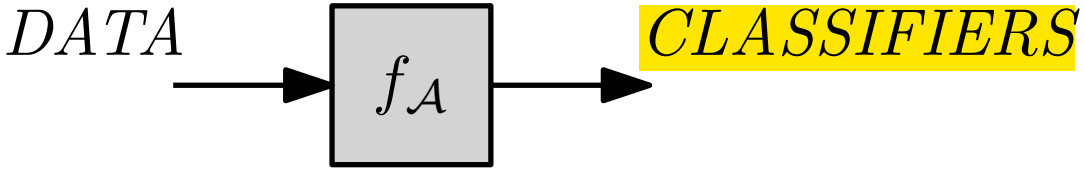



- Is it that **learning problemes**, among the most crucial tasks in AI, can be seen as computational problems?
- The answer is positive: any learning algorithm $\mathcal{A}$ actually computes a function $f_{\mathcal{A}}$ whose input is a finite sequence of *labelled data* and whose output can be seen as a *classifier*:



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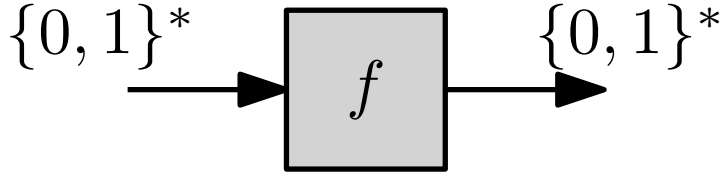


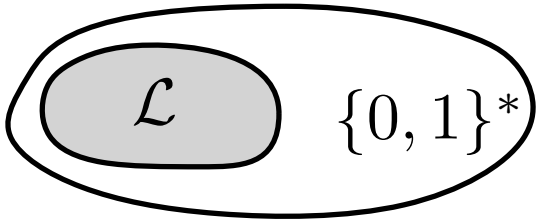


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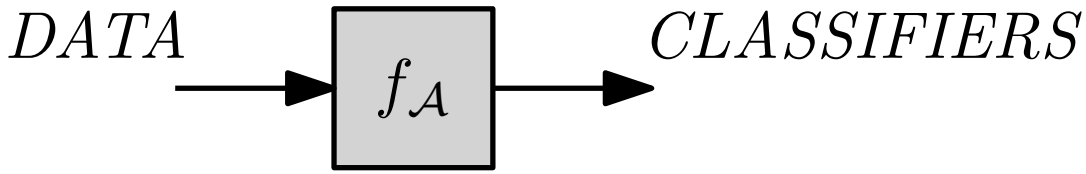
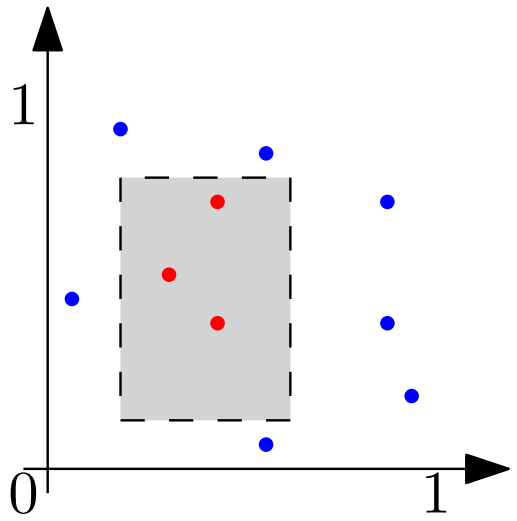



  - Could data and classifiers be encoded as strings, thus turning $f_{\mathcal{A}}$ as a function of the kind we know?
  - How could we formalize the fact that $\mathcal{A}$ correctly solves a given learning task?

# An Example Problem

Suppose that the data the algorithm $\mathcal{A}$ takes in input are points $(x, y) \in \mathbb{R}_{[0,1]} \times \mathbb{R}_{[0,1]}$ (where $\mathbb{R}_{[0,1]}$ is the set of real numbers between 0 and 1). These are labelled as positive or negative depending on they being inside a rectangle



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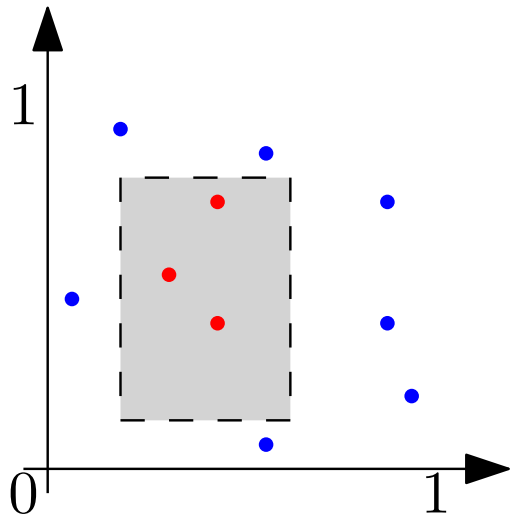


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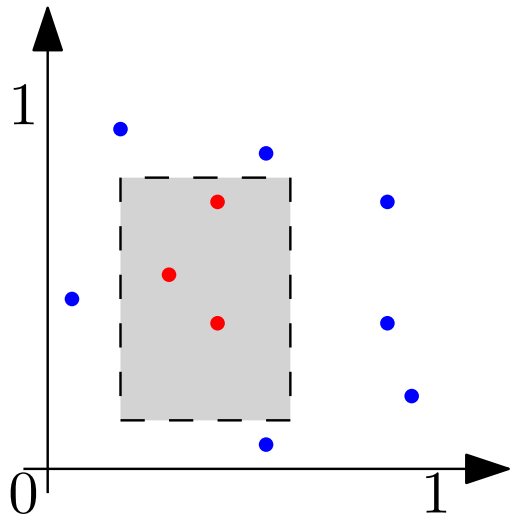


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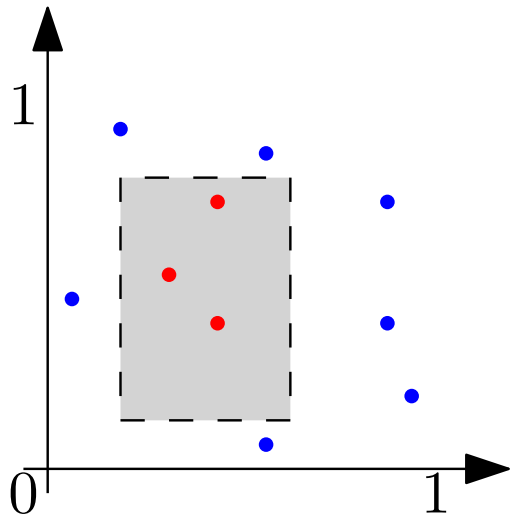


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- The algorithm $\mathcal{A}$ cannot guess the rectangle $R$ with perfect accuracy if the data it receives in input are too few. As the data in $D$ grow in number, we would expect the rectangle $f_{\mathcal{A}}(D)$ to converge to $R$, wouldn't we?

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  - It is supposed to "do the job" for each possible distribution $\mathbf{D}$.
- $\mathcal{A}$ is an ordinary algorithm.
  - Ultimately, it can be seen as a TM.
  - We thus assume that real numbers can be appropriately approximated as binary strings.
  - In some cases, it is useful to assume $\mathcal{A}$ to have the possibility to "flip a coin", i.e., to be a randomized algorithm.

# The Algorithm $\mathcal{A}_{\mathsf{BFP}}$

- We could define an Algorithm $\mathcal{A}_{\mathsf{BFP}}$ as follows:
  1. Given the data $((x_1, y_1), p_1), \ldots, ((x_n, y_n), p_n)$;
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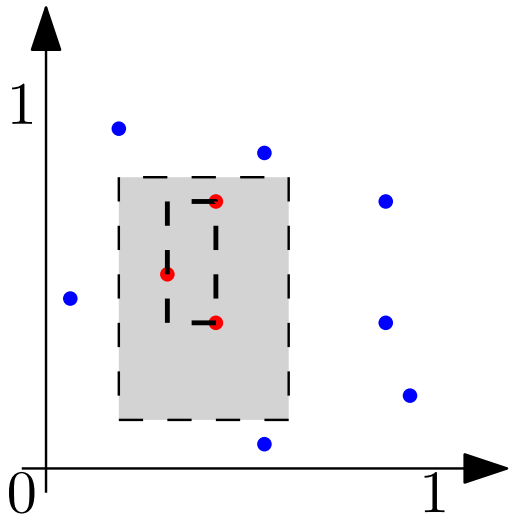

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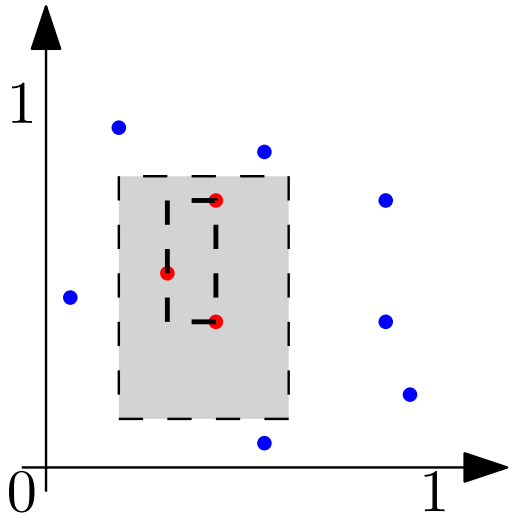


- The output classifier is a rectangle, which can be easily represented as a pair of coordinates.

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- As the number of samples in $D$ grows, the result $\mathcal{A}_{\mathsf{BFP}}(D)$ does *not* necessarily approach the target rectangle, but its probability of error approaches zero.

**Theorem**

*For every distribution* $\mathbf{D}$, *for every* $0 < \varepsilon < \frac{1}{2}$ *and for every* $0 < \delta < \frac{1}{2}$, *if* $m \geq \frac{4}{\varepsilon} \ln\left(\frac{4}{\delta}\right)$, *then*

$$\Pr_{D \sim \mathbf{D}^m}[error_{\mathbf{D},T}(\mathcal{A}_{\mathsf{BFP}}(T(D)) < \varepsilon] > 1 - \delta$$

# The General Model — Terminology

- We assume to work within an **instance space** $X$.
  - $X$ is the set of (encodings) of instances of objects the learner wants to classify.
  - Data from the instance spaces are generated through a distribution $\mathbf{D}$, unknown to the learner.
  - In the example, $X = \mathbb{R}^2_{[0,1]}$.

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- We assume to work within an **instance space** $X$.
  - $X$ is the set of (encodings) of instances of objects the learner wants to classify.
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  - In the example, $X = \mathbb{R}^2_{[0,1]}$.
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  - In the example, concepts are arbitrary subsets of $X = \mathbb{R}^2_{[0,1]}$, i.e. arbitrary regions within $\mathbb{R}^2_{[0,1]}$.
- A **concept class** $\mathcal{C}$ is a collection of concepts, namely a subset of $\mathcal{P}(X)$. These are the concepts which are sufficiently simple to describe, and that algorithms can handle.
  - The concept class $\mathcal{C}$ we work with in the example is the one of rectangles whose sides are parallel to the axes.
  - The **target concept** $c \in \mathcal{C}$ is the concept the learner wants to build a classifier for.

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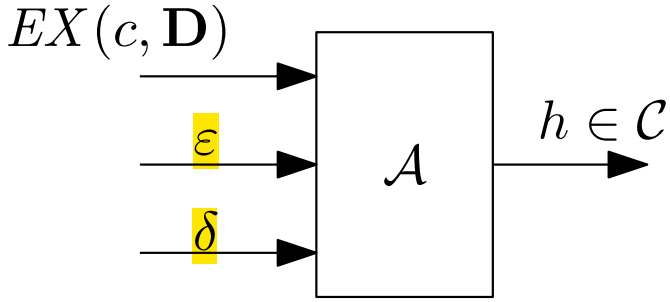


where:

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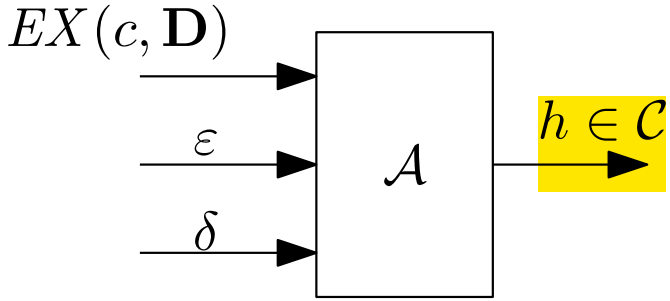


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- The **error** of any $h \in \mathcal{C}$ is defined as $error_{\mathbf{D},c} = \Pr_{x \sim \mathbf{D}}[h(x) \neq c(x)]$.

# The General Model — PAC Concept Classes

- Let $\mathcal{C}$ be a concept class over the instance space $X$. We say that $\mathcal{C}$ is **PAC learnable** iff there is an algorithm $\mathcal{A}$ such that for every $c \in \mathcal{C}$, for every distribution $\mathbf{D}$, for every $0 < \varepsilon < \frac{1}{2}$ and for every $0 < \delta < \frac{1}{2}$, then

$$\Pr[error_{\mathbf{D},c}(\mathcal{A}(EX(c,\mathbf{D}),\varepsilon,\delta)) < \varepsilon] > 1 - \delta$$

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## Corollary

*The concept-class of axis-aligned rectangles over $\mathbb{R}^2_{[0,1]}$ is efficiently PAC-learnable.*

# Thank You!

Questions?

# Representation Classes

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- Examples
  - $X_n$ could be $\{0,1\}^n$, the set of **boolean vectors** of of (fixed!) length $n$, and $\mathcal{C}_n$ is the set of all subsets of $\{0,1\}^n$ *represented by CNFs.*
  - $X_n$ could rather be $\mathbb{R}^n$, the set of **vectors of real numbers** of length $n$, while $\mathcal{C}_n$ are say, the subsets of $\mathbb{R}^n$ *represented by some form of neural network* with $n$ inputs and 1 output.

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- In many cases (e.g. SGD), one has a *single* learning algorithm that work for every value of $n$. In that case, we allow (in the definition of efficient PAC learning) the algorithm $\mathcal{A}$ to take time polynomial in $n$, $size(c)$, $\frac{1}{\varepsilon}$ and $\frac{1}{\delta}$.

# Boolean Functions as a Representation Class

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  - As an example, the conjunction

    $$x_1 \wedge \neg x_2 \wedge x_4,$$

    defines a subset of $\{0,1\}^4$.
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    defines a subset of $\{0,1\}^4$.
  - *Not all* subsets of $\{0,1\}^n$ can be captured.
- A **second example** of a representation class for $X$ is a class we know, namely the class $\mathbf{CNF}_n$ of CNFs over $x_1, \ldots, x_n$, which are conjunction *of disjunctions* of literals.
  - CNFs are normal forms of any boolean functions.
  - *All* subsets of $\{0,1\}^*$ can be captured this way.
  - We could even consider $k\mathbf{CNF}_n$ rather than arbitrary one, but this way we would lose universality.

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**Theorem**

*The representation class of boolean conjuctions of literals is efficiently PAC-learnable.*

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  - A **3-term DNF formula** over $n$ bits is a propositional formula in the form $T_1 \vee T_2 \vee T_3$, where each $T_i$ is a conjunction of literals over $x_1, \ldots, x_n$.
  - In a sense, this class is the *dual* to 3CNFs!
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**Theorem**

*If* **NP** $\neq$ **RP**, *then the representation class of 3-term DNF formulas is not efficiently PAC learnable.*

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- **Definitely No!** Actually, we have just *scratched the surface* of computational learning theory.
- Models and results we did not have time to talk about include:
  - The VC Dimension.
  - The Fundamental Theorem of Learning.
  - The No-Free-Lunch Theorem.
  - Occam's Razor.
  - Positive and negative results about neural networks.
  - ...
- More information can be found in of the many excellent books on CLT, e.g.
  - Mehryar Mohri, Afshin Rostamizadeh, and Ameet Talwalkar *Foundations of Machine Learning* Second Edition. The MIT Press. 2018
  - Shai Shalev-Shwartz and Shai Ben-David. *Understanding Machine Learning: from Theory to Algorithms* Cambridge University Press. 2014.
  - Michael Kearns and Umesh Vazirani. *An Introduction to Computational Learning Theory* The MIT Press. 1994.

# Example Results about Neural Networks (from Kearns and Vazirani's Book)

**Theorem 3.7** *Let $G$ be any directed acyclic graph, and let $C_G$ be the class of neural networks on an architecture $G$ with indegree $r$ and $s$ internal nodes. Then the number of examples required to learn $C_G$ is*

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**Theorem 6.6** *Under the Discrete Cube Root Assumption, there is fixed polynomial $p(\cdot)$ and an infinite family of directed acyclic graphs (architectures) $G=\{G_{n^2}\}_{n\geq 1}$ such that each $G_{n^2}$ has $n^2$ boolean inputs and at most $p(n)$ nodes, the depth of $G_{n^2}$ is a fixed constant independent of $n$, but the representation class $C_G=\cup_{n\geq 1}C_{G_{n^2}}$ (where $C_{G_{n^2}}$ is the class of all neural networks over $\Re^{n^2}$ with underlying architecture $G_{n^2}$) is not efficiently PAC learnable (using any polynomially evaluatable hypothesis class). This holds even if we restrict the networks in $C_{G_{n^2}}$ to have only binary weights.*

# Thank You!

Questions?