Propositional Logic

Credits

- Part of the content of the following slides is from:
  - van Dalen's book "Logic and structure"
  - Claudio Sacertoti's slides

Truth

- (Wikipedia) Truth is most often used to mean being in accord with fact or reality, or fidelity to an original or standard. Truth is also sometimes defined in modern contexts as an idea of "truth to self", or authenticity.
- Physics, chemistry, biology …. Truth associated to the sensible world
- A repeatable experiment defines a truth
- Changes in the sensible world changes the truth
- What happens when there is no sensible world (e.g. in mathematics)?

Truth in mathematics

Your probable truth:

- The sum of the internal angles of a triangle il 180^0, or in an equivalent way
- there exists one and only one line which goes through a given point and is parallel to a given line (V postulate of Euclidean geometry)
- However …

Truth in non euclidean geometries

Spherical geometry (after Riemann geometry, 1854)

- Point = point on a sphere.
- Line = maximum circle on a sphere, obtained by intersecting the sphere with a plane passing through the centre of the sphere.
- Given two points there exists only one line that goes through them
- Any two lines always have at least two points in common.
- The sum of the inner angles of a triangle is greater than 180 degrees.

Consequences

- The axiomatic method is not a description of the sensible world
- Geometry (mathematics) does not seem to provide a priori synthetic judgments
- Rather it provides the description of a set of worlds (models) in which
  1. One can interpret the entities (points, straight, etc.) as s/he wish, provided that the axioms are respected
  2. different worlds have different properties
  3. only some of these properties can be demonstrated from the axioms (incompleteness)

Consequences 2

- Impact on the vision of mathematics
- A theory should not be discarded/accepted if satisfied (banally) by the sensitive world, but if allows some models
- A theory that admits models is never inconsistent (that is, one cannot derive absurd)
- The absolute notion of truth (given by the sensible world) replaced by the concept of

  logical consequence

Summary

- Truth of the sensible world: the truth changes as the world changes (restricted mutations)
- Truth of mathematics: the truth changes by adopting new worlds (not restricted mutations)
- In addition, the truth changes by interpreting the words used in a sentence in different ways
- What remains of the notion of truth if the world changes arbitrarily?

Logical Consequence

- Let $\Gamma = F_1, \ldots, F_1$ be a set of sentences (hypothesis or premises) and F a be a sentence (conclusion)
- F is a logical consequence of Γ, written Γ F if, for different interpretations of the formulas in the various possible worlds, it is always true that: if all the formulas in Γ are true (in the world under examination) then also F is true.

Logical Equivalence

- F logically equivalent to G iff F logical consequence of G and G logical consequence of F
- Exercise: Prove that this is an equivalence relation

Conclusion about truth

- Truth is always defined w.r.t. a world and a given intepretation of symbols in that world
- Truth of a single sentence is not interesting (changes when the world changes)
- What is interesting is the notion of logical consequence and logical equivalenc e (do not change with the world)

Propositional Logic

Propositional logic is the simplest logic- illustrates basic ideas using **propositions**

- P1 Snow is white
- P2 Today it is raining
- P3 This course is boring

Pi is an atom or atomic formula
Each Pi can be either **true** or **false** but never both
The values true or false assigned to each proposition is called **truth value** of the proposition

Connectives

Many connectives used in natural language: and, or, not, but, since, if ... then ... etc. Why in logic we use a few of them?

- Max is a Marxist, but he is not humourless
- π irrational, but it is not algebraic

"But" in the first statement carries some emotional meaning (surprise) while in the second statement such a surprise is less immediate (one has to know that almost all irrationals are algebraic) and we can use "and".

We stick to some principle of economy and we select connectives which arde useful for expressing proofs. such as "and" and "or" . However also in this case we have some ambiguities ...

Connectives problems 1

- John drove on and hit a pedestrian.
- John hit a pedestrian and drove on.

Problems:

- It seems that "and" 'and' may have an ordering function in time, while this is not the case in mathematics (and logic)

Need to create an artificial language with precise meanings of connectives.

Connectives problems 2

1. If I open the window then we'll have fresh air.
2. If I open the window then $1 + 3 = 4$.
3. If $1 + 2 = 4$, then we'll have fresh air.
4. John is working or he is at home.
5. Euclid was a Greek or a mathematician.

Problems:

- From 1: there is a relation between the premise and conclusion
- From 2 and 3: no relation between the premise and conclusion. Perfectly adequate in mathematics though, often called *material implication*
- We tend to accept 4 and to reject 5.

Need to create an artificial language with precise meanings of

Alphabet for Propositional Logic

Definition 1.1.1 *The language of propositional logic has an alphabet consisting of*

(i) *proposition symbols :* $p_0, p_1, p_2, \ldots,$
(ii) *connectives :* $\wedge\,, \vee\,, \rightarrow\,, \neg\,, \leftrightarrow\,, \bot\,,$
(iii) *auxiliary symbols :* (,).

The connectives carry traditional names:

$\wedge$ - *and*	- *conjunction*
$\vee$ - *or*	- *disjunction*
$\rightarrow$ - *if ..., then ...*	- *implication*
$\neg$ - *not*	- *negation*
$\leftrightarrow$ - *iff*	- *equivalence, bi-implication*
$\bot$ - *falsity*	- *falsum, absurdum*

The proposition symbols and $\bot$ stand for the indecomposable propositions, which we call *atoms*, or *atomic propositions*.

Inductive definition of syntax of propositional logic

Definition 1.1.2 *The set $PROP$ of propositions is the smallest set X with the properties*
(i) $p_i \in X (i \in N)$, $\perp \in X$,
(ii) $\varphi, \psi \in X \Rightarrow (\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi), (\varphi \leftrightarrow \psi) \in X$,
(iii) $\varphi \in X \Rightarrow (\neg\varphi) \in X$.

Useful for proving (by induction) properties of formulae

Inductive definition of syntax

Theorem 1.1.3 (Induction Principle) *Let A be a property, then $A(\varphi)$ holds for all $\varphi \in PROP$ if*
(i) $A(p_i)$, *for all i,and* $A(\perp)$,
(ii) $A(\varphi)$, $A(\psi) \Rightarrow A((\varphi \square \psi))$,
(iii) $A(\varphi) \Rightarrow A((\neg\varphi))$.

Proof. Let $X = \{\varphi \in PROP \mid A(\varphi)\}$, then X satisfies (i), (ii) and (iii) of definition 1.1.2. So $PROP \subseteq X$, i.e. for all $\varphi \in PROP$ $A(\varphi)$ holds. $\square$

Where $\square$ indicate any binary connective