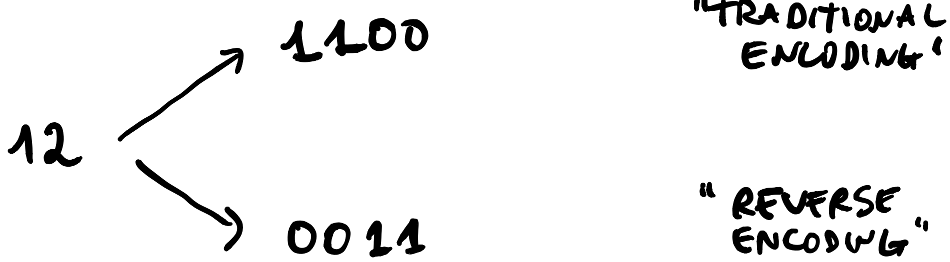
EXERCISE 1

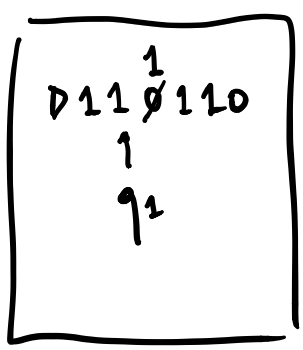
SHOW THAT THE FUNCTION $INC: \mathbb{N} \to \mathbb{N}$ SUCH THAT $INC(n) = n+1$ CAN BE COMPUTED IN LINEAR TIME BY GIVING AN <u>EXPLICIT</u> CONSTRUCTION OF A TM COMPUTING THE FUNCTION.

- How should we encode the natural number or strings?

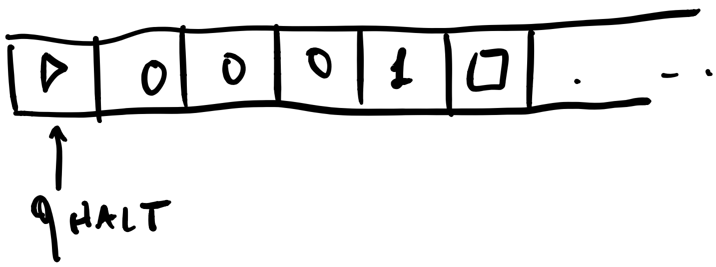


- We can, if we use the reverse encoding, give a TM working with just <u>ONE</u> tape, which serves as an input, output and work tape. Let's write its transition function.

$$(q_{INIT}, \triangleright) \xmapsto{\delta} (q_0, \triangleright, R)$$
$$(q_0, 0) \xmapsto{\delta} (q_1, 1, L)$$
$$(q_0, 1) \xmapsto{\delta} (q_0, 1, R)$$
$$(q_0, \square) \xmapsto{\delta} (q_2, 1, L)$$
$$(q_1, 1) \xmapsto{\delta} (q_1, 0, L)$$
$$(q_1, \triangleright) \xmapsto{\delta} (q_{HALT}, \triangleright, S)$$



$$(q_2, 1) \xmapsto{\delta} (q_2, 0, L)$$
$$(q_2, \triangleright) \xmapsto{\delta} (q_{HALT}, \triangleright, S)$$



EXERCISE 2

GIVEN A BINARY STRING $x \in \{0,1\}^*$, WE INDICATE AS $|x|_0$ AND AS $|x|_1$, RESPECTIVELY, THE NUMBER OF 0s AND 1s OCCURRING IN x, E.G.

$$|00101|_0 = 3 \qquad |00101|_1 = 2$$

SHOW THE FOLLOWING LANGUAGE TO BE DECIDABLE IN TIME PROPORTIONAL TO $n \lg n$:

$$\mathcal{L}_= = \{x \in \{0,1\}^* \mid |x|_1 = |x|_0\}$$

- We can solve this exercise by describing a 4-tape TM which works as follows:
  - → in the two work-tapes, the machine keeps track of the number of 0s and 1s read so far from the input (initially they

one 0).

- → The machine proceeds by scanning from left to right, the input and when reading a 0 (respectively a 1) it updates the first worn-tape (respectively, the second) using a "procedure" very similar to the one we have employed in the previous exercise
- → When the scan of the input is over, the TM just compare the strings in the second and third tapes, bit by bit. If there is any position in which they differ, the machine writes 0 in the output tape and halts, otherwise it writes 1 in the output tapes and halts.

· How about the complexity of this construction?

- → The initial setup of the machine consists in writing "0" in the second and third tapes, and thus taxes constant time
- → The input is then scanned once, and for every symbol in the input, one of the two worn-tapes needs to be updated, which taxes linear

time in its content. Since, however, the numbers in the two work-tapes are always smaller than the size of the input their representations as strings are at most $\lg|x|$-long. Altogether, then the scan takes

$$O(|x| \cdot \lg |x|)$$

steps

→ At the end of the scan, the second and third tapes need to be compared and this takes altogether, time

$$O(\lg |x|)$$

→ Altogether, then, the time needed to compute the result is

$$O(1) + O(|x| \lg |x|) + O(\lg |x|)$$
$$\in O(|x| \lg |x|)$$

QED.