LOGIC = IT'S A FORMAL* LANGUAGE. IT HAS SYNTAX AND SEMANTIC. ITS AIM IS TO PROOF VALIDITY OF INFERENCES BY METHODS WHICH EXPLOIT ONLY SYNTACTIC RULES.
(* = SYNTAX + SEMANTIC + PROOF SYSTEM.)

THERE ARE MANY DIFFERENT LOGIC FORMS:

CLASSICAL, INTUITIONISTIC, LINEAR, EPISTEMIC, TEMPORAL, MATHEMATICAL ...
- PROPOSITIONAL
- FOL = PREDICATE LOGIC → WE'LL SEE ONLY THIS
- 2° ORDER ..
- ---

WHY DO WE STUDY IT?

1) COMPUTER SCIENCE DERIVES FROM IT

2) IT IS THE FUNDATION OF SOME DECLARATIVE LANGUAGES

PROGRAMMING LANGUAGES
- IMPERATIVE (WRITE HOW TO ACHIEVE THE GOAL)
  - PROCEDURAL C, PASCAL, FORTRAN
  - OBJECT ORIENTED JAVA, C++
    - (SCALA, PYTHON)
- DECLARATIVE (WRITE WHAT IS YOUR GOAL)
  - FUNCTIONAL LISP, ML, HASKELL
  - LOGIC PROLOG
    - CLP

3) FOUNDATION OF THEORETICAL COMPUTER SCIENCE
- STUDY OF LIMITS OF HUMAN KNOWLEDGE

4) PROVING CORRECTNESS OF SOFTWARE / LANGUAGE
- MOST PROPERTIES ARE NOT DECIDABLE, FOR INSTANCE YOU CAN'T SAY IF A PROGRAM NEVER TERMINATES BECAUSE YOU SHOULD HAVE INFINITE TIME.
  - → LOGIC HELP US.

5) LOGIC IS A LANGUAGE "PARADOX FREE"
(UNLIKE NATURAL LANGUAGE AND MATHEMATIC AND SOME PROGRAMMING LANGUAGE)

- AI STUDIES HOW TO IMPLEMENT "KNOWLEDGE BASED AGENTS" WHICH EMULATE HUMANS: GIVEN A KNOWLEDGE OF THE WORLD/ENVIRONMENT THEY ARE ABLE TO ACHIEVE THE GOAL THANKS TO REASONING. BETWEEN THE GIVEN KNOWLEDGE AND THE INFERENCE OF SOMETHING ELSE THERE'S THE REASONING.

THE INITIAL/GIVEN KNOWLEDGE OF AN AGENT IS ALSO CALLED "BACKGROUND KNOWLEDGE". THE SENTENCES WHICH COMPOSE IT ARE CALLED "AXIOMS" (= NOT PROVEN, TAKEN AS TRUE). THEN THANKS TO INFERENCE THE AGENT MUST BE ABLE OF ADD NEW SENTENCES and ASK IF A SENTENCE IS TRUE. THE SET OF SENTENCES IS ALSO CALLED KB = KNOWLEDGE BASE.

SENTENCES CAN ALSO BE ADDED THANKS TO THE OUTPUT OF SENSING ACTIONS.

HUMANS DO REASONING/INFERENCES IN MANY WAYS:

- LOGICAL INFERENCES (DEDUCTIVE REASONING, INDUCTIVE REASONING, ABDUCTIVE REASONING)
- BY ANALOGIES
- PROBABILITY AND STATISTIC

...

WE'RE INTERESTED ON IMPLEMENT LOGICAL INFERENCES.

- IN THE NEXT PAGE: Γ = SET OF FORMULAS i.e. $\{F_1, F_2, F_3, \dots\}$ F = FORMULA = e.g. $P_1 \vee P_2 \wedge \neg P_3 \dots$

FUNDAMENTAL CONCEPTS (INDIPENDENT FROM THE PARTICULAR LOGIC FORM)

(OF CLASSICAL LOGIC!)

- TRUTH = IN ACCORD WITH FACT/REALITY, IT DEPENDS ON THE WORLD/MODEL CONSIDERED
- MODEL = "POSSIBLE WORLD CONSIDERED". (→ THIS ESTABLISH THE INTERPRETATION TO GIVE TO THE FORMULA) GIVEN A MODEL m and a FORMULA F we say: m is a MODEL OF F = m SATISFIES F = F IS TRUE IN m.

  WE INDICATE WITH $M(F)$ THE SET OF ALL MODELS WHICH SATISFY F.
- THEORY = SET OF AXIOMS ON WHICH CAN BE BASED MANY MODELS. (e.g. MATHEMATICAL THEORY HAS EUCLIDEAN MODEL, PROJECTIVE MODEL...)

  (IN CLASSICAL LOGIC THEORY A SENTENCE CAN BE ONLY TRUE OR FALSE).

  (THE KB IS ALSO CALLED THEORY)
- WE WANT SOMETHING WHICH IS INDIPENDENT FROM THE MODEL CONSIDERED: LOGIC CONSEQUENCE & LOGIC EQUIVALENCE

LOGIC CONSEQUENCE

Γ = SET OF SENTENCES

SEMANTIC CONSEQUENCE = ENTAILMENT

$$\Gamma \underset{\text{"TRUTH"}}{\overset{\text{SEMANTIC}}{\models}} F$$

iff $M(\Gamma) \subseteq M(F)$

- IT MEANS THAT $\Gamma \models F$ iff F is TRUE IN EACH MODEL IN WHICH Γ IS TRUE.
- THE STUDY OF THIS KIND OF CONSEQUENCE IS CALLED "MODEL THEORY"
- THANKS TO THE "DEDUCTION THEOREM" WE CAN SAY THIS: $F_1 \models F_2$ iff $\models (F_1 \to F_2)$
- $\models F$ MEANS THAT F IS VALID = TRUE ∀ MODEL
- SEMANTIC CONSEQUENCE IS STUDIED BY "MODEL THEORY" = (= "MODEL CHECKING") SHOWING THAT $M(\Gamma) \subseteq M(F)$ (OR STH WHICH IS PROVED TO BE EQUIVALENT) BY TRYING ALL THE INTERPRETATIONS.

SYNTACTIC CONSEQUENCE = INFERENCE = DERIVATION

$$\Gamma \underset{j(i)}{\overset{\text{SYNTACTIC}}{\vdash}} F$$

"DERIVABILITY" = "PROVABILITY"

- IT MEANS THAT ∃ A DERIVATION WITH HYPOTHESIS IN Γ WITH CONCLUSION F, WHICH MEANS THAT: F IS SYNTATICALLY DERIVABLE FROM Γ
- $\vdash F$ MEANS THAT F IS A THEOREM
- SYNTACTIC CONSEQUENCE IS STUDIED BY "PROOF THEORY", WHICH IS BASED ON THE DEMOSTRATION OF $\Gamma \vdash F$ ONLY BY SYNTACTIC RULES, FOR INSTANCE "NATURAL DEDUCTION", OR "HILBERT-STYLE SYSTEMS", "RESOLUTION" (= "THEOREM PROVING")

THANKS TO THE THEOREM OF COMPLETENES

$\Gamma \models F \iff \Gamma \vdash F$ ("IS EQUIVALENT TO")

WE CAN USE BOTH TECHNIQUES, REGARDLESS IF WE WANT TO PROVE $\Gamma \models F$ OR WE WANT TO PROVE $\Gamma \vdash F$

(DON'T BE CONFUSED WITH THE PROPERTY OF COMPLETENESS OF A DEDUCTIVE SYSTEM)

• LOGIC EQUIVALENCE

$F_1 \equiv F_2$ iff $F_1 \models F_2$ & $F_2 \models F_1$

OR

$F_1 \Leftrightarrow F_2$

(which means "iff THEY ARE TRUE IN THE SAME MODELS"
which can be written like this "iff $M(F_1) = M(F_2)$
WHICH CAN BE EXPRESSED LIKE $\models (F_1 \leftrightarrow F_2)$)

• THE MAIN IDEA IS THIS :
WE'LL DISCUSS THE "MODEL CHECKING" APPROACHES (SOME DIFFERENT WAYS TO IMPLEMENT THEM AND SOME PROBLEMS/LIMITATIONS) THEN WE'LL DISCUSS THE "THEOREM PROVING" APPROACHES (= DEDUCTIVE SYSTEMS = INFERENCE ALGORITHMS = INFERENCE PROCEDURES, ...).
THE SET OF POSSIBLE DEDUCTIVE SYSTEMS APPLICABLE TO A CERTAIN SYNTAX DEPENDS ON THE FORM OF LOGIC CONSIDERED, THAT'S WHY WE'LL TALK AT FIRST ONLY OF GENERAL IDEAS AND PROPERTIES THEN WE'LL SEE THE DEDUCTIVE SYSTEMS OF PROPOSITIONAL AND FIRST ORDER LOGIC.
THE MODEL CHECKING APPROACHES HAVE HIGHER COMP. COST, THAT'S WHY WE'LL FOCUS MOSTLY ON DEDUCTIVE SYSTEMS.

• WHY DO WE CARE SO MUCH ABOUT LOGIC CONSEQUENCES?
- WE COULD USE IT TO ADD KNOWLEDGE IN AN AGENT
- DEDUCTIVE PROCEDURES CAN BE USED TO OBTAIN PLANS (DEDUCTIVE PLANNING)
- CREATE NEW WAYS OF SOLVING PROBLEMS (DECLARATIVE LANGUAGES)

• WHEN YOU READ ABOUT A PROOF STRATEGY, OR A WAY TO DERIVE SMTH, IT IS MEANT TO TALK ABOUT SYS WHICH WORK ON SYNTAX, SO A THEOREM PROVING APPROACH.

(TAKE TIME TO READ AND UNDERSTAND WELL THESE TWO PAGES BEFORE TO CONTINUE BECAUSE ARE THE CORE OF EVERYTHING.)

KEEPING IN MIND OUR GOAL ABOUT THE USE OF LOGIC, LET'S DISCUSS SOME NOTIONS which are TRUE for all CLASSICAL LOGIC, THEN WE'LL TALK IN PARTICULAR ABOUT PROPOSITIONAL L. AND FOL.

- **SYNTAX**: SET OF RULES which DEFINES WHICH EXPRESSIONS ARE LEGAL IN THAT LANGUAGE.

  e.g.
  - $x+4=y$ (IS LEGAL IN ALGEBRA)
  - $x+4=$!!!?$=0+-$ (IS ILLEGAL IN ALGEBRA)
  - HOW CAKE PEN ME PARK? (IS ILLEGAL IN NATURAL LANGUAGE)
  - SHREK IS MOTHER OF COWS WITH PURPLE IDEAS. (IS LEGAL IN NATURAL LANGUAGE ALSO IF IT DOESN'T MAKE SENSE! THAT'S A PROBLEM OF SEMANTIC)

- **SEMANTIC**: IS THE MEANING OF A SYNTACTICALLY WELL FORMED EXPRESSION IN THAT LANGUAGE. WHICH IN OTHER WORDS MEANS TO GIVE AN INTERPRETATION TO AN EXPRESSION.

  NOTE:
  - ALSO SYMBOLS HAVE A SEMANTIC IN A LANGUAGE, FOR INSTANCE "+" IN ALGEBRA HAS A MEANING while "+" WHEN DEALING WITH STRINGS IN PYTHON HAS ANOTHER MEANING.
  - THERE'S A DIFFERENCE BETWEEN THE SEMANTIC OF A LANGUAGE / THEORY AND OF A MODEL.

    Ex. THE SEMANTIC OF LINEAR ALGEBRA IS THAT YOU MUST INTERPRETE THIS $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ AS A MATHEMATICAL ENTITY. AND THIS IS THE SEMANTIC OF THE THEORY.

    BUT THESE ARE TWO POSSIBLE INTERPRETATIONS:
    - IN THE EUCLIDEAN MODEL THAT'S A 2D POINT IN THE COORDINATE (1,1)
    - IN THE PROJECTIVE MODEL THAT'S A POINT AT INFINITY IN THE DIRECTION (1,1).

    SO IN THE SAME WAY IN LOGIC WE HAVE THE SEMANTIC OF THE THEORY WHICH SAYS THAT TO A SENTENCE YOU CAN ONLY ASSOCIATE THE MEANING TRUE OR FALSE. BUT THIS ASSOCIATION DEPENDS ON THE MODEL USED TO INTERPRETATE THAT SENTENCE

• NOTE THAT TO DESCRIBE A LANGUAGE WE NEED ANOTHER LANGUAGE. WE ARE USING NATURAL LANGUAGE TO DESCRIBE LOGIC. SO IN THIS CASE THE NATURAL LANGUAGE IS CALLED "META LANGUAGE". 6

SOMETIMES CAN BE CONFUSING:

LOGIC HAS SYNTACTIC SYMBOLS $\underbrace{\text{"t" and "F"}}_{\text{\{ TRUTH VALUES \}}}$ TO which we associate THE INTERPRETATION "TRUE" and "FALSE" which has a meaning in NATURAL LANGUAGE. (some books indicate "t" and "F" IN DIFFERENT WAYS, THE MOST CONFUSING ONES GIVE USE THE WORDS "TRUE" AND "FALSE" AS SYNTACTIC ELEMENTS)

NOTE THAT THE SAME THING HAPPENS FOR "&" "OR"...

• SO TO DESCRIBE THE SEMANTIC WE USE A "META LANGUAGE" INSTEAD TO DESCRIBE THE SYNTAX WE USE A "META SYNTAX" / "GRAMMAR". THE MOST USED METASYNTAX FOR LOGIC AND PROGRAMMING LANGUAGES IS:

BNF = BACKUS NAUR FORM = BACKUS NORMAL FORM

IS EASIER TO UNDERSTAND IT WITH AN EXAMPLE: BNR FOR SIMPLE ARITHMETIC EXPRESSION:

EXPR → EXPR OPERATOR EXPR | (EXPR) | NUMBER ← "REWRITE RULES"

NUMBER → DIGIT | NUMBER DIGIT

DIGIT → 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9

OPERATOR → + | − | ÷ | ×

"NON TERMINAL SYMBOLS" = CAN BE REPLACED BY OTHER NON TERMINAL OR TERMINAL SYMBOLS

"TERMINAL SYMBOLS" = CAN'T BE REPLACED BY ANY RULE OF THE GRAMMAR → OF THE LANGUAGE

"→" also WRITTEN "::=" and "|" are SYMBOLS OF THE BNF (= META SYNTAX OF DESCRIBED LANG) (SOME BOOK EXPRESS THEM IN DIFF. WAYS. (ALSO THE OTHER KIND OF SYMBOLS)

THE FIRST SYMBOL ON THE TOP LEFT SIDE IS CALLED "START SYMBOL" IT IS THE NON TERMINAL SYMBOL WHICH DENOTES THE WHOLE SET OF STRINGS OF THE LANGUAGE.
(IN MOST NATURAL LANGUAGES AND CLASSICAL LOGIC THE START SYMBOL IS "SENTENCE",
IN PROGRAMMING LANGUAGE IT IS "PROGRAMME".)

HOW TO READ THE RULES: EXPR → EXPR OPERATOR EXPR | (EXPR) | NUMBER

means that an EXPR consists of an EXPR FOLLOWED BY AN OPERATOR FOLLOWED BY AN EXPR OR "(" FOLLOWED BY AN EXPR FOLLOWED BY ")" OR A NUMBER.

- PROPERTY OF A SENTENCE :

- AXIOM = POSTULATE = ASSUMPTION

  IT'S A SENTENCE TAKEN TO BE TRUE, TO SERVE AS A PREMISE FOR REASONING / INFERENCES.

- THEOREM

  IT IS A SENTENCE WHICH HAS BEEN PROVED: A LOGICAL CONSEQUENCE OF AXIOMS OR OF OTHER THEOREMS, PROVED BY A DEDUCTIVE SYSTEM.

  ACTUALLY ANY STATEMENT CAN BE THOUGHT AS A THEOREM IN MATHEMATICS (?)

  BUT IN LOGIC IS MORE PRECISE WHAT WE MEAN:

  $\vdash F$ INDICATES THAT F IS A THEOREM WHICH ONLY PREMISES ARE AXIOMS BUT IS SHORTLY CALLED THEOREM. IN NATURAL DEDUCTION A THEOREM IS SMTH WHICH ALL PREMISES ARE DISCARDABLE [ ]

  - WE CALL DERIVATION SMTH DERIVED THANKS TO HYPOTHESIS ([illegible]) $\Gamma \vdash F$ ([illegible] IN MATHS WOULD STILL BE A THEOREM)

- VALID = TAUTOLOGY = TRUE $\forall$ MODEL

  $\models F$ MEANS THAT F IS VALID = F IS TRUE IN ALL MODELS = F IS TRUE FOR ALL THE POSSIBLE "TRUTH VALUE" ASSIGNMENTS OF THE VARIABLES WHICH COMPOSE IT. EXAMPLE: $P \vee \neg P$

  HOW DO WE PROVE IT? TRYING ALL POSSIBLE ASSIGNMENTS. THERE ARE ALGORITHMS TO DO THIS TRYING TO BE EFFICIENT, BUT IF THE KB IS HUGE THE COMPUT. COST. IS HUGE.

  WHY DO WE CARE? BECAUSE THANKS TO THE DEDUCTION THEOREM:

  $F_1 \models F_2$ iff $\models (F_1 \to F_2)$. SO WE CAN OBTAIN A LOGICAL CONSEQUENCE BY CHECKING VALIDITY. THIS WILL BE DISCUSSED ON "MODEL THEORY".

- SATISFIABLE = CONSISTENT

  F IS SATISFIABLE IF THERE IS AT LEAST AN ASSIGNMENT (= A MODEL = A TRUTH VALUE ASSIGNMENT = AN INTERPRETATION) WHICH MAKES F TRUE

  = $\exists$ MODEL s.t. F is TRUE.

  - ALSO THIS ONE IS CHECKED BY TRYING ALL ASSIGNMENTS
  - ALSO THIS HAS MANY USES LIKE FINDING BUGS IN PROGRAMS (MAYBE IN THE FUTURE), SOLVING CSP, LOGICAL CONSEQUENCES...

- UNSATISFIABLE = INCONSISTENT

  F is INCONSISTENT iff IT'S FALSE $\forall$ MODEL/INTERPRETATION

- INVALID

  F is NOT TRUE $\forall$ MODEL / INTERPRETATION

A formula is **invalid** if there is at least an interpretation in which the formula is false (true)

- CONSISTENCY OF Γ:

Γ is CONSISTENT if $\Gamma \nvdash \bot$

which MEANS THAT Γ IS A THEORY/KB IN WHICH IS NOT POSSIBLE TO OBTAIN A CONTRADDICTION

THEORIES/ ALGORITHMS PROPERTIES:

(FOR BOTH "MODEL THEORY" AND "PROOF THEORY")

SOUND = CORRECT = TRUTH-PRESERVING

AN ALGORITHM/THEORY IS SOUND IF WHENEVER IT GIVES YOU AN ANSWER, THAT'S CORRECT.

- ALL MODEL-CHECKING ALGORITHMS ARE SOUND (BUT NOT FINITE IN FOL)
- FOR PROOF THEORY IS USEFUL THE SOUNDNESS THEOREM which says:

$$\Gamma \vdash F \Rightarrow \Gamma \models F$$

which MEANS THAT IF I OBTAIN THE LOGICAL CONSEQUENCE (between Γ and F) SYNTACTICALLY THEN IT IS ALSO AN ENTAILMENT (= SEMANTIC CONSEQUENCE) which is WHAT WE MEAN WITH CORRECT.

- IN GENERAL ANY ANSWER MUST BE AN ENTAILMENT.

COMPLETE

AN ALGORITHM/THEORY IS COMPLETE IF IT IS ABLE TO OBTAIN ANY POSSIBLE ENTAILMENT.

- IT'S MORE DIFFICULT TO OBTAIN. SOMETIMES COMPLETENESS MISSES, TO OBTAIN A FASTER ALGORITHM.
- FOR PROOF THEORY IS USEFUL THE COMPLETENESS THEOREM which says:

$$\Gamma \vdash F \Leftrightarrow \Gamma \models F$$

which MEANS THAT THE ALGO CAN DERIVE SYNTACTICALLY ANY ENTAILMENT (= SEMANTIC CONSEQUENCE)

NOTE:

IT'S WRONG TO SAY THAT PROP. LOGIC OR FOL ARE COMPLETE/SOUND
IT DEPENDS ON THE ALGORITHM USED IN CASE OF MODEL CHECKING
AND DEPENDS ON THE INFERENCE RULES USED IN CASE OF DEDUCTIVE SYSTEMS

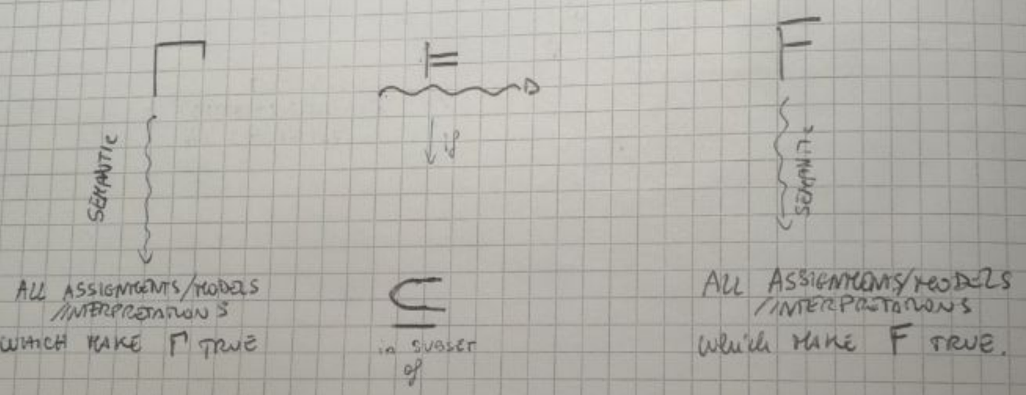
ANYWAY:

- MODEL CHECKING in FOL also if can be SOUND, IT'S NOT FINITE BECAUSE YOU HAVE INFINITE INTERPRETATIONS.
- RESOLUTION in PROP. LOGIC is COMPLETE (so ALSO SOUND)

MODEL THEORY

WE'VE ALREADY SAID MANY TIMES THAT WE CAN OBTAIN A SEMANTIC CONSEQUENCE $\Gamma \models F$ BY CHECKING IF $M(\Gamma) \subseteq M(F)$ SO IF ALL THE "TRUTH VALUE" ASSIGNMENTS which MAKES Γ TRUE, MAKE TRUE ALSO F.
TO DO THIS WE HAVE TO TRY ALL MODELS AND CHECK IF IT IS SATISFIED. (THIS IS WHAT COMPLETE ALGORITHMS DO, BUT TO REDUCE THE COMP. COST MANY ALGORITHMS SACRIFICE THIS PROPERTY).

THE IDEA IS THIS:



SOME ALGORITHMS INSTEAD OF CHECKING DIRECTLY THIS, EXPLOIT THE DEDUCTION THEOREM: $F_1 \models F_2$ iff $\models F_1 \rightarrow F_2$

SO THE PROBLEM BECAMES A VALIDITY PROBLEM.
AND SINCE VALIDITY IS CONNECTED WITH SATISFIABILITY, OF COURSE WE CAN USE ALSO SATISFIABILITY AND ALGORITHMS.
(OR USING REFUTATION CAN BECOME A SATISFACTION PROBLEM).
BUT ANYWAY THE COST IS TOO HIGH. WE PREFER DEDUCTIVE SYSTEMS (=THEOREM PROVING = BASED ON SYNTAX)

WE'LL SEE SOME EXAMPLES TALKING ABOUT PROPOSITION LOGIC, BUT IT'S THE SAME FOR ALL CLASSICAL LOGIC.

PROPOSITIONAL LOGIC

SYNTAX:

DESCRIBED WITH BNF:

Sentence → ATOMIC_SENTENCE | COMPLEX_SENTENCE

ATOMIC SENTENCE → True | False | P | Q | R | ...

in SOME CONTEXTS FALSE CAN BE WRITTEN "⊥" TRUE "⊤"

COMPLEX SENTENCE → (SENTENCE)
| [SENTENCE]
| SENTENCE ∧ SENTENCE
| SENTENCE ∨ SENTENCE
| SENTENCE → SENTENCE
| SENTENCE ⟷ SENTENCE
| ¬ SENTENCE

∧, ∨, →, ⟷, ¬ are called logic connectives.

WE CALL LITERAL AN ATOMIC SENTENCE OR ITS NEGATION.

WE NEED ALSO TO GIVE PRECEDENCES TO THE CONNECTIVES:

"IMPLICATION" "BICONDITIONAL"

¬ , ∧ , ∨ , → , ⟷

HIGH PRIORITY ⟷ LOW PRIORITY

ALSO PARENTHESIS GIVES PRIORITY LIKE IN MATHEMATIC.

- DON'T DO CONFUSION:
- PROPOSITION = ATOMIC FORMULA
- FORMULA = SENTENCE
- TRUTH VALUE = OR TRUE OR FALSE

not, ⇒, and, or, ⟺ are metalanguage symbols while ¬, ∧, ∨, →, ⟷ are syntax symbols.

- ⟷ IS DIFFERENT FROM THE CONCEPT OF LOGICAL EQUIVALENCE.
  BECAUSE THE LOGICAL EQUIVALENCE SAYS THAT $F_1 \leftrightarrow F_2$ FOR ANY INTERPRETATION!

ANYWAY LOGICAL EQUIVALENCE CAN BE EXPRESSED LIKE THIS: $\models (F_1 \leftrightarrow F_2)$

- UNLIKELY "→" and "⊨" and "⊢" are ALSO all called "LOGICAL CONSEQUENCE" BUT WITH DIFFERENT MEANING:

→: SYNTACTIC ELEMENT DEPENDS ON THE INTERPRETATION

⊨: SEMANTIC CONSEQ. = ENTAILMENT (MODEL INDEPENDENT)

⊢: SYNTACTIC CONSEQ. = DERIVABILITY = PROVABILITY (MODEL INDEPENDENT)

SEMANTIC

TRUTH TABLE

P_1	P_2	$\neg P_1$	$P_1 \land P_2$	$P_1 \lor P_2$	$P_1 \to P_2$	$P_1 \leftrightarrow P_2$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

note: • $P_1 \to P_2$ IS TRUE WHENEVER IS SATISFIED THIS: IF P_1 IS TRUE ALSO P_2 MUST BE, IF P_1 ISN'T TRUE P_2 CAN BE WHATEVER

• $P_1 \leftrightarrow P_2$ IS TRUE WHEN IS SATISFIED IN BOTH DIRECTIONS

• EACH ROW IS A MODEL/INTERPRETATION

TO UNDERSTAND THE MEANING OF INTERPRETATION/MODEL:

CONSIDER THE INTERPRETATION/MODEL $M(P) = F$, $M(Q) = T$ $M(R) = T$

SAY FOR WHICH SENTENCES M IS A MODEL (=SATISFIE IT = MAKES IT TRUE):

1) $(P \to \neg Q) \lor (R \land Q)$

SOLUTION: $(F \to F) \lor (T \land T) \equiv T$ $\Rightarrow$ M IS A MODEL FOR THIS SENTENCE

2) $(\neg P \lor \neg Q) \to (P \lor \neg R)$

SOLUTION: $(T \lor F) \to (F \lor F) \equiv T \to F \equiv F$ $\Rightarrow$ M IS NOT A MODEL FOR THIS SENTENCE

3) $\neg(\neg P \to \neg Q) \land R$

SOLUTION: $\neg(T \to F) \land T \equiv T$ $\Rightarrow$ M IS A MODEL FOR THIS SENTENCE

4) $\neg P \land Q \lor R \land \neg Q$

PAY ATTENTION TO PRIORITIES!

SOLUTION: $\equiv (\neg P \land Q) \lor (R \land \neg Q) \equiv (T \land T) \lor (T \land F) \equiv T$ M IS A MODEL...

. note that to ask to write the truth table of a formula is like asking to write all possible interpretations.

GIVEN A FORMULA CONTAINING m DISTINCT ATOMS, HOW MANY DISTINCT INTERPRETATIONS DOES IT HAVE? 2^m

. AN INTERPRETATION $I = \{P, \neg Q\}$ IS THE SAME OF SAYING $I = \{P = T, Q = F\}$

. GIVEN A FORMULA $P \land \neg Q \lor R$ AND ASKED TO BE EVALUATED YOU HAVE 3 CHOICES:

a) SAY THAT YOU CAN'T BECAUSE AN INTERPRETATION MUST BE GIVEN

b) SUPPOSE AN INTERPRETATION

c) BEST CHOICE: WRITE THE TRUTH TABLE (ALL POSSIBLE ASSIGNMENTS) = in THIS CASE 2^3

• LET SEE IF YOU REMEMBER THE CONCEPT OF ENTEILMENT:

PROVE THAT $(P \to Q) \land P \models Q$ $\models (A \to B)$

SOLUTION:
THE ENTAILMENT $F_1 \models F_2$ MEANS THAT F_2 IS TRUE IN ALL MODELS OF F_1.
I HAVE AT LEAST 3 WAYS TO PROVE IT (WITH MODEL CHECKING):

a) I PROVE ALL THE INTERPRETATIONS: WRITE THE TRUTH TABLE

P	Q	$(P \to Q) \land P$	Q	$((P \to Q) \land P) \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	F	T

I PROVED THAT $(P \to Q) \land P \models Q$ BECAUSE $\forall$ MODEL/INTERPRETATION OF $(P \to Q) \land P$ WHICH MAKES IT TRUE, ALSO Q IS TRUE!

b) USE DEDUCTION THEOREM: !!(IT HAS NOTHING TO DO WITH NATURAL DEDUCTION!)!!

DEDUCTION THEOREM:

$$\underbrace{F_1 \land F_2 \land \ldots \land F_m \models G}_{A} \quad \text{iff} \quad \underbrace{\models F_1 \land F_2 \land \ldots \land F_m \to G}_{B}$$

PROOF; DIRECT PROOF JUST USING THE DEFINITION OF "$\to$"

1) I PROVE "IF" $\Rightarrow$ (PROVE B FROM A)

a) A IS SAYING THAT FOR ANY INTERPRETATION I_1 WHICH MAKES $F_1 \land \ldots F_m$ TRUE ALSO G IS TRUE $\Rightarrow I_1 \models F_1 \land \ldots F_m \to G$

b) THAT'S EVERYTHING I NEED BECAUSE THE REMAINING INTERPRETATIONS ARE I_2, THOSE FOR WHICH $F_1 \land F_2 \ldots \land F_m$ IS FALSE, BUT FOR DEFINITION OF "$\to$" $I_2 \models F_1 \land \ldots \land F_m \to G$ IS SATISFIED ANYWAY.
SO $F_1 \land \ldots F_m \to G$ IS SATISFIED FOR ANY INTERPRETATION:
$\models F_1 \land F_2 \land \ldots F_m \to G$

2) I PROVE ONLY IF $\Leftarrow$ PROVE A FROM B

EASY: $\models (F_1 \land \ldots F_m) \to G$ means that FOR ANY INTERPRETATION IS TRUE THAT $(F_1 \land \ldots F_m) \to G$ SO FOR DEF of "$\to$" I CAN SAY THAT $\forall I$ which MAKES $F_1 \land \ldots F_m$ TRUE ALSO G IS TRUE, WHICH CAN BE
WRITTEN [illegible]

which MEANS THAT I MUST SHOW THAT $\models (P \to Q) \land P \to Q$
which MEANS TO SHOW THAT THE TRUTH TABLE OF
$(P \to Q) \land P \to Q$ IS TRUE IN EACH ROW (= EACH INTERPRETATION):

P	Q	$(P \to Q) \land P \to Q$
T	T	T
T	F	T
F	T	T
F	F	T

→ THIS MEANS THAT $(P \to Q) \land P \to Q$ IS VALID WHICH IS WAS I WANTED.

c) USE REFUTATION

$A \to B \equiv \neg A \lor B \equiv \neg(A \land \neg B)$

REFUTATION THEOREM

$(F_1 \land F_2 \ldots \land F_m) \models G$ iff $F_1 \land F_2 \ldots \land F_m \land \neg G$ IS INCONSISTENT (= UNSATISFIABLE)

PROVED SIMPLY BY SAYING THAT $(F_1 \land F_2 \ldots \land F_m) \models G$ MEANS THAT G MUST BE TRUE ∀ INTERPRETATION which MAKES $F_1 \land F_2 \ldots F_m$ TRUE. THIS HOLDS iff THERE IS NO INTERPRETATION FOR WHICH $F_1 \land F_2 \ldots F_m$ IS TRUE AND G IS FALSE; THIS HAPPENS IF $F_1 \land F_2 \ldots F_m \land \neg G$ IS FALSE FOR ANY INTERPRETATION (= IS INCONSIST.)

SO WE MUST SHOW THIS:

P	Q	$(P \to Q) \land P \land \neg Q$
T	T	F
T	F	F
F	T	F
F	F	F

→ THIS MEANS THAT $(P \to Q) \land P \land \neg Q$ IS INCONSISTENT which IS what WE WANTED.

ANOTHER PROPOSED EXAMPLE: $(P \to Q) \land \neg Q \models \neg P$

(→ LOOK HOW COOL: TO PROVE THAT AN ARGUMENT IS INVALID YOU COULD SHOW THE TRUTH TABLE)!?

CHECK IF THESE ARGUMENTS ARE VALID:

1) I AM WEALTHY, THEN I AM HAPPY. I AM HAPPY. THEREFORE I AM WEALTHY.

SOLUTION: I AM HAPPY = P . I AM WEALTHY = Q. SO:

i) $Q \rightarrow P$

ii) P

$\Rightarrow Q$ (IS NOT VALID, INDEED FOR DEFINITION OF "$\rightarrow$" Q CAN BE FALSE WHEN P IS TRUE.)

MUST BE PROVEN LIKE THIS:

THE SENTENCE IS $\equiv$ $((Q \rightarrow P) \land P) \rightarrow Q$

SO I SHOW THAT IS NOT VALID:

P	Q	$((Q \rightarrow P) \land P) \rightarrow Q$
T	T	T
T	F	F → IS NOT VALID.
F	T	T
T	T	T

IMPORTANT:

VERIFY THE VALIDITY/CORRECTNESS OF THIS:

IF P AND S THEN E, WHILE IF P AND ¬S THEN ¬E.
THUS, IF P, EITHER S AND P, OR ¬S AND ¬P

FORMULE

1) $P \land S \rightarrow E$ 2) $P \land \neg S \rightarrow \neg E$ CONCLUSION: $P \rightarrow (S \land P) \lor (\neg S \land \neg P)$

TO VERIFY THE VALIDITY MEANS SHOW THAT $\boxed{1) \land 2) \models \text{CONCL.}}$

WHICH MEANS TO DO THE TRUTH TABLE OF $1) \land 2)$ AND SHOW THAT
. IS THE SAME OF THE " " OF THE CONCLUSION

. OR SHOW THAT $\models 1) \land 2) \rightarrow$ CONCLUSION AS DONE IN THE PREVIOUS EXAMPLE.

NOW LET'S TALK ABOUT EQUIVALENCES.
WE HAVE ALREADY DEFINED IT, SO LET'S CHECK IT:

$\neg(P \land Q) \equiv \neg P \lor \neg Q$

WE HAVE MANY WAYS:

a) SHOW THAT $\neg(P \land Q) \models \neg P \land \neg Q$ and $\neg P \lor \neg Q \models \neg(P \land Q)$ WITH ALL THE METHODS DISCUSSED UNTIL NOW

b) MORE INTELLIGENTLY NOTE THAT IT MEANS THAT THEY ARE THE SAME IN ANY INTERPRETATION $\rightarrow$ SIMPLY DO THE TRUTH TABLE

c) PROVE THAT $\models (\neg(P \land Q)) \longleftrightarrow (\neg P \lor \neg Q)$ which IS THE MEANING OF EQUIVALENCE, SHOWING THE VALIDITY WITH THE TRUTH TABLE.

d) USE THE WELL KNOWN PROPERTIES / TAUTOLOGIES OF PROPOS. LOGIC.

TO TRANSFORM FORULAE INTO CNF OR DNF we need in particular THESE PROPERTIES:

- DISTRIBUTIVITY $\begin{cases} A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C) \\ A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C) \end{cases}$
- COMUTATIVITY $A \vee B \equiv B \vee A$ $\quad A \wedge B \equiv B \wedge A$
- ASSOCIATIVITY remove PARENTHESIS
- DE MORGAN NEGATION of $\wedge$ and $\vee$ $\quad \neg(A \wedge B) \equiv \neg A \vee \neg B$
- IDEMPOTENCE
- DOUBLE NEGATION

ABOUT $\wedge, \vee, \neg$

FROM $\leftrightarrow$ [illegible]
- ABOUT $\leftrightarrow$: $\quad (A \leftrightarrow B) \equiv (A \rightarrow B) \wedge (B \rightarrow A)$ ("BICONDITIONAL ELIMINATION")
- FROM $\rightarrow$: $\quad (A \rightarrow B) \equiv \neg A \vee B$ ("IMPLICATION ELIMINATION")
- "remove" "$\rightarrow$": WITH NOT $\quad A \vee B \equiv \neg A \rightarrow B$

FOR NATURAL DEDUCTION:

$\neg A \equiv A \rightarrow \bot$

$\bot \equiv A \wedge \neg A$

$A \rightarrow B \equiv \neg B \rightarrow \neg A$ "CONTRAPOSITION"

SOME ALGORITHMS WORKS GIVEN THE FORMULA IN A SPECIFIC FORM.

CNF = CLAUSAL FORM $\qquad \underbrace{(A \vee B)}_{\text{CLAUSE = DISJUNCT. OF LITERALS}} \wedge (\neg B \vee C) \wedge \dots$

CONJUNC. NORMAL FORM

$\wedge$

$(\neg A \vee B) \wedge (B \vee C)$

$(A \vee B) \wedge (\neg B \vee C \vee A)$ CNF

$(\neg B \vee C)$

$(\neg B \vee C) \wedge A$

DNF $\qquad (A \wedge B) \vee (B \wedge \neg A) \vee \dots$

DISJ. NORMAL FORM

$\vee$

SAME EXAMPLES SWAPPING $\wedge$ WITH $\vee$ and [illegible]

$(A \vee \neg B \vee C) \wedge (A \vee \neg D \vee E)$ IS A CNF!!

NOTE: $\neg(A \wedge B) \vee (B \wedge C)$ IS NOT DNF: YOU CAN HAVE ONLY DISJUNCTION OF CONJUNCTION OF LITERALS.

SIMILARLY FOR THE CNF. THE SAME THING.

note: $(A \wedge B) \vee C$ IS A DNF

ANY FORMULA CAN BE TRANSFORMED IN NORMAL FORM BY USING THE PROPERTIES/ EQUIVALENCES OF THE PREVIOUS PAGE.

LET'S TRY IT:

WRITE $\neg(A \leftrightarrow B)$ IN CNF:

$$\equiv \neg\big((A \to B) \wedge (B \to A)\big) \equiv \neg\big((\neg A \vee B) \wedge (\neg B \vee A)\big) \equiv (A \wedge \neg B) \vee (B \wedge \neg A)$$

$$\equiv \big(A \vee (B \wedge \neg A)\big) \wedge \big(\neg B \vee (B \wedge \neg A)\big) \equiv \big((A \vee B) \wedge (A \vee \neg A)\big) \wedge \big((\neg B \vee B) \wedge (\neg B \vee \neg A)\big) \equiv$$

$$\equiv (A \vee B) \wedge (\neg B \vee \neg A)$$

HOW DO YOU VERIFY AN EQUIVALENCE?
MUST HAVE THE SAME TRUTH TABLE

A	B	$\neg(A \leftrightarrow B)$		$(A \vee B) \wedge (\neg B \vee \neg A)$
T	T	F	=	F
T	F	T	=	T
F	T	T	=	T
F	F	F	=	F

✓OK.

WRITE $(\neg A \rightarrow B) \rightarrow (C \rightarrow \neg D)$ IN CNF AND DNF

$\equiv (A \vee B) \rightarrow (\neg C \vee \neg D) \equiv \neg(A \vee B) \vee (\neg C \vee \neg D) \equiv$

$\equiv (\neg A \wedge \neg B) \vee (\neg C \vee \neg D) \equiv (\neg A \vee (\neg C \vee \neg D)) \wedge (\neg B \vee (\neg C \vee \neg D)) \equiv$

CHOICE POINT

~~$\equiv \neg A \wedge (\neg B \vee (\neg C \vee \neg D))$ ...~~ NO ?

THIS IS A CNF!!!

(JUST REMOVE THE INTERNAL PARENTHESIS)

TRY TO SOLVE FIRST WHAT HAS PRIORITY :

~~$\equiv \neg(\neg A \rightarrow B) \vee (C \rightarrow \neg D) \equiv \neg(A \vee B) \vee (\neg C \vee \neg D)$~~ $\equiv$ SAME PROBLEM

THE RULE IS THIS:

1) REMOVE "$\rightarrow$" AND "$\leftrightarrow$"

2) APPLY DE MORGAN UNTIL YOU OBTAIN A NNF (ALL THE "$\neg$" MUST BE APPLIED ONLY TO ATOMIC FORMULA)

3) APPLY DISTRIBUTIVE LAW UNTIL THE RESULT COMES OUT.

↳ FOR CNF APPLY THE ONE WHICH RESULT IS $F_1 \wedge F_2$ OF COURSE, FOR DNF $F_1 \vee F_2$.

~~SO, WE REACHED THE NNF : $(\neg A \wedge \neg B) \vee (\neg C \vee \neg D)$~~

~~SO IT WAS CORRECT :~~

~~$\equiv (\neg A \vee (\neg C \vee \neg D)) \wedge (\neg B \vee (\neg C \vee \neg D))$ ...~~

I CAN CONSIDER ~~$(P_1 \wedge P_2) \vee P_3$~~ OR $P_1 \vee (P_2 \vee P_3)$

↳ NO DISTRIBUTIVITY

~~$(P_1 \vee P_3) \wedge (P_2 \vee P_3)$~~ $= (\neg A \vee (\neg C \vee \neg D)) \wedge (\neg B \vee (\neg C \vee \neg D))$

$(P_1 \vee P_2) \wedge (P_3 \vee P_4)$

...

EXAM EXAMPLE:

$A \vee (\neg B \wedge (C \rightarrow \neg A)) \equiv A \vee (\neg B \wedge (\neg C \vee \neg A)) \equiv (A \vee \neg B) \wedge (A \vee (\neg C \vee \neg A))$

$\equiv (A \vee \neg B) \wedge (A \vee \neg C \vee \neg A)$ CNF.

FORMALIZATION: FROM NATURAL LANGUAGE TO LOGIC:

P = PAOLA IS HAPPY Q = PAOLA PAINTS A PICTURE R = RENZO IS HAPPY

FORMALIZE THESE:

1) IF PAOLA IS HAPPY AND PAINTS A PICTURE THEN RENZO IS HAPPY : $(P \wedge Q) \rightarrow R$

2) PAOLA IS HAPPY ONLY IF SHE PAINTS A PICTURE : $P \leftrightarrow Q$?

= PAOLA ISN'T HAPPY IF SHE DOESN'T PAINT A PICTURE: $\neg Q \rightarrow \neg P$

$P \rightarrow Q$

⇓ THERE ARE RULES! :

GIVEN PROPOSITIONS A and B :

LOGIC

IF A THEN B $A \rightarrow B$

A ONLY IF B $A \rightarrow B$

A	B	$A \rightarrow B$	$B \rightarrow A$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

TWO WAYS TO UNDERSTAND IT:

1) LOOK AT THE TRUTH TABLE: $B \rightarrow A$ IS TRUE ALSO when B=False and A=True WHICH IS NOT WHAT WE WANT

2) THE SENTENCE SAYS THAT YOU DON'T WANT THAT $A \wedge \neg B$ HAPPENS, SO YOU WANT $\neg(A \wedge \neg B) \equiv \neg A \vee B$ which is $\equiv A \rightarrow B$

3) ACCORDING TO THE DEFINITION OF "$\rightarrow$", SAYING $A \rightarrow B$ IS LIKE SAYING THAT A CAN BE TRUE ONLY IF B IS TRUE.

B IF A

IT MEANS THAT A IMPLIES B

IT'S LIKE SAYING IF A THEN B

$A \rightarrow B$

A IF AND ONLY IF B

$A \leftrightarrow B$

A SUFFICIENT CONDITION FOR B $A \to B$

A NECESSARY CONDITION FOR B $B \to A$

I DON'T WANT THIS TO HAPPEN: $\neg A \wedge B$

SO $\neg(\neg A \wedge B) \equiv (A \vee \neg B) \equiv B \to A$

NEITHER A NOR B $\neg A \wedge \neg B$

A: A GOES TO FLOR. B = B GOES TO FLOR. C = .. D = ..

A, B and C go to FLOR. IF and ONLY IF D doesn't go, BUT IF NEITHER A NOR B GO, THEN $\underbrace{\text{D GOES ONLY IF C GOES}}$

I WANT THAT $\neg(D \wedge \neg C) \equiv (\neg D \vee C) \equiv D \to C$

2 SENTENCES:

1) $A \wedge B \wedge C \iff \neg D$

2) $(\neg A \wedge \neg B) \to (D \to C)$

"ALL THE A are B" : $A \to B$

"NONE OF A are not B" : $A \to B$

"NONE OF $\neg A$ is B" $\neg(\neg A \wedge B) \equiv A \vee \neg B \equiv B \to A$ $(\equiv \neg A \to \neg B)$

NATURAL DEDUCTION

- IT DOES PROOFS JUST BY USING SYNTAX (BELONGS TO "THEOREM PROVING" "PROOF THEORY")
- FROM KB YOU DERIVE NEW KNOWLEDGE: $KB \vdash F$

IF THE INFERENCE RULES USED ARE SOUND $\Rightarrow$ $KB \models F$

$\Rightarrow$ YOU KNOW THAT $\models (KB \to F)$ = FOR ANY INTERPRETATION KB IMPLIES F.

- IT'S CALLED NATURAL BECAUSE SIMULATE THE "NATURAL" WAY OF HUMAN REASONING
- IT USES A SET OF RULES CALLED "INFERENCE RULES", WHICH INTRODUCE OR ELIMINATE CONNECTIVES. THE COMBINATION OF THEM ALLOWS TO DO DERIVATIONS

LET'S SEE SOME INFERENCE RULES:

$$\frac{A \to B \quad A}{B}$$

$\Rightarrow$ PREMISES: THINGS THAT HOLDS IN YOUR KB

$\Rightarrow$ WHAT HAS BEEN DERIVED

MODUS PONENS — name: MP — $\to E$ IS THE SAME

$$\frac{A \to B \quad \neg B}{\neg A}$$

MODUS TOLENS — name: MT

$$\frac{A \to B \quad B \to C}{A \to C}$$

SYLLOGISM — name: S

$$\frac{A \quad B}{A \wedge B}$$

AND INTRODUCTION — name: $\wedge I$

$$\frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

AND ELIMINATION — name: $\wedge E$

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B}$$

"$\to$" INTRODUCTION — name: $\to I$

IT MEANS THAT IF B CAN BE DERIVED FROM A THEN $A \to B$.

([A] IS CALLED HYPOTHESIS DISCARDED!! IT MEANS THAT THE FINAL CONCLUSION DOES NOT DEPEND ON THE ASSUMPTION [A]. INDEED A CAN ALSO BE FALSE, AND $A \to B$ HOLDS ANYWAY.)

$$\dfrac{A \qquad A \to B}{B}$$

"$\to$" ELIMINATION — name: $\to E$

$\longrightarrow$ EXAMPLE TO CLARIFY DOUBTS

TO DO DERIVATIONS/PROOFS IN OUR EXERCISES WE'LL NEED ONLY $\to I$, $\to E$, $\wedge I$, $\wedge E$ INFERENCE RULES AND TO REMEMBER SOME TAUTOLOGIES LIKE:

- $\neg A \equiv A \to \bot$ (indeed $A \to \bot \equiv \neg A \vee \bot \equiv \neg A$)
- $\bot \equiv A \wedge \neg A$

... WILL NEED ALSO

(TO BE COMPLETE WE DISCUSS ALSO THESE TWO IMPORTANT INFERENCE RULES:

$$\dfrac{\bot}{A}$$

"EX FALSO SEQUITUR QUODLIBET" name: $\bot$

which means that FROM AN INCONSISTENT FORMULA I CAN DERIVE ANY OTHER FORMULA (WHICH IS WRONG AND USELESS, SO PAY ATTENTION TO DON'T HAVE INCONSISTENT ASSUMPTION OTHERWISE ISN'T GUARANTEED TO BE CORRECT YOUR DERIVATION. (INCONSISTENT IN THIS CASE MEANS THAT YOUR KB SAYS THAT $\bot$ IS TRUE)) BUT ANYWAY THIS IS AN USEFUL INFERENCE RULE!

$$\dfrac{\begin{matrix}[\neg A] \\ \vdots \\ \bot\end{matrix}}{A}$$

"REDUCTIO AD ABSURDUM" name: RAA

IT MEANS THAT IF FROM THE ASSUMPTION OF $[\neg A]$ YOU OBTAIN A CONTRADICTION (WHICH IS $\bot$ IS TRUE) THEN YOU PROVED/DERIVED A.

(ALL THESE ARE SOUND/CORRECT INFER. RULE, SO ANYTHING OBTAINED)

THAT $KB \vdash F$ IS also $KB \models F$

THESE INFERENCE RULES CAN BE USED TO DO DERIVATIONS:

EXAMPLE:

LET SEE FROM WHAT WE COULD DERIVE/PROOF/INFER "$A \wedge B \rightarrow B \wedge A$"

WHICH IS LIKE ASKING WHAT SHOULD I HAVE ON NUMERATOR: $\dfrac{?}{A \wedge B \rightarrow B \wedge A}$

TO KNOW IT WE MUST APPLY THE RULES:

$$\dfrac{B \wedge A}{A \wedge B \rightarrow B \wedge A}\ (\rightarrow I)$$

SO WE COULD SAY THAT GIVEN THAT $B \wedge A$ HOLDS (= IS TRUE) THEN WE CAN SAY THAT ALSO $A \wedge B \rightarrow B \wedge A$ HOLDS. WHICH MEANS:

$B \wedge A \vdash A \wedge B \rightarrow B \wedge A$.

WE OBTAINED THIS THANKS TO THE INFERENCE RULE "$\rightarrow I$".

BUT LET'S SEE IF CONTINUING IN THIS WAY WE OBTAIN SOMETHING MORE INTERESTING:

$$\dfrac{\dfrac{B \qquad A}{B \wedge A}\ (\wedge I)}{A \wedge B \rightarrow B \wedge A}\ (\rightarrow I)$$

OR EVEN MORE:

$$\dfrac{\dfrac{\dfrac{A \wedge B}{B} \qquad \dfrac{A \wedge B}{A}\ (\wedge E)}{B \wedge A}\ (\wedge I)}{A \wedge B \rightarrow B \wedge A}\ (\rightarrow I)$$

NOW WE NOTICE THAT IT'S HAPPENING THIS:

$$\dfrac{\begin{matrix}[A \wedge B] \\ \vdots \\ B \wedge A\end{matrix}}{A \wedge B \rightarrow B \wedge A}\ \rightarrow I$$

SO WE CAN WRITE THIS:

$$\dfrac{\dfrac{\dfrac{[A \wedge B]}{B} \qquad \dfrac{[A \wedge B]}{A}}{B \wedge A}}{A \wedge B \rightarrow B \wedge A}$$

WHICH MEANS THAT THE HYPOTHESIS $A \wedge B$ IS DISCARDED.

WHICH MEANS THAT $A \wedge B \rightarrow B \wedge A$ DOESN'T DEPEND ON $A \wedge B$.

IF ALL THE HYPOTHESIS ARE DISCARDABLE THE CONCLUSION IS A THEOREM.

∴ YOU MUST NOT DECIDE **HOW** TO DISCARD PREMISES/HYPOTHESIS. IT IS INDICATED BY THE INFERENCE RULES!

→ APPLY THE RULE "→I":

$$\dfrac{\begin{matrix}[A \wedge B] \\ \vdots \\ B \wedge A\end{matrix}}{A \wedge B \rightarrow B \wedge A}\ (\rightarrow I)$$

SO NOW WHAT WE NEED TO DO IS TO SUBSTITUTE THOSE POINTS $\vdots$ WITH PASSAGES TO OBTAIN $B \wedge A$ FROM $[A \wedge B]$!!

YOU'RE OBLIGED TO DO THIS OTHERWISE YOU CAN'T APPLY "→I" !!!!!

SOLUTION:

$$\dfrac{\dfrac{\begin{matrix}[A \wedge B] \\ \vdots \\ B\end{matrix} \qquad A}{B \wedge A}\ (\wedge I)}{A \wedge B \rightarrow B \wedge A}\ (\rightarrow I) \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{[A \wedge B]}{B} \qquad \dfrac{[A \wedge B]}{A}}{B \wedge A}}{A \wedge B \rightarrow B \wedge A}$$

(NOTE THAT YOU CAN USE A PREMISE IN **ALL** BRANCHES BELONGING TO THE RULE WHICH GENERATE THAT PREMISE)

SO YOU PROVED THAT $[A \wedge B] \vdash ((A \wedge B) \rightarrow (B \wedge A))$ WHERE $[A \wedge B]$ IS SAID DISCARDABLE SO: $\vdash ((A \wedge B) \rightarrow (B \wedge A))$ WHICH MEANS THAT $A \wedge B \rightarrow B \wedge A$ IS A DERIVATION WITH NO PREMISES = A THEOREM.

INDEED ALSO IF $A \wedge B$ IS FALSE, $(A \wedge B) \rightarrow (B \wedge A)$ IS TRUE / HOLDS.

LOOK VAN DALEN BOOK FOR EXAMPLES

EXAMPLE: PROVE THAT $A \to (B \to A)$ IS A THEOREM:

$$\dfrac{\begin{matrix}[A]_1 \\ \vdots \\ B \to A\end{matrix}}{A \to (B \to A)} \to I_1 \quad ; \quad \dfrac{\dfrac{\begin{matrix}[B]_2 \\ \vdots \\ [A]_1\end{matrix}}{B \to A} \to I_2}{A \to (B \to A)} \to I_1$$

NOT USED, DOESN'T MATTER. INDEED YOU COULD HAVE WRITTEN

$$\dfrac{\dfrac{\dfrac{\dfrac{[A] \quad [B]}{A \wedge B}}{A}}{B \to A}}{A \to (B \to A)}$$

① TO INDICATE FROM WHICH FORMULA IT COMES BUT ACTUALLY SOMETIMES IS USELESS

PROVE $\vdash (P \to Q) \to ((P \wedge R) \to (Q \wedge R))$

$$\dfrac{\begin{matrix}[P \to Q]_1 \\ \vdots \\ (P \wedge R) \to (Q \wedge R)\end{matrix}}{(P \to Q) \to ((P \wedge R) \to (Q \wedge R))} \to I_1 \quad ; \quad \Rightarrow \quad \dfrac{\dfrac{\begin{matrix}[P \wedge R]_1 \quad [P \to Q]_2 \\ \vdots \\ Q \wedge R\end{matrix}}{(P \wedge R) \to (Q \wedge R)} \to I_2}{(P \to Q) \to ((P \wedge R) \to (Q \wedge R))} \to I_1$$

NOW THE REASONING IS:

$Q \wedge R$ CAN ONLY BECOME $\dfrac{Q \quad R}{Q \wedge R}$,

AND FOR BOTH Q and R I

CAN USE BOTH PREMISES, SO:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}[P \wedge R], [P \to Q] \\ \vdots \\ Q\end{matrix} \qquad \begin{matrix}[P \wedge R], [P \to Q] \\ \vdots \\ R\end{matrix}}{Q \wedge R} \wedge I}{(P \wedge R) \to (Q \wedge R)} \to I}{(P \to Q) \to ((P \wedge R) \to (Q \wedge R))} \to I \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[P \wedge R]}{P} \wedge E \quad [P \to Q]}{Q} \to E \qquad \dfrac{[P \wedge R]}{R} \wedge E}{Q \wedge R} \wedge I}{(P \wedge R) \to (Q \wedge R)} \to I}{(P \to Q) \to ((P \wedge R) \to (Q \wedge R))} \to I$$

IF THESE HYPOTHESIS COME FROM A RULE APPLIED TO A BRANCH BELONGING TO BOTH, AS IN THIS CASE.

EXAMPLE:

$$\dfrac{\dfrac{\begin{matrix}[A]_1 \\ \vdots\end{matrix}}{\vdots} \to I_1 \qquad \dfrac{}{} \to I_2}{}$$

IT COMES FROM HERE SO IT BELONGS ONLY TO THE LEFT BRANCH.

MORE PECULIAR CASES: $\vdash \neg\neg A \leftrightarrow A$ WITHOUT USING INF. RULES OF "$\leftrightarrow$". YOU CAN'T SAY $\neg\neg A \equiv A$ BECAUSE IT'S EXACTLY WHAT YOU WANT TO PROVE.

WE NEED TO TRANSFORM IT THANKS TO TAUTOLOGIES:

$$\vdash (\neg\neg A \to A) \land (A \to \neg\neg A)$$

SO:

2) 1)

$$\dfrac{\dfrac{\begin{matrix}[\neg\neg A]\\ \vdots\\ A\end{matrix}}{\neg\neg A \to A}\to I \qquad \dfrac{\begin{matrix}[A]\\ \vdots\\ \neg\neg A\end{matrix}}{A \to \neg\neg A}\to I}{(\neg\neg A \to A)\land(A \to \neg\neg A)}\land I$$

LET'S CONSIDER SEPARATELY THE TWO BRANCHES:

1)

$$\dfrac{\begin{matrix}[A]\\ \vdots\\ \neg\neg A\end{matrix}}{A \to \neg\neg A}\to I$$

REMEMBER THAT $\neg A \equiv A \to \perp$

SO $\neg\neg A \equiv \neg A \to \perp$

$$\Rightarrow \quad \dfrac{\dfrac{\begin{matrix}[A] \quad [\neg A]\\ \vdots \qquad \vdots\\ \perp\end{matrix}}{\neg A \to \perp}\to I}{A \to \neg\neg A}\to I \qquad \xRightarrow{\neg A \equiv A \to \perp} \qquad \dfrac{\dfrac{\dfrac{[A] \quad [A \to \perp]}{\perp}E\to}{\neg A \to \perp}}{A \to \neg\neg A}$$

2)

$$\dfrac{\begin{matrix}[\neg\neg A]\\ \vdots\\ A\end{matrix}}{\neg\neg A \to A} \quad \Rightarrow \quad \dfrac{\begin{matrix}[\neg A \to \perp]\\ \vdots\\ A\end{matrix}}{\neg\neg A \to A}$$

YOU HAVE NO CHOICE: YOU MUST ADD A PREMISE SOMEHOW ⇒ WITH THE RULES GIVEN YOU CAN ONLY APPLY "RAA"!

$$\Rightarrow \quad \dfrac{\dfrac{\begin{matrix}[\neg A]_1 \quad [\neg A \to \perp]\\ \vdots\\ \perp\end{matrix}}{A}RAA_1}{\neg\neg A \to A} \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{[\neg A]_1 \quad [\neg A \to \perp]}{\perp}\to E}{A}RAA_1}{\neg\neg A \to A}\to I$$

OK

WE PROVED

$\neg\neg A \leftrightarrow A$

1 $\vdash A \to (\neg A \to B)$

$$\dfrac{\begin{matrix}[A] \\ \vdots \\ \neg A \to B\end{matrix}}{A \to (\neg A \to B)} \to I \quad \Rightarrow \quad \dfrac{\dfrac{\begin{matrix}[\neg A] \quad [A] \\ \vdots \\ B\end{matrix}}{\neg A \to B}}{A \to (\neg A \to B)} \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{[A \to \bot] \quad [A]}{\bot} \to E}{B} \bot}{\neg A \to B} \to I}{A \to (\neg A \to B)} \to I$$

$\wedge$

P $\vdash (A \to B) \to ((B \to C) \to (A \to C))$

$$\dfrac{\begin{matrix}[A \to B] \\ \vdots \\ (B \to C) \to (A \to C)\end{matrix}}{(A \to B) \to ((B \to C) \to (A \to C))} \to I \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix}[A \to B] \quad [B \to C] \quad [A] \\ \vdots \\ C\end{matrix}}{A \to C}}{(B \to C) \to (A \to C)}}{(A \to B) \to ((B \to C) \to (A \to C))} \Rightarrow$$

$$\Rightarrow \quad \dfrac{\dfrac{\dfrac{[A] \quad [A \to B]}{B} \quad [B \to C]}{C}}{A \to C} \quad \vdots$$

Vok.

$\vdash (\neg A \to \neg B) \to (A \to B)$

TRY TO DO IT IN ONLY ONE STEP:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg B] \quad [\neg A \to \neg B]}{\neg A \equiv A \to \bot} \to E \quad [A]}{\bot} \to E}{B} \text{RAA}}{A \to B} \to I}{(\neg A \to \neg B) \to (A \to B)} \to I$$

OF COURSE IT COULD BE ASKED TO PROVE LOGICAL CONSEQUENCES WITH NATURAL DEDUCTION:

$A \to (B\wedge C) \vdash (A\to B)\wedge(A\to C)$ SO YOU WANT $\dfrac{\begin{array}{c}A\to(B\wedge C)\\ \vdots\end{array}}{(A\to B)\wedge(A\to C)}$

$\Leftarrow$

1) from $A\to(B\wedge C)$ THE ONLY THING YOU CAN DO IS $\dfrac{A \quad A\to B\wedge C}{B\wedge C}$ SO YOU NEED A SOMEHOW $\vdots$

$$\dfrac{\begin{array}{c}\dfrac{[A] \quad A\to B\wedge C}{B\wedge C}\\ \vdots\end{array}}{(A\to B)\wedge(A\to C)} \quad \Rightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{[A]_1 \quad A\to B\wedge C}{B\wedge C}\ \to I}{B}\ \wedge E}{A\to B}\ \to I_1 \qquad A\to C}{(A\to B)\wedge(A\to C)}$$

$A\to(B\wedge C)$ IS A PREMISE $\Rightarrow$ CAN BE USED IN ALL BRANCHES, NOT ONLY ONE!!! SO:

$$\dfrac{\dfrac{\dfrac{\dfrac{[A]_1 \quad A\to B\wedge C}{B\wedge C}}{B}\ \wedge E}{A\to B}\ \to I_1 \qquad \dfrac{\dfrac{\dfrac{[A]_2 \quad A\to B\wedge C}{B\wedge C}}{C}}{A\to C}\ \to I_2}{(A\to B)\wedge(A\to C)}$$

✓ok.

$B \vee (A \wedge C) \vdash (A \wedge B)$

WE DON'T USE INFERENCE RULES WITH "OR" SO YOU MUST TRANSFORM IT:

$\varphi \to \psi \equiv \neg\varphi \vee \psi$ SO $B \vee (A \wedge C) \equiv \neg B \to (A \wedge C)$

SO:

$$\dfrac{\dfrac{\dfrac{[\neg B]_1 \quad \neg B \to (A \wedge C)}{A \wedge C}}{A} \quad \dfrac{\perp}{B}\,\text{RAA}_1}{A \wedge B}$$

⇒ NOW THE PROBLEM IS TO OBTAIN

$$\begin{array}{c}\neg B \to (A \wedge C)\\ \vdots \\ \perp\end{array}$$

SO PROBABLY IT'S NOT DERIVABLE, LET'S VERIFY IT:

A	B	C
T	T	T
T	T	F
T	F	T
F	T	T
⋮		

-- TOO LONG.

BETTER REASON TO FIND AN INTERPRETATION IN WHICH $B \vee (A \wedge C) \neq A \wedge B$

; if B = TRUE THE LEFT SIDE FOR SURE IS TRUE INSTEAD THE RIGHT SIDE "$A \wedge B$" CAN BE FALSE

⇒ B = TRUE A = FALSE SHOW THAT THE DERIVATION DOESN'T HOLD.

REMEMBER THAT ~~GIVEN~~ $F_1 \vdash F_2$ MEANS THAT YOU'RE PROVING THAT $F_1 \to F_2$ SYNTACTICALLY. SO TO FIND A COUNTER EXAMPLE YOU MUST FIND THAT when F_1 = TRUE F_2 = FALSE.
IF YOU FIND F_2 = TRUE and F_1 = FALSE YOU HAVEN'T PROVED THAT $F_1 \nvdash F_2$

EXAM EXERCISES:

1) $\vdash A \rightarrow (B \rightarrow (A \wedge B))$

SOLUTION:

$$\dfrac{\dfrac{\dfrac{[A] \quad [B]}{A \wedge B}}{B \rightarrow (A \wedge B)}}{A \rightarrow (B \rightarrow (A \wedge B))}$$

2) $\vdash \neg A \rightarrow ((A \wedge \neg B) \rightarrow (A \wedge \neg B))$

(IT'S VISIBLE THAT $\cancel{\neg A}$ → THAUTOLOGY
⇒ ¬A → TRUE ⇒ ALWAYS TRUE)

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg A] \quad [A \wedge \neg B]}{\neg A \wedge (A \wedge \neg B)}\wedge I}{A \wedge \neg B}\wedge E}{(A \wedge \neg B) \rightarrow (A \wedge \neg B)}\rightarrow I}{\neg A \rightarrow ((A \wedge \neg B) \rightarrow (A \wedge \neg B))}\rightarrow I$$

ALTERNATIVE WAY: CONSIDER $\neg A \equiv A \rightarrow \bot$

$$\dfrac{\dfrac{\dfrac{\dfrac{[\neg A] \quad \dfrac{[A \wedge \neg B]}{A}\wedge E}{\bot}\rightarrow E \text{ (CONSIDERING } \neg A \equiv A \rightarrow \bot\text{)}}{A \wedge \neg B}\bot}{(A \wedge \neg B) \rightarrow (A \wedge \neg B)}\rightarrow I}{\neg A \rightarrow ((A \wedge \neg B) \rightarrow (A \wedge \neg B))}\rightarrow I$$

RESOLUTION in PROP. LOGIC

- WITH EXTENSIONS IS APPLICABLE ALSO IN FOL
- WORKS BY CONTRADICTION: NEGATE WHAT YOU WANT TO PROVE AND DERIVE FALSE.
- ONLY ONE INFERENCE RULE CALLED "RESOLUTION RULE": $\dfrac{A \lor B \quad \neg B \lor C}{A \lor C}$
- IT IS SOUND AND COMPLETE
- EFFICIENT ON COMPUTERS.
- NEEDS CNF FORMULAS AS INPUT. $\Rightarrow$ ONCE YOU HAVE $A \land B \land C \ldots$ YOUR KB = $\begin{cases} A \\ B \\ C \\ \vdots \end{cases}$

EXAMPLE: $(P \lor Q) \land (\neg P \lor R) \land (\neg Q \lor R)$

$KB = \begin{cases} P \lor Q \\ \neg P \lor R \\ \neg Q \lor R \\ \neg R \end{cases}$ YOU WANT TO PROVE **R** ↯ SO YOU ADD $\neg R$ TO THE GIVEN SENTENCES. ↯

(ALREADY IN CNF FORM)

DO THIS:

1) APPLY RESOLUTION RULE (CHOICE POINT: TO WHICH SENTENCES DO YOU APPLY IT)
2) ADD WHAT YOU DERIVE (TO THE GIVEN SENTENCES)
3) REPEAT UNTIL YOU HAVE $\rightarrow \perp \Rightarrow$ YOU'VE PROVED R
   $\rightarrow$ YOU CAN'T APPLY RESOLUTION RULE $\Rightarrow$ YOU HAVEN'T PROVED R

LET'S DO IT:

1) APPLY RESOLUTION RULE:

a) TO WHICH SENTENCES FIRST? IT'S A CHOICE POINT WHICH MAKES THE DERIVATION SHORTER OR LONGER.

b) DO I HAVE TO SUBSTITUTE $A = P \lor Q$ IN THE ~~INFERENCE~~ RESOLUTION RULE OR CAN I DO THIS: $\dfrac{P \lor Q \quad \neg P \lor R}{Q \lor R}$? BOTH, ANOTHER CHOICE POINT

2) $\Rightarrow$ I ADD $Q \lor R$ TO MY KB.

1) AGAIN: WITH THE FORMULAS $\neg P \lor R$ and $\neg R$, I CAN USE THEM LIKE THIS:

$\dfrac{A \lor B \quad \neg B \lor C}{A \lor C} \xrightarrow{\text{when}} \dfrac{\neg P \lor R \quad \neg R \lor \perp}{\neg P \lor \perp} \equiv \dfrac{\neg P \lor R \quad \neg R}{\neg P}$

2) SO WE PROVED $\neg P$ WHICH IS ADDED TO OUR KB.

AGAIN:

1) $\dfrac{A \vee B \quad \neg B \vee C}{A \vee C}$ $\longrightarrow$ $\dfrac{\neg Q \vee R \quad \neg R}{\neg Q}$ $\Rightarrow$ $\neg Q$ IS ADDED

AGAIN:

1) $\dfrac{A \vee B \quad \neg B \vee C}{A \vee C}$ $\longrightarrow$ $\dfrac{R \vee Q \quad \neg Q}{R}$

FINALLY:

1) $\dfrac{A \vee B \quad \neg B \vee C}{A \vee C}$ $\longrightarrow$ $\dfrac{\perp \vee R \quad \neg R \vee \perp}{\perp \vee \perp}$ $\equiv \perp \equiv$ "EMPTY CLAUSE" $\equiv$ CONTRADDICTION

WE PROVED R IN THIS WAY.

- THERE'S A PROBLEM: FROM A CONTRADICTION YOU CAN DERIVE ANYTHING: IF IN YOUR KB YOU HAVE P and $\neg P$ YOU CAN USE THEM TO DERIVE ANYTHING. (THIS MEANS THAT RESOLUTION IS NOT ROBUST TO NOISE) WHICH MEANS THAT IF YOU HAVE SMTH WRONG IN YOUR KB YOU COULD DO WRONG DERIVATIONS.

FIRST ORDER LOGIC

- EVERYTHING SAID UNTIL NOW FOR PROP. LOGIC IS TRUE. HERE WE ADD THINGS:

- THERE ARE VARIABLES, AND THEY CAN BE QUANTIFIED (SOME OF THEM, ALL OF THEM, NONE OF THEM, AT LEAST ONE...) THANKS TO QUANTIFIERS "∀", "∃"

- IN PROPOSITIONAL LOGIC WE ONLY COULD TREAT SENTENCES !! WITH A VERB! INSTEAD IN FOL: FACTS, OBJECTS, RELATIONS.

BNF GRAMMAR EXPLANATION:

FORMULA/SENTENCE → ATOMIC SENTENCE | COMPLEX SENTENCE

ATOMIC SENTENCE → PREDICATE | PREDICATE (TERM, ..) | TERM = TERM

COMPLEX SENTENCE → (SENTENCE)
| [SENTENCE]
| ¬ SENTENCE | SENTENCE → SENTENCE | SENTENCE ⟷ SENTENCE
| SENTENCE ∧ SENTENCE | SENTENCE ∨ SENTENCE
| QUANTIFIER VARIABLE, ... SENTENCE

TERM → FUNCTION(TERM, ..)
| CONSTANT
| VARIABLE

QUANTIFIER → ∀ | ∃

CONSTANT → NAME$_1$ | JOHN | a | b | ... | e | ...

VARIABLE → X | Y | Z | ...

PREDICATE → TRUE | FALSE | AFTER | LOVES | PLAY | MOTHER | p | q | ... | v | ...

FUNCTION → MOTHER | LEFTLEG | f | g | .. | m |

OPERATOR PRECEDENCE: ¬ ∀ ∃, =, ∧, ∨, →, ⟷

HIGH PREC. ... LOW PREC.

MORE DETAIL:

- TERMS REFERS TO OBJECTS. THEY DO NOT CONTAIN VERBS!

example; MOTHER (X) IF THIS IS A TERM (FUNCTION) THIS OBJECT INDICATES THE MOTHER OF X.

- OBJECTS ARE ANY THINGS, THEY COME FROM A SET (FINITE OR INFINITE) CALLED UNIVERSE / DOMAIN OF DISCOURSE

- FORMULA/ SENTENCE HAVE RELATION/ PREDICATES / VERBS! AND CAN HAVE FUNCTIONS.

example: MOTHER (X, Y) IF THIS IS A SENTENCE MEANS X IS MOTHER OF Y OR SOMETHING LIKE THIS.

- note that PREDICATE AND ATOMIC SENTENCES ARE ALMOST THE SAME THING
- note that a PREDICATE CAN HAVE ZERO OR M INPUTS.
- IF A PREDICATE HAS INPUTS, THE OUTPUT CAN BE ONLY TRUE OR FALSE
- SENTENCE = FORMULA

- QUANTIFIERS ARE APPLIED TO VARIABLES.
- $\forall$ = UNIVERSAL QUANTIFIER $\exists$ = EXISTENCE QUANTIFIER.
- THEY "BOUND" VARIABLES, SO ARE NOT "FREE" BUT RESTRICTED TO THE FORMULA SCOPE

GIVEN:

a, b CONSTANTS f, g FUNCTIONS p, r, q predicates x, y VARIABLES

SAY WHICH ARE WELL FORMED FORMULAS/SENTENCE, AND WHICH ARE TERMS; WHICH NOTHING

- $q(a)$ FORMULA
- $p(y)$ FORMULA
- $p(g(b))$ FORMULA
- $\neg r(x, a)$ FORMULA
- $f(g(y))$ TERM
- $\neg f(x)$ NOTHING: YOU CAN'T APPLY CONNECTIVES TO FUNCTIONS
- $f(p(y))$ YOU CAN'T HAVE PREDICATES INTO FUNCTION
- $\forall x. \neg p(x)$ FORMULA
- $a \to p(b)$ CONNECTIVES CAN BE APPLIED ONLY TO SENTENCES/FORMULAS OR VARIABLES
- $\neg\neg p(a)$ FORMULA
- $a = b$ FORMULA

FIND FREE VARIABLES:

$p(x) \wedge \neg r(y, a)$ F.V. = x, y

$\exists x. r(y, x)$ F.V. = y

$\exists x. p(x) \to \exists y. q(x, y)$ F.V. = X because appears in a formula which is not bounded to $\exists x$! THIS IS TRICKY!

$\forall x \exists y. (p(x, y) \to q(x, y))$ F.V. = none

1 $\forall x \exists y. p(x, y) \to q(x, y)$ F.V. = x and y

$\forall x. (p(x) \to \exists y. q(y, x))$ F.V. = none

$\forall x. \exists y. p(x, y) \to r(y, x))$ F.V. = y

2 $\forall x. (\exists y. (p(x, y) \to r(y, x)))$ F.V. = none

SEMANTIC.

IS A LITTLE BIT MORE COMPLEX THAN PROP. LOGIC BECAUSE FOL HAS A BIGGER DESCRIPTIVE POWER.

INTERPRETATION OF FUNCTIONS:

$I(f): U^m \rightarrow U$ means that they take m INPUTS FROM U and RETURN AN OBJECT OF U.

EXAMPLE where U is ~~My~~ FAMILY: father (ANDREA, ANNA, LAURA, EMMY)
THE OUTPUT IS "GUIDO" $\in U$.

NOTE THAT THE SAME FUNCTION CAN HAVE DIFFERENT MEANINGS DEPENDING ON THE UNIVERSE.

EXAMPLE THE FUNCTION SUM:

$I(SUM): U^m \rightarrow U$ CAN BE SUM(1,2) OUTPUT = 3 where $U = \mathbb{R}$
CAN BE SUM("a", "la") OUTPUT = ala where U = STRINGS

~~INTERPRETATION~~ EVALUTATION OF VARIABLES: SIMPLY THE SUBSTITUTION WITH AN OBJECT $\in U$.

INTERPRETATION OF A TERM

- IF THE TERM IS A VARIABLE, THE INTERPRETATION IS ITS VALUATION.
- IF IT IS A FUNCTION $\Rightarrow$ INTERPRETATE IT AND ITS INPUTS/TERMS.

SO TO INTERPRETATE A TERM YOU NEED TO MAKE IT GROUND = EVALUATE ALL ITS VARIABLES.

EXAMPLE: $I(SUM(x,3)) \xrightarrow{x=2} 5$

INTERPRETATION OF A PREDICATE / RELATIONS

$I(\text{PREDICATE}) \subseteq U^m$

IT MEANS THAT THE OUTPUT OF A PREDICATE IS A SUBSET OF U^m where m is the n° INPUTS OF THE PREDICATE.

EXAMPLE: GREATER_THAN (x, y) where U = INTEGERS.

$$I(\text{GREATER\_THAN}) = \{(2,1), (3,2), (3,1) \ldots\}$$

- IT DOES NOT DEPEND ON THE INPUTS AND THEIR VALUATION!! ONLY THE n° OF INPUTS, AND DOMAIN.

INTERPRETATION OF A SENTENCE/FORMULA

- IT IS TRUE OR FALSE DEPENDING ON THE INTERPRETATION AND VALUATION OF EVERYTHING IT CONTAINS (PREDICATES, TERMS..)

EXAMPLE: $I(\text{GREATER\_THAN}(x, y))$

CAN BE INTERPRETED GIVEN:

$$I(\text{GREATER\_THAN}) = \{(2,1), (3,2), (3,1) \ldots\}$$

$x = 5$

$y = 3$

=> THE INTERPRETATION IS TRUE, because $(5,3)$ belongs TO THE INTERPRETATION OF "GREATER_THAN". (BUT GREATER THAN COULD HAVE HAD INFINITE INTERPRETATIONS..)

NOTE THAT INTERPRETATION IS STILL ~~A CONCEPT~~ THE SAME CONCEPT OF MODEL IN PROPOS. LOGIC.

WE SAY THAT AN INTERPRETATION MAKES TRUE A FORMULA = SATISFIES THE FORMULA = IS A MODEL FOR THAT FORMULA = $I \models F$ *

=> THE NOTATION IS A LITTLE BIT DIFFERENT WRT TO PROP. LOGIC.

FORMULATION OF SENTENCES IN FOL

"LUIS LOVES EVERYBODY WHO GIVES HIM FOOD"

MEANING — FORMALIZATION — GIVEN:

$$\forall x. \left(P_1(x, L, F) \longrightarrow P_2(L, x) \right)$$

	I(.)
U	ALL OBJECTS
L	"LUIS"
F	"FOOD"
P_1	" _ given _ _ "
P_2	" _ loves _ "

SIMILARLY:

ALL EMPLOYES HAVE INCOME $\rightarrow$ $\forall x. \left(EMPLOYE(x) \rightarrow HAS\text{-}INCOME(x) \right)$

SOME EMPLOYES ARE ON HOLIDAYS $\rightarrow$ $\exists x \left(EMPLOYE(x) \wedge ONHOLIDAYS(x) \right)$

NO EMPLOYES ARE UNEMPLOYED $\rightarrow$ $\forall x \left(EMPLOYES(x) \rightarrow \neg UNEMPLOYED(x) \right)$

EVERYONE BOUGHT SOMETHING $\rightarrow$ $\forall x \exists y.\ bought(x, y)$

SOMEONE BOUGHT EVERYTHING $\rightarrow$ $\exists x \forall y.\ bought(x, y)$

QUANTIFIER ORDER MATTERS !!

EVERY STUDENT LOVES SOME STUDENT $\rightarrow$ $\cancel{\forall x. \exists y. (STUDENT(x) \rightarrow LOVES(x, y))}$

$$\forall x. \left(STUDENT(x) \rightarrow \exists y. \left(STUDENT(y) \wedge LOVES(x, y) \right) \right)$$

EVERY STUDENT LOVES SOME OTHER STUDENT $\rightarrow$

$$\forall x. \left(STUDENT(x) \rightarrow \exists y. \left(\neg(x = y) \wedge STUDENT(y) \wedge LOVES(x, y) \right) \right)$$

I ALWAYS TRAVEL SOMEWHERE NOT FAR FROM THE CITY $\rightarrow$ $\forall x. \left(IS\_TRAVEL(x) \rightarrow \neg is\_long(x) \right)$

"A NEPHEW IS A SIBLING SON" $\rightarrow$ $\forall x \forall y. \left(NEPHEW(x, y) \longleftrightarrow \exists z. \left(SIBLING(y, z) \wedge SON(x, z) \right) \right)$

(NEPHEW, SIBLING AND SON ARE ALL BINARY RELATIONS!)

GIVE AN INTUITIVE MEANING TO THESE FORMULAS:

IT

- bought(FRANK, dvd) : FRANK HAS BOUGHT A DVD
- $\exists x.$ bought(FRANK, x) : FRANK " " SMTH
- $\forall x.$ (bought (FRANK, x) $\rightarrow$ bought (Susan, x)) : SUSAN BOUGHT EVERYTHING THAT FRANK BOUGHT
- $\forall x.$ bought(FRANK, x) $\rightarrow \forall x.$ bought(Susan, x) : IF FRANK BOUGHT EVERYTHING $\rightarrow$ SUSAN BOUGHT EVERYTHING

WHAT'S THE DIFFERENCE ON FUNCTION AND PREDICATE INTERPRETATION ?

- THE INTERPRETATION OF A PREDICATE IS A SUBSET OF U^M WHICH ALLOWS US TO DO A MAP FROM $U^M \rightarrow$ TRUE OR FALSE
- THE INTERPRETATION OF A FUNCTION IS A MAPPING FROM $U^M \rightarrow U$ SO THE OUTPUT IS AN OBJECT OF U !!

ABOUT INTERPRETATIONS:

* NOTE THE DIFFERENCE:

PROPOS. LOGIC: $\boxed{\Gamma \models F}$

THIS IS HOW WE EXPRESS SEMANTIC CONSEQUENCE

Where Γ is a set of formulas. means that ALL THE INTERPRETATIONS / MODELS OF Γ ARE ALSO MODELS OF F.

WHERE AN INTERPRETATION IS SIMPLY THE ASSIGNMENT OF TRUE OR FALSE: 2^m POSSIBLE ASSIGNATION.

(ALL POSSIBLE INTERPRETATIONS!)

$\boxed{M(F)}$

THIS IS HOW WE EXPRESS WHAT IS A MODEL OF F

FOL $\left((I, \eta \models F\right) \equiv \boxed{I \models F}$

Where I is AN INTERPRETATION, SO SOMETHING WHICH DOES A MEANING TO ALL TERMS AND PREDICATES!!! WHICH ARE NOT ONLY TRUE AND FALSE THINGS!

→ IT MEANS THAT F IS SATISFIED BY I WITH ANY POSSIBLE VALUATION OF THE VARIABLES

(ONLY ONE INTERPRETATION! ALL VALUATIONS)

PROP. LOGIC

VALID FORMULA: $\models F$

$\models F$: SATISFIED $\forall$ INTERPRETATION

INVALID:

F not SATISFIED BY SOME I

FOL

VALID FORMULA:

$I \models F \; \forall I$: SATISFIED $\forall I$

FALSIABLE: (= "INVALID")

F not SATISFIED BY SOME I

So HOW DO WE EXPRESS SEMANTIC CONSEQUENCE IN FOL?

in THE SAME WAY: $\Gamma \models F$!!!

BUT in PROP. we had 2^m INTERPRETATIONS → ALWAYS POSSIBLE TO EVALUATE IT.

IN FOL WE HAVE INFINITE INTERPRETATIONS! $\Rightarrow \Gamma \models F$ IS UNDECIDABLE.

(BUT WE COULD PROVE THAT IS NOT A LOG. CONS. BY FINDING A CASE IN WHICH IT ISN'T)

BUT THANKS TO HERBRAND WE CAN KNOW IF $I \models F \; \forall I$ (SO IF F IS VALID)

(IN PROP. VALIDITIES = TAUTOLOGIES, IN FOL BY TAUTOLOGIES WE ONLY MEAN SOME PARTICULAR USEFUL VALIDITIES :

TAUTOLOGIES

↓

EACH PROP. LOGIC TAUTOLOGY IN WHICH YOU SUBSTITUTE A PROPOSITION WITH A FOL FORMULA, IS A TAUTOLOGY IN FOL

$\forall x. \forall y. P \equiv \forall y. \forall x. P$
$\exists x. \exists y. P \equiv \exists y. \exists x. P$
} QUANTIFICATORI DELLO STESSO TIPO COMMUTANO !!!

$\neg \forall x. A \equiv \exists x. \neg A$

$\neg \exists x A \equiv \forall x. \neg A$

(TO PROVE THEM YOU MUST THINK ABOUT THE MEANING : TRANSLATE IT IN ENGLISH, PARAPHRASE THEM AND THEN RETRANSFORM IN FOL

$\forall x (A \wedge B) \equiv \forall x A \wedge \forall x B$

$\exists x (A \vee B) \equiv \exists x A \vee \exists x B$

- FOL IS UNDECIDABLE : YOU CAN'T PROOF SEMANTICALLY THE VALIDITY OF A FORMULA.
- BUT YOU CAN SYNTACTICALLY ! e.g. USE RESOLUTION WITH KB = { } TO PROVE THE VALIDITY OF A FORMULA.
- HOW CAN YOU PROVE $\models \forall x (A \wedge B) \Leftrightarrow \forall x A \wedge \forall x B$

HOW TREAT IT?

YOU MUST TRANSFORM IT : $\models \forall x (A \wedge B) \quad \wedge \quad \models (\forall x A \wedge \forall x B)$
THANKS TO THE DEFINITION OF "$\Leftrightarrow$"

- mode: $\neg(\exists x (A \rightarrow B)) \equiv \neg(\exists x (\neg A \vee B)) \equiv \forall x \, \neg(\neg A \vee B) \equiv \forall x (A \wedge \neg B)$
- ! mode: $\neg(\forall x (A \rightarrow B)) \equiv \exists x (\neg(\neg A \vee B)) \equiv \exists x (A \wedge \neg B)$

TO SHOW THAT $\forall x (A \lor B) \not\equiv \forall x A \lor \forall x B$

TO PROVE IT I NEED A PROPERTY A WHICH MAKES $\neg B$ AND VICEVERSA:

$\forall x$ (is_EVEN(x) $\lor$ is_ODD(x)) IS ALWAYS TRUE.

$\forall x$ is_EVEN(x) $\lor$ $\forall x$ is_ODD IS ALWAYS FALSE.

HERBRAND ("THE CONSTANTS ARE THEMSELVES THE OBJECTS CONSIDERED")

WE INTRODUCE SOME DEFINITIONS WHICH ALLOWS US TO DO EASIER PROOFS AND THEOREMS: INSTEAD OF ANY UNIVERSE WE ONLY CARE ABOUT THE "HERBRAND UNIVERSE" AND INSTEAD OF ALL INTERPRETATIONS (TO PROOF THE VALIDITY OF A FORMULA FOR EXAMPLE) WE ONLY LOOK AT THE "HERBRAND INTERPRETATION".
WHY? BECAUSE THE HERBRANDT THEOREM TELLS THAT IF A FORMULA IS VALID IN THE "HERBRAND UNIVERSE" THANKS TO THE "HERBRAND" INTERPRETATION" THEN IT IS VALID IN ANY INTERPRETATION.

WHY IS THIS USEFUL? BECAUSE THE "HERBRAND UNIVERSE" AND "HERBRAND INTERPRETATION" ARE SO EASY THAT ALLOWS US TO FOCUS ON SYNTAX.

HERBRAND UNIVERSE = SET OF GROUND TERMS.

HERBRAND INTERPRETATION =

- COSTANTS ARE INTERPRETED AS THEMSELVES. $a \to a$
- FUNCTIONS ARE INTERPRETED AS A MAPPING FROM $\underbrace{t_1, t_2 .. t_n}$ (ITS INPUTS) TO $f(t_1, t_2 .. t_n)$ when
  ↳ HAVE BEEN GROUNDED.
- PREDICATES ARE INTERPRETED AS ~~A~~ SUBSET OF HERBRAND UNIVERSE.

(HERBRAND BASE = SET OF ATOMIC FORMULAS).

(HERBRAND MODEL = HERBRAND INTERPRETATION which MAKES F TRUE).

* HE DIDN'T EXPLAIN HOW WE USE IT. THIS PART IS NOT CLEAR.

HERBRAND THEOREM:

Γ is a set of sentences. THE FOLLOWING ARE EQUIVALENT:

1. Γ HAS AN HERBRAND MODEL
2. Γ HAS A MODEL
3. GROUND(Γ) IS SATISFIABLE.

PROOF:

1 → 2 OBVIOUS

2 → 3 EVERY GROUND(Γ) IS A LOGICAL CONSEQUENCE OF Γ.
SO EVERY MODEL OF Γ IS A MODEL OF GROUND(Γ).

3 → 1 IF GROUND(Γ) IS SATISFIABLE THEN IT HAS A HERBRAND MODEL "A".
LET BE M A MODEL OF GROUND(Γ).
SO WE CAN define A in THE USUAL WAY FOR FUNCTION SYMBOLS,
WHILE FOR ATOMIC FORMULAS $A \models p(t_1 \dots t_m)$ iff $M \models p(t_1 \dots t_m)$
So A is also A MODEL OF Γ...

NORMAL FORMS (CNF & DNF)

WE'LL SEE RESOLUTION WHICH WORKS WITH NORMAL FORMS

IN FOL WE OBTAIN CNF AND DNF IN THE SAME WAY OF PROP. LOGIC, THINKING ABOUT QUANTIFIERS ONLY AT THE END!

↳ AND USING WITH THESE TAUTOLOGIES FROM LEFT TO RIGHT:

$$1)\ \neg\forall x.F \equiv \exists x.\neg F$$

$$2)\ \neg\exists x F \equiv \forall x \neg F$$

GREAT EXAMPLE WITH ANY CASISTIC:

$$(\exists x \forall y\, p(x,y)) \vee \neg\exists y (q(y) \rightarrow \forall z\, r(z))$$

1) REMOVE IMPLICATIONS:

$$(\exists x \forall y\, p(x,y)) \vee \neg\exists y (\neg q(y) \vee \forall z\, r(z))$$

2) MOVE $\neg$ INTO LITERALS TO OBTAIN NNF = NEGATION NORMAL FORM:

$$(\exists x \forall y\, p(x,y)) \vee \forall y\, \neg(\neg q(y) \vee \forall z\, r(z))$$

$$(\exists x \forall y\, p(x,y)) \vee \forall y\, (q(y) \wedge \neg\forall z\, r(z))$$

$$(\exists x \forall y\, p(x,y)) \vee \forall y\, (q(y) \wedge \exists z\, \neg r(z))$$

3) STANDARDIZE VARIABLES $\doteq$ "RENAME VARIABLES APART"
GIVE DIFFERENT NAMES TO VARIABLES WHICH ARE NOT BOUNDED TO THE SAME QUANTIFIER:

$$(\exists x \forall y\, p(x,y)) \vee \forall T\, (q(T) \wedge \exists z\, \neg r(z))$$

4) OBTAIN "PRENEX FORM" = IT SIMPLY MEANS TO HAVE ALL QUANTIFIERS AT THE BEGINNING. AFTER DOING 1) 2) 3) YOU SIMPLY NEED TO MOVE THEM: DO NOT CHANGE THE ORDER!

$$\exists x \forall y \forall T \exists z\, \big(p(x,y) \vee (q(T) \wedge \neg r(z))\big)$$

5) SK

• WHAT

• F

• F

THIS

$\exists x$

whe

• X

• Z

(TO BE
THAT

6) SI

P

7) APP

(P

8) RE
A

5) SKOLEMIZATION = REMOVE $\exists$ QUANTIFIERS.

- WHAT WE OBTAIN IS CALLED SKOLEMIZED FORM OF F
- F IS SATISFIABLE iff F SKOLEMIZED IS SATISFIABLE
- F $\not\equiv$ F SKOLEMIZED

THIS IS HOW YOU OBTAIN THE SKOLEMIZED FORM AFTER 1), 2), 3), 4):

$$\exists x \forall y \forall T \exists z \left(p(x,y) \vee (q(T) \wedge \neg r(z)) \right)$$

$\Downarrow$ SKOLEMIZATION

$$\forall y \forall T \left(p(a,y) \vee (q(T) \wedge \neg r(f(y,T)) \right)$$

where the $\exists x$ and $\exists z$ have been removed and their variables substituted

- $x \mapsto a$ — "x" becomes simply "a", which is called SKOLEM CONSTANT, BECAUSE x WASN'T INTO THE SCOPE OF A UNIVERSAL QUANTIFIER.

- $z \mapsto f(y,T)$ — "z" becomes "$f(y,T)$", where f is CALLED SKOLEM FUNCTION, BECAUSE z WAS INTO THE SCOPE OF THE UNIVERSAL QUANTIFIERS $\forall y \forall T$.

(TO BE INTO THE SCOPE WHEN YOU HAVE REACHED THE PRENEX FORM SIMPLY MEANS THAT IT'S ON THE RIGHT OF THE QUANTIFIER)

6) SIMPLY REMOVE THE $\forall$ QUANTIFIERS:

(WE DO THIS BECAUSE WE'LL USE THIS SENTENCE AS PREMISE IN PROOFS WITH RESOLUTION. THE PROOF DOESN'T CHANGE IF WE REMOVE THE "$\forall$")

$$p(a,y) \vee (q(T) \wedge \neg r(f(y,T)))$$

7) APPLY DISTRIBUTIVITY TO HAVE THE CNF:

$$(p(a,y) \vee q(T)) \wedge (p(a,y) \vee \neg r(f(y,T)))$$

(8) RENAME APART VARIABLES TO DO RESOLUTION → AS WE'LL SEE)

MORE EXAMPLES:

$F \equiv \exists x\, P(x) \vee \exists y\, P(y) \vee \forall z\, P(z)$

MANY WAYS:

① $\exists x\, \exists y\, \forall z\, (P(x) \vee P(y) \vee P(z))$ $\quad x \mapsto a,\ y \mapsto b$

$\forall z\, (P(a) \vee P(b) \vee P(z))$

② NOTICE THAT YOU CAN RENAME $y = x$

$\Rightarrow \exists x\, P(x) \vee \exists x\, P(x) \vee \forall z\, P(z)$ $\quad$ IDEMPOTENCE $A \vee A \equiv A$

$\equiv \exists x\, P(x) \vee \forall z\, P(z)$

$\equiv \forall z\, (P(a) \vee P(z))$ $\quad x \mapsto a$

OUR AIM IS TO DO RESOLUTION, BUT BEFORE WE STILL NEED SOME NOTIONS:

UNIFICATION

= IT'S A PROCEDURE WHICH FINDS SUBSTITUTIONS THAT MAKES TWO LITERALS LOOK IDENTICAL, SYNTACTICALLY.

EXAMPLE: UNIFY $(p(x,a), p(b,y)) \Rightarrow \{x/b, y/a\}$ "

(WHICH MEANS THAT THANKS TO THE SUBSTITUTIONS $x \mapsto b$ and $y \mapsto a$ THE TWO LITERALS ARE SINTACTICALLY IDENTICAL. USUALLY THE LIST OF SUBSTITUTION IS INDICATED LIKE $\sigma = \{x/b, y/a\}$ $\Rightarrow$ ALSO CALLED "UNIFIER". IT MUST APPLIED TO BOTH TO OBTAIN THE UNIFICATION.)

BUT THERE ARE SOME PROBLEMS:

- TO UNIFY $P(x,a)$ and $P(b,x)$ I SHOULD SAY $\{x/b, x/a\}$ WHICH IS NOT POSSIBLE. THIS HAPPENS BECAUSE THE TWO FORMULAS USE SAME NAMES FOR SOME VARIABLES. SO BEFORE DOING UNIFICATION CHANGE THAT, RENAME THEM.

- THERE COULD BE MANY SUBSTITUTIONS WHICH MAKE UNIFICATION POSSIBLE, WHICH ONE I AM LOOKING FOR?

  ↳ MGU = MOST GENERAL UNIFIER

  WHICH IS THE SUBSTITUTION WHICH HAS THIS PROPERTY:

  g IS A MGU OF F_1 and F_2 IFF:

  $\forall$ UNIFIERS μ, $\exists\, \mu'$ s.t. $(F_1)\mu = ((F_1)g)\mu'$ and $(F_2)\mu = ((F_2)g)\mu'$

(= ANY OTHER UNIFIER CAN BE EXPRESSED AS A SUBSTITUTION APPLIED TO F_1 and F_2 ALREADY SUBSTITUTED BY g)

EXAMPLES:

FORMULA OR TERM	FORMULA OR TERM	MGU
$P(x)$	$P(A)$	$\{x/A\}$
$P(f(x), y, G(x))$	$P(f(x), x, G(x))$	$\{x/y\}$ or $\{y/x\}$
$P(f(x), y, g(y))$	$P(f(x), z, g(x))$	$\{y/x, z/x\}$
$P(x, B, B)$	$P(A, y, z)$	$\{x/A, y/B, z/B\}$
$P(g(f(v)), g(u))$	$P(x, x)$	$\{x/g(f(v)), u/f(v)\}$
$P(x, f(x))$	$P(x, x)$	NO UNIFICATION POSSIBLE

THE RULES TO OBTAIN IT ARE THESE:

- SCAN THE F_1 and F_2 TERM by TERM SIMOULTANEOUSLY.
- COMPARE THE TWO TERMS IN THIS WAY:

F_1 TERM	F_2 TERM	OUTPUT
$f(\dots)$	$g(\dots)$	FAILURE because different FUNCTIONS
a	a	NOTHING
a	b	FAILURE because different CONSTANTS (FUNCTIONS)
x	y	ADD THIS TO THE SUBSTITUTIONS $\{x \mapsto y\}$ (or $y \mapsto x$)
$f(a, x)$	$f(y, b)$	ADD THIS: $\{y/a, x/b\}$
$f(g(a, x), y)$	$f(c, x)$	FAILURE BECAUSE $g(a, x)$ IS A DIFF. FUNC. from c.
$f(g(a, x), h(c))$	$f(g(a, b), y)$	ADD THIS: $\{x/b, y/h(c)\}$
$f(g(a, x), h(y))$	$f(g(a, b), y)$	FAILURE BECAUSE THE VARIABLE OCCURS INTO THE TERM I WANT TO SUBSTITUTE WITH. (I SHOULD DO $y/h(y)$ WHICH IS AN INFINITE SUBSTITUTION)

NOTE THAT YOU CAN ONLY SUBSTITUTE

VARIABLE / FUNCTION OR CONSTANT OR VARIABLE

FINALLY WE CAN TALK ABOUT

RESOLUTION REFUTATION PROCEDURE
= RESOLUTION PROCEDURE

- IT'S A SOUND AND COMPLETE INFERENCE PROCEDURE
- IT'S APPLIABLE ALSO IN PROP. LOGIC, BUT MORE POWERFUL IN FOL.
- EFFICIENT TO BE IMPLEMENTED IN COMPUTERS
- BASED ONLY ON 1 INFERENCE RULE :→ "RESOLUTION RULE" = "RR"

- IT WORKS BY CONTRADDICTION.
- IT NEEDS FORMULAS IN CNF : ONCE YOU HAVE $A \wedge B \wedge C \ldots \Rightarrow KB = \{A, B, C\}$
(THIS IS WHY WE NEEDED SKOLEMIZATION)
- WE NEED A WAY TO DEAL WITH VARIABLES, THAT'S THE DIFFERENCE WITH RESOLUTION FOR PROP. LOGIC
↓
THIS IS UNIFICATION

REMEMBER THAT THIS IS THE RESOLUTION RULE (FOR PROP. LOGIC): $\dfrac{A \vee B \quad \neg B \vee C}{A \vee C}$

THE ONLY DIFFERENCES WITH RESOLUTION FOR PROP. LOGIC ARE 2:

1) IT'S MORE DIFFICULT TO OBTAIN THE CNF FORM, WITHOUT QUANTIFIERS. THAT'S WHAT WE DID WITH SKOLEMIZATION.

2) WE HAVE TO DEAL WITH VARIABLES, THANKS TO UNIFICATION:

RESOLUTION RULE FOR FOL:

$$\frac{A \vee B \quad \neg B' \vee C}{(A \vee C)\,\theta} \qquad \text{where } \theta = \text{MGU OF } B, B'$$

THAT'S IT. YOU NEED TO UNIFY B AND B' WITH THEIR MGU (= θ), AND THEN THE DERIVED FORMULA IS A V C TO WHICH YOU MUST APPLY THE MGU (= θ).

(WITH THE SAME REASONING COULD BE USED ANOTHER RULE:
"FACTORING RULE" = $\dfrac{A \vee B \vee B'}{(A \vee B)\sigma}$ when $\sigma = MGU(B, B')$)

- YOU CAN ALSO USE RESOLUTION FOR PROOF VALIDITY
(= FORMULA SATISFIED FOR ANY INTERPRETATION), BY USING $KB = \{\ \}$)

1) EASY EXAMPLE OF RESOLUTION RULE:

AFTER SKOLEMIZATION etc. YOU OBTAIN $(p(x) \vee q(x)) \wedge \neg p(a)$

$\Rightarrow KB = \begin{cases} p(x) \vee q(x) \\ \neg p(a) \end{cases}$

(YOU CAN THINK OF p AS $p \vee \perp$ TO APPLY THE RULE)

YOU CAN OBTAIN FOLLOWING $\dfrac{A \vee B \quad \neg B' \vee C}{(A \vee C)\sigma}$: $\dfrac{q(x) \vee p(x) \quad \neg p(a) \vee \perp}{(q(x) \vee \perp)\sigma}$

where $\sigma = MGU(p(x), p(a)) = \{x/a\}$

$\Rightarrow$ YOU OBTAIN $q(a) \vee \perp \equiv \boxed{q(a)}$

2) THE RESOLUTION PROCEDURE INSTEAD USES THE "RR" RULE FOR PROOF BY CONTRADICTION:

GIVEN THE $KB = \begin{cases} \vdots \\ \vdots \end{cases}$ IF YOU WANT TO OBTAIN THE PROOF OF P, ADD $\neg P$ TO THE KB AND DERIVE $\perp$

(EXAMPLE IN RESOLUTION FOR PROP. LOGIC).

IN ORDER TO DON'T MAKE CONFUSION WITH THE UNIFICATION, BEFORE TO DO IT "RENAME APART" THE VARIABLES. $\ast$
/ STANDARDIZE THE VARIABLES

(example: YOU MUST UNIFY $p(x, a)$ and $p(b, x)$.
BEFORE TO DO IT CHANGE VARIABLE NAMES:
$p(x, a)$ and $p(b, y)$)