First Order Logic II

Prof. Mauro Gaspari
Dipartimento di Informatica Scienze e Ingegneria (DISI)
mauro.gaspari@unibo.it

Reasoning in FOL

- Two are the main concerns:
- Inference steps are more complex: we need **unification and substitutions** to perform them.
- The set of all the possible clauses is not finite any more. Thus **termination becomes an issue**.

Termination

- The set of all possible formulas to explore during the reasoning process is now infinite.
- A functional symbol of arity one and a constant are enough to obtain an infinite set of terms and thus an infinite set of formulas.
- Example:
    - s(x) as a functional symbol (successor).
    - 0 as a constant (0)
    - This allows us to represent all the infinite positive integers as follows: $0 = 0$, $1 = s(0)$, $2 = s(s(0))$, $3 = s(s(s(0)))$

Natural Deduction in FOL

- All of the rules from propositional logic carry over to predicate logic, and there are four new rules for (introduction and elimination of quantifiers).

$$\frac{A(x)}{\forall y\ A(y)}\ \forall I \qquad \frac{\forall x\ A(x)}{A(t)}\ \forall E$$

If a predicate is true for all members of a domain, then it is also true for a specific one t. Also known as universal instantiation.

$$\frac{A(t)}{\exists x\ A(x)}\ \exists I$$

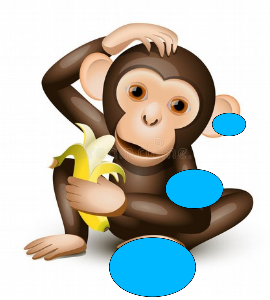
$$\frac{\exists x\ A(x) \qquad \begin{matrix}\overline{A(y)}^{\,1} \\ \vdots \\ B\end{matrix}}{B}\ 1\ \exists E$$

Existential elimination

$$\dfrac{\exists x.\ P(x) \qquad \begin{array}{c}[P(a)]\\ \vdots\\ q\end{array} \qquad (a \text{ arbitrary, a variable})}{q \quad (a \text{ not free in } q)}$$

- If P(x) holds for some arbitrary individual x, let that individual be called a (so P(a) holds).
- If q follows from P(a) and proving q doesn't involve our choice of a.
- Then q holds regardless of which individual has made P true.
- Thus, the proof of q from P(a) must work for any individual in place of a.

Considering Equality



- In FOL we can use the equality symbol to state that two terms refer to the same object.
- Examples:

  mother(john) = mary

  ∀x ∀y brother(x, mauro) ∧ brother(y, mauro)

  ∀x ∀y brother(x, mauro) ∧ brother(y, mauro)∧¬(x=y)

It does not mean that x and y are different. Adding equality it works

- Natural Deduction can be extended to deal with deduction adding the following rules:

$$\frac{}{t = t}\,\text{refl} \qquad \frac{s = t}{t = s}\,\text{symm} \qquad \frac{r = s \qquad s = t}{r = t}\,\text{trans}$$

$$\frac{s = t}{r(s) = r(t)}\,\text{subst} \qquad \frac{s = t \qquad P(s)}{P(t)}\,\text{subst}$$

Natural Deduction in FOL

- Soundness: IF KB $\vdash^{ND}$ F THEN KB $\models$ F
- Completeness: IF KB $\models$ F THEN KB $\vdash^{ND}$ F
- Semidecidable.
- Efficient: NO
- Expressive Power: GOOD

RR in FOL

- It is possible to adapt the engine based on refutation and resolution to FOL:
  - Trasforming formulas in CNF is more complex. The main difference concerns variables and identifier.
  - The resolution tree becomes infinite.
  - The resolution rules must be extended with unification and substitutions.

CNF in FOL

- We can use equivalences that hold for propositional logic to transform a FOL KB in CNF form.
- However, the presence of quantifiers makes this transformation process more complex and additional equivalences are needed.
- The transformation process can be divided in two phases:

1. Transform the KB in **prenex normal form**: where all quantifiers are moved at the front of a formula:

$$\exists x \forall y \exists z\ (R(x, y, z) \land P(x) \land (\neg P(x) \lor Q(x,y))).$$

2. Eliminating quantifiers (**skolemization**).

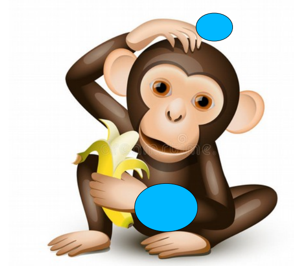
Equivalences for Quantifiers

- Infinitary de Morgan laws:
  - $\neg(\forall x\ A) \equiv \exists x\ \neg A$
  - $\neg(\exists x\ A) \equiv \forall x\ \neg A$
- Moving quantifiers through conjunction and disjunction
  - $(\forall x\ A) \land B \equiv \forall x\ (A \land B)$
  - $(\forall x\ A) \lor B \equiv \forall x\ (A \lor B)$
  - $(\exists x\ A) \land B \equiv \exists x\ (A \land B)$
  - $(\exists x\ A) \lor B \equiv \exists x\ (A \lor B)$

Equivalences for Quantifiers

- Distributive laws:
  - $(\forall x\ A) \land (\forall x\ B) \equiv \forall x\ (A \land B)$
  - $(\exists x\ A) \lor (\exists x\ B) \equiv \exists x\ (A \lor B)$
- Implication when x is not free in B:
  - $(\forall x\ A) \to B \equiv \exists x\ (A \to B)$
  - $(\exists x\ A) \to B \equiv \forall x\ (A \to B)$
- Expansion as infinitary conjunction and disjunction
  - $\forall x\ A \equiv (\forall x\ A) \land A[x/t]$
  - $\exists x\ A \equiv (\exists x\ A) \lor A[x/t]$

Skolemization



- **Esistential instantiation rule**: for any formula P, variable x and constant symbol k, which does not appear elsewhere in the knowledge base.

$$\frac{\exists x\ P(x)}{P(k)[x/k]}$$

The same concept is used in the existential elimination rule of natural deduction.

- Example, from $\exists x\ course(x) \land involved(x, mauro)$
- We can infer the formula: $course(k) \land involved(k, mauro)$
- k is defined as **skolem constant** and it must not appear elsewhere in the knowledge base (otherwise it could match a course in which mauro is not actually involved).

Observations

- Universal instantiation can be applied many times in reasoning to produce many different consequences.
- Existential instantiation can be applied once in reasoning and the existential quantifier must be discarted.
- As a result the new KB is not longer equivalent to the old one.
- But the new KB is **inferentially equivalent**: it is satisfiable when the original KB is satisfiable.
- **Intuition:** whe can always chose an interpretation where the skolem constant is associated to the right object.

Skolemization

- Consider the following example:

$$\forall x\ \exists y\ loves(x, y) \quad \overset{\rightarrow}{\underset{\leftarrow}{???}} \quad \exists y\ \forall x\ loves(x, y)$$

- Does the first imply the second or viceversa?

Skolem functions

- $\exists y \forall x$ loves(x, y) $\rightarrow$ $\forall x \exists y$ loves(x, y) holds!
- Viceversa does not hold.
- Considering: $\forall x \exists y$ loves(x, y)
  - If an existential quantified variable like y is after an universal quantified ones, x, there is a dependence between x and y.
  - For each x that we can fix, there is a y that makes the loves predicate true. But, it is not the same y for all the x.
  - In this case we need a **Skolem function** which represents the above intuition: $\forall x$ loves(x, k(x))

Skolemization

- Skolemization replaces every existentially bound variable by a Skolem constant or function.
- Then all the universal quantifier are removed and all the remaining variables are cosidered to be universaly quantified.
- This transformation does not preserve the meaning of a formula.
- **It does preserve inconsistency**: resolution works by detecting contradictions!

Skolemization Example

$\underline{\exists u}\ \forall v\ \exists w\ \exists x\ \forall y\ \exists z\ ((P(h(\underline{u}, v)) \lor Q(w)) \land R(x, h(y, z)))$

- Eliminate the $\exists u$ using the Skolem constant c:

$\forall v\ \underline{\exists w}\ \exists x\ \forall y\ \exists z\ ((P(h(\mathbf{c}, v)) \lor Q(\underline{w})) \land R(x, h(y, z)))$

- Eliminate the $\exists w$ using the 1-place Skolem function f :

$\forall v\ \underline{\exists x}\ \forall y\ \exists z\ ((P(h(c, v)) \lor Q(\mathbf{f(v)})) \land R(\underline{x}, h(y, z)))$

- Eliminate the $\exists x$ using the 1-place Skolem function g:

$\forall v\ \forall y\ \underline{\exists z}\ ((P(h(c, v)) \lor Q(f(v))) \land R(\mathbf{g(v)}, h(y, \underline{z})))$

- Eliminate the $\exists z$ using the 2-place Skolem function s:

$\forall v\ \forall y\ ((P(h(c, v)) \lor Q(f(v))) \land R(g(v), h(y, \mathbf{s(v, y)})))$

- Remove universal quantifiers:

$(P(h(c, v)) \lor Q(f(v))) \land R(g(v), h(y, \mathbf{s(v, y)}))$

Depth First RR in FOL

- Soundness: IF $KB \cup \{\neg F\} \vdash^{RR} \Box$ THEN $KB \models F$
- Completeness: NO
- Semidecidable.
- Efficient: NO
- Expressive Power: GOOD

Observations

- The same resolution strategies presented for propositional logic can be applied in FOL.
- However, the branching factor increases a given clause can be applied several times (universal instantiation). Breadth first search cannot be used because of the large frontier.
- The resolution tree is not finite, thus depth first search strategies are not complete anymore.
- Theorem provers use depth-first iterative deepening strategies which are efficient and complete and can be used in many cases. However, the worst case is still un-tractable

Depth-first iterative deepening RR in FOL

- Soundness: IF $KB \cup \{\neg F\} \vdash^{RR} \Box$ THEN $KB \models F$
- Completeness: IF $KB \models F$ THEN $KB \cup \{\neg F\} \vdash^{RR} \Box$
- Semidecidable.
- Efficient: NO
- Expressive Power: GOOD

Tools

- **Z3 Theorem prover**, it works also with FOL.
- **The E Theorem Prover**: full first-order logic with equality.
  - https://wwwlehre.dhbw-stuttgart.de/~sschulz/E/E.html
- **PTTP - Prolog Technology Theorem Prover:** unification with the occurs check; depth-first iterative deepening; model elimination rule.
  - http://www.ai.sri.com/~stickel/pttp.html
- **Otter and Mace2** - Organized Techniques for Theorem proving and Efficient Reasoning: uses set of support and a form of best first search.
  - https://www.cs.unm.edu/~mccune/otter/
- Stepchowfun theorem-prover: a python based theorem prover,
  - https://github.com/stepchowfun/theorem-prover

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