MAXIMUM INDEPENDENT SET

- Given a graph $G=(V,E)$, determine $W \subseteq V$ such that

  $N(v) = \{w \in V \mid \exists \{v,w\} \in E\}$

  1. $\forall v \in W \quad N(v) \cap W = \emptyset$
  2. $|W|$ is maximum among all the subsets of V having the property 1.

- an algorithm for MIS can be defined as follows, as a piece of pseudocode

```
INPUT:  G = (V,E)
OUTPUT: |W|, WHERE W ⊆ V SATISFIES 1-2 ABOVE
max ← 0
Z ← ∅
foreach W ⊆ V do
    ind ← True
    foreach w ∈ W
        if N(w) ∩ W ≠ ∅ then
            ind ← False
    if ind then
        if max < |W| then
            max ← |W|
            Z ← W
return |Z|
```

THE NUMBER OF SUBSETS OF V IS $O(2^{|V|})$

$O(1)$

$O(|V|^2)$

$O(1)$

$O(2^{|V|} \cdot p(|V|))$ WHERE p IS A POLYNOMIAL

$\in O(2^{|V|} \cdot 2^{|V|})$

THIS PERFECTLY MATCHES

$= O(2^{2|N|}) \in O(2^{|N|^2}) \in O(2^{n^2})$ THE DEF. OF FEXP.

WHERE n IS THE SIZE OF THE INPUT. THIS IMPLIES THAT $\boxed{\text{MIS} \in \text{FEXP}}$

- Can we see MIS as a language?
  Yes, we can, because we can define DMIS as the language

$$\{x \in \{0,1\}^* \mid x = \llcorner \underbrace{(V,E)}_{\text{GRAPH}}, \underbrace{K}_{\text{THRESHOLD}} \lrcorner \wedge \text{MIS}(V,E) \leq K\}$$

MIS $\in$ FEXP $\Longrightarrow$ DMIS $\in$ EXP

THEOREM

$P \subsetneq EXP$

PROOF

- Of course we know that $P \subseteq EXP$, because any TM working in polynomial time also works in exponential time.
- What remains to be proved is that $EXP \neq P$.
- To prove this, we actually state and prove a stronger result, and to do that we need the o-notation.
  → Given $f, g : \mathbb{N} \to \mathbb{N}$, we say that

$f(n)$ is $o(g(n))$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

EXAMPLES
$n \in o(n^2)$
$n \notin o(n)$

- The (stronger) auxiliary result we need is the following: if f, g are time-constructible and $f(n) \lg(f(n))$ is $o(g(n))$ then

$$\mathrm{DTIME}(f(n)) \subsetneq \mathrm{DTIME}(g(n))$$

- We prove an example of the HT, namely that $\mathrm{DTIME}(n) \subsetneq \mathrm{DTIME}(n^2)$

⇒ Consider the following TM D.
D, on input x, run for $|x|^{3/2}$ steps the UTM $\mathcal{U}$ on the input (x, x). If $\mathcal{U}$ outputs (in less that $|x|^{3/2}$ steps) the bit b, then D outputs $1-b$; otherwise, D outputs 0.

⇒ By definition, D halts in $n^{7/4}$ steps, the gap with $|x|^{3/2}$ accounting for the overhead in the simulation. Thus the language L decided by D is in $\mathrm{DTIME}(n^2)$

⇒ What we want to prove now is that $L \notin DTIME(n)$. Suppose, by way of contradiction, that there is some TM N that, in a number of steps $c|x|$, outputs $D(x)$

⇒ We need to analyse the time it takes the UTM to simulate N on input x. This is at most

$$c' \cdot c\,|x| \log |x|$$

for c' independent on $|x|$.

⇒ Now, there is some $n_0 \in \mathbb{N}$ such that

$$n^{3/2} > c' \cdot c \cdot n \log n$$

for every $n \geq n_0$

⇒ Let now x_0 be a string representing N and whose length is above n_0 (this string exists because we assumed all TM to be represented by infinitely many strings).

⇒ Up to now, we know that

- $|x_0| \geq n_0$
- what N should behave like D on input x_0
- what D on input x_0, would simulate UTM on input (x_0, x_0),
- x_0 is however the code of N
- D on input x_0, would simulate N, but returning $1 - N(x_0)$ because

$$|x_0|^{3/2} \geq c' \cdot c \cdot |x_0| \lg |x_0|$$

THESE TWO FACTS ARE IN CONTRADDICTION

THIS PROVES THE HT!

- How could we derive that $P \neq EXP$? Consider the two classes $DTIME(2^n)$ and $DTIME(2^{2n})$. Take $f(n) = 2^n$ and $g(n) = 2^{2n}$. We easily derive that

$$f(n) \lg f(n) = 2^n \lg 2^n = n 2^n$$

$$\in o(g(n))$$

$$\boxed{\lim_{n \to \infty} \frac{n 2^n}{2^{2n}} = \lim_{n \to \infty} \frac{n \cancel{2^n}}{2^n \cdot \cancel{2^n}} = \lim_{n \to \infty} \frac{n}{2^n} = 0}$$

- Thus, due to HT

$$DTIME(2^n) \subsetneq DTIME(2^{2n})$$

- Then:

$$P \subseteq DTIME(2^n) \subsetneq DTIME(2^{2n}) \subseteq EXP$$

THIS IS THE SEPARATION WE WERE LOOKING FOR.