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- ▶  $X_n$  could be  $\{0,1\}^n$ , the set of **boolean vectors** of of (fixed!) length n, and  $C_n$  is the set of all subsets of  $\{0,1\}^n$  represented by CNFs.
- $\searrow X_n$  could rather be  $\mathbb{R}^n$ , the set of **vectors of real numbers** of length n, while  $\mathcal{C}_n$  are say, the subsets of  $\mathbb{R}^n$  represented by some form of neural network with n inputs and 1 output.

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- In many cases (e.g. SGD), one has a *single* learning algorithm that work for every value of n. In that case, we allow (in the definition of efficient PAC learning) the algorithm  $\mathcal{A}$  to take time polynomial in n, size(c),  $\frac{1}{\varepsilon}$  and  $\frac{1}{\varepsilon}$

# Boolean Functions as a Representation Class

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- Not all subsets of  $\{0,1\}^n$  can be captured.
- ▶ A **second example** of a representation class for X is a class we know, namely the class  $\mathbf{CNF}_n$  of CNFs over  $x_1, \ldots, x_n$ , which are conjunction of disjunctions of literals.
  - ► CNFs are normal forms of any boolean functions.
  - ▶ All subsets of  $\{0,1\}^*$  can be captured this way.
  - ightharpoonup We could even consider  $k\mathbf{CNF}_n$  rather than arbitrary one, but this way we would lose universality.

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#### Theorem

The representation class of boolean conjuctions of literals is efficiently PAC-learnable.

# Intractability of Learning DNFs

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### Intractability of Learning DNFs

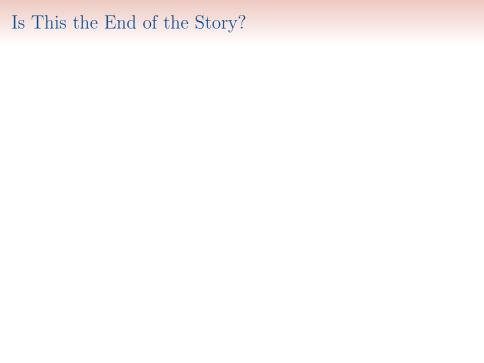
- ▶ We know that conjunctions of literals are efficiently learnable. But they are highly incomplete as a way to represent boolean functions.
- Let us take a look at a *slight generalization* of conjunctions of literals as a representation class.
  - ▶ A 3-term DNF formula over n bits is a propositional formula in the form  $T_1 \vee T_2 \vee T_3$ , where each  $T_i$  is a conjunction of literals over  $x_1, \ldots, x_n$ .
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#### Theorem

If  $\mathbf{NP} \neq \mathbf{RP}$ , then the representation class of 3-term DNF formulas is not efficiently PAC learnable.



#### Is This the End of the Story?

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- ▶ **Definitely No!** Actually, we have just *scratched the surface* of computational learning theory.
- ▶ Models and results we did not have time to talk about include:
  - ► The VC Dimension.
  - ► The Fundamental Theorem of Learning.
  - ► The No-Free-Lunch Theorem.
  - Occam's Razor.
  - ▶ Positive and negative results about neural networks.
  - ▶ ..
- ▶ More information can be found in of the many excellent books on CLT, e.g.
  - Mehryar Mohri, Afshin Rostamizadeh, and Ameet Talwalkar Foundations of Machine Learning Second Edition. The MIT Press. 2018
  - ▶ Shai Shalev-Shwartz and Shai Ben-David. *Understanding Machine Learning: from Theory to Algorithms* Cambridge University Press. 2014.
  - ▶ Michael Kearns and Umesh Vazirani. An Introduction to Computational Learning Theory The MIT Press. 1994.

Example Results about Neural Networks (from Kearns and Vazirani's Book)

**Theorem 3.7** Let G be any directed acyclic graph, and let  $C_G$  be the class of neural networks on an architecture G with indegree r and s internal nodes. Then the number of examples required to learn  $C_G$  is

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Theorem 6.6 Under the Discrete Cube Root Assumption, there is fixed polynomial  $p(\cdot)$  and an infinite family of directed acyclic graphs (architectures)  $G = \{G_{n^2}\}_{n\geq 1}$  such that each  $G_{n^2}$  has  $n^2$  boolean inputs and at most p(n) nodes, the depth of  $G_{n^2}$  is a fixed constant independent of n, but the representation class  $C_G = \bigcup_{n\geq 1} C_{G_{n^2}}$  (where  $C_{G_{n^2}}$  is the class of all neural networks over  $\Re^n$  with underlying architecture  $G_{n^2}$ ) is not efficiently PAC learnable (using any polynomially evaluatable hypothesis class). This holds even if we restrict the networks in  $C_{G_{n^2}}$  to have only binary weights.

Thank You!

Questions?