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 - ▶ We define a polynomial-time transformation $x \mapsto \varphi_x$ from strings to CNFs such that

$$\varphi_x \in \text{SAT} \Leftrightarrow \exists y \in \{0, 1\}^{p(|x|)}. \mathcal{M}(x, y) = 1$$

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- ▶ Finally, we need to show that $\mathbf{SAT} \leq_p 3\mathbf{SAT}$.

Proving a Problem Hard

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- ▶ By proving that $\mathcal{L} \in \mathbf{NP}$?
 - ▶ Again, this **does not** mean much.
- ▶ By proving that \mathcal{L} is **NP**-complete?
 - ▶ **Yes**, this way you prove that the problem is not so hard (being in **NP**), but not so easy either (unless $\mathbf{P} = \mathbf{NP}$).

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2. That any other language $\mathcal{H} \in \mathbf{NP}$ is such that $\mathcal{H} \leq_p \mathcal{L}$.

- ▶ We can of course prove the statement directly.
- ▶ More often (e.g. when showing **3SAT** **NP**-complete), one rather proves that $\mathcal{J} \leq_p \mathcal{L}$ for a language \mathcal{J} which is already known to be **NP**-complete.
- ▶ This is correct, simply because \leq_p is transitive:

$$\begin{array}{ccccc} \vdots & & & & \\ \mathcal{H} & \xrightarrow{\leq_p} & \mathcal{J} & \xrightarrow{\leq_p} & \mathcal{L} \\ \vdots & & & & \end{array}$$

NP-complete Problems: Examples

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Theorem

The Decisional 0/1 Linear Programming Problem ILP is NP-complete

- ▶ There is an easy polytime reduction from SAT to ILP.

The Graph of **NP**-complete Problems

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$$\mathcal{L} \leq_p \mathcal{H} \quad \mathcal{H} \leq_p \mathcal{L}$$

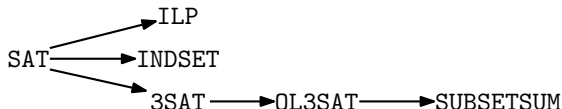
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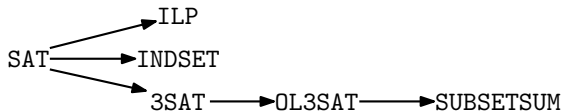


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- ▶ In fact, this graph is **huge**: thousands of different problems are known to be **NP**-complete

Mitigating the Computational Difficulty of **NP**-complete Problems

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- ▶ This is often **very useful**, because specialised tools for **SAT**, called SAT-solvers do exist.
 - ▶ They do not work in polynomial time.
 - ▶ Concretely, they work extremely well on a relatively large class of formulas.

Thank You!

Questions?