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Abstract

For the numerical solution of stochastic differential equations explicit economical three-stage Runge-Kutta schemes of weak second order have been proposed in a previous work. Here numerical stability of these methods is studied and some examples are presented to support the theoretical results.

Mathematics Subject Classification: 60H10

Keywords: Stochastic differential equations, Stochastic Runge-Kutta methods, Weak approximation, Economical methods

1 Introduction.

Many physical phenomena and biological systems are modeled by stochastic differential equations (SDEs), which are obtained by including random effects into the ordinary differential equations. Models of this type offer a more realistic representation of the real physical systems than deterministic models. However, most of the SDEs cannot usually be solved analytically, so numerical methods are needed.

We consider the scalar Itô autonomous SDE

$$\begin{aligned} &dy(t) = a(t, y(t))dt + b(t, y(t))dW_t \qquad t \in [t_0, T] \\ &y(t_0) = y_0 \end{aligned} \tag{1}$$

where $a, b : [t_0, T] \times \mathbb{R} \to \mathbb{R}$ are respectively the drift and the diffusion coefficient, $W = \{W_t, 0 \le t \le T\}$ is a one-dimensional standard Wiener process, and the initial value $y_0 \in \mathbb{R}$ is nonrandom. We assume that a and b are defined and measurable in $[t_0, T] \times R$ and satisfy both Lipschitz and linear growth bound conditions in y. These requirements ensure existence and uniqueness of solution of the SDE (1).

Equation (1) can be written in the integral form

$$y(t) = y_0 + \int_{t_0}^{t} a\left(s, y(s)\right) ds + \int_{t_0}^{t} b\left(s, y(s)\right) dW_s \tag{2}$$

where the first integral is a regular Riemann-Stiltjes integral and the second one is a stochastic integral with respect to the Wiener process $W(t)$.

By using the truncated Taylor series expansion for the process (2), different numerical methods can be constructed ([5]). But this approach can be very expensive in terms of computational cost since many partial derivatives are requested when many stochastic integral terms are used. In order to derive derivative-free methods, the extension of classical Runge-Kutta methods to SDEs has been introduced ([5, 1]). In [8] and [10] new classes of stochastic Runge-Kutta schemes of weak second-order were derived.

For deterministic differential equations the so-called economical Runge-Kutta methods have been proposed ([2]) which are less expensive then the classical Runge-Kutta methods in terms of function evaluations. In [3] and [4] the authors extended the idea of deterministic economical Runge-Kutta methods to the solution of SDEs, by constructing respectively a weak second order Runge-Kutta type method, which is the economical version of a method proposed in [10], and a strong global order one method. In [6] an explicit three-stage economical Runge-Kutta methods with weak order 2 is presented.

The outline of the paper is the following: in Section 2 we quote deterministic (classical and economical) Runge-Kutta methods and in Section 3 stochastic Runge-Kutta methods. Moreover we present examples of families of weak second order SDEs. Then, in Section 4, we analyze the MS stability of the proposed methods. In the last Section numerical examples are given.

2 Deterministic Runge-Kutta scheme

A deterministic explicit s-stage Runge-Kutta method for the numerical solution of the differential equation $y'(t) = f(t, y)$ with initial condition $y\left(t_0\right) = y_0$ is

$$\begin{cases} Y_i^n & = y_n + h_n \displaystyle\sum_{j=1}^{i-1} a_{ij} f\left(t_n + c_j h_n, Y_j^n\right) \\ y_{n+1} & = y_n + h_n \displaystyle\sum_{i=1}^{s} b_i f\left(t_n + c_i h_n, Y_i^n\right) \qquad n = 0, 1, \ldots, N-1\,. \end{cases} \tag{3}$$

where $b_i, a_{ij} \in \mathbb{R}$, $h_n = t_{n+1} - t_n$, $c_1 = 0$ and, usually, $c_i = \displaystyle\sum_{j=1}^{i-1} a_{ij}$ $i = 1, \ldots, s$.

For a method of class A^p, in [2] the authors proposed the following class of methods

$$\begin{cases} Y_i^n &= y_n + h_n \sum_{j=2}^{i-1} a_{ij} f\left(t_n + c_j h_n, Y_j^n\right) + a_{i1} f\left(t_n + c_1 h_n, Y_s^{n-1}\right) \\ y_{n+1} &= y_n + h_n \sum_{i=2}^{s} b_i f\left(t_n + c_i h_n, Y_i^n\right) \qquad n = 0, 1, \ldots, N-1 \end{cases} \tag{4}$$

with $Y_s^{-1} = Y_1^0 = y_0$.

We remember that a Runge-Kutta method of order $p \geq 3$ belongs to class A^p if $b_1 = 0$ and $c_s = 1$ ([2]).

Observe that in scheme (4) at each step one function evaluation is saved. For this reason it is called *economical* Runge-Kutta method.

3 Stochastic Runge-Kutta schemes

Let's consider an equidistant discretization $\{t_0, \ldots, t_N\}$ of $[t_0, T]$ with stepsize $\Delta = (T - t_0)/N$. An s-stage stochastic Runge-Kutta scheme for the solution of (1), in the case of one Wiener process, has the following form

$$\begin{cases} Y_i = y_n + \sum_{j=1}^{s} Z_{ij}^{(0)} a\left(t_n + \mu_j \Delta, Y_j\right) + \sum_{j=1}^{s} Z_{ij}^{(1)} b\left(t_n + \mu_j \Delta, Y_j\right) \quad i = 1, \ldots, s \\ y_{n+1} = y_n + \sum_{j=1}^{s} z_j^{(0)} a\left(t_n + \mu_j \Delta, Y_j\right) + \sum_{j=1}^{s} z_j^{(1)} b\left(t_n + \mu_j \Delta, Y_j\right). \end{cases} \tag{5}$$

$Z^{(1)}$ and $z^{(1)}$ are respectively arbitrary matrix and vector whose elements are random variables and $Z^{(0)}$ and $z^{(0)}$ are respectively the parameter matrix and vector associated with the deterministic components. If both $Z^{(0)}$ and $Z^{(1)}$ are strictly lower triangular, then (5) is said to be explicit, otherwise it is implicit.

A class of SRK methods given by (5) can be characterized by

$$Z^{(0)} = \Delta A, \quad z^{(0)^T} = \Delta \alpha^T, \quad Z^{(1)} = \Delta W_n B, \quad z^{(1)^T} = \Delta W_n \gamma^T \tag{6}$$

where $A = (\lambda_{ij})$ and $B = (\gamma_{ij})$ are $s \times s$ real matrices, $\alpha^T = (\alpha_1 \ldots, \alpha_s)$ and $\gamma^T = (\gamma_1 \ldots, \gamma_s)$ are row vectors in $\mathbb{R}^s$.

Analogously with the deterministic case, to obtain the order conditions we have to match the truncated Runge-Kutta scheme with the stochastic Taylor series expansion of the exact solution over one step assuming exact initial values ([5]).

For a given function $f = f(t, y)$ with $t, y \in \mathbb{R}$ we denote $f = f_{00} = f\left(t_n, y_n\right)$ and $f_{ij} = \dfrac{\partial^{i+j}}{\partial t^i \partial y^j} f\left(t_n, y_n\right)$.

With this notation, if we replace the Gaussian increments ΔW in the Itô-Taylor expansion of order 2 ([9]) of the Itô process y_{n+1} by some simpler random variables $\Delta\hat{W}$ with appropriate moment properties (for example $N(0, \Delta)$ Gaussian random variables), the simplified order two weak Taylor scheme is given by

$$\begin{aligned} y_{n+1} = y_n + a\Delta + b\Delta\hat{W} + \frac{1}{2}bb_{01}\left((\Delta\hat{W})^2 - \Delta\right) \\ +\tfrac{1}{2}\left(b_{10} + ab_{01} + \tfrac{1}{2}b^2 b_{02} + ba_{01}\right)\Delta\,\Delta\hat{W} + \tfrac{1}{2}\left(a_{10} + aa_{01} + \tfrac{1}{2}b^2 a_{02}\right)\Delta^2 + R \end{aligned} \tag{7}$$

where R is the remainder term.

In [6] economical 3-stage explicit Runge-Kutta methods for SDEs have been derived:

$$\begin{cases} Y_i^n = y_n + \Delta \displaystyle\sum_{j=2}^{i-1} \lambda_{ij} a\left(c_{nj}, Y_j^n\right) + \Delta\lambda_{i1} a\left(c_{n1}, Y_3^{n-1}\right) + \Delta W_n \sum_{j=1}^{i-1} \gamma_{ij} b\left(c_{nj}, Y_j^n\right) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, 3 \\ y_{n+1} = y_n + \Delta \displaystyle\sum_{j=2}^{3} \alpha_j a\left(c_{nj}, Y_j^n\right) + \Delta W_n \sum_{j=1}^{3} \beta_j b\left(c_{nj}, Y_j^n\right) \end{cases} \tag{8}$$

with $c_{nj} = t_n + \mu_j \Delta$, $\mu_1 = 0$, or, equivalently,

$$\begin{cases} K_j^n = a\left(c_{nj}, y_n + \Delta\lambda_{j1} K_3^{n-1} + \Delta \displaystyle\sum_{i=2}^{j-1} \lambda_{ji} K_i^n + \Delta W_n \sum_{i=1}^{j-1} \gamma_{ji} \overline{K}_i^n\right) \quad j = 2, 3 \\ \overline{K}_j^n = b\left(c_{nj}, y_n + \Delta\lambda_{j1} K_3^{n-1} + \Delta \displaystyle\sum_{i=2}^{j-1} \lambda_{ji} K_i^n + \Delta W_n \sum_{i=1}^{j-1} \gamma_{ji} \overline{K}_i^n\right) \quad j = 1, \ldots, 3 \\ y_{n+1} = y_n + \displaystyle\sum_{j=2}^{3} \alpha_j K_j^n + \sum_{j=1}^{3} \beta_j \overline{K}_j^n \end{cases} \tag{9}$$

with $K_3^{-1} = a\left(t_0, y_0\right)$.

But a gap occurs in the expansion of y_{n+1}. Here we present the correct expression of this expansion.

For a process $f(t+\Delta, X_t+\Delta X)$ the order 2 truncated expansion with respect to Δ and $\Delta X = X_{t+\Delta} - X_t$ is ([9])

$$\begin{aligned} f(t+\Delta, X_t+\Delta X) \overset{(2)}{\simeq} \quad & f_{00} + f_{10}\Delta + f_{01}\Delta X + \\ & + \left(f_{20} + b^2 f_{12} + b^3 b_{01} f_{03} + \frac{b^4}{4} f_{04}\right)\frac{\Delta^2}{2} + \left(f_{11} + \frac{b^2}{2} f_{03}\right)\Delta\,\Delta X + f_{02}\frac{(\Delta X)^2}{2}\,. \end{aligned} \tag{10}$$

The notation $A \overset{(2)}{\simeq} B$ means that replacing the variable A by B in a second-order approximation leads to an equivalent approximation.

If ΔW^1 and ΔW^2 are independent random variables with distribution $N(0,\Delta)$, then ([9]) $\Delta\,\Delta W^1\,\Delta W^2 \overset{(2)}{\simeq} 0$. Hence $\Delta\,\Delta W_n\,\Delta W_{n-1} \overset{(2)}{\simeq} 0$. From this and (10) we get that the truncated expansion of (8) is 2-equivalent to

$$\begin{aligned} y_{n+1} = \quad & y_n + (\alpha_2+\alpha_3)\,a\Delta + (\beta_1+\beta_2+\beta_3)\,b\Delta\hat{W}_n \\ & + (\alpha_2\mu_2+\alpha_3\mu_3)\,a_{10}\Delta^2 + (\alpha_2\lambda_{21}+\alpha_3(\lambda_{31}+\lambda_{32}))\,aa_{01}\Delta^2 \\ & + \frac{1}{2}\left(\alpha_2\gamma_{21}^2 + \alpha_3(\gamma_{31}+\gamma_{32})^2\right) a_{02}b^2\Delta^2 + (\beta_2\mu_2+\beta_3\mu_3)\,b_{10}\Delta\,\Delta\hat{W}_n \\ & + (\beta_2\lambda_{21}+\beta_3(\lambda_{31}+\lambda_{32}))\,ab_{01}\Delta\,\Delta\hat{W}_n + \frac{1}{2}\left(\beta_2\gamma_{21}^2+\beta_3(\gamma_{31}+\gamma_{32})^2\right) b^2b_{02}\Delta\,\Delta\hat{W}_n \\ & + (\alpha_2\gamma_{21}+\alpha_3(\gamma_{31}+\gamma_{32}))\,a_{01}b\Delta\,\Delta\hat{W}_n + 3\beta_3\gamma_{21}\gamma_{32}bb_{01}^2\Delta\,\Delta\hat{W}_n \\ & + (\beta_2\gamma_{21}+\beta_3(\gamma_{31}+\gamma_{32}))\,bb_{01}(\Delta\hat{W}_n)^2 \\ & + (\beta_2\lambda_{21}\gamma_{21}+\beta_3(\lambda_{31}+\lambda_{32})(\gamma_{31}+\gamma_{32}))\,abb_{02}\Delta^2 \\ & + \frac{1}{2}\beta_3\gamma_{21}\gamma_{32}\left(\gamma_{21}+6(\gamma_{31}+\gamma_{32})\right) b^2b_{01}b_{02}\Delta^2 \\ & + \beta_3\mu_2\gamma_{32}b_{10}b_{01}\Delta^2 + \beta_3\lambda_{21}\gamma_{32}ab_{01}^2\Delta^2 + \gamma_{21}\left(\alpha_3\gamma_{32}+\beta_3\lambda_{32}\right) a_{01}bb_{01}\Delta^2 \\ & + (\beta_2\mu_2\gamma_{21}+\beta_3\mu_3(\gamma_{31}+\gamma_{32}))\,b\left(b_{11}+\frac{b^2}{2}b_{03}\right)\Delta^2 + \overline{R}\,. \end{aligned} \tag{11}$$

Expansions (7) and (11) coincide if the coefficients in (11) satisfy the following system

$$
\begin{cases}
\alpha_2+\alpha_3=1 & \beta_1+\beta_2+\beta_3=1\\
\alpha_2\mu_2+\alpha_3\mu_3=\dfrac{1}{2} & \alpha_2\lambda_{21}+\alpha_3(\lambda_{31}+\lambda_{32})=\dfrac{1}{2}\\
\alpha_2\gamma_{21}^2+\alpha_3(\gamma_{31}+\gamma_{32})^2=\dfrac{1}{2} & \beta_2\mu_2+\beta_3\mu_3=\dfrac{1}{2}\\
\beta_2\lambda_{21}+\beta_3(\lambda_{31}+\lambda_{32})=\dfrac{1}{2} & \beta_2\gamma_{21}^2+\beta_3(\gamma_{31}+\gamma_{32})^2=\dfrac{1}{2}\\
\alpha_2\gamma_{21}+\alpha_3(\gamma_{31}+\gamma_{32})=\dfrac{1}{2} & \beta_3\gamma_{21}\gamma_{32}=0\\
\beta_2\lambda_{21}\gamma_{21}+\beta_3(\lambda_{31}+\lambda_{32})(\gamma_{31}+\gamma_{32})=0 & \beta_3\gamma_{21}\gamma_{32}\left(\gamma_{21}+6(\gamma_{31}+\gamma_{32})\right)=0\\
\beta_3\mu_2\gamma_{32}=0 & \beta_3\lambda_{21}\gamma_{32}=0\\
\beta_2\mu_2\gamma_{21}+\beta_3\mu_3(\gamma_{31}+\gamma_{32})=0 & \gamma_{21}\left(\alpha_3\gamma_{32}+\beta_3\lambda_{32}\right)=0\\
\beta_2\gamma_{21}+\beta_3(\gamma_{31}+\gamma_{32})=0 &
\end{cases}
\tag{12}
$$

and $\overline{R}=\dfrac{1}{2}bb_{01}\left((\Delta\hat{W}_n)^2-\Delta\right)$.

Each solution of the above system leads to a weak order 2 scheme. Observe that in order to have a solution there must be $\alpha_2,\alpha_3,\beta_2,\beta_3\neq 0$.

Some solutions are presented in the following tables. The corresponding methods will be denoted respectively by *ERK2-1* and *ERK2-2*.

$$
\begin{array}{c|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0\\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0\\
\hline
 & 0 & \frac{2+\sqrt{2}}{4} & \frac{2-\sqrt{2}}{4} & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

Table 1: *ERK2-1* method.

$$
\begin{array}{c|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0\\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0\\
\hline
 & 0 & \frac{2-\sqrt{2}}{4} & \frac{2+\sqrt{2}}{4} & 0 & \frac{1}{2} & \frac{1}{2}
\end{array}
$$

Table 2: *ERK2-2* method

As an alternative, we can replace the last equation of the system (12) with

$$\beta_2\gamma_{21}+\beta_3(\gamma_{31}+\gamma_{32})=\frac{1}{2}$$

and consider $\overline{R} = -\frac{1}{2} b b_{01} \Delta$. In this case we obtain the following one parameter families.

Family F1

$$\begin{cases} \lambda_{21} = \frac{1}{2} - \frac{\beta}{2\left(1-\beta_3\right)} & \lambda_{31} = \frac{1}{2} + \frac{\beta}{2\beta_3}, \quad \lambda_{32} = 0 \\ \gamma_{21} = \frac{1}{2} + \frac{\beta}{2\left(1-\beta_3\right)} & \gamma_{31} = \frac{1}{2} - \frac{\beta}{2\beta_3}, \quad \gamma_{32} = 0 \\ \mu_2 = \lambda_{21}, \quad \mu_3 = \lambda_{31} & \\ \alpha_2 = 1 - \beta_3, \quad \alpha_3 = \beta_3, \quad \beta_1 = 0, \quad \beta_2 = \alpha_2 & \end{cases}$$

Family F2

$$\begin{cases} \lambda_{21} = \frac{1}{2} + \frac{\beta}{2\left(1-\beta_3\right)} & \lambda_{31} = \frac{1}{2} - \frac{\beta}{2\beta_3}, \quad \lambda_{32} = 0 \\ \gamma_{21} = \frac{1}{2} - \frac{\beta}{2\left(1-\beta_3\right)} & \gamma_{31} = \frac{1}{2} + \frac{\beta}{2\beta_3}, \quad \gamma_{32} = 0 \\ \mu_2 = \lambda_{21}, \quad \mu_3 = \lambda_{31} & \\ \alpha_2 = 1 - \beta_3, \quad \alpha_3 = \beta_3, \quad \beta_1 = 0, \quad \beta_2 = \alpha_2 & \end{cases}$$

with $\beta = \sqrt{\beta_3\left(1-\beta_3\right)}$, $0 < \beta_3 < 1$.

4 Stability analysis

In order to study the stability domains of a method, we consider the Itô test equation

$$dy = \lambda\, y\, dt + \mu\, y\, dW_t \qquad t > t_0 \quad \lambda, \mu \in \mathbb{C} \tag{13}$$

with nonrandom initial conditions $y\left(t_0\right) = y_0 \in \mathbb{R}, y_0 \neq 0$.

We now study what conditions must be imposed in order that the numerical solution $\{y_n\}_{n\in\mathbf{N}}$ generated by a scheme with equidistant stepsize applied to test equation (13) satisfies $\lim\limits_{n\to\infty} \|y_n\| = 0$.

When we apply to (13) the *ERK2-1* method in form (9) we obtain the one-step difference equation

$$\begin{cases} y_{n+1} & = P y_n + Q K_3^{n-1} \\ K_3^n & = L y_n + M K_3^{n-1} \end{cases} \tag{14}$$

with $K_3^{-1} = 0$ and

$$P = 1 + \Delta\lambda + \mu\Delta W + \frac{1}{2}\Delta\lambda\mu\Delta W \qquad Q = \frac{\Delta^2\lambda}{2} + \frac{\Delta\mu\Delta W}{2}$$
$$L = \lambda\left(1 - \frac{\mu\Delta W}{\sqrt{2}}\right) \qquad M = \frac{\Delta\lambda}{2}.$$

Equation (14) can also be written as $u_{n+1} = Au_n$ where $u_n = \left[y_n, K_3^{n-1}\right]^T$ and $A = \begin{bmatrix} P & Q \\ L & M \end{bmatrix}$. When we calculate the components of the second moment of u_n, we have the following difference equation

$$X_{n+1} = \Omega X_n$$

where

$$X_n = \begin{bmatrix} X_n^1 \\ X_n^2 \\ X_n^3 \end{bmatrix} = \begin{bmatrix} E\left(u_n^1\right)^2 \\ E\left(u_n^2\right)^2 \\ E\left(u_n^1 u_n^2\right)^2 \end{bmatrix}$$

and $\Omega = \Omega(\lambda, \mu)$ is the stability matrix.

It is evident that $\lim\limits_{n\to\infty} \|X_n\| = 0$ iff

$$\|\Omega\| < 1. \tag{15}$$

Hence a scheme is asymptotically mean square stable w.r.t. $\|\cdot\|$ for the values of λ and μ satisfying (15) and the set $S = \{(\lambda, \mu) : \|\Omega(\lambda, \mu)\| < 1\}$ is the MS-stability *domain* of the scheme.

The entries of the stability matrix Ω can be determined by direct computation.

Theorem 4.1 *When ERK2-1 method is applied to the test equation (13), the stability matrix Ω is given by*

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix}$$

where

$$\begin{aligned}
\Omega_{11} &= 1 + \frac{1}{4}\Delta^3\lambda^2\mu^2 + \Delta^2\lambda(\lambda+\mu^2) + \Delta(2\lambda+\mu^2) \\
\Omega_{12} &= \frac{\Delta^4\lambda^2}{4} + \frac{\Delta^3\mu^2}{4} & \Omega_{13} &= \Delta^2(\lambda+\mu^2) + \frac{1}{2}\Delta^3\lambda(2\lambda+\mu^2) \\
\Omega_{21} &= \lambda^2 + \frac{1}{2}\Delta\lambda^2\mu^2 & \Omega_{22} &= \frac{1}{4}\Delta^2\lambda^2 \qquad \Omega_{23} = \Delta\lambda^2 \\
\Omega_{31} &= \lambda - \frac{1}{2\sqrt{2}}\Delta^2\lambda^2\mu^2 + \frac{1}{2}\Delta\lambda(2\lambda - \sqrt{2}\mu^2) \\
\Omega_{32} &= \frac{1}{4}\Delta^3\lambda^2 & \Omega_{33} &= \frac{1}{2}\Delta\lambda + \frac{1}{4}\Delta^2\lambda(4\lambda - \sqrt{2}\mu^2)\ .
\end{aligned}$$

The scheme is asymptotically MS-stable with respect to $\|\cdot\|_\infty$ if

$$\max\{A, B, C\} < 1$$

where

$$\begin{aligned} A &= |\Omega_{11}| + |\Omega_{12}| + |\Omega_{13}| \\ B &= |\Omega_{21}| + |\Omega_{22}| + |\Omega_{23}| \\ C &= |\Omega_{31}| + |\Omega_{32}| + |\Omega_{33}| \end{aligned}$$

and Ω_{ij} are as given in Theorem 1.

Since λ and μ are complex value, the visualization of the stability domain requires the four-dimensional space.

An approach is to restrict attention only to real values of λ and μ ([7]). In this case we call the stability domain stability *region* and we may obtain a graphical representation of it in the real plane. In Figure 1 the regions of mean square stability of scheme *ERK2-1* for several values of Δ are showed. Figures 2 and 3 show the stability regions of schemes of family *F1* and *F2*

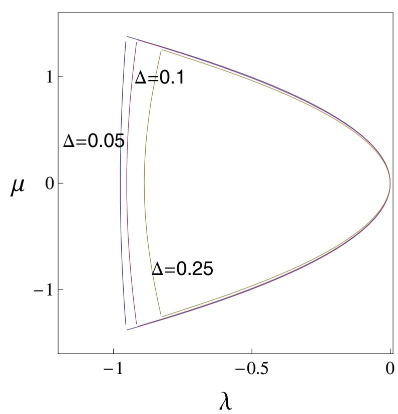

Figure 1: Mean square stability regions of $ERK2-1$ method.

respectively with $\beta_3 = 0.1$ (left), $\beta_3 = 0.5$ (center), $\beta_3 = 0.9$ (right). We can see how the regions of stability for the schemes of family *F1* and *F2* increase when the parameter β_3 increases.

5 Numerical results

In this section numerical examples are presented to compare the *ERK2* methods to method proposed by Soheili in [8] which has the same order and the Butcher tableau in Table 3.

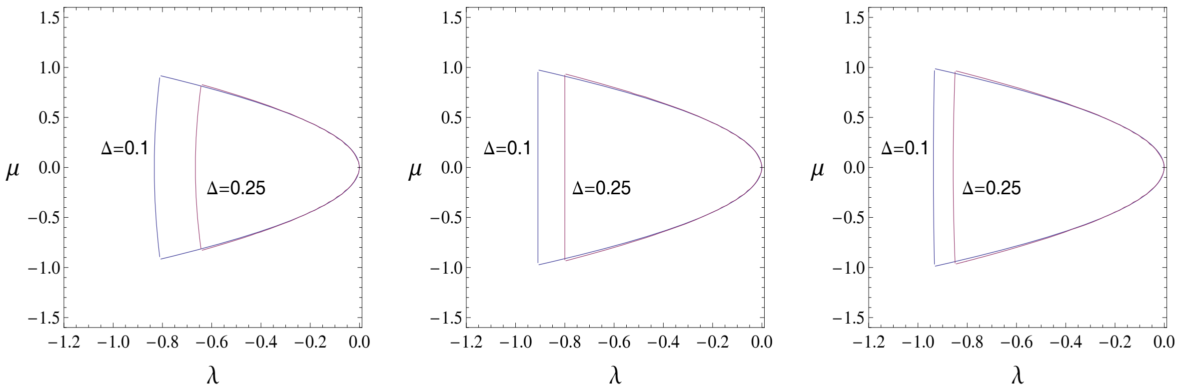

Figure 2: Mean square stability regions of methods of family *F1*.

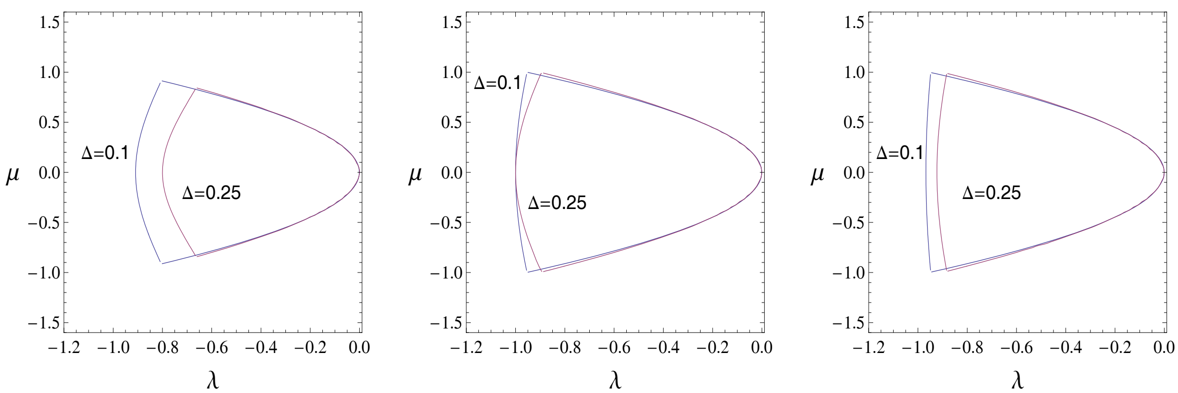

Figure 3: Mean square stability regions of methods of family *F2*.

$$
\begin{array}{c|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 0 \\
\frac{1}{2} & \frac{1}{5} & 0 & 0 & \frac{1}{3} & 0 & 0 \\
\hline
 & \frac{1}{2} & \frac{1}{2} & 0 & -\frac{5}{2} & -\frac{1}{4} & \frac{15}{4}
\end{array}
$$

Table 3: *Soheili* method

All methods are implemented with constant step size on problems for which the exact solution in terms of a Wiener process is known. For each example we have used 5000 independent simulations for different stepsizes.

In order to simulate the Gaussian variable ΔW_n with distribution $N(0, \Delta)$ we have used the Matlab random number generator `randn`.

In the economical schemes at each step we use a function call of the previous step. Therefore the cost of *ERK2* methods is less than the cost of Soheili method.

The implementation determines the average error for each stepsize h at the end of the interval of integration. This error and the computational work (the number of function evaluations *nfc*) for each problem are summarized in

Tables 4-6.

Example 1. Consider the linear SDE ([8])

$$\begin{cases} dy = -a^2 y\left(1-y^2\right)dt + a\left(1-y^2\right)dW & t \in [0,1] \\ y(0) = y_0 \end{cases} \tag{16}$$

with solution $y(t) = \tanh\left(aW(t) + \operatorname{arctanh}\left(y_0\right)\right)$.
For $a = 1.0$ and $y_0 = 0$ we have the following results

Δ	*Soheili method*		*ERK2 methods*		
	error	nfc	error *ERK2-1*	error *ERK2-2*	nfc
2^{-1}	9.4795e-1	30000	2.3144e-1	2.3725e-1	25001
2^{-2}	9.9745e-1	90000	1.6959e-1	1.6460e-1	75001
2^{-3}	2.0123	210000	1.0815e-1	1.0581e-1	175001
2^{-4}	2.6240e-1	450000	7.4935e-2	7.4522e-2	375001
2^{-5}	2.6919e-1	930000	5.0450e-2	5.0261e-2	775001

Table 4: Error and computational work in the approximation of (16).

Example 2 Consider the nonlinear SDE ([8])

$$\begin{cases} dy = \left(\frac{1}{3}y^{\frac{1}{3}} + 6y^{\frac{2}{3}}\right)dt + y^{\frac{2}{3}}dW & t \in [0,1] \\ y(0) = 1 \end{cases} \tag{17}$$

The solution is $y(t) = \left(2t + 1 + \frac{W(t)}{3}\right)^3$ and the exact value of the first moment is $E[y] = 28$ at point $t = 1$. The obtained results are summarized in Table 5.

Δ	*Soheili method*			*ERK2 methods*				
				ERK2-1		*ERK2-2*		
	error	st. dev.	nfc	error	st. dev.	error	st. dev.	nfc
2^{-3}	17.682	18.303	210000	2.623	8.5359	2.685	8.5820	175001
2^{-4}	17.558	18.296	450000	1.738	8.8808	1.772	8.8782	375001
2^{-5}	17.610	18.308	930000	1.150	8.8715	1.160	8.8676	775001
2^{-6}	17.662	18.441	1890000	6.938e-1	8.9720	6.961e-1	8.9795	1575001
2^{-7}	17.817	18.407	9810000	3.920e-1	9.0386	3.928e-1	9.0406	3175001

Table 5: Error, standard deviation and computational work in the approximation of (17).

Example 3. Consider the SDE ([4])

$$\begin{cases} dy = -\left(\alpha + \beta^2 y\right)\left(1 - y^2\right)dt + \beta\left(1 - y^2\right)dW \qquad t \in [0,1] \\ y(0) = 0 \end{cases} \tag{18}$$

with solution $y(t) = \dfrac{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) + 1 - y_0}$.
For $\alpha = 1.0, \beta = 0.01$ the results are given in Table 6.

Δ	*Soheili method*		*ERK2 methods*		
	error	nfc	error *ERK2-1*	error *F1* ($\beta_3 = \frac{1}{5}$)	nfc
2^{-1}	3.2407e-1	30000	2.8943e-1	2.7761e-1	25001
2^{-2}	1.3481e-1	90000	1.2960e-1	1.1496e-1	75001
2^{-3}	5.9752e-2	210000	5.9333e-2	5.1537e-2	175001
2^{-4}	2.8056e-2	450000	2.8000e-2	2.4312e-2	375001
2^{-5}	1.3596e-2	930000	1.3567e-2	1.1802e-2	775001

Table 6: Error and computational work in the approximation of (18).

6 Conclusion

In this paper we have studied a class of economical stochastic Runge-Kutta methods, already proposed in [6]. After some corrections from the previous version, we studied the stability domains of the new methods, so-called *ERK2* methods. The *ERK2* methods are of second order accuracy in the weak sense and are less expensive in terms of function evaluations than the classical Runge-Kutta methods. We compared them with another weak second order method proposed by Soheili in [8]. As we can see from the above examples, the proposed methods are also more efficient than the Soheili method as far as the magnitude of the error is concerned. It is left for future work the extension of the proposed methods to SDEs with more than one Wiener process.

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