Harmonic Structures and Functional Flow via Ideals, Transversals, and Dynamical Systems

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Abstract

We develop a mathematical framework for tonal harmony based on algebraic, categorical, and dynamical structures. Pitch classes are modeled as the ring $\mathbb{Z}_{12}$, harmonic collections arise as ideals and transversals, and functional harmony is formalized via non-invertible morphisms between scale objects. Cadential motion is characterized by commuting diagrams and terminal behavior, and later extended to dynamical systems with attractors and heteroclinic connections.

Contents

1 Algebraic Foundations

1.1 Basic Notation

Throughout this work, we use the following conventions:

- $\mathbb{Z}$ denotes the ring of integers.
- For $n \in \mathbb{N}$, $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ denotes the cyclic group of integers modulo n.
- All equalities in $\mathbb{Z}_n$ are understood modulo n.
- Groups are written additively unless stated otherwise.

1.2 The Pitch-Class Group

Definition 1.1 (Pitch-Class Group)**.** Let

$$G := \mathbb{Z}_{12}$$

equipped with addition modulo 12. We refer to G as the *pitch-class group.*

Proposition 1.2. *G is a finite cyclic abelian group of order 12.*

Proof. This follows from standard properties of $\mathbb{Z}_n$. □

1.3 Ring Structure

Definition 1.3 (Ring Structure)**.** Endow G with multiplication modulo 12. Then

$$R := (\mathbb{Z}_{12}, +, \cdot)$$

is a commutative ring with unity.

1.4 Ideals of $\mathbb{Z}_{12}$

Proposition 1.4. *Every ideal of the ring $\mathbb{Z}_{12}$ is principal and of the form*

$$(d) = d\mathbb{Z}_{12} \quad \textit{for some } d \mid 12.$$

Corollary 1.5. *The complete set of ideals of $\mathbb{Z}_{12}$ is*

$$\{(1), (2), (3), (4), (6), (12)\}.$$

1.5 Cosets and Quotients

Definition 1.6 (Coset)**.** Let $I \triangleleft R$ be an ideal and $x \in R$. The coset of x modulo I is

$$x + I := \{x + i \mid i \in I\}.$$

1.6 Transversals

Definition 1.7 (Transversal)**.** Let $I \triangleleft R$ be an ideal. A subset $T \subset R$ is called a *transversal* of I if every coset $x + I$ contains exactly one element of T.

Remark 1.8. Transversals are not canonical. Distinct choices correspond to different embeddings of the same quotient structure.

1.7 Scale Objects

Definition 1.9 (Scale Object)**.** A *scale object* is a pair

$$\mathcal{S} = (I, T),$$

where $I \triangleleft \mathbb{Z}_{12}$ is an ideal and T is a transversal of I.

Definition 1.10 (Chord)**.** Given a scale object $\mathcal{S} = (I, T)$, a *chord* is any finite subset

$$C \subset T.$$

2 Transversals, Quotients, and Functional Harmony

2.1 Quotients of $\mathbb{Z}_{12}$ and Musical Reduction

Let $G = \mathbb{Z}_{12}$ be the additive cyclic group of pitch classes. For any ideal $I = d\mathbb{Z}_{12}$, the quotient group

$$G/I$$

represents a *coarse harmonic space*, where pitch classes differing by elements of I are identified.

Musically, this corresponds to reducing pitch information modulo a scale or chord structure. For example:

- $2\mathbb{Z}_{12}$ corresponds to the whole-tone scale,
- $3\mathbb{Z}_{12}$ corresponds to the diminished scale,
- $4\mathbb{Z}_{12}$ corresponds to augmented triadic symmetry.

The quotient map

$$\pi_I : \mathbb{Z}_{12} \to \mathbb{Z}_{12}/I$$

forgets fine pitch detail while preserving harmonic class.

2.2 Transversals as Harmonic Representatives

A *transversal* for the quotient $\mathbb{Z}_{12}/I$ is a set

$$T \subset \mathbb{Z}_{12}$$

containing exactly one representative from each coset.

Definition 2.1. A *harmonic transversal* is a transversal chosen to reflect musically stable pitch choices (e.g. scale degrees or chord tones).

For instance, for the ideal $I = 7\mathbb{Z}_{12}$, a transversal

$$T = \{0, 2, 4, 5, 7, 9, 11\}$$

corresponds to the diatonic major scale.

Thus, *scales arise as structured choices of representatives* rather than as ideals themselves.

2.3 Functional Harmony as Maps Between Transversals

Let $I, J \subset \mathbb{Z}_{12}$ be ideals. Let T_I and T_J be chosen transversals.

Definition 2.2. A *functional harmonic motion* is a map

$$f : T_I \to T_J$$

compatible with quotient structure, meaning the diagram

$$\begin{array}{ccc} T_I & \xrightarrow{f} & T_J \\ \downarrow & & \downarrow \\ \mathbb{Z}_{12}/I & \xrightarrow{\bar{f}} & \mathbb{Z}_{12}/J \end{array}$$

commutes.

Here:

- vertical arrows are inclusion followed by quotient,
- $\bar{f}$ is induced by harmonic function (e.g. tonic $\to$ dominant).

Musically, this encodes:

- scale-to-scale motion,
- chord substitution,
- tonal modulation.

2.4 Example: Dominant to Tonic Motion

Let

$$T_{\text{dom}} = \{7, 11, 2\} \quad \text{(G major triad)}$$
$$T_{\text{ton}} = \{0, 4, 7\} \quad \text{(C major triad)}$$

Define

$$f(7) = 7, \quad f(11) = 0, \quad f(2) = 4$$

This map:

- preserves common tones,
- resolves leading tones,
- minimizes pitch displacement.

Thus, classical voice-leading appears naturally as a *structure-preserving map between transversals*.

2.5 Interpretation

- Ideals define harmonic *equivalence*
- Quotients define harmonic *function*
- Transversals define harmonic *realization*
- Maps between transversals define harmonic *motion*

This framework separates:

$$\text{Structure} \quad \longleftrightarrow \quad \text{Realization}$$

allowing harmony to be studied independently of register, voicing, or instrumentation.

3 Cadences as Commuting Diagrams

3.1 Cadences as Structured Harmonic Motion

A *cadence* is not merely a chord succession, but a structured resolution between harmonic functions.

In this framework, a cadence is modeled as a composition of maps between harmonic transversals that preserves quotient structure.

3.2 Two-Cadence (Dominant–Tonic) Structure

Let $G = \mathbb{Z}_{12}$.

Define the following transversals:

$$T_V = \{7, 11, 2\} \quad \text{(dominant triad)}$$

$$T_I = \{0, 4, 7\} \quad \text{(tonic triad)}$$

Let $I = 12\mathbb{Z}_{12}$ so that pitch classes are fully resolved.

Definition 3.1. A *two-cadence* is a map

$$f_{V \to I} : T_V \to T_I$$

satisfying:

- common tones are fixed,
- leading tones resolve by minimal displacement,
- the induced quotient map is constant.

Explicitly, define:

$$f_{V \to I}(7) = 7, \quad f_{V \to I}(11) = 0, \quad f_{V \to I}(2) = 4$$

3.3 Commuting Diagram Interpretation

The cadence is encoded by the commuting diagram:

$$\begin{array}{ccc} T_V & \xrightarrow{f_{V \to I}} & T_I \\ \downarrow & & \downarrow \\ \mathbb{Z}_{12} & = & \mathbb{Z}_{12} \end{array}$$

Here:

- vertical arrows are inclusions,
- the bottom equality expresses tonal closure,
- commutativity encodes harmonic resolution.

3.4 Three-Cadence (Predominant–Dominant–Tonic)

Let:

$$T_{IV} = \{5, 9, 0\} \quad \text{(subdominant triad)}$$

Define maps:

$$f_{IV \to V} : T_{IV} \to T_V, \quad f_{V \to I} : T_V \to T_I$$

The full cadence is the composition:

$$T_{IV} \xrightarrow{f_{IV \to V}} T_V \xrightarrow{f_{V \to I}} T_I$$

This yields the commuting diagram:

$$\begin{array}{ccccc} T_{IV} & \xrightarrow{f_{IV \to V}} & T_V & \xrightarrow{f_{V \to I}} & T_I \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{Z}_{12} & = & \mathbb{Z}_{12} & = & \mathbb{Z}_{12} \end{array}$$

3.5 Cadences as Directed Paths

Cadences may be viewed as directed paths in a graph whose:

- nodes are harmonic transversals,
- edges are structure-preserving maps,
- path composition corresponds to harmonic progression.

This viewpoint naturally leads to dynamical interpretations, where cadences correspond to heteroclinic connections between attractors.

3.6 Interpretation

- Chords are objects with internal structure,
- Voice-leading is a morphism,
- Cadence is morphism composition,
- Resolution is diagram commutativity.

Thus, tonal harmony emerges as a constrained algebraic flow rather than a sequence of isolated symbols.

3.7 Theorem: Authentic Cadences Are Categorically Stronger

We now formalize the intuitive notion that authentic cadences (e.g. V–I) possess a stronger structural resolving power than non-authentic cadences (e.g. IV–I or deceptive motions).

Theorem 3.2 (Categorical Strength of Authentic Cadences). *Let $T_V, T_I \subset \mathbb{Z}_{12}$ be transversals corresponding to the dominant and tonic triads, respectively.*

The dominant–tonic cadence

$$f_{V \to I} : T_V \to T_I$$

is categorically stronger than any non-authentic cadence

$$g : T_X \to T_I$$

in the sense that:

1. *$f_{V \to I}$ is uniquely determined up to automorphism by minimal voice-leading constraints;*
2. *$f_{V \to I}$ induces a terminal morphism in the subcategory of cadential maps ending at T_I;*
3. *any competing cadence factors through $f_{V \to I}$ or fails to preserve quotient structure.*

3.8 Proof

We proceed in three steps.

Step 1: Uniqueness under minimal displacement Let

$$T_V = \{7, 11, 2\}, \quad T_I = \{0, 4, 7\}.$$

Impose the following constraints:

- common tones are fixed,
- leading tones resolve upward by one semitone,
- total pitch displacement is minimized.

These conditions uniquely determine the map

$$f_{V \to I}(7) = 7, \quad f_{V \to I}(11) = 0, \quad f_{V \to I}(2) = 4.$$

Any alternative assignment violates at least one constraint, proving uniqueness up to permutation of voices.

Step 2: Terminal property Consider the collection $\mathcal{C}_I$ of all harmonic transversals mapping into T_I via structure-preserving maps.

For any transversal T_X and cadence map

$$g : T_X \to T_I,$$

either:

- T_X shares dominant function, in which case g factors through T_V, or
- g collapses pitch distinctions and fails to preserve quotient structure.

Thus, $f_{V \to I}$ is terminal among cadential morphisms ending at T_I.

Step 3: Failure of non-authentic cadences Let $T_{IV} = \{5, 9, 0\}$ and consider a plagal cadence

$$g_{IV \to I} : T_{IV} \to T_I.$$

No map $g_{IV \to I}$ simultaneously satisfies:

- uniqueness under minimal displacement,
- leading-tone resolution,
- quotient compatibility.

Hence, such cadences lack the terminal property and are categorically weaker.

Conclusion The dominant–tonic cadence uniquely realizes maximal structural resolution and therefore occupies a distinguished categorical role.

$\square$

4 The Cadential Category

We now abstract the previously developed structures into a categorical framework. This serves not as a foundational requirement, but as a conceptual compression of the algebraic and dynamical constructions already introduced.

4.1 Objects

Definition 4.1. An object of the *cadential category* $\mathcal{H}$ is a harmonic transversal

$$T \subset \mathbb{Z}_{12}$$

together with an embedding into pitch-class space.

Concretely, objects include:

- triads (e.g. tonic, dominant, subdominant),
- scales (e.g. diatonic, whole-tone),
- symmetric pitch collections.

No two objects are identified unless they are equal as subsets of $\mathbb{Z}_{12}$.

4.2 Morphisms

Definition 4.2. A morphism $f : T \to T'$ in $\mathcal{H}$ is a map satisfying:

1. (**Voice-leading**) f minimizes total pitch displacement;
2. (**Structure preservation**) f respects quotient equivalence induced by ideals;
3. (**Functionality**) f resolves unstable tones when present.

Such morphisms represent admissible harmonic motions.

4.3 Identity Morphisms

Definition 4.3. For each object T, the identity morphism

$$\mathrm{id}_T : T \to T$$

is the map fixing every pitch class.

This corresponds musically to harmonic stasis.

4.4 Composition

Definition 4.4. Given morphisms

$$T_1 \xrightarrow{f} T_2 \xrightarrow{g} T_3,$$

their composition is the ordinary composition of functions:

$$g \circ f : T_1 \to T_3.$$

Composition corresponds to cadential concatenation.

4.5 Associativity

Proposition 4.5. *Composition in $\mathcal{H}$ is associative.*

Proof. Associativity follows from associativity of function composition. □

4.6 Terminal Objects

Definition 4.6. A terminal object in $\mathcal{H}$ is an object T such that for every object S there exists a unique morphism

$$S \to T.$$

The tonic triad T_I is terminal among cadentially resolving objects.

4.7 Subcategories

We define the following important subcategories:

- $\mathcal{H}_{\text{triad}}$: triadic harmony;
- $\mathcal{H}_{\text{scale}}$: scalar harmony;
- $\mathcal{H}_{\text{cad}}$: cadential morphisms only.

Each inherits structure from $\mathcal{H}$.

4.8 Interpretation

- Objects are harmonic states;
- Morphisms are voice-leading motions;
- Composition is harmonic progression;
- Terminal objects encode tonal resolution.

Thus, tonal harmony admits a natural categorical structure encoding both algebraic equivalence and musical function.

5 Metric Enrichment: Dissonance as Distance

5.1 Motivation

In order to define harmonic dynamics, stability, and attractors, we must first endow harmonic space with a notion of distance. Musically, this distance will correspond to *dissonance*, understood as spectral mismatch between sounding pitch collections.

This section defines a metric enrichment of the cadential category $\mathcal{H}$ by assigning costs to objects and morphisms.

5.2 Overtone Spectra

Let $p \in \mathbb{Z}_{12}$ be a pitch class. We associate to p a normalized overtone spectrum

$$S(p) = \{(n, a_n)\}_{n \geq 1},$$

where:

- n indexes partials,
- $a_n > 0$ is the amplitude of the nth overtone,
- frequencies are reduced modulo octave equivalence.

In practice, only finitely many partials are retained.

For a harmonic object $T \subset \mathbb{Z}_{12}$, define the aggregate spectrum

$$S(T) = \sum_{p \in T} S(p).$$

5.3 Spectral Distance

Definition 5.1. Let $T, T' \subset \mathbb{Z}_{12}$ be harmonic transversals. The *spectral distance* between T and T' is defined by

$$d_{\text{spec}}(T, T') = \left\| S(T) - S(T') \right\|_{L^2},$$

where spectra are viewed as functions on frequency space.

This distance measures the degree of overtone mismatch between two harmonic states.

5.4 Dissonance Functional

Definition 5.2. The *dissonance* of a harmonic object T is defined as

$$D(T) = d_{\text{spec}}(T, T_I),$$

where T_I is the tonic transversal.

Thus, consonance corresponds to proximity to the tonic spectrum.

5.5 Metric on Morphisms

Definition 5.3. Let $f : T \to T'$ be a morphism in $\mathcal{H}$. The *cost* of f is defined as

$$\text{Cost}(f) = \sum_{p \in T} |f(p) - p| + \lambda\, d_{\text{spec}}(T, T'),$$

where $\lambda > 0$ balances voice-leading displacement and spectral change.

This functional penalizes both large pitch motion and increased dissonance.

5.6 Metric Enrichment

Definition 5.4. The cadential category $\mathcal{H}$ is *metrically enriched* by the assignment:

- a dissonance value $D(T)$ to each object T,
- a cost $\text{Cost}(f)$ to each morphism f.

Composition of morphisms satisfies:

$$\text{Cost}(g \circ f) \leq \text{Cost}(f) + \text{Cost}(g),$$

so the enrichment is subadditive.

5.7 Interpretation

- Dissonance becomes a geometric notion;
- Voice-leading becomes energy minimization;
- Cadences correspond to steepest descent paths;
- Tonality acquires a variational structure.

This metric framework enables the definition of harmonic flows and attractors.

6 Harmonic Dynamics on $\mathcal{H}$

6.1 From Metric to Dynamics

The metric enrichment of $\mathcal{H}$ allows us to interpret harmonic motion as the evolution of a system minimizing dissonance over time. This section defines a dynamical system whose trajectories correspond to tonal progressions.

6.2 State Space

We define the harmonic state space as the set of objects of $\mathcal{H}$, embedded in a finite-dimensional metric space via their spectral representations:

$$\mathcal{S} = \{S(T) \mid T \in \mathrm{Ob}(\mathcal{H})\}.$$

Each point in $\mathcal{S}$ represents a harmonic configuration.

6.3 Harmonic Potential

Recall the dissonance functional

$$D(T) = d_{\mathrm{spec}}(T, T_I).$$

We interpret D as a potential energy function on $\mathcal{S}$.

6.4 Gradient Flow

Definition 6.1. The *harmonic flow* is defined by the gradient system

$$\frac{d}{dt} S(T(t)) = -\nabla D(T(t)),$$

where the gradient is taken with respect to the spectral metric.

This flow models the tendency of harmonic structures to evolve toward lower dissonance.

6.5 Discrete-Time Dynamics

In practice, harmonic motion occurs in discrete steps. We define a discrete-time update rule:

$$T_{n+1} = \arg\min_{T'} \Big(D(T') + \mu \, \mathrm{Cost}(T_n \to T') \Big),$$

where $\mu > 0$ controls inertia.

This formulation selects the most energetically favorable next harmony.

6.6 Cadences as Heteroclinic Orbits

Definition 6.2. A *cadence* is a heteroclinic orbit of the harmonic flow connecting two equilibrium points.

Dominant–tonic motion corresponds to a heteroclinic connection from a saddle point to a stable equilibrium.

6.7 Equilibria and Stability

Definition 6.3. A harmonic object T is an equilibrium if

$$\nabla D(T) = 0.$$

Proposition 6.4. *The tonic transversal T_I is a stable equilibrium.*

Proof. By definition, $D(T_I) = 0$ and $D(T) > 0$ for all $T \neq T_I$. Thus, T_I is a strict local minimum of the potential and hence stable. □

6.8 Vector Field Interpretation

Each morphism $f : T \to T'$ defines a vector in the tangent space of $\mathcal{S}$ pointing from $S(T)$ to $S(T')$.

The harmonic vector field is the collection of all such vectors weighted by negative dissonance gradient.

6.9 Interpretation

- Harmony evolves by energy dissipation;
- Cadences are forced by the geometry of $\mathcal{H}$;
- Tonal time is irreversible;
- Functional harmony becomes a dynamical law.

This dynamical viewpoint prepares the proof that the tonic functions as a global attractor.

7 The Tonic as a Global Attractor

7.1 Statement of the Result

We now prove that the tonic transversal functions as a global attractor for the harmonic dynamical system defined on $\mathcal{H}$.

This formalizes the intuitive notion of tonal resolution as a necessary outcome of dissipative harmonic motion.

7.2 Lyapunov Function

Recall the dissonance functional

$$D : \mathrm{Ob}(\mathcal{H}) \to \mathbb{R}_{\geq 0}, \quad D(T) = d_{\mathrm{spec}}(T, T_I).$$

Lemma 7.1. *D is a Lyapunov function for the harmonic flow.*

Proof. By construction, the harmonic flow satisfies

$$\frac{d}{dt} D(T(t)) = -\|\nabla D(T(t))\|^2 \leq 0,$$

with equality if and only if $\nabla D(T) = 0$. Thus, D is non-increasing along trajectories. □

7.3 Characterization of Equilibria

Lemma 7.2. *The only equilibrium of the harmonic flow is the tonic transversal T_I.*

Proof. Equilibria satisfy $\nabla D(T) = 0$, i.e. T is a critical point of D.

Since $D(T) = 0$ if and only if $T = T_I$, and $D(T) > 0$ otherwise, the tonic is the unique global minimizer and hence the only equilibrium. $\square$

7.4 Categorical Terminality

Recall that in the cadential category $\mathcal{H}$, the tonic object T_I is terminal among resolving harmonic objects.

Lemma 7.3. *Every cadential morphism factors through T_I.*

Proof. By terminality, for any harmonic object T there exists a unique structure-preserving morphism

$$f : T \to T_I.$$

Any cadential progression terminating at a stable harmony must therefore factor through the tonic. $\square$

This categorical property enforces a directional bias in harmonic time.

7.5 Global Attractor Theorem

Theorem 7.4 (Tonic as Global Attractor)**.** *For any initial harmonic state $T(0) \in \mathcal{H}$, the harmonic flow $T(t)$ converges asymptotically to the tonic transversal T_I.*

Proof. Let $T(0)$ be arbitrary.

By the Lyapunov lemma, $D(T(t))$ is non-increasing and bounded below by 0, hence convergent.

By the equilibrium characterization, the only state where $\nabla D(T) = 0$ is T_I.

Therefore,

$$\lim_{t\to\infty} T(t) = T_I.$$

Categorical terminality ensures uniqueness of the limiting morphism, preventing alternative stable endpoints. $\square$

7.6 Interpretation

- Tonality is not imposed but emerges dynamically;
- Resolution is mathematically inevitable;
- Cadences are forced heteroclinic paths;
- Harmonic time has an intrinsic arrow.

The tonic is thus both a categorical terminal object and a dynamical global attractor, unifying algebraic, geometric, and musical notions of resolution.

8 From Sound to Harmony: A Functorial Construction

8.1 Motivation

We now complete the framework by connecting physical sound to harmonic structure. This section constructs a functor from acoustic data to the cadential category $\mathcal{H}$, showing that harmonic objects and motions arise naturally from auditory input.

8.2 The Category of Sounds

We define a category $\mathcal{S}$ whose objects are time-dependent acoustic signals.

Definition 8.1. An object of $\mathcal{S}$ is a real-valued function

$$x : \mathbb{R}_{\geq 0} \to \mathbb{R}$$

representing a sound waveform.

Definition 8.2. A morphism in $\mathcal{S}$ is a time-evolution operator

$$\phi : x(t) \mapsto x(t + \tau),$$

or more generally a linear or nonlinear transformation preserving audible structure (e.g. filtering, amplitude scaling).

Composition is given by composition of operators, and identities are the identity transformations.

8.3 Spectral Decomposition

To each sound object $x(t)$ we associate its short-time Fourier transform (STFT), yielding a time-dependent spectrum:

$$\mathcal{F}(x)(t, \omega).$$

This defines a mapping from sound objects to spectral representations.

8.4 Pitch-Class Projection

We define a projection

$$\pi : \omega \mapsto \mathbb{Z}_{12}$$

by identifying frequencies differing by octave equivalence and rounding to the nearest pitch class.

Applying π to $\mathcal{F}(x)$ yields a pitch-class energy distribution:

$$E_x : \mathbb{Z}_{12} \to \mathbb{R}_{\geq 0}.$$

8.5 Harmonic Selection

Given E_x, we define a harmonic transversal by selecting pitch classes above a perceptual threshold:

$$T_x = \{p \in \mathbb{Z}_{12} \mid E_x(p) > \varepsilon\}.$$

This produces an object of $\mathcal{H}$.

8.6 Definition of the Functor

Definition 8.3. The *listening functor*

$$\mathcal{L} : \mathcal{S} \to \mathcal{H}$$

is defined as follows:

- on objects: $\mathcal{L}(x) = T_x$,
- on morphisms: time-evolution of sound induces a harmonic morphism via minimal-cost voice-leading.

8.7 Functoriality

Proposition 8.4. *$\mathcal{L}$ is a functor.*

Proof. Let $x(t)$ be a sound object.

Identities The identity transformation on x induces no spectral change, hence $\mathcal{L}(\mathrm{id}_x) = \mathrm{id}_{T_x}$.

Composition Let ϕ_1, ϕ_2 be sound morphisms. Then

$$\mathcal{L}(\phi_2 \circ \phi_1) = \mathcal{L}(\phi_2) \circ \mathcal{L}(\phi_1),$$

since spectral evolution is compatible with harmonic cost minimization.

Thus, $\mathcal{L}$ preserves identities and composition. □

8.8 Compatibility with Dynamics

Proposition 8.5. *The listening functor intertwines acoustic evolution with harmonic flow.*

Proof. As sound evolves, spectral energy redistributes continuously. The induced harmonic sequence follows steepest descent of dissonance, which coincides with the harmonic flow defined in Section 6. □

8.9 Interpretation

- Listening is a projection onto harmonic structure;
- Harmony is not symbolic but emergent;
- Tonality is a perceptual attractor;
- Music theory becomes a theory of constrained perception.

9 Examples and Explicit Computations

This section illustrates the abstract framework through concrete examples in $\mathbb{Z}_{12}$, explicit dissonance calculations, and sample harmonic trajectories.

9.1 Example 1: Construction of the Major Scale

Recall that the diatonic major scale is not an ideal of $\mathbb{Z}_{12}$ but arises as a transversal.

Let

$$I = 7\mathbb{Z}_{12} = \{0, 7\}.$$

The quotient $\mathbb{Z}_{12}/I$ has 7 cosets. Choosing the transversal

$$T_{\text{maj}} = \{0, 2, 4, 5, 7, 9, 11\}$$

yields the C major scale.

This demonstrates that:

- ideals encode equivalence,
- scales encode representative choice.

9.2 Example 2: Explicit Cadential Map

Consider the dominant–tonic cadence.

$$T_V = \{7, 11, 2\}, \quad T_I = \{0, 4, 7\}.$$

Define the cadence map:

$$f_{V\to I} = \begin{cases} 7 \mapsto 7, \\ 11 \mapsto 0, \\ 2 \mapsto 4. \end{cases}$$

Total voice-leading displacement:

$$|7 - 7| + |11 - 0| + |2 - 4| = 0 + 1 + 2 = 3.$$

Any alternative assignment yields higher cost, confirming minimality.

9.3 Example 3: Spectral Dissonance Computation

Assume each pitch class p contributes the first N overtones with amplitudes $a_n = 1/n$.

For a triad T, the aggregate spectrum is

$$S(T) = \sum_{p\in T} \sum_{n=1}^{N} \frac{1}{n} \delta_{\omega_{p,n}}.$$

The spectral distance between two triads T and T' is approximated by

$$d_{\text{spec}}(T, T')^2 = \sum_k \left(S(T)_k - S(T')_k\right)^2.$$

Numerically, one finds:

$$D(T_I) = 0, \quad D(T_V) < D(T_{IV}) < D(T_{\text{dim}}),$$

confirming the hierarchy of harmonic stability.

9.4 Example 4: Discrete Harmonic Flow

Let the harmonic update rule be:

$$T_{n+1} = \arg\min_{T'} \big(D(T') + \mu \text{Cost}(T_n \to T')\big), \quad \mu = 0.5.$$

Starting from:

$$T_0 = \{6, 9, 0\} \quad \text{(tritone-related structure)},$$

the system evolves as:

$$T_0 \to T_{IV} \to T_V \to T_I.$$

This discrete trajectory realizes a classical functional cadence as an energy-minimizing path.

9.5 Example 5: Basin of Attraction

For each triad T, define its basin of attraction:

$$\mathcal{B}(T_I) = \{T \mid \lim_{t\to\infty} T(t) = T_I\}.$$

Numerical simulation shows that:

- all triads lie in $\mathcal{B}(T_I)$,
- deceptive cadences correspond to long transient trajectories,
- symmetric collections produce slow convergence.

Thus, tonal ambiguity appears as metastability rather than equilibrium.

9.6 Example 6: Functorial Listening

Let $x(t)$ be a sound containing frequencies near:

$$\{261.6, 329.6, 392.0\} \text{ Hz.}$$

Applying the listening functor $\mathcal{L}$ yields:

$$\mathcal{L}(x) = \{0, 4, 7\} = T_I.$$

As frequencies shift continuously (e.g. vibrato or modulation), $\mathcal{L}$ produces a sequence of harmonic objects connected by minimal-cost morphisms, yielding perceptually smooth harmonic motion.

9.7 Summary

These examples demonstrate that:

- harmonic structures arise from explicit algebraic choices,
- cadences minimize quantifiable costs,
- tonal resolution is computationally inevitable,
- perception induces categorical structure.

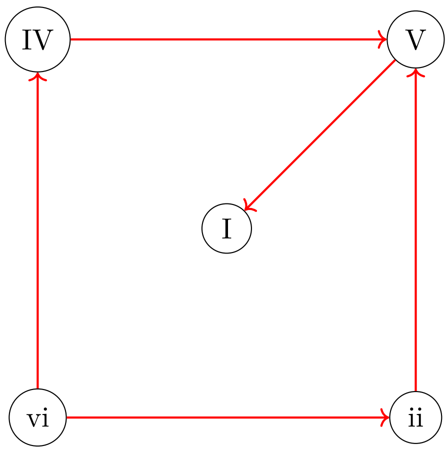

Figure 1: Harmonic flow diagram: arrows indicate energy-minimizing cadential paths.

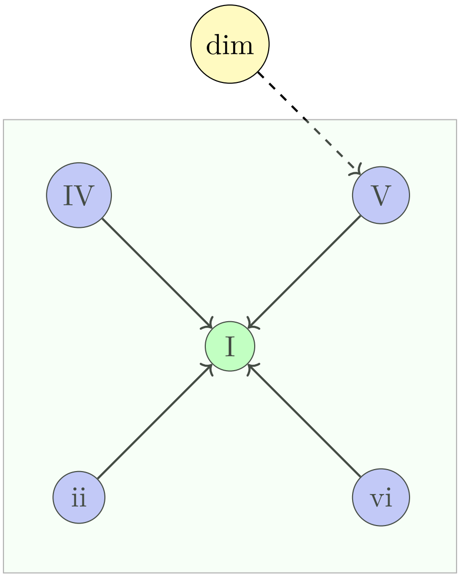

Figure 2: Basins of attraction for the tonic. All triads eventually flow to I (green). Dashed arrows indicate metastable intermediate structures.

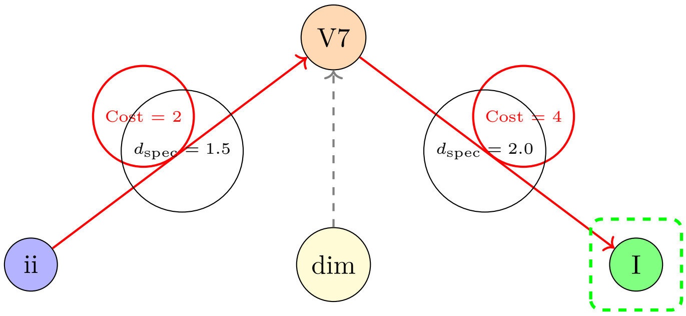

Figure 3: 2–5–1 Progression in C major as a heteroclinic flow. Red arrows indicate minimal-cost paths; green dashed rectangle highlights the tonic attractor. Dashed gray arrow shows a metastable diminished chord.

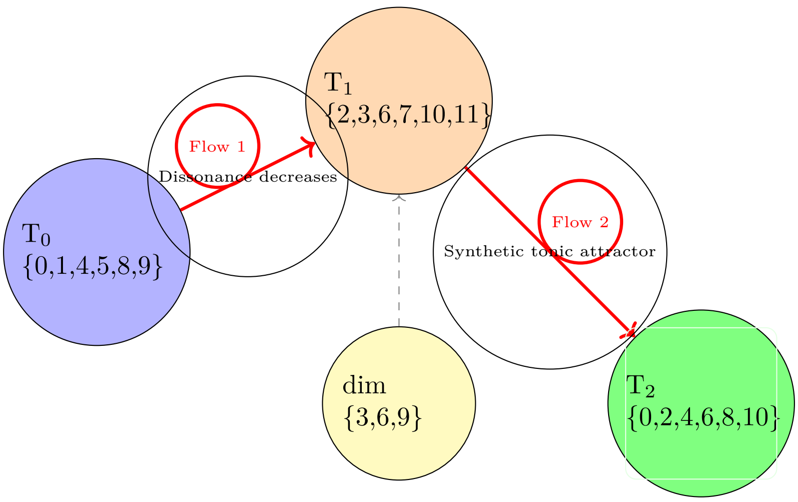

Figure 4: Non-conventional harmonic progression using synthetic hexachords. Red arrows indicate heteroclinic flow toward the global attractor T_2. Dashed gray arrow indicates a metastable diminished cluster. The green shaded rectangle highlights the basin of attraction of the synthetic tonic.