# CPEN 400Q Lecture 06
# Teleportation and Measurement II
# (expectation values)

Wednesday 24 January 2024

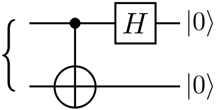
# Announcements

- Assignment 1 due Fri 02 Feb at 23:59
- Midterm in class on Wed 31 Jan - see Piazza post 14 for practice strategies; covers "the basics", i.e., lectures 01-07
- Quiz 3 at beginning of class on Monday; Monday is a hands-on class

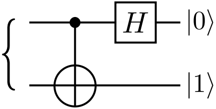
# Last time

We measured two-qubit states in the Bell basis.

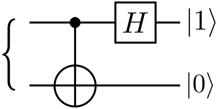
$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$



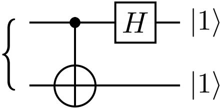

$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$




$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

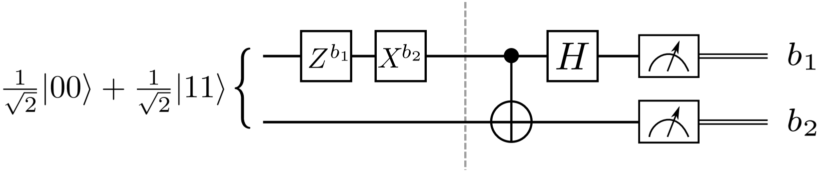



$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$



# Last time

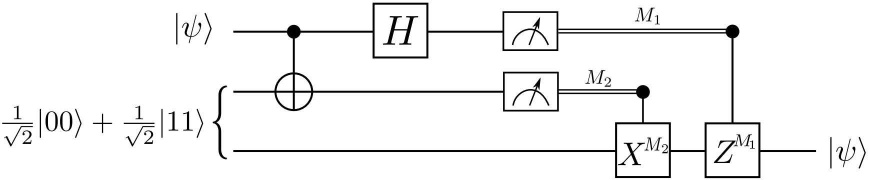
We implemented superdense coding.



# Last time

$$U\left(|\psi\rangle \otimes |s\rangle\right) = |\psi\rangle \otimes |\psi\rangle$$

We learned quantum states cannot be cloned, but can be teleported.



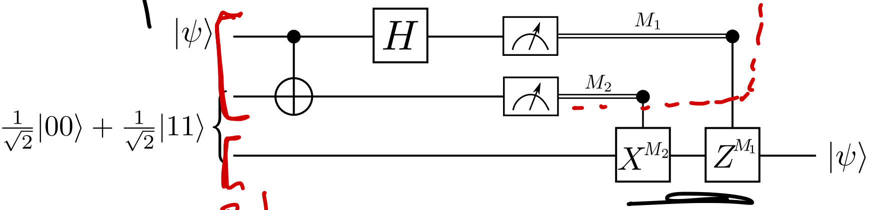
# Learning outcomes

- teleport a qubit
- define observables and expectation values
- compute expectation values of an observable after running a circuit
- use expectation values to indicate a qubit's state on the Bloch sphere

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

We cannot clone arbitrary qubit states, but we *can* teleport them!

Alice




Bob

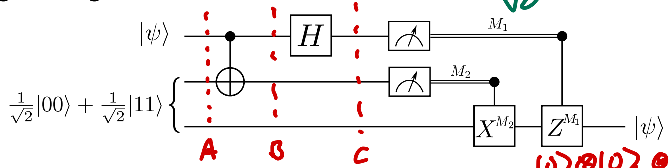
$\boxed{\text{Factory}}$

$|\psi\rangle$ $\quad \downarrow \quad \downarrow \quad \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

A A B

Let's go one gate at a time.

$(H \otimes I)|00\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$




$|0\rangle \otimes |0\rangle \otimes |0\rangle$

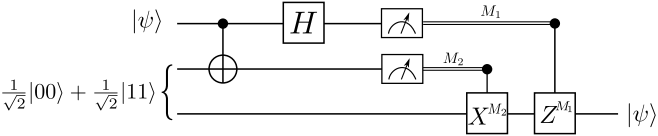
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

A: $(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}\left[\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\right]$

B: $\frac{1}{\sqrt{2}}\left[\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\right]$

C: $\frac{1}{2}\left[\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\right]$

Let's go one gate at a time.




$$\frac{1}{2}\Big[\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\Big]$$

$$= \frac{1}{2}\Big[|00\rangle(\alpha|0\rangle + \beta|1\rangle) + |01\rangle(\alpha|1\rangle + \beta|0\rangle) + |10\rangle(\alpha|0\rangle - \beta|1\rangle) + |11\rangle(\alpha|1\rangle - \beta|0\rangle)\Big]$$

$\Rightarrow$ apply X

$\downarrow$ apply Z

X, then Z

Before measurements, the combined state of the system is (removing the $\frac{1}{2}$ for readability):

$$\begin{array}{lcl} |00\rangle & \otimes & (\alpha|0\rangle + \beta|1\rangle) + \\ |01\rangle & \otimes & (\alpha|1\rangle + \beta|0\rangle) + \\ |10\rangle & \otimes & (\alpha|0\rangle - \beta|1\rangle) + \\ |11\rangle & \otimes & (\alpha|1\rangle - \beta|0\rangle) \end{array}$$

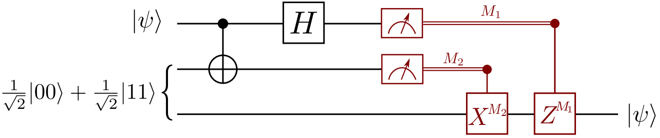
This is a *uniform* superposition of 4 distinct terms. If we measure the first two qubits in the computational basis, we are equally likely to obtain each of the four outcomes.

You can see that Bob's state is always some variation on the original state of Alice:

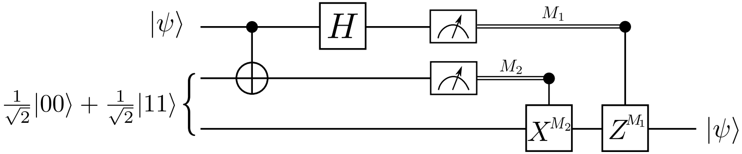
$$
\begin{aligned}
|00\rangle &\otimes (\alpha|0\rangle + \beta|1\rangle) + \\
|01\rangle &\otimes (\alpha|1\rangle + \beta|0\rangle) + \\
|10\rangle &\otimes (\alpha|0\rangle - \beta|1\rangle) + \\
|11\rangle &\otimes (\alpha|1\rangle - \beta|0\rangle)
\end{aligned}
$$

Alice measures in the computational basis and sends her results to Bob. Once Bob knows the results, he knows exactly what term of the superposition they had, and can adjust his state accordingly.

$$
\begin{aligned}
00 &: I(\alpha|0\rangle + \beta|1\rangle) = (\alpha|0\rangle + \beta|1\rangle)\\
01 &: X(\alpha|1\rangle + \beta|0\rangle) = (\alpha|0\rangle + \beta|1\rangle)\\
10 &: Z(\alpha|0\rangle - \beta|1\rangle) = (\alpha|0\rangle + \beta|1\rangle)\\
11 &: ZX(\alpha|1\rangle - \beta|0\rangle) = (\alpha|0\rangle + \beta|1\rangle)
\end{aligned}
$$



# Hands on: let's teleport a state



# Observables

Generally, we are interested in measuring real, physical quantities. In physics, these are called observables.

Observables are represented mathematically by Hermitian matrices. An operator (matrix) $H$ is Hermitian if

$$H = H^\dagger$$

**Why Hermitian?** The possible measurement outcomes are given by the eigenvalues of the operator, and eigenvalues of Hermitian operators are **real**.

Example:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$Z$ is Hermitian:

$$Z^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

Its eigensystem is

$$\lambda_1 = +1 \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$$

$$\lambda_2 = -1 \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$$

Example:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$X$ is Hermitian and its (normalized) eigensystem is

$$\lambda_1 = +1 \qquad |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = |+\rangle$$

$$\lambda_2 = -1 \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle$$

Example:

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$Y$ is Hermitian and its (normalized) eigensystem is

$$\lambda_1 = +1 \qquad |\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = |p\rangle$$

$$\lambda_2 = -1 \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} = |m\rangle$$

# Expectation values

When we measure $X$, $Y$, or $Z$ on a state, for each shot we will get one of the eigenstates (/eigenvalues).

If we take multiple shots, what do we expect to see *on average*?

Analytically, the **expectation value** of measuring the observable M given the state $|\psi\rangle$ is

$$\langle M \rangle = \langle \psi | M | \psi \rangle$$

**Exercise**: consider the quantum state

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Y|0\rangle = i|1\rangle \qquad Y|1\rangle = -i|0\rangle$$

$$|\psi\rangle = \frac{1}{2}|0\rangle - i\frac{\sqrt{3}}{2}|1\rangle.$$

$$\langle Y\rangle = \underbrace{\langle\psi| \, Y|\psi\rangle}$$

Compute the expectation value of $Y$:

$$Y|\psi\rangle = \frac{1}{2}i|1\rangle - i\frac{\sqrt{3}}{2}(-i|0\rangle) = -\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle$$

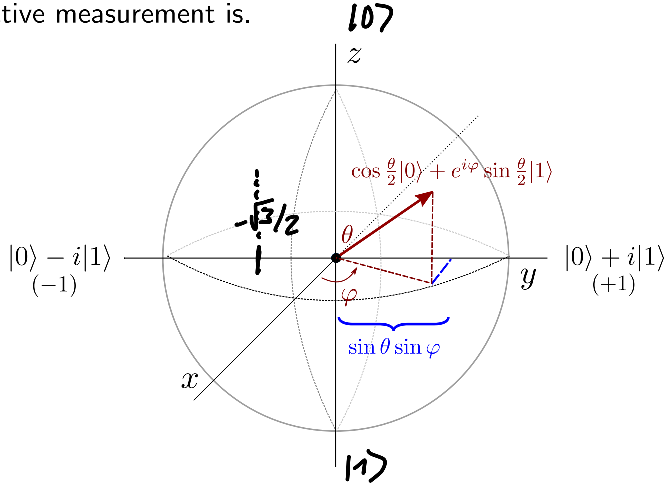
$$\langle\psi|Y|\psi\rangle = \left(\frac{1}{2}\langle 0| + i\frac{\sqrt{3}}{2}\langle 1|\right)\left(-\frac{\sqrt{3}}{2}|0\rangle + \frac{i}{2}|1\rangle\right)$$

$$= -\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}$$

$$= -\frac{\sqrt{3}}{2}$$

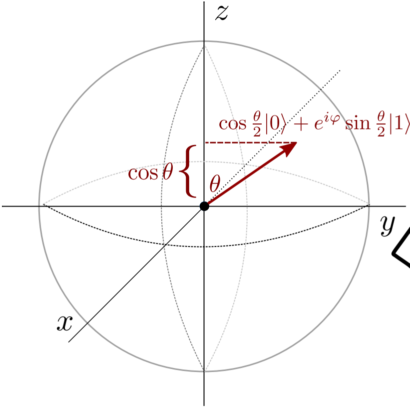
## Expectation values and the Bloch sphere

The Bloch sphere offers us some more insight into what a projective measurement is.




Exercise: derive the expression in blue by computing $\langle\psi| Y |\psi\rangle$.

## Expectation values and the Bloch sphere




$\langle Z \rangle$

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

$$Z|\psi\rangle = \cos\frac{\theta}{2}|0\rangle - e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

$$\langle\psi|Z|\psi\rangle = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$= \cos\theta$$

# Expectation values: from measurement data

Let's compute the expectation value of $Z$ for the following circuit using 10 samples:

```
dev = qml.device('default.qubit', wires=1, shots=10)

@qml.qnode(dev)
def circuit():
    qml.RX(2*np.pi/3, wires=0)
    return qml.sample()
```

Results might look something like this:

```
[1, 1, 1, 0, 1, 1, 1, 0, 1, 1]
```

# Expectation values: from measurement data

The expectation value pertains to the measured eigenvalue; recall $Z$ eigenstates are

$$\lambda_1 = +1, \qquad |\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\lambda_2 = -1, \qquad |\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So when we observe $|0\rangle$, this is eigenvalue $+1$ (and if $|1\rangle$, $-1$). Our samples shift from

```
[1, 1, 1, 0, 1, 1, 1, 0, 1, 1]
```

to

```
[-1, -1, -1, 1, -1, -1, -1, 1, -1, -1]
```

# Expectation values: from measurement data

The expectation value is the weighted average of this, where the weights are the eigenvalues:

$$\langle Z \rangle = \frac{1 \cdot n_1 + (-1) \cdot n_{-1}}{N}$$

where

- $n_1$ is the number of $+1$ eigenvalues
- $n_{-1}$ is the number of $-1$ eigenvalues
- $N$ is the total number of shots

For our example,

$$\langle Z \rangle = \text{exercise}$$

# Expectation values

Let's do this in PennyLane instead:

```
dev = qml.device('default.qubit', wires=1)

@qml.qnode(dev)
def measure_z():
    qml.RX(2*np.pi/3, wires=0)
    return qml.expval(qml.PauliZ(0))
```

# Multi-qubit expectation values

Example: operator $Z \otimes Z$.

Eigenvalues are computational basis states:

$$
\begin{aligned}
(Z \otimes Z)|00\rangle &= |00\rangle \\
(Z \otimes Z)|01\rangle &= -|01\rangle \\
(Z \otimes Z)|10\rangle &= -|10\rangle \\
(Z \otimes Z)|11\rangle &= |11\rangle
\end{aligned}
$$

To compute an expectation value from data:

$$\langle Z \otimes Z \rangle = \frac{1 \cdot n_1 + (-1) \cdot n_{-1}}{N}$$

$a+bi \to a-bi$

Example: operator $X \otimes I$.

Eigenvalues of $X$ are the $|+\rangle$ and $|-\rangle$ states:

$$
\begin{aligned}
(X \otimes I)|+0\rangle &= |+0\rangle \\
(X \otimes I)|+1\rangle &= |+1\rangle \\
(X \otimes I)|-0\rangle &= -|-0\rangle \\
(X \otimes I)|-1\rangle &= -|-1\rangle
\end{aligned}
$$

*Fun fact: All Pauli operators have an equal number of* $+1$ *and* $-1$ *eigenvalues!*

# Multi-qubit expectation values

How to compute expectation value of $X$ from data, when we can only measure in the computational basis?

Basis rotation: apply $H$ to first qubit

$$
\begin{aligned}
(H \otimes I)(X \otimes I)|+0\rangle &= |00\rangle \\
(H \otimes I)(X \otimes I)|+1\rangle &= |01\rangle \\
(H \otimes I)(X \otimes I)|-0\rangle &= -|10\rangle \\
(H \otimes I)(X \otimes I)|-1\rangle &= -|11\rangle
\end{aligned}
$$

When we measure and obtain $|10\rangle$ or $|11\rangle$, we know those correspond to the $-1$ eigenstates of $X \otimes I$.

# Hands-on: multi-qubit expectation values

Multi-qubit expectation values can be created using the `@` symbol:

```
@qml.qnode(dev)
def circuit(x):
    qml.Hadamard(wires=0)
    qml.CRX(x, wires=[0, 1])
    return qml.expval(qml.PauliZ(0) @ qml.PauliZ(1))
```

# Hands-on: multi-qubit expectation values

Can also return *multiple* expectation values, if there are no shared qubits.

```
@qml.qnode(dev)
def circuit(x):
    qml.Hadamard(wires=0)
    qml.CRX(x, wires=[0, 1])
    return qml.expval(qml.PauliZ(0)), qml.expval(qml.PauliZ(
                                          1))
```

# Recap

- teleport a qubit
- define observables and expectation values
- measure single-qubit expectation values
- use expectation values to indicate a qubit's state on the Bloch sphere

# Next time

Content:

- Hands-on: measuring expectation values for the variational quantum classifier

Action items:

1. Assignment 1
2. Quiz 3 on Monday
3. Study for midterm

Recommended reading:

- All of the Codebook module I
- `https://pennylane.ai/qml/demos/tutorial_variational_classifier.html`