

# Notes on Atiyah-MacDonald

## (Intro to Commutative Algebra)

Kat Watson  
<https://pqcfox.dev>

### Abstract

These notes are from late 2024, and are intended as a review of commutative algebra in preparation to cover more of the in-depth sections of Hartshorne. In these notes, I make arguments from the text as explicit as possible, even when they may be “obvious” in order to get additional practice—it’s been a while since my undergrad. Exercise solutions are included, and are selected to cover the material as well as possible.

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## 1. Rings and ideals

### 1.1. Prime and maximal ideals (page 3)

The claim is made that for a ring  $A$  with  $\mathfrak{p}$  and  $\mathfrak{m}$  ideals of  $A$ ,

$\mathfrak{p}$  is prime  $\Leftrightarrow A/\mathfrak{p}$  is an integral domain;  
 $\mathfrak{m}$  is maximal  $\Leftrightarrow A/\mathfrak{m}$  is a field (by (1.1) and (1.2)).

Indeed,  $A/\mathfrak{p}$  is a domain iff for all  $x, y \in A$ ,  $\overline{xy} = 0$  implies either  $\overline{x} = 0$  or  $\overline{y} = 0$ . Since

$$\overline{xy} = 0 \Leftrightarrow xy \in \mathfrak{p}, \quad \overline{x} = 0 \Leftrightarrow x \in \mathfrak{p}, \quad \overline{y} = 0 \Leftrightarrow y \in \mathfrak{p}$$

we have that this is equivalent to  $\mathfrak{p}$  being prime.

For the other claim, (1.2) states that  $A/\mathfrak{m}$  is a field iff the only ideals in  $A$  are 0 and (1). By (1.1), there is a one-to-one order-preserving correspondence between the ideals of  $A/\mathfrak{m}$  and the ideals of  $A$  containing  $\mathfrak{m}$  sending  $0, (1) \subseteq A/\mathfrak{m}$  to  $\mathfrak{m}, (1) \subseteq A$  respectively. Thus, we have

$$\begin{aligned} A/\mathfrak{m} \text{ is a field} &\Leftrightarrow \text{the only ideals of } A/\mathfrak{m} \text{ are } 0 \text{ and } (1) \\ &\Leftrightarrow \text{the only ideals of } A \text{ containing } \mathfrak{m} \text{ are } \mathfrak{m} \text{ and } (1) \\ &\Leftrightarrow \mathfrak{m} \text{ is maximal.} \end{aligned}$$

### 1.2. Pullbacks of prime ideals (page 3)

The claim is also made that

If  $f : A \rightarrow B$  is a ring homomorphism and  $\mathfrak{q}$  is a prime ideal in  $B$ , then  $f^{-1}(\mathfrak{q})$  is a prime ideal in  $A$ , for  $A/f^{-1}(\mathfrak{q})$  is isomorphic to a subring of  $B/\mathfrak{q}$  and hence has no zero-divisor  $\neq 0$ .

First, to construct this isomorphism, compose  $f$  with the quotient map  $\pi : B \rightarrow B/\mathfrak{q}$  to get  $\pi \circ f : A \rightarrow B/\mathfrak{q}$  and note that  $\ker(\pi \circ f)$  is precisely  $f^{-1}(\mathfrak{q})$ , so that by the first isomorphism theorem, we have an injective map  $\overline{\pi \circ f} : A/f^{-1}(\mathfrak{q}) \rightarrow B/\mathfrak{q}$ . Since the image of any ring homomorphism is a subring of the codomain, this map makes  $A/f^{-1}(\mathfrak{q})$  isomorphic to a subring of  $B/\mathfrak{q}$  as desired.

### 1.3. Every non-zero ring has a maximal ideal (Proposition 1.3)

As part of the proof, it is claimed that for a chain  $(\mathfrak{a}_\alpha)$  in  $\Sigma$ ,  $\mathfrak{a} := \bigcup_\alpha \mathfrak{a}_\alpha$  is an ideal. Indeed, take  $x, y \in \mathfrak{a}$ . We have that for some  $\alpha$  and  $\beta$ ,  $x \in \mathfrak{a}_\alpha$  and  $y \in \mathfrak{a}_\beta$ . WLOG, assume  $\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta$ . Then  $x, y \in \mathfrak{a}_\beta$ , meaning  $x + y \in \mathfrak{a}_\beta \subseteq \mathfrak{a}$ .

Similarly, if  $a \in A$  is a general element of  $A$  and  $x \in \mathfrak{a}_\alpha$ , then we have  $ax \in \mathfrak{a}_\alpha \subseteq \mathfrak{a}$ , so that  $\mathfrak{a}$  is an ideal of  $A$ .

### 1.4. Every ideal is contained in a maximal ideal (Corollary 1.4)

In the proof of Corollary 1.4, the full argument is

Apply (1.3) to  $A/\mathfrak{a}$ , bearing in mind (1.1). Alternatively, modify the proof of (1.3).

We expand on this: applying (1.3) to  $A/\mathfrak{a}$ , we get a maximal ideal of  $A/\mathfrak{a}$ , which by the correspondence in (1.1) corresponds to a maximal ideal of  $A$  containing  $\mathfrak{a}$ .

### 1.5. Criterion for a ring to be local (Proposition 1.6(ii))

The proof elides the detail that for any  $x, y \in A$ , if  $xy$  is a unit then  $x$  is a unit. This follows from definition: if  $xy$  is a unit, then there exists  $z \in A$  such that  $(xy)z = 1$ , but then  $x(yz) = 1$ , meaning  $x$  is a unit.

### 1.6. Irreducible polynomials generate prime ideals (Example 1, page 4)

As an example of a prime ideal, page 4 gives

$A = k[x_1, \dots, x_n]$ ,  $k$  a field. Let  $f \in A$  be an irreducible polynomial. By unique factorization, the ideal  $(f)$  is prime.

Making this explicit, assume that we have  $g, h \in A$  such that  $gh \in (f)$ , i.e.  $f \mid gh$ . Then since  $f$  is irreducible, we have by unique factorization that either  $f \mid g$  or  $f \mid h$ , i.e.  $g \in (f)$  or  $h \in (f)$ , meaning  $(f)$  is prime.

### 1.7. Ideals of $\mathbb{Z}$ are all principal (Example 2, page 4)

As another example on page 4, there is a claim that

Every ideal in  $\mathbb{Z}$  is of the form  $(m)$  for some  $m \geq 0$ .

Indeed, take an arbitrary non-zero ideal  $\mathfrak{a} \subseteq \mathbb{Z}$ , and take the smallest positive element of  $\mathfrak{a}$ : denote this as  $m$ . Clearly  $(m) \subseteq \mathfrak{a}$ . Now, take any  $n \in \mathfrak{a}$ . By Bezout's identity, we have  $\gcd(m, n) \in \mathfrak{a}$ . Since  $\gcd(m, n) \leq m$ , but  $m$  is the smallest positive element of  $\mathfrak{a}$ , we have  $\gcd(m, n) = m$ , i.e.  $m \mid n$ . Thus  $\mathfrak{a} \subseteq (m)$ , and the two ideals are equal.

As implicitly noted in the text, a similar argument can be applied to show that  $k[x]$  is a principal ideal domain, based on the Euclidean algorithm for univariate polynomials.

### 1.8. The modular law for ideals (page 6)

The following is stated as the closest approximate we have in general to  $\cap$  and  $+$  distributing over each other in a general ring:

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$$\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c} \text{ if } \mathfrak{a} \supseteq \mathfrak{b} \text{ or } \mathfrak{a} \supseteq \mathfrak{c}$$

To see this, WLOG let  $\mathfrak{a} \supseteq \mathfrak{b}$ . We show each side of the desired equality contains the other.

$\subseteq$ : By definition, we have  $\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) \subseteq \mathfrak{a}$ , and since  $\mathfrak{b} \subseteq \mathfrak{a}$ , we have  $\mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c}) \subseteq \mathfrak{b} + \mathfrak{c} \subseteq \mathfrak{a} + \mathfrak{c}$ . This means that any  $x \in \mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c})$  both lies in  $\mathfrak{a}$  and can be written as  $x = y + z$  for some  $y \in \mathfrak{b}$  and some  $z \in \mathfrak{c}$ . But then,  $z = x - y \in \mathfrak{a}$ , meaning  $z \in \mathfrak{a} \cap \mathfrak{c}$  and therefore

$$x = y + z \in \mathfrak{a} + \mathfrak{a} \cap \mathfrak{c} = \mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c}.$$

$\supseteq$ : Any element  $x \in \mathfrak{a} \cap \mathfrak{b} + \mathfrak{a} \cap \mathfrak{c}$  can be written as  $x = y + z$  for  $y \in \mathfrak{a} \cap \mathfrak{b}$  and  $z \in \mathfrak{a} \cap \mathfrak{c}$ . Since  $y, z \in \mathfrak{a}$ ,  $x = y + z \in \mathfrak{a}$ , and since  $y \in \mathfrak{b}$  and  $z \in \mathfrak{c}$ , we have  $x = y + z \in \mathfrak{b} + \mathfrak{c}$  as well, meaning  $x \in \mathfrak{a} \cap (\mathfrak{b} + \mathfrak{c})$  as desired.

### 1.9. End of chapter exercises

#### 1.9.1. Exercise 1

Take  $n > 0$  such that  $x^n = 0$ . Then  $(1 + x)(1 - x + x^2 - x^3 + \dots \pm x^{n-1}) = 1 \pm x^n = 1$ .  $\square$

#### 1.9.2. Exercise 6

By Proposition 1.9, it suffices to show that for any  $x \in A$ , if  $1 - xy$  is a unit for every  $y \in A$ , then  $x$  lies in the nilradical. Assume to a contraction that some such  $x$  is not in the nilradical. By the premise of the problem,  $(x)$  would then contain some non-zero idempotent  $e$ . Take  $y$  such that  $e = xy$ . Then  $1 - xy = 1 - e$ , which is idempotent:  $(1 - e)^2 = 1 - 2e + e^2 = 1 - e$ .

If  $1 - e$  was a unit, then multiplying both sides of the previous equality by its inverse would give  $1 - e = 1$ , i.e.  $e = 0$ , contradicting our choice of  $e$ .  $\Rightarrow \Leftarrow$   $\square$

#### 1.9.3. Exercise 7

Fix an arbitrary prime ideal  $\mathfrak{p} \subseteq A$ , and take  $x \in A - \mathfrak{p}$  with image  $\bar{x} \in A/\mathfrak{p}$ . Since  $x^n = x$  for some  $n > 1$ , we have  $(x^{n-1} - 1)x = 0$ . Passing to the quotient, this gives  $(\bar{x}^{n-1} - 1)\bar{x} = 0$ .

Since  $x \notin \mathfrak{p}$ , we have  $\bar{x} \neq 0$ . Because  $A/\mathfrak{p}$  is an integral domain, this means  $\bar{x}^{n-1} - 1 = 0$ , implying that  $\bar{x}$  is a unit. As  $x \in A - \mathfrak{p}$  was arbitrary, this means that any nonzero element of  $A/\mathfrak{p}$  is a unit, i.e.  $A/\mathfrak{p}$  is a field and thus  $\mathfrak{p}$  is maximal.  $\square$

#### 1.9.4. Exercise 8

Let  $\Sigma$  be the set of all prime ideals in  $A$ , ordered by reverse inclusion.  $\Sigma$  is non-empty, since  $0 \in \Sigma$ . Moreover, if  $(\mathfrak{a}_\alpha)$  is a chain of ideals in  $\Sigma$ , we claim that  $\cap_\alpha \mathfrak{a}_\alpha$  is an upper (or rather, lower) bound to  $(\mathfrak{a}_\alpha)$ . It suffices to show that  $\mathfrak{a} := \cap_\alpha \mathfrak{a}_\alpha$  is prime, and indeed if  $x, y \notin \mathfrak{a}$ , then we have that there are some  $\alpha, \beta$  such that  $x \notin \mathfrak{a}_\alpha$  and  $y \notin \mathfrak{a}_\beta$ . Since  $(\mathfrak{a}_\alpha)$  is a chain, we can assume WLOG that  $\mathfrak{a}_\alpha \subseteq \mathfrak{a}_\beta$ . Thus,  $x, y \notin \mathfrak{a}_\beta$ , meaning  $xy \notin \mathfrak{a}_\beta$  and thus  $xy \notin \mathfrak{a}$ , showing that  $\mathfrak{a}$  is prime.

#### 1.9.5. Exercise 9

We handle both directions separately:

$\Rightarrow$ : If  $\mathfrak{a} = r(\mathfrak{a})$ , then any  $x \in A$  such that  $x^n \in \mathfrak{a}$  for some  $n > 0$  must lie in  $\mathfrak{a}$ . Thus, for any  $\bar{x} \in A/\mathfrak{a}$  such that  $\bar{x}^n = 0$ , we have  $\bar{x} = 0$ , meaning in  $A/\mathfrak{a}$  that the nilradical is the zero ideal, i.e.  $\bigcap_{\mathfrak{p} \in \mathcal{F}} \mathfrak{p} = 0$  for a family of prime ideals  $\mathcal{F}$  in  $A/\mathfrak{a}$ . Now, take  $\pi : A \rightarrow A/\mathfrak{a}$  to be the natural projection of  $A$  onto  $A/\mathfrak{a}$ . Because taking preimages preserves intersections and sends prime ideals to prime ideals, we have that  $\pi^{-1}\mathcal{F}$  is a family of prime ideals whose intersection is  $\pi^{-1}(0) = \mathfrak{a}$ , i.e.  $\mathfrak{a}$  is an intersection of prime ideals.

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$\Leftarrow$ : Let  $\mathfrak{a}$  be the intersection of some family  $\mathcal{F}$  of prime ideals in  $A$ . If  $x^n \in \mathfrak{a}$  for some  $n > 0$ , then  $x^n \in \mathfrak{p}$  for each  $\mathfrak{p} \in \mathcal{F}$ . Since prime ideals are radical, this means that  $x \in \mathfrak{p}$  for each  $\mathfrak{p} \in \mathcal{F}$ , and thus  $x \in \mathfrak{a}$  as desired.  $\square$

### 1.9.6. Exercise 10

We prove a cycle of implications.

(i)  $\Rightarrow$  (ii): If  $A$  has exactly one prime ideal, then it must equal  $\mathfrak{N}$ , being the intersection of every prime ideal. Because  $\mathfrak{N}$  is the maximal ideal in a local ring, every element not in  $\mathfrak{N}$  must be a unit, i.e. every element is nilpotent or a unit.

(ii)  $\Rightarrow$  (iii): Since every non-nilpotent element of  $A$  is a unit, we have that  $\mathfrak{N}$  is maximal by Proposition 1.6(i) and thus  $A/\mathfrak{N}$  is a field.

(iii)  $\Rightarrow$  (i): If  $A/\mathfrak{N}$  is a field, then  $\mathfrak{N}$  is maximal. For any prime ideal  $\mathfrak{p} \subseteq A$ , we have  $\mathfrak{N} \subseteq \mathfrak{p}$  as the nilpotent is the intersection of all prime ideals, but then by maximality of  $\mathfrak{N}$  we must have  $\mathfrak{N} = \mathfrak{p}$ . Thus, there can only be one prime ideal in  $A$ .  $\square$

### 1.9.7. Exercise 11

- (i) For any  $x \in A$ , we have  $x + 1 = (x + 1)^2 = x^2 + 2x + 1 = 3x + 1$ , meaning  $2x = 0$ .
- (ii) For a given prime ideal  $\mathfrak{p} \subseteq A$ , take any  $\bar{x} \in A/\mathfrak{p}$ . Since  $x^2 = x$  in  $A$ , i.e.  $x(x - 1) = 0$ , we have  $\bar{x}(\bar{x} - 1) = 0$  in  $A/\mathfrak{p}$ , meaning either  $\bar{x} = 0$  or  $\bar{x} = 1$  as  $A/\mathfrak{p}$  is a domain. Thus,  $A/\mathfrak{p}$  is a field with two elements, implying  $\mathfrak{p}$  is maximal.
- (iii) Take an arbitrary finitely generated ideal  $\mathfrak{a} = (x_1, x_2, \dots, x_n) \subseteq A$ . Let  $y = \dots$

TODO: finish this!

### 1.9.8. Exercise 12

Let  $\mathfrak{m}$  be the maximal ideal of a local ring  $A$ . Note that any  $x \notin \mathfrak{m}$  must be a unit, otherwise by Corollary 1.5 it would lie in a maximal ideal distinct from  $\mathfrak{m}$ .

We show that the only idempotent element of  $\mathfrak{m}$  is 0. Assume some  $e \in \mathfrak{m}$  satisfies  $e^2 = e$ . Then  $1 - e \notin \mathfrak{m}$ , otherwise we would have  $1 - e + e = 1 \in \mathfrak{m}$ . Thus  $1 - e$  is a unit. But  $(1 - e)^2 = 1 - 2e + e^2 = 1 - e$ , so then multiplying both sides by  $(1 - e)^{-1}$ , we get  $1 - e = 1$ , i.e.  $e = 0$ .

The only idempotent  $e \notin \mathfrak{m}$  is 1, since  $e \notin \mathfrak{m}$  implies  $e$  is a unit, meaning  $e^2 = e$  implies  $e = 1$ . Since either  $e \in \mathfrak{m}$  or  $e \notin \mathfrak{m}$ , our only options for an idempotent  $e \in A$  are 0 and 1.  $\square$

## 2. Modules