NOTES ON CARLIP'S LECTURE ON SPIN NETWORKS

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Consider the simplest spin network, we need two nodes and three edges (if we had two nodes and two edges, it'd be an identity). This simplest nontrivial spin network is shown in figure (1). We don't want to have spin 0 for an edge, if we did the propagator would be the identity. For each of these lines (which we labeled c_1, c_2, and c_3 for reference) we have the Wilson line

$$\mathcal{U} = \mathcal{P}\exp\left[-\int A\right] \tag{1}$$

We have the Wilson lines for each of these edges: $\mathcal{U}^{m_1}_{1,n_1}$, $\mathcal{U}^{m_2}_{2,n_2}$, $\mathcal{U}^{m_3}_{3,n_3}$. The $\mathcal{U}$'s tell us how a spin-half state rotates in the spin space. We have $m_1 = -1/2,\ 1/2$ for spin up and spin down (respectively). We see m_2 is also spin half, but m_3 is spin 1 with possible values of -1, 0, 1. We can now use Clebsch-Gordon Coefficients

$$\langle j\, m | j_1\, m_1,\ j_2\, m_2\rangle \tag{2}$$

and find

$$\sum_{\substack{m_1,m_2,m_3\\ n_1,n_2,n_3}} \mathcal{U}^{m_1}_{1,n_1}\mathcal{U}^{m_2}_{2,n_2}\mathcal{U}^{m_3}_{3,n_3}\langle 1\, m_3|\frac{1}{2}\, m_1,\ \frac{1}{2}\, m_2\rangle\langle 1\, n_3|\frac{1}{2}\, n_1,\ \frac{1}{2}\, n_2\rangle \tag{3}$$

which is a function of A.

Now, we can take a surface Σ and ask "What is the area of this surface?" Suppose we have some spin network that "goes through" our surface Σ. We won't consider any edge of the spin networking "grazing" the surface (that it, just touching it in one spot) or lying on the surface, we will only suppose that the edges pierce the

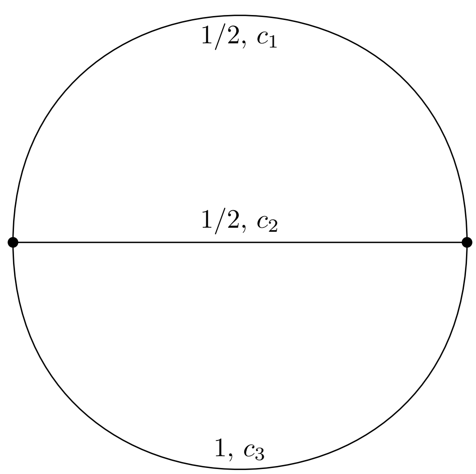

FIGURE 1. A Simple Spin Network.

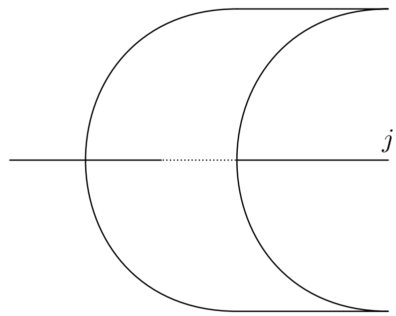

FIGURE 2. A spin network edge piercing a "small" surface.

surface. We will only really consider a simple example choosing a surface where $x^3 = 0$. The area of the surface would be classically

$$A = \int_\Sigma \sqrt{{}^{(2)}g} \tag{4}$$

where ${}^{(2)}g$ is the determinant of the metric induced on the surface. We see that

$$ {}^{(2)}g = g_{11}g_{22} - g_{12}^2 = gg^{33} = \widetilde{E}^3_{\hat{I}}\widetilde{E}^{3\hat{I}}. \tag{5}$$

So the area is

$$A = \int_\Sigma \sqrt{\widetilde{E}^3_{\hat{I}}\widetilde{E}^{3\hat{I}}}. \tag{6}$$

Consider a more general surface with intrinsic coordinates σ^1, σ^2. Then in this general setting, we have the correspondence

$$\widetilde{E}^3_{\hat{I}} \to \varepsilon_{ijk}\frac{\partial x^i}{\partial\sigma^1}\frac{\partial x^j}{\partial\sigma^2}\widetilde{E}^k_{\hat{I}}. \tag{7}$$

Lets define

$$\widetilde{E}_{\hat{I}} := \int_{\text{small region}} d\sigma^1 d\sigma^2 \varepsilon_{ijk}\frac{\partial x^i}{\partial\sigma^1}\frac{\partial x^j}{\partial\sigma^2}\widetilde{E}^k_{\hat{I}} \tag{8}$$

In the classical arena, the notion of a "small region" is ill defined. However, in the quantum world, it's just a region where one edge of a spin network pierces it. We can turn this into an operator

$$\widetilde{E}_{\hat{I}} := \int_{\text{small region}} d\sigma^1 d\sigma^2 \varepsilon_{ijk}\frac{\partial x^i}{\partial\sigma^1}\frac{\partial x^j}{\partial\sigma^2}\frac{\delta}{\delta A^{\hat{I}}_k} \tag{9}$$

We need to consider

$$\frac{\delta}{\delta A^{\hat{I}}_k}\mathcal{U} = ??? \tag{10}$$

So we remember

$$\mathcal{U} = \mathcal{P}\exp\left[-\int A^{\hat{I}}_i \tau_{\hat{I}}\frac{dx^i}{ds}ds\right] \tag{11}$$

If we didn't have the path ordering, this would be trivial, but we need to be more careful since things don't commute. Consider the path as shown in figure (3). We

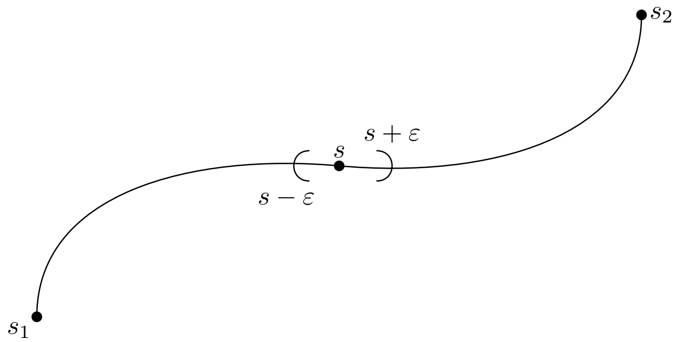

FIGURE 3. An example path.

have

$$\mathcal{U}(s_2, s_1) = \mathcal{U}(s_2, s)\mathcal{U}(s, s_1) \tag{12a}$$
$$= \mathcal{U}(s_2, s+\varepsilon)\mathcal{U}(s+\varepsilon, s-\varepsilon)\mathcal{U}(s-\varepsilon, s_1) \tag{12b}$$

If ε is "small enough", we have

$$\frac{\delta}{\delta A_i^{\hat{I}}(s)}\mathcal{U}(s_2, s_1) = \mathcal{U}(s_2, s)\left(-\tau_{\hat{I}}\frac{dx^i}{ds}\right)\mathcal{U}(s, s_1) \tag{13}$$

More generally we can write

$$\frac{\delta}{\delta A_i^{\hat{I}}(x)}\mathcal{U}(s_2, s_1) = \int ds\delta^{(3)}(C(s), x)\mathcal{U}(s_2, s)\left(-\tau_{\hat{I}}\frac{dx^i}{ds}\right)\mathcal{U}(s, s_1) \tag{14}$$

where $C(s)$ is our path parametrized by s, we see this is 0 if x is not on the path. We see now that

(15)

$$E_{\hat{I}}\mathcal{U}(s_2, s_1) = 8\pi G\gamma\int d\sigma^1 d\sigma^2 ds\varepsilon_{ijk}\frac{\partial x^i}{\partial\sigma^1}\frac{\partial x^j}{\partial\sigma^2}\frac{\partial x^k}{\partial s}\delta^{(3)}(C(s), x)\mathcal{U}(s_2, s)\left(-\tau_{\hat{I}}\frac{dx^i}{ds}\right)\mathcal{U}(s, s_1)$$

We also see that

$$\int d\sigma^1 d\sigma^2 ds\varepsilon_{ijk}\frac{\partial x^i}{\partial\sigma^1}\frac{\partial x^j}{\partial\sigma^2}\frac{\partial x^k}{\partial s}\delta^{(3)}(C(s), x) \tag{16}$$

is called the "oriented circle number" (it's ± 1 if $C(s)$ intersects Σ, 0 otherwise).

The moral of the story is that the oriented intersection number $I(C, \Sigma)$ is used to find $E_{\hat{I}}\mathcal{U}(s_2, s_1) = kI(C, \Sigma)\mathcal{U}(s_2, s)\tau_{\hat{I}}\mathcal{U}(s, s_1)$ where s is the point of intersection, and $k = 8\pi\gamma G$.

Lets consider

$$E_{\hat{I}}E^{\hat{I}}\mathcal{U}(s_2, s_1) = (8\pi\gamma G)^2\mathcal{U}(s_2, s)\tau_{\hat{I}}\tau^{\hat{I}}\mathcal{U}(s, s_1) \tag{17}$$

We see that for SU(2), $\tau_{\hat{I}}\tau^{\hat{I}}$ is the quadratic Casimir (it's not too surprising, it's kind of like J^2 from quantum mechanics). So we can plug in $j(j+1)$ instead and we end up with

$$E_{\hat{I}}E^{\hat{I}}\mathcal{U}(s_2, s_1) = (8\pi\gamma G)^2 j(j+1)\mathcal{U}(s_2, s_1). \tag{18}$$

This is assuming there is an intersection, of course. We can write a spin network state $|s\rangle$ so

$$E_{\hat{I}}E^{\hat{I}}|s\rangle = \sum_{\text{intersections}} (8\pi\gamma G)^2 j(j+1)|s\rangle \tag{19}$$

We can now define the area operator

$$\widehat{A} = \sum_{\text{small regions of }\Sigma} \sqrt{E_{\hat{I}}E^{\hat{I}}} \tag{20}$$

Classically we had

$$A = \int (E^3_{\widehat{I}}E^{3\widehat{I}})^{1/2} \tag{21}$$

and an integral is nothing more than a continuous sum, so we see that this is a sensible definition in the classical limit. Given this area operator, we see that when it acts on a spin network that

$$\widehat{A}_\Sigma|s\rangle = \sum_{\text{intersections}} 8\pi\gamma G\sqrt{j(j+1)}|s\rangle \tag{22}$$

The spectrum of the area is discrete (j comes in half integer values). The spacing between the high j's is "smaller" than the spacing between the low j's. There are some attempts at a number theoretic explanation.

The volume operator can also be defined similarly using the product of 3 $\widetilde{E}$'s instead of 2, but its construction is horrible. The area operator has contributions from edges, but the volume operator has contributions from vertices "with enough edges" (4 edges at a node should be viewed as dual to a tetrahedron, 3 edges has area but no volume). At this point, the volume spectrum is not well understood.

If we have a cubic network, if we have too many edges the metric "looks flat". There are angle operators with fairly odd properties. One fairly old but beautiful reference is [1].

References

[1] C. Rovelli and P. Upadhya, "Loop quantum gravity and quanta of space: A primer," `arXiv:gr-qc/9806079`.