NOTES ON REAL ANALYSIS

ALEX NELSON

Contents

1. Properties of Real Numbers

There are some important properties of the real numbers that we need to introduce, since we are introducing real analysis.

Proposition 1.1 (Triangle Inequality)**.** For any $a, b \in \mathbb{R}$, we have

$$|a+b| \leq |a| + |b|. \tag{1}$$

Remark 1.2. More generally, in metric topology, when we have a metric

$$d : X \times X \to \mathbb{R},$$

the identity

$$d(x,z) + d(z,y) \geq d(x,y) \tag{2}$$

holds for all $x, y, z \in X$. It is called the "triangle inequality".

2. Continuity

Definition 2.1. Let $f : D \to \mathbb{R}$ be a real valued function whose domain is a subset of $D \subset \mathbb{R}$. Then f is said to be **continuous at** $x_0 \in D$ iff for each $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$|x - x_0| < \delta \ \text{ implies } \ |f(x) - f(x_0)| < \varepsilon. \tag{3}$$

Remark 2.2. The intuition behind this definition is if we "wiggle" around in the range by some amount less than ε about $f(x_0)$, then we should "wiggle" about in the domain by some amount less than δ. If we end up have δ be too large as ε gets too small, there is a discontinuity (a jump permitting a lot of wiggling). We can see a diagram illustrating this point in fig (1). Observe that in the diagram, as we make $|f(x) - f(x_0)|$ decrease, $|x - x_0|$ decreases faster. In general, we just want a relationship expressing some positive δ in terms of a given $\varepsilon > 0$. This relationship should be injective(?) and such that as ε decreases, δ decreases too. So we can have an arbitrarily small ε, and δ goes to zero as well. This is fundamentally all a limit really is.

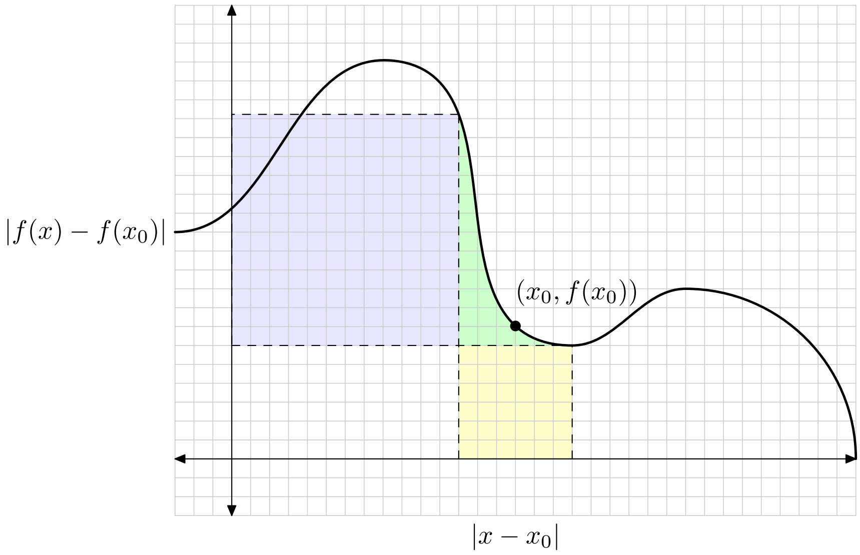

FIGURE 1. An example of the notion of continuity. The yellow region is the δ neighborhood of x_0, the blue region is the ε neighborhood of $f(x_0)$.

There is an alternate definition using sequences which is less intuitive.

Definition 2.3. Let f be a real function, we say f is continuous at x_0 if for every sequence (x_n) that converges to x_0, then

$$\lim_{n\to\infty} f(x_n) = f(x_0). \tag{4}$$

(We assume that the sequence all belongs to the domain of f.)

Remark 2.4. This can be worded slightly differently as "*Continuity preserves convergence of sequences.*"

Example 2.5. We will show that $f(x) = 2x^2 + 1$ is continuous. So, we want to write δ in terms of ε. This will allow us to state "For each ε, there is a δ such that..." which implies continuity. So, let x_0 be some arbitrary real number, we have

$$|f(x) - f(x_0)| = \left| \left[2x^2 + 1\right] - \left[2x_0^2 + 1\right] \right| = 2|x^2 - x_0^2| < \varepsilon \tag{5}$$

We can rewrite this as

$$2|x^2 - x_0^2| \leq 2|x - x_0||x + x_0| < 2\delta|x + x_0| \tag{6}$$

We also have a bound that $|x| < |x_0| + \delta$. We plug this in, we set the result equal to ε so

$$2\delta|x + x_0| \leq 2\delta(|x| + |x_0|) < 2\delta(2|x_0| + \delta) = \varepsilon. \tag{7}$$

We rearrange terms to find

$$\delta^2 + 2|x_0|\delta - \frac{1}{2}\varepsilon = 0. \tag{8}$$

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As we learned from primary school, this is a quadratic equation which has precisely two solutions, we choose the positive one:

$$\delta = \sqrt{x_0^2 + \frac{1}{2}\varepsilon} - |x_0|. \tag{9}$$

This is always greater than zero provided $\varepsilon > 0$, $x_0 \in \mathbb{R}$. Thus f is continuous. QEF.

Example 2.6. Consider

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases} \tag{10}$$

We can "see" that when $x \neq 0$, g is continuous. At $x = 0$, we need to prove it's continuous. So, we see that for each $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$|x - 0| < \delta \Rightarrow |g(x) - 0| < \varepsilon. \tag{11}$$

We can rewrite the relations to be (since $\sin(x) \leq 1$)

$$|g(x)| \leq x^2 = (x)^2 < \delta^2 = \varepsilon. \tag{12}$$

So $\delta = \sqrt{\varepsilon}$ is always positive and nonzero. Thus $g(x)$ is continuous at $x = 0$. QEF.

Theorem 2.7. Let

$$f : D \to \mathbb{R} \tag{13}$$

where $D \subseteq \mathbb{R}$. If f is continuous at $x_0 \in D$, then $|f|$ and kf are continuous at x_0 (where $k \in \mathbb{R}$ is arbitrary).

Proof. Since f is continuous, then for each $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \tag{14}$$

We see that

$$|kf(x) - kf(x_0)| = |k||f(x) - f(x_0)| < |k|\varepsilon \tag{15}$$

so we choose $\varepsilon' = |k|\varepsilon$ for nonzero k. We see then that for each $\varepsilon' > 0$ there is a corresponding $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |kf(x) - kf(x_0)| < \varepsilon' \tag{16}$$

which is necessarily true if f is continuous.

Observe that, for some $a, b \in \mathbb{R}$,

$$|b| \leq |a| + |b - a|. \tag{17}$$

Let $b = f(x)$, $a = f(x_0)$, then

$$|f(x)| < |f(x_0)| + \varepsilon \tag{18}$$

Similarly

$$|a| \leq |b| + |a - b| \Rightarrow |f(x_0)| < |f(x)| + \varepsilon \tag{19}$$

We have

$$-\varepsilon < |f(x)| - |f(x_0| < \varepsilon \tag{20}$$

which implies

$$||f(x)| - |f(x_0)|| < \varepsilon. \tag{21}$$

This implies continuity. $\square$

Theorem 2.8. Let f, g be real functions continuous at x_0. Then $f+g$, fg and (if $g(x_0) \neq 0$) f/g are continuous at x_0.

Proof. Let ε_1, δ_1 be for f, ε_2, δ_2 be for g. So for each $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there is a corresponding $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon_1 \tag{22}$$

and

$$|x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \varepsilon_2. \tag{23}$$

Since ε_i $(i = 1, 2)$ is arbitrary, we will impose the condition that $\varepsilon_i < 1$. We see that by the triangle inequality

$$|(f(x) + g(x)) - (f(x_0) + g(x_0))| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)| < \varepsilon_1 + \varepsilon_2 \tag{24}$$

We can also see that

$$|f(x)g(x) - f(x_0)g(x_0)| < |f(x_0)|\varepsilon_2 + |g(x)| \tag{25}$$

$\square$

References

[1] Kenneth A. Ross. *Elementary Analysis: The Theory of Calculus*. Springer Science and Business Media, 2000.