NOTES ON TOPOLOGY

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ABSTRACT. Topology is the study of boundaries. We usually think of an "open set" as not containing its boundary, and a "closed set" as one that contains its boundary. We generalize these notions in topology suitably.

CONTENTS

1. INTRODUCTION

Loosely speaking, basic topology is concerned with boundaries. What is an open set? When is a set "in one piece"? When can we "put" a set "inside a box"? What is a "boundary"? What is a "neighborhood"? And so on.

The basic approach to introducing topology will be done in three acts. The first act will introduce what is a topology, basic definitions, various operations on sets, neighborhoods, and bases for topologies. The second act will deal with the construction of topological spaces, specifically subspaces and product spaces. The third will deal with a variety of properties a topological space can have (e.g. compactness, connectedness, countability and seperation axioms, etc.).

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Part 1. Act I: Topological Spaces

2. Properties of Sets

Just a review of a few properties of sets, and some basic approaches to proofs.

Proposition 2.1. For any set A

(1) The empty set is a subset of A:
$$\emptyset \subseteq A, \qquad \forall A.$$
(2) The union of A with the empty set is A
$$A \cup \emptyset = A, \qquad \forall A.$$
(3) The intersection of A with the empty set is the empty et
$$A \cap \emptyset = \emptyset, \qquad \forall A.$$
(4) The Cartesian product of A and the empty set is empty
$$A \times \emptyset = \emptyset, \qquad \forall A.$$
(5) The only subset of the empty set is itself
$$A \subseteq \emptyset \Rightarrow A = \emptyset, \qquad \forall A$$
(6) The power set of the empty set is a set containing only the empty set
$$2^{\emptyset} = \{\emptyset\}$$
(7) The empty set has cardinality zero (it has zero elements). Moreover, the empty set is finite
$$|\emptyset| = 0.$$

Given some statement
$$\text{If } P, \text{ then } Q. \tag{2.1}$$
its **contrapositive** is
$$\text{If } Q \text{ is not true, then } P \text{ is not true.} \tag{2.2}$$
Its **converse** is
$$\text{If } P, \text{ then } Q. \tag{2.3}$$
Sometimes it's easier to prove the converse than to prove a given proposition.

To show that two sets A, B are equal, we need to show $A \subset B$ and $B \subset A$.

3. Topological Spaces

Problem. The fundamental aim of topology, for all practical purposes, is to generalize real analysis to sets other than $\mathbb{R}$. We begin by generalising the notion of what an "open set" is. That is, previously we thought of an open set as a set that doesn't include its boundary, for example
$$S = \{x \in \mathbb{R} : -1 < x < 1\} \tag{3.1}$$
is an open interval since it doesn't include -1 or 1. We wish to generalize this notion, so lets begin by introducing a collection of open subsets of a given set X called a "*topology on X*".

Definition 3.1. A **Topology** on a set X is a collection $\mathcal{T}$ of subsets of X having the following properties:

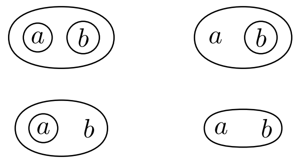

FIGURE 1. An example of all the possible topologies on the set $X = \{a, b\}$ corresponding to (from top to bottom, left to right) the discrete topology, $\{\emptyset, \{b\}, X\}$, $\{\emptyset, \{a\}, X\}$, and the trivial topology.

(1) $\emptyset$ and X are in $\mathcal{T}$
(2) The union of the elements of any subcollection of $\mathcal{T}$ is in $\mathcal{T}$
(3) The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

A set X for white a topology has $\mathcal{T}$ been specified is called a **topological space**.

We generalize the notion of an open set to be any subset $U \subset X$ such that $U \in \mathcal{T}$. This "weakens" our previous notion of an open set, i.e. this topological definition includes our previous intuition about open sets and extends beyond it. We call elements of X "*points*" of the space. In fact, we will *define* open sets as elements of $\mathcal{T}$:

Definition 3.2. Let $(X, \mathcal{T})$ be a topological space. Then the members of $\mathcal{T}$ are defined as **open sets**.

Now this definition of a topological space is rather abstract. Lets consider a few examples using finite sets.

Example 3.3. Consider the set $X = \{a, b\}$ where a, b are arbitrary objects. The possible topologies on this are

(1) $\{\emptyset, \{a, b\}\}$ This is trivially a topology, since it includes X, $\emptyset$. Their union $X \cup \emptyset = X$, their intersection $X \cap \emptyset = \emptyset$ are both in the topology. This is dubbed the "**trivial topology**". The only open sets are empty and X.
(2) $\{\emptyset, \{a\}, \{a, b\}\}$ This is also a topology, since $\{a\} \cap \{a, b\} = \{a\}$, $\{a\} \cup X = X$, $\{a\} \cup \emptyset = \{a\}$, $\{a\} \cap \emptyset = \emptyset$. All of these are in the topology.
(3) $\{\emptyset, \{b\}, \{a, b\}\}$ This is just switching a with b, which is also a topology.
(4) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ This is also a topology, it's all possible subsets of X. This is given a special name known as the "*power set*" or "**discrete topology**". Here, *every* subset is open.

We can see a diagrammatic scheme of these topologies in fig (1).

Example 3.4. This is a nonexample. Consider the set $X = \{a, b, c\}$. The collection

$$\mathcal{T} = \{\emptyset, \{a, b\}, \{b, c\}, X\}. \tag{3.2}$$

Observe that

$$\{a, b\} \cap \{b, c\} = \{b\} \notin \mathcal{T} \tag{3.3}$$

So it's not closed under finite intersections, so $\mathcal{T}$ isn't a topology.

Proposition 3.5. If we have some topological space $(X, \mathcal{T})$, and for each $x \in X$ we have $\{x\} \in \mathcal{T}$, then $\mathcal{T}$ is the discrete topology.

Proof. We want to show that $\mathcal{T}$ is the power set of X. We know that arbitrary unions of subsets of X in $\mathcal{T}$ is also in $\mathcal{T}$, and by hypothesis all singletons are elements of $\mathcal{T}$. It follows that any arbitrary subset of $U \subset X$ can be written as an arbitrary union of singletons

$$U = \bigcup_{x\in U} \{x\}. \tag{3.4}$$

Since this is any arbitrary subset of X, it follows that *every* subset of X is in $\mathcal{T}$. Thus $\mathcal{T}$ is the discrete topology. □

Definition 3.6. Let $\mathcal{T}$, $\mathcal{T}'$ be two topologies on the same set X. We say that

$$\mathcal{T} \supset \mathcal{T}' \iff \begin{cases} \mathcal{T} \text{ is } \textbf{finer} \text{ than } \mathcal{T}' \\ \mathcal{T}' \text{ is } \textbf{coarser} \text{ than } \mathcal{T}. \end{cases} \tag{3.5}$$

Remark 3.7. This notion of "finer" and "coarser" topologies are worthy of explanation. The typical explanation is to think of sand paper, or gravel. Munkres [3] gives the example of having a topology be visualized by a collection of large rocks. If we break it up into finer gravel, we can "union" them together to get back the "coarser topology" and it takes up less volume. Intuitively, this is a "finer" configuration of pebbles.

4. Basis For Topological Spaces

Problem. Oftentimes, a topology is cumbersome to work with, so we want to make life easier. In linear algebra, we don't work with vectors aribtrarily, we work with linear combinations of basis vectors. We use an analogous concept for topological spaces. We work with basis elements when discussing topological properties, it simplifies life a bit. For us, however, our basis elements are not vectors! Instead, they are a selected collection of open subsets of a given topological space X with some extra properties. They are the "basic building blocks" for a topology, so intuitively if all the "building blocks" share a property, the topology should have it too.

Definition 4.1. If X is a set, a **basis for a topology** on X is a collection $\mathscr{B}$ of subsets of X (or "**basis elements**") such that

(1) For each $x \in X$, there is at least one basis element B containing x.
(2) If x belongs to the intersection of two basis elements B_1, B_2, then there is a basis element B_3 such that

$$x \in B_3 \subset B_1 \cap B_2. \tag{4.1}$$

If $\mathscr{B}$ satisfies these two conditions, then we define the **topology generated by the basis** $\mathscr{B}$ by first saying an open set $U \subseteq X$ is in the "topology generated" by $\mathscr{B}$ iff $\forall x \in U$, there is a $B \in \mathscr{B}$ such that $x \in B \subseteq U$.

Remark 4.2. We can alternatively say that if an open set U in the "topology generated" by $\mathscr{B}$, then

$$U = \bigcup_{\alpha\in J} B_\alpha \tag{4.2}$$

where J is some indexing set, $B_\alpha \in \mathscr{B}$ are basis elements.

Remark 4.3. A few notes on the properties of the basis for a topology.

(1) The first property is basically saying the union of all our basis elements covers X. There are no "holes" in our topology.
(2) The second property is a bit more abstract. We want intersections to be covered by some number of basis elements. If we didn't have this property, the intersection of two basis elements wouldn't be an open set, which implies that basis elements are not open sets. This is bad, it would imply that the "topology generated" by a basis isn't really a topology!

Problem. Note that the definition of the "topology generated" is not yet shown to be a topology, we need to prove that it is a topology.

Theorem 4.4. Let $\mathscr{B}$ be a basis for a topology on X. Then $\mathcal{T}$, the "topology generated" by $\mathscr{B}$, is a topology.

Proof. We need to prove that the "topology generated" by a given basis satisfies the properties of a topology. We will do this property by property. Let $\mathscr{B}$ be the basis for a topology on X, $\mathcal{T}$ by the "topology generated" by $\mathscr{B}$.

(1) (Contains $\emptyset$, X)
   (a) We see that $\emptyset \in \mathcal{T}$ is vacuously true.
   (b) We see that for each $x \in X$ there is a $B_x \in \mathscr{B}$ such that $x \in B_x$. It follows that
   $$X = \bigcup_{x \in X} \{x\} \subseteq \bigcup_{x \in X} B_x \tag{4.3}$$
   Since each $B_x \subseteq X$, it follows that
   $$\bigcup_{x \in X} B_x \subseteq X \tag{4.4}$$
   Thus the union of all basis elements is X.

   So both $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.
(2) (Closure under arbitrary unions) We see that if $\{U_\alpha\}_{\alpha \in J}$ is a collection of open sets in $\mathcal{T}$ and J is some index set, then we can write
   $$\bigcup_{\beta \in I_\alpha} B_\beta = U_\alpha \tag{4.5}$$
   by virtue of the definition of an open set in the "topology generated" by $\mathscr{B}$. If
   $$I = \bigcup_{\alpha \in J} I_\alpha \tag{4.6}$$
   then
   $$\bigcup_{\beta \in I} B_\beta = \bigcup_{\alpha \in J} \left(\bigcup_{\beta \in I_\alpha} B_\beta \right) = \bigcup_{\alpha \in J} U_\alpha \tag{4.7}$$
   is also, by definition, a set in $\mathcal{T}$.
(3) (Closure under finite intersections) Let $\{U_\alpha\}_{\alpha=1}^{n}$ be some finite collection of open sets in $\mathcal{T}$. We see that by definition of an open set in the "topology generated" by $\mathscr{B}$
   $$U_\alpha = \bigcup_{\beta \in I_\alpha} B_\beta \tag{4.8}$$

for some index set I_α, and basis elements $B_\alpha \in \mathscr{B}$. We see then that the finite intersection is then

$$\bigcap_{\alpha=1}^{n} U_\alpha = \bigcap_{\alpha=1}^{n} \left(\bigcup_{\beta \in I_\alpha} B_\beta \right). \tag{4.9}$$

Let

$$I = \bigcap_{\alpha=1}^{n} I_\alpha \tag{4.10}$$

then

$$\bigcap_{\alpha=1}^{n} U_\alpha = \bigcup_{\beta \in I} B_\beta. \tag{4.11}$$

But this is by definition an open set in the "topology generated" by $\mathscr{B}$. Thus the "topology generated" $\mathcal{T}$ by $\mathscr{B}$ satisfies all the properties of a topology. □

Lemma 4.5. Let X be a set, let $\mathscr{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T}$ equals the collection of all unions of elements of $\mathscr{B}$.

Proof. Given any arbitrary $U \in \mathcal{T}$, we can write it as

$$U = \bigcup_{x \in U} B_x \tag{4.12}$$

where $x \in B_x \subseteq U$ is a basis element. We know such an element exists by property 1 of a basis element, and we know it is contained in U by property 2 of a basis element.

We have that any element of $\mathcal{T}$ can be written as an arbitrary union of basis elements, we need to show that it contains every union of basis elements. We know by definition, every basis element is open. We also know that arbitrary union of open sets is open. By definition, open sets are elements of a specified topology. Bases are defined for a given topology, which implies the arbitrary union of basis elements is in the given topology. So every union of basis elements is in the topology $\mathcal{T}$. □

Remark 4.6. This lemma basically says that any open set U can be written as a union of basis elements. Note that it does not specify this union is necessarily *unique*. This stands in stark contrast to linear algebra, where a vector *is* a unique linear combination of basis elements. It is an unfortunate fact of life that topologists choose poor definitions.

Remark 4.7. So, just to review, if we have a basis $\mathscr{B}$ for the topological space $(X, \mathcal{T})$, if $U \subset X$, and if for each $x \in U$ there is a $B_x \in \mathscr{B}$ such that $x \in B_x \subset U$, then U is a union of basis elements.

4.1. **Given a Topological Space, Finding a Basis.**

Problem. But now that we have some notion of a "basis" for a topology, the question is "How can we find a basis for a given topology?"

Lemma 4.8. Let $(X, \mathcal{T})$ be a topological space. Suppose that $\mathscr{C}$ is a collection of open sets of X such that for each open set U of X and each $x \in U$, there is an element $C \in \mathscr{C}$ such that $x \in C \subset U$. Then $\mathscr{C}$ is a basis of X.

Proof. We need to do two things. First we need to show that $\mathscr{C}$ is a basis. Second, we need to show that the topology generated by $\mathscr{C}$, call it $\mathcal{T}'$, is really just $\mathcal{T}$.

(1) We see that given each $x \in X$, there is at least one $C \in \mathscr{C}$ containing it. This is, by hypothesis, true. We need to show that if $x \in C_1 \cap C_2$, then there is a $C_3 \in \mathscr{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. We see that $C_1 \cap C_2$ is open, so by hypothesis there is a $C_3 \subseteq C_1 \cap C_2$ such that $x \in C_3$. This means that $\mathscr{C}$ is a basis.

(2) We need to show that $\mathcal{T}' = \mathcal{T}$.
   (a) We see that any element $U \in \mathcal{T}$ is such that for some $x \in U$, there is a $C \in \mathscr{C}$ such that $x \in C \subset U$ so $U \in \mathcal{T}'$ by definition.
   (b) We also see that any element $V \in \mathcal{T}'$ is such that it is the union of basis elements in $\mathscr{C}$, by lemma (4.5). But each $C \in \mathscr{C}$ is such that $C \in \mathcal{T}$. This means that V (as a union of elements of $\mathcal{T}$) is also in $\mathcal{T}$.

   This implies that $\mathcal{T} = \mathcal{T}'$, as desired.

Thus $\mathscr{C}$ is a basis for the topology $\mathcal{T}$ on X. □

4.2. **Example of Usefulness of Bases.**

Problem. In the beginning of this section, we mentioned that it is useful to use bases instead of topologies since they allow us to prove properties faster. That is, if we can prove some topological property holds on each basis element (or equivalently, an arbitrary basis element), then it holds for the topology. We will illustrate the use of bases in making such a claim.

Lemma 4.9. Let $\mathscr{B}$, $\mathscr{B}'$ be bases for the topologies $\mathcal{T}$, $\mathcal{T}'$ (respectively) on X. Then the following are equivalent:

(1) $\mathcal{T}'$ is finer than $\mathcal{T}$.
(2) For each $x \in X$ and each basis element $B \in \mathscr{B}$ containing x, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$.

Proof. (2) $\Rightarrow$ (1). We see that an element $U \in T$ can be written as the union of basis elements $\{B_\alpha\}_{\alpha \in J}$. But since, for each $x \in U$, there is a $B'_\alpha \in \mathscr{B}'$ such that $x \in B'_\alpha \subset B_\alpha$. Then $B'_\alpha \subset U$, which implies $U \in \mathcal{T}'$.
(1) $\Rightarrow$ (2). Given $x \in X$, and a $B \in \mathscr{B}$ that contains x. We see that $B \in \mathcal{T}$. But by (1), $\mathcal{T} \subset \mathcal{T}'$ so $B \in \mathcal{T}'$. So consider B as an open set that coincidentally contains a point x. Then by definition of $\mathscr{B}'$ as a basis, there is a basis element $B' \in \mathscr{B}'$ such that $x \in B' \subset B$. □

Remark 4.10. The gist of this lemma boils down to this: given two bases on a given set, and one is "finer" than the other, the topology generated by the "finer" basis is finer than the topology generated by the "coarser" basis.

4.3. **A "Sub-Basis".**

Definition 4.11. Let $(X, \mathcal{T})$ be a topological space. A **Subbasis** for the topology on X is the collection of subsets of X whose union equals X. The **Topology Generated by a Subbasis** is defined as the collection $\mathcal{T}$ of all unions of finite intersections of subbasis elements.

Remark 4.12. In the author's opinion, the previous definition is a bad one. It appeals too much to linear algebraic intuition of a subbasis as a basis of a subspace, and that is not the case here. Instead, it should really have the intuition of a *pre*basis. That is, after a bit of manipulation we can get a basis. But it is not as though anyone can change this definition now!

5. Continuous Functions

Problem. One of the interesting properties we had with functions in real analysis was the notion of continuity. Is there a way to generalize this notion to a topological setting?

Definition 5.1. (Real Analysis Definition of Continuous) Let $f : X \to Y$ be a function, X and Y be sets. We say that f is "**Continuous**" if for each $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon \tag{5.1}$$

Or in other words, for each ε neighborhood of $f(x_0)$, there is a δ neighborhood of x_0.

Remark 5.2. Observe that what we are doing is specifying a neighborhood in the range of a certain size, then demanding there is a corresponding neighborhood in the domain of a certain size. We can do this in real analysis since the reals are sufficiently nice (they are a metric space which is a really really strong condition for a topological space). We want to generalize this for topological spaces so when we work with metric spaces we recover our previous notion of continuity.

Definition 5.3. Let X, Y be topological spaces, $f : X \to Y$ be a function. We say that "f **is continuous at** x_0" if for every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subset V$.

Remark 5.4. This notion of continuity is nearly identical to the notion we previously introduced from real analysis. The difference is that we are not using a metric to keep track of every open neighborhood in the domain and the range.

But observe we demand each open neighborhood V of $f(x)$ has a corresponding neighborhood U in the domain such that the image of U is contained in V. This is precisely what we did in the real analysis case, we specified an ε neighborhood, and demanded that for each ε neighborhood there is a corrsponding δ neighborhood in the domain such that the image of the δ neighborhood is contained in the ε neighborhood.

Theorem 5.5. Let X and Y be topological spaces, if $f : X \to Y$ is such that for each open set $V \subset Y$ its preimage $f^{-1}(V) \subset X$ is open, then f is continuous.

Proof. Trivial. □

Remark 5.6. (Inverse Functions) Note that contrary to appearances *continuity does not demand the function be invertible!* The preimage of a set *is not* the same as the function's inverse. Consider $f : X \to Y$, and $Z \subset Y$ is some open set. Then

$$f^{-1}(Z) = \{x \in X : \; f(x) \in Z\} \tag{5.2}$$

which does not say anything about the existence of an inverse for f.

Definition 5.7. Let X, Z be topological spaces, and $Y \subset X$ be a subspace. A map

$$\begin{array}{rccc} i: & Y & \hookrightarrow & X \\ & x & \mapsto & i(x) = x \end{array} \tag{5.3}$$

is defined as the "**Inclusion Map**". Similarly, a continuous map

$$
\begin{array}{rccl} r: & X & \hookrightarrow & Y \\ & x & \mapsto & r(x) = x \text{ if } x \in Y \end{array} \tag{5.4}
$$

is defined as the "**Retraction**" if the retraction of i to Y is the identity on A.

Remark 5.8. We use the inclusion map and the retraction to extend and restrict functions (respectively) by composing them with the functions of interest. Note that the restriction of a continuous function (i.e. composing it with a retraction) is the composition of two continuous functions and thus continuous.

Definition 5.9. Let X, Z be topological spaces, $Y \subset X$ be a subspace. If $f : X \to Z$ is a function, we can define the "**Restriction of f to Y**" as

$$
f \circ r : X \to Z \tag{5.5}
$$

the composition of the retraction map (which is just id_{Y} on Y) with f.

Remark 5.10. Observe that the restriction of a function is just composition of functions.

Lemma 5.11. (Pasting Lemma) Let X and Y be topological spaces, and U, V be open sets in X such that $X = U \cup V$. Suppose $f : U \to Y$ and $g : V \to Y$ are continuous and

$$
f(x) = g(x) \quad \text{for all } x \in U \cap V \tag{5.6}
$$

Then the function $h : X \to Y$ defined by

$$
h(x) = \begin{cases} f(x), & \text{if } x \in U \\ g(x), & \text{if } x \in V \end{cases} \tag{5.7}
$$

is continuous.

6. Closed Sets

Problem. We defined what it means for a set to be "open" by just demanding it be in a topology on a space. So how do we define a set to be "closed"? Intuitively, it's an "open set that contains its boundary", but what is a "boundary" topologically?

Definition 6.1. Let X be a topological space, $A \subset X$ be some subset. We say that A is "**Closed**" if $X - A$ is open.

Remark 6.2. This is kind of weaseling out of the problem, just define a set to be closed if its compliment is open. This doesn't seem intuitive or immediately clear why one would want to define it this way, but we will see why later on.

Definition 6.3. Let X be a topological space, $Y \subset X$.

(1) The **Closure** of Y (denoted $\overline{Y}$ or closure(Y)) is the intersection of all closed sets containing Y.
(2) The **Interior** of Y (denoted Y^0 or int(Y)) is the union of all open sets contained in Y.

Remark 6.4. Observe that the closure of a set is closed, the interior of a set is open.

Remark 6.5. The closure of a set is the "smallest" closed set containing it, but we conveniently avoided the problem of "What do we mean by 'smallest'?" by using intersections of closed sets. Similarly, the interior of a set is the "largest" open set contained in it.

Remark 6.6. A set Y is closed iff $Y = \overline{Y}$ and it is open iff $Y = Y^0$.

Definition 6.7. Let X be a topological space, $A \subset X$ be some subset. We define the "**Boundary**" of A (denoted ∂A) to be

$$\overline{A} \cap \overline{(X - A)} \tag{6.1}$$

the intersection of the closure of A with the closure of the compliment of A.

Example 6.8. Consider the set $X = \{a, b, c\}$. Consider the topology

$$\mathcal{T} = \{\emptyset,\ \{a\},\ \{b\},\ \{a, b\},\ X\} \tag{6.2}$$

We see that $\{a, c\}$ is closed since its compliment $\{b\}$ is open, and we also see that $\{b, c\}$ is closed since its compliment $\{a\}$ is open. The intersection of these two closed sets $\{c\}$ is closed, since its compliment is open and the intersection of closed sets is closed.

Observe the closure of $\{a\}$ is given by the intersection of all closed sets containing a:

$$\{a, c\} \cap \{a, b, c\} = \{a, c\} \tag{6.3}$$

and the closure of $\{b, c\}$ (that is, the compliment of $\{a\}$) is itself, i.e. $\{b, c\}$ is a closed set so its closure is itself. The boundary of $\{a\}$ is then the intersection of these two sets

$$\partial\{a\} = \{b, c\} \cap \{a, c\} = \{c\} \tag{6.4}$$

which we could not have found if we didn't define the boundary in a topological way!

Remark 6.9. Let X be a topological space, $A \subset X$ be a subset. The definition of the boundary of A is the same as the closure of A minus its interior

$$\partial A = \overline{A} - A^0. \tag{6.5}$$

How can we see this? Well, the interior of A is the union of all open sets contained in A. Its compliment would be the closure of $X - A$. We see that the intersection of $\overline{(X - A)}$ with $\overline{A}$ is just the closure of A intersected with the compliment of its interior, i.e.

$$\overline{A} \cap \overline{(X - A)} = \overline{A} \cap \left(A^0\right)^C = \overline{A} - A^0. \tag{6.6}$$

This is by virtue of the property of compliments

$$(A - B)^C = A \cap B^C \tag{6.7}$$

where A and B are subsets of some set U.

Definition 6.10. Let X be a topological space, $Y \subset X$. A point $x \in X$ is a **Limit Point** of Y if every neighborhood of x intersects $Y - \{x\}$.

Remark 6.11. This definition makes more sense given the notion of what a "continuous function" is, since we defined continuity in the real analysis situation as the limit as $f(x)$ approaches $f(x_0)$. We take this notion, and use the topological notion of continuity to concoct a *topological* notion of a limit.

Part 2. Act II: Construction of Topological Spaces

7. The Order Topology

Problem. Topologies seem more or less abstract. Isn't there some way to simplify specifying open sets? We are going to introduce the notion of specifying open sets using open intervals. How do we specify open intervals? By using an order relation $a < b$, we can specify open intervals.

If X is a simply ordered set, there is a standard topology for it defined using the order relation. It's called the **order topology**.

We should probably first generalize the notion of intervals familiar from real analysis. Since X is a simply ordered set, there is a (simple) order relation $<$. So suppose we have $a, b \in X$ such that $a < b$, then we have 4 possible subsets of X:

$$(a,b) = \{x | a < x < b\} \tag{7.1a}$$

$$(a,b] = \{x | a < x \leq b\} \tag{7.1b}$$

$$[a,b) = \{x | a \leq x < b\} \tag{7.1c}$$

$$[a,b] = \{x | a \leq x \leq b\}. \tag{7.1d}$$

We call the eq (7.1a) an "open interval", eqs (7.1b) (7.1c) "clopen intervals", and lastly eq (7.1d) a "closed interval".

Definition 7.1. Let X be a set with a simple order relation. Suppose X has more than one element. Let $\mathscr{B}$ be the collection of all sets of the following types:

1. All open intervals (a, b) in X.
2. All intervals of the form $[a_0, b)$, where a_0 is the smallest element (if any) of X.
3. All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X.

The collection $\mathscr{B}$ is a basis for a topology on X, which is called the **order topology**.

Remark 7.2. If X has no smallest element, there are no sets of type (2), and if X has no largest element there are no sets of type (3).

Definition 7.3. If X is an ordered set, and a is an element of X, there are four subsets of X that are called the **rays** determined by a. They are the following

$$(a,+\infty) = \{x | x > a\} \tag{7.2a}$$

$$(-\infty,a) = \{x | x < a\} \tag{7.2b}$$

$$[a,+\infty) = \{x | x \geq a\} \tag{7.2c}$$

$$(-\infty,a] = \{x | x \leq a\} \tag{7.2d}$$

The first two are **open rays**, the last two are **closed rays**.

8. The Product Topology

Problem. If we have two topological spaces X and Y, can we "glue" them together? That is, given two topological spaces, we wish to construct a new one using only what we know of X and Y.

Definition 8.1. Let X, Y be topological spaces. The **Product Topology** on $X \times Y$ is the topology having as basis the collection $\mathscr{B}$ of all sets of the form $U \times V$, where U is an open subset of X and V is an open subset of Y.

Remark 8.2. Being rigorous, we should probably verify that this $\mathscr{B}$ beast really is a basis. For any $x \times y \in X \times Y$, we see that $X \times Y \in \mathscr{B}$ so the first condition is trivially satisfied. The second condition, let $(x \times y) \in U_1 \times V_1$ and $(x \times y) \in U_2 \times V_2$. We see that $\mathscr{B}$ is the collection of the product of all open subsets, which allows us to see that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2) \tag{8.1}$$

is also the product of open sets, so it's a basis element. Thus the second property of a basis is satisfied.

Theorem 8.3. *If $\mathscr{B}$ is a basis for the topology of X and $\mathscr{C}$ is a basis for the topology of Y, then the collection*

$$\mathscr{D} = \{B \times C | B \in \mathscr{B}, C \in \mathscr{C}\} \tag{8.2}$$

is a basis for the topology of $X \times Y$.

Proof. We will use lemma 4.8 to prove this. That is, for some open set $W \subset X \times Y$ and each $x \times y \in W$ there is a basis element $U \in \mathscr{B}$ such that $x \in U$ and a basis element $V \in \mathscr{C}$ such that $y \in V$, so $x \times y \in U \times V \subset W$ and $U \times V \in \mathscr{D}$. Thus $\mathscr{D}$ is a basis. □

Definition 8.4. Let $\pi_1 : X \times Y \to X$ be defined by the equation

$$\pi_1(x, y) = x; \tag{8.3}$$

let $\pi_2 : X \times Y \to Y$ be defined by the equation

$$\pi_2(x, y) = y. \tag{8.4}$$

The maps π_1 and π_2 are called the **projections** of $X \times Y$ onto its first and second factors, respectively.

Remark 8.5. Observe that projections are surjective provided that both X and Y are nonempty. If one is empty, $X \times Y$ is empty too, and everything becomes trivial.

Remark 8.6. Note that if $U \subset X$ is open, $\pi_1^{-1}(U) = U \times Y$. Similarly, if $V \subset Y$ is open, $\pi_2^{-1}(V) = X \times V$. Their intersection is $U \times V$.

Theorem 8.7. *The collection*

$$\mathscr{S} = \{\pi_1^{-1}(U) | U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) | V \text{ is open in } Y\} \tag{8.5}$$

is a subbasis for the product topology on $X \times Y$.

Proof. Let $\mathcal{T}$ be the topology on $X \times Y$, $\mathcal{T}'$ be the topology generated by our subbasis. We see that each element $W \in \mathscr{S}$ is an open subset of the product topology, which means that the topology generated by $\mathscr{S}$ is contained in $\mathcal{T}$, i.e.

$$\mathcal{T}' \subset \mathcal{T}. \tag{8.6}$$

We need to show that $\mathcal{T} \subset \mathcal{T}'$. We see given a basis element $U \times V$ for $\mathcal{T}$ that

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V) \tag{8.7}$$

which is nothing more than a finite intersection of subbasis elements. This implies that $\mathcal{T} \subset \mathcal{T}'$ as desired. □

Part 3. Act III: Properties of Topological Spaces

9. Connectedness

Problem. How can we tell if a space is "connected" or not? That is, if it's "disconnected", then we should be able to "break it up" into independent pieces. How do we make this notion of "disconnectedness" rigorous?

Definition 9.1. Let X be a topological space. A **Seperation** of X is a pair U, V of disjoint, nonempty, open sets whose union is X. The space X is said to be "**Connected**" if there is no seperation in it.

Remark 9.2. Observe that we didn't really define connectedness. Instead we defined what it means for it to have a seperation, and then proceeded to define "connected" as "not seperated". So, to prove a topological space is connected is equivalent to disproving the existence of a seperation. Consequently, we will use proof by contradiction a lot when dealing with connectedness.

Remark 9.3. Observe that if X is connected and Y is homeomorphic to X, then Y is necessarily connected.

Remark 9.4. A space X is connected if and only if the only closed subsets of X that are both open and closed are X and $\emptyset$. Otherwise if we had some additional set U that is closed and open, then $X - U$ is also closed and open. But U and $X - U$ are disjoint, nonempty, open subsets of X whose union is X. This couldn't happen if X were connected, so U needs to be empty or equal to all of X.

Lemma 9.5. If Y is a subspace of X, a seperation of Y is a pair of disjoint nonempty sets A and B whose union is Y, neither of which contains a limit point of the other. The space Y is connected if there exists no seperation of Y.

Proof. Let A, B form a seperation of Y. We see that

$$A = \overline{A} \cap Y \subset Y \tag{9.1}$$

and

$$A^C = (\overline{A} \cap Y)^C = B, \tag{9.2}$$

which implies

$$(\overline{A} \cap Y) \cap B = \emptyset. \tag{9.3}$$

We can argue similarly for $(\overline{B} \cap Y) \cap A = \emptyset$. □

Proposition 9.6. Let $\mathcal{T}$, $\mathcal{T}'$ be two topologies on X. If $\mathcal{T} \subset \mathcal{T}'$, then

(1) the existence of a seperation in $\mathcal{T}$ implies existence of a seperation in $\mathcal{T}'$;
(2) connectedness in $\mathcal{T}'$ implies connectedness in $\mathcal{T}$.

Proof.

(1) Observe that if A, B forms a seperation in the $\mathcal{T}$ topology, then $A, B \in \mathcal{T} \subset \mathcal{T}'$. So the seperation is in the $\mathcal{T}'$ topology too.
(2) Observe that if X is connected in the $\mathcal{T}'$ topology, then there is no seperation (no pair A, B that form a seperation of X that live in $\mathcal{T}'$). This means that no such A, B exist in $\mathcal{T}$, which implies there is no seperation of X in the $\mathcal{T}$ topology. So X is connected in the $\mathcal{T}$ topology.

□

Problem. How can we *construct* connected topological spaces? This seems especially daunting since we only can really know if a topological space is *disconnected*.

Theorem 9.7. If the sets C and D form a seperation of X, and if Y is a connected subspace of X, then Y lies entirely in C or entirely in D.

Proof. Assume for contradiction that Y can lie in both C and D. Then we can observe that $C \cap Y$ and $D \cap Y$ are subsets of Y that are both open and closed, as well as disjoint and their union would be Y. How can we see this last point? Well, we know since C and D form a seperation of X that

$$C \cup D = X \tag{9.4}$$

so it follows that

$$(C \cap Y) \cup (D \cap Y) = (C \cup D) \cap Y = X \cap Y = Y. \tag{9.5}$$

We have our contradiction, it implies Y has a seperation. We reject our assumption and say that Y has to lie entirely in C or entirely in D. □

Theorem 9.8. The union of connected subspaces of X with a common shared point is connected.

Proof. Let $\{A_\alpha\}$ be our collection of connected subspaces,

$$Y = \bigcup_\alpha A_\alpha \tag{9.6}$$

be the subspace we are trying to show is connected,

$$\{y_0\} = \bigcap_\alpha A_\alpha \tag{9.7}$$

be the point common to all connected subspaces.

Assume for contradiction Y is disconnected. Then there exists a pair C, D of disjoint, nonempty, open subsets of Y whose union is Y. Since A_α is connected, by our previous theorem, it must lie entirely in C or D.

Since all A_α share a common point, it implies that all A_α lie entirely in C or they all lie entirely in D. If some lived entirely in C while the rest entirely in D, there would be no shared point y_0 since C and D are disjoint.

So it follows that their union lies entirely in one. But this means that either $C = \emptyset$ and $D = Y$, or $C = Y$ and $D = \emptyset$, which is a contradiction of our assumption that $C \neq \emptyset$ and $D \neq \emptyset$. □

Theorem 9.9. Let A be a connected subspace of X. If $A \subset B \subset \overline{A}$, then B is connected.

Proof. Assume for contradiction that B has a seperation. More precisely, there is a pair of nonempty disjoint open sets C, D whose union is B. Since $A \subset B$, and A is connected, then A is contained entirely in C or it is contained entirely in D. Suppose it is contained entirely in C, then

$$C \cup D \subset \overline{A} \;\Rightarrow\; D \subset A' \tag{9.8}$$

where A' is the set of limit points of A. Recall a limit point x is such that

$$x \in \overline{A - \{x\}}. \tag{9.9}$$

But observe that

$$x \in \overline{A - \{x\}} \subset \overline{C - \{x\}} \subset \overline{C} \tag{9.10}$$

but this implies that

$$A' \subset \overline{C} \;\Rightarrow\; B \subset \overline{A} \subset \overline{C} \tag{9.11}$$

or equivalently, $D = \emptyset$ since we previously proven $D \cap \overline{C} = \emptyset$, which is a contradiction. □

Theorem 9.10. The image of a connected space under a continuous mapping is connected.

Proof. Let $f : X \to Y$ be continuous, X be connected. Assume for contradiction there is a seperation (i.e. a pair A, B of nonempty, open, disjoint subsets whose union is Y) in Y. Their preimage would be a seperation in X, which contradicts our hypothesis that X is connected. We reject our assumption that Y has a seperation and conclude it is connected. □

Theorem 9.11. A finite Cartesian product of connected spaces is connected.

Proof. We will do a sort of inductive proof by contradiction.
Base Case ($n = 2$)**:** Lets consider two connected spaces X_1 and X_2. Let

$$Y = X_1 \times X_2. \tag{9.12}$$

For contradiction, assume Y is disconnected. Then there is a pair A, B of disjoint, nonempty subsets of Y whose union is Y. This means that either $\pi_1(A)$, $\pi_1(B)$ forms a seperation of X_1 or $\pi_2(A)$, $\pi_2(B)$ forms a seperation of X_2. But we assumed that X_i $(i = 1, 2)$ was connected. So we have a contradiction, reject our assumption that Y is disconnected, and conclude Y is connected.
Inductive Hypothesis (arbitrary n): Suppose this works for

$$Y = X_1 \times \cdots \times X_n. \tag{9.13}$$

Inductive Cast ($n + 1$)**:** Lets consider $X_1, \ldots, X_{n+1}$ connected spaces, let

$$X = X_1 \times \cdots \times X_n \tag{9.14}$$

and

$$Y = X \times X_{n+1}. \tag{9.15}$$

Then this is just the base case, and Y was shown to be connected. □

Definition 9.12. A space is "**Totally Disconnected**" if its only connected subspaces are one-point sets.

Theorem 9.13. If X is a topological space equipped with the discrete topology, then it is totally disconnected.

Proof. Since X has the discrete topology, we see that every subset of X is open and consequently its compliment (being a subset of X) is open, so every subset is also closed. The only subspaces of X which has the only subsets be both open and closed would necessarily be the one-point sets.

(Observe that an arbitrary set $Y \subset X$ with more than one element could be partitioned into two subsets, which are both open and closed. This implies that Y is not connected.)

It then follows by definition that X is totally disconnected. □

Proposition 9.14. Let $Y \subset X$, let X and Y be connected. If A and B form a seperation of $X - Y$, then $A \cup Y$ and $B \cup Y$ are connected.

Proof. We will prove that if $A \cup Y$ has a seperation, it means that X has a seperation in the sense of a seperation in a subspace (it is more general this way – if X isn't a subspace, it's still a kosher seperation).

(1) Let $A \cup B = X - Y$, assume for contradiction that $Z := A \cup Y$ has a seperation $C \cup D = Z$.
(2) We will show that B, $C \cup D$ form a seperation of X. Observe first that $C \cup D = A \cup Y$.
(3) We see that $\overline{B} \cap A = B \cap \overline{A} = \emptyset$ since A, B form a seperation of $X - Y$.
(4) We see that Y is closed since $X - Y = A \cup B$ is the union of open sets (thus open).
(5) We see that $\overline{Y} \cap B = Y \cap B = \emptyset$.
(6) We see that $\overline{B} \subset X - Y$ so $\overline{B} \cap Y = \emptyset$.
(7) Observe that

$$\overline{Z} \cap B = (\overline{A \cup Y}) \cap B = (\overline{A} \cup \overline{Y}) \cap B = \emptyset \tag{9.16}$$

and similarly

$$Z \cap \overline{B} = (A \cup Y) \cap \overline{B} = \emptyset. \tag{9.17}$$

(8) We see that B and $Z = A \cup Y$ form a seperation of X, implying X is disconnected. This is a contradiction. We have to reject the assumption that Z has a seperation.

A similar argument holds for $B \cup Y$ being connected. □

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