FUNCATIONAL METHODS IN QUANTUM FIELD THEORY

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Contents

1. Path Integrals In Quantum Mechanics

1.1. **Definition via Unitary Time Evolution.** Consider a nonrelativistic particle in one dimension. The Hamiltonian for this system is generically given as

$$H = \frac{p^2}{2m} + V(x). \tag{1.1}$$

Suppose we want to consider the probability amplitude for this particle moving from x_0 to x_1 in some time interval T. We'll denote such an amplitude $U(x_0, x_1; T)$. It is the position representation of the Schrodinger time-evolution operator. In the canonical Hamiltonian formalism, U is computed by the expression

$$U(x_0, x_1; T) = \langle x_1 | e^{-i\widehat{H}T/\hbar} | x_0 \rangle. \tag{1.2}$$

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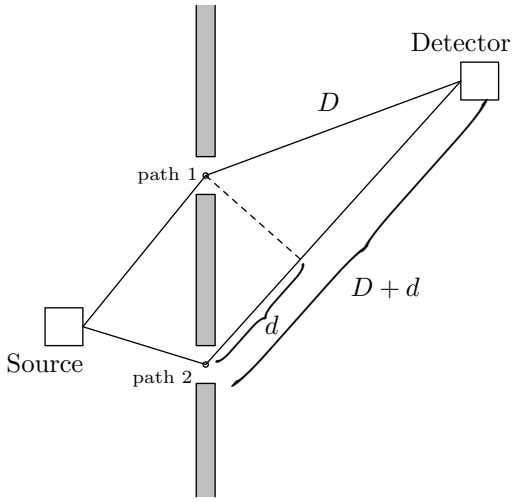

FIGURE 1. The double slit experiment.

The right hand side is read from right to left, like Chinese, and the is interpreted as "The quantum process moving from some initial state $|x_0\rangle$ to some final state $\langle x_1|$ by means of the physical process described by the operator $\exp(-i\hat{H}T/\hbar)$". In the path integral formalism, however, our expression for U is remarkably different. We'll try to give a motivation of the path integral expression of U and then prove it is equal to (1.2).

The fundamental principle that we will be using in the derivation of the path integral formalism is the superposition principle. What do we mean by this? Well, we take the observation summed up thus:

> **Superposition Principle:** when a process can take place in more than one way, its total probability amplitude is the "coherent sum" of the amplitudes of each possible way.

A simple, nontrivial example is the double slit experiment. The amplitude for an electron to arrive at the detector from the source is the sum of the amplitudes of the two paths drawn in figure (1). As the two paths differ in length, there is necessarily interference.

For a general quantum system, we might try to write the total amplitude for traveling from x_a to x_b as:

$$U(x_a, x_b; T) = \sum_{\substack{\text{all paths} \\ \text{from} \\ x_a \text{ to } x_b}} e^{i\cdot(\text{phase})} = \int \mathcal{D}x(t) e^{i\cdot(\text{phase})}. \tag{1.3}$$

Now the physicist's explanation is the following: to be democratic, we have each path be written as a pure phase, so there is no interference and thus no path is "more important than others". There is probably some kernel of truth to this, but it's unrigorous — there's little sign that *a priori* nature is either fair or democratic. A probably more realistic explanation is that any two distinct paths are "orthogonal" in some sense, so they are each different "ways" from the initial state to the final state. Note that the symbol $\int \mathcal{D}x(t)$ should be intuitively interpreted as "sum over

all paths". For our example, all paths that start at x_a and ends at x_b. This sum (in our situation) is a continuous one, i.e. an integral.

Recall that a "*functional*" is a mathematical gadget that "eats in" a function and "spits out" a number. The integrand in (1.3) is a functional, since it "eats in" a path $x(t)$ and it "spits out" a complex number. The argument of a functional will be written in square brackets e.g. $F[x(t)]$, it usually is convention to do so. And just as a function $y(t)$ can be integrated over a set of points, a functional can be integrated over a set of functions. The measure of such a "functional integral" is notationally written with a prefix $\mathcal{D}$. We can also differentiate a functional with respect to its argument, and this "functional derivative" is denoted by $\delta F/\delta x(t)$.

1.2. **Functional Derivatives.** Lets take some time to develop these notions a bit further. We'll set up the analogous situation in vector calculus. Consider the following differential situations

$$\frac{\partial}{\partial x_i} x_j = \delta_{ij}, \quad \text{or} \quad \frac{\partial}{\partial x_i} \sum_j x_j k_j = k_i. \tag{1.4}$$

When we think of a vector as a function from a discrete (usually finite) set of ordinal numbers, we see that the indices are sort of analogous to variables. Now consider the functional $F[x]$ where x is either a single variable or a vector of variables, it doesn't matter (the generalization to many variables is straightforward). We expect by analogy that (in four dimensions)

$$\frac{\delta}{\delta f(x)} f(y) = \delta^{(4)}(x-y), \quad \text{or} \quad \frac{\delta}{\delta f(x)} \int f(y)\phi(y) d^4y = \phi(x). \tag{1.5}$$

This gives us some intuitive understanding of the functional derivative. The notion of a functional integral is questionable in terms of mathematical rigor, but a good reference on the subject is DeWitt-Morette et al. [1]

Now we have the bothersome problem that should be plaguing us: in (1.3) what the devil is that "phase" term?! In answering this question, we should bear in mind we want some limit which will reproduce classical mechanics. We thus expect that only one path — the classical one — contributes to the total amplitude. We may hope to evaluate the functional integral by method of stationary phase, identifying the classical path $x_{cl}(t)$ by the stationary condition

$$\frac{\delta}{\delta x(t)} \left(\text{phase}[x(t)]\right)\Big|_{x_{cl}} = 0. \tag{1.6}$$

But the classical path is the one that satisfies the principle of least action

$$\frac{\delta}{\delta x(t)} \left(S[x(t)]\right)\Big|_{x_{cl}} = 0, \tag{1.7}$$

where $S = \int L dt$ is the classical action. It is very tempting to say that the phase is just S up to some constant. Since the stationary-phase approximation should be valid in the classical limit — i.e. when $S \gg \hbar$ — we need to demand that the phase is $S/\hbar$...or at least that's what we will use. We end up with the conclusion that unitary time evolution in the path integral formalism should thus be

$$U(x_a, x_b; T) = \left\langle x_b \left| e^{i\widehat{H}t/\hbar} \right| x_a \right\rangle = \int \mathcal{D}x(t) e^{iS[x(t)]/\hbar} \tag{1.8}$$

based off of our hand-wavy reasoning involving orthogonal democratic paths and the superposition principle.

1.3. **Connecting Back to Canonical Formalism.** Now, we should be worried that this hand-waviness is complete hog wash. We need to check somehow that we're on the right track, and the usual way we do this is by recovering something from the usual theory (i.e. the Hamiltonian formalism of quantum mechanics). Consider our favorite double slit experiment as in figure 1. The action for both paths would be $(1/2)mv^2t$, the kinetic energy multiplied by time. For path 1, once it gets to the slit, it has to travel a distance D in time t, so it has a velocity

$$v_1 \approx \frac{D}{t} \tag{1.9}$$

so the phase would thus be

$$S \approx \frac{mD^2}{2\hbar t}. \tag{1.10}$$

For path 2, it travels a distance $d + D$ from the slit in time t, so it has velocity

$$v_2 \approx \frac{D+d}{t}, \tag{1.11}$$

and similarly the phase would be

$$S \approx \frac{m(D+d)^2}{2\hbar t}. \tag{1.12}$$

We may assume for argument's sake that $d \ll D$ so that

$$(D+d)^2 = D^2 + 2dD + \mathcal{O}(d^2) \approx D^2 + 2dD \tag{1.13}$$

and moreover $v_1 \approx v_2$. But in our approximation, the excess phase for path 2 is "merely"

$$\frac{mDd}{\hbar t} = \left(\frac{mD}{t}\right)\frac{d}{\hbar} \approx \frac{pd}{\hbar} \tag{1.14}$$

where p is momentum. This is exactly what we should expect from the de Broglie wave relations

$$p = \frac{h}{\lambda} \tag{1.15}$$

which is a good sign that we're on to something!

1.4. **Limit of Discretized Scheme.** Recall for the usual integral, when it is first introduced in college, we partition the domain of integration thus breaking the integral up into a discrete sum. To further investigate the functional integral, we'll do a similar discretization of the time interval $[0, T]$. We actually did this albeit it sloppily, we split it in two and demanded that there are only two possible physical processes (through slit 1 or slit 2). By generalizing to finitely many possible time steps and continuously many positions, we expect to obtain more information of how the functional integral works. We will divide the time interval into N intervals of duration ε, and approximate the trajectory $x(t)$ by a sequence of straight lines (one line that starts and ends at each time slice). Upon discretization, the action becomes

$$S = \int_0^T \left(\frac{m}{2}\dot{x}^2 - V(x)\right) dt \mapsto \sum_k \left[\frac{m}{2}\frac{(x_{k+1}-x_k)^2}{\varepsilon^2} - V\left(\frac{x_{k+1}+x_k}{2}\right)\right]\varepsilon \tag{1.16}$$

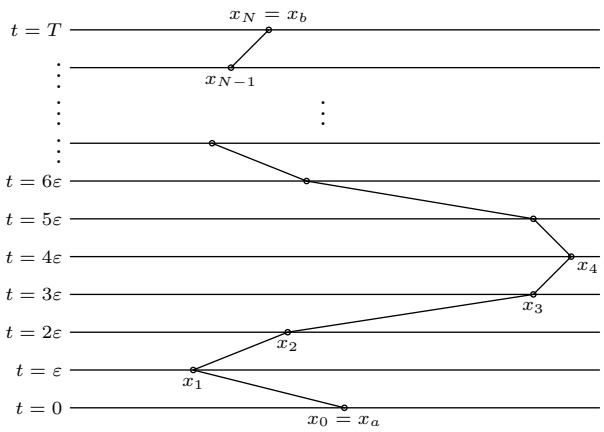

FIGURE 2. The discretized approximation of the path integral

We then define the path integral by taking the limit $\varepsilon \to 0$ of

$$\int \mathcal{D}x(t) \equiv \frac{1}{C(\varepsilon)} \int \frac{dx_1}{C(\varepsilon)} \int \frac{dx_2}{C(\varepsilon)} (\cdots) \int \frac{dx_{N-1}}{C(\varepsilon)} = \frac{1}{C(\varepsilon)} \prod_k \int \frac{dx_k}{C(\varepsilon)} \tag{1.17}$$

where $C(\varepsilon)$ is some constant we'll worry about later. We see this discretization depicted in figure 2. This scheme of setting up a discretized partition then taking the continuum limit is used a lot in functional quantization.

1.5. **Equivalence of Definitions.** Now the astute reader should be asking themselves "Wait, we have defined the function integral twice, what's up with that?" This is true, we first defined it in (1.3) as

$$U(x_a, x_b; T) = \sum_{\substack{\text{all paths} \\ \text{from} \\ x_a \text{ to } x_b}} e^{i\cdot(\text{phase})} = \int \mathcal{D}x(t) e^{i\cdot(\text{phase})}.$$

and again in eq (1.17). Using the second definition, we will prove the validity of the first. Here's the sketch of the proof: we'll show that both are obtained by integrating the same differential equation with the same initial condition. How we'll do this is by considering an individual subinterval in our partition of time as we take the $\varepsilon \to 0$ limit.

To derive the differential equation in question, we'll start by considering the last time slice in our discrete approximation to the path integral. According to both of our definitions, we expect to have

$$U(x_a, x_b; T) = \int_{-\infty}^{\infty} \frac{dx'}{C(\varepsilon)} \exp\left(\frac{i}{\hbar} \frac{m(x_b - x')^2}{2\varepsilon} - \frac{i\varepsilon}{\hbar} V\left(\frac{x_b + x'}{2}\right)\right) \times U(x_a, x'; T - \varepsilon) \tag{1.18}$$

The integral over x' is just the contribution from the last time slice, while the exponential is from the $e^{iS/\hbar}$ of that slice. The contributions from the prior slices are contained in $U(x_a, x'; T - \varepsilon)$.

Now, we take the $\varepsilon \to 0$ limit, and as we do so the rapid oscillations of the first term in the exponential constrains x' to be "very close" to x_b. We can expand eq

(1.18) to be

$$U(x_a,x_b;t)=\int_{-\infty}^{\infty}\frac{dx'}{C(\varepsilon)}\exp\left(\frac{i}{\hbar}\frac{m(x_b-x')^2}{2\varepsilon}\right)[1-\frac{i\varepsilon}{\hbar}V(x_b)+\cdots] \tag{1.19}$$
$$\times[1+(x'-x_b)\frac{\partial}{\partial x_b}+\frac{1}{2}(x'-x_b)^2\frac{\partial^2}{\partial x_b^2}+\cdots]U(x_a,x_b;T-\varepsilon)$$

Observe that we make use of Taylor expanding $U(x_a,x_b;T-\varepsilon)$ on the right hand side, since $x_b-x'\ll 1$. We then look up a few Gaussian integrals

$$\int e^{-b\xi^2}d\xi=\sqrt{\frac{\pi}{b}},\quad \int \xi e^{-b\xi^2}d\xi=0,\quad \int \xi^2 e^{-b\xi^2}d\xi=\frac{1}{2b}\sqrt{\frac{\pi}{b}} \tag{1.20}$$

and then apply these to our time slice to find

$$U(x_a,x_b;T)=\left(\frac{1}{C(\varepsilon)}\sqrt{\frac{2\pi\hbar\varepsilon}{-im}}\right)\left[1-\frac{i\varepsilon}{\hbar}V(x_b)+\frac{i\varepsilon\hbar}{2m}\frac{\partial^2}{\partial x_b^2}+\mathcal{O}(\varepsilon^2)\right] \tag{1.21}$$
$$\times U(x_a,x_b;T-\varepsilon).$$

Observe that the parenthetic term causes problems as we take the limit $\varepsilon\to 0$. We demand that

$$C(\varepsilon)=\sqrt{\frac{2\pi\hbar\varepsilon}{-im}} \tag{1.22}$$

so the parenthetic term becomes unity, and the rest of the expression is well defined. If we now compare terms of order ε and multiply both sides by $i\hbar$ we find

$$i\hbar\frac{\partial}{\partial T}U(x_a,x_b;T)=\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x_b^2}+V(x_b)\right]U(x_a,x_b;T) \tag{1.23a}$$
$$=\hat{H}U(x_a,x_b;T). \tag{1.23b}$$

This is the Schrodinger equation! This is great news, we have another connection to canonical Quantum Mechanics. Observe that U as defined in eq (1.2) satisfies the same Schrodinger equation.

As $T\to 0$ the left hand side of (1.3) tends to $\delta(x_a-x_b)$. Compare this to how our other definition behaves in this limit, specifically in the case of one time slice:

$$\frac{1}{C(\varepsilon)}\exp\left[\frac{i}{\hbar}\frac{m(x_b-x_a)^2}{2\varepsilon}+\mathcal{O}(\varepsilon)\right] \tag{1.24}$$

This is just the peaked exponential of eq (1.19)

$$U(x_a,x_b;t)=\int_{-\infty}^{\infty}\frac{dx'}{C(\varepsilon)}\exp\left(\frac{i}{\hbar}\frac{m(x_b-x')^2}{2\varepsilon}\right)[1-\frac{i\varepsilon}{\hbar}V(x_b)+\cdots]$$
$$\times[1+(x'-x_b)\frac{\partial}{\partial x_b}+\frac{1}{2}(x'-x_b)^2\frac{\partial^2}{\partial x_b^2}+\cdots]U(x_a,x_b;T-\varepsilon)$$

and it also tends to $\delta(x_a-x_b)$ as $\varepsilon\to 0$. Thus we have both of our definitions of the path integral satisfy the same differential equation with the same initial condition, which necessarily implies they are the same. Moreover, we have shown that the canonical expression with the Hamiltonian yields the same results, so we have an explicit connection from the path integral formalism back to the canonical formalism.

1.6. **Generalization to Multiple Dimensions.** We have considered thus far working in one dimension, but that was purely for simplicity (the physics shouldn't change if we work in one or one million dimensions; the math is just simpler in one). We are now going to generalize the path integral to arbitrarily many coordinates and momenta. We have q^i for the position, and p_i for the momentum. We denote the Hamiltonian by $H(q,p)$. Note that when we don't use indices like this, it's to indicate that we are working with all of the position or momenta (depending on the context). The transition amplitude in the canonical formalism should be

$$U(q_a, q_b; T) = \langle q_b | e^{-i\widehat{H}T} | q_a \rangle. \tag{1.25}$$

Note: we have been remarkably careful about explicitly stating $\hbar$, we will now use the standard lazy convention of setting $\hbar = 1$.

We perform the same song and dance, beginning with a discretization in time, specifically N slices of duration ε. We can write

$$e^{-i\widehat{H}T} = e^{-i\sum \varepsilon \widehat{H}} = \underbrace{e^{-i\varepsilon\widehat{H}} e^{-i\varepsilon\widehat{H}} \cdots e^{-i\varepsilon\widehat{H}}}_{(N \text{ times})} \tag{1.26}$$

The trick is to insert a complete set of intermediate states between each of these factors. Here "complete" in the sense that

$$\mathbf{1} = \left(\prod_j \int dq_k^j \right) |q_k\rangle\langle q_k|. \tag{1.27}$$

Note that subscripts refer to the time slice, and superscripts refer to the components of the vector. Inserting these factors for $k = 1, \ldots, N-1$, we are left with a product of factors of the form

$$\langle q_{k+1} | e^{-i\widehat{H}\varepsilon} | q_k \rangle \xrightarrow[\varepsilon \to 0]{} \langle q_{k+1} | 1 - i\varepsilon \hat{H} + \mathcal{O}(\varepsilon^2) | q_k \rangle. \tag{1.28}$$

To express the first and last factors as this form, we define $q_0 = q_a$ and $q_N = q_b$.

Now we will investigate $\hat{H}$. Specifically, we are concerned with what terms it contains. The simplest Hamiltonian to consider is when it is a function of position only, not of momenta. The matrix element of such a term would be

$$\langle q_{k+1} | f(q) | q_k \rangle = f(q_k) \prod_j \delta(q_k^j - q_{k+1}^j). \tag{1.29}$$

It'd be convenient to write this as

$$\langle q_{k+1} | f(q) | q_k \rangle = f\left(\frac{q_k + q_{k+1}}{2} \right) \left(\prod_j \int \frac{dp_k^j}{2\pi} \right) \exp[i \sum_j p_k^j (q_{k+1}^j - q_k^j)], \tag{1.30}$$

for reasons that will hopefully be obvious soon.

We will, for the sake of symmetry, consider a Hamiltonian which is purely a function of momenta. We introduce a complete set of momentum eigenstates to obtain

$$\langle q_{k+1} | f(p) | q_k \rangle = \left(\prod_j \int \frac{dp_k^j}{2\pi} \right) f(p_k) \exp[i \sum_j p_k^j (q_{k+1}^j - q_k^j)] \tag{1.31}$$

So if $\hat{H}$ contains only terms of the form $f(q)$ and $f(p)$, its matrix element can be written

$$\begin{aligned} &\langle q_{k+1}|\hat{H}(q,p)|q_k\rangle = \\ &\left(\prod_j \int \frac{dp_k^j}{2\pi}\right) \hat{H}\left(\frac{q_{k+1}+q_k}{2}, p_k\right) \exp[i\sum_j p_k^j(q_{k+1}^j - q_k^j)]. \end{aligned} \tag{1.32}$$

In general, eq (1.32) is wrong. We usually have mixed terms of order $\mathcal{O}(q^\alpha p^\beta)$ in the Hamiltonian. So why bother considering Hamiltonians of such a form? Because we can pick a sufficiently nice ordering that makes it true in general. For example, this combination

$$\langle q_{k+1}|\frac{1}{4}(q^2p^2 + 2qp^2q + p^2q^2)|q_k\rangle = \left(\frac{q_{k+1}+q-k}{2}\right)^2 \langle q_{k+1}|p^2|q^k\rangle \tag{1.33}$$

works out as desired. This is due to the q's appearing symmetrically on the left hand side and the right hand side in just the right way. When this happens, we say that the Hamiltonian is "*Weyl ordered*". Any Hamiltonian can be Weyl ordered by commuting p's and q's; in general this procedure will introduce extra terms which need to be put on the right hand side of eq (1.32). For more on Weyl ordering, see Ticciatic [2] (specifically pages 335 et seq.).

Supposing that from now on (unless explicitly states otherwise) $\hat{H}$ is Weyl ordered, our typical matrix element from (1.28) can be expressed as

$$\langle q_{k+1}|e^{-i\varepsilon\widehat{H}}|q_k\rangle = \left(\prod_j \int \frac{dp_k^j}{2\pi}\right) \exp\left[-i\varepsilon\hat{H}\left(\frac{q_{k+1}+q_k}{2}, p_k\right)\right] \exp[i\sum_j p_k^j(q_{k+1}^j - q_k^j)]. \tag{1.34}$$

(We use the well known that that as $\epsilon \to 0$, $\epsilon \ll 1$.) To obtain $U(q_a, q_b; T)$ we multiply N such factors together (one for each k) and integrate on our intermediate coordinates q_k:

$$\begin{aligned} U(q_0, q_N; T) = &\left(\prod_{j,k} \int dq_k^j \int \frac{dp_k^j}{2\pi}\right) \\ &\times \exp\left[i\sum_k \left(\sum_j p_k^j(q_{k+1}^j - q_k^j) - \varepsilon\hat{H}(\frac{q_{k+1}+q_k}{2}, p_k)\right)\right]. \end{aligned} \tag{1.35}$$

There is one momentum integral for each k from 0 to $N-1$, and one coordinate integral for each k from 1 to $N-1$.

This expression, we deduce, is the discretized form of

$$U(q_a, q_b; T) = \left(\prod_j \int \mathcal{D}q(t)\mathcal{D}p(t)\right) \exp\left[i\int_0^T (\dot{q}^i p_i - H(q,p))dt\right], \tag{1.36}$$

where the functions $q(t)$ are constrained at the endpoints, but the functions $p(t)$ are not. Note that we don't have any whacky constants in the integration measure $\mathcal{D}q$, unlike the situation in one dimension. The functional measure in eq (1.36) is

just the product of the standard integral over the phase space

$$\prod_j \int \frac{dq^j dp_j}{2\pi} \tag{1.37}$$

at each point in time. We can conclude that eq (1.36) is the most general formula for computing transition amplitudes via functional integration for practical purposes.

2. Path Integral Quantization of Harmonic Oscillator

The functional quantization of the Harmonic oscillator provides a powerful example when generalizing to fields, so it is worth while to first study it in detail. Recall the Hamiltonian of the Harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2 \tag{2.1}$$

In the presence of an external force, this yeilds the path integral expression

$$\langle 0|0\rangle_f = \int \mathcal{D}p\mathcal{D}q \exp\left[i\int_{-\infty}^{\infty}(p\dot{q} - (1-i\varepsilon)H + fq)dt\right] \tag{2.2}$$

where ε is an "infinitesimal" quantity. We see by inspection that multiplication of H by $(1-i\varepsilon)$ is completely equivalent as *BOTH*

$$m^{-1} \mapsto (1-i\varepsilon)m^{-1} \tag{2.3a}$$

$$\frac{m}{2}\omega^2 q^2 \mapsto \frac{(1-i\varepsilon)m}{2}\omega^2 q^2. \tag{2.3b}$$

So this means that

$$\frac{p^2}{2m} = \frac{m}{2}\dot{q}^2 \mapsto \frac{(1+i\varepsilon)m}{2}\dot{q}^2 \tag{2.4}$$

thus transforming the path integral expression to be

$$\langle 0|0\rangle_f = \int \mathcal{D}q \exp\left[i\int_{-\infty}^{\infty}(\frac{m}{2}(1-i\varepsilon)\dot{q}^2 - (1-i\varepsilon)\frac{m}{2}\omega^2 q^2 + fq)dt\right]. \tag{2.5}$$

We will henceforth set $m = 1$.

We find the Fourier transformed quantities

$$\widetilde{q}(E) = \int_{-\infty}^{\infty} q(t)e^{iEt}dt, \qquad q(t) = \int_{-\infty}^{\infty}\widetilde{q}(E)e^{-iEt}\frac{dE}{2\pi} \tag{2.6}$$

Thus we can compute

$$q(t)^2 = \left(\int_{-\infty}^{\infty}\widetilde{q}(E)e^{iEt}\frac{dE}{2\pi}\right)\left(\int_{-\infty}^{\infty}\widetilde{q}(E')e^{iE't}\frac{dE'}{2\pi}\right) \tag{2.7a}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(E+E')t}\widetilde{q}(E)\widetilde{q}(E')\frac{dE}{2\pi}\frac{dE'}{2\pi} \tag{2.7b}$$

and

$$\dot{q}(t)^2 = \int_{-\infty}^{\infty} -EE'e^{-i(E+E')t}\widetilde{q}(E)\widetilde{q}(E')\frac{dE}{2\pi}\frac{dE'}{2\pi}. \tag{2.8}$$

We also compute

$$f(t)q(t) = \frac{1}{2}(f(t)q(t) + q(t)f(t)) \tag{2.9a}$$

$$= \frac{1}{2}\int \left(\widetilde{f}(E)\widetilde{q}(E') + \widetilde{f}(E')\widetilde{q}(E)\right) e^{-i(E+E')t}\frac{dE}{2\pi}\frac{dE'}{2\pi}. \tag{2.9b}$$

We can plug these into eq (2.5) to find

$$\langle 0|0\rangle_f = \int \mathcal{D}q \exp\Big[\int [\left(\frac{(1+i\varepsilon)m}{2}(-EE') - \frac{(1-i\varepsilon)m\omega^2}{2}\right)\widetilde{q}(E)\widetilde{q}(E')$$

$$+\frac{1}{2}\left(\widetilde{f}(E)\widetilde{q}(E') + \widetilde{f}(E')\widetilde{q}(E)\right)]\underbrace{e^{-i(E+E')t}dt}_{=\delta(E+E)}\frac{dE}{2\pi}\frac{dE'}{2\pi}\Big] \tag{2.10a}$$

$$= \int \mathcal{D}q \exp\Big[\int [\left(\frac{(1+i\varepsilon)m}{2}(-EE') - \frac{(1-i\varepsilon)m\omega^2}{2}\right)\widetilde{q}(E)\widetilde{q}(E')$$

$$+\frac{1}{2}\left(\widetilde{f}(E)\widetilde{q}(E') + \widetilde{f}(E')\widetilde{q}(E)\right)]\delta(E+E')\frac{dE}{2\pi}\frac{dE'}{2\pi}\Big] \tag{2.10b}$$

$$= \int \mathcal{D}q \exp\left[\int [\left(\frac{(1+i\varepsilon)m}{2}(E^2) - \frac{(1-i\varepsilon)m\omega^2}{2}\right)\widetilde{q}(E)^2 + \frac{1}{2}\widetilde{f}(E)\widetilde{q}(E)]\frac{dE}{2\pi}\right] \tag{2.10c}$$

This simplifies things slightly.

Lets make a change of variables:

$$\widetilde{x}(E) = \widetilde{q}(E) + \frac{\widetilde{f}(E)}{E^2 - \omega^2 + i\varepsilon} \tag{2.11}$$

which allows us to write the action as

$$S = \frac{1}{2}\int_{-\infty}^{\infty}\left[\widetilde{x}(E)(E^2 - \omega^2 + i\varepsilon)\widetilde{x}(-E) - \frac{\widetilde{f}(E)\widetilde{f}(-E)}{E^2 - \omega^2 + i\varepsilon}\right]\frac{dE}{2\pi} \tag{2.12}$$

But how does this change the measure? Well, according to mathematica, if we suppose that the expression for the external force looks like

$$\widetilde{f}(E) = const \tag{2.13}$$

then the measure changes by

$$dx = \left(1 + \frac{d|q|}{dq}e^{\sqrt{-\operatorname{sgn}(k)}k|q|}\sqrt{-\frac{pi}{2k^2}}\right)dq \tag{2.14}$$

which is just a shift by a constant really. So we find that

$$\mathcal{D}x = \mathcal{D}q. \tag{2.15}$$

We see that this makes the path integral expression

$$\langle 0|0\rangle_f = \underbrace{\exp\left[\frac{-i}{2}\int_{-\infty}^{\infty}\frac{\widetilde{f}(E)\widetilde{f}(-E)}{E^2 - \omega^2 + i\varepsilon}\right]}_{\text{dependent on } \widetilde{f}(E)\text{, external force}}\underbrace{\int \mathcal{D}x \exp\left[\frac{i}{2}\int_{-\infty}^{\infty}\widetilde{x}(E)(E^2 - \omega^2 + i\varepsilon)\widetilde{x}(-E)\frac{dE}{2\pi}\right]}_{\text{independent of external force}} \tag{2.16}$$

So here's the trick: we have two factors, one dependent of the external force, the other independent of the external force. So in a vacuum (i.e. a free Harmonic oscillator) one has $f = 0$. This occurs for $\langle 0|0\rangle_f$. But a Harmonic oscillator that is

not acted on by an external force remains in its ground state $\langle 0|0\rangle_f = 1$. Thus we find

$$\langle 0|0\rangle_f = \exp\left[\frac{-i}{2}\int_{-\infty}^{\infty}\frac{\widetilde{f}(E)\widetilde{f}(-E)}{E^2-\omega^2+i\varepsilon}\right]. \tag{2.17}$$

We want a more intuitive interpretation of this, so it requires performing the Fourier transform to the time domain, yielding

$$\langle 0|0\rangle_f = \exp\left[\frac{i}{2}\int f(t)G(t-t')f(t')dt\ dt'\right] \tag{2.18}$$

where we have introduced the shorthand notation

$$G(t-t') = \int \frac{e^{-iE(t-t')}}{E^2-\omega^2+i\varepsilon}\frac{dE}{2\pi} \tag{2.19}$$

which is the Green's function for the Harmonic oscillator equation of motion. That is, it satisfies

$$\left(\frac{d^2}{dt^2}+\omega^2\right)G(t-t') = \delta(t-t'), \tag{2.20}$$

which can be computed and verified by direct substitution, then taking the $\varepsilon \to 0$ limit. We can also use the method of residues to find

$$G(t-t') = 2\pi i \lim_{E\to\omega}(E-\omega)\frac{1}{2\pi}\frac{e^{-iE(t-t')}}{(E+\omega)(E-\omega)} = \frac{i}{2\omega}\exp\big[-i\omega|t-t'|\big]. \tag{2.21}$$

The trick with plugging this into the Harmonic oscillator equation of motion is that the absolute value, when differentiated, yields the Step function...which when differentiated yields a delta function.

3. Functional Quantization of Scalar Fields

We would like to generalize our functional methods to field theory by generalizing the parametrization from a 1 dimensional parameter (time t) to a 4 dimensional parametrization x^μ. So this means that

$$\begin{aligned} t &\to x^\mu \\ \frac{d}{dt} &\to \partial_\mu \\ q(t) &\to \varphi(x^\mu) \end{aligned}$$

In fact we can set up a dictionary for our generalization

mechanics	field theory	
$q(t)$	$\varphi(\bar{x},t)$	classical field
$\widehat{Q}$	$\varphi(\bar{x},t)$	operator field
$f(t)$	$J(\bar{x},t)$	classical source

We can write the Hamiltonian density for the free scalar field as

$$\mathcal{H}_0 = \underbrace{\frac{1}{2}\Pi^2+\frac{1}{2}(\nabla\varphi)^2}_{\substack{=\partial^\mu\varphi\partial_\mu\varphi \\ =\text{kinetic energy}}} + \underbrace{\frac{1}{2}m^2\varphi^2}_{\text{potential}} \tag{3.1}$$

The trick we've employed so many times before using infinitesimals is done precisely the same way using the Hamiltonian density. That is, $\mathcal{H}_0 \mapsto (1-i\varepsilon)\mathcal{H}_0$. For

simplicity, we tacitly assume we always mean $(1 - i\varepsilon)m^2$ whenever we write m^2, it's completely equivalent to $\mathcal{H}_0 \mapsto (1 - i\varepsilon)\mathcal{H}_0$.

Consider the functional integral

$$Z[J] = \langle 0|0\rangle_J = \int \mathcal{D}\varphi \exp\left[i \int [\mathcal{L}_0 + J\varphi] d^4x\right] \tag{3.2}$$

where

$$\mathcal{L}_0 = \frac{1}{2}\partial^\mu \varphi \partial_\mu \varphi - \frac{1}{2}m^2\varphi^2 \tag{3.3}$$

is the Lagrangian density and

$$\mathcal{D}\varphi \propto \prod_{\substack{\text{space-}\\ \text{time}\\ \text{points } x}} d\varphi(x) \tag{3.4}$$

is the functional measure.

3.0.1. *Remark on Peskin and Schroeder's "Derivation".* In Chapter 9 of Peskin and Schroeder [3] there is a "derivation" of the functional quantization of the scalar field using the limit of a discrete lattice. There is some mathematically dubious aspects of this derivation, for one they use a Fourier series when they should be using a discrete Fourier transform. What difference does this make? Well, the intuition of the discrete Fourier transform is that the field exists only on the nodes of the lattice; the intuition of a Fourier series is that it exists everywhere continuously in a finite universe. The latter is not in spirit with taking the continuum limit! In fact, this beautifully motivates us to read the paper by Wise on chain field theory [4], which resolves this problem by a topological approach which yields the correct continuum limit by taking advantage of recent work on discrete differential geometry.

3.1. **Slick Method of Computing Correlation Functions.**

3.1.1. *Outline.* As promised, we'll introduce a slicker way to compute Feynman rules using functional derivatives. It's a lot more mathematically rigorous (and simpler) than the discretization scheme. The method uses a mathematical gadget which generalizes the notion of a generating function — the generating *functional*. Recall the generating function is used to compute constants and other useful numbers by taking its n^{th} derivative and evaluating it at 0. We *functionally* do the same thing, take the functional derivative of the generating functional and evaluate it at 0.

3.1.2. *Properties of the Functional Derivative.* First lets try to review the properties of the functional $\delta/\delta J(x)$. The functional derivative obeys the basic property (in four dimensions)

$$\frac{\delta}{\delta J(x)} J(y) = \delta^{(4)}(x - y), \quad \text{or} \quad \frac{\delta}{\delta J(x)} \int J(y)\phi(y) d^4y = \phi(x). \tag{3.5}$$

This can be viewed as a continuous generalization of the vector calculus derivative

$$\frac{\partial}{\partial x_i} x_j = \delta_{ij} \quad \text{or} \quad \frac{\partial}{\partial x_i} \sum_j x_j k_j = k_i. \tag{3.6}$$

To take the functional derivatives of more complicated situations, we use the basic properties of the chain rule and the product rule. **Warning:** we **assume without**

proof that these properties hold, we'll not divulge into the proof here. So we have situations like the following:

$$\frac{\delta}{\delta J(x)} \exp\left[i\int J(y)\phi(y)d^4y\right] = i\phi(x)\exp\left[i\int J(y)\phi(y)d^4y\right]. \tag{3.7}$$

When the functional depends on the derivative of J we integrate by parts — and for all practical purposes, we always can integrate by parts in quantum field theory — then apply the functional derivative as follows:

$$\frac{\delta}{\delta J(x)}\int V^\mu(y)\partial_\mu J(y)d^4y = \frac{\delta}{\delta J(x)}\left(\text{bdryterms} + \int J(y)\partial_\mu V^\mu(y)d^4y\right) = -\partial_\mu V^\mu(x). \tag{3.8}$$

This concludes our review of basic properties that we'll use later on.

3.1.3. *Generating Functional.* As alluded to earlier, the basic object of interest is the generating functional of correlation functions. We denote this object of interest by $Z[J]$. In a scalar field theory, it's defined as

$$Z[J] \stackrel{\text{def}}{=} \int \mathcal{D}\phi \exp\left[i\int[\mathcal{L} + J(x)\phi(x)]d^4x\right]. \tag{3.9}$$

This is a functional integral over ϕ. We've merely added to $\mathcal{L}$ in the exponent an extra term $J(x)\phi(x)$, which we usually refer to as the "*source term*".

3.1.4. *Derivation of Correlation Function.* Now we use it to compute the generating functions for the Klein-Gordon field (the free scalar field). For example the two-point function is

$$\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = \frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)Z[J]\Big|_{J=0} \tag{3.10}$$

where $Z_0 = Z[0]$. Each functional derivative brings down a factor of ϕ in the numerator of $Z[J]$; setting $J = 0$ we recover our desired expression. In more explicit detail, we can compute

$$\frac{\delta}{\delta J(x_2)}Z[J] = \frac{\delta}{\delta J(x_2)}\int \mathcal{D}\phi \exp\left[i\int[\mathcal{L} + J(x)\phi(x)]d^4x\right] \tag{3.11a}$$

$$= \int \mathcal{D}\phi \frac{\delta}{\delta J(x_2)}\exp\left[i\int[\mathcal{L} + J(x)\phi(x)]d^4x\right] \tag{3.11b}$$

$$= \int \mathcal{D}\phi i\phi(x_2)\exp\left[i\int[\mathcal{L} + J(x)\phi(x)]d^4x\right] \tag{3.11c}$$

This is the effect of one functional derivative, we multiply by $-i$ to finish one functional operation. We need to do another to get the expression

$$\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)Z[J] = \int \mathcal{D}\phi\phi(x_1)\phi(x_2)\exp\left[i\int[\mathcal{L} + J(x)\phi(x)]d^4x\right]. \tag{3.12}$$

To get the final expression, we need to divide by Z_0 and evaluate at $J = 0$ to get

$$\frac{1}{Z_0}\left(-i\frac{\delta}{\delta J(x_1)}\right)\left(-i\frac{\delta}{\delta J(x_2)}\right)Z[J]\Big|_{J=0} = \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\exp\left[i\int[\mathcal{L}]d^4x\right]}{\int \mathcal{D}\phi\exp\left[i\int\mathcal{L}d^4x\right]}$$

which is precisely what is expected.

3.1.5. *Slicker Way to Compute Correlation Functions.* We've seen that eq (3.9) recovers the expected expression for two-point functions. It's pretty nifty for us since the free scalar field can be written fairly easily. It's explicitly written after integrating by parts (on the first term)

$$\int [\mathcal{L}_0(\phi) + J\phi] d^4x = \int [\frac{1}{2}\phi(-\partial^2 - m^2 + i\varepsilon)\phi + J\phi] d^4x. \tag{3.13}$$

The factor of $i\varepsilon$ is to guarantee convergence. We complete the square by introducing a shifted scalar field

$$\phi'(x) \stackrel{\text{def}}{=} \phi(x) - i\int D_F(x-y)J(y)d^4y \tag{3.14}$$

where $D_F(x-y)$ is the Feynman propagator — i.e. the Green's function of the Klein-Gordon operator, we find that our original expression (3.13) becomes

$$\begin{aligned} &\int [\mathcal{L}_0(\phi) + J\phi] d^4x = \\ &\qquad \int [\frac{1}{2}\phi'(-\partial^2 - m^2 + i\varepsilon)\phi'] d^4x - \int \frac{1}{2} J(x) \left[-iD_F(x-y)\right] J(y) d^4y. \end{aligned} \tag{3.15}$$

More symbolically, we could rewrite the change of variables as

$$\phi' \stackrel{\text{def}}{=} \phi + (-\partial^2 - m^2 + i\varepsilon)^{-1} J, \tag{3.16}$$

and the result becomes

$$\int [\mathcal{L}_0 + J\phi] d^4x = \int \left[\frac{1}{2}\phi'(-\partial^2 - m^2 + i\varepsilon)\phi' - \frac{1}{2} J(-\partial^2 - m^2 + i\varepsilon)^{-1} J\right] d^4x. \tag{3.17}$$

This looks nasty, but we are not done yet. We have a few tricks left.

When we change variables from ϕ to ϕ', it's just a shift, so the Jacobian in the functional integral definition of $Z[J]$ is the identity. The result is

$$\underbrace{\int \mathcal{D}\phi' \exp\left[i\int \mathcal{L}_0(\phi') d^4x\right]}_{Z_0} \underbrace{\exp\left[-i\int \frac{1}{2} J(x)[-iD_F(x-y)J(y)] d^4x d^4y\right]}_{\text{independent of } \phi'} \tag{3.18}$$

As noted, the second integral is independent of ϕ' and the first is precisely Z_0. The generating functional of the free scalar field is thus

$$Z[J] = Z_0 \exp\left[\frac{-1}{2}\int J(x) D_F(x-y) J(y) d^4x d^4y\right] \tag{3.19}$$

Lets use (3.9) and (3.19) to compute some correlation functions. The two-point function is by definition

$$\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = \frac{-\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \exp\left[\frac{-1}{2}\int J(x) D_F(x-y) J(y) d^4x d^4y\right]\Big|_{J=0} \tag{3.20}$$

We can evaluate one functional derivative to find

$$\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = \frac{-\delta}{\delta J(x_1)} \left[\frac{-1}{2}\int D_F(x_2-y) J(y) d^4y - \frac{1}{2}\int J(x) D_F(x-x_2) d^4x\right] \frac{Z[J]}{Z_0}\Big|_{J=0} \tag{3.21}$$

We can see that this is just what happens after evaluating one functional derivative of the definition of the two-point function, which allows us to conclude that

$$\langle 0|T\{\phi(x_1)\phi(x_2)\}|0\rangle = D_F(x_1 - x_2). \tag{3.22}$$

This is good because it connects back to what we should already know.

3.1.6. *Example: Four Point Correlation Functions.* This is a rather space-consuming computation, so we need to introduce some abbreviations. Namely we'll use the conventions that arguments of functions are subscripts: $\phi_1 = \phi(x_1)$, $J_x = J(x)$, $D_{x4} = D_F(x - x_4)$, etc. Repeated subscripts will be integrated over implicitly (like a continuous Einstein's summation convention). The four-point function is thus

(3.23a)

$$\begin{aligned}
\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle &= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}[-J_x D_{x4}]e^{-\frac{1}{2}J_x D_{xy} J_y}\Big|_{J=0} \\
&= \frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}[-D_{34} + J_x D_{x4} J_y D_{y3}]e^{-\frac{1}{2}J_x D_{xy} J_y}\Big|_{J=0} \qquad (3.23b) \\
&= \frac{\delta}{\delta J_1}[D_{34} J_x D_{x2} + D_{24} J_y D_{y3} + J_x D_{x4} J_y D_{23}]e^{-\frac{1}{2}J_x D_{xy} J_y}\Big|_{J=0} \qquad (3.23c) \\
&= D_{34}D_{12} + D_{24}D_{13} + D_{14}D_{23} \qquad (3.23d)
\end{aligned}$$

which is precisely what we expect by Wick's theorem.

3.1.7. *The Beauty of the Generating Functional.* The beauty of the situation is that these calculations are completely independent of whether things are free or interacting. The catch is the $Z[J = 0]$ factor is not trivial in the interacting situation. It gives us the sum of the vacuum diagrams.

4. Functional Quantization of Electromagnetic Field

We assert the Feynman rule for the photon propagator to be

$$\sim\sim\sim\sim = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon}. \tag{4.1}$$

Now that we have the basic tools of functional quantization, lets try to prove it.

Consider the functional integral

$$\int \mathcal{D}A\, e^{iS[A]} \tag{4.2}$$

where $S[A]$ is the action for the free electromagnetic field. The functional integral is taken over the four components, i.e. we have $\mathcal{D}A = \mathcal{D}A^0\mathcal{D}A^1\mathcal{D}A^2\mathcal{D}A^3$. We integrate by parts and Fourier expand to find the action to be

$$\begin{aligned}
S &= \int \left[\frac{-1}{4}F_{\mu\nu}F^{\mu\nu}\right] d^4x \qquad (4.3a) \\
&= \frac{1}{2}\int A_\mu(x)\left[\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu\right] A_\nu(x) d^4x \qquad (4.3b) \\
&= \frac{-1}{2}\int \tilde{A}_\mu(k)\left[-k^2 g^{\mu\nu} + k^\mu k^\nu\right] \tilde{A}_\nu(-k)\frac{d^4k}{(2\pi)^4}. \qquad (4.3c)
\end{aligned}$$

Observe that if $\tilde{A}_\mu(k) = k_\mu \alpha(k)$ for some scalar function α, the integrand becomes

$$\tilde{A}_\mu(k)\left[-k^2 g^{\mu\nu} + k^\mu k^\nu\right]\tilde{A}_\nu(-k) = \alpha(k)k_\mu\left[-k^2 g^{\mu\nu} + k^\mu k^\nu\right](-k_\nu)(\alpha(-k)) \tag{4.4a}$$

$$= \alpha(k)\alpha(-k)\left[k_\mu k_\nu k^2 g^{\mu\nu} - k_\mu k_\nu k^\mu k^\nu\right] \tag{4.4b}$$

$$= \alpha(k)\alpha(-k)[0] = 0. \tag{4.4c}$$

This integrand vanishes for **any** choice of scalar function α. In this situation, the integrand of the functional integral $\int \mathcal{D}A e^{iS[A]}$ is 1, which more importantly implies it is a badly divergent functional integral. We deduce that

$$\begin{aligned} &(\partial^2 g_{\mu\nu} - \partial_\mu\partial_\nu)D_F^{\nu\rho}(x-y) = i\delta_\mu{}^\rho \delta^{(4)}(x-y) \\ &\text{or} \quad (-k^2 g_{\mu\nu} + k_\mu k_\nu)\tilde{D}_F^{\nu\rho}(k) = i\delta_\mu{}^\rho \end{aligned} \tag{4.5}$$

(which importantly defines the Feynman propagator $D_F^{\mu\rho}$) has no solution. This shouldn't surprise anyone since $(-k^2 g_{\mu\nu} + k_\mu k_\nu)$ has a singularity.

The real problem child here is gauge invariance. Recall that $F_{\mu\nu}$ (and thus $\mathcal{L}$) is invariant under a general U(1) gauge transformation of the form

$$A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \tag{4.6}$$

where $\alpha(x)$ is any scalar function. This transformation means that potentials of the form $A_\mu(x) = \frac{1}{e}\partial_\mu\alpha(x)$ is gauge equivalent to 0. The functional integral is badly defined because we are redundantly integrating over a continuous infginity of physically equivalent field configurations. That is, we haven't gauge-fixed the action yet! We need to count each physically interesting state once.

We can accomplish this gauge fixing by a method due to Faddeev and Popov (for the original paper, see [5]). Let $G(A)$ be a sort of generalized "indicating functional" which is zero for a certain gauge condition, e.g. for the Lorenz gauge we have

$$G(A) = \partial_\mu A^\mu(x). \tag{4.7}$$

We can constrain the functional integral to be when $G(A) = 0$ by using a delta function $\delta(G(A))$. Geometrically we could have the intuition that we assign a delta function at each point x. To do this legally we insert 1 in the functional integral of the form

$$1 = \int \mathcal{D}\alpha(x)\delta\left(G(A^\alpha)\right)\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) \tag{4.8}$$

where we have $A_\mu^\alpha(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)$. We see that this condition (4.8) generalizes the identity

$$1 = \left(\prod_j \int da_j\right)\delta^{(n)}(\bar{g}(\bar{a}))\det(\partial g_j/\partial a_k) \tag{4.9}$$

for discrete n-dimensional vectors. In the Lorenz gauge we have

$$G(A^\alpha) = \partial^\mu A_\mu + \frac{1}{e}\partial^2\alpha \tag{4.10}$$

so the functional determinant in our situation (due to our case being the vacuum) is precisely $\det(\partial^2/e)$. For our situation the only thing that matters is that the functional determinant is independent of A, i.e. that it's like a constant term in our functional integral.

After inserting (4.8) the functional integral becomes

$$\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\int\mathcal{D}\alpha\int\mathcal{D}Ae^{iS[A]}\delta\left(G(A^\alpha)\right) \tag{4.11}$$

Now we will change the variable of integration from A to A^α. This is a simple shift so $\mathcal{D}A=\mathcal{D}A^\alpha$, and by gauge invariance $S[A]=S[A^\alpha]$. We then obtain

$$\int\mathcal{D}Ae^{iS[A]}=\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)\int\mathcal{D}\alpha\int\mathcal{D}A^\alpha e^{iS[A^\alpha]}\delta\left(G(A^\alpha)\right). \tag{4.12}$$

The functional integral over A is now restricted by the delta function to physically distinct inequivalent states, as desired. The infinity comes from a divergent integral over $\alpha(x)$ which simply gives an infinite multiplicative factor.

To go any further in our investigation, we have to gauge fix $G(A)$. We choose the general class of functions

$$G(A)=\partial^\mu A_\mu(x)-\omega(x) \tag{4.13}$$

where $\omega(x)$ is any scalar function. Setting this $G(A)$ zero occurs as a generalization of the Lorenz gauge. The functional determinant is the same as in the Lorenz gauge

$$\det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)=\det(\partial^2/e). \tag{4.14}$$

Thus the functional integral becomes:

$$\int\mathcal{D}Ae^{iS[A]}=\det(\partial^2/e)\left(\int\mathcal{D}\alpha\right)\left(\int\mathcal{D}A^\alpha e^{iS[A^\alpha]}\delta\left(\partial^\mu A^\alpha_\mu(x)-\omega(x)\right)\right). \tag{4.15}$$

This equality holds for any $\omega(x)$, so it will hold if we replace the right hand side with any properly normalized linear combination involving different functions $\omega(x)$. Our final trick will be integrating over all $\omega(x)$ with a Gaussian weighting function centered at $\omega=0$. Our expression will become

$$N(\xi)\int\mathcal{D}\omega\ \exp\left[-i\int\frac{\omega^2}{2\xi}d^4x\right]\det(\frac{\partial^2}{e})\int\mathcal{D}\alpha\int\mathcal{D}A^\alpha\ e^{iS[A^\alpha]}\delta(\partial^\mu A^\alpha_\mu(x)-\omega(x))=$$

$$N(\xi)\det(\partial^2/e)\int\mathcal{D}\alpha\int\mathcal{D}A^\alpha\ e^{iS[A^\alpha]}\exp\left[-i\int\frac{(\partial^\mu A^\alpha_\mu(x))^2}{2\xi}d^4x\right]$$

where $N(\xi)$ is a normalization constant and we have used the delta function to perform the integral over ω. We can choose ξ to be any finite constant. What we have really done is we've effectively added a new term $-(\partial^\mu A^\alpha_\mu)^2/(2\xi)$ to the Lagrangian.

What we have done thus far in our functional quantization of the electromagnetic field is we have worked with the denominator of our formula for the correlation functions:

$$\langle\Omega|T\{\mathcal{O}(A)\}|\Omega\rangle=\lim_{T\to\infty(1-i\varepsilon)}\frac{\int\mathcal{D}A\ \mathcal{O}(A)\exp\left[-i\int_{-T}^{T}\mathcal{L}d^4x\right]}{\int\mathcal{D}A\ \exp\left[-i\int_{-T}^{T}\mathcal{L}d^4x\right]}. \tag{4.16}$$

We can do the same manipulations for the numerator, provided that the operator $\mathcal{O}(A)$ is gauge invariant. (If it isn't, the change of variables trick $A\to A^\alpha$ won't

work.) Assuming that the operator $\mathcal{O}(A)$ is in fact gauge invariant, we find that its correlation function becomes

$$\begin{aligned}&\langle\Omega|T\{\mathcal{O}(A)\}|\Omega\rangle \\ &= \lim_{T\to\infty(1-i\varepsilon)} \frac{\displaystyle\int \mathcal{D}A\ \mathcal{O}(A)\exp\left[-i\int_{-T}^{T}\left(\mathcal{L}-\frac{1}{2\xi}(\partial^\mu A_\mu)^2\right)d^4x\right]}{\displaystyle\int \mathcal{D}A\ \exp\left[-i\int_{-T}^{T}\left(\mathcal{L}-\frac{1}{2\xi}(\partial^\mu A_\mu)^2\right)d^4x\right]}.\end{aligned} \tag{4.17}$$

Mathemagically the awkward constants we had previously are all canceled out. The only trace left behind of our meddling is the extra ξ-term that is added to the action.

We concluded that eq (4.5) implies that it is not sensible to obtain a photon propagator from the action $S[A]$. With our new ξ-term, however, that equation becomes

$$(-k^2 g_{\mu\nu} + (1-\frac{1}{\xi})k_\mu k_\nu)\tilde{D}_F^{\nu\rho}(k) = i\delta_\mu{}^\rho, \tag{4.18}$$

which has the solution

$$\tilde{D}_F^{\mu\nu}(k) = \frac{-i}{k^2+i\varepsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right). \tag{4.19}$$

This is our desired expression for the photon propagator. The $i\varepsilon$ term in the denominator arises exactly in the same way as in the Klein-Gordon free field case.

In practice, one usually chooses a specific value of ξ to actually perform computations. Two choices that are often convenient are

$$\begin{aligned}\xi &= 0 \qquad \text{Landau Gauge} \\ \xi &= 1 \qquad \text{Feynman Gauge}\end{aligned}$$

In our notes on the subject of Feynman diagrams, we have used the Feynman gauge.

The Faddeev-Popov procedure guarantees the value of any correlation function of gauge invariant operators computed from Feynman diagrams *will be independent of the value of ξ used in the calculation* (provided the value of ξ is used consistently; we can't do half of one computation using the Landau gauge, then finish up the rest in the Feynman gauge because it makes things nicer!). It's easy to show that in QED this assertion is true (I suspect due to the fact that it is an Abelian gauge group). Note in eq (4.19) that ξ multiplies a term in the photon propagator proportional to $k^\mu k^\nu$. According to the Ward-Takahashi identity, the replacement in a Green's function of any photon propagator by $k^\mu k^\nu$ yields zero, except for terms involving external off-shell fermions. These terms are equal and opposite for particle and anti-particle, and vanish when the fermions are grouped into gauge-invariant combinations.

Now, QED isn't just photon propagators. We have something else we need to compute: S-matrix elements from correlation functions of non-gauge-invariant operators $\psi(x)$, $\overline{\psi}(x)$ and $A_\mu(x)$. We will assert that the S-matrix elements are correctly computed by this procedure. The S-matrix is really defined between asymptotic states, we can compute S-matrix elements in a formalism where the coupling constant is "turned off" some "infinitely long time ago" in the past and far in the future. In the zero coupling limit, there is a clean seperation between gauge-invariant and gauge-variant states. On the one hand, single-particle states

containing one electron, one positron, or one transversely polzarized photon are gauge-invariant; while, on the other, states with timelike and longitudinal photon polarizations transform under gauge motions. We can thus define a gauge-invariant S-matrix in the following way: Let S_{FP} be the S-matrix between general asymptotic states, computed from the Faddeev-Popov procedure. The matrix is unitary but not gauge invariant (proof?). Let P_0 be a projection onto the subspace of the space of asymptotic states in which all particles are either electrons, positrons, or transverse photons. Let

$$S = P_0 S_{FP} P_0. \tag{4.20}$$

This S-matrix is gauge-invariant by construction, *it is projected onto gauge-invariant states.* Now it is not obvious that it is unitary.

In a handwavy way, we'll summarize the reasoning that this matrix is in fact unitary. Any matrix element $\mathcal{M}^\mu \epsilon^*_\mu$ for photon emission satisfies

$$\sum_{i=1,2} \epsilon^*_{i\mu} \epsilon_{i\nu} \mathcal{M}^\mu \mathcal{M}^{*\nu} = (-g_{\mu\nu}) \mathcal{M}^\mu \mathcal{M}^{*\nu}, \tag{4.21}$$

where the sum on the left hand side runs over transverse polarizations only. (The same argument applies if $\mathcal{M}^\mu$ and $\mathcal{M}^{*\nu}$ are distinct amplitudes, as long as they satisfy the Ward identity.) This is exactly the information we need to see that

$$S S^\dagger = P_0 S_{FP} P_0 S^\dagger_{FP} P_0 = P_0 S_{FP} S^\dagger_{FP} P_0. \tag{4.22}$$

Now, we can use the unitarity of S_{FP} to see that S is unitary, $SS^\dagger = 1$ on the subspace of gauge-invariant states. It is easy to check explicitly that the formula (4.20) for the S-matrix is independent of ξ: *the Ward identity implies that any QED matrix element with all external fermions on-shell is unchanged if we add to the photon propagator $D^{\mu\nu}(q)$ any term proportional to q^μ.*

5. Functional Quantization of Spinor Fields

The functional methods we have considered so far allows us to compute correlation functions of fields obeying the canonical commutation relations. To generalize to spinor fields (i.e. ones obeying canonical anticommutation relations) we must do something different: we must represent even the classical fields by anticommuting ("fermionic") numbers. Lets first review some of the properties of these fermionic numbers.

5.1. **Fermionic Numbers.**

We will define "Fermionic" numbers (or *Grassmann numbers*) by giving algebraic properties for their manipulation.

The basic feature for Grassmannians is that they *anticommute*. For any two such numbers θ, η,

$$\theta\eta = -\eta\theta. \tag{5.1}$$

In particular this implies for $\eta = \theta$ that

$$\theta^2 = 0. \tag{5.2}$$

A product of two Grassmannians $(\theta\eta)$ commutes with other Grassmannians. We really have a so-called $\mathbb{Z}$-grading, and the commuting terms are the "even" terms, the anticommuting ones are the "odd" terms (corresponding to bosonic and fermionic notions, respectively).

We can define integrals of Grassmannians. A function $f(\theta)$ of one Grassmannian is really of the form

$$f(\theta) = f(0) + f'(0)\theta \tag{5.3}$$

by Taylor expansion. This presupposes we have some notion of "differentiation" of Grassmannians. We really do, it's simply

$$\frac{d}{d\theta}(\eta\theta) = \frac{d}{d\theta}(-\theta\eta) = -\eta. \tag{5.4}$$

Integration is "dually"

$$\int f(\theta)d\theta = \int \Big(f(0) + f'(0)\theta\Big)d\theta = f'(0) \tag{5.5}$$

which is curious, as

$$\frac{d}{d\theta}f(\theta) = \frac{d}{d\theta}\Big(f(0) + f'(0)\theta\Big) = f'(0). \tag{5.6}$$

In this situation, as in complex analysis, we can write the derivative in terms of the integral. We adopt the convention

$$\int d\theta \int d\eta\ \eta\theta = +1 \tag{5.7}$$

performing the innermost integral first, as is usual in calculus.

When we have the Dirac field, we need to introduce complex Grassmannians. We think of them as sort of like operators, so complex conjugation "acts like" Hermitian conjugation:

$$(\theta\eta)^* \stackrel{\text{def}}{=} \eta^*\theta^* = -\theta^*\eta^*. \tag{5.8}$$

To integrate over complex Grassmannians, lets define

$$\theta = \frac{\theta_1 + i\theta_2}{\sqrt{2}}, \qquad \theta^* = \frac{\theta_1 - i\theta_2}{\sqrt{2}}. \tag{5.9}$$

We can treat θ, θ^* as independent Grassmann numbers, and use the convention $\int d\theta^* d\theta\ (\theta\theta^*) = 1$.

We can consider a Grassmann Gaussian

$$\int d\theta^* d\theta e^{-\theta^* b\theta} = \int d\theta^* d\theta(1 - \theta^* b\theta) = \int d\theta^* d\theta - (-\theta\theta^*)b = b. \tag{5.10}$$

If θ were some ordinary, everyday complex number, this integral would instead be $2\pi/b$. The factor of 2π is more or less unimportant, the main difference is the fact that one is the inverse of the other (more or less). However, observe that by adding another factor of $\theta\theta^*$, we find

$$\int d\theta^* d\theta\ \theta\theta^* e^{-\theta^* b\theta} = 1. \tag{5.11}$$

So now, the extra factor of $\theta\theta^*$ "cancels" the b factor, which should look familiar from our regular, old-fashioned Gaussian integrals.

To perform higher dimensional Grassmann Gaussian integrals, we first have to observe the behavior of unitary transformations acting on Grassmannians. Consider

n complex Grassmannians θ_i, and a unitary matrix $U = U_j{}^i$. Let $\theta'_i = U_i{}^j\theta_j$. Then

$$\prod_i U_i{}^j\theta_j = \frac{1}{n!}\epsilon^{ij\cdots\ l}\theta'_i\theta'_j(\cdots)\theta'_l \tag{5.12a}$$

$$= \frac{1}{n!}\epsilon^{ij(\cdots)l}(U_i{}^{i'}\theta_{i'})(U_j{}^{j'}\theta_{j'})(\cdots)(U_l{}^{l'}\theta_{l'}) \tag{5.12b}$$

$$= \frac{1}{n!}\epsilon^{ij(\cdots)l}U_i{}^{i'}U_j{}^{j'}(\cdots)U_l{}^{l'}\epsilon_{i'j'(\cdots)l'}\left(\prod_i\theta_i\right) \tag{5.12c}$$

$$= \det(U)\left(\prod_i\theta_i\right). \tag{5.12d}$$

In general, we find that

$$\left(\prod_i\int d\theta_i^* d\theta_i\right) f(\theta), \tag{5.13}$$

the only term that survives is proportional to $(\prod\theta_i)(\prod\theta_i^*)$. If we replace θ by $U\theta$, we end up with an extra factor of $\det(U)\det(U)^* = 1$, which doesn't change anything.

We can evaluate a general Grassmann Gaussian involving a Hermitian matrix B with eigenvalues b_i:

$$\left(\prod_i\int d\theta_i^* d\theta_i\right) e^{-\theta_i^* B^{ij}\theta_j} = \left(\prod_i\int d\theta_i^* d\theta_i\right) e^{-\Sigma_i\theta_i^* b_i\theta_i} = \prod_i b_i = \det(B). \tag{5.14}$$

Similarly we can see that

$$\left(\prod_i\int d\theta_i^* d\theta_i\right)\theta_k\theta_l^* e^{-\theta_i^* B^{ij}\theta_j} = \Big(\det(B)\Big)(B^{-1})_{kl} \tag{5.15}$$

where $(B^{-1})_{kl}$ is a component from the inverse for B. As a general rule of thumb, Gaussian Grassmann integrals behave similar to Gaussian integrals, with the exception that the determinant is in the numerator (as opposed to the denominator).

5.2. **Dirac Propagator.** A Grassmann *field* is a function of spacetime whose values are anticommuting numbers. More precisely we can define a Grassmann field $\psi(x)$ in terms of any set of orthonormal basis functions:

$$\psi(x) = \sum_i \psi_i\phi_i(x) \tag{5.16}$$

where the basis functions $\phi_i(x)$ are ordinary functions, while the coefficients ψ_i are Grassmann numbers. To describe the Dirac field, we take the ϕ_i to be a basis of four-component spinors.

Surprisingly enough, we have enough tools to start evaluating functional integrals (and thus correlations functions) involving fermions. Consider, for example, the two-point function given by

$$\langle 0|T\{\psi(x_1)\bar\psi(x_2)\}|0\rangle = \frac{\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left[i\int\bar\psi(i\partial\!\!\!/ - m)\psi\ d^4x\right]\psi(x_1)\bar\psi(x_2)}{\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp[i\int\bar\psi(i\partial\!\!\!/ - m)\psi\ d^4x]} \tag{5.17}$$

Note that $\mathcal{D}\bar\psi$ is unitarily equivalent to $\mathcal{D}\psi^*$.

Appendix A. Notes on the Functional Determinant

A.1. **Introduction.** We review the zeta regularisation approach to calculating the functional determinant of differential operators. First we review a few properties of the finite dimensional determinant. These include the 'exp(tr(A)) = det(exp(A))' identity, and the usefulness of eigenvalues when calculating determinants.

We move on to the general case of infinite dimensions. We generalize the determinant "definition" as the infinite product series, and use the aforementioned identity to use the Riemann zeta function in calculating determinants of differential operators.

Then we compute a few examples from various fields. First, the example of the heat equation. Then we compute the determinant of the Hamiltonian for a quantum Harmonic oscillator.

The interested reader can refer to Elizalde [6] for a similar review with more references.

A.2. **Review of Finite Dimensional Determinants.** Recall that given a (square) $n \times n$ matrix A, we can write it as

$$A = P^{-1}DP \tag{A.1}$$

where D is a diagonal matrix with entries being eigenvalues, and P is an orthogonal matrix whose columns are the corresponding eigenvectors. We find its determinant to be:

$$\begin{aligned} \det(A) &= \det(P^{-1}DP) \\ &= \det(P^{-1})\det(D)\det(P) \\ &= \det(D) \\ &= \prod_{j=1}^{n} \lambda_j \end{aligned} \tag{A.2}$$

where λ_j is the j^{th} eigenvalue of A.

There is one identity that would be nice to cover before we move on (because I am most familiar with the so-called 'zeta-regularisation' scheme, this is a necessary identity). If we represent A with the diagonalized matrix D, we find that

$$\operatorname{trace}(\ln(D)) = \ln(\lambda_1) + (\ldots) + \ln(\lambda_n) \tag{A.3}$$

and we find

$$\begin{aligned} \exp(\operatorname{trace}(\ln(D))) &= \exp(\ln(\lambda_1) + \ln(\ldots) + \ln(\lambda_n)) \\ &= \lambda_1(\ldots)\lambda_n \\ &\overset{\text{def}}{=} \det(A). \end{aligned} \tag{A.4}$$

One may see this and say "Huh, that's neat but seemingly useless" and you would be right! Well, so far you would be, but you will see that it's useful!

A.3. **Zeta-Regularisation Scheme.** Recall from differential equations, that we may treat a derivative as an "infinite dimensional" linear operator (and recall from linear algebra all linear operators may be represented as a square matrix!). So we may apply Eq. (A.2) to a differential operator in general. The first thought that

comes to my mind is "You're crazy, man! That would diverge like...a...divergent product series!" That is true, if we naively write

$$\det(\mathcal{D}) = \prod_{n=1}^{\infty} \lambda_n. \tag{A.5}$$

(Where $\mathcal{D}$ is a differential operator). But as we have just seen, we may use the identity derived in Eq. (A.4) to find:

$$\exp(\text{trace}(\ln(\mathcal{D}))) := \det(\mathcal{D}) \tag{A.6}$$

so the question is to find an equation that gives us

$$(\text{trace}(\ln(\mathcal{D}))) = \sum_{n=0}^{\infty} \ln(\lambda_n). \tag{A.7}$$

Let us now introduce the Riemann zeta function

$$\zeta_R(s) = \sum_{n=1}^{\infty} n^{-s} \text{ with } \text{Re}(s) > 1. \tag{A.8}$$

It is something well studied in number theory, modern analysis, and a number of other subjects. Note that

$$\zeta_R'(s) = -\sum_{n=1}^{\infty} \frac{\ln(n)}{n^s} \tag{A.9}$$

and what physicists like to do is then set $s = 0$, so we find

$$\zeta_R'(0) = -\sum_{n=1}^{\infty} \ln(n). \tag{A.10}$$

This is remarkably similar to what we're looking for!

What we do is we take the so-called "zeta-trace" of our differential operator $\mathcal{D}$:

$$\zeta_D(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}. \tag{A.11}$$

Then we just take its derivative and set $s = 0$. Usually we'd want to write this in terms of the Riemann zeta function, since we know a lot more about the Riemann zeta function in general (such as what its value of its derivative at 0 is!).

A.4. **Example: Heat Equation.** Recall the infamous heat equation

$$\frac{\partial^2}{\partial x^2} U(x,t) = \frac{\partial}{\partial t} U(x,t) \tag{A.12}$$

with the boundary conditions

$$\begin{aligned} U(x,0) =& f(x) \quad \forall x \in [0,L] \\ U(0,t) =& U(L,t) = 0 \quad \forall t > 0 \end{aligned} \tag{A.13}$$

By the famous calculation, we seperate the variables and we are know dealing with a sort of eigenvalue problem

$$\frac{\partial^2}{\partial x^2} u(x) = -\lambda u(x) \tag{A.14}$$

which tells us that

$$\begin{aligned} u(x) =& B\sin\left(\sqrt{\lambda}x\right) \\ \lambda_n =& (n\pi/L)^2 \quad \forall n \in \mathbb{N} \end{aligned} \tag{A.15}$$

So lets figure out its determinant!

Well, we first note that the zeta trace of the differential operator is

$$\zeta_A(s) = \sum_{n=1}^{\infty} \frac{L^{2s}}{(n\pi)^{2s}} \tag{A.16}$$

We can factor this out to be

$$\zeta_A(s) = \frac{L^{2s}}{\pi^{2s}} \sum_{n=1}^{\infty} n^{-2s} \tag{A.17}$$

which is tricky to work with, but we'd really like to write it in terms of the Riemann zeta function! We find:

$$\zeta_A(s) = \left(\frac{L}{\pi}\right)^{2s} \zeta_R(2s) \tag{A.18}$$

and thus

$$\zeta_A'(s) = -2\ln(L/\pi)\left(\frac{L}{\pi}\right)^{2s} \zeta_R(2s) + 2\left(\frac{L}{\pi}\right)^{2s} \zeta_R'(2s). \tag{A.19}$$

We find

$$\zeta_A'(0) = -2\ln(L/\pi)\zeta_R(0) + 2\zeta_R'(0). \tag{A.20}$$

Because I don't know the Riemann Zeta Function like the back of my hand (unfortunately), I look up its values[1]:

$$\begin{aligned} \zeta_R(0) =& \frac{-1}{2} \\ \zeta_R'(0) =& -\frac{1}{2}\ln(2\pi) \end{aligned} \tag{A.21}$$

So we find then

$$\zeta_A'(0) = \ln(L/\pi) - \ln(2\pi). \tag{A.22}$$

and then, by defintion, the determinant of our operator A is

$$\det(A) := \exp\left(-\zeta_A'(0)\right) = \frac{2\pi^2}{L}. \tag{A.23}$$

This concludes this section.

[1]From `http://mathworld.wolfram.com/RiemannZetaFunction.html`

A.5. **Example: Quantum Harmonic Oscillator.** In the one dimensional case, a particle of mass m has the potential

$$V(x) = \frac{1}{2}m\omega^2 x^2 \tag{A.24}$$

where $m\omega^2 = k$ is called the "spring stiffness coefficient" or sometimes the "force constant", and ω is the circular frequency. We find the Hamiltonian to be

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{A.25}$$

and so the Schrodinger equation is

$$\frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2|\psi\rangle = E|\psi\rangle. \tag{A.26}$$

We will not show all the work, but through solving this differential equation, the eigenvalues for the Hamiltonian are

$$E_n = \hbar\omega(n + \frac{1}{2}) \text{ with } \forall n \in \mathbb{N} \tag{A.27}$$

which will be used in our computation of the determinant of the Hamiltonian.

We find that the zeta trace of the Hamiltonian is thus

$$\zeta_H(s) = \sum_{n=1}^{\infty} E_n^{-s} = (\hbar\omega)^{-s}\sum_{n=1}^{\infty}(n + \frac{1}{2})^{-s} \tag{A.28}$$

and we find this to be

$$\zeta_H(s) = (\hbar\omega)^{-s}\sum_{n=1}^{\infty}(n + \frac{1}{2})^{-s} = (\hbar\omega/2)^{-s}\sum_{n=1}^{\infty}(2n+1)^{-s}. \tag{A.29}$$

We can refer to the Hurwitz zeta-function

$$\zeta(s, q) = \sum_{k=0}^{\infty}(k + q)^{-s} \tag{A.30}$$

and rewrite our zeta trace for the Hamiltonian to be

$$\zeta_H(s) = (\hbar\omega/2)^{-s}\left(\zeta(s, 3) - 2\zeta(s, 1) + 2^{-s}\right). \tag{A.31}$$

Again, referring to a table[2] we find

$$\begin{array}{c} \zeta(0, a) = \frac{1}{2} - a \\ \frac{d}{ds}\zeta(0, a) = \ln(\Gamma(a)) - \frac{1}{2}\ln(2\pi) \end{array} \tag{A.32}$$

So we have

$$\begin{aligned} \zeta_H'(0) &= -\ln(\hbar\omega/2)\left(\zeta(0, 3) - 2\zeta(0, 1) + 1\right) + \left(\ln(6) + \frac{1}{2}\ln(2\pi) - \ln(2)\right) \\ &= -\ln(\hbar\omega/2)\left(\frac{-1}{2}\right) + \left(\ln(6) + \frac{1}{2}\ln(2\pi) - \ln(2)\right) \\ &= \ln(\sqrt{\hbar\omega/2}) + \ln(3) + \ln(\sqrt{2\pi}) \end{aligned} \tag{A.33}$$

We can now compute

$$\det(\hat{H}) = \exp\left(-\zeta_H'(0)\right) = \frac{1}{3}\sqrt{\frac{1}{2\pi\hbar\omega}} \tag{A.34}$$

[2]More precisely, Eqs (9) and (16) of `http://mathworld.wolfram.com/HurwitzZetaFunction.html`

which is the determinant of the above differential operator.

Appendix B. Glossary

A small glossary of various terms used in quantum field theory.

Abelian Group: when the group is commutative, we call it "Abelian".

BRST Method: a generalization of the Faddeev-Popov procedure for non-Abelian gauge symmetries, it involves Ghosts.

Correlation Function: the two-point correlation function corresponds to the amplitude for propagation of a particle or excitation between y and x; Wick's theorem tells us we can express higher order correlation functions in terms of two-point correlation functions.

Faddeev-Popov Procedure: a method to avoid redundant infinities in the functional integral when we have (Abelian) gauge symmetries.

Gauge Symmetry: a change in coordinates which leaves the equations of motion invariant.

Ghosts: a sort of "particle" which serves as negative degrees of freedom to cancel the effects of the unphysical timelike and longitudinal polarization states of gauge bosons.

LSZ Reduction Formula: the multiparticle generalization of the Lorentz-invariant formula $\langle k|k'\rangle = (2\pi)^3 2\omega\delta^{(3)}(\bar{k}-\bar{k}')$.

Non-Abelian Group: when the group's multiplication operation is not commutative, it is Non-Abelian. Contrast to an Abelian Group.

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