COMPLEX ANALYSIS CHEAT SHEET CONT'D

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Contents

1. LAPLACE TRANSFORM

Definition 1. (Laplace Transform) The **Laplace Transform** of a function $f(t)$ (for $t \geq 0$) is the function $\widetilde{f}(z)$ defined by

$$\widetilde{f}(z) = \mathcal{L}\{f\}(z) = \int_0^\infty e^{-zt} f(t) dt \tag{1.1}$$

where z is complex.

Proposition 1. (Asymptotic Behavior of Laplace Transform) Suppose g is analytic in a region containing the positive real axis and is bounded on the positve real axis. Let the Taylor series for g centered at 0 be

$$\sum_{n=0}^\infty a_n z^n \tag{1.2}$$

and let

$$\widetilde{g}(z) = \int_0^\infty e^{-zt} g(t) dt. \tag{1.3}$$

Then

$$\widetilde{g}(z) \sim \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{2a_2}{z^3} + \cdots + \frac{n!a_n}{z^{n+1}} + \cdots \tag{1.4}$$

as $z \to \infty$, $\arg(z) = 0$.

Date: June 10, 2009.

Proposition 2. Suppose g is infinitely differnetiable on the positive real axis and that g and each of its derivatives are of exponential order. That is, there are constants A_n and B_n such that

$$(1.5) \qquad |g^{(n)}(t)| \leq A_n e^{B_n t}$$

for $t \geq 0$. Let

$$(1.6) \qquad \widetilde{g}(z) = \int_0^\infty e^{-zt} g(t) dt.$$

Then

$$(1.7) \qquad \widetilde{g}(z) \sim \frac{g(0)}{z} + \frac{g'(0)}{z^2} + \frac{g''(0)}{z^3} + \cdots + \frac{g^{(n)}(0)}{z^{n+1}} + \cdots$$

as $z \to \infty$, $\arg(z) = 0$.

Theorem 1. (Convergence Theorem for Laplace Transform) Assume

$$(1.8) \qquad f : (0, \infty) \to \mathbb{C}$$

is of exponential order and let

$$(1.9) \qquad \widetilde{f}(z) = \int_0^\infty e^{-zt} f(t) dt.$$

There exists a unique number σ, $-\infty \leq \sigma < \infty$ such that this integral converges if $\mathrm{Re}(z) > \sigma$ and diverges if $\mathrm{Re}(z) < \sigma$. Furthermore if $\widetilde{f}$ is analytic on the set

$$(1.10) \qquad A = \{z \,|\, \mathrm{Re}(z) > \sigma\}$$

and we have

$$(1.11) \qquad \frac{d}{dz} \widetilde{f}(z) = -\int_0^\infty t e^{-zt} f(t) dt$$

for $\mathrm{Re}(z) > \sigma$. The number σ is called the "**Abscissa of Convergence**" and if we define the number ρ by

define ρ

$$(1.12) \qquad \rho = \inf\{B \in \mathbb{R} \,|\, \text{there exists an } A > 0 \text{ such that } |f(t)| \leq A e^{Bt}\}$$

then $\sigma \leq \rho$.

Theorem 2. (Laplace Transforms) Suppose that the functions f and h are continuous and that $\widetilde{f}(z) = \widetilde{h}(z)$ for $\mathrm{Re}(z) > \gamma_0$ for some γ_0. Then $f(t) = h(t)$ for all $t \in (0, \infty)$.

Proposition 3. Let $f(t)($ be continuous on $(0, \infty)$ and piecewise C^1. Then for $\mathrm{Re}(z) > \rho$

$$(1.13) \qquad \widetilde{\left(\frac{df}{dt}\right)}(z) = z\widetilde{f}(z) - f(0).$$

Proposition 4. Let

$$(1.14) \qquad g(t) = \int_0^t f(\tau) d\tau$$

Then for $\mathrm{Re}(z) > \max[0, \rho(f)]$,

$$(1.15) \qquad \widetilde{g}(z) = \frac{\widetilde{f}(z)}{z}.$$

Theorem 3. (First Shifting Theorem)Fix $a \in \mathbb{C}$ and let $g(t) = e^{-at}f(t)$. Then for $\mathrm{Re}(z) > \sigma(f) - \mathrm{Re}(a)$, we have

$$(1.16) \qquad\qquad \widetilde{g}(z) = \widetilde{f}(z + a).$$

Theorem 4. (Second Shifting Theorem)Let $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \geq 1$ be the **Step Function** or **Heaviside Step Function**. Let $a \geq 0$ and let $g(t) = f(t - a)H(t - a)$; that is, $g(t) = 0$ if $t < a$ while $g(t) = f(t - a)$ if $t \geq a$. Then for $\mathrm{Re}(z) > 0$ we have

$$(1.17) \qquad\qquad \widetilde{g}(z) = e^{-az}\widetilde{f}(z).$$

Definition 2. (Convolution)The "**Convolution**" of two functions $f(t)$ and $g(t)$ is defined for $t \geq 0$ by

$$(1.18) \qquad\qquad (f * g)(t) = \int_0^\infty f(t - \tau)g(\tau)d\tau$$

where we set $f(t) = 0$ if $t < 0$.

Theorem 5. (Convolution Theorem)The equalities

$$(1.19) \qquad\qquad (f * g)(t) = (g * f)(t)$$

whenever $\mathrm{Re}(z) > \max[\rho(f), \rho(g)]$.

1.1. **Table of Properties of the Laplace Transform.** Let $u(t)$ be the Heaviside step function.

$$(1.20) \qquad\qquad u(t) = \int_{-\infty}^t \delta(\tau)d\tau$$

where δ is the delta function we all know and love.

Linearity	$af(t) + bg(t)$	$a\widetilde{f}(z) + b\widetilde{g}(z)$
Frequency Differentiation	$tf(t)$	$-\widetilde{f}'(z)$
Frequency Differentiation	$t^n f(t)$	$(-1)^n \widetilde{f}^n(z)$
Differentiation	$f'(t)$	$z\widetilde{f}(z) - f(0)$
Differentiation	$f''(t)$	$z^2\widetilde{f}(z) - zf(0) - f'(0)$
Differentiation	$f^{(n)}(t)$	$z^n\widetilde{f}(z) - z^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
Frequency Integration	$f(t)/t$	$\int_z^\infty \widetilde{f}(\omega)d\omega$
Integration	$\int_0^t f(\tau)d\tau = (u * f)(t)$	$\widetilde{f}(z)/z$
Scaling	$f(at)$	$\widetilde{f}(z/a)/|a|$
Frequency Shifting	$e^{at}f(t)$	$\widetilde{f}(z - a)$
Time shifting	$f(t - a)u(t - a)$	$e^{-az}\widetilde{f}(z)$
Convolution	$(f * g)(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
Periodic Function	$f(t)$	$(\int_0^T e^{-zt}f(t)dt)/(1 - e^{-Tz})$

1.2. **List of Properties of the Laplace Transform. Definition**

$$(1.21) \qquad \widetilde{f}(z) = \int_0^\infty e^{-zt} f(t) dt.$$

(1) $\widetilde{g}(z) = -\dfrac{d}{dz} \widetilde{f}(z)$ where $g(t) = tf(t)$.

(2) $\mathcal{L}\{af + bg\} = a\widetilde{f} + b\widetilde{g}$

(3) $\widetilde{\left(\dfrac{df}{dt}\right)}(z) = z\widetilde{f}(z) - f(0)$.

2. GAMMA FUNCTION

So for n a positive integer, we have

$$(2.1) \qquad \Gamma(n) = (n-1)!$$

2.1. **List of Properties of the Gamma Function.** Remember that it is defined as

$$(2.2) \qquad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

or equivalently as an infinite product

$$(2.3) \qquad \Gamma(z) = \frac{1}{ze^{\gamma z} \left[\prod_{n=1}^\infty \left(1 + \frac{z}{n}\right) e^{-z/n}\right]}$$

where

$$(2.4) \qquad \begin{aligned} \gamma &= \lim_{n \to \infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n)\right) \\ &\approx 0.5772156649015328606065120900082 \end{aligned}$$

It has the following properties:

(1) Γ is meromorphic with simple poles at $0, -1, -2, \ldots$

(2) $\Gamma(z+1) = z\Gamma(z)$ for $z \neq 0, -1, -2, \ldots$

(3) $\Gamma(n+1) = n!$ for $n = 0, 1, \ldots$

(4) $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$

(5) $\Gamma(z) \neq 0$ for all z

(6) $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$, $\Gamma\left(n + \dfrac{1}{2}\right) = \dfrac{1 \cdot 3 \cdot (\cdots) \cdot (2n-1)}{2n}\sqrt{\pi}$

(7) $\Gamma(z) = \dfrac{1}{z} \prod_{n=1}^\infty \left[\left(1 + \dfrac{1}{n}\right)^z \left(1 + \dfrac{z}{n}\right)^{-1}\right]$

(8) $\Gamma(z) = \lim_{n \to \infty} \dfrac{n! n^z}{z(z+1)(\cdots)(z+n)}$

(9) $\Gamma(z)\Gamma\left(z + \dfrac{1}{n}\right)(\cdots)\Gamma\left(z + \dfrac{n-1}{n}\right) = (2\pi)^{(n-1)/2} n^{(1/2)-nz} \Gamma(nz)$

(10) $2^{2z-1}\Gamma(z)\Gamma\left(z + \dfrac{1}{2}\right) = \sqrt{\pi}\Gamma(2z)$

(11) The residue of $\Gamma(z)$ at $z = -m$ is equals $(-1)^m/m!$

(12) (Euler's Integral) $\Gamma(z) = \displaystyle\int_0^\infty t^{z-1} e^{-t} dt$ for $\text{Re}(z) > 0$. The convergence is uniform and absolute for $-\pi/2 + \delta \leq \arg(z) \leq \pi/2 + \delta$ $(\delta > 0)$ and for $\varepsilon \leq |z| \leq R$ where $0 < \varepsilon < R$.

(13) $\dfrac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \dfrac{1}{z} + \displaystyle\sum_{n=1}^\infty \left(\dfrac{1}{n} - \dfrac{1}{z+n} \right) = \int_0^\infty \left(\dfrac{e^{-t}}{t} - \dfrac{e^{-zt}}{1 - e^{-t}} \right) dt.$

(14) $\pi^{-z/2} \, \Gamma\left(\dfrac{z}{2}\right) \zeta(z) = \pi^{-\frac{1-z}{2}} \, \Gamma\left(\dfrac{1-z}{2}\right) \zeta(1-z)$. (Where $\zeta(s)$ is the Riemann zeta function)

(15) $\zeta(z) \, \Gamma(z) = \displaystyle\int_0^\infty \dfrac{u^{z-1}}{e^u - 1} \, du$ which holds for $\text{Re}(z) > 1$.

3. Zeta Function

The definition for the Riemann zeta function is

(3.1)
$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

It is holomorphic everwhere except for a simple pole at $s = 1$ with residue 1.

For any positive even integer $2n$, we have

(3.2)
$$\zeta(2n) = (-1)^{n+1} \frac{B_{2n}(2\pi)^{2n}}{2(2n)!}$$

where B_{2n} is a Bernoulli number, and for negative integers we have

(3.3)
$$\zeta(-n) = \frac{-B_{n+1}}{n+1}$$

for $n \geq 1$.

Let

(3.4)
$$f(x) = \frac{x}{e^x - 1}$$

then the Bernoulli numbers may be found from

(3.5)
$$B_n = \lim_{x \to 0} \frac{d^n}{dx^n} \frac{x}{(e^x - 1)}.$$

Observe that for $n = 1$

(3.6)
$$f'(x) = \left(\frac{1}{e^x - 1} \right) \left(1 - \frac{f(x)}{e^x - 1} \right)$$

and now observe that

(3.7)
$$\frac{d}{dx} \left(\frac{1}{e^x - 1} \right) = -e^x \left(\frac{1}{e^x - 1} \right)^2$$

and we can use the product rule to find all of our favorite Bernoulli numbers.

We have a table of the first few Bernoulli numbers:

n	B_n
0	1
1	-1/2
2	1/6
4	-1/30
6	1/42
8	-1/30
10	5/66≈ 0.07575757576
12	-691/2730≈-0.25311355311
14	7/6
16	-3617/510≈ -7.09125686275
18	43867/798≈ 54.9711779448

The zeta function satisfies the functional equation

$$(3.8) \qquad \zeta(s) = 2^s \pi^{s-1} \, \sin\left(\frac{\pi s}{2}\right) \, \Gamma(1-s) \, \zeta(1-s),$$

valid for all $s \in \mathbb{C}$. An equivalent relationship may be expressed as a sum

$$(3.9) \qquad \zeta(s)(1 - 2^{1-s}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

3.1. **Mellin Transform.** The Mellin transform of a function $f(x)$ is defined as

$$(3.10) \qquad \int_0^\infty f(x) x^{s-1} \, dx,$$

when defined. We can relate the zeta function to one million and one things this way, we have

$$(3.11) \qquad \Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{\exp(x) - 1} dx,$$

where Γ is our favorite gamma function, and

$$(3.12) \qquad 2\sin(\pi s)\Gamma(s)\zeta(s) = i \oint_C \frac{(-x)^{s-1}}{\exp(x) - 1} dx$$

for all s where the contour C begins and ends at $+\infty$ and circles the origin once.

3.2. **Laurent Series.** Since the zeta function has a single simple pole at $s = 1$ we can expand it around the singular point. The series is

$$(3.13) \qquad \zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n \, (s-1)^n.$$

where γ_n are the Stieltjes constants, defined by the limit

$$(3.14) \qquad \gamma_n = \lim_{m\to\infty} \left(\left(\sum_{k=1}^m \frac{(\log k)^n}{k} \right) - \frac{(\log m)^{n+1}}{n+1} \right).$$

where the constant $n = 0$ term in the Laurent series is just γ_0 the Euler-Mascheroni constant.

References

[1] J. E. Marsden and M. J. Hoffman, *Basic Complex Analysis*. W. H. Freeman, third ed., 1998.

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