NOTES ON LADDER OPERATORS

ALEX NELSON

1. Angular Momentum in Quantum Mechanics

In classical mechanics, we described the angular momentum of a body by

$$\vec{x} \times \vec{p} = \vec{L}. \tag{1.1}$$

In quantum mechanics, we do the same thing more or less, but we need to make the replacement

$$p \to \widehat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x} \tag{1.2}$$

and so on. We will recklessly switch notations at random between using hats to denote operators and not using hats. It is understood, unless otherwise specified, we will work henceforth with operators.

Here it is convenient to note that we can write the cross product in a slick way using matrix multiplication. Observe

$$\vec{a} \times \vec{b} = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}. \tag{1.3}$$

Also observe that

$$\begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}^2 = -\begin{bmatrix} a_2^2 + a_3^2 & -a_1 a_2 & -a_1 a_3 \\ -a_1 a_2 & a_1^2 + a_3^2 & -a_2 a_3 \\ -a_1 a_3 & -a_2 a_3 & a_1^2 + a_2^2 0 \end{bmatrix}. \tag{1.4}$$

If we perform that computations, we find that (using, apologies to the physicists struggling with it, abstract index notation)

$$L^2 = \vec{L} \cdot \vec{L} \tag{1.5a}$$

$$= \sum_{i,j,k,m,n} \epsilon^{ijk} \epsilon^{imn} x^j p_k x^m p_n \tag{1.5b}$$

$$= \sum_{i,j,k,m,n} (\delta^{jm}\delta^{kn} - \delta^{jn}\delta^{km}) x^j p_k x^m p_n \tag{1.5c}$$

$$= \sum_{i,j,k,m,n} x^j p_k x^j p_k - x^j p_k x^k p_j. \tag{1.5d}$$

We can using the commutation relations

$$[\widehat{x}^i, \widehat{p}_j] = i\delta^i{}_j \tag{1.6}$$

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to reorder terms, for example

$$p_k x_j = x_j p_k - i\delta_{kj} \tag{1.7a}$$

$$x_j p_k = p_k x_j + i\delta_{kj}. \tag{1.7b}$$

We can rewrite the angular momentum squared as

$$L^2 = x^j(xjp_k - i\delta^j{}_k)p_k - (p_k x^j + i\delta^j{}_k)x_k p_j \tag{1.8a}$$

$$= x^j x_j p^k p_k - ix^j p_j - p_k x^k x^j p_j - ix^j p_j \tag{1.8b}$$

$$= x^j x_j p^k p_k - 2ix^j p_j - (x^k p_k - i\delta^k{}_k)x^j p_j. \tag{1.8c}$$

2. Rotations in Two and Three Dimensions

Recall in 2 dimensions, we can use complex analysis to simplify rotations. That is, due to Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \tag{2.1}$$

where $i^2 = -1$, we can write any ordered pair (x, y) as

$$x + iy = re^{i\theta_0}. \tag{2.2}$$

If we want to rotate by some angle θ anticlockwise, we simply multiply by $\exp(i\theta)$ to find

$$(x + iy)e^{i\theta} = (x\cos(\theta) - y\sin(\theta)) + i(x\sin(\theta) + y\cos(\theta)). \tag{2.3}$$

We can write this in matrix form as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2.4}$$

We denote the rotated coordinates with primes.

For three dimensions, we can rotate about the x, y, or z axis. These have similar forms as their two dimensional counterparts. Namely a rotation about the x axis demands $x' = x$, so

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix} \tag{2.5}$$

which is intuitively clear since we treat the $y - z$ plane as a two dimensional plane and "rotate in it". Similarly, for the rotation about the z axis we have for anticlockwise rotations by an angle γ

$$R_z(\gamma) = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.6}$$

For rotations about the y axis, it is a bit more tricky if we wish to maintain sign. That is, the sign of the sine changes, otherwise we'd do a clockwise rotation. (Read twice, and prove this important fact to yourself. It will become evident later on.) For an anticlockwise rotation about the y axis by an angle β we have

$$R_y(\beta) = \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} \tag{2.7}$$

Observe the change of signs on sine. Remember too that $\sin(-x) = -\sin(x)$ for a huge hint why.

An additional property of rotations that are worthy of note is that we may compose them. That is

$$R(\alpha)R(\beta) = R(\alpha + \beta) \tag{2.8}$$

for some rotation $R(\cdot)$. This means we can write a rotation by an angle θ as

$$R(\theta) = \left[R\left(\frac{\theta}{N}\right)\right]^N \tag{2.9}$$

for some $N \in \mathbb{N}$. For large N we make the approximations

$$\cos(\theta/N) \approx 1 \tag{2.10a}$$

$$\sin(\theta/N) \approx \frac{\theta}{N}. \tag{2.10b}$$

So we can rewrite our rotation matrices as the sum of two matrices

$$R_x\left(\frac{\alpha}{N}\right) = \left[I + \left(\frac{\alpha}{N}\right)T_x\right]^N \tag{2.11a}$$

$$R_y\left(\frac{\beta}{N}\right) = \left[I + \left(\frac{\beta}{N}\right)T_y\right]^N \tag{2.11b}$$

$$R_z\left(\frac{\gamma}{N}\right) = \left[I + \left(\frac{\gamma}{N}\right)T_z\right]^N \tag{2.11c}$$

where we have silently introduced the matrices

$$T_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.12a}$$

$$T_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \tag{2.12b}$$

$$T_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.12c}$$

Observe that by formally taking the limit $N \to \infty$, we have for some rotation operator R

$$R(\theta) = \lim_{N\to\infty}\left[I + \frac{\theta}{N}T\right]^N = \exp(\theta T) \tag{2.13}$$

where $\exp(\cdot)$ here is matrix exponentiation. We simply plug in the matrix into the Taylor series of e^x, using matrix multiplication and matrix addition.