NOTES ON CATEGORY THEORY

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1. Introduction

What is category theory and what can we do with it? This is a rather big question that deserves easily a thousand books to give a finished answer. We will approach this question in two parts: the mathematician's approach, and the physicist's approach.

Starting from the mathematician's approach, we will define what a category is. Unfortunately, (as in most of math) the shorter the explanation, the more technically incoherent it becomes.

We will spend some time on our overly technical summary, and give the technical conditions for a category to be a category. In the immortal words of Douglas Adams, we offer the following advice to the category theory novice: **don't panic!**

After this brief introduction to the definition (or more precisely, its statement), we will try to define precisely the notion of a "mathematical object". What is such a creature and why do we want to define it? Well, it can't be defined precisely *a priori*, but we can give examples. A vector space, a group, a topology, a Sigma algebra, a Lie algebra, a Ring, a Field, a manifold – all of these are mathematical objects. In general, anything that is ever defined is a mathematical object. Why do we want to define it? Well, this is what mathematicians study, so it's good to have some idea of what it is. Additionally, we wish to construct mathematical objects with categories. We first would need to figure out how to express mathematical objects categorically.

So, having defined a category and a mathematical object, we have the perfect storm to start expressing mathematical objects categorically.

This concludes the mathematician's approach.

The physicist's approach starts off immediately by solving problems categorically without defining categories. This is a bit of a mixed bag that Lawvere [?] pioneered. We will be looking at a lot of problems that plagued early physicists in classical mechanics, specifically Galileo's work and (time permitting) Kepler's Laws.

We do this by first examining the problem of how to determine the trajectory of a flying bird.

We then present how to set up basic problems categorically, and provide worked examples.

Part 1. A Mathematician's Approach (*Overture*)

2. Mathematical Objects: Stuff, Structure, and Properties

Problem. Suppose we wanted to describe all of math by some common "meta-structure". How would we do it? It has to be general enough to encompass all of the diverse aspects of every mathematical field, but the danger is becoming *too general*. This scares the physicists.

An alternate way to think of the problem is perhaps this programming problem: we want to program an object oriented computer algebra system with a common base class, with functions/homomorphisms/operators/etc as instances of a `function` class which extends our base class. Is there an elegant way to solve this problem?

So, to answer the initial problem, how can we describe some generic mathematical object? We will follow the examples of Baez [?], Baez and Wise [?], Baez and Shulman [?], and Baez and Dolan [?] in the answer to this question. Well, in linear algebra when we introduce the notion of a vector space, it is defined as the set of vectors. Similarly, in abstract algebra, a group is defined as a set with a binary operator with various properties. The recurring theme appears to be that there is some set underlying the mathematical object, but lets not be so strict. Lets instead give the following proposition:

Proposition 2.1. A "**Mathematical Object**" is defined by at least specifying some underlying "*stuff*" (e.g. a set, several sets, etc.).

Well, returning to our example of a vector space, what else makes this set a vector space? There is some "extra structure" that makes it so. Specifically we can do two things: 1) we can add two vectors together to get a third vector, 2) we can multiply a vector by a scalar to get another vector. These are binary operators, which really are just functions

$$f : X \times Y \to Z \tag{2.1}$$

where X, Y, and Z are the underlying sets, f is the binary operator in question. It's just that in practice, it looks kinda funny writing $+(\vec{x}, \vec{y})$ for vector addition. This suggestion of binary operators as functions is – at first glance – foreign. So, being general (again), we suggest that there are functions, relations, some specified elements, etc. all fit in this "extra structure" description. In e.g. a topology or a sigma algebra, we are worried about collections of subsets, which should be taken into account as "extra structure" as well since we are working with distinguished subsets. So to be fully general we propose the following revised proposition

Proposition 2.2. A "**Mathematical Object**" is defined by at least specifying

(1) some underlying "*stuff*" (e.g. a set, several sets, etc.)
(2) equipping this "stuff" with some "*structure*" (e.g. functions, binary operators, collections of subsets, distinguished elements, etc.).

This still is not enough. We can't just have any old "structure", we need to specify conditions our desired structure satisfies. If we are considering a binary operator, is our "stuff" closed under this binary operator? Is there some identity element e so when we consider the binary operator applied to e and some x (i.e. $f(e,x)$) we end up with our x? If so, does every x in our stuff have an inverse? And so on, the list is endless.

These conditions are really just demanding certain equations (or in some cases inequalities or inclusions) holds. These are just algebraically described as "*properties*" of our structure. This is enough to give a fully general account of a mathematical object:

Definition 2.3. A "**Mathematical Object**" is defined by:
(1) specifying some underlying "*stuff*" (e.g. a set, several sets, etc.)
(2) equipping this "stuff" with some "*structure*" (e.g. functions, binary operators, collections of subsets, distinguished elements, etc.).
(3) with this "structure" satisfying certain "*properties*" (e.g. equations, inequalities, etc.).

This is a sufficient generalized notion of a mathematical object. We can use this in the definition of a category.

Remark 2.4. If one really pushed, I doubt that any of these conditions can be rigorously defined (what do you mean by "equations"? "functions"? etc.). The point is, as with every definition, to give some intuition behind the concept as well as some defining characteristics of the object. But if one really pushed, I suspect this is where Godel's incompleteness comes into play.

Remark 2.5. As Baez and Wise point out [**?**], we can always *check* the properties – they are either true or false. We can also *choose* structures from a set of possibilities. And we can *choose* stuff from a category of possibilities. But each step depends on the following ones. Structure depends on stuff, and properties depends on structure. This should be somewhat intuitive, we can't demand conditions on structure we don't have, nor can we have structure depend on stuff we don't have.

Proposition 2.6. Every definition in math is defining one of the following: a mathematical object, some "stuff", some "structure", or some "property".

Conjecture 2.7. In relating definitions to mathematical objects, we conjecture:
(1) The definition of some "stuff" or "structure" can be reformulated as a definition of a mathematical object.
(2) The definition of some "property" can be reformulated as a definition of some "structure"; moreover, it can be reformulated as a definition of a mathematical object.

Or every definition defines a mathematical object — defining a mathematical object is sometimes more circuitous and not as interesting or useful.

This conjecture is summarized in the single phrase "*everything defined in math is-a mathematical object*".

3. Examples of Mathematical Objects

Now that we have introduced the notion of a "mathematical object", perhaps we should start examining mathematical objects. Well, it turns out that it's all of math, so perhaps we should consider certain examples. It's only really necessary to look at a few examples, it turns out that in categories the objects play second fiddle to the morphisms.

3.1. **A Topology on a Set.**

Topologies are interesting, they specify the open subsets of a given set X. Recall that a topology $\mathcal{T}$ on X is a collection of subsets of X having the following properties:

1. $\emptyset$ and X are in $\mathcal{T}$
2. The union of the elements of any subcollection of $\mathcal{T}$ are in $\mathcal{T}$
3. The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$.

This is the definition of a topology on X.

Now we want to demonstrate that it is a mathematical object.

- **Stuff:** The underlying set X is the stuff topologies are made of...
- **Structure:** We have some extra structure by considering a collection of subsets of X.
- **Properties:** We just listed what the properties of this structure should be! It's closed under arbitrary union of elements of $\mathcal{T}$, finite closure of elements of $\mathcal{T}$, and the collection contains both the empty set and X itself.

So we see that a topology is a mathematical object.

3.2. **Binary Operator on a Set.**

Let S be a set, a binary operation on S is a binary relation that maps elements of the Cartesian product $S \times S$ to S:

$$f : S \times S \to S. \tag{3.1}$$

The underlying argument is a generalization of the argument that a function is a mathematical object. In fact, the argument is exactly the same replacing X with $S \times S$, and Y with S. But since a function was demonstrated as the first example of a mathematical object in this paper, it follows that a binary operator on a set is a mathematical object.

3.3. **Groups.**

Let $(G, \cdot)$ be a group with elements G and binary operator represented as multiplication. A group is closed under its binary operator and necessarily has inverses exist. We will show that this is clearly a mathematical object:

- **Stuff:** The underlying set G is the stuff.
- **Structure:** The binary operator acting on G is our structure.
- **Properties:** We demand certain properties hold. First for any $x \in G$, there is a unique $e \in G$ such that $xe = ex = x$. Second, for each $x \in G$, there is a corresponding $y \in G$ such that $xy = yx = e$. Third, for each $x, y \in G$, we demand that $xy \in G$.

The pattern is kind of clear how to get from a definition to our grocery list of "stuff-structure-properties". This is how most people do abstract math, with such grocery lists.

4. Quick and Painful Introduction to Categories

The motivating problem is that we want to look at mappings between mathematical objects, but it may not always make sense. So we want to have two things: (1) mappings between objects, and (2) a setting where it make sense.

The heart of mathematics is definitions, so we will begin by bluntly defining a category:

Definition 4.1. A "**Category**" $\mathbf{C}$ consists of

(1) a collection $\mathrm{Ob}(\mathbf{C})$ of "**Objects**";
(2) for any pair of objects $x, y \in \mathrm{Ob}(\mathbf{C})$, we have a set $\mathrm{Hom}(x, y)$ of "**morphisms**" from x to y (so if $f \in \mathrm{Hom}(x, y)$, then $f : x \to y$)

equipped with

(1) for any object $x \in \mathrm{Ob}(\mathbf{C})$, an "**identity morphism**" $\mathrm{id}_x : x \to x$;
(2) for any pair of morphisms $f : x \to y$ and $g : y \to z$, a morphism $g \circ f : x \to z$ called the "**composition**" of f and g (note it's written and read from right to left, like Chinese, to confuse undergraduates)

such that

(1) for any morphism $f : x \to y$, the "**left and right unit laws**" hold: $f \circ \mathrm{id}_x = f = \mathrm{id}_y \circ f$;
(2) for any triple of morphisms $f : w \to x$, $g : x \to y$, $h : y \to z$, the "**associative law**" holds: $(h \circ g) \circ f = h \circ (g \circ f)$.

This is not all too enlightening, at least immediately. It'd be a pity to end here, state "We will not insult the intelligence of the reader with examples or theorems, since everything follows immediately", and pronounce the reader a *bona fide* expert category theorist. We will gently introduce various examples before moving on to start thinking categorically.

Remark 4.2. The composition of morphisms $g \circ f$ should be read from right to left (like Chinese or bra-ket notation in quantum mechanics) and it should be intuitively regarded as

$$\text{first do } f \text{ then do } g. \tag{4.1}$$

So do not confuse it with everyday multiplication, *order matters here!*

Remark 4.3. With the arrows, or "morphisms", we have

$$\mathrm{Source}(f) \xrightarrow{f} \mathrm{Target}(f) \tag{4.2}$$

or (more categorically, with duality in mind)

$$\mathrm{Domain}(f) \xrightarrow{f} \mathrm{Codomain}(f) \tag{4.3}$$

where f is a morphism, and the domain/source of f is an object, the target/codomain of f is also an object.

Wait, we just gave a definition! We can show, as with our conjecture that everything "is-a" mathematical object, that a category and a morphism both are mathematical objects! We will only show that a category is an object for now (something to bear in mind: what's a morphism *between* categories?).

We want to have a taxonomy of morphisms, so we begin very simply by supposing what happens when we have the domain be the codomain.

Definition 4.4. Let $\mathbf{C}$ be a category, and $x \in \mathrm{Ob}(\mathbf{C})$. We define an "**endomorphism**" to be a morphism $f \in \mathrm{Hom}(x, x)$. That is

$$f : x \to x \tag{4.4}$$

so its domain is its codomain.

Now, intuitively the notion of composition of morphisms should be likened to multiplication. We have a notion of an inverse operation for multiplication, we call it division. What about the notion of an inverse for a morphism? That is, given some category $\mathbf{C}$, some objects $x, y \in \mathbf{C}$, and a morphism $f \in \mathrm{Hom}(x, y)$, can we find a $g : y \to x$ such that

$$f \circ g = \mathrm{id}_y \tag{4.5}$$

and if so, is it true that

$$g \circ f = \mathrm{id}_x? \tag{4.6}$$

Here the analogy to commutative multiplication should be discarded. The morphisms represent processes, and the order of processes matter. Consider for example h being the morphism of mixing the ingredients of a cake, and i being the process of pouring the contents of the mixing bowl into a pan and placing the pan in an oven preheated to 350 degrees Fehrenheit. Well, $i \circ h$ is baking the normal way, $h \circ i$ is putting the pan in the oven, and then mixing the ingredients together.

Consequently we don't expect Eq (4.5) to necessarily imply Eq (4.6) (nor do we expect Eq (4.6) to necessarily imply Eq (4.5)). We have two new definitions:

Definition 4.5. Let $\mathbf{C}$ be a category, $x, y \in \mathrm{Ob}(\mathbf{C})$, $f \in \mathrm{Hom}(x, y)$. Then we define:

1. f be a "**monomorphism**" (or "**monic**") if for morphisms $g_1, g_2 : x \to \omega$ we have $f \circ g_1 = f \circ g_2$ imply $g_1 = g_2$;
2. a "**epimorphism**" (or "**epic**") if for morphisms $g_1, g_2 : z \to x$ we have $g_1 \circ f = g_2 \circ f$ imply $g_1 = g_2$;
3. the "**left inverse**" or "**retraction of** f" to be a morphism $g \in \mathrm{Hom}(y, x)$ such that $g \circ f = \mathrm{id}_x$;
4. the "**right inverse**" or "**section of** f" to be a morphism $h \in \mathrm{Hom}(y, x)$ such that $f \circ h = \mathrm{id}_y$.

Remark 4.6. Epic is more general than having a right inverse, and monic is more general than having a left inverse. Consider the following situation: we have a category with two objects A, B, three morphisms (the identity morphisms id_A, id_B, and $f : A \to B$). The morphism f is epic since it can be composed on the left by exactly one morphism — id_B. Similarly, when it can be composed on the right by exactly one morphism id_A. This is more or less a trivial situation that is true when we can only compose by the identity on the left or on the right. Whenever this is not true, and a morphism is both epic and monic, it has left and right inverses.

There is one last important type of morphism to consider, it's a way to state that two objects are "the same" in some sense. The idea is if there is an invertible morphism between two objects, it's necessarily "equivalent" in some sense. Why should this be so? Well, an invertible mapping requires having the left and right inverses (a) exist and (b) be the same.

Definition 4.7. Let $\mathbf{A}$ be a category, $X, Y \in \mathrm{Ob}(\mathbf{A})$, and $f : X \to Y$. We say f is an "**isomorphism**" if there is a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

We also have an invertible endomorphism being given a special name:

Definition 4.8. Let $\mathbf{A}$ be a category, $X \in \mathrm{Ob}(\mathbf{A})$, and $f : X \to X$ be an isomorphism. Then f is called an "**automorphism**".

To summarize this section, we have this handy table of morphism properties:

Morphism Property	**What it means**
Automorphism	Invertible endomorphism.
Endomorphism	The domain and codomain are the same.
Epimorphism	Generalization of having a right inverse.
Isomorphism	Generalization of equivalence of domain with codomain.
Monomorphism	Generalization of having a left inverse.
Retraction of f	Left Inverse of f
Section of f	Right Inverse of f

Table 1. A Table of Morphism Properties

4.1. **Categories Are Also Mathematical Objects.** It seems trivial to demonstrate that categories are mathematical objects, but for the sake of rigor we will demonstrate it.

Stuff: What is the stuff? Well, there is the collection of objects. Additionally, for any pair of objects, we have a set of morphisms.

Structure: The structure of a category is simply this: 1) for any object, we have an identity morphism; 2) for any pair of morphisms, we have a composition of morphisms.

Properties: Each component of our structure has a property to satisfy. The identity morphism satisfies the left and right unit laws. The composition of morphisms satisfies the associativity law. We demand nothing more, nothing less.

This is precisely a mathematical object description of a category.

4.2. **A Grocery List of Examples.**

Example 4.9. We have a few examples to start with that are all pretty similar.

0 the category with zero objects and zero morphisms. Well, it's trivially a category, albeit a really boring one.

1 the category with a single object, and a single morphism (the identity morphism), is a category.

2 the category with two objects a and b, and just one morphism $a \to b$ that *is not* an identity.

3 the category with three objects a, b, and c whose morphisms are $a \to b$, $b \to c$ and $c \to a$ (none are identities).

$\downdownarrows$ the category with two objects a, b and two morphisms $a \rightrightarrows b$ that aren't the identity.

A few properties of categories are worth noting here and now.

Definition 4.10. A category is said to be "**Thin**" if there is at most one morphism from an object to another.

Definition 4.11. A category is said to be "**Discrete**" if every morphism is an identity.

Discrete categories are really just sets since each object has exactly one morphism, and aside from that there are no additional morphisms.

We can continue introducing examples, one large class of examples that the reader may be inclined to notice is that we have a lot of mathematical objects that are "sets equipped with structure", so why not have a category of such "sets equipped with the same structure"? Like the category of all groups, or of all rings, or of all sets? Such categories are called "constructs" or "concrete" depending on the author.

Definition 4.12. A category $\mathbf{C}$ is called "**Concrete**" iff there is a functor $U : \mathbf{C} \to \mathbf{Set}$ to the underlying sets of $\mathbf{C}$ called the "**Forgetful Functor**" (it is faithful, we'll come back to this later).

Example 4.13. Consider **Mon**, the category whose objects are monoids, and whose morphisms are monoidal homomorphisms. It is concrete.

Example 4.14. Consider **Grp**, the category whose objects are groups and whose morphisms are group homomorphisms. It is concrete.

Example 4.15. Consider **Set**, the category whose objects are sets and whose morphisms are functions. It is concrete trivially.

We can also describe a group using category theory. That is given some group, we can turn it into a category defined by a single object and the elements of the group become morphisms in the category. We'll give a few examples of these.

4.3. **Example: Complex Group.** Consider the group generated by $\{1, i = \sqrt{-1}\}$ equipped with multiplication. Explicitly, we have the multiplication table

(4.7)

$\times$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
$-i$	$-i$	i	1	-1
i	i	$-i$	-1	1

We have this categorified in the following diagram

$$\begin{tikzcd}
* \arrow[r, "-i"] \arrow[d, "i"'] \arrow[dr, "-1"] \arrow[rr, bend left, "1"] & * \arrow[r, "i"] & * \\
* \arrow[r, "i"'] & * \arrow[u, "i"'] \arrow[ur, "-1"'] \arrow[r, "i"'] & * \arrow[u, "i"']
\end{tikzcd} \tag{4.8}$$

This encodes all the information about the group generated by $\{1, i\}$. We can do the diagram chasing to see that the multiplication is really encoded in our category theoretic doodling.

4.4. **Example: Quaternion Group.** Consider the quaternions, which were characterized by the equation

$$i^2 = j^2 = k^2 = ijk = -1 \tag{4.9}$$

We have the multiplication table

(4.10)

$\times$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

We have various ways to encode the information in the quaternions into diagrams, e.g.

$$jk = i \quad \iff \quad \begin{tikzcd} * \arrow[d, "j"'] \arrow[dr, "i"] & \\ * \arrow[r, "k"'] & * \end{tikzcd} \tag{4.11}$$

We can cycle through the rest of the identities, or we can skip ahead to the final diagram

$$\begin{tikzcd} * \arrow[r, "i"] \arrow[rr, bend left, "-1"] \arrow[d, "j"'] \arrow[dr, "k"] & * \arrow[r, "i"] \arrow[d, "j"] & * \\ * \arrow[urr, bend right=60, "j"'] & * \arrow[l, "i"] \arrow[ur, "k"] & \end{tikzcd} \tag{4.12}$$

With the understanding that $-1 \circ -1 = 1 \equiv \mathrm{id}_*$.

5. Functors

Consider this: we have morphisms be generalizations of mappings from one object to another, and we have categories be a mathematical object. What is a mapping from one category to another? We call such things *functors* and they encode mathematical procedures, or assigning information to objects and morphisms.

Definition 5.1. Given categories $\mathbf{C}, \mathbf{D}$ a "**functor**" $F : \mathbf{C} \to \mathbf{D}$ consists of

- a function $F : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D})$;
- for any pair of objects $X, Y \in \mathrm{Ob}(\mathbf{C})$, a function $F : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(F(X), F(Y))$;

such that

- F **preserves identities:** for any $X \in \mathbf{C}$, $F(1_X) = 1_{F(X)}$;
- F **preserves composition:** for any pair of morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathbf{C}$, $F(fg) = F(f)F(g)$.

Remark 5.2. It is not uncommon to see expressions like

$$F\left(X \xrightarrow{f} Y\right) = F(X) \xrightarrow{F(f)} F(Y) \tag{5.1}$$

in practice, which tells us how the functor behaves.

We can compose two functors in the obvious way, if $F : \mathbf{B} \to \mathbf{C}$ and $G : \mathbf{C} \to \mathbf{D}$, then $G \circ F : \mathbf{B} \to \mathbf{D}$ is defined by

$$G \circ F\left(X \xrightarrow{f} Y\right) = (G \circ F)(X) \xrightarrow{(G\circ F)(f)} (G \circ F)(Y) \tag{5.2}$$

componentwise composition of the functor on objects and the functor on morphisms.

Also, we have the intuition that a functor is a "morphism between categories". So we'll often use the adjectives and definitions for morphisms for functors, e.g. an endofunctor has its codomain and domain be the same category, an isomorphism between categories is an invertible functor, etc.

Theorem 5.3. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor, and $f : A \to B$ be an isomorphism in $\mathbf{C}$. Then $F(f)$ is an isomorphism in $\mathbf{D}$

Proof. The proof is straightforward, we see that

$$F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) \tag{5.3a}$$

$$= F(\mathrm{id}_B) = \mathrm{id}_{F(B)} \tag{5.3b}$$

and similarly $F(f^{-1}) \circ F(f) = \mathrm{id}_{F(A)}$ which implies that $F(f^{-1})$ is the two-sided inverse of $F(f)$ — i.e. $F(f)$ is an isomorphism in $\mathbf{D}$. □

This is a powerful tool, since we often will take the approach in disproving something is a functor by showing that f is an isomorphism but $F(f)$ is not.

NOTE: just because $F(k)$ may be an isomorphism, it *in no way implies ANYTHING* about k being an isomorphism. It may possibly be an isomorphism, it may not. This property ($F(k)$ is an iso $\Rightarrow$ k is an iso) is called "reflection" of isomorphisms, it is not necessarily true for functors.

We have a number of examples and properties of functors. The most important example to consider is the identity functor.

Example 5.4. Let $\mathbf{C}$ be some category, with $X, Y \in \mathbf{C}$ and $f : X \to Y$. Let $F = \mathrm{id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ be the identity functor. Then

$$F\left(X \xrightarrow{f} Y\right) = X \xrightarrow{f} Y \tag{5.4}$$

just as one expects.

Why is this an important example? Well, we can introduce the notion of inverse functors now. That is, if $F : \mathbf{C} \to \mathbf{D}$ we can ask if there is a $G : \mathbf{D} \to \mathbf{C}$ such that $G \circ F = \mathrm{id}_{\mathbf{C}}$ and $F \circ G = \mathrm{id}_{\mathbf{D}}$? If we can find such a G then F is invertible or more precisely an "*isomorphism*". How do we know when we are dealing with isomorphisms? Well, the intuition to have is that they are analogies. That is, we can set up tables of objects corresponding uniquely to objects, and morphisms corresponding uniquely to morphisms.

Definition 5.5. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor such that there is a functor $F^{-1} : \mathbf{D} \to \mathbf{C}$ satisfying (1) $F^{-1} \circ F = \mathrm{id}_{\mathbf{C}}$, (2) $F \circ F^{-1} = \mathrm{id}_{\mathbf{D}}$. Then we define F to be an "**Isomorphism**" and say that $\mathbf{C}$ is "**isomorphic**" to $\mathbf{D}$.

Note that isomorphic categories are "essentially the same". We have analogous objects and analogous morphisms between them. The intuition one should have is that an isomorphism functor sets up an analogy between two categories.

Example 5.6. We have a functor $X : \mathbf{1} \to \mathbf{C}$ be a "**point**", that is it picks out a single object $X(*) \in \mathbf{C}$. Most of the times, we use the notation that $X(*) = X$ the point's name is the object it points out. We then have an interesting point of view that *an object is a functor.*

Example 5.7. We have $F : \mathbf{2} \to \mathbf{C}$ be a functor which is then just defined by

$$F\left(a \xrightarrow{f} b\right) = F(a) \xrightarrow{F(f)} F(b) \tag{5.5}$$

which just specifies a morphism in $\mathbf{C}$. So the morphisms from $\mathbf{2}$ to $\mathbf{C}$ picks out morphisms.

Example 5.8. We have $F : \mathbf{3} \to \mathbf{C}$ be a functor. What does it do? Well, it takes in the category with 3 objects and 3 distinct, non-identity morphisms, and it spits out at most 3 objects and 3 morphisms in $\mathbf{C}$. The functor has to obey the composition of morphisms, so this specifies one composite morphism in $\mathbf{C}$. It is then just defined by

$$F\left(a \xrightarrow{f} b \xrightarrow{g} c\right) = F(a) \xrightarrow{F(f)} F(b) \xrightarrow{F(g)} F(c) \tag{5.6}$$

which just specifies a morphism in $\mathbf{C}$. This defines a composite morphism $F(g) \circ F(f) : F(a) \to F(c)$. So the morphisms from $\mathbf{3}$ to $\mathbf{C}$ picks out composite morphisms and its components.

Example 5.9. Consider some arbitrary category $\mathbf{C}$. We can construct a functor from $\mathbf{C} \to \mathbf{Set}$ by choosing some object $C \in \mathbf{C}$ and then considering

$$\mathrm{Hom}_{\mathbf{C}}(C, -)\left(A \xrightarrow{f} B\right) = \mathrm{Hom}(C, A) \xrightarrow{\mathrm{Hom}(C,f)} \mathrm{Hom}(C, B) \tag{5.7}$$

where $\mathrm{Hom}(C, A)$ and $\mathrm{Hom}(C, B)$ are sets of morphisms from C to A (resp. B) and $\mathrm{Hom}(C, f)$ is a function given by $g \mapsto f \circ g$ for each $g \in \mathrm{Hom}(C, A)$. The functor $\mathrm{Hom}(C, -) : \mathbf{C} \to \mathbf{Set}$ is called the "**Hom-functor**".

We have a few useful definitions we should cover before getting to examples. Our taxonomy of functors are:

Definition 5.10. Let $F : \mathbf{A} \to \mathbf{B}$ be a functor. F is called an "**embedding**" provided that F is injective on morphisms.

Remark 5.11. This is for the obvious reason. If we think of a directed subgraph embedded in a directed graph, there is an injective relation between the edges of the subgraph to the graph.

Definition 5.12. Let $F : \mathbf{A} \to \mathbf{B}$ be a functor. F is called "**faithful**" provided that all the hom-set restrictions

$$F : \mathrm{Hom}_{\mathbf{A}}(A, A') \to \mathrm{Hom}_{\mathbf{B}}(F(A), F(A')) \tag{5.8}$$

are injective.

Definition 5.13. Let $F : \mathbf{A} \to \mathbf{B}$ be a functor. F is called "**full**" provided all the hom-set restrictions are surjective.

Definition 5.14. Let $F : \mathbf{A} \to \mathbf{B}$ be a functor. F is called "**amnestic**" provided that an $\mathbf{A}$-isomorphism f is an identity whenever $F(f)$ is an identity.

Proposition 5.15. Observe that a functor is:

(1) an embedding if and only if it is injective on objects and it is faithful;
(2) an isomorphism if and only if it is bijective on objects, full, and faithful.

Corollary 5.16. A embedding F is an isomorphism iff it is surjective on objects and full.

We can also specify functors as covariant or contravariant. Most of the times it's covariant, it's as we defined it. When it's contravariant, the domain is the dual category. That is $F : \mathbf{C} \to \mathbf{D}$ is covariant, then $F' : \mathbf{C}^{op} \to \mathbf{D}$ is contravariant. In other words, we can define it thus:

Definition 5.17. A "**Contravariant Functor**" is a functor $F : \mathbf{C}^{op} \to \mathbf{D}$.

So to each morphism $f : Y \to X$ in $\mathbf{C}$, we have $F(f) : F(X) \to F(Y)$ in $\mathbf{D}$. We also have the compositions go the other way, that is for $f : X \to Y$, $g : Y \to Z$, we have $F(g \circ f) = F(f) \circ F(g)$.

We summarize our taxonomy of functors in the following table:

Functor Property	**What it means**
Amnestic	Reflects identities.
Equivalence	Full, Faithful, and essentially surjective.
Embedding	Injective on morphisms.
Essentially Surjective	For each object B in the codomain, there is an A in the domain s.t. $F(A)$ is isomorphic to B.
Faithful	Hom-set restrictions are injective
Full	Hom-set restrictions are surjective
Isomorphism	Bijective on morphisms and objects.
	(Alternatively) It's invertible.
	(Alternatively) It's an equivalence relation on the conglomerate of all categories (i.e. the domain and codomain are "essentially the same").

Table 2. A Table of Functor Properties.

5.1. **Functors are Mathematical Procedures.** To give an example of how a functor encodes mathematical procedures, consider the following:

Example 5.18. Consider $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ be the powerset functor. That is, it takes objects (sets) to sets of all possible subsets of it, and functions $f : A \to B$ to functions between the powersets $\mathcal{P}(A)$ and $\mathcal{P}(B)$. It's a functor since (1) it maps the identity of A to the identity $\mathcal{P}(A)$, and (2) it maps the composition to the composition of morphisms in the obvious way.

Example 5.19. Consider $F : \mathbf{Set} \to \mathbf{Mon}$ the functor which associates to each set $S \in \mathbf{Set}$ the monoid freely generated by it $F(S) \in \mathbf{Mon}$. What happens is if we have a set $\{a, b\}$, the monoid freely generated by it consists of strings whose letters are a or b, the operator concatenates two strings together. That is it takes one string $(aababbbaababa)$ and another $(bbabbaabbaa)$ and the operator then

$$(aababbbaababa) * (bbabbaabbaa) = (aababbbaababababbabbaabbaa) \tag{5.9}$$

just glues the two strings together in the obvious way. The identity element is just the empty string $()$ since

$$(ababaabbaaa) * () = (ababaabbaaa) \tag{5.10}$$

attaches nothing to the end of the string, and it attaches nothing to the end of every string, so it "does nothing".

5.2. **Functors as Assigning Information: Sheaves as Example.** This is one use of functors, to embody mathematical procedures. The other use is to assign information to each object in a category. This could be viewed as one particular mathematical procedure, the intuition should be the same as that of a vector field assigning a vector to each point in a space. The most famous generalization of this is known as "sheaves" and "presheaves".

Recall in topology [**?**], we defined a topology (in an example in subsection 3.1) $\mathcal{T}$ over a set X to be a sufficiently nice collection of subsets of a given set. We defined these subsets to be "open subsets" and elements of X to be "points" of X.

Our real aim is to generalize the notion of a vector field (a mathematical object that assigns to each point of $\mathbb{R}^n$ a vector). We need to use the aforementioned notions from topology to generalize "where" we assign mathematical objects. That is, we proceed in our generalization by assigning to each open subset U of our space X some "local data" i.e. mathematical object defined on U.

As this is a collection of notes on category theory, we will approach the problem categorically. First we realize that open subsets $V \subseteq U \subseteq X$ we have an inclusion mapping

$$i_{V,U} : V \to U \tag{5.11}$$

which is continuous.

Now, what would assign a mathematical object to an open subset? Well, we conveniently used insight that an open subset is an object in a category $\mathbf{O}(X)$ which is our topological space as a category. Why not claim that a functor $F : \mathbf{O}(X) \to \mathbf{V}$ assigns to each open subset $U \in \mathbf{O}(X)$ some information $F(U)$? We have to be careful about consistency problems. It would be bad if two local regions that overlap end up having inconsistent data on the overlap[1]. We then specify that for each inclusion of open sets $V \subseteq U$, we need a restriction morphism

$$\rho_{V,U} : F(U) \to F(V) \tag{5.12}$$

in the category $\mathbf{V}$. This morphism has two properties of importance:

(1) for each open $U \subseteq X$, the restriction morphism $\rho_{U,U} : F(U) \to F(U)$ is the identity morphism on $F(U)$;
(2) if we have $W \subseteq V \subseteq U$ open, then $\rho_{W,V} \circ \rho_{V,U} = \rho_{W,U}$.

[1]It'd be unimaginable, for example, that upon hearing the temperature throughout California ranges from 80 to 100 degrees that Davis, California is 110 degrees. We don't have consistency on the overlap!

This is a step towards ensuring consistency on overlaps.

BUT This means that our first attempt at a definition is WRONG! Observe the arrows *go the wrong way in the codomain!* This means that our functor is really a *contravariant* functor $F : \mathbf{O}(X)^{op} \to \mathbf{V}$, but we did this rather quickly and subtle details are swept under the rug so lets explain why this is a good definition. First observe a contravariant functor behaves as follows:

$$F\left(U \xrightarrow{f} V\right) = F(V) \xrightarrow{F(f)} F(U) \tag{5.13}$$

which is not good, *UNLESS* we reverse the arrow again! That is, we have in our category $\mathbf{O}(X)$ a term

$$U \xrightarrow{f} V$$

so if we first reverse this arrow by using the category dual to it $\mathbf{O}(X)^{op}$ we have our arrows be

$$V \xrightarrow{f} U$$

which means a contravariant functor would behave thusly:

$$F\left(V \xrightarrow{f} U\right) = F(U) \xrightarrow{F(f)} F(V) \tag{5.14}$$

which is precisely what is desired! The cost, however, is to have the functor take in the dual category as its domain. But we are rich enough to pay.

We can now define a presheaf on some general category thus:

Definition 5.20. Let $\mathbf{V}$ be some category, then a $\mathbf{V}$-valued "**Presheaf**" F on a category $\mathbf{C}$ is a functor

$$F : \mathbf{C}^{op} \to \mathbf{V}. \tag{5.15}$$

If we restrict our attention to the category $\mathbf{O}(X)$ of open subsets of X, we recover the intuitive generalization of vector fields we introduced in this section.

If F is a $\mathbf{V}$-valued presheaf on X, and U is an open subset of X, then we call $F(U)$ the "**sections of F over U**". If $\mathbf{V}$ is a concrete category, then each element of $F(U)$ is called a "**section**". A section over X is called a "**global section**". We adopt the notion that the restriction of a section

$$\rho_{V,U}(s) = s|_V \tag{5.16}$$

This is somewhat similar to sections with fiber bundles.

We will now quickly define a sheaf in the most intuitively appealing way:

Definition 5.21. A "**sheaf with value in a concrete category V**" is a presheaf with values in $\mathbf{V}$ such that

1. (Normalization) $F(\emptyset)$ is the terminal object of $\mathbf{V}$;
2. (Local Identity) If (U_i) is an open covering of an open set U, and if $s, t \in F(U)$ are such that when restricted on each U_i of the open covering $s|_{U_i} = t|_{U_i}$, then $s = t$; and
3. (Gluing) If (U_i) is an open covering of an open set U, and if for each i there is a section s_i of F over U_i such that for each pair U_i, U_j of the covering sets, the restrictions s_i and s_j agree on the overlaps $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is a section $s \in F(U)$ such that $s|_{U_i} = s_i$ for each i.

This definition is really not as general as it could be, since the normalization property assumes the objects are open subsets in a topology.

At any rate, the section s whose existence is guaranteed by the third property is usually called the "gluing", "concatenation", or "collation" of the section s_i. We see that it is unique (by our local identity property). Sections s_i satisfying the condition of gluing property are often called "**compatible**"; so when we look at the gluing and local identity properties, we can summarise their meaning thus: *compatible sections can be uniquely glued together.* This guarantees consistency on overlap.

5.3. **Quantization (Elaborate Non-Example of a Functor).** (For finite dimensional symplectic vector spaces over finite fields, there actually is a quantization functor [**?**]. But for the general case in physics, there is no such functor – there is a no go theorem due to Groenewald and van Hove. The interested reader is referred to [**?**, **?**].)

Consider the canonical procedure to quantizing classical systems. Mathematically, we describe classical systems by symplectic manifolds, and quantum systems by Hilbert spaces. So, quantization should naively be a functor

$$\mathcal{Q} : \mathbf{Symp} \to \mathbf{Hilb} \tag{5.17}$$

where **Symp** is the category of symplectic manifolds whose morphisms are symplectic maps (or in the jargon of Hamiltonian mechanics, "canonical transformations"); **Hilb** is the category of Hilbert spaces, whose morphisms are unitary mappings.

This should seem somewhat unnatural for a host of reasons. First, classical mechanics should be *obtained from quantum mechanics* in some appropriate limit. Why would would expect to obtain a unique quantum theory from a given classical description of a system?

Ignoring this, we have a problem that requires some explanation. Suppose we had such a functor. Let $u^i = (q, p)$ be a choice of variables for an object in **Symp** with symplectic structure ω_{ij}. Recall the symplectic structure is such that

$$\{u_i, u_j\} = \omega_{ij} \quad \Rightarrow \quad \{f, g\} = \frac{\partial f}{\partial u_i}\omega_{ij}\frac{\partial g}{\partial u_j} \tag{5.18}$$

where $\{-,-\}$ is the Poisson bracket. Then we have some properties for $\mathcal{Q}$:

1. (Linearity) $\mathcal{Q}(c_1 f + c_2 g) = c_1\mathcal{Q}(f) + c_2\mathcal{Q}(g)$;
2. (Identity Preserved) $\mathcal{Q}(1) = I$;
3. (Poisson Bracket) $\mathcal{Q}(\{f,g\}) = (-i/\hbar)[\mathcal{Q}(f), \mathcal{Q}(g)]$;
4. (Irreducible) the operators $\mathcal{Q}(\mathbf{x})$ and $\mathcal{Q}(\mathbf{p})$ are irreducibly represented.

The domain of such a mapping is called the "*space of quantizable observables*". The irreducibility condition is somewhat enigmatic. It can be reworded as:

> Let $\mathcal{H}$ be the Hilbert space $\mathcal{Q}(\mathbf{x})$ and $\mathcal{Q}(\mathbf{p})$ act on. Then there are no subspaces $\mathcal{H}_0 \subset \mathcal{H}$ (other than $\{0\}$ and $\mathcal{H}$ itself) that are stable under the action of all the operators $\mathcal{Q}(\mathbf{x})$ and $\mathcal{Q}(\mathbf{p})$.

Now, here comes the problems: lets work in 1 dimension, and let

$$f(p,q) = p^2q^2 = (pq)^2 \tag{5.19}$$

We can write

$$(p+q)^2 = 2pq + p^2 + q^2 \quad \Rightarrow \quad pq = \frac{(p+q)^2 - p^2 - q^2}{2}. \tag{5.20}$$

When we quantize $\mathcal{Q}(pq)$ we can quantize this:

$$\mathcal{Q}(pq) = \frac{1}{2}\big((\mathcal{Q}(p)+\mathcal{Q}(q))(\mathcal{Q}(p)+\mathcal{Q}(q)) - \mathcal{Q}(q)^2 - \mathcal{Q}(p)^2\big) \tag{5.21a}$$

$$= \frac{\mathcal{Q}(p)\mathcal{Q}(q)+\mathcal{Q}(q)\mathcal{Q}(p)}{2}. \tag{5.21b}$$

Similarly we can write

$$p^2q^2 = \frac{(p^2+q^2)^2 - p^4 - q^4}{2} \quad \Rightarrow \quad p^2q^2 = \frac{(p^2+q^2)^2 - p^4 - q^4}{2}. \tag{5.22}$$

and by the same reasoning as before, when quantized we get

$$\mathcal{Q}(p^2q^2) = \frac{\mathcal{Q}(p^2)\mathcal{Q}(q^2)+\mathcal{Q}(q^2)\mathcal{Q}(p^2)}{2} \tag{5.23}$$

Now, our argument is summarized in the following diagram

$$\begin{tikzcd}
p^2q^2 \arrow[r, "id"] \arrow[d, "\mathcal{Q}"'] & (pq)^2 \arrow[d, "\mathcal{Q}"] \\
\left(\dfrac{(\mathcal{Q}(p)^2+\mathcal{Q}(q)^2)^2 - \mathcal{Q}(p)^4 - \mathcal{Q}(q)^4}{2}\right) & \left(\dfrac{\mathcal{Q}(p)\mathcal{Q}(q)+\mathcal{Q}(q)\mathcal{Q}(p)}{2}\right)^2
\end{tikzcd}$$

where *id* is just the identity morphism. Equivalently, we could write

$$\mathcal{Q}\left(\frac{(p^2+q^2)^2 - p^4 - q^4}{2} \xrightarrow{id} \left[\frac{pq+qp}{2}\right]^2\right) \tag{5.24}$$

$$= \left(\frac{(\mathcal{Q}(p)^2+\mathcal{Q}(q)^2)^2 - \mathcal{Q}(p)^4 - \mathcal{Q}(q)^4}{2}\right) \xrightarrow{???} \left(\frac{\mathcal{Q}(p)\mathcal{Q}(q)+\mathcal{Q}(q)\mathcal{Q}(p)}{2}\right)^2$$

What is the morphism on the right hands side of this equation?

By our specification of quantization, we should expect an identity morphism to make the diagram commute. But clearly

$$\frac{\mathcal{Q}(p^2)\mathcal{Q}(q^2)+\mathcal{Q}(q^2)\mathcal{Q}(p^2)}{2} \neq \left(\frac{\mathcal{Q}(p)\mathcal{Q}(q)+\mathcal{Q}(q)\mathcal{Q}(p)}{2}\right)^2 \tag{5.25}$$

In other words, our diagram doesn't commute! So the property

$$\mathcal{Q}(id) = id \tag{5.26}$$

doesn't always hold, which is a critical property of a functor. We then conclude that quantization *is not a functor.*

6. Natural Transformations

We might start thinking: we had objects and morphisms between them. We had categories and morphisms (called "functors") between them. Now we have functors, do we have morphisms between them? It turns out we can, this is the importance of natural transformations. Before getting to them, it should be noted that Category Theory was invented to investigate natural transformations. It is more useful than functors, but it is trickier to start thinking in terms of them.

Definition 6.1. Given two functors $S, T : \mathbf{A} \to \mathbf{B}$, a "**Natural Transformation**" is a function $\tau : S \Rightarrow T$ which assigns to each object $X \in \mathrm{Ob}(\mathbf{A})$ an arrow $\tau_X = \tau(X) : S(X) \to T(X)$ of $\mathbf{B}$ in such a way that every arrow $f : X \to Y$ in $\mathbf{A}$ yields a diagram

$$\begin{tikzcd} S(X) \arrow[r, "S(f)"] \arrow[d, "\tau_X"] & S(Y) \arrow[d, "\tau_Y"] \\ T(X) \arrow[r, "T(f)"] & T(Y) \end{tikzcd} \tag{6.1}$$

which is commutative. This condition that the diagram described by eq (6.1) is commutative is called the "**naturality condition**"; when this holds, we also say that $\tau_X : S(X) \to T(X)$ is "**natural**" in X. We call τ_X, τ_Y the "**components**" of the natural transformation.

Remark 6.2 (Notation)**.** We have a natural transformation $\tau : F \Rightarrow G$ usually denoted with the $F \Rightarrow G$ instead of $F \to G$.

There are probably more than two ways to picture a natural transformation. We'll focus on the two obvious ones: as a deformation of one functor into another, or — if we view functors as assigning information to objects of a category — as a natural way to transform information assigned to objects of a category.

One way is as a "deformation" (in the homotopy-theoretic sense of the word) of one functor into another. But recall with a homotopy, we had specified one path $\gamma_1 : [0,1] \to \mathbb{C}$ is homotopic to another $\gamma_2 : [0,1] \to \mathbb{C}$ if we have a continuous function

$$H(s,t) = (1-s)\gamma_1(t) + s\gamma_2(t) \tag{6.2}$$

such that $H(0,t) = \gamma_1(t)$ and $H(1,t) = \gamma_2(t)$. To construct an analogous deformation, we need some category analogous to the $s \in [0,1]$ term. This is precisely $\mathbf{2}$ category. But a natural transformation from $F : \mathbf{C} \to \mathbf{D}$ to $G : \mathbf{C} \to \mathbf{D}$ becomes a functor

$$\alpha : \mathbf{C} \times \mathbf{2} \to \mathbf{D} \tag{6.3}$$

where $\mathbf{2}$ is the categorical analog to the "interval".

An aside on product categories, if the reader is unfamiliar with them, one should envision the objects of $\mathbf{C} \times \mathbf{2}$ being ordered pairs $(C,0)$ and $(C,1)$ for all $C \in \mathbf{C}$. That is, we end up with two copies of the category $\mathbf{C}$. The morphisms are also ordered pairs (g,f) where $f : 0 \to 1$ and $g : C \to D$. We can "break them up" in the sense that the diagram

$$\begin{tikzcd} (C,0) \arrow[rr, "(g{,}\mathrm{id}_0)"] \arrow[d, "(\mathrm{id}_C{,}f)"'] \arrow[drr, "(g{,}f)"] & & (D,0) \arrow[d, "(\mathrm{id}_D{,}f)"] \\ (C,1) \arrow[rr, "(g{,}\mathrm{id}_1)"'] & & (D,1) \end{tikzcd} \tag{6.4}$$

commutes. So (id, f) transaltes from one copy to the other, and $(g, \mathrm{id}_)$ acts on one copy.

Observe that if we let $0, 1 \in \mathbf{2}$ and $f : 0 \to 1$ so for some fixed object denoted by $(\cdot)$ we have $\alpha(\cdot, 0) = F(\cdot)$, $\alpha(\cdot, 1) = G(\cdot)$, and

$$\alpha(\cdot, 0 \xrightarrow{f} 1) = F(\cdot) \xrightarrow{\alpha(\cdot,f)} G(\cdot) \tag{6.5}$$

which should begin looking vaguely familiar. We shouldn't be too surprised since the category really looks like

$$\begin{tikzcd} \mathbf{C} \times 0 & \cdot \arrow[r] \arrow[d, "f"'] & \cdot \arrow[d, "f"] \\ \mathbf{C} \times 1 & \cdot \arrow[r] & \cdot \end{tikzcd} \tag{6.6}$$

which is two copies of $\mathbf{C}$. We have specifically, since it's a functor, we see how it acts on the diagram

$$\begin{tikzcd} \alpha(\mathbf{C} \times 0) & F(X) \arrow[r, "F(g)"] \arrow[d, "\alpha(X,f)"'] & F(Y) \arrow[d, "\alpha(Y,f)"] \\ \alpha(\mathbf{C} \times 1) & G(X) \arrow[r, "G(g)"] & G(Y) \end{tikzcd} \tag{6.7}$$

which is precisely a naturality condition! It's not too far a stretch to state that $\alpha(X, f)$ and $\alpha(Y, f)$ are components of the natural transformation.

The other intuition to have is to think of a functor as assigning some mathematical object to each object of its domain. We have some intuition of how to transform objects into other objects via morphisms and functors. A natural transformation, then, is nothing more than changing the information assigned to each object in the domain. But this is done in some "natural" way. It's specifically natural if it satisfies the naturality condition.

Think for a moment about the importance of the diagram commuting. What this means is that

$$T(f) \circ \tau_X = \tau_Y \circ S(f) \tag{6.8}$$

or intuitively "translate how to assign information, then translate information = translate information, then translate how to assign information." This means the result of a mathematical process from one category translated into another category is the same as translating the "ingredients" from one category then applying the other process. Or, in terms of our problem, both recipes yield cake.

Now, we can think of categories as consisting of diagrams. The above definition gives us instructions how to "translate" from a morphism in the category described by $S(\mathbf{A})$ to the corresponding morphism in the category described by $T(\mathbf{A})$. This is precisely what the intuition behind the definition of a natural transformation as a functor from $S(\mathbf{A}) \times \mathbf{2} \to T(\mathbf{A})$ is!

We have diagrams "built" from morphisms, and we know how to "translate" each morphism individually, so it's not too hard of a stretch to figure out how to "translate" a diagram. We can also have the intuition that a natural transformation is a "*morphism of functors*".

Example 6.3. Let S be a fixed set, X^S be the set of all functions $h : S \to X$. We want to show

(1) $X \mapsto X^S$ is the object function of a functor $\mathbf{Set} \to \mathbf{Set}$, and that

(2) evaluation $e_X : X^S \times S \to X$ (defined by $e(h,s) = h(s)$) is a natural transformation.

The overall scheme of things is we wish to assign some information on each set in **Set**. We wish to assign on $X \in \mathbf{Set}$ the information $\mathrm{Hom}(S,X) \times \mathrm{id}_S$ on the one hand, and X itself on the other. We wish to go from the first to the second in the "natural" way of evaluating functions. It seems that this should be a natural transformation, but we should show it in two steps.

(1) Consider the category of sets **Set**. The Covariant-Hom functor (recall example 5.9) is:

$$\mathrm{Hom}(S,-) : \mathbf{Set} \to \mathbf{Set} \tag{6.9}$$

defined by

$$\mathrm{Hom}(S,-)\left(B \xrightarrow{f} C\right) = \mathrm{Hom}(S,B) \xrightarrow{\mathrm{Hom}(S,f)} \mathrm{Hom}(S,C) \tag{6.10}$$

maps $X \mapsto X^S$ where $X^S = \mathrm{Hom}(S,X)$.

(2) We wish to show that evaluation of a function in the obvious way is a natural transformation. The naive way to set up the commutative diagram describing our naturality condition would be thus:

$$\begin{tikzcd} S(X) \arrow[r, "S(f)"] \arrow[d, "\tau_X"] & S(Y) \arrow[d, "\tau_Y"] \\ T(X) \arrow[r, "T(f)"] & T(Y) \end{tikzcd} \tag{6.11}$$

This is assuming, of course, that the functor in question is $\mathrm{Hom}(S,-) \times \mathrm{id}_S$. What would be the codomain of our natural transformation? Well, we would have to describe evaluation as

$$e : \mathrm{Hom}(S,-) \times \mathrm{id}_S \to \mathrm{id}_{\mathbf{Set}} \tag{6.12}$$

but this is precisely a natural transformation. This concludes our example

Remark 6.4. Observe that here our natural transformation was really just transforming information on each set $X \in \mathbf{Set}$ assigned via the functors $\mathrm{Hom}(S,X) \times \mathrm{id}_S$ and $\mathrm{id}_{\mathbf{Set}}$. The first functor assigns all functions mapping S to X and the set S, the second is just the identity. What's the natural thing to do? Simply take a function and "feed in" S. This transforms the first functor into the second.

Definition 6.5. Let $F, G : \mathbf{A} \to \mathbf{B}$ be functors, a natural transformation $\tau : F \Rightarrow G$ with every component $\tau(X)$ invertible in $\mathbf{B}$ is called a "**Natural Equivalence**" or (better) a "**Natural Isomorphism**". We denote this with the special symbol $\tau : F \cong G$. In such a case, $(\tau(Y))^{-1}$ in $\mathbf{B}$ are the components of a natural transformation $\tau^{-1} : G \Rightarrow F$.

Definition 6.6. We wish to give a natural transformation definition for an "**equivalence**". That is, a functor $F : \mathbf{A} \to \mathbf{B}$ is an "equivalence" if it has a "**weak inverse**", i.e. a functor $G : \mathbf{B} \to \mathbf{A}$ such that there exists natural isomorphisms $\alpha : G \circ F \cong 1_{\mathbf{A}}$, $\beta : F \circ G \cong 1_{\mathbf{B}}$.

We can also introduce several ways to construct more natural transformations:

Proposition 6.7. Given functors $F, G : \mathbf{C} \to \mathbf{D}$, $H : \mathbf{D} \to \mathbf{E}$, $K : \mathbf{B} \to \mathbf{C}$, and natural transformation $\eta : F \Rightarrow G$, we can construct:

- a natural transformation $H\eta : HF \Rightarrow HG$ by defining $(H\eta)_X = H_{\eta(X)}$;
- a natural transformation $\eta K : FK \Rightarrow GK$ by defining $(\eta K)_X = \eta_{K(X)}$.

This turns out to be used a number of times later on in category theory. The intuition is that we can "compose" a natural transformation with a functor "componentwise" (i.e. with the domain and codomain functors), resulting with a natural transformation.

We can also compose natural transformations with natural transformations "in the obvious way" (componentwise). More precisely, we can sketch it out in a proposition:

Proposition 6.8. Let $F, G, H : \mathbf{C} \to \mathbf{D}$ be functors, $\varepsilon : F \Rightarrow G$ and $\eta : G \Rightarrow H$ be natural transformations. Then we can compose the natural transformations to be $\eta \circ \varepsilon : F \Rightarrow H$ whose components are $(\eta \circ \varepsilon)_X = \eta_X \circ \varepsilon_X$ componentwise composed.

Part 2. A Mathematician's Approach (Movement One *Allegretto*)

7. Adjunctions and Adjoints

7.1. Definition via Hom-Set Adjunction. Consider two functors $F : \mathbf{D} \to \mathbf{C}$ and $G : \mathbf{C} \to \mathbf{D}$, and the natural isomorphism

$$\Phi : \mathrm{Hom}_{\mathbf{C}}(F-, -) \cong \mathrm{Hom}_{\mathbf{D}}(-, G-). \tag{7.1}$$

This specifies a family of bijections for each pair of objects $X \in \mathbf{C}$ and $Y \in \mathbf{D}$

$$\Phi_{X,Y} : \mathrm{Hom}_{\mathbf{C}}(F(Y), X) \cong \mathrm{Hom}_{\mathbf{D}}(Y, G(X)). \tag{7.2}$$

In this situation, we say that F is "**Left Adjoint**" to G and G is "**Right Adjoint**" to F.

7.2. Definition via Unit and Counit. We'll introduce the notion of an adjunction as a weaker form of equivalence. That is, we have two categories $\mathbf{C}, \mathbf{D}$. We have a small hierarchy so far of the notion of $\mathbf{C}$ being "the same" as $\mathbf{D}$. We have the notion they are isomorphic if there is an isomorphism $F : \mathbf{C} \to \mathbf{D}$, which happens if it is invertible i.e. there is a $G : \mathbf{D} \to \mathbf{C}$ such that $F \circ G = \mathrm{id}_{\mathbf{D}}$ and $G \circ F = \mathrm{id}_{\mathbf{C}}$.

We can weaken this to the notion of $\mathbf{C}$ and $\mathbf{D}$ are "equivalent" if there are two functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that we have two natural isomorphisms

$$F \circ G \cong \mathrm{id}_{\mathbf{C}} \tag{7.3}$$

and

$$\mathrm{id}_{\mathbf{D}} \cong G \circ F. \tag{7.4}$$

Now, why did we choose to write it this way?

Well, we can weaken the notion of an equivalence of two categories even further. Instead of demanding that we have a pair of natural isomorphisms, we can demand we have "some" arbitrary pair of natural transformations

$$\varepsilon : F \circ G \Rightarrow \mathrm{id}_{\mathbf{C}} \tag{7.5}$$

and

$$\eta : \mathrm{id}_{\mathbf{D}} \Rightarrow G \circ F. \tag{7.6}$$

We call them the counit and unit (respectively). We demand they satisfy the demand that

$$F \xrightarrow{F\eta} F \circ G \circ F \xrightarrow{\varepsilon F} F \tag{7.7}$$

and

$$G \xrightarrow{\eta G} G \circ F \circ G \xrightarrow{G\varepsilon} G \tag{7.8}$$

when composed are the identity natural transformations on F and G (respectively). We call eqs (7.7) (7.8) the "**Triangle Identities**". When this happens we say that F is "**Left Adjoint**" to G, and G is "**Right Adjoint**" to F.

7.3. **Equivalence of Two Definitions.** This derivation is really inspired from Baez [**?**]. We have the definition of an "adjunction" as two functors

$$L : \mathbf{C} \to \mathbf{D} \quad \text{and} \quad R : \mathbf{D} \to \mathbf{C} \tag{7.9}$$

and a natural isomorphism between $\mathrm{Hom}_{\mathbf{D}}(Lc, d)$ and $\mathrm{Hom}_{\mathbf{C}}(c, Rd)$. So it's just a pair of functors L, R and a natural isomorphism.

What happens if we take $c = Rd$? This could only affect one of three things (either one of the functors or the natural isomorphism). We see that our natural isomorphism becomes

$$\mathrm{Hom}(LRd, d) \cong \mathrm{Hom}(Rd, Rd) \tag{7.10}$$

which is interesting. We know there is a special object in $\mathrm{Hom}(Rd, Rd)$, namely the identity morphism $\mathrm{id}_{Rd} : Rd \to Rd$. This implies that there is a special object in $\mathrm{Hom}(LRd, d)$ since the two objects are isomorphic. Specifically we'll denote

$$e_d : LRd \to d \tag{7.11}$$

denote this special object.

What is this "special object"? What does it do, what intuition should we have when we see it? Well, take $L : \mathbf{Set} \to \mathbf{Mon}$ be the functor which associates to each set $S \in \mathbf{Set}$ the free monoid generated by it. It behaves in the obvious way, the elements of $L(S)$ are lists of elements in S, and the binary operator of $L(S)$ simply concatenates two lists together. We'll let $R : \mathbf{Mon} \to \mathbf{Set}$ be the forgetful functor. It simply forgets the binary operator, and we have – to no great surprise – the set of elements of the monoid. Now our "special morphism" maps LRd to d. Step by step we see that Rd is the set underlying d and $L(Rd)$ to be the free monoid generated by Rd. So the morphism maps LRd to d, what can do this? Well, our binary operator is usually written tacitly as just multiplication, so if we look at these lists as strings of elements of d, what could map strings of elements of d to d? The simplest answer would be to evaluate the string of elements of d, that is carry out the multiplication. This necessarily yields an element in d. That is what our "special morphism" does.

In fact, this is not just the "simplest" choice of morphisms, it's fairly (dare I say) "*natural*" to choose such a morphism. It's not too much of a stretch to say that the morphism e_d defines a natural transformation

$$e : LR \Rightarrow \mathrm{id}_{\mathbf{D}} \tag{7.12}$$

where $\mathrm{id}_{\mathbf{D}}$ is the identity functor on $\mathbf{D}$.

We can similarly ask what happens if we take $d = Lc$? Then we have a natural transformation between $\mathrm{Hom}(c, RLc)$ and $\mathrm{Hom}(Lc, Lc)$. As before, we have a "special" morphism in $\mathrm{Hom}(Lc, Lc)$ which is id_{Lc}. This gives a special object in $\mathrm{Hom}(c, RLc)$. We'll denote this by

$$i_c : c \to RLc. \tag{7.13}$$

Again, we ask "What does it do?"

Using notation from the previous example, we have i_c taking a set of "stuff" c to the set underlying the free monoid generated by this "stuff".

As before, this yields a natural transformation

$$i : \mathrm{id}_{\mathbf{C}} \Rightarrow RL. \tag{7.14}$$

In other words, we end up with a pair of natural transformations from our definition of adjoint functors. When one sees this notion of a pair of natural transformations, one should be reminded of an equivalence of categories. With adjunctions, we have a sort of weaker form of equivalence. We no longer demand that the natural transformations are invertible. (Just as a group is a monoid, so too is an equivalence an adjunction?)

Recall, an equivalence of categories $\mathbf{C}$ and $\mathbf{D}$ is defined as a pair of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ equipped with a pair of natural transformations $e : FG \Rightarrow \mathrm{id}_{\mathbf{D}}$ and $i : \mathrm{id}_{\mathbf{C}} \Rightarrow GF$ such that these natural transformations are invertible.

Remark 7.1. Some people define adoint functors in this manner as a sort of distant cousin to equivalences, others prefer using the "pair of functors equipped with a natural isomorphism" definition. We see that we have derived the former from the latter.

8. Product Categories

Problem. We wish to introduce some notion of a "product" of two categories $\mathbf{A}$ and $\mathbf{B}$, which has the intuition that the objects are "ordered pairs" of objects from $\mathbf{A}$ and $\mathbf{B}$ respectively, and whose morphisms are "ordered pairs" of morphisms from $\mathbf{A}$ and $\mathbf{B}$ respectively. Composition, and everything in general, is done componentwise — at least, intuitively. We would like to abandon the use of set theory, and exclusively use category theory instead.

We'll begin by leading by misexample and use set theoretic terms to make our intuitions more precise.

Lets first begin by introducing a set theoretic definition of product categories:

Definition 8.1. Given two categories $\mathbf{A}$ and $\mathbf{B}$, we defined the "**Product Category**" $\mathbf{A} \times \mathbf{B}$ consisting of

(1) all pairs of objects (X_A, Y_B) where $X_A \in \mathrm{Ob}(\mathbf{A})$, $Y_B \in \mathrm{Ob}(\mathbf{B})$, and
(2) all pairs of objects (X_A, Y_B), (X'_A, Y'_B) a set of morphisms (f, g) where $f : X_A \to X'_A$, $g : Y_B \to Y'_B$.

equipped with

(1) for any object (X_A, Y_B), an identity morphism $(\mathrm{id}_{X_A}, \mathrm{id}_{Y_B})$;
(2) for any pair of morphisms

$$(T, U) \xrightarrow{(f,g)} (T', U') \xrightarrow{(f',g')} (T'', U'')$$

we have a morphism

$$(T,U) \xrightarrow{(f',g')\circ(f,g)} (T'',U'')$$

which is calculated componentwise $(f',g')\circ(f,g) = (f'\circ f, g'\circ g)$.

Now that we have introduced the notion of taking the "product" of two categories, we can introduce "projection functors" to recover the components of the product.

Definition 8.2. Let $\mathbf{A}$, $\mathbf{B}$ be two categories. We define the "**Projection Functors**" P, Q to be given by

$$\mathbf{A} \xleftarrow{P} \mathbf{A}\times\mathbf{B} \xrightarrow{Q} \mathbf{B} \tag{8.1}$$

which is specified by

$$P\left((X,Y) \xrightarrow{(f,g)} (X',Y')\right) = X \xrightarrow{f} X'$$

and

$$Q\left((X,Y) \xrightarrow{(f,g)} (X',Y')\right) = Y \xrightarrow{g} Y'.$$

Now we want to generalize this notion of a product to a more general setting, and in a more category theoretic manner. We have the product category for $\mathbf{Set}\times\mathbf{Set}$ defined as above, the Cartesian product of elements and morphisms with everything done componentwise. This is too set theoretic. We would like to relax the notion of a product a wee bit.

Definition 8.3. Let $\mathbf{C}$ be a category, $X_1, X_2 \in \mathbf{C}$, we define the "**product**" of X_1 and X_2 to be an object $P \in \mathbf{C}$ with two morphisms $\pi_1 : P \to X_1$ and $\pi_2 : P \to X_2$ such that for every object $C \in \mathbf{C}$ equipped with the morphisms

$$f_1 : C \to X_1, \quad \text{and} \quad f_2 : C \to X_2 \tag{8.2}$$

we have that there is a unique function $f : C \to P$ such that the following diagram commute:

$$\begin{tikzcd} & C \arrow[dl, "f_2"'] \arrow[d, dashed, "f"] \arrow[dr, "f_1"] & \\ X_2 & P \arrow[l, "\pi_2"] \arrow[r, "\pi_1"'] & X_1 \end{tikzcd} \tag{8.3}$$

Sometimes we denote $P = X_1 \times X_2$ and $f = \langle f_1, f_2\rangle$.

This may possibly seem like an arbitrary condition to let such a diagram hold, but it secretly contains two lifting problems whose solution is f.

Part 3. A Mathematician's Approach (Movement Two *Andante*)

9. Introduction to Movement Two: Constructions in Categories

We introduced the basic notions in category theory (objects, morphisms, functors, and natural transformations), and we introduced the notion of objects as "stuff" equipped with "structure" such that "properties" hold. We went out of our way to show that categories are objects. Now we are interested in some structure we equip on categories and properties we demand categories to obey.

The first major concept we want to tackle is the notion of universal arrows. But this is too deep a concept to be tackled "head on", we need to cover a few preliminary notions first.

We introduce the notion of duality, and specifically how to find dual properties in category theory. Intuitively, we find the "dual" to something by "reversing the direction of the arrows".

Following this principle, we'll move on to initial and terminal objects. Put simply, initial objects have exactly one arrow to every object in the category. Terminal objects are "dual" to this (every object has exactly one arrow to the terminal object).

We then proceed to introduce the notion of comma categories. That is, a category has objects and morphisms, and morphisms are themselves (in a sense) objects. The logical question is: can we have a category whose objects are morphisms? This is precisely what comma categories formalize.

These notions provide sufficient structure and properties to introduce the notion of universal arrows. That is, in math we come across the recurring phrase "... *there exists* a *unique* function such that..." which is concerned with: (1) existence, i.e. define entities; and (2) uniqueness, i.e. prove properties. It turns out that a universal arrow is an initial object in a comma category.

We'll cover a few examples of universal arrows and its usefulness, but that concludes this movement of the mathematician's approach.

10. Duality Principle

Duality is a very useful notion in category theory. We use it to get "two objects for the price of one definition".

Definition 10.1. Given a category $\mathbf{A}$, we can define a "**dual category of A**" consists of

(1) a collection of objects $\mathrm{Ob}(\mathbf{A})$, and
(2) for each pair of objects $X, Y \in \mathrm{Ob}(\mathbf{A})$ the set $\mathrm{Hom}_{\mathbf{A}^{op}}(A, B) = \mathrm{Hom}_{\mathbf{A}}(B, A)$.

All the structure and properties of a dual category are inherited from a category in the obvious way (loosely, "by reversing the arrows").

Usually, we use the prefix *co-* for dual objects, hence why we use codomain — when we reverse the direction of f, we get

$$f^{\mathrm{op}} : y \to x \tag{10.1}$$

where f^{op} is the dual to f, and y is the domain of the dual to f *or the "codomain"*.

Remark 10.2. Observe that the dual some property $\mathcal{P}^{\mathrm{op}}$ is $\mathcal{P}$ (reversing the direction of reversed arrows returns the original direction, kind of like squaring -1 yields 1).

We will give the general procedure for finding the dual of a property about objects X in $\mathbf{A}$:

Example 10.3. Consider the property of objects X in $\mathbf{A}$:

$$\begin{aligned}\mathcal{P}_{\mathbf{A}}(X) \equiv\ & \textit{For any } \mathbf{A}\textit{-object } A \textit{ there exists is exactly one}\\ & \mathbf{A}\textit{-morphism } f : A \to X\end{aligned}$$

Step 1: In $\mathcal{P}_{\mathbf{A}}(X)$, replace all occurrences of $\mathbf{A}$ by $\mathbf{A}^{op}$, thus yielding the property

$$\begin{aligned}\mathcal{P}_{\mathbf{A}^{op}}(X) \equiv &\ \textit{For any } \mathbf{A}^{op}\textit{-object } A \textit{ there exists is exactly one}\\ &\ \mathbf{A}^{op}\textit{-morphism } f : A \to X\end{aligned}$$

Step 2: "Translate it into the logically equivalent statment." That is translate it into the equivalent statement, translating the dual category into the original category, dual morphisms into the original ones, etc.:

$$\begin{aligned}\mathcal{P}^{op}_{\mathbf{A}}(X) \equiv &\ \textit{For any } \mathbf{A}\textit{-object } A \textit{ there exists is exactly one}\\ &\ \mathbf{A}\textit{-morphism } f : X \to A.\end{aligned}$$

Similarly, there is a procedure for finding the dual for a property about morphisms:

Example 10.4. Consider the property for a morphism $X \xrightarrow{f} Y$ in $\mathbf{A}$

$$\begin{aligned}\mathcal{Q}_{\mathbf{A}}(f) \equiv &\textit{There exists an } \mathbf{A}\textit{-morphism } Y \xrightarrow{g} X\\ &\textit{with } X \xrightarrow{f} Y \xrightarrow{g} X = X \xrightarrow{id} X\end{aligned}$$

Step 1: In $\mathcal{P}_{\mathbf{A}}(X)$, replace all occurrences of $\mathbf{A}$ by $\mathbf{A}^{op}$, thus yielding the property

$$\begin{aligned}\mathcal{Q}_{\mathbf{A}^{op}}(f) \equiv &\textit{There exists an } \mathbf{A}^{op}\textit{-morphism } Y \xrightarrow{g} X\\ &\textit{with } X \xrightarrow{f} Y \xrightarrow{g} X = X \xrightarrow{id} X\end{aligned}$$

Step 2: Translate $\mathcal{Q}_{\mathbf{A}^{op}}(f)$ into the logically equivalent statement:

$$\begin{aligned}\mathcal{Q}^{op}_{\mathbf{A}}(f) \equiv &\textit{There exists an } \mathbf{A}\textit{-morphism } X \xrightarrow{g} Y\\ &\textit{with } Y \xrightarrow{g} X \xrightarrow{f} Y = Y \xrightarrow{id} Y\end{aligned}$$

We now introduce one of the most foundational concepts in category theory:

Dual Principle for Categories. *Whenever a property $\mathcal{P}$ holds for all categories, then the property $\mathcal{P}^{op}$ holds for all categories.*

We can observe several properties,

(1) $(\mathbf{A}^{op})^{op} = \mathbf{A}$, and
(2) $\mathcal{P}^{op}(\mathbf{A})$ holds iff $\mathcal{P}(\mathbf{A}^{op})$ holds.

Further, we say a property $\mathcal{P}$ is "**Self-Dual**" if $\mathcal{P}^{op} = \mathcal{P}$.

11. Initial and Terminal Objects

The notions of initial and terminal objects are very useful later on when thinking about universal arrows — that is, whenever we have phrases like "…*there exists* a *unique* function such that…" should ring an alarm that we're working with a universal arrow, which is an initial object in some category.

Definition 11.1. An object $0 \in \mathbf{C}$ is called an "**Initial Object**" if, for every object $A \in \mathbf{C}$, there is exactly one arrow $0 \xrightarrow{!} A$.

This is fairly simple as a definition: an initial object is mapped to every object in the category, including itself (by the identity morphism).

Definition 11.2. An object $1 \in \mathbf{C}$ is called a "**Terminal Object**" if, for every object $A \in \mathbf{C}$, there is exactly one arrow from $A \xrightarrow{!} 1$.

Note we denote morphisms to initial (respectively, from terminal) objects by !.

Example 11.3. In **Set**, the empty set $\emptyset = \{\}$ is the only initial object. For every set S, the empty function is the unique function from $\emptyset \to S$.

In **Set**, each one-element set is a terminal object. Why? Well, for each set $S \in \mathbf{Set}$ there is a function from S to a one element set $\{x\}$ mapping every element of S to x (the constant function).

Example 11.4. In **Grp**, the trivial group $G = \{e\}$ is the initial object, since there is exactly one group homomorphism from G to every group $G' \in \mathbf{Grp}$.

Example 11.5. In **Cat**, the category **0** is the initial object and **1** is the terminal object for the exact same reasoning that $\emptyset$ and $\{x\} \in \mathbf{Set}$ are initial (terminal) objects (respectively).

Example 11.6. If we think of a topological space (X, T) (where T is a topology on X) as a category whose objects are open sets $U \in T$, we have the morphisms be inclusions — i.e. $U \to V$ iff $U \subseteq V$. Then $\emptyset$ is an initial object, and X is a terminal object.

Definition 11.7. If $C \in \mathbf{C}$ is both an initial and a terminal object, then it is called a "**Null Object**".

12. Comma Category

12.1. **Category of objects over B and under A.** If $B \in \mathbf{C}$ is an object, we can construct a "**Category of Objects Under B**" is the category $(B \downarrow \mathbf{C})$ with:

- objects be ordered pairs (f, C) where $f : B \to C$;
- arrows $h : (f, C) \to (f', C')$ where $h : C \to C'$ is such that $h \circ f = f'/$

In other words, the objects are arrows from B to $C \in \mathbf{C}$, and arrows are commutative triangles with the top vertex be B. Or diagramatically

$$\text{Objects } (f,C) : \begin{tikzcd} B \arrow[d, "f"] \\ C \end{tikzcd} \quad \text{Arrows } (f,C) \xrightarrow{h} (f',C') : \begin{tikzcd} & B \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ C \arrow[rr, "h"] & & C' \end{tikzcd} \tag{12.1}$$

The composition of arrows in $(B \downarrow \mathbf{C})$ is just the composition in $\mathbf{C}$ of the base arrows h of these triangles.

Example 12.1. Consider any one-point set denoted by $*$, let $X \in \mathbf{Set}$. Well, each function $* \to X$ is an element of X; hence the category of objects under $*$, $(* \downarrow \mathbf{Set})$, is the category of pointed sets.

We can similarly (letting $A \in \mathbf{C}$ be an object in a category $\mathbf{C}$) define a "**Category of Objects Over A**" denoted by $(\mathbf{C} \downarrow A)$ as sort of dual to $(A \downarrow \mathbf{C})$. We diagramatically note that it has

$$\text{Objects } (f,C) : \begin{tikzcd} C \arrow[d, "f"] \\ A \end{tikzcd} \quad \text{Arrows } (f,C) \xrightarrow{h} (f',C') : \begin{tikzcd} C \arrow[rr, "h"] \arrow[dr, "f"'] & & C' \arrow[dl, "f'"] \\ & A & \end{tikzcd} \tag{12.2}$$

This is "dual" in the sense that it has its objects be arrows with *codomain* A as opposed to *domain* B.

Example 12.2. In **Set**, one-point sets $*$ are terminal, so there is exactly one arrow from each object $S \in \mathbf{Set}$ to $*$. That is, $S \to *$ is unique. So $(\mathbf{Set} \downarrow *)$ is isomorphic to **Set**.

12.2. **Category of Objects F-Under B and G-Over A.** If $B \in \mathbf{C}$ is an object of the category $\mathbf{C}$, and $F : \mathbf{D} \to \mathbf{C}$ is a functor, we can introduce the notion of a "**Category of Objects F-Under B**" which has as

- objects all ordered pairs (f, D) where $D \in \mathbf{D}$ and $f : B \to F(D)$;
- arrows $h : (f, D) \to (f', D')$ all arrows $h : D \to D'$ in $\mathbf{D}$ for which $f' = S(h) \circ f$.

In pretty pictures, we can write them as

$$\text{Objects: } \begin{tikzcd} B \arrow[d, "f"] \\ F(D) \end{tikzcd} \quad \text{Arrows: } \begin{tikzcd} & B \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ F(D) \arrow[rr, "F(h)"] & & F(D') \end{tikzcd} \tag{12.3}$$

As before, arrow composition takes place in $\mathbf{D}$.

If $A \in \mathbf{C}$ is an object of the category $\mathbf{C}$ and $G : \mathbf{D} \to \mathbf{C}$ is a functor, we can analogously construct the "**Category of Objects G-Over A**". We leave it as an exercise to the reader.

12.3. **Comma Categories.** We'll describe the basic construction. Given functors and categories

$$\mathbf{E} \xrightarrow{T} \mathbf{C} \xleftarrow{S} \mathbf{D} \tag{12.4}$$

the "**Comma Category**" $(T \downarrow S)$ — also written sometimes as (T, S) — has as

- objects all triples (E, D, f) with $E \in \mathbf{E}$, $D \in \mathbf{D}$ and $f : T(E) \to S(D)$
- arrows $(E, D, f) \to (E', D', f')$ all pairs (k, h) of arrows $k : E \to E'$ in $\mathbf{E}$ and $h : D \to D'$ in $\mathbf{D}$, such that $f' \circ T(k) = S(h) \circ f$.

In prettier pictures

$$\text{Objects } (D, E, f) : \begin{tikzcd} T(E) \arrow[d, "f"] \\ S(D) \end{tikzcd} \quad \text{Arrows } (k, h) : \begin{tikzcd} T(E) \arrow[r, "T(k)"] \arrow[d, "f"] & T(E') \arrow[d, "f'"] \\ S(D) \arrow[r, "S(h)"'] & S(D') \end{tikzcd} \tag{12.5}$$

with the square commutative. The composition of arrows is done componentwise $(k', h') \circ (k, h) = (k' \circ k, h' \circ h)$, when defined.

Note that this notion of a comma category is more general than the notions previously introduced, and actually embodies them quite naturally. Any object $C \in \mathbf{C}$ can be considered as a functor from $C : \mathbf{1} \to \mathbf{C}$. So we can construct, easily, the category of objects over C, or under C, or S-under C or T-over C. The comma category embodies it all quite naturally.

13. Universal Arrows

As previously mentioned, universal arrows arise whenever phrases like "... *there exists* a *unique* function such that..." occur. We're concerned about (1) existence, and (2) uniqueness. We'll start by defining a universal arrow.

Definition 13.1. If $F : \mathbf{D} \to \mathbf{C}$ is a functor and $C \in \mathbf{C}$ is an object, a "**Universal Arrow from** C **to** F" is a pair (D, u) consisting of

- an object $D \in \mathbf{D}$;
- an arrow $u : C \to F(D)$ of $\mathbf{C}$;

such that

- to every pair (D', f) with $D' \in \mathbf{D}$ and $f : C \to F(D')$ an arrow of $\mathbf{C}$, there is a unique arrow $f' : D \to D'$ with $F(f') \circ u = f$.

Or in other words, every arrow f to F factors uniquely through the universal arrow u as in the commutative diagram

$$\begin{tikzcd} C \arrow[r, "u"] \arrow[d, equal] & F(D) \arrow[d, dashed, "F(f')"] & D \arrow[d, dashed, "f'"] \\ C \arrow[r, "f"] & F(D') & D' \end{tikzcd} \tag{13.1}$$

As previously mentioned, we can come up with an equivalent definition using initial objects instead. We say that $u : C \to F(D)$ is universal from C to F when the pair (D, u) is an initial object in the comma category $(C \downarrow F)$ whose objects are arrows $C \to F(D')$. As with any initial object, it turns out that (D, u) is unique up to isomorphism in $(C \downarrow F)$. This is the typical use of the comma categories.

Like most of category theory, this definition is best illuminated with many novel examples.

Example 13.2 (Tensor Algebra)**.** Let V be a vector space over $\mathbb{K}$, and A be an algebra over $\mathbb{K}$. To construct the tensor algebra over V we typically do a mathematical procedure, let:

$$T^k V = V^{\otimes k} = V \otimes V \otimes \cdots \otimes V. \tag{13.2}$$

then we can define the tensor algebra over V as

$$T(V) = \bigoplus_{k=0}^{\infty} T^k V = K \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots . \tag{13.3}$$

Our intuition about category theoretic descriptions of mathematical procedures should be alerted: we have $T(V)$ as a mathematical procedure! It's a *functor!*

Now, the question that should come to mind next is *Suppose the Tensor Algebra is a functor, what categories are its domain and codomain?* Excellent question! We should suspect since it acts on V, a finite dimensional vector space over $\mathbb{K}$, that its domain category should be $\mathbf{Vect}_{\mathbb{K}}$ and its codomain category is yet to be known. It wouldn't be too much of a stretch to suppose its codomain category would be the category of graded algebras over $\mathbb{K}$. Why graded? Well, observe that our direct sum is taken over $\mathbb{N}$, so thinking of it as a graded algebra is natural. So we can state that

$$T : \mathbf{Vect}_{\mathbb{K}} \to \mathbf{Alg}_{\mathbb{K}} \tag{13.4}$$

is a functor encoding our procedure.

Lets stop and think for a second: we have a functor T and an object $V \in \mathbf{Vect}_{\mathbb{K}}$. How can we construct a universal arrow with this information? Well, we can rephrase the question thus: what is an initial object in the comma category $(V \downarrow T)$?

Any linear transformation $f : V \to A$ can be uniquely extended to an algebra homomorphism $\widetilde{f}$ from $T(V)$ to A, diagramatically depicted as

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & T(V) \arrow[d, dashed, "\widetilde{f}"] \\ & A \end{tikzcd} \tag{13.5}$$

where i is the canonical inclusion of V into $T(V)$. Or to phrase it in the more familiar manner: for each linear transformation $f : V \to A$ *there exists* an *unique* algebra homomorphism $\widetilde{f} : T(V) \to A$ *such that* the diagram in eq (13.5) commutes. Note this uniqueness is upto isomorphism. This concludes our lengthy example.

14. Yoneda Lemma

We can rephrase the notion of universality with hom-sets, which we summarize in the following proposition:

Proposition 14.1. For a functor $F : \mathbf{D} \to \mathbf{C}$ a pair $(D, u : C \to F(D))$ is universal from C to F iff the function sending each $f' : D \to D'$ into $S(f') \circ u : C \to F(D)$ is a bijection of hom-sets

$$\mathrm{Hom}_{\mathbf{D}}(D, D') \cong \mathrm{Hom}_{\mathbf{C}}(C, F(D')). \tag{14.1}$$

This bijection is natural in D'. Conversely, given D and C, any natural isomorphism (14.1) is determined in this way by a unique arrow $u : C \to F(D)$ such that (D, u) is universal from C to F.

Proof. The statement that (D, u) is universal is basically the same as $f' \mapsto F(f') \circ u = f$ is bijective. Why can we state this? Well, for each f there is a corresponding f' such that $F(f') \circ u = f$. This is a one-to-one correspondence, implying there is a bijection. This is also natural in D, what does this mean? Well, if we had a $g : D' \to D''$, then we would have $F(g \circ f') \circ u = F(g) \circ (F(f') \circ u)$.

Conversely, a natural isomorphism (14.1) gives for each $D' \in \mathbf{D}$ a bijection $\varphi_{D'} : \mathrm{Hom}_{\mathbf{D}}(D, D') \cong \mathrm{Hom}_{\mathbf{C}}(C, F(D'))$. In particular, choosing $D' = D$, we end up with $\varphi_D : \mathrm{Hom}_{\mathbf{D}}(D, D) \cong \mathrm{Hom}_{\mathbf{C}}(C, F(D))$ and we know there is a special element of $\mathrm{Hom}_{\mathbf{D}}(D, D)$ — the identity $\mathrm{id}_D \in \mathrm{Hom}_{\mathbf{D}}(D, D)$! This means there is a corresponding special element in $\mathrm{Hom}_{\mathbf{C}}(C, F(D))$, by our natural isomorphism! Lets consider what happens in our naturality condition, for any $f' : D \to D''$ the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{D}}(D, D) \arrow[r, "\varphi_D"] \arrow[d, "\mathrm{Hom}_{\mathbf{D}}(D, f')"'] & \mathrm{Hom}_{\mathbf{C}}(C, F(D)) \arrow[d, "\mathrm{Hom}_{\mathbf{C}}(C, F(f'))"] \\ \mathrm{Hom}_{\mathbf{D}}(D, D'') \arrow[r, "\varphi_{D''}"'] & \mathrm{Hom}_{\mathbf{C}}(C, F(D'')) \end{tikzcd} \tag{14.2}$$

commutes by the naturality of φ. But by the top right of the diagram, id_D is mapped to $F(f') \circ u$, and to the bottom left of the diagram it's mapped to $\varphi_{D''}(f')$.

Since φ is a bijection, this states that each $f : C \to F(D)$ has the form $f = F(f') \circ u$ for some corresponding (unique) f'. This is precisely stating (D, u) is universal, so this concludes our proof. □

Note that this technique, when we have a natural isomorphism from $\mathrm{Hom}_{\mathbf{C}}(X, X')$ to $\mathrm{Hom}_{\mathbf{D}}(Y, F(X'))$, to pick $X = X'$ and have the insight to deduce that there is a special element in $\mathrm{Hom}_{\mathbf{D}}(Y, F(X))$ corresponding to $\mathrm{id}_X \in \mathrm{Hom}_{\mathbf{C}}(X, X)$ is one recurring pattern in category theoretic proofs.

Observe if $\mathbf{C}$, $\mathbf{D}$ have small hom-sets, then our proposition is precisely stating that the functor $\mathrm{Hom}_{\mathbf{C}}(C, F(-)) : \mathbf{D} \to \mathbf{Set}$ is naturally isomorphic to a covariant hom-functor $\mathrm{Hom}(D, -) : \mathbf{D} \to \mathbf{Set}$. Before we can really say "Woah, awesome!" we should really introduce a new notion:

Definition 14.2. Let $\mathbf{D}$ have small hom-sets. A "**Representation of a Functor**" $K : \mathbf{D} \to \mathbf{Set}$ is a pair (D, ψ) with $D \in \mathbf{D}$ and

$$\psi : \mathrm{Hom}_{\mathbf{D}}(D, -) \cong K \tag{14.3}$$

a natural isomorphism. The object D is called the "**Representing Object**". The functor K is said to be "**Representable**" when such a representation exists.

Up to isomorphism, a representable functor is just a covariant hom-functor $\mathrm{Hom}_{\mathbf{D}}(D, -)$. Very interesting, we began with universal arrows, and ended up with a notion of representations of functors. Perhaps we can make the connection more explicit?

Proposition 14.3. Let $*$ be any one-point set, $\mathbf{D}$ have small hom-sets. If $(D, u : * \to K(D))$ is a universal arrow from $*$ to $K : \mathbf{D} \to \mathbf{Set}$, then

(1) the function ψ which (for each $D' \in \mathbf{D}$) sends the arrow $f' : D \to D'$ to $K(f')(u*) \in K(D')$ is a representation of K;
(2) every representation of K is obtained this way from exactly one such universal arrow.

Proof. For any set X, the function $f : * \to X$ is determined by the element $f(*) \in X$. The correspondence $f \mapsto f(*)$ is a bijection $\mathrm{Hom}_{\mathbf{Set}}(*, X) \to X$, natural in $X \in \mathbf{Set}$. How can we see this claim? Well, $\mathrm{Hom}_{\mathbf{Set}}(*, X)$ is the set of all function $f : * \to X$. These are in one-to-one correspondence to each element of X. This means there is such a bijection.

Now, we can compose with K to obtain a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Set}}(*, K(-)) \cong K. \tag{14.4}$$

This should be fairly straightforward to see, by proposition 6.7.

This together with the representation ψ gives (by definition of a representation of a functor):

$$\mathrm{Hom}_{\mathbf{Set}}(*, K(-)) \cong K \cong \mathrm{Hom}_{\mathbf{D}}(D, -). \tag{14.5}$$

A representation of K amounts to a natural isomorphism $\mathrm{Hom}_{\mathbf{Set}}(*, K(-)) \cong \mathrm{Hom}_{\mathbf{D}}(D, -)$. The rest of the proposition holds from proposition 14.1. □

These propositions allows us to consider one of the most important lemmas in category theory: the Yoneda Lemma. It basically states that, when studying a small category $\mathbf{D}$, we should study the category of all functors from $\mathbf{D}$ to $\mathbf{Set}$.

That is, the objects are functors $F, F' : \mathbf{D} \to \mathbf{Set}$ and the morphisms are natural transformations $\alpha : F \Rightarrow F'$. This notion (that the functors from $\mathbf{D} \Rightarrow \mathbf{Set}$ determines everything of interest about $\mathbf{D}$) is similar to the remark we made earlier, about having expressions like $\mathrm{Hom}_{\mathbf{C}}(C, C') \cong \mathrm{Hom}_{\mathbf{D}}(D, F(C'))$ be determined completely by choosing $C' = C$ and figuring out what happens to id_C.

Yoneda Lemma. If $K : \mathbf{D} \to \mathbf{Set}$ is a functor from $\mathbf{D}$, and $D \in \mathbf{D}$ is an object in a category $\mathbf{D}$ with small hom-sets, then there is a bijection

$$y : \mathrm{Nat}(\mathrm{Hom}_{\mathbf{D}}(D, -), K) \cong K(D) \tag{14.6}$$

which sends each natural transformation $\alpha : \mathrm{Hom}_{\mathbf{D}}(D, -) \Rightarrow K$ to $\alpha_D \, \mathrm{id}_D$, the image of the identity $D \to D$.

Proof. Trivial, it follows from the commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{D}}(D, D) \arrow[r, "\alpha_D"] \arrow[d, "f_* = \mathrm{Hom}_{\mathbf{D}}(D,f)"'] & K(D) \arrow[d, "K(f)"] & D \arrow[d, "f"'] \\ \mathrm{Hom}_{\mathbf{D}}(D, D') \arrow[r, "\alpha_{D'}"] & K(D'), & D'. \end{tikzcd} \tag{14.7}$$

□

Corollary 14.4. For objects $A, B \in \mathbf{D}$, each natural transformation

$$\mathrm{Hom}_{\mathbf{D}}(A, -) \Rightarrow \mathrm{Hom}_{\mathbf{D}}(B, -)$$

has the form $\mathrm{Hom}_{\mathbf{D}}(h, -)$ for a unique arrow $h : B \to A$.

First remark to make about the Yoneda Lemma: note the Yoneda mapping y basically states the natural transformations

$$\mathrm{Hom}_{\mathbf{D}}(D, -) \Rightarrow K$$

is "the same as" (naturally isomorphic to) $K(D)$. It may be hard to read "on the fly", so one should digest it here.

Now, the Yoneda map y is "natural" in K and D. To be more precise, consider $K \in \mathbf{Set}^{\mathbf{D}}$ as an object in the functor category. We basically want to show that "evaluation" as a functor $E : \mathbf{Set}^{\mathbf{D}} \times \mathbf{D} \to \mathbf{Set}$ and natural transformations as a functor $N : \mathbf{Set}^{\mathbf{D}} \times \mathbf{D} \to \mathbf{Set}$ (it maps each object (K, D) in its domain to the set of natural transformations $\mathrm{Nat}(\mathrm{Hom}_{\mathbf{D}}(D, -), K)$) are naturally isomorphic.

Consider further both the domain and codomain of the map y as functors of the pair (K, D), and consider this pair as an object in the category $\mathbf{Set}^{\mathbf{D}} \times \mathbf{D}$. The codomain of y is then merely the evaluation functor E which maps (K, D) to the value $K(D)$ of K at D. The domain is the functor N which maps the object (K, D) to the set $\mathrm{Nat}(\mathrm{Hom}_{\mathbf{D}}(D, -), K)$ of all natural transformations from $\mathrm{Hom}_{\mathbf{D}}(D, -)$ to K and which maps arrows $F : K \to K'$, $f : D \to D'$, to $\mathrm{Nat}(\mathrm{Hom}_{\mathbf{D}}(f, -), F)$. With these observations, we may prove an addendum to the Yoneda Lemma:

Lemma 14.5. The bijection in eq (14.6) is a natural isomorphism $y : N \cong E$ between the functors $E, N : \mathbf{Set}^{\mathbf{D}} \times \mathbf{D} \to \mathbf{Set}$.

Part 4. A Mathematician's Approach (Movement Three *Minuet*)

15. Introduction: Constructions with Cones and Limits

We continue this movement by introducing some more structure on a category. We'll do this in a sort of unique way: first we'll introduce the notions of cones and limits. Then we'll introduce a number of interesting structure as *examples* of cones and limits!

In particular, the examples which we'll cover are products of categories, pullbacks, and equalizers (as well as terminal objects). The manner which we'll cover these exotic structures is (appropriately enough) in the manner of a minuet — we'll dance back and forth over the definition, giving examples of each structure, and (hopefully) some examples reasoning with such things.

16. Diagrams, Cones, and Limits

16.1. **Diagrams.** We have an intuition in our heart of hearts of what a diagram in category theory is, but is there a rigorous way to describe it? The answer is "Yes, there is a rigorous way to describe it." What we do is we typically have a category which describes the shape of the diagram. It is typically finite (or at least small), we shall denote the shape category by $\mathbb{I}$.

Now, the diagram is actually a functor which embeds the shape into our category $\mathbf{C}$. That is, a diagram

$$D : \mathbb{I} \to \mathbf{C}$$

is a functor that identifies the diagram in $\mathbf{C}$.

16.2. **Cones.** We need to first introduce the constant functor before leaping to cones. Consider a functor $\Delta_U : \mathbb{I} \to \mathbf{C}$ be such that

$$\Delta_U\left(A \xrightarrow{f} B\right) = U \xrightarrow{\mathrm{id}_U} U \tag{16.1}$$

or in other words, every object is mapped to U and every morphism is mapped to id_U. This shouldn't be too surprising behavior for a functor dubbed "the constant functor"!

Now, we can define a cone:

Definition 16.1. Let $D : \mathbb{I} \to \mathbf{C}$ be a diagram in $\mathbf{C}$ and $\Delta_U : \mathbb{I} \to \mathbf{C}$ be the constant functor. A "**Cone over** D **with Vertex** U" is a natural transformation $\Delta_U \Rightarrow D$ with components (for all $I \in \mathbb{I}$)

$$\begin{tikzcd} U \arrow[r, "P_I"] \arrow[d, "\mathrm{id}_U"'] & D(I) \arrow[d, "D(f)"] \\ U \arrow[r, "P_{I'}"'] & D(I') \end{tikzcd} \tag{16.2}$$

Observe that this is secretly a triangle with vertex U! The naturality condition encodes all the information about the cone.

(We can also have the notion of a *cocone* which is dual to the cone; we just have the *target* of the arrows be U instead of having the *source* of the arrows be U.)

Now, the reason for calling it a cone is somewhat clear, it secretly is a triangle with vertex U after all. But the reason why it's important is not so clear at the moment. What happens is that we have a diagram and an object U. We basically have morphisms from U to each object in the diagram, and demand the resulting larger diagram commutes. But this larger diagram can be broken up into many smaller diagrams similar to eq (16.2). Why is this a good thing? Why should we care about such a gadget that has morphisms from U to each object in a diagram? It turns out that we can express many notions (structures we can equip a category with) in category theory as a cone (or more precisely, a "limit"). We will now turn our focus to limits.

16.3. **Limits.** Now in analysis, we typically have limits of sequences of numbers

$$\lim_{n\to\infty} x_n = x \tag{16.3}$$

or we have a limit of a function

$$\lim_{x\to x_0} f(x) = L. \tag{16.4}$$

We (should) have a good intuition about these procedures. But in category theory, what can we take the limit of? How about taking the limit *of diagrams?*

It turns out that a limit for a diagram in $\mathbf{C}$ is a universal cone. What do we mean by a "universal cone"? Well, universal objects usually means that every other object "factors uniquely" through it. So what would this mean for our precious cone? It means that if we have a universal cone with vertex U, and another cone with vertex V, that V factors through U, i.e. there is a unique morphism $h : V \to U$ such that

$$\begin{tikzcd} V \arrow[rr, dashed, "h"] \arrow[dr, "f"'] & & U \arrow[dl, "g"] \\ & X & \end{tikzcd} \tag{16.5}$$

commutes for all X. This might seem innocent enough, but it turns out that many many important notions in category theory can be expressed as limits or colimits over some vertex and some simple diagram. We will turn our attention to the rest of the chapter to examples.

17. Pullbacks and Pushouts

We can describe pullbacks (and its dual, pushouts) using limits (respectively colimits). We'll unfortunately have to dance around the intuitive notion of a pullback, its rigorous definition, and various examples (hence why it's in the Minuet chapter!).

17.1. **Limit Point of View.** Consider the situation when we have a pair of morphisms

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{17.1}$$

A "**Pullback**" is a universal cone with vertex U over this diagram. That means we have a pair of projection maps $p : U \to A$ and $q : U \to B$ such that

$$\begin{tikzcd} U \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd} \tag{17.2}$$

commutes. This is sort of like a product, at least intuitively, but if we have another object V and pair of projection morphisms $t : V \to B$ and $s : V \to A$, then we see we have to demand

$$\begin{tikzcd} V \arrow[dr, dashed, "!"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\ & U \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd} \tag{17.3}$$

commutes. We denote this pullback by $A \times_C B$. It's sometimes known as a fibred product or a Cartesian square.

Appendix

Appendix A. Grocery List of Categories

We have categorical description of the various different mathematical objects, here's some selected examples:

(Concrete) If the category's objects are "sets with structure" (e.g. **Mon** has monoids, **Top** has topological spaces, etc.), then the category is "concrete".
(Discrete Categories) The only morphisms are identity morphisms.
(Groups) If the category is a monoid with the extra condition the morphisms are invertible.
(Monoids) The category has just one object, so it's completely determined by its morphisms.
(Thin) If there is at most one morphism from an object to another.

Here we simply give a grocery list of categories that common come up. The following are commonly used categories, given by their shorthand notation commonly used in the literature.

0: is the empty category (no objects, no arrows).
1: is the singleton category, with one object and only one arrow – the identity morphism.
2: is the category with two objects a, b and exactly one morphism $a \to b$ which is not the identity.
3: is the category with three objects a, b, c and exactly three morphisms $a \to b$, $b \to c$, $c \to a$, none identities.

$\downdownarrows$: is the category with two objects a, b and two morphisms $a \rightrightarrows b$ none identities.

$\mathbf{Mat_K}$: Matrices over a commutative ring K, we have objects be positive integers $m, n, \ldots$ and each $m \times n$ matrix A eats in a vector in K^n and spits out a vector in K^m, so it's a morphism $A : n \to m$.

Set: Objects are all small sets, and morphisms are functions between them.

Cat: Objects are all small categories, and morphisms are functors.

Mon: Objects are small monoids, morphisms are morphisms of monoids.

Grp: Objects are small groups, morphisms are morphisms of groups.

Ab: Objects are small Abelian groups, morphisms are morphisms between them.

Rng: Objects are small rings, morphisms are morphisms of rings.

CRng: Objects are small commutative rigns, morphisms are morphisms between them.

R-Mod: All small left modules over the ring R, with linear maps.

Mod-R: Small right R-modules.

Top: Small topological spaces and continuous maps.

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