CLASSICAL HAMILTONIAN FIELD THEORY

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Abstract. This is a set of notes introducing Hamiltonian field theory, with focus on the scalar field.

Contents

1. Lagrangian Field Theory

As with Hamiltonian mechanics, wherein one begins by taking the Legendre transform of the Lagrangian, in Hamiltonian field theory we "transform" the Lagrangian field treatment. So lets review the calculations in Lagrangian field theory.

Consider the classical fields $\phi^a(t,\bar{x})$. We use the index a to indicate which field we are talking about. We should think of $\bar{x}$ as another index, except it is *continuous*. We will use the confusing short hand notation ϕ for the column vector $\phi^1,\dots,\phi^n$. Consider the Lagrangian

$$L(\phi) = \int_{\text{all space}} \mathcal{L}(\phi,\partial_\mu\phi)d^3\bar{x} \tag{1.1}$$

where $\mathcal{L}$ is the *Lagrangian density*. Hamilton's principle of stationary action is still used to determine the equations of motion from the action

$$S[\phi] = \int L(\phi,\partial_\mu\phi)dt \tag{1.2}$$

where we find the Euler-Lagrange equations of motion for the field

$$\frac{d}{dx^\mu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)} = \frac{\partial\mathcal{L}}{\partial\phi^a} \tag{1.3}$$

where we note these are evil second order partial differential equations. We also note that we are using Einstein summation convention, so there is an implicit sum over μ but not over a. So that means there are n independent second order partial differential equations we need to solve.

But how do we really know these are the correct equations? How do we really know these are the Euler-Lagrange equations for classical fields? We can obtain it directly from the action S by functional differentiation with respect to the field. Taking ϕ to be a single scalar

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field (for simplicity's sake, it doesn't change anything if we work with n fields), functional differentiation can be defined by

$$\frac{\delta S}{\delta\phi(x)} \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(S\left[\phi(y)+\varepsilon\delta^{(4)}(x-y)\right]-S\left[\phi(y)\right]\right) \tag{1.4}$$

where $\delta^{(4)}(y-x)$ is the 4-dimensional densitized Dirac delta function. Note that we will often use the shorthand notation $\delta^{(4)}_x = \delta^{(4)}(y-x)$. Applying this to the action yields

$$\frac{\delta S}{\delta\phi(x)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\left[\mathcal{L}\left(\phi+\varepsilon\delta^{(4)}_x, \partial_\mu\phi+\varepsilon\partial_\mu\delta^{(4)}_x\right)-\mathcal{L}(\phi,\partial_\mu\phi)\right]d^4y \tag{1.5a}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\left[\mathcal{L}(\phi,\partial_\mu\phi)+\frac{\partial\mathcal{L}}{\partial\phi}\delta^{(4)}_x\varepsilon+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta^{(4)}_x\varepsilon+\mathcal{O}(\varepsilon^2)-\mathcal{L}(\phi,\partial_\mu\phi)\right]d^4y \tag{1.5b}$$

$$= \lim_{\varepsilon\to 0}\int\left[\frac{\partial\mathcal{L}}{\partial\phi}\delta^{(4)}_x+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta^{(4)}_x+\mathcal{O}(\varepsilon)\right]d^4y \tag{1.5c}$$

$$= \int\left[\frac{\partial\mathcal{L}}{\partial\phi}\delta^{(4)}_x+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta^{(4)}_x\right]d^4y \tag{1.5d}$$

$$= \int\left[\frac{\partial\mathcal{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta^{(4)}_x d^4y \tag{1.5e}$$

Where we justify the second line by Taylor expanding to first order, then in the third line we factor through by the $(1/\varepsilon)$ factor, in the fourth line we take the limit, and integrate by parts to yield the last line. Note also that we factored out the delta function to make the last line prettier. Now the last line is zero if and only if

$$\frac{\partial\mathcal{L}}{\partial\phi^a}-\frac{d}{dx^\mu}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}=0 \tag{1.6}$$

which is precisely the Euler-Lagrange equations of motion!

Why are we working with these delta functions? Well, we are working with something a little more than just time. We are working with points in space. Locality means, mathematically, we work with vectors sharing the same base point. Or in the jargon of differential geometry, we are working in the tangent space $T_p\mathcal{M}$ where $\mathcal{M}$ is our manifold, and $p\in\mathcal{M}$ is our base point. If we work with multiple base points at a time, not only is it mathematically not well defined, but it is *nonlocal* which results in *a loss of causality!* Needless to say this is bad, so we try to work with the evolution of the field at a specified (but arbitrary) tangent space. If time permits, we will revisit this notion of spatial coordinates as an "index" in the appendix.

More precisely, we have the "base space" be the manifold $\mathcal{M}$ representing spacetime. We have our fields assign to each point $p\in\mathcal{M}$ some "physical information" $\phi(x)$. The question presents itself "Where does this 'information' live?" It lives in a generalization of the tangent space, called the "fiber". In the Lagrangian setting, we work with ordered pairs $(x^\mu, \phi(x^\mu))$ which is actually something called the "section" of the fiber bundle. That is, the field is a mapping

$$\phi : \mathcal{M}\to\mathcal{M}\times F \tag{1.7}$$

where F is "where" the fields "live", i.e. it's the fiber.

2. Hamiltonian Field Theory

If we want to begin the canonical treatment of fields, we need to start by finding the canonically conjugate momenta. We need to first "foliate" spacetime into space and time. Why do we do this? Well, we are concerned about the time derivatives of the field (how it changes in time), and we need to foliate spacetime into constant-time surfaces to have this be meaningful in the obvious way.

Let ϕ be a (column) vector of scalar fields. So to find the canonically conjugate momenta to ϕ^a, we follow our nose to find

$$\Pi_a = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi^a)}. \tag{2.1}$$

We will write Π for the row vector $\Pi = (\Pi_1, \Pi_2, \ldots, \Pi_n)$. If the mapping

$$(\phi, \partial_0\phi, \partial_1\phi, \partial_2\phi, \partial_3\phi) \mapsto (\Pi, \phi, \partial_1\phi, \partial_2\phi, \partial_3\phi) \tag{2.2}$$

is invertible (bijective, but when wouldn't it be?[1]), then we define the Hamiltonian density function by

$$\mathcal{H}(\Pi, \phi, \partial_1\phi, \partial_2\phi, \partial_3\phi) = \Pi\cdot\partial_0\phi - \mathcal{L} \tag{2.3}$$

which is a generalization of the Legendre transformation.

By direct computation we can find Hamilton's equations to be

$$\dot{\phi}^a = \frac{\partial\mathcal{H}}{\partial\Pi_a}, \quad \text{and} \quad \dot{\Pi}_a = -\frac{\partial\mathcal{H}}{\partial\phi^a} + \frac{d}{dx^m}\frac{\partial\mathcal{H}}{\partial(\partial_m\phi^m)} \tag{2.4}$$

where the equation on the right hand side has an implicit sum over $m = 1, 2, 3$. These equations naturally folow from Hamilton's principle applied to

$$S = \int(\Pi\cdot\partial_0\phi - \mathcal{H})d^4x \tag{2.5}$$

the action in canonical form.

Example 1. (Working in the East coast convention $- + ++$) Consider the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{\mu^2}{2}\phi^2 \tag{2.6}$$

where μ is some "mass" scalar. We find the equations of motion being

$$\frac{\delta S}{\delta\phi} = 0 \;\Rightarrow\; \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0. \tag{2.7}$$

We find by direct computation

$$\frac{\partial\mathcal{L}}{\partial\phi} = +\mu^2\phi \tag{2.8a}$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial_\mu\left(\partial^\mu\phi\right) = \partial^2\phi \tag{2.8b}$$

which yields the equations of motion to be

$$\left(\partial^\mu\partial_\mu + \mu^2\right)\phi = 0. \tag{2.9}$$

This is precisely the Klein-Gordon equation!

We find that the canonically conjugate momenta is

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi = -\partial_0\phi. \tag{2.10}$$

[1] The condition of this being invertible is equivalent to Henneaux and Teiteilboim's criteria of "regularity" it seems...

Thus we can find the Hamiltonian density to be

$$\mathcal{H} = \Pi \cdot \partial_0\phi - \mathcal{L} \tag{2.11a}$$
$$= -\partial^0\phi\partial_0\phi - \left(\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{\mu^2}{2}\phi^2\right) \tag{2.11b}$$
$$= \frac{-1}{2}\partial^0\phi\partial_0\phi - \frac{1}{2}\vec{\nabla}\phi\cdot\vec{\nabla}\phi - \frac{\mu^2}{2}\phi^2 \tag{2.11c}$$
$$= -\frac{1}{2}\left(\Pi\cdot\Pi + \vec{\nabla}\phi\cdot\vec{\nabla}\phi + \mu^2\phi^2\right) \tag{2.11d}$$

Note our convention makes the sign of the μ^2 term different than what is conventionally used in most particle physics texts. Because of this choice of signature convention, our energy is *negative*.

We can find Hamilton's equations for the Klein Gordon scalar field quite easily. One we already found accidentally, it is

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi = -\partial_0\phi. \tag{2.12}$$

The other is more interesting, it is

$$\dot{\Pi} = -\frac{\partial\mathcal{H}}{\partial\phi} + \frac{d}{dx^m}\frac{\partial\mathcal{H}}{\partial(\partial_m\phi)} \tag{2.13a}$$
$$= -\mu^2\phi - \partial_m(\partial^m\phi) \tag{2.13b}$$
$$= -(\mu^2 + \nabla^2)\phi \tag{2.13c}$$

Note that the sign of our results differs from what we derived due to our choice of signature convention!

Now the Hamiltonian approach to fields, as previously mentioned, is slightly diffferent because we work with a one-parameter family of spatial hypersurfaces instead of spacetime. To consider the scalar field $\phi(x)$ in this setting, we let it become $\phi(\bar{x};t)$ where we use the semicolon to remind ourselves that we are working with "some parameter" t. We obtain the Hamiltonian simply by integrating the density over all space

$$H(\Pi,\phi) = \int_{\text{all space}} \mathcal{H}(\Pi,\phi,\partial_1\phi,\partial_2\phi,\partial_3\phi)d^3\bar{x}. \tag{2.14}$$

We need to consider how the functional derivative behaves under this change from spacetime to "space plus time". The functional derivative with respect to $\phi(\bar{x};t)$ is

$$\frac{\delta A[\phi(\bar{y};t)]}{\delta\phi(\bar{x};t)} = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(A[\phi_t + \delta^{(3)}_{\bar{x}}] - A[\phi_t]\right) \tag{2.15}$$

where $\phi_t(\bar{y}) = \phi(\bar{y};t) = \phi_t$, and $\delta^{(3)}_{\bar{x}} = \delta^{(3)}(\bar{x}-\bar{y})$.

We obtain the equations of motion from the principle of stationary action applied to the canonical form of the action. The Euler-Lagrange equations encoded in the canonical formalism are

$$\dot{\phi}^a = \frac{\partial\mathcal{H}}{\partial\Pi_a}, \quad \text{and} \quad \dot{\Pi}_a = -\frac{\partial\mathcal{H}}{\partial\phi^a} + \frac{d}{dx^m}\frac{\partial\mathcal{H}}{\partial(\partial_m\phi^m)} \tag{2.16}$$

as we previously mentioned.

We can directly compute that

$$\frac{\delta H}{\delta\phi(\bar{x};t)} = \int \frac{\delta\mathcal{H}}{\delta\phi(\bar{x};t)} d^4y \tag{2.17a}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\Big[\mathcal{H}(\Pi,\phi_t+\varepsilon\delta^{(3)}_{\bar{x}},\partial_k\phi_t+\varepsilon\partial_k\delta^{(3)}_{\bar{x}}) - \mathcal{H}(\Pi,\phi_t,\partial_m\phi_t)\Big]\,d^3\bar{y} \tag{2.17b}$$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int\Big[\mathcal{H}(\Pi,\phi_t,\partial_m\phi_t)+\varepsilon\frac{\partial\mathcal{H}}{\partial\phi}\delta^{(3)}_{\bar{x}}+\varepsilon\frac{\partial\mathcal{H}}{\partial(\partial_m\phi)}\partial_m\delta^{(3)}_{\bar{x}} \tag{2.17c}$$

$$+\mathcal{O}(\varepsilon^2)-\mathcal{H}(\Pi,\phi_t,\partial_m\phi_t)\Big]\,d^3\bar{y}$$

$$= \int\left[\frac{\partial\mathcal{H}}{\partial\phi}\delta^{(3)}_{\bar{x}}+\frac{\partial\mathcal{H}}{\partial(\partial_m\phi)}\partial_m\delta^{(3)}_{\bar{x}}\right]d^3\bar{y} \tag{2.17d}$$

$$= \int\left[\frac{\partial\mathcal{H}}{\partial\phi}-\partial_m\frac{\partial\mathcal{H}}{\partial(\partial_m\phi)}\right]\delta^{(3)}_{\bar{x}}\,d^3\bar{y} \tag{2.17e}$$

$$= -\int\left[-\frac{\partial\mathcal{H}}{\partial\phi}+\partial_m\frac{\partial\mathcal{H}}{\partial(\partial_m\phi)}\right]\delta^{(3)}_{\bar{x}}\,d^3\bar{y} \tag{2.17f}$$

$$= -\int\dot{\Pi}\delta^{(3)}_{\bar{x}}\,d^3\bar{y} \tag{2.17g}$$

$$= -\dot{\Pi}(x). \tag{2.17h}$$

This is precisely one of Hamilton's equations, and by similar reasoning we find

$$\dot{\phi} = \frac{\delta H}{\delta\Pi_t(\bar{x})}, \quad \text{and} \quad \dot{\Pi}_t(\bar{x}) = -\frac{\delta H}{\delta\phi_t(\bar{x})} \tag{2.18}$$

are both generalizations of Hamilton's equations to field theoretic setting. This motivates us to define an analogous Poisson bracket:

$$\{A,B\} \stackrel{\text{def}}{=} \int\left(\frac{\delta A}{\delta\phi_t(\bar{x})}\frac{\delta B}{\delta\Pi_t(\bar{x})}-\frac{\delta B}{\delta\phi_t(\bar{x})}\frac{\delta A}{\delta\Pi_t(\bar{x})}\right)d^3\bar{x}. \tag{2.19}$$

This allows us to recover the equations of motion in a familiar way

$$\{A,H\} = \int\left(\frac{\delta A}{\delta\phi_t(\bar{x})}\dot{\phi}_t(\bar{x})+\dot{\Pi}_t(\bar{x})\frac{\delta A}{\delta\Pi_t(\bar{x})}\right)d^3\bar{x}. \tag{2.20}$$

Similarly, the canonical commutation relations hold classically as

$$\{\phi^a(\bar{x};t),\Pi_b(\bar{y};t)\} = \delta^a{}_b\delta^{(3)}(\bar{x}-\bar{y}). \tag{2.21}$$

Note that this only makes sense if the two variables are considered on the same time slice. If we were to consider two different timeslices, we'd necessarily have the commutation relations vanish due to locality constraints.

References

[1] R. Ticciati, "Quantum field theory for mathematicians,". Cambridge, UK: Univ. Pr. (1999) 699 p.

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