NOTES ON DIFFERENTIAL FORMS

ALEX NELSON

1. An Introduction To Grassmann Variables

We want to have variables which satisfy the relation

$$\varepsilon^i \varepsilon^j = -\varepsilon^j \varepsilon^i \tag{1.1}$$

which implies when $i = j$ that

$$(\varepsilon^i)^2 = 0. \tag{1.2}$$

So we basically will have variables that look like

$$a_0 + \sum_i b_i \varepsilon^i + \sum i, j c_{ij} \varepsilon^i \varepsilon^j + \dots \tag{1.3}$$

where a_0, b_i, $c_i j$, etc. are coefficients. By our condition (1.2) we see that we can have at most, with n different grassmann "generators" (meaning we have ε^i and $i = 1, ..., n$), we can have

$$1 + \binom{n}{1} + \binom{n}{2} + \dots = n^2 \tag{1.4}$$

terms total, where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} \tag{1.5}$$

is the binomial coefficients.

2. Exterior Derivative

With n grassmann generators, we can write

$$d = \sum_i \varepsilon^i \frac{\partial}{\partial x^i} \tag{2.1}$$

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where in cartesian coordinates $x^1 = x$, $x^2 = y$, $x^3 = z$, etc. Observe that by direct computation we find

$$d^2 = \left(\sum_i \varepsilon^i \frac{\partial}{\partial x^i}\right)^2. \tag{2.2}$$

We will perform a proof by induction that

$$d^2 = 0. \tag{2.3}$$

We will use the notation

$$\partial_i = \frac{\partial}{\partial x^i}. \tag{2.4}$$

Proof. **Base Case** We see that with

$$\begin{aligned}
\left(\varepsilon^1\partial_1 + \varepsilon^2\partial_2\right)^2 &= (\varepsilon^1\partial_1)^2 + (\varepsilon^2\partial_2)^2 + (\varepsilon^1\varepsilon^2\partial_1\partial_2) + (\varepsilon^2\varepsilon^1\partial_1\partial_2) && \text{(2.5a)}\\
&= (0) + (0) + (\varepsilon^1\varepsilon^2 + \varepsilon^2\varepsilon^1)\partial_1\partial_2 && \text{(2.5b)}\\
&= (0)\partial_1\partial_2 && \text{(2.5c)}\\
&= 0. && \text{(2.5d)}
\end{aligned}$$

So it is true with $n = 2$.

Inductive Hypothesis We assume that it works with n.

Inductive Case With $n+1$ we see that

$$\left(\varepsilon^1\partial_1 + \ldots + \varepsilon^n\partial_n + \varepsilon^{n+1}\partial_{n+1}\right)^2 = (z + \varepsilon^{n+1}\partial_{n+1})^2 \tag{2.6}$$

where we define

$$z = \varepsilon^1\partial_1 + \ldots + \varepsilon^n\partial_n. \tag{2.7}$$

By the inductive hypothesis, we assumed that

$$z^2 = 0 \tag{2.8}$$

so we find

$$(z + \varepsilon^{n+1}\partial_{n+1})^2 = \cancelto{0}{z^2} + \cancelto{0}{(\varepsilon^{n+1}\partial_{n+1})^2} + z\varepsilon^{n+1}\partial_{n+1} + \varepsilon^{n+1}\partial_{n+1}z. \tag{2.9}$$

So we have a simple expression, and it's one we all know and love! It's simply

$$z\varepsilon^{n+1}\partial_{n+1} + \varepsilon^{n+1}\partial_{n+1}z = \left(\sum_{i=1}^{n} \varepsilon i\partial_i\right)\varepsilon^{n+1}\partial_{n+1} + \varepsilon^{n+1}\partial_{n+1}\left(\sum_{i=1}^{n} \varepsilon i\partial_i\right) \tag{2.10}$$

but by our expression (1.1), we see that these two terms cancel each other out! That is, we get a number of expressions of the form

$$\sum_{i=1}^{n} \varepsilon^{n+1}\varepsilon^{i} + \varepsilon^{i}\varepsilon^{n+1} = \sum_{i=1}^{n}(0) = 0. \tag{2.11}$$

This completes our proof by induction! □

Appendix A. A Note on Differentials

We have seen that

$$d = \sum_{i} \varepsilon^{i} \frac{\partial}{\partial x^{i}} \tag{A.1}$$

but how exactly do we end up with dx, dy, dz, etc.?

Well, we take in this case

$$\varepsilon^{i} = dx^{i} \tag{A.2}$$

and we can rewrite the exterior derivative as

$$d = \sum_{i} dx^{i} \frac{\partial}{\partial x^{i}}. \tag{A.3}$$

A k-form is then

$$\omega = \partial_{i_1} \dots \partial_{i_k} f \varepsilon^{i_1} \dots \varepsilon^{i_k} \tag{A.4}$$

and integration of a one form is simply

$$\int \omega d^{n}x d^{n}\varepsilon = \int \partial_{i_1} \dots \partial_{i_n} f d^{n}x. \tag{A.5}$$

Furthermore change of coordinates is absolutely trivial, if

$$x^{i} = \psi^{i}(\tilde{x}) \tag{A.6}$$

then

$$\int f(x) dx^{1} \dots dx^{n} = \int f(\psi(\tilde{x})) \underbrace{\prod (\frac{\partial x^{i}}{\partial \tilde{x}^{j}} d\tilde{x}^{j})}_{\text{Super-Jacobian}} = \int f(\psi(\tilde{x})) d^{n}\tilde{x}. \tag{A.7}$$

Note that the super Jacobian is 1.