NOTES ON LADDER OPERATORS

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Abstract. We review ladder operators.

1. Commutation Relations

We have the operators $\widehat{X}$ and $\widehat{N}$ such that

$$[\widehat{N}, \widehat{X}] = c\widehat{X} \tag{1.1}$$

where c is a scalar value. We have the eigenstates of $\widehat{N}$ such that

$$\widehat{N}|n\rangle = n|n\rangle \tag{1.2}$$

where we have abused notation letting $|n\rangle$ be a vector and n be a scalar (the eigenvalue of the $|n\rangle$ vector). We see that

$$\widehat{N}\widehat{X}|n\rangle = \left(\widehat{X}\widehat{N} + [\widehat{N}, \widehat{X}]\right)|n\rangle \tag{1.3a}$$

$$= \left(\widehat{X}\widehat{N} + c\widehat{X}\right)|n\rangle \tag{1.3b}$$

$$= \widehat{X}\widehat{N}|n\rangle + c\widehat{X}|n\rangle \tag{1.3c}$$

$$= \widehat{X}n|n\rangle + c\widehat{X}|n\rangle \tag{1.3d}$$

$$= (n + c)\widehat{X}|n\rangle. \tag{1.3e}$$

This means that if $|n\rangle$ is an eigenstate of $\widehat{N}$ with eigenvalue n, then $(\widehat{X}|n\rangle)$ is an eigenstate of $\widehat{N}$ with eigenvalue $n + c$. If $c > 0$ then $n + c > n$, so the operator $\widehat{X}$ is called the "**Creation Operator**".

If $\widehat{N}$ is a physical observable, it's necessarily self adjoint. This implies that c is real, since n is real and $n + c$ is an eigenvalue of a self-adjoint operator. We should remember from linear algebra the eigenvalues of a self-adjoint operator is always real. The Hermitian adjoint of $\widehat{X}$ satisfies

$$[\widehat{N}, \widehat{X}^\dagger] = -c\widehat{X}^\dagger \tag{1.4}$$

We call $\widehat{X}^\dagger$ an "**Annihilation Operator**" if $\widehat{X}$ is a creation operator.

Proposition 1. Let $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ be operators. Then

$$[\widehat{A}, \widehat{B}\widehat{C}] = [\widehat{A}, \widehat{B}]\widehat{C} + \widehat{B}[\widehat{A}, \widehat{C}]. \tag{1.5}$$

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Proof. We see by direct computation

$$[\widehat{A},\widehat{B}\widehat{C}] = \widehat{A}\widehat{B}\widehat{C} - \widehat{B}\widehat{C}\widehat{A} \tag{1.6a}$$

$$\widehat{B}[\widehat{A},\widehat{C}] = \widehat{B}(\widehat{A}\widehat{C} - \widehat{C}\widehat{A}) \tag{1.6b}$$

$$[\widehat{A},\widehat{B}]\widehat{C} = (\widehat{A}\widehat{B} - \widehat{B}\widehat{A})\widehat{C} \tag{1.6c}$$

so when we add equation (1.6b) to equation (1.6c) we get

$$\widehat{A}\widehat{B}\widehat{C} - \widehat{B}\widehat{C}\widehat{A} = [\widehat{A},\widehat{B}\widehat{C}] \tag{1.7}$$

as desired. □

Proposition 2. Let $\widehat{A}$, $\widehat{B}$ be operators. If their commutator vanishes

$$[\widehat{A},\widehat{B}] = 0 \tag{1.8}$$

then the two operators are equal up to a constant.

Proof. Trivial. □

Proposition 3. Given these two propositions and the commutations relations, we can find that

$$[\widehat{N},\widehat{X}\widehat{X}^{\dagger}] = 0. \tag{1.9}$$

Or equivalently

$$\widehat{N} = \widehat{X}^{\dagger}\widehat{X} \tag{1.10}$$

up to some constant.

Proof. How to prove this? By direct computation

$$[\widehat{N},\widehat{X}^{\dagger}\widehat{X}] = [\widehat{N},\widehat{X}^{\dagger}]\widehat{X} + \widehat{X}^{\dagger}[\widehat{N},\widehat{X}] \tag{1.11a}$$

$$= (-c\widehat{X}^{\dagger})\widehat{X} + \widehat{X}^{\dagger}(c\widehat{X}) \tag{1.11b}$$

$$= (c-c)\widehat{X}^{\dagger}\widehat{X} \tag{1.11c}$$

$$= 0. \tag{1.11d}$$

This means, up to some constant and ordering,

$$\widehat{N} = \widehat{X}^{\dagger}\widehat{X} \tag{1.12}$$

just as desired. □

References

[1] P. A. M. Dirac, *Principles of Quantum Mechanics*. Oxford Science Publications, fourth ed., 2007.

[2] O. L. de Lange and R. E. Raab, "Ladder Operators for Orbital Angular Momentum," *American Journal of Physics* **54** (1986) no. 4, 372–375. http://dx.doi.org/10.1119/1.14625.

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