NOTES ON THE QUANTUM HARMONIC OSCILLATOR

ALEX NELSON

Abstract. We briefly introduce the notion of creation and annihilation operators through a change of coordinates. We introduce the number operator, its relation to the Hamiltonian operator, and find the vacuum state.

1. Introduction

Recall for classical mechanics, the Harmonic oscillator potential is

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 \tag{1.1}$$

where ω is the angular velocity. We plug this into Schrodinger's equation

$$\left[\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}kx^2\right]|\psi\rangle = E|\psi\rangle. \tag{1.2}$$

How to solve this? Well, there are two ways: the smart way and the stupid way. We'll do it the smart way.

We introduce a change of variable[1]:

$$Q = \sqrt{\frac{m\omega}{2\hbar}}, \qquad \widehat{P} = \frac{\partial}{\partial Q} = \sqrt{\frac{-1}{2m\omega\hbar}}\widehat{p}. \tag{1.3}$$

Note these are dimensionless and simplify computations significantly. We can factor the Hamiltonian, since in these new variables we have

$$\hbar\omega\left[Q^2 - \frac{\partial^2}{\partial Q^2}\right] = \widehat{H}. \tag{1.4}$$

We want to use the coordinates

$$a = \left[Q + \frac{\partial}{\partial Q}\right], \qquad a^\dagger = \left[Q - \frac{\partial}{\partial Q}\right] \tag{1.5}$$

which we call "**annihilation and creation operators**" respectively.

Observe that

$$2a^\dagger a = \left[Q + \frac{\partial}{\partial Q}\right]\left[Q - \frac{\partial}{\partial Q}\right] \tag{1.6a}$$

$$= Q^2 + \frac{\partial}{\partial Q}Q - Q\frac{\partial}{\partial Q} - \frac{\partial^2}{\partial Q^2} \tag{1.6b}$$

$$= Q^2 - \frac{\partial^2}{\partial Q^2} + [\widehat{P}, Q]. \tag{1.6c}$$

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[1]Note that this is done *classically*, that is *before* quantization. After quantization changing coordinates is always a fuzzy subject. These two steps are done tacitly in most derivations, but it should be known in the back of one's mind what's going on.

All computation is by definition and substitution. Nothing too fancy so far. We can now write the Hamiltonian operator as

$$\widehat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}[Q, \widehat{P}]\right) \tag{1.7}$$

Observe that

$$[Q, \widehat{P}] = 1 \tag{1.8}$$

since Q and $\widehat{P}$ are the dimensionless counterparts to x and $\widehat{p}$ which implies we set $\hbar \to 1$. We end up with the form of the Hamiltonian operator

$$\widehat{H} = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \tag{1.9}$$

It would then be logical to investigate how these creation and annihilation operators behave.

2. Ladder Operators

We see first of all that there is a state $|0\rangle$ which we will call the "**vacuum state**". It is such that

$$a|0\rangle = 0|0\rangle = 0. \tag{2.1}$$

That is, it is the lowest energy eigenstate. We find that in explicit coordinates the function $f(Q)$ that does this is

$$\langle Q|a|0\rangle = a(f) \tag{2.2a}$$

$$= \left[Q + \frac{\partial}{\partial Q}\right] f(Q) \tag{2.2b}$$

$$\Rightarrow -Qf(Q) = f'(Q) \tag{2.2c}$$

$$\Rightarrow -Q = \frac{f'(Q)}{f(Q)} \tag{2.2d}$$

$$\Rightarrow C_1 - \frac{Q^2}{2} = \ln(f(Q)) \tag{2.2e}$$

$$\Rightarrow f(Q) = c\exp(-Q^2/2) \tag{2.2f}$$

where c, C_1 are constants of integration. We find by normalization

$$\langle 0|0\rangle = 1 \quad \Longleftrightarrow \quad f(Q) = \langle Q|0\rangle = \frac{1}{\sqrt[4]{\pi}} e^{-Q^2/2}. \tag{2.3}$$

The energy of this eigenstate is then open to question. We would *like* it to be zero, as this is the vacuum, but we need to *prove* it.

First we should define the **Number Operator** as

$$N = a^\dagger a. \tag{2.4}$$

We see that

$$N|n\rangle = n|n\rangle \tag{2.5}$$

and note the abuse of notation here: $n|n\rangle$ is the scalar n multiplied with the ket $|n\rangle$. We chose this notation for the ket because it gives us complete information about its number eigenvalue (the scalar number n).

We can rewrite the Hamiltonian operator in terms of the number operator

$$\widehat{H} = \hbar\omega\left(N + \frac{1}{2}\right). \tag{2.6}$$

Observe then that the energy eigenvalue of the vacuum is

$$\widehat{H}|0\rangle = \frac{\hbar\omega}{2}|0\rangle. \tag{2.7}$$

This is not really too good, since the *vacuum has positive energy.* But what it also means is that we can write the energy eigenvalues as

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{2.8}$$

where $(n = 0, 1, 2, \ldots)$.

References

[1] Jun John Sakurai. *Modern Quantum Mechanics.* Addison-Wesley Publishing Company, revised edition, 1994.

E-mail address: pqnelson@gmail.com