NOTES ON CANONICAL GRAVITY

ALEX NELSON

Abstract. A review of the ADM Hamiltonian formalism for classical General Relativity.

Just a few remarks on notation. Latin indices $i, j, \ldots$ will be for spatial components of four-tensors, Greek indices $\alpha, \beta, \ldots$ will be for spacetime components of four-tensors.

We will approach the subject of the Hamiltonian formulation of general relativity (also known as canonical formulation of gravity, canonical dynamics for general relatvity, canonical gravity, among countless other names – canonical here refers to the use of Hamiltonian formalism as opposed to the Lagrangian formalism) by the following process. We will perform the decomposition of spacetime into space plus time, or the ADM form of the metric. Given such a decomposition, we look at how the Lagrangian gives way to the Hamiltonian formalism. Then we review the constraints of General Relativity, both the Hamiltonian and Momentum constraints.

1. A Review of Hypersurfaces

Lets briefly turn our attention to the geometry of hypersurfaces in four dimensions. In a four dimensional manifold, a hypersurface is a three dimensional submanifold that can be timelike, spacelike, or null. We may consider a particular hypersurface Σ by restricting the coordinates

$$C(x^\alpha) = 0 \tag{1.1}$$

or by parametrizing the coordinates

$$x^\alpha = x^\alpha(y^a) \tag{1.2}$$

where y^a $(a = 1, 2, 3)$ are the coordinates intrinsic to the hypersurface. Think of the sphere in three dimensions, we can specify it by the restriction of the coordinates

$$C(x, y, z) = R^2 - x^2 - y^2 - z^2 = 0 \tag{1.3}$$

or in parametric form

$$x(\theta, \phi) = R\sin(\theta)\cos(\phi) \tag{1.4a}$$

$$y(\theta, \phi) = R\sin(\theta)\sin(\phi) \tag{1.4b}$$

$$z(\theta, \phi) = R\cos(\theta) \tag{1.4c}$$

where θ, ϕ are the coordinates intrinsic to the sphere.

Now, if we consider the vector $\partial_\alpha C$, it is normal to the hypersurface. This is Unit normal

Date: June 05, 2009.

due to the observation the only value of C changes in a direction orthogonal to Σ. If the surface is not null, we can introduce the unit normal n_α defined such that

$$n^\alpha n_\alpha = s = \begin{cases} +1, & \text{if } \Sigma \text{ is timelike} \\ -1, & \text{if } \Sigma \text{ is spacelike.} \end{cases} \tag{1.5}$$

We demand the condition that n^α point in the direction of increasing C,

$$n^\alpha \partial_\alpha C > 0. \tag{1.6}$$

We see that given this condition on the sign of the unit normal, and the fact that $\partial_\alpha C$ points in the direction of the unit normal, that

$$n_\alpha = \frac{s\partial_\alpha C}{|g^{\mu\nu}\partial_\mu C \partial_\nu C|^{1/2}} \tag{1.7}$$

provided that the hypersurface is not null (i.e. it must be either timelike or spacelike). If it were null, then the denominator vanishes:

$$g^{\mu\nu}\partial_\mu C \partial_\nu C = 0 \tag{1.8}$$

which is bad.

Induced Metric

The metric intrinsic to the hypersurface is obtained by restricting the line element ds^2 to displacements confined to the hypersurface. Remember we parametrized the four dimensional coordinates of the surface by the equations

$$x^\alpha = x^\alpha(y^a). \tag{1.9}$$

We see that the vectors

$$e^\alpha{}_a = \frac{\partial x^\alpha}{\partial y^a} \tag{1.10}$$

are tangent to the curves contained in Σ. Why? Well, the relations described by Eq (1.9) describe curves contained entirely in Σ, parametrized in y^a, so differentiating with respect to the parameters would yield the tangent vectors.

Now, for displacements contained in Σ, we can write the infinitesimal line element as

$$ds^2_\Sigma = g_{\alpha\beta} dx^\alpha dx^\beta = g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a} dy^a\right)\left(\frac{\partial x^\beta}{\partial y^b} dy^b\right) = h_{ab} dy^a dy^b. \tag{1.11}$$

where

$$h_{ab} = g_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^a}\frac{\partial x^\beta}{\partial y^b} = g_{\alpha\beta} e^\alpha{}_a e^\beta{}_b \tag{1.12}$$

is called the "induced metric". It behaves as a scalar when we change coordinates $x^\mu \to x^{\mu'}$ but it behaves as a tensor when we change coordinates $y^m \to y^{m'}$ intrinsic to the surface. We'll call such things "**three-tensors**".

1.1. **Lie Derivative of the Metric Along a Vector.** The Lie derivative of the metric along a vector ξ^a is

$$\mathscr{L}_\xi g_{ab} = g_{ac}\partial_b \xi^c + g_{bc}\partial_a \xi^c + \xi^c \partial_c g_{ab}. \tag{1.13}$$

Observe that

$$g_{bc}\nabla_a \xi^c = g_{bc}(\partial_a \xi^c + \Gamma^c_{ad}\xi^d) \tag{1.14}$$

where Γ is the Christoffel symbol, ∇ is the covariant derivative. We specifically find

$$g_{bc}\Gamma^{c}_{ad}\xi^{d} = \Gamma_{bad}\xi^{d}. \tag{1.15}$$

For the affine connection, we have

$$\partial_c g_{ab} = \Gamma_{acb} + \Gamma_{bca} = 0. \tag{1.16}$$

So we plug this into eq (1.13) to find

$$\mathscr{L}_\xi g_{ab} = g_{ac}\partial_b\xi^c + g_{bc}\partial_a\xi^c + \xi^c\left(\Gamma_{acb} + \Gamma_{bca}\right) \tag{1.17}$$

By the properties of the Christoffel symbol, specifically

$$\Gamma_{cab} = \Gamma_{cba} \tag{1.18}$$

we can rewrite eq (1.14) as

$$g_{bc}\nabla_a\xi^c = g_{bc}\partial_a\xi^c + \Gamma_{bad}\xi^d. \tag{1.19}$$

Now observe the Lie derivative of the metric along our vector ξ^a can be grouped in terms

$$\mathscr{L}_\xi g_{ab} = (g_{ac}\partial_b\xi^c + \Gamma_{abc}\xi^c) + (g_{bc}\partial_a\xi^c + \Gamma_{bac}\xi^c) \tag{1.20}$$

since the c index is summed over, it's a dummy index. We can rewrite this in more familiar terms

$$\mathscr{L}_\xi g_{ab} = (g_{ac}\partial_b\xi^c + \Gamma_{abd}\xi^d) + (g_{bc}\partial_a\xi^c + \Gamma_{bad}\xi^d) \tag{1.21}$$

thus

$$\mathscr{L}_\xi g_{ab} = g_{ac}\nabla_b\xi^c + g_{bc}\nabla_a\xi^c. \tag{1.22}$$

Since the connection is metric compatible, we can "bring the metric inside the derivative"

$$g_{ab}\nabla_c(\cdots) \to \nabla_c(g_{ab}\cdots)$$

since $\nabla g_{ab} = 0$. Now we can rewrite our Lie derivative as

$$\mathscr{L}_\xi g_{ab} = \nabla_b\xi_a + \nabla_a\xi_b. \tag{1.23}$$

This is precisely the Killing equation.

2. ADM Form of the Metric

So when turning to the canonical formalism, we need to split spacetime into space and time. Although this "goes against the spirit of special relativity", there is a theorem from Geroch [1] proving it's completely kosher:

> *Property 7: Let S be a Cauchy surface for the space-time M. Then M is topologically $S\times\mathbb{R}$. In particular, if M is connected, so is S.*
>
> [...]
>
> *Theorem 11: A spacetime M is globally hyperbolic if and only if it has a Cauchy surface.*[1]

---

[1] Or equivalently

If the spacetime is globally hyperbolic, then it is necessarily topologically $M = \mathbb{R}\times S$, where M is our spacetime manifold, $\mathbb{R}$ is "time" and S is our spatial hypersurface.

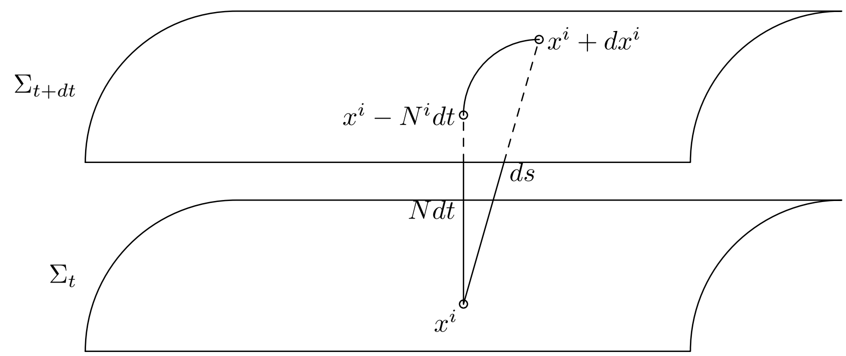

FIGURE 1. ADM decomposition of spacetime.

If it turns out that spacetime is indeed Lorentzian, it is necessarily hyperbolic. It follows that such a decomposition is always allowable, at least *classically.* We need to "lift" this condition to the quantum case, so there are no "rips" or "mending of holes" in spatial hypersurfaces in the quantum case. We will not worry about that too much for now.

Now that we know it's kosher to split spacetime up into space and time, lets start by considering a foliation of spacetime. Let M be the manifold we use for spacetime, $\Sigma_t := X_t(S)$ be (for each $t \in \mathbb{R}$) an embedding (a globally injective immersion) $X_t : S \to M$. So how this should appear intuitively, it's "layering" spacetime with "spatial surfaces" (time slices) where time is constant. Each of these time slices has local coordinates x_i and an induced metric $g_{ij}(x,t)$[2]. We can reconstruct all four dimensions by examining how one time slice described by Σ_t to a nearby time slice Σ_{t+dt} fit together. Consider a point x^i on Σ_t and displacing it in time by an infinitesimal amount in the direction of the normal of the surface. We write the resulting change in proper time as

$$d\tau = N dt = \begin{pmatrix}\text{lapse of proper time}\\ \text{between lower and}\\ \text{upper hypersurfaces}\end{pmatrix} \tag{2.1}$$

where $N(t,x^i)$ is called the "lapse function". Such a displacement is only a shift in the time coordinate, to be fully general we should consider a generic shift in both spatial and time coordinates. A general shift in spatial coordinates would be described by

$$x^i(t+dt) = x^i(t) - N^i dt, \tag{2.2}$$

where $N^i(t,x^i)$ is the "shift vector". By the Lorentzian pythagorean theorem, we see graphically in figure 1, the interval between (t,x^i) and $(t+dt, x^i+dx^i)$ is

$$\begin{aligned} ds^2 &= -N^2 dt^2 + {}^{(3)}g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \\ &= -\begin{pmatrix}\text{proper time from}\\ \text{lower to upper}\\ \text{3-geometry}\end{pmatrix}^2 + \begin{pmatrix}\text{proper distance in}\\ \text{base-3 geometry}\end{pmatrix}^2 \end{aligned} \tag{2.3}$$

This is the ADM form of the metric.

[2]We will denote the induced metric by latin indices or ${}^{(3)}g_{ij}$.

We can observe from

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.4}$$

that we can write the metric tensor in block form

$$g_{\mu\nu} = \begin{bmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{bmatrix} = \left[\begin{array}{c:c} (N_i N^i - N^2) & N_j \\ \hdashline N_i & {}^{(3)}g_{ij} \end{array}\right] \tag{2.5}$$

where q_{ij} is the induced metric on the time slice. How to find the inverse of the four-metric using this decomposition? We can observe that we can find the inverse by

$$g_{\mu\nu} g^{\nu\rho} = \begin{bmatrix} (N_i N^i - N^2) & N_j \\ N_i & {}^{(3)}g_{ij} \end{bmatrix} \begin{bmatrix} \alpha & \beta^k \\ \beta^j & \gamma^{jk} \end{bmatrix} = \delta^\rho{}_\mu. \tag{2.6}$$

We end up with, through matrix multiplication, four conditions

$$\alpha N_i + {}^{(3)}g_{ij} \beta^j = 0 \tag{2.7a}$$

$$(N_i N^i - N^2)\beta^k + N_j \gamma^{jk} = 0 \tag{2.7b}$$

$$(N_i N^i - N^2)\alpha + N_j \beta^j = 1 \tag{2.7c}$$

$$N_i \beta^k + {}^{(3)}g_{ij} \gamma^{jk} = \delta^k{}_i \tag{2.7d}$$

Let ${}^{(3)}g^{ik}$ be the inverse of the induced metric, i.e.

$${}^{(3)}g^{ik\,(3)}g_{kj} = {}^{(3)}\delta^i{}_j \tag{2.8}$$

where ${}^{(3)}\delta^i{}_j$ is the 3 by 3 Kronecker delta. We find from eq (2.7a) that

$$\beta^k = -\alpha N_i{}^{(3)}g^{ik} = -\alpha N^k. \tag{2.9}$$

We plug this into eq (2.7c) to find that

$$\alpha = \frac{-1}{N^2}. \tag{2.10}$$

This implies

$$\beta^k = \frac{N^k}{N^2}. \tag{2.11}$$

We plug this into eq (2.7d) to find

$${}^{(3)}g_{ij}\gamma^{jk} = \delta_i{}^k - \frac{N_i N^k}{N^2} \Rightarrow \gamma^{kl} = {}^{(3)}g^{kl} - \frac{N^l N^k}{N^2} \tag{2.12}$$

by multiplying both sides by ${}^{(3)}g^{il}$. We can now recombine our results to find the inverse of the metric tensor to be

$$g^{\mu\nu} = \frac{1}{N^2} \left[\begin{array}{c:c} -1 & N^k \\ \hdashline N^j & {}^{(3)}g^{jk} N^2 - (N^j N^k) \end{array}\right] \tag{2.13}$$

Observe that there is a difference between the spatial component of the 4-metric ${}^{(4)}g^{ij} = {}^{(3)}g^{ij} + \alpha N^i N^j$ and the inverse of the induced metric ${}^{(3)}g^{ij}$.

Lets consider some geometry in this ADM form of the metric. Let $n_\alpha = -N\delta_{0\alpha}$ be the unit normal to the time slice. We define the first fundamental form as

$$q_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu. \tag{2.14}$$

Intuitively $q^\alpha{}_\beta$ be thought of as a sort of "projection", i.e. it projects any index into a "purely spatial" one (more precisely, a "purely spatial" index is one which – when contracted with n^α or n_α – vanishes). We then define the extrinsic curvature Extrinsic Curvature Tensor $K_{\alpha\beta}$

tensor

$$K_{\mu\nu} = q_\mu{}^\rho q_\nu{}^\sigma \nabla_\rho n_\sigma. \tag{2.15}$$

We assert that $K_{\alpha\beta}$ is symmetric in its indices. How can we see this? From observing the Lie derivative of the metric along a vector in section 1.1 we see that

$$\mathscr{L}_n g_{\alpha\beta} = \nabla_\alpha n_\beta + \nabla_\beta n_\alpha. \tag{2.16}$$

We see the left hand side is symmetric in its indices which implies the right hand side is symmetric in its indices. We see that

$$q_\mu{}^\alpha q_\nu{}^\beta \mathscr{L}_n g_{\alpha\beta} = K_{\mu\nu} + K_{\nu\mu} \neq 0. \tag{2.17}$$

It's nonvanishing since n_α is not a Killing vector. We see that

$$q_\mu{}^\alpha q_\nu{}^\beta \mathscr{L}_n g_{\alpha\beta} - K_{\nu\mu} = K_{\mu\nu} \tag{2.18}$$

which implies

$$q_{[\mu}{}^\alpha q_{\nu]}{}^\beta \mathscr{L}_n g_{\alpha\beta} - K_{[\nu\mu]} = K_{[\mu\nu]}. \tag{2.19}$$

We also see that

$$q_\mu{}^\alpha q_\nu{}^\beta \mathscr{L}_n g_{\alpha\beta} - 2K_{\nu\mu} = K_{\mu\nu} - K_{\nu\mu} \tag{2.20}$$

Setting these two equations equal yields

$$q_\mu{}^\alpha q_\nu{}^\beta \mathscr{L}_n g_{\alpha\beta} - 2K_{\nu\mu} = q_\mu{}^\alpha q_\nu{}^\beta \mathscr{L}_n g_{\alpha\beta} - K_{\nu\mu} - q_\nu{}^\alpha q_\mu{}^\beta \mathscr{L}_n g_{\alpha\beta} + K_{\mu\nu} \tag{2.21}$$

which

3. Einstein Hilbert Action to Canonical Variables

Recall that for General Relativity, the Einstein Hilbert action is

$$I = \frac{1}{16\pi G_N} \int d^4x \sqrt{|g|} R \tag{3.1}$$

where G_N is Newton's constant, g is the determinant of the metric, and R is the Ricci scalar. We demand the action vanishes upon variation of the metric

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu} \tag{3.2}$$

and we obtain Einstein's field equations this way. We use a very specific variation in the metric, namely a diffeomorphism invariant variation

$$\delta g_{\mu\nu} = \nabla_\nu \xi_\mu + \nabla_\mu \xi_\nu \tag{3.3}$$

which is precisely the killing equation for diffeomorphism invariance.

Naively, one could suppose that we can do the canonical formalism in a covariant way by sleight of hand. We just pretend the canonical position is the 4 metric $g_{\mu\nu}$ and find its time derivative, then apply Dirac's constrained dynamics scheme. This was the approach of Pirant, Schild, and Skinner [2]. Their analysis was not complete, as Kiriushcheva et al noted [3] the time development of secondary constraints were not considered. Neither was the closure of the Dirac procedure demonstrated. Dirac attacked this problem [4] and came to the conclusion that four dimensions should be split into three plus one dimensions.

4. Canonical Dynamics

Now, we can write the action in terms of our new canonical variables as

$$I = \int dt\, d^3x \left[\pi^{ij}\dot{q}_{ij} - N\mathscr{H} - N_i\mathscr{H}^i\right] \tag{4.1}$$

where

$$\mathscr{H} = \frac{16\pi G_N}{\sqrt{q}}\left(\pi^{ij}\pi_{ij} - \pi^2\right) - \frac{\sqrt{q}}{16\pi G_N}\sqrt{q}^{(3)}R \tag{4.2}$$

and

$$\mathscr{H}^i = -2D_j\pi^{ij}. \tag{4.3}$$

Eq (4.2) is a Hamiltonian for General Relativity based off of a certain set of variables – the metric for a spatial hypersurface as the position variable, and its time derivative as the canonically conjugate momenta.

We now find the dynamics in the usual way, by using the Poisson bracket. We see that

$$\{q_{ij}(x), \pi^{kl}(x')\} = \frac{1}{2}\left(\delta^k_i\delta^l_j + \delta^k_j\delta^l_i\right)\tilde{\delta}^{(3)}(x - x') \tag{4.4}$$

where $\widetilde{\delta}^{(3)}$ is the densitized delta function, i.e. the delta function such that

$$\int \tilde{\delta}^{(3)}(x)d^3x = 1 \tag{4.5}$$

so we won't need $\sqrt{q}$. Now, this is a completely constrained system, with the momentum constraints generating spatial change of coordinates. Consider the Poisson bracket of the momentum constraints with the spatial metric:

$$\left\{\int \xi^i\mathscr{H}_i(x)d^3x,\ q_{kl}(x')\right\} = \left\{-2\int \xi^i D^j\pi_{ij}(x)d^3x,\ q_{kl}(x')\right\} \tag{4.6a}$$

$$= \left\{\int (D_i\xi_j + D_j\xi_i)\pi^{ij}(x)d^3x,\ q_{kl}(x')\right\} \tag{4.6b}$$

$$= -(D_k\xi_l + D_l\xi_k) \tag{4.6c}$$

$$= -\mathscr{L}_\xi q_{kl} \tag{4.6d}$$

where $\mathscr{L}_\xi$ is the Lie derivative. This means that $\mathscr{H}_i$ are the generators of spatial coordinate transformations. The Poisson bracket for the momentum constraints and the π^{ij} are a bit more complicated.

We are working on spatial hypersurfaces, so the question of *what "$\mathscr{H}$ generates time translations" means* needs to be investigated. Again, the easy Poisson bracket to consider is with the spatial metric. Technically, these are "spatial deformations" yielded by such a bracket.

References

[1] R. Geroch, "Domain of dependence," *Journal of Mathematical Physics* **11** (1970) 437. `http://link.aip.org/link/?JMAPAQ/11/437/1`.

[2] F. A. E. Pirant, A. Schild, and R. Skinner, "Quantization of Einstein's Gravitational Field Equations. 2," *Phys. Rev.* **87** (1952) 452–454. `http://link.aps.org/doi/10.1103/PhysRev.87.452`.

[3] N. Kiriushcheva, S. V. Kuzmin, C. Racknor, and S. R. Valluri, "Diffeomorphism Invariance in the Hamiltonian formulation of General Relativity," *Phys. Lett.* **A372** (2008) 5101, `arXiv:0808.2623 [gr-qc]`.

[4] P. A. M. Dirac, "The Theory of gravitation in Hamiltonian form," *Proc. Roy. Soc. Lond.* **A246** (1958) 333–343.
[5] K. V. Kuchar, "Canonical quantum gravity," `arXiv:gr-qc/9304012`.
[6] R. L. Arnowitt, S. Deser, and C. W. Misner, "The dynamics of general relativity," `arXiv:gr-qc/0405109`.
[7] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. WH Freeman & Company, 1973.
[8] V. A. Franke, "Different canonical formulations of Einstein's theory of gravity," *Theor. Math. Phys.* **148** (2006) 995–1010, `arXiv:0710.4953 [gr-qc]`.
[9] D. Giulini and C. Kiefer, "The canonical approach to quantum gravity: General ideas and geometrodynamics," *Lect. Notes Phys.* **721** (2007) 131–150, `arXiv:gr-qc/0611141`.
[10] T. Thiemann, "Introduction to modern canonical quantum general relativity," `arXiv:gr-qc/0110034`.

E-mail address: `pqnelson@gmail.com`