NOTES ON CHAIN FIELD THEORY

ALEX NELSON

Abstract. Scratch work double checking Dr Wise's calculations in chain field theory.

1. Trouser Diagrams

We first set up the trousers diagram, as doodled on the right. It basically is a cobordism from one circle to two (disjoint) circles. The boundaries (well, the notion of a circle to be more precise) consist of 1 edge and 1 vertex (each). So e_i is the edge that starts and ends at v_i (where $i = 1, 2, 3$). We have two additional edges which connects the initial state (the e_1, v_1 circle) to the terminal state (the e_2, e_3 circles). These edges define the trousers diagram. We are interested in calculating the various algebraic quantities which will be used in the homological calculations, which we use motivated by discrete differential geometry.

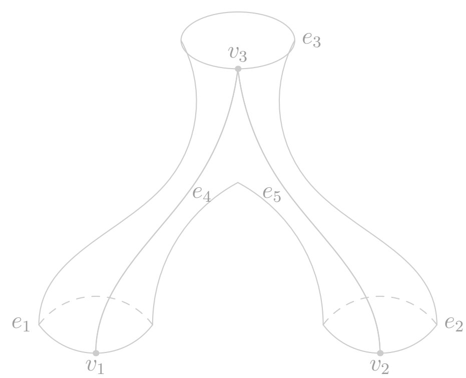

Figure 1. Trouser Diagram

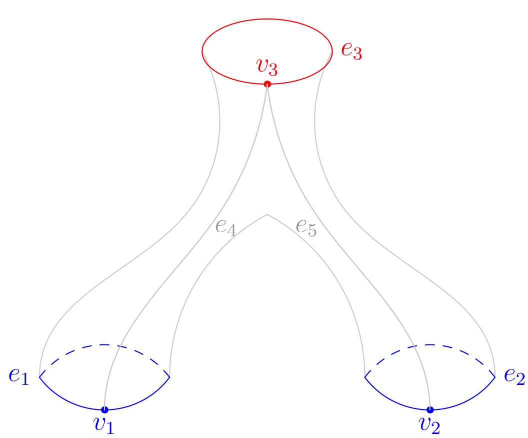

Figure 2. Initial/terminal states highlighted.

To begin setting up a chain to describe the initial and terminal states (doodled on the left in red and blue, respectively), we should consider the number of edges and vertices. We see that we don't need to consider anything "higher" than vertices and edges since there are no p-cells. Let r_v be the number of red vertices, r_e be the number of red edges. We see that the chain describing the initial state is $0 \leftarrow C_0 \leftarrow C_1$ where $C_0 \cong \mathbb{Z}^{r_v}$ and $C_1 \cong \mathbb{Z}^{r_e}$ are the free groups generated by the vertices and edges in the initial state (respectively).

We thus have our chain describing our initial state be:

$$0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \tag{1.1}$$

Now, we would like a corresponding chain describing the final state. We see, similarly, that the chain would be

$$0 \leftarrow C'_1 \leftarrow C'_2 \tag{1.2}$$

where $C'_1 \cong \mathbb{Z}^{b_v}$ and $C'_2 \cong \mathbb{Z}^{b_e}$, b_v is the number of blue vertices, b_e is the number of blue edges. We see by inspection that $b_v = 2$ and $b_e = 2$, thus the chain describing the final state is

$$0 \leftarrow \mathbb{Z}^2 \leftarrow \mathbb{Z}^2. \tag{1.3}$$

We would like a chain complex to describe the cobordism altogether.

The general scheme for the cobordism is

$$\begin{tikzcd} \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \\ \mathcal{M}_1 & \mathcal{M}_2 \arrow[l] & \mathcal{M}_3 \arrow[l] \\ \mathbb{Z}^2 \arrow[u] & \mathbb{Z}^2 \arrow[l] \arrow[u] & \end{tikzcd} \tag{1.4}$$

where we are trying to find $\mathcal{M}_1$ which corresponds to the free group generated by *all* of the vertices in the diagram, and $\mathcal{M}_2$ corresponds to the gree group generated by *all* of the edges in the diagram.

The $\mathcal{M}_3$ corresponds to the free group generated by the "skin" of the cobordism, if we think of the edges as the "bones" the cobordism is somewhat analogous to a tent. We see that there are only three edges in total in our diagram. They are doodled on the left. The initial vertices are in red, the terminal vertices are in blue. So we see that there are 2+1=3 vertices telling us that $\mathcal{M}_1 \cong \mathbb{Z}^3$, which solves one part of our problem. We are left with trying to deduce what the other aspects of the chain complex could be.

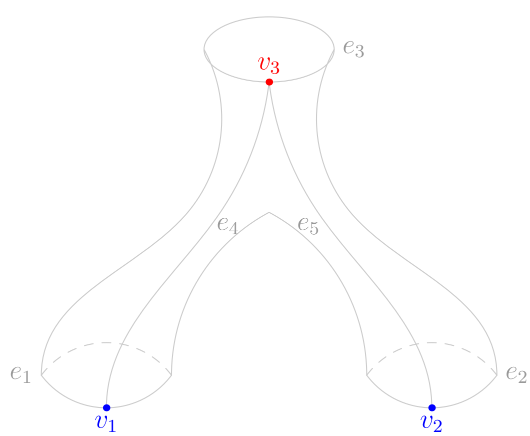

FIGURE 3. All vertices highlighted

We are worried about the edges, since we already deduced that $\mathcal{M}_3 \cong \mathbb{Z}$. There is only one "skin" to the diagram. We can fill in the parts of the chain complex that we know:

$$\begin{tikzcd} \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \\ \mathbb{Z}^3 & \mathcal{M}_2 \arrow[l] & \mathbb{Z} \arrow[l] \\ \mathbb{Z}^2 \arrow[u] & \mathbb{Z}^2 \arrow[l] \arrow[u] & \end{tikzcd} \tag{1.5}$$

We need to deduce what $\mathcal{M}_2$ is.

Doodled on the right is the diagram with *all* of the edges highlighted. The initial edge is in red, the terminal edge is in blue, and the intermediate edges are in purple. We see that there is a total of 1+2+2=5 edges, which allows us to deduce that $\mathcal{M}_2 \cong \mathbb{Z}^5$. This is the last part of the computation of the chain complex, the rest of the calculation for this particular cobordism is strictly manipulation via the functor **nChain**→**Hilb**. (This won't require too much algebraic manipulation since we are working with regular, old fashioned electromagnetism, so we are concerned with assigning information from U(1) to edges; the compactness of U(1) simplifies life significantly.) To summarize, the chain calculation is finished with

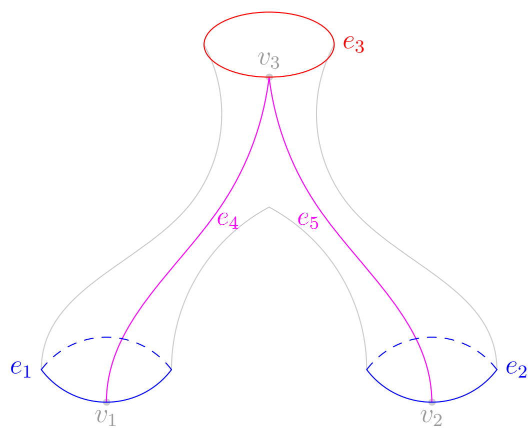

FIGURE 4. All edges highlighted.

$$\begin{tikzcd} \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \\ \mathbb{Z}^3 & \mathbb{Z}^5 \arrow[l] & \mathbb{Z} \arrow[l] \\ \mathbb{Z}^2 \arrow[u] & \mathbb{Z}^2 \arrow[l] \arrow[u] & \end{tikzcd} \tag{1.6}$$

Let

$$Z : \textbf{nChain} \to \textbf{Hilb} \tag{1.7}$$

be the chain field theory describing regular, old-school electromagnetism (i.e. the connections are defined on the edges, the gauge is U(1), etc.). The time evolution of our doodle is described by the morphism

$$Z(\mathbb{Z}) \to Z(\mathbb{Z}^2). \tag{1.8}$$

With gauge systems, we typically find the physically meaningful states by taking the orbit of the gauge group modulo the stabilizer. Similarly, the physically meaningful states would be

$$Z(C) \cong L^2\left(\frac{\mathcal{A}(C)}{\mathcal{G}(C)}\right) \tag{1.9}$$

where $\mathcal{A}(C) := C^p =$ group of p-connections, and $\mathcal{G}(C) := C^{p-1} =$ gauge group, $C^p := \hom(C_p, U(1))$. For those of us interested in old-school electromagnetism this is 1-connections. We see that

$$\frac{\mathcal{A}(S)}{\mathcal{G}(S)} = \frac{C^1(S, U(1))}{B^1(S, U(1))} \tag{1.10}$$

where $B^q := \mathrm{ran}(d_{q-1})$ is the space of q-coboundaries. We have used the shorthand $C^p(S, G) = \hom(C_p(S), G)$.

We see that if $\omega \in C^0$, then $d_0\omega \in C^1$ is defined by $d_0\omega(x) = \omega(\partial_1 x)$. But for the circle, $\partial_1 e = t(e) - s(e) = v - v = 0$, which means that $d_0\omega(e) = \omega(t(e)) - \omega(s(e)) = 0$ (justified by page 9 of [1]). So we mod out by $\{0\}$ which doesn't change anything. We end up with

$$Z(C) \cong L^2\Big(\mathcal{A}(C)/\{0\}\Big) \cong L^2\Big(\mathcal{A}(C)\Big) \tag{1.11}$$

which for us is

$$Z(C) \cong L^2\Big(\hom(\mathbb{Z}^q, U(1))\Big) \tag{1.12}$$

which leaves us to figure out what this is equal to.

Proposition 1. We have the following isomorphism

$$\hom(\mathbb{Z}, U(1)) \cong U(1). \tag{1.13}$$

Proof. The proof is more or less roundabout. We know that an element of $U(1)$ looks like $\exp(i\theta)$ for some $\theta \in \mathbb{R}$. We know we can construct $\mathbb{R}$ from $\mathbb{Q}$, and we can construct $\mathbb{Q}$ from $\mathbb{Z}$. So we consider a family of homomorphisms

$$\phi_m(n) \in \hom(\mathbb{Z}, U(1)) \tag{1.14}$$

which really gives us two degrees of freedom to play around with: m and n. This allows us to embed

$$\hom(\mathbb{Q}, U(1)) \subseteq \hom(\mathbb{Z}, U(1)) \tag{1.15}$$

and by constructing the slice category of **Grp** over U(1), we see that there is an isomorphism from $\hom(\mathbb{Q}, U(1))$ to $\hom(\mathbb{R}, U(1))$. Thus by our deduction, there is an isomorphism from $\hom(\mathbb{R}, U(1))$ to a subset of $\hom(\mathbb{Z}, U(1))$, and there is the obvious isomorphism $\hom(\mathbb{R}, U(1)) \cong U(1)$ which proves the hypothesis. □

By our proposition, we have that

$$Z(\mathbb{Z}) \cong L^2(U(1)). \tag{1.16}$$

Similarly, we have for the target

$$Z(\mathbb{Z}^2) \cong L^2\Big(U(1) \otimes U(1)\Big). \tag{1.17}$$

Thus our cobordism give the time evolution by the functor

$$Z(M) : L^2(U(1)) \to L^2\Big(U(1) \otimes U(1)\Big). \tag{1.18}$$

The question we want to answer is: *how exactly does it work?*

References

[1] D. K. Wise, "p-form electromagnetism on discrete spacetimes," *Class. Quant. Grav.* **23** (2006) 5129–5176.

E-mail address: `pqnelson@gmail.com`