NOTES ON REAL ANALYSIS

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Contents

1. Properties of Real Numbers

There are some important properties of the real numbers that we need to introduce, since we are introducing real analysis.

Proposition 1 (Triangle Inequality). For any $a, b \in \mathbb{R}$, we have

$$|a + b| \leq |a| + |b|. \tag{1}$$

Remark 1. More generally, in metric topology, when we have a metric

$$d : X \times X \to \mathbb{R},$$

the identity

$$d(x, z) + d(z, y) \geq d(x, y) \tag{2}$$

holds for all $x, y, z \in X$. It is called the "triangle inequality".

2. Continuity

Definition 1. Let $f : D \to \mathbb{R}$ be a real valued function whose domain is a subset of $D \subset \mathbb{R}$. Then f is said to be **continuous at** $x_0 \in D$ iff for each $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$|x - x_0| < \delta \ \text{ implies } \ |f(x) - f(x_0)| < \varepsilon. \tag{3}$$

Remark 2. The intuition behind this definition is if we "wiggle" around in the range by some amount less than ε about $f(x_0)$, then we should "wiggle" about in the domain by some amount less than δ. If we end up have δ be too large as ε gets too small, there is a discontinuity (a jump permitting a lot of wiggling). We can see a diagram illustrating this point in fig (1). Observe that in the diagram, as we make $|f(x) - f(x_0)|$ decrease, $|x - x_0|$ decreases faster. In general, we just want a relationship expressing some positive δ in terms of a given $\varepsilon > 0$. This relationship should be injective(?) and such that as ε decreases, δ decreases too. So we can have an arbitrarily small ε, and δ goes to zero as well. This is fundamentally all a limit really is.
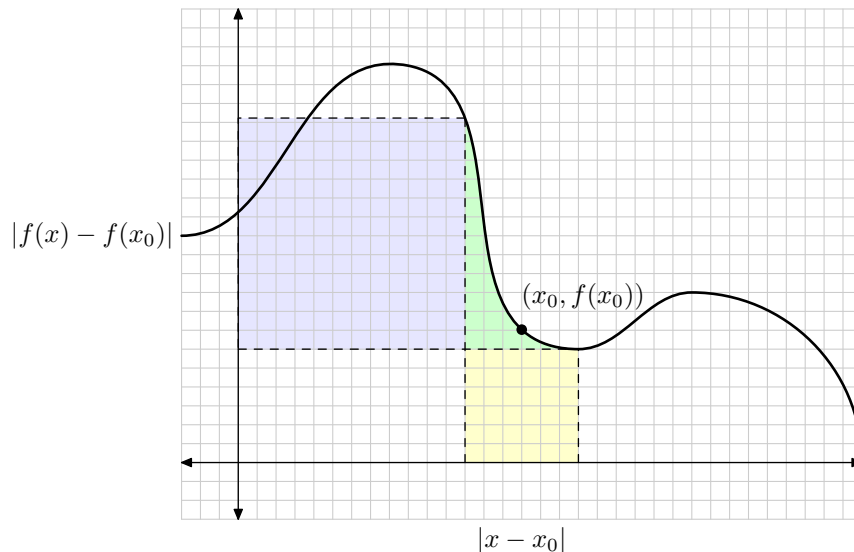
FIGURE 1. An example of the notion of continuity. The yellow region is the δ neighborhood of x_0, the blue region is the ε neighborhood of $f(x_0)$.

There is an alternate definition using sequences which is less intuitive.

Definition 2. Let f be a real function, we say f is continuous at x_0 if for every sequence (x_n) that converges to x_0, then

$$(4) \qquad \lim_{n \to \infty} f(x_n) = f(x_0).$$

(We assume that the sequence all belongs to the domain of f.)

Remark 3. This can be worded slightly differently as "*Continuity preserves convergence of sequences.*"

Example 1. We will show that $f(x) = 2x^2 + 1$ is continuous. So, we want to write δ in terms of ε. This will allow us to state "For each ε, there is a δ such that..." which implies continuity. So, let x_0 be some arbitrary real number, we have

$$(5) \qquad |f(x) - f(x_0)| = \left| \left[2x^2 + 1 \right] - \left[2x_0^2 + 1 \right] \right| = 2|x^2 - x_0^2| < \varepsilon$$

We can rewrite this as

$$(6) \qquad 2|x^2 - x_0^2| \le 2|x - x_0||x + x_0| < 2\delta|x + x_0|$$

We also have a bound that $|x| < |x_0| + \delta$. We plug this in, we set the result equal to ε so

$$(7) \qquad 2\delta|x + x_0| \le 2\delta(|x| + |x_0|) < 2\delta(2|x_0| + \delta) = \varepsilon.$$

We rearrange terms to find

$$(8) \qquad \delta^2 + 2|x_0|\delta - \frac{1}{2}\varepsilon = 0.$$

As we learned from primary school, this is a quadratic equation which has precisely two solutions, we choose the positive one:

$$\delta = \sqrt{x_0^2 + \frac{1}{2}\varepsilon} - |x_0|. \tag{9}$$

This is always greater than zero provided $\varepsilon > 0$, $x_0 \in \mathbb{R}$. Thus f is continuous. QEF.

Example 2. Consider

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases} \tag{10}$$

We can "see" that when $x \neq 0$, g is continuous. At $x = 0$, we need to prove it's continuous. So, we see that for each $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$|x - 0| < \delta \implies |g(x) - 0| < \varepsilon. \tag{11}$$

We can rewrite the relations to be (since $\sin(x) \leq 1$)

$$|g(x)| \leq x^2 = (x)^2 < \delta^2 = \varepsilon. \tag{12}$$

So $\delta = \sqrt{\varepsilon}$ is always positive and nonzero. Thus $g(x)$ is continuous at $x = 0$. QEF.

Theorem 3. Let

$$f : D \to \mathbb{R} \tag{13}$$

where $D \subseteq \mathbb{R}$. If f is continuous at $x_0 \in D$, then $|f|$ and kf are continuous at x_0 (where $k \in \mathbb{R}$ is arbitrary).

Proof. Since f is continuous, then for each $\varepsilon > 0$ there is a corresponding $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon. \tag{14}$$

We see that

$$|kf(x) - kf(x_0)| = |k||f(x) - f(x_0)| < |k|\varepsilon \tag{15}$$

so we choose $\varepsilon' = |k|\varepsilon$ for nonzero k. We see then that for each $\varepsilon' > 0$ there is a corresponding $\delta > 0$ such that

$$|x - x_0| < \delta \implies |kf(x) - kf(x_0)| < \varepsilon' \tag{16}$$

which is necessarily true if f is continuous.

Observe that, for some $a, b \in \mathbb{R}$,

$$|b| \leq |a| + |b - a|. \tag{17}$$

Let $b = f(x)$, $a = f(x_0)$, then

$$|f(x)| < |f(x_0)| + \varepsilon \tag{18}$$

Similarly

$$|a| \leq |b| + |a - b| \Rightarrow |f(x_0)| < |f(x)| + \varepsilon \tag{19}$$

We have

$$-\varepsilon < |f(x)| - |f(x_0)| < \varepsilon \tag{20}$$

which implies

$$||f(x)| - |f(x_0)|| < \varepsilon. \tag{21}$$

This implies continuity.   □

Theorem 4. Let f, g be real functions continuous at x_0. Then $f + g$, fg and (if $g(x_0) \neq 0$) f/g are continuous at x_0.

Proof. Let ε_1, δ_1 be for f, ε_2, δ_2 be for g. So for each $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there is a corresponding $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$(22) \qquad\qquad |x - x_0| < \delta_1 \Rightarrow |f(x) - f(x_0)| < \varepsilon_1$$

and

$$(23) \qquad\qquad |x - x_0| < \delta_2 \Rightarrow |g(x) - g(x_0)| < \varepsilon_2.$$

Since ε_i ($i = 1, 2$) is arbitrary, we will impose the condition that $\varepsilon_i < 1$. We see that by the triangle inequality

$$(24) \quad |(f(x) + g(x)) - (f(x_0) + g(x_0))| \leq |f(x) - f(x_0)| + |g(x) - g(x_0)| < \varepsilon_1 + \varepsilon_2$$

We can also see that

$$(25) \qquad\qquad |f(x)g(x) - f(x_0)g(x_0)| < |f(x_0)|\varepsilon_2 + |g(x)|$$

<div align="right">□</div>

References

[1] Kenneth A. Ross. *Elementary Analysis: The Theory of Calculus.* Springer Science and Business Media, 2000.