NOTES ON CHAIN FIELD THEORY

ALEX NELSON (16 AUGUST 2009)

Abstract. Scratch work double checking Dr Wise's calculations in chain field theory.

1. Trouser Diagrams

We first set up the trousers diagram, as doodled on the right. It basically is a cobordism from one circle to two (disjoint) circles. The boundaries (well, the notion of a circle to be more precise) consist of 1 edge and 1 vertex (each). So e_i is the edge that starts and ends at v_i (where $i = 1, 2, 3$). We have two additional edges which connects the initial state (the e_1, v_1 circle) to the terminal state (the e_2, e_3 circles). These edges define the trousers diagram. We are interested in calculating the various algebraic quantities which will be used in the homological calculations, which we use motivated by discrete differential geometry.

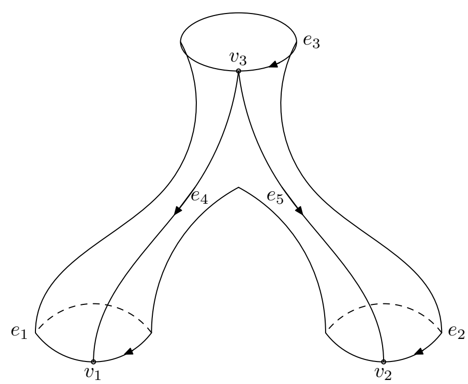


Figure 1. Trouser Diagram

To begin setting up a chain to describe the initial and terminal states (doodled in figure 2 in red and blue, respectively), we should consider the number of edges and vertices. We see that we don't need to consider anything "higher" than vertices and edges since there are no p-cells. Let r_v be the number of red vertices, r_e be the number of red edges. We see that the chain describing the initial state is $0 \leftarrow C_0 \leftarrow C_1$ where $C_0 \cong \mathbb{Z}^{r_v}$ and $C_1 \cong \mathbb{Z}^{r_e}$ are the free groups generated by the vertices and edges in the initial state (respectively).

We thus have our chain describing our initial state be:

$$0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \tag{1.1}$$

Now, we would like a corresponding chain describing the final state. We see, similarly, that the chain would be

$$0 \leftarrow C'_1 \leftarrow C'_2 \tag{1.2}$$

where $C'_1 \cong \mathbb{Z}^{b_v}$ and $C'_2 \cong \mathbb{Z}^{b_e}$, b_v is the number of blue vertices, b_e is the number of blue edges. We see by inspection that $b_v = 2$ and $b_e = 2$, thus the chain describing

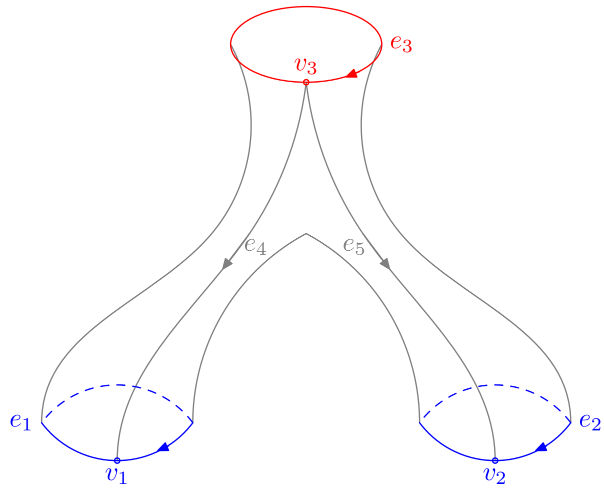


FIGURE 2. Initial/terminal states highlighted.

the final state is

$$0 \leftarrow \mathbb{Z}^2 \leftarrow \mathbb{Z}^2. \tag{1.3}$$

We would like a chain complex to describe the cobordism altogether.

The general scheme for the cobordism is

$$\begin{tikzcd} \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \\ \mathcal{M}_1 & \mathcal{M}_2 \arrow[l] & \mathcal{M}_3 \arrow[l] \\ \mathbb{Z}^2 \arrow[u] & \mathbb{Z}^2 \arrow[l] \arrow[u] & \end{tikzcd} \tag{1.4}$$

where we are trying to find $\mathcal{M}_1$ which corresponds to the free group generated by *all* of the vertices in the diagram, and $\mathcal{M}_2$ corresponds to the gree group generated by *all* of the edges in the diagram.

The $\mathcal{M}_3$ corresponds to the free group generated by the "skin" of the cobordism, if we think of the edges as the "bones" the cobordism is somewhat analogous to a tent. We see that there are only three edges in total in our diagram. They are doodled on the left. The initial vertices are in red, the terminal vertices are in blue. So we see that there are 2+1=3 vertices telling us that $\mathcal{M}_1 \cong \mathbb{Z}^3$,

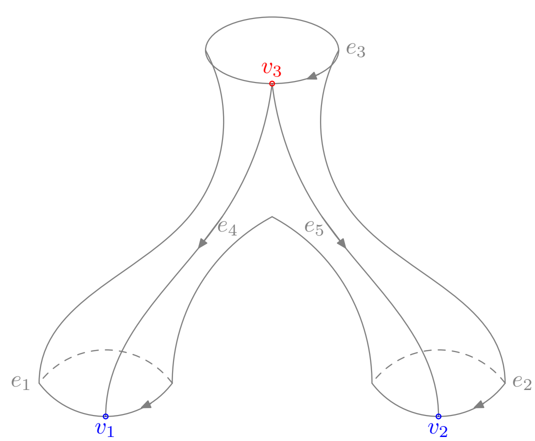


FIGURE 3. All vertices highlighted

which solves one part of our problem. We are left with trying to deduce what the other aspects of the chain complex could be.

We are worried about the edges, since we already deduced that $\mathcal{M}_3 \cong \mathbb{Z}$. There is only one "skin" to the diagram. We can fill in the parts of the chain complex that we know:

$$\begin{tikzcd} \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \\ \mathbb{Z}^3 & \mathcal{M}_2 \arrow[l] & \mathbb{Z} \arrow[l] \\ \mathbb{Z}^2 \arrow[u] & \mathbb{Z}^2 \arrow[l] \arrow[u] & \end{tikzcd} \tag{1.5}$$

We need to deduce what $\mathcal{M}_2$ is.

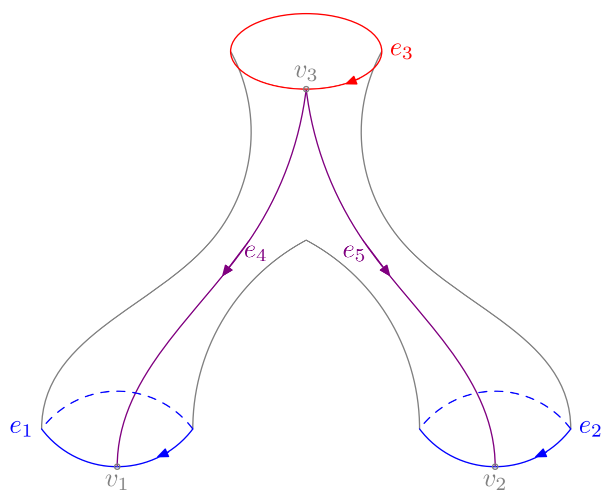


FIGURE 4. All edges highlighted.

Doodled above in figure (4) is the diagram with *all* of the edges highlighted. The initial edge is in red, the terminal edge is in blue, and the intermediate edges are in purple. We see that there is a total of 1+2+2=5 edges, which allows us to deduce that $\mathcal{M}_2 \cong \mathbb{Z}^5$. This is the last part of the computation of the chain complex, the rest of the calculation for this particular cobordism is strictly manipulation via the functor **nChain**→**Hilb**. (This won't require too much algebraic manipulation since we are working with regular, old fashioned electromagnetism, so we are concerned with assigning information from U(1) to edges; the compactness of U(1) simplifies life significantly.) To summarize, the chain calculation is finished with

$$\begin{tikzcd}
\mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \\
\mathbb{Z}^3 & \mathbb{Z}^5 \arrow[l] & \mathbb{Z} \arrow[l] \\
\mathbb{Z}^2 \arrow[u] & \mathbb{Z}^2 \arrow[l] \arrow[u] &
\end{tikzcd} \tag{1.6}$$

Let

$$Z : \mathbf{nChain} \to \mathbf{Hilb} \tag{1.7}$$

be the chain field theory describing regular, old-school electromagnetism (i.e. the connections are defined on the edges, the gauge is U(1), etc.). The time evolution of our doodle is described by the morphism

$$Z(\mathbb{Z}) \to Z(\mathbb{Z}^2). \tag{1.8}$$

With gauge systems, we typically find the physically meaningful states by taking the orbit of the gauge group modulo the stabilizer. Similarly, the physically meaningful states would be

$$Z(C) \cong L^2\left(\frac{\mathcal{A}(C)}{\mathcal{G}(C)}\right) \tag{1.9}$$

where $\mathcal{A}(C) := C^p =$ group of p-connections, and $\mathcal{G}(C) := C^{p-1} =$ gauge group, $C^p := \hom(C_p, U(1))$. For those of us interested in old-school electromagnetism this is 1-connections. We see that for compact groups we don't have to mod out by the gauge transformations, so we integrate over the connections on the manifold.

Remark 1. Remember that connections are elements of $\hom(\mathbb{Z}^{X_p}, U(1))$. That is, functions assigning to elements of the free group generated by the p-cells X_p data from our group U(1).

We end up with

$$Z(C) \cong L^2\left[\hom\Big(\mathbb{Z}, U(1)\Big)\right] \tag{1.10}$$

since there is only one edge in the initial state.

Proposition 1. We have the following isomorphism

$$\hom(\mathbb{Z}, U(1)) \cong U(1). \tag{1.11}$$

Proof. We see that for $\varphi \in \hom(\mathbb{Z}, U(1))$ that

$$\varphi(1) = \alpha \quad \Rightarrow \quad \varphi(n) = a^n \tag{1.12}$$

since $n = 1 + \cdots + 1$ which uses the law of composition in $\mathbb{Z}$ as a free group. This is preserved by a homomorphism φ and becomes multiplication (the law of composition) in the group U(1). Since the choice of $a = \exp(i\theta)$ is arbitrary (for some $\theta \in [0, 2\pi)$), we can choose a different φ for each θ mapping all of $\mathbb{Z}$ to all of U(1). $\square$

By our proposition, we have that

$$Z(\mathbb{Z}) \cong L^2(U(1)). \tag{1.13}$$

Similarly, we have for the target

$$Z(\mathbb{Z}^2) \cong L^2\Big(U(1) \otimes U(1)\Big). \tag{1.14}$$

Thus our cobordism give the time evolution by the functor

$$Z(M) : L^2(U(1)) \to L^2\Big(U(1) \otimes U(1)\Big). \tag{1.15}$$

The question we want to answer is: *how exactly does it work?*

The strategy is that we'll consider the object $Z(M)$ as a mathematical operator, and we recall from linear algebra to compute the entries of an operator we have to compute how it acts on basis elements.

First we need to pick some elements of $L^2(U(1))$ and $L^2(U(1))^{\otimes 2}$. Remember that we are working with connections, i.e. functions from C_p to $U(1)$. So we should choose for $L^2(U(1))$ functions of the form

$$\psi(A) = \exp(ikA). \tag{1.16}$$

Similarly for $L^2(U(1))^{\otimes 2}$ we have a "tensor product" of two such functions yielding one that looks like

$$\phi(A_1, A_2) = \exp(ik_1 A_1)\exp(ik_2 A_2). \tag{1.17}$$

Note that k_1, k_2 are two distinct parameters, but for simplicity we'll just denote it by k in this calculation (it does make a difference as we will see in the next example!). We will use such functions to figure out the components of the operator in question.

Proposition 2. We have by Fourier expansion:

$$\sum_{n\in\mathbb{Z}} \exp\left(\frac{-1}{2e^2V}(A + 2n\pi)^2\right) = \sum_{n\in\mathbb{Z}} \exp\left(\frac{-e^2Vn^2}{2}\right) e^{inA} \tag{1.18}$$

(up to some constant coefficient).

Well, we find by eq (16) of [1] that

$$\langle \overbrace{\phi}^{\in L^2(U(1))^{\otimes 2}}, Z(M)\underbrace{\psi}_{\in L^2(U(1))}\rangle = \int_{\substack{\text{connections}\\ \text{on } \mathcal{M}}} \overline{\phi(A|_{S'})}\psi(A|_S)e^{-S(A)}\mathcal{D}A \tag{1.19}$$

where $S(A)$ is the action. This allows us to compute entries in $Z(M)$, kind of like how we compute entries in the S-matrix. We see in section 7 (et seq) of [1] that the path integral should consider

$$\begin{pmatrix} p\text{-connections} \\ \text{on } \mathcal{M} \end{pmatrix} = U(1)^{X_p}. \tag{1.20}$$

For us $p = 1$ and $|X_1| = 5$, so the measure $\mathcal{D}A$ in our case becomes

$$\mathcal{D}A = \frac{dA_1}{2\pi}\frac{dA_2}{2\pi}\frac{dA_3}{2\pi}\frac{dA_4}{2\pi}\frac{dA_5}{2\pi} \tag{1.21}$$

To calculate the action, we need to pick an orientation for the manifold.

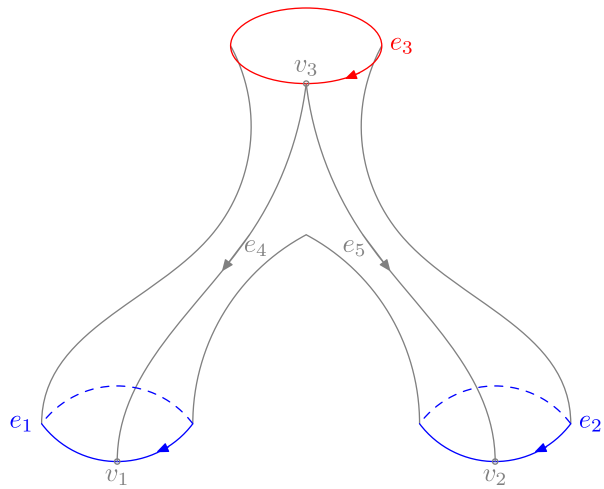


FIGURE 5. The manifold with orientation picked arbitrarily. The curvature calculations depend on the parts of the figure colored in. Red contributions are positive, blue contributions are negative

We do this in figure 5. Explicitly, it's like peeling the "skin" from the tent by cutting it. We cut it by starting at v_3 and moving against the orientation of e_3 (contributing a term $-A(e_3)$), then down e_4 (contributing a term $+A(e_4)$), then around e_1 in its orientation (contributing a term $+A(e_1)$), then back up e_4 (contributing a term $-A(e_4)$), then down e_5 (contributing a term $+A(e_5)$), then around e_2 in its orientation (contributing a term $+A(e_2)$), then up e_5 to where we started (contributing a term $-A(e_5)$). We thus calculate the curvature of the entire manifold to be:

$$\begin{aligned} F &= -A(e_3) - A(e_4) + A(e_1) + A(e_4) - A(e_5) + A(e_2) + A(e_5) \\ &= -A(e_3) + A(e_1) + A(e_2) \end{aligned} \tag{1.22}$$

We end up with the integrand being

$$\overline{\phi(A|_{S'})}\psi(A|_S)e^{-S(A)} = e^{-ikA_1}e^{-ikA_2}e^{ikA_3}\sum_{n\in\mathbb{Z}}\exp\left(\frac{-h}{2e^2}(F+2\pi n)^2\right) \tag{1.23}$$

where we sum over $\mathbb{Z}^{X_{p+1}} = \mathbb{Z}^1$. We can now rewrite our integral to be

$$\begin{aligned} \langle\phi, Z(M)\psi\rangle = &\iint_0^{2\pi}\frac{dA_4}{2\pi}\frac{dA_5}{2\pi} \\ &\times\iiint_0^{2\pi} e^{-ikA_1}e^{-ikA_2}e^{ikA_3}\sum_{n\in\mathbb{Z}}\exp\left(\frac{-h}{2e^2}(F+2\pi n)^2\right)\frac{dA_1}{2\pi}\frac{dA_2}{2\pi}\frac{dA_3}{2\pi} \\ = &\iiint e^{-ikA_1}e^{-ikA_2}e^{ikA_3}\sum_{n\in\mathbb{Z}}\exp\left(\frac{-h}{2e^2}(F+2\pi n)^2\right)\frac{dA_1}{2\pi}\frac{dA_2}{2\pi}\frac{dA_3}{2\pi} \end{aligned}$$

The factor $\iint dA_4 dA_5$ end up contributing a factor of 1.

By proposition (2) we see that (up to a constant) we can simplify our expression to a friendler one:
(1.24)

$$\langle \phi, Z(M)\psi\rangle = \iiint_0^{2\pi} e^{-ikA_1}e^{-ikA_2}e^{ikA_3}\sum_{n\in\mathbb{Z}}\exp\left(\frac{-e^2Vn^2}{2}\right)\exp(inF)\frac{dA_1}{2\pi}\frac{dA_2}{2\pi}\frac{dA_3}{2\pi}.$$

We just need to plug in our expression for the curvature F and manipulate it a little to get the solution. Recall first that the kronecker delta is defined by

$$\delta_{x,n} = \int_0^{2\pi} e^{i(x-n)\theta}d\theta \tag{1.25}$$

this will come in handy in a few moments.

By rearranging terms and expanding out the expression for the curvature, we end up with

$$\begin{aligned}\langle \phi, Z(M)\psi\rangle &= \iiint_0^{2\pi} e^{-ikA_1}e^{-ikA_2}e^{ikA_3}\sum_{n\in\mathbb{Z}}\exp\left(\frac{-e^2Vn^2}{2}\right)\exp(inF)\frac{dA_1}{2\pi}\frac{dA_2}{2\pi}\frac{dA_3}{2\pi}\\ &= \iint_0^{2\pi} e^{-ikA_1}e^{-ikA_2}\sum_{n\in\mathbb{Z}}\exp\left(\frac{-e^2Vn^2}{2}\right)\exp(inA_1+inA_2)\frac{dA_1}{2\pi}\frac{dA_2}{2\pi}\int_0^{2\pi} e^{i(-n+k)A_3}\frac{dA_3}{2\pi}\end{aligned}$$

We see that our careful studying of the curvature yields a factor which turns out to be the Kronecker delta! We explicitly obtain for our calculations
(1.26)

$$\langle \phi, Z(M)\psi\rangle = \iint_0^{2\pi} e^{-ikA_1}e^{-ikA_2}\sum_{n\in\mathbb{Z}}\exp\left(\frac{-e^2Vn^2}{2}\right)\exp(inA_1+inA_2)\frac{dA_1}{2\pi}\frac{dA_2}{2\pi}\underbrace{\int_0^{2\pi} e^{i(n-k)A_3}\frac{dA_3}{2\pi}}_{\delta_{k,n}}$$

It allows us to reduce this to (by plugging in the Kronecker delta):
(1.27)

$$\langle \phi, Z(M)\psi\rangle = \iint_0^{2\pi} e^{-ikA_1}e^{-ikA_2}\underbrace{\left[\sum_{n\in\mathbb{Z}}\exp\left(\frac{-e^2Vn^2}{2}\right)\exp(inA_1+inA_2)\delta_{n,k}\right]}_{\text{identify as } Z(M)|\psi\rangle}\frac{dA_1}{2\pi}\frac{dA_2}{2\pi}$$

The bracketed factor is identified with how the basis vector $\exp(ikA_3)$ behaves when acted on by the operator $Z(M)$. We can simplify things to summarize it thus:

$$Z(M) : e^{ikA_3} \mapsto \exp\left(\frac{-e^2Vk^2}{2}\right)\exp(ikA_1+ikA_2) \tag{1.28}$$

which is precisely the desired result.

2. Upside Down Trouser Diagram

This is nearly identical to the rightside up trousers. However, since the initial state has two (disjoint) circles, and the final state has one circle, the chain diagram

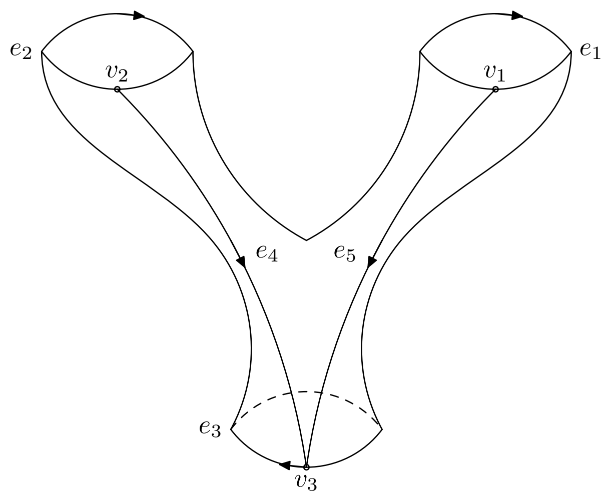


FIGURE 6. The upside down trousers diagram.

becomes

$$\begin{array}{ccccc} \mathbb{Z}^2 & \longleftarrow & \mathbb{Z}^2 & & \\ \downarrow & & \downarrow & & \\ \mathcal{M}_1 & \longleftarrow & \mathcal{M}_2 & \longleftarrow & \mathcal{M}_3 \\ \uparrow & & \uparrow & & \\ \mathbb{Z} & \longleftarrow & \mathbb{Z} & & \end{array} \tag{2.1}$$

In fact we can fill in the missing pieces using the same arguments as for the rightside up trousers:

$$\begin{array}{ccccc} \mathbb{Z}^2 & \longleftarrow & \mathbb{Z}^2 & & \\ \downarrow & & \downarrow & & \\ \mathbb{Z}^3 & \longleftarrow & \mathbb{Z}^5 & \longleftarrow & \mathbb{Z} \\ \uparrow & & \uparrow & & \\ \mathbb{Z} & \longleftarrow & \mathbb{Z} & & \end{array} \tag{2.2}$$

since there are 3 vertices, 5 edges, and 1 3-cell.

We use the same reasoning to suggest that the time evolution operator is described by

$$Z(m) : Z(\mathbb{Z}^2) \to Z(\mathbb{Z}) \tag{2.3}$$

where (by the same arguments as last time)

$$Z(\mathbb{Z}) \cong L^2(U(1)), \quad \text{and} \quad Z(\mathbb{Z}^2) \cong L^2(U(1))^{\otimes 2}. \tag{2.4}$$

We pick out some basis vectors in the initial and final states, specifically

$$\psi = e^{ik_1 A_1} e^{ik_2 A_2} \tag{2.5}$$

for the initial state, and

$$\phi = e^{ikA_3} \tag{2.6}$$

for the final state (where we use shorthand $A_i = A(e_i)$). We calculate the transition probability to deduce what the operator for time evolution is

$$\langle \phi, Z(m)\psi\rangle = \int_{\mathcal{A}(m)} \overline{\phi(A|_S)}\psi(A|_S)e^{-S(A)}\mathcal{D}A \tag{2.7a}$$

$$= \int_{\mathcal{A}(m)} e^{-ikA_3}e^{ik_1A_1}e^{ik_2A_2}\sum_{n\in\mathbb{Z}} e^{\frac{-1}{2e^2V}(F+2n\pi)^2}\mathcal{D}A \tag{2.7b}$$

$$= \iiint_0^{2\pi} e^{-ikA_3}e^{ik_1A_1}e^{ik_2A_2}\sum_{n\in\mathbb{Z}} e^{\frac{-e^2Vn^2}{2}}e^{iFn}\frac{dA_1}{2\pi}\frac{dA_2}{2\pi}\frac{dA_3}{2\pi} \tag{2.7c}$$

where F is the curvature (field strength tensor), $\mathcal{A}(m)$ is the space of connections on m, and the last line is (of course) up to some constant.

To compute the field strength we need to calculate the curvature of the connection cochains. If we start at v_1, go against the orientation of e_1 (add term of $-A(e_1)$), then down e_4 (add term of $+A(e_4)$), around e_3 in the direction towards its orientation (add a term of $+A(e_3)$), up e_3 (add a factor of $-A(e_5)$), around e_2 against its orientation (add a term of $-A(e_2)$), back down e_5 (add a term of $+A(e_5)$), then up e_4 (add a term of $-A(e_4)$) to end our computation. This results in the curvature being

$$F = A_3 - A_1 - A_2 \tag{2.8}$$

where $A_i = A(e_i)$. We plug this into our calculations to find

$$\langle \phi, Z(m)\psi\rangle = \int_{\mathcal{A}(m)} \overline{\phi(A|_S)}\psi(A|_S)e^{-S(A)}\mathcal{D}A$$

(2.9a)

$$= \iiint_0^{2\pi} e^{-ikA_3}e^{ik_1A_1}e^{ik_2A_2}\sum_{n\in\mathbb{Z}} e^{\frac{-e^2Vn^2}{2}}e^{i(A_3-A_1-A_2)n}\frac{dA_1}{2\pi}\frac{dA_2}{2\pi}\frac{dA_3}{2\pi}$$

(2.9b)

$$= \int e^{-ikA_3}\sum_{n\in\mathbb{Z}} e^{inA_3}e^{-e^2Vn^2/2}\left(\int_0^{2\pi} e^{i(n-k_1)A_1}\frac{dA_1}{2\pi}\right)\left(\int_0^{2\pi} e^{i(n-k_2)A_2}\frac{dA_2}{2\pi}\right)\frac{dA_3}{2\pi}$$

(2.9c)

$$= \int e^{-ikA_3}\sum_{n\in\mathbb{Z}} e^{inA_3}e^{-e^2Vn^2/2}\delta_{k_1,n}\delta_{k_2,n}\frac{dA_3}{2\pi}$$

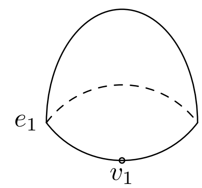
It's easy to tell by inspection that we expect

$$Z(m) : e^{ik_1A_1}e^{ik_2A_2} \mapsto e^{inA_3}e^{-e^2Vn^2/2}\delta_{k_1,n}\delta_{k_2,n} \tag{2.10}$$

to be how basis vectors in $L^2(U(1))^{\otimes 2}$ behave, just as we desire.

3. Cap

We see for the situation as doodled on the right where we start from "nothing" and end up with the final state being a "cap". This is a little tricky, not because it is difficult but because it is so vacuous. We can see that the initial state consists of no vertices, no edges, no p-cells. That means that $X_p = \emptyset$ for the initial state's data. The free group generated by this is then



$$\mathbb{Z}^{\emptyset} = \{0\} \tag{3.1}$$

since there is exactly one map $\emptyset \to \mathbb{Z}$, it generates the trivial group. (As we are working with commutative groups, the group operation is addition, and in this context the identity element is generically denoted by 0 — thus the trivial group is $\{0\}$.) We are working in two dimensions, the chain in this context for the initial state would be

$$\mathbb{Z}^{\emptyset} \leftarrow \mathbb{Z}^{\emptyset} = \{0\} \leftarrow \{0\}. \tag{3.2}$$

This is the boring part of the chain calculations.

The more interesting part of the chain calculations comes from the final state. We have 1 edge and 1 vertex, which implies the chain complex would be for the final state

$$\mathbb{Z} \leftarrow \mathbb{Z} \tag{3.3}$$

and thus for the entire process as a whole

$$\begin{tikzcd} \{0\} \arrow[d] & \{0\} \arrow[l] \arrow[d] & \\ \mathbb{Z} & \mathbb{Z} \arrow[l] & \mathbb{Z} \arrow[l] \\ \mathbb{Z} \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u] & \end{tikzcd} \tag{3.4}$$

This is completely trivial.

Now we should worry about the Hilbert spaces for the initial and final states. For the initial state, we have a Hilbert space for $L^2(\text{connections on } \emptyset)$. More precisely we can write this as

$$L^2(\text{connections on } \emptyset) = L^2(\hom(\mathbb{Z}^{\emptyset}, U(1))) \tag{3.5a}$$
$$= L^2(\hom(\langle e\rangle, U(1))) \tag{3.5b}$$
$$= L^2(\langle e\rangle) \tag{3.5c}$$
$$= \mathbb{C} \tag{3.5d}$$

The last step is justified since the set of complex valued functions on a singleton is all complex numbers.

The Hilbert space for the final state is — as we have seen two times before — really just

$$L^2(\hom(\mathbb{Z}, U(1))) = L^2(U(1)). \tag{3.6a}$$

This concludes the portion of deducing what the Hilbert spaces are.

Now we consider the expression describing the probability amplitude for the time evolution of this particular process. Observe

$$\langle \overbrace{\phi}^{\in L^2(U(1))},\ Z(\iota)\underbrace{\psi}_{\in\mathbb{C}} \rangle = \int_{\substack{\text{connections}\\ \text{on } \iota}} \overline{\phi(A|_{S'})}\psi(A|_{\emptyset})e^{-S(A)}\mathcal{D}A. \tag{3.7}$$

We choose our basis vectors to test this expression on to be $\psi = 1$ and $\phi = \exp(ikA)$. Note that this equation implies that

$$Z(\iota) : \mathbb{C} \to L^2(U(1)). \tag{3.8}$$

We plug in our expression for $\mathcal{D}A = dA/2\pi$, among our deductions above, to get the following:

$$\langle e^{ikA}, Z(\iota)1\rangle = \int_0^{2\pi} e^{-ikA}\cdot 1 e^{-S(A)}\frac{dA}{2\pi}. \tag{3.9}$$

We will manipulate this as before to deduce how $Z(\iota)$ behaves.

As before, we compute

$$e^{-S(A)} = \sum_{n\in\mathbb{Z}} e^{\frac{-1}{2e^2}(F+2n\pi)^2/V}. \tag{3.10}$$

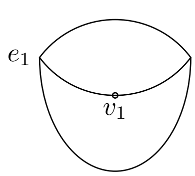
We need to calculate the curvature, which is easy in our situation. It is, up to an arbitrary sign, $F = A$. Putting this into our expression, we deduce

$$\langle e^{ikA}, Z(\iota)1\rangle = \int_0^{2\pi} e^{-ikA}\sum_{n\in\mathbb{Z}} e^{\frac{-1}{2e^2}(A+2n\pi)^2/V}\frac{dA}{2\pi} \tag{3.11a}$$

$$= \int_0^{2\pi} e^{-ikA}\underbrace{\sum_{n\in\mathbb{Z}} e^{\frac{-n^2e^2V}{2}}e^{inA}}_{=Z(\iota)1}\frac{dA}{2\pi}. \tag{3.11b}$$

We identify the underbracketed term with $[Z(\iota)1] \in L^2(U(1))$.

4. Cup



For the next situation, we'll consider the doodle on the left. That is, it's...well, a cup. Dual to the picture before, where from the void there came about a cap, here from a cup begath the void. We expect this to be remarkably similar to the calculation we just performed. We should consider the initial state to be described by the chain

$$\mathbb{Z}^{n_v} \leftarrow \mathbb{Z}^{n_e}$$

where n_v, n_e are the number of vertices and edges (respectively). The final state is the vacuous one

$$\mathbb{Z}^{\emptyset} \leftarrow \mathbb{Z}^{\emptyset}$$

since there are no edges or vertices, we have to work with the empty set! Putting together everything we know, we get the chain complex

$$\begin{tikzcd}
\mathbb{Z}^1 & \mathbb{Z}^1 \arrow[l] \arrow[d] & \\
\mathbb{Z}^1 \arrow[from=u] & \mathbb{Z}^1 \arrow[l] & \mathbb{Z}^1 \arrow[l] \\
\mathbb{Z}^\emptyset \arrow[u] & \mathbb{Z}^\emptyset. \arrow[l] \arrow[u] &
\end{tikzcd} \tag{4.1}$$

Where we have taken advantage of the obvious: there is one vertex and one edge in the initial state.

We see that the Hilbert spaces describing the initial and the final states are precisely those from the corresponding component from the previous computation. That is, the initial state is $L^2(U(1))$ and the final state is described by the Hilbert space $\mathbb{C}$. We can similarly choose our favorite states from each space (e^{ikA} and 1 respectively). We describe the time evolution operator as

$$Z(\varepsilon) : L^2(U(1)) \to \mathbb{C}. \tag{4.2}$$

To do so we need to consider the inner product

$$\langle \phi, Z(\varepsilon)\psi \rangle = \int_{\mathcal{A}(\varepsilon)} \overline{\phi(A|_S)}\psi(A|_{S'})e^{-S(A)}\mathcal{D}A \tag{4.3}$$

where $\phi \in \mathbb{C}$ and $\psi \in L^2(U(1))$.

We can plug in our choices $\phi = 1$ and $\psi = \exp(ikA)$:

$$\langle 1, Z(\varepsilon)e^{ikA} \rangle = \int_{\mathcal{A}(\varepsilon)} 1 \cdot e^{ikA}e^{-S(A)}\mathcal{D}A \tag{4.4}$$

To compute the action, we need the curvature but the curvature is simply $F = A$. We can plug this into the result of our proposition 2 to find

$$\langle 1, Z(\varepsilon)e^{ikA} \rangle = \int_{\mathcal{A}(\varepsilon)} 1 \cdot e^{ikA} \sum_{n\in\mathbb{Z}} e^{\frac{-e^2n^2V}{2}} e^{inA}\mathcal{D}A \tag{4.5}$$

up to a constant. We see that

$$\mathcal{D}A = \frac{dA}{2\pi} \tag{4.6}$$

and rearranging terms we can find

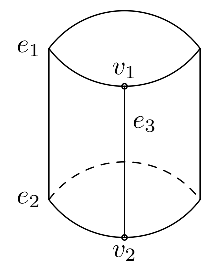
$$\langle 1, Z(\varepsilon)e^{ikA} \rangle = \sum_{n\in\mathbb{Z}} e^{\frac{-e^2n^2V}{2}} \underbrace{\int_0^{2\pi} e^{i(n+k)A}\frac{dA}{2\pi}}_{\delta_{-k,n}} \tag{4.7}$$

where the underbracketed factor turns out to be the Kronecker delta, as noted in the equation. Rearranging terms we find that

$$Z(\varepsilon)e^{ikA} = \sum_{n\in\mathbb{Z}} e^{-e^2n^2V/2}\delta_{-k,n} = \exp\left(\frac{-e^2k^2V}{2}\right) \tag{4.8}$$

This is precisely what we desire to deduce.

5. Cylinder Cobordism



Our last consideration is the cylinder cobordism. We should consider the chains describing the initial and final states. Both consist of 1 edge and 1 vertex. We describe these initial and terminal states by the sequence $\mathbb{Z} \leftarrow \mathbb{Z}$. We see that there are, on the whole, 2 vertices and *3 edges*, which allows us to fill in the rest of the complex describing the situation. We have the cobordism be described by the chain $\mathbb{Z}^2 \leftarrow \mathbb{Z}^3 \leftarrow \mathbb{Z}$, where we justify the first $\mathbb{Z}$ by the observation the cobordism consists of a single 2-cell. We summarize this entire observation by the following chain complex:

$$\begin{tikzcd} \mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[l] \arrow[d] & \\ \mathbb{Z}^2 & \mathbb{Z}^3 \arrow[l] & \mathbb{Z} \arrow[l] \\ \mathbb{Z} \arrow[u] & \mathbb{Z} \arrow[l] \arrow[u] & \end{tikzcd} \tag{5.1}$$

We now are concerned about the Hilbert spaces describing the initial and final states.

By our reasoning from the previous examples, we have the Hilbert space describing the initial state be $L^2(U(1))$. Similarly for the final state, we use the Hilbert space $L^2(U(1))$. Our time evolution operator is then

$$Z(C) : L^2(U(1)) \to L^2(U(1)) \tag{5.2}$$

a linear operator. To see how it behaves, we should consider the expectation value $\langle \phi, Z(C)\psi \rangle$ for some initial state vector $\psi \in L^2(U(1))$ and some final state vector $\phi \in L^2(U(1))$.

References

[1] D. K. Wise, "p-form electromagnetism on discrete spacetimes," *Class. Quant. Grav.* **23** (2006) 5129–5176. `http://math.ucdavis.edu/%7Ederek/pform/pform.pdf`.

E-mail address: `pqnelson@gmail.com`