NOTES ON EQUATIONS OF MOTION FROM THE STRESS ENERGY TENSOR

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1. Geodesic Equation

Geometrically, we wish to find the shortest distance between two given points. That is the point of the geodesic equation, it is a differential equation that has solutions which are the shortest distance between two points! So, how can we do this? Well, we usually write the distance between two neighboring points (that is the distance between x^μ and $x^\mu + dx^\mu$) as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

where $g_{\mu\nu}$ is the metric tensor. This is almost what we are looking for, kind of, but this relation holds for *neighboring points!* What if the distance is greater than dx^μ? We need to do something else.

Being physicists (or mathematical physicists, or "even just" mathematicians) we can find the desired trajectories from a Lagrangian approach. We demand that

$$I = \int ds = \int \sqrt{ds^2} \tag{1.2}$$

vanish under arbitrary variations. We let

$$\mathcal{L} = E^{1/2} = \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \tag{1.3}$$

be our Lagrangian, and λ be an "affine parametrization" of the trajectory. To physicists, this means $\lambda = a\tau + b$ where $a, b \in \mathbb{R}$, τ is the proper time, and $a \neq 0$. We find that our integral becomes

$$I = \int \mathcal{L} d\lambda \tag{1.4}$$

and it has variation

$$\delta I = \delta \int \mathcal{L} d\lambda \tag{1.5a}$$

$$= \frac{1}{2} \int E^{-1/2} \delta E d\lambda \tag{1.5b}$$

Let $dx^\mu / d\lambda = \dot{x}^\mu$. We find that the variation of E is explicitly

$$\delta E = \delta \left(g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \right) \tag{1.6a}$$

$$= (\delta g_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu + g_{\mu\nu} \left(\frac{d\delta x^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} + \frac{dx^\mu}{d\lambda} \frac{d\delta x^\nu}{d\lambda} \right) \tag{1.6b}$$

$$= (\delta g_{\mu\nu}) \dot{x}^\mu \dot{x}^\nu + 2 g_{\mu\nu} \frac{d\delta x^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \tag{1.6c}$$

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We justify the last step by index gymnastics.

We find that the variation of the metric along the curve is

$$\delta g_{\mu\nu} = \left(\frac{\partial}{\partial x^\sigma} g_{\mu\nu}\right)\delta x^\sigma \quad \text{(Chain Rule)} \tag{1.7a}$$

$$= \partial_\sigma g_{\mu\nu}\delta x^\sigma. \tag{1.7b}$$

We now have

$$\delta E = 2g_{\mu\nu}\frac{d\delta x^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} + \delta x^\sigma(\partial_\sigma g_{\mu\nu})\dot{x}^\mu\dot{x}^\nu \tag{1.8}$$

We plug this expression back into our variation of the integral, and perform integration by parts on the first term to find

$$\delta I = \frac{1}{2}\int\left[-2\delta x^\mu\frac{d}{d\lambda}\left(E^{-1/2}g_{\mu\nu}\frac{dx^\nu}{d\lambda}\right) + E^{-1/2}\delta x^\mu\dot{x}^\alpha\dot{x}^\beta\partial_\mu g_{\alpha\beta}\right] \tag{1.9a}$$

$$= 0 \tag{1.9b}$$

where we have changed indices on the second term to make the variations of x both have the same index, so we can factor it out. We then end up with the Geodesic equations

$$\frac{d}{d\lambda}\left(E^{-1/2}g_{\mu\nu}\frac{dx^\nu}{d\lambda}\right) - \frac{1}{2}E^{-1/2}\partial_\mu g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0. \tag{1.10}$$

Since the parametrization was chosen with some λ, we can choose $\lambda = s$ so $E = 1$ giving us

$$\boxed{\frac{d}{ds}\left(g_{\alpha\beta}\frac{dx^\beta}{ds}\right) - \frac{1}{2}\partial_\alpha g_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0} \tag{1.11}$$

We see that

$$\frac{d}{ds}\left(g_{\alpha\beta}\frac{dx^\beta}{ds}\right) = \frac{dg_{\alpha\beta}}{ds}\frac{dx^\beta}{ds} + g_{\alpha\beta}\frac{d^2x^\beta}{ds^2} \tag{1.12a}$$

$$= \partial_\gamma g_{\alpha\beta}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} + g_{\alpha\beta}\frac{d^2x^\beta}{ds^2} \tag{1.12b}$$

$$= \frac{1}{2}\left(\partial_\beta g_{\alpha\gamma} + \partial_\gamma g_{\alpha\beta}\right)\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} + g_{\alpha\beta}\frac{d^2x^\beta}{ds^2} \tag{1.12c}$$

by the product rule, chain rule, and index gymnastics respectively. We plug this into the geodesic equation to find

$$g_{\alpha\beta}\frac{d^2x^\beta}{ds^2} + \frac{1}{2}(\partial_\gamma g_{\alpha\beta} + \partial_\beta g_{\alpha\gamma} - \partial_\alpha g_{\beta\gamma})\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0. \tag{1.13}$$

We can identify the parenthetic term as the Christoffel symbol $\Gamma_{\alpha\beta\gamma}$, multiply through by $g^{\alpha\mu}$ to get

$$\boxed{\frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\beta\gamma}\frac{dx^\beta}{ds}\frac{dx^\gamma}{ds} = 0} \tag{1.14}$$

which is the famous geodesic equation.

References

[1] J. N. Islam. *Rotating Fields in General Relativity.* Cambridge University Press, 1985.

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