NOTES ON COMPOSING COBORDISMS IN CHAIN FIELD THEORY

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1. The "Torus"

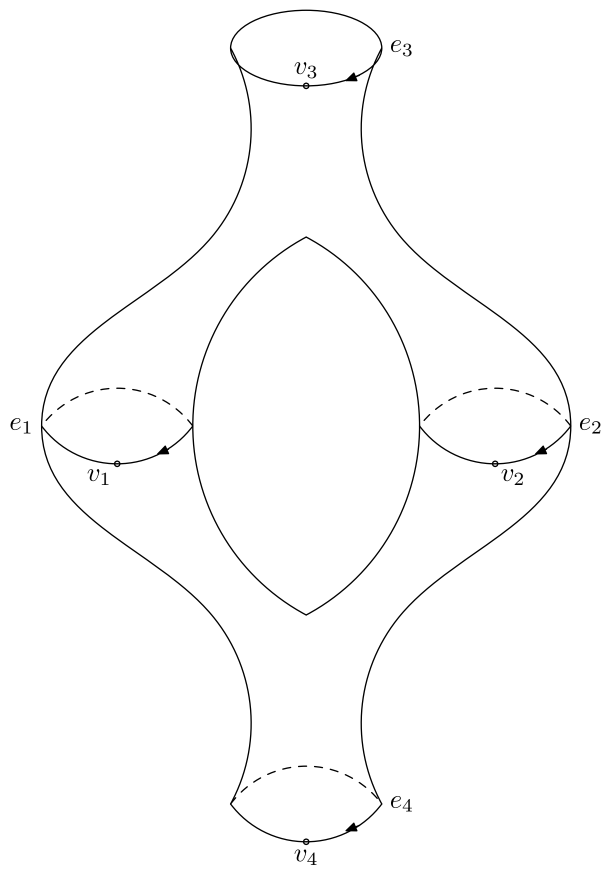

Figure 1. Composition of trousers with upside down trousers

Consider the composition as doodled in figure 1. We can consider the time evolution operation described by $Z(m) \circ Z(\Delta)$, or — read from right to left — "the time evolution operator for the rightside up trousers followed by the time evolution

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operator for the upside down trousers". We see that this behavior is determined by

$$\langle\phi, Z(m)\circ Z(\Delta)\psi\rangle = \int(\cdots)\mathcal{D}A \tag{1.1}$$

where $\phi,\psi\in L^2(U(1))$ describes the initial and final state vectors, respectively. We recall that

$$Z(\Delta) : e^{ikA}\mapsto e^{-k^2e^2V/2}e^{ikA_1}e^{ikA_2} \tag{1.2}$$

and

$$Z(m) : e^{ikA_1}e^{ikA_2}\mapsto \delta_{k_1,k}\delta_{k_2,k}e^{-k^2e^2V/2}e^{ikA} \tag{1.3}$$

so it would logically seem that

$$[Z(m)\circ Z(\Delta)](e^{ikA}) = Z(m)\left(e^{-k^2e^2V/2}e^{ikA_1}e^{ikA_2}\right) \tag{1.4a}$$

$$= e^{-k^2e^2V/2}Z(m)\left(e^{ikA_1}e^{ikA_2}\right) \tag{1.4b}$$

$$= e^{-k^2e^2V/2}\delta_{k_1,k}\delta_{k_2,k}e^{-k^2e^2V/2}e^{ikA} \tag{1.4c}$$

$$= e^{-k^2e^2V}e^{ikA}\delta_{k_1,k}\delta_{k_2,k} \tag{1.4d}$$

where we justify the second step by linearity.

Compare this to the time evolution operator described by the cylinder with the same amount of volume, i.e. $2V$. We see that

$$[Z(C)]\left(e^{ikA}\right) = e^{-k^2e^2(2V)/2}e^{ikA} = e^{-k^2e^2V}e^{ikA} \tag{1.5}$$

which looks surprisingly familiar (hint hint). In fact the only difference between (1.4d) and our cylinder is a factor of $\delta_{k_1,k}\delta_{k_2,k}$; should this be interpreted as a residue of topological origin? Some sign that there was a "hole"? The answer is, unsurprisingly, no. It wouldn't be testable. Its contributions would be either 1 or 0, and in the case of the former it matches exactly with the cylinder. In the case of the latter, we wouldn't know anything happened. So it is as though that factor is 1.

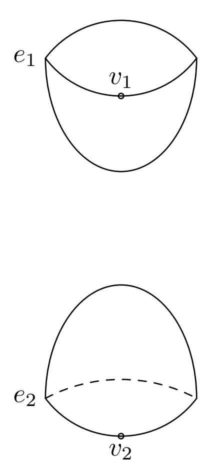

FIGURE 2. A cup followed by a cap

We can also compare this to the cobordism described by figure 2. We consider how the operator $Z(\iota)\circ Z(\varepsilon)$ acts on a basis state vector $\exp(ik'A)$. We see that

$$\langle e^{ikA_2}, Z(\iota)\circ Z(\varepsilon)e^{ik'A_1}\rangle = \int e^{-ikA_2}e^{ik'A_1}\sum_n e^{-n^2e^2(2V)/2}e^{in(A_2-A_1)}\mathcal{D}A \tag{1.6a}$$

$$= \int_0^{2\pi} e^{-ikA_2}\sum_n e^{-n^2e^2(2V)/2}e^{inA_2}\int_0^{2\pi} e^{i(k'-n)A_2}\frac{dA_1}{2\pi}\frac{dA_2}{2\pi} \tag{1.6b}$$

$$= \int_0^{2\pi} e^{-ikA_2}\sum_n e^{-n^2e^2(2V)/2}e^{inA_2}\delta_{n,k'}\frac{dA_2}{2\pi} \tag{1.6c}$$

$$= \int_0^{2\pi} e^{-ikA_2}e^{-(k')^2e^2(2V)/2}e^{ik'A_2}\frac{dA_2}{2\pi} \tag{1.6d}$$

which implies that

$$Z(\iota)\circ Z(\varepsilon) : \exp(ik'A) \mapsto e^{-(k')^2e^2V}\exp(ik'A). \tag{1.7}$$

References

[1] D. K. Wise, "p-form electromagnetism on discrete spacetimes," *Class. Quant. Grav.* **23** (2006) 5129–5176. http://math.ucdavis.edu/%7Ederek/pform/pform.pdf.

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