Complex Analysis Cheat Sheet Cont'd

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1 Laplace Transform

Definition 1.1 (Laplace Transform)**.** The **Laplace Transform** of a function $f(t)$ (for $t \geq 0$) is the function $\widetilde{f}(z)$ defined by

$$\widetilde{f}(z) = \mathcal{L}\{f\}(z) = \int_0^\infty e^{-zt} f(t)dt \tag{1.1}$$

where z is complex.

Proposition 1.2 (Asymptotic Behavior of Laplace Transform)**.** *Suppose g is analytic in a region containing the positive real axis and is bounded on the positve real axis. Let the Taylor series for g centered at 0 be*

$$\sum_{n=0}^\infty a_n z^n \tag{1.2}$$

and let

$$\widetilde{g}(z) = \int_0^\infty e^{-zt} g(t)dt. \tag{1.3}$$

Then

$$\widetilde{g}(z) \sim \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{2a_2}{z^3} + \cdots + \frac{n!a_n}{z^{n+1}} + \cdots \tag{1.4}$$

as $z \to \infty$, $\arg(z) = 0$.

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Proposition 1.3. *Suppose g is infinitely differnetiable on the positive real axis and that g and each of its derivatives are of exponential order. That is, there are constants A_n and B_n such that*

$$|g^{(n)}(t)| \leq A_n e^{B_n t} \tag{1.5}$$

for $t \geq 0$. Let

$$\widetilde{g}(z) = \int_0^\infty e^{-zt} g(t) dt. \tag{1.6}$$

Then

$$\widetilde{g}(z) \sim \frac{g(0)}{z} + \frac{g'(0)}{z^2} + \frac{g''(0)}{z^3} + \cdots + \frac{g^{(n)}(0)}{z^{n+1}} + \cdots \tag{1.7}$$

as $z \to \infty$, $\arg(z) = 0$.

Theorem 1.4 (Convergence Theorem for Laplace Transform)**.** *Assume*

$$f : (0, \infty) \to \mathbb{C} \tag{1.8}$$

is of exponential order and let

$$\widetilde{f}(z) = \int_0^\infty e^{-zt} f(t) dt. \tag{1.9}$$

There exists a unique number σ, $-\infty \leq \sigma < \infty$ such that this integral converges if $\mathrm{Re}(z) > \sigma$ and diverges if $\mathrm{Re}(z) < \sigma$. Furthermore if $\widetilde{f}$ is analytic on the set

$$A = \{z | \mathrm{Re}(z) > \sigma\} \tag{1.10}$$

and we have

$$\frac{d}{dz}\widetilde{f}(z) = -\int_0^\infty t e^{-zt} f(t) dt \tag{1.11}$$

*for $\mathrm{Re}(z) > \sigma$. The number σ is called the "**Abscissa of Convergence**" and if we define the number ρ by* define ρ

$$\rho = \inf\{B \in \mathbb{R} | \textit{there exists an } A > 0 \textit{ such that } |f(t)| \leq A e^{Bt}\} \tag{1.12}$$

then $\sigma \leq \rho$.

Theorem 1.5 (Laplace Transforms)**.** *Suppose that the functions f and h are continuous and that $\widetilde{f}(z) = \widetilde{h}(z)$ for $\mathrm{Re}(z) > \gamma_0$ for some γ_0. Then $f(t) = h(t)$ for all $t \in (0, \infty)$.*

Proposition 1.6. *Let $f(t)($ be continuous on $(0, \infty)$ and piecewise C^1. Then for $\mathrm{Re}(z) > \rho$*

$$\widetilde{\left(\frac{df}{dt}\right)}(z) = z\widetilde{f}(z) - f(0). \tag{1.13}$$

Proposition 1.7. *Let*

$$g(t) = \int_0^t f(\tau) d\tau \tag{1.14}$$

Then for $\mathrm{Re}(z) > \max[0, \rho(f)]$,

$$\widetilde{g}(z) = \frac{\widetilde{f}(z)}{z}. \tag{1.15}$$

Theorem 1.8 (First Shifting Theorem)**.** *Fix $a \in \mathbb{C}$ and let $g(t) = e^{-at} f(t)$. Then for $\mathrm{Re}(z) > \sigma(f) - \mathrm{Re}(a)$, we have*

$$\widetilde{g}(z) = \widetilde{f}(z + a). \tag{1.16}$$

Theorem 1.9 (Second Shifting Theorem)**.** *Let $H(t) = 0$ if $t < 0$ and $H(t) = 1$ if $t \geq 1$ be the* ***Step Function*** *or* ***Heaviside Step Function****. Let $a \geq 0$ and let $g(t) = f(t-a)H(t-a)$; that is, $g(t) = 0$ if $t < a$ while $g(t) = f(t-a)$ if $t \geq a$. Then for $\mathrm{Re}(z) > 0$ we have*

$$\widetilde{g}(z) = e^{-az}\widetilde{f}(z). \tag{1.17}$$

Definition 1.10 (Convolution)**.** The "**Convolution**" of two functions $f(t)$ and $g(t)$ is defined for $t \geq 0$ by

$$(f * g)(t) = \int_0^\infty f(t-\tau)g(\tau)d\tau \tag{1.18}$$

where we set $f(t) = 0$ if $t < 0$.

Theorem 1.11 (Convolution Theorem)**.** *The equalities*

$$(f * g)(t) = (g * f)(t) \tag{1.19}$$

whenever $\mathrm{Re}(z) > \max[\rho(f), \rho(g)]$.

1.1 Table of Properties of the Laplace Transform

Let $u(t)$ be the Heaviside step function.

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \tag{1.20}$$

where δ is the delta function we all know and love.

Linearity	$af(t) + bg(t)$	$a\widetilde{f}(z) + b\widetilde{g}(z)$
Frequency Differentiation	$tf(t)$	$-\widetilde{f}'(z)$
Frequency Differentiation	$t^n f(t)$	$(-1)^n \widetilde{f}^n(z)$
Differentiation	$f'(t)$	$z\widetilde{f}(z) - f(0)$
Differentiation	$f''(t)$	$z^2\widetilde{f}(z) - zf(0) - f'(0)$
Differentiation	$f^{(n)}(t)$	$z^n\widetilde{f}(z) - z^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
Frequency Integration	$f(t)/t$	$\int_z^\infty \widetilde{f}(\omega)d\omega$
Integration	$\int_0^t f(\tau)d\tau = (u * f)(t)$	$\widetilde{f}(z)/z$
Scaling	$f(at)$	$\widetilde{f}(z/a)/\|a\|$
Frequency Shifting	$e^{at}f(t)$	$\widetilde{f}(z-a)$
Time shifting	$f(t-a)u(t-a)$	$e^{-az}\widetilde{f}(z)$
Convolution	$(f * g)(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
Periodic Function	$f(t)$	$(\int_0^T e^{-zt}f(t)dt)/(1-e^{-Tz})$

1.2 List of Properties of the Laplace Transform

Definition. The Laplace transform of $f(t)$ is given by:

$$\widetilde{f}(z) = \int_0^\infty e^{-zt}f(t)dt. \tag{1.21}$$

It is such that:

1. $\widetilde{g}(z) = -\dfrac{d}{dz}\widetilde{f}(z)$ where $g(t) = tf(t)$.
2. $\mathcal{L}\{af + bg\} = a\widetilde{f} + b\widetilde{g}$
3. $\widetilde{\left(\dfrac{df}{dt}\right)}(z) = z\widetilde{f}(z) - f(0)$.

2 Gamma Function

So for n a positive integer, we have

$$\Gamma(n) = (n-1)! \tag{2.1}$$

2.1 List of Properties of the Gamma Function

Remember that it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt. \tag{2.2}$$

or equivalently as an infinite product

$$\Gamma(z) = \frac{1}{ze^{\gamma z}\left[\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}\right]} \tag{2.3}$$

where

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln(n)\right)\\ &\approx 0.5772156649015328606065120900082\end{aligned} \tag{2.4}$$

It has the following properties:

1. Γ is meromorphic with simple poles at 0, -1, -2, $\ldots$
2. $\Gamma(z+1) = z\Gamma(z)$ for $z \neq 0, -1, -2, \ldots$
3. $\Gamma(n+1) = n!$ for $n = 0, 1, \ldots$
4. $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$
5. $\Gamma(z) \neq 0$ for all z
6. $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$, $\Gamma\left(n+\dfrac{1}{2}\right) = \dfrac{1\cdot 3\cdot(\cdots)\cdot(2n-1)}{2n}\sqrt{\pi}$
7. $\Gamma(z) = \dfrac{1}{z}\displaystyle\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}\right]$
8. $\Gamma(z) = \displaystyle\lim_{n\to\infty}\frac{n!n^z}{z(z+1)(\cdots)(z+n)}$
9. $\Gamma(z)\Gamma\left(z+\dfrac{1}{n}\right)(\cdots)\Gamma\left(z+\dfrac{n-1}{n}\right) = (2\pi)^{(n-1)/2}n^{(1/2)-nz}\Gamma(nz)$
10. $2^{2z-1}\Gamma(z)\Gamma\left(z+\dfrac{1}{2}\right) = \sqrt{\pi}\Gamma(2z)$
11. The residue of $\Gamma(z)$ at $z=-m$ is equals $(-1)^m/m!$
12. (Euler's Integral) $\Gamma(z) = \displaystyle\int_0^\infty t^{z-1}e^{-t}dt$ for $\mathrm{Re}(z) > 0$. The convergence is uniform and absolute for $-\pi/2+\delta \leq \arg(z) \leq \pi/2+\delta$ $(\delta > 0)$ and for $\varepsilon \leq |z| \leq R$ where $0 < \varepsilon < R$.
13. $\dfrac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \dfrac{1}{z} + \displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{z+n}\right) = \int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-zt}}{1-e^{-t}}\right)dt.$

14. $\pi^{-z/2}\,\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{-\frac{1-z}{2}}\,\Gamma\left(\frac{1-z}{2}\right)\,\zeta(1-z)$. (Where $\zeta(s)$ is the Riemann zeta function)

15. $\zeta(z)\,\Gamma(z) = \int_0^\infty \frac{u^{z-1}}{e^u - 1}\,du$ which holds for $\mathrm{Re}(z) > 1$.

3 Zeta Function

The definition for the Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{3.1}$$

It is holomorphic everwhere except for a simple pole at $s = 1$ with residue 1.

For any positive even integer $2n$, we have

$$\zeta(2n) = (-1)^{n+1}\frac{B_{2n}(2\pi)^{2n}}{2(2n)!} \tag{3.2}$$

where B_{2n} is a Bernoulli number, and for negative integers we have

$$\zeta(-n) = \frac{-B_{n+1}}{n+1} \tag{3.3}$$

for $n \geq 1$.

Let

$$f(x) = \frac{x}{e^x - 1} \tag{3.4}$$

then the Bernoulli numbers may be found from

$$B_n = \lim_{x\to 0}\frac{d^n}{dx^n}\frac{x}{(e^x - 1)}. \tag{3.5}$$

Observe that for $n = 1$

$$f'(x) = \left(\frac{1}{e^x - 1}\right)\left(1 - \frac{f(x)}{e^x - 1}\right) \tag{3.6}$$

and now observe that

$$\frac{d}{dx}\left(\frac{1}{e^x - 1}\right) = -e^x\left(\frac{1}{e^x - 1}\right)^2 \tag{3.7}$$

and we can use the product rule to find all of our favorite Bernoulli numbers.

We have a table of the first few Bernoulli numbers:

n	B_n
0	1
1	-1/2
2	1/6
4	-1/30
6	1/42
8	-1/30
10	5/66≈ 0.07575757576
12	-691/2730≈-0.25311355311
14	7/6
16	-3617/510≈ -7.09125686275
18	43867/798≈ 54.9711779448

The zeta function satisfies the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \ \sin\left(\frac{\pi s}{2}\right) \ \Gamma(1-s) \ \zeta(1-s), \tag{3.8}$$

valid for all $s \in \mathbb{C}$. An equivalent relationship may be expressed as a sum

$$\zeta(s)(1-2^{1-s}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}. \tag{3.9}$$

3.1 Mellin Transform

The Mellin transform of a function $f(x)$ is defined as

$$\int_0^\infty f(x) x^{s-1} \, dx, \tag{3.10}$$

when defined. We can relate the zeta function to one million and one things this way, we have

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{\exp(x)-1} dx, \tag{3.11}$$

where Γ is our favorite gamma function, and

$$2\sin(\pi s)\Gamma(s)\zeta(s) = i \oint_C \frac{(-x)^{s-1}}{\exp(x)-1} dx \tag{3.12}$$

for all s where the contour C begins and ends at $+\infty$ and circles the origin once.

3.2 Laurent Series

Since the zeta function has a single simple pole at $s = 1$ we can expand it around the singular point. The series is

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \gamma_n \ (s-1)^n. \tag{3.13}$$

where γ_n are the Stieltjes constants, defined by the limit

$$\gamma_n = \lim_{m\to\infty} \left(\left(\sum_{k=1}^{m} \frac{(\log k)^n}{k} \right) - \frac{(\log m)^{n+1}}{n+1} \right). \tag{3.14}$$

where the constant $n = 0$ term in the Laurent series is just γ_0 the Euler-Mascheroni constant.

References

[MH98] J. E. Marsden, M. J. Hoffman. *Basic Complex Analysis*. W. H. Freeman, third edition (1998).