SPONTANEOUS SYMMETRY BREAKING IN CONFORMAL WEYL GRAVITY

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ABSTRACT. By spontaneously breaking conformal symmetry, a scalar field emerges which we identify as the cosmological constant. The first instance is in the Schwarzschild solution for the Weyl Gravity Field Equations. The second is in the homogeneous and isotropic universe with fermionic and bosonic matter. In the second case, an effective gravitational constant emerges, as well as a positive cosmological constant, both give theoretical justification within the conformal gravity framework of an accelerating universe.

1. Introduction

Recall, in General Relativity, a heuristic derivation usually begins with the Newtonian gravity's Poisson equation

$$\nabla^2\Phi_N = 4\pi G_N\rho \tag{1.1}$$

where Φ_N is the Newtonian potential, G_N is the gravitation constant, and ρ is the mass-density. It has a solution of

$$\Phi_N(r) = \frac{c_0}{r} \tag{1.2}$$

where c_0 is constant. From here, one typically identifies the right hand side of Eq (1.1) as the time-time component of stress-energy tensor, and the left hand side is identified as the time-time component of the Ricci tensor. This is how most approaches to gravity begin.

In conformal gravity, we begin with a different Poisson equation. Instead of a second order one, we begin with a fourth order one [Man94]

$$\nabla^4 B(r) = f(r) \tag{1.3}$$

which has the general solution for a spherical source

$$B(r) = \frac{-r}{2}\int_0^R dx f(x)x^2 - \frac{1}{6r}\int_0^R dx f(x)x^4 = \widetilde{c}_0 r + \frac{\widetilde{c}_1}{r}. \tag{1.4}$$

Observe that when $r \ll 1$, Eq (1.4) has $\widetilde{c}_1/r$ be the dominant term and $\widetilde{c}_0 r \to 0$. Thus for small r, we can recover the Newtonian Poisson equation (1.1).

At first, this may be startling to see Eq (1.4) as being proposed for the gravitational potential. It is counter-intuitive to propose adding an $\mathcal{O}(r)$ term, as we don't observe it at "small" scales (dropping an apples behaves as being in a $\mathcal{O}(1/r)$ potential!). However, at such scales, the potential for a fourth order Poisson equation behaves as the potential for a second order one. Additionally, there is observational problems with gravity that departs from a second order Poisson equation at *large* distances. So the fourth order approach modifies only what is expected at *large* distances, and agrees with what we expect at *small* distances.

Here, we must intervene and confess that there is a terribly strong no-go theorem: Ostrogradski's theorem (for a beautiful introduction, see section 2 of Woodard [Woo07]). With Lagrangians involving terms of second order (or higher) time derivatives of the position term is unstable. Woodard notes that in Lagrangians of the form $L(q_i, \dot{q}_j, \ddot{q}_k)$ "there is not even any barrier to decay". Adding insult to injury, the situation does not improve if we add in higher order derivatives!

However, Adler [Adl82] has proposed recovering Einstein's General Relativity from Spontaneous Symmetry Breaking[1] and Zee [Zee83] has proposed using spontaneously breaking conformal invariance to give rise to masses in Weyl gravity. This is very much analogous to how spontaneous symmetry breaking in the weak force generates the Fermi constant. The aim of Adler and Zee is that in breaking symmetry, there is a sort of "macroscopic/low energy" limit in which we can recover Newtonian gravity (or some generalization of it).

If instead we consider conformal gravity as a theory in and of itself, as Mannheim suggests, we find that – from the solution for a static, spherically symmetric body – scale invariance is spontaneously broken to give rise to a nonzero cosmological constant term. Further, if we consider working with a stress energy tensor involving a spontaneous symmetry breaking boson (which is to be expected if the standard model of particle physics is correct), we get a sort of "induced conformal cosmology" model which avoids a Big Bang singularity. So the plan is to first examine the case of the static, spherically symmetric gravitating body which resembles (up to some negligibly small terms at the solar-system level) the Schwarzschild solution. Then we will proceed to consider symmetry breaking at the cosmological scale and review the consequences in conformal cosmology. Included is an appendix which constitutes an extremely brief ("five-minute") introduction to spontaneous symmetry breaking.

2. Action Principle and Field Equations

One begins with the conformally-invariant fourth order action

$$I_W = -\alpha_g \int d^4x\sqrt{-g}C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \tag{2.1}$$

where α_g is the coupling constant and $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor. This action is invariant under conformal transformations of the metric

$$g_{\mu\nu}(x) \to e^{2\alpha(x)}g_{\mu\nu}(x)$$

(Originally Weyl considered α in the action (2.1) to be used for the both the conformal and the electromagnetic gauge transformations. This way one has for the covariant derivative $\nabla_\mu g_{\alpha\beta} = A_\mu g_{\alpha\beta}$ where A_μ is the electromagnetic 4-potential. The problem with this approach is that conformal invariance implied the particles are massless, which is observably false.) We can now simplify this Lagrangian a bit.

[1] In all fairness, Mannheim credits Adler's work with recovering Einstein's general relativity, but it appears that Adler's paper is more related to the role of spontaneous symmetry breaking in the context of avoiding some problems in induced gravity like the logarithmic and quadratic divergence of the effective Gravitation constant.

By plugging in the definition of the Riemann tensor, and recalling that any contraction of any pair of indices of the Weyl tensor vanishes, we see

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} + 2R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2 \tag{2.2}$$

where R is the Ricci scalar. Rearranging terms, we have

$$C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2. \tag{2.3}$$

Before beginning, note that the quantity [KM91, Lan38]

$$\sqrt{-g}\left(R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2\right) \tag{2.4}$$

is a total divergence. So instead of having our Lagrangian be

$$L = \sqrt{-g}C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu} \tag{2.5}$$

we can *equivalently* use the Lagrangian

$$L = -2\sqrt{-g}(R_{\alpha\beta}R^{\alpha\beta} - R^2/3) \tag{2.6}$$

since we would be working with an extra term (by Stoke's theorem a surface integral), and by demanding the variation vanishes on the boundary the only nonzero contribution would be this Lagrangian.

Now, De Witt [Wit64] explicitly calculates out the equations of motion for two Lagrangians:

$$L_2 = \sqrt{-g}R^2, \qquad \text{and} \qquad L_1 = \sqrt{-g}R^{\mu\nu}R_{\mu\nu} \tag{2.7}$$

Our Lagrangian is a linear combination of these two, so we use a linear combination of the variation of their respective actions

$$\frac{\delta S_2}{\delta g^{\mu\nu}} = \frac{g_{\mu\nu}}{2}\nabla^\beta\nabla_\beta(R^\alpha{}_\alpha) + \nabla^\beta\nabla_\beta R_{\mu\nu} - \nabla_\beta\nabla_\nu R_\mu{}^\beta - \nabla_\beta\nabla_\mu R_\nu{}^\beta - 2R_{\mu\beta}R_\nu{}^\beta + \frac{g_{\mu\nu}}{2}R_{\alpha\beta}R^{\alpha\beta} \tag{2.8}$$

(where ∇_μ is the covariant derivative operator) and

$$\frac{\delta S_1}{\delta g^{\mu\nu}} = 2g_{\mu\nu}\nabla^\beta\nabla_\beta R^\alpha{}_\alpha - 2\nabla_\mu\nabla_\nu R^\alpha{}_\alpha - 2R^\alpha{}_\alpha R_{\mu\nu} + g_{\mu\nu}R^2/2 \tag{2.9}$$

where S_1 and S_2 are the actions of the Lagrangians L_1 and L_2 respectively. In the literature, these two quantities are typically referred to as $W^{(2)}_{\mu\nu} = \delta S_2/\delta g_{\mu\nu}$ and $W^{(1)}_{\mu\nu} = \delta S_1/\delta g_{\mu\nu}$. From them, we can construct the quantity

$$2\alpha_g W_{\mu\nu} = 2\alpha(W^{(2)}_{\mu\nu} - \frac{1}{3}W^{(1)}_{\mu\nu}) \tag{2.10}$$

which is precisely the variation of the conformal action. So we end up with the field equations being

$$4\alpha_g W_{\mu\nu} = T_{\mu\nu} \tag{2.11}$$

where $T_{\mu\nu}$ is the stress-energy tensor we all know and love.

Although this is an intimidating system of coupled fourth order, nonlinear partial differential equations, there are a few solutions calculated out. Mannheim and Kazanas [MK89] have computed the exact solution exterior to a static, spherically symmetric gravitating source, which is

$$-g_{00} = 1/g_{rr} = 1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2 \tag{2.12}$$

where the parameters β, γ and k are three dimensionful integrations constants which appear in the solution but not in the equations of motion. They spontaneously break the scale symmetry. This should look familiar as it resembles the Schwarzschild solution with a cosmological constant

$$-g_{00} = 1/g_{rr} = 1 + \frac{\Lambda}{3}r^2 - \frac{2m}{r} \tag{2.13}$$

in units where $G_N = 1$ and $c = 1$. The only difference is a constant term and a term that linearly depends on r.

Breaking symmetry gives information about cosmological constant

Here we need to reiterate so one appreciates the beauty of the situation. In a Lagrangian of the form (2.1) which has no boundary term or constant term added in by hand, makes no assumptions about the cosmological constant, one can solve for the spherically symmetry, static gravitating body and one *naturally* gets a term which yields information about the cosmological constant and a term which breaks symmetry to give masses to the massless particles.

3. Conformal Cosmology

We begin by thinking about breaking symmetry differently (read: in the naive way) by considering[2] the Lagrangian of matter conformally coupled to gravity [Man01, Man07]

$$I_M = -\int d^4x\sqrt{-g}\left[\underbrace{\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{12}\phi^2 R + \lambda\phi^4}_{\text{scalar}} + \underbrace{i\bar{\psi}\gamma^\mu(x)[\partial_\mu + \Gamma_\mu(x)]\psi}_{\text{fermion}} - \underbrace{g\phi\bar{\psi}\psi}_{\text{interaction}}\right] \tag{3.1}$$

where $\Gamma_\mu(x)$ is the fermion spin connection, λ and g are the dimensionless coupling constants, $\phi(x)$ is the (symmetry breaking) scalar field and ψ is a fermionic field.

We will demonstrate that the scalar spontaneously breaks symmetry. Observe that the potential term for the scalar field is

$$V(\phi) = \frac{\phi^2 R}{12} - \lambda\phi^4 \tag{3.2}$$

we take its derivative

$$V'(\phi) = \frac{\phi R}{6} - 4\lambda\phi^3 \tag{3.3}$$

then set it to zero and solve for ϕ. The resulting value is

$$v = \pm\sqrt{\frac{R}{24\lambda}} \tag{3.4}$$

then we plug it back into the potential to find

$$V\left(\pm\sqrt{\frac{R}{24\lambda}}\right) = \left(\frac{R}{24\lambda}\right)\frac{R}{12} - \lambda\left(\frac{R}{24\lambda}\right)^2 = \frac{R^2}{576\lambda} \tag{3.5}$$

which is nonzero, which implies that symmetry is spontaneously broken. Note that if we included the fermion-scalar interaction term, the results would not have changed as it would have been equivalent to adding a term linear in ϕ into the

[2] Note that this is for De-Sitter spacetime, to make this anti-de-Sitter spacetime we need to change the sign of the ϕ^4 term. This has been calculated in [EFP06].

potential (for explicit calculations refer to appendix B). (Observe the dependence on R is directly proportional too.)

When the scalar field $\phi(x)$ in I_M obtains a nonzero mass (which we are free to rotate to some "spacetime constant" ϕ_0 due to conformal invariance), the fermion then obeys the curved space Dirac equations

$$i\hbar\bar{\psi}\gamma^{\mu}(x)(\partial_{\mu}+\Gamma_{\mu}(x))\psi = \hbar g\phi_0\psi \tag{3.6}$$

and acquires a mass $\hbar g\phi_0$. The scalar field's equation of motion is

$$\nabla_{\mu}\nabla^{\mu}\phi + \frac{\phi R}{6} - 4\lambda\phi^3 + g\bar{\psi}\psi = 0. \tag{3.7}$$

The corresponding stress-energy tensor to (3.1) is

$$\begin{aligned} T^{\mu\nu} &= \hbar\Big[i\bar{\psi}\gamma^{\mu}(\partial^{\nu}+\Gamma^{\nu})\psi + \frac{2}{3}\nabla^{\mu}\phi\nabla^{\nu}\phi - \frac{g^{\mu\nu}}{6}\nabla^{\alpha}\phi\nabla_{\alpha}\phi - \frac{\phi\nabla^{\mu}\nabla_{\nu}\phi}{3} \\ &\quad + \frac{g^{\mu\nu}\phi\nabla^{\alpha}\nabla_{\alpha}\phi}{3} - \frac{\phi^2}{6}(R^{\mu\nu} - \frac{g^{\mu\nu}}{2}R) - g^{\mu\nu}\lambda\phi^4\Big] \end{aligned} \tag{3.8}$$

which can be rewritten grouping terms in a more elegant manner. If we think of

$$\rho u^{\mu}u^{\nu} = i\hbar\bar{\psi}\gamma^{\mu}(\partial^{\nu}+\Gamma^{\nu})\psi + \frac{\hbar}{2}\nabla^{\mu}\phi\nabla^{\nu}\phi \tag{3.9}$$

where ρ is the "pressure" of an ideal fluid, u^{μ} is thought of as the worldline (so it satisfies $g_{\mu\nu}u^{\mu}u^{\nu} = -1$), and

$$pu^{\mu}u^{\nu} = \frac{-\hbar}{3}\phi\nabla^{\mu}\nabla^{\nu}\phi + \frac{\hbar}{6}\nabla^{\mu}\phi\nabla^{\nu}\phi \tag{3.10}$$

where p is the "pressure" of an ideal fluid, then the stress energy tensor may be written as

$$T_{\mu\nu} = (\rho+p)U_{\mu}U_{\nu} + pg_{\mu\nu} - \frac{1}{6}\phi_0^2\left(R_{\mu\nu} - \frac{\hbar}{2}g_{\mu\nu}R\right) - g_{\mu\nu}\hbar\lambda\phi_0^4. \tag{3.11}$$

This is the right hand side of our fourth order field equations.

By working in an isotropic and homogeneous geometry, the left hand side of (3.11) necessarily vanishes, giving us the equation

$$\frac{1}{6}\phi^2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = (\rho+p)U_{\mu}U_{\nu} + pg_{\mu\nu} - g_{\mu\nu}\lambda\phi^4 \tag{3.12}$$

Thus conformal cosmology looks like the standard cosmology with a perfect matter fluid and a nonzero cosmological constant with the important exception that Newton's constant has been replaced by an "effective" constant of the form

Replace Newton's G_N with an effective one

$$G_{\text{eff}} = \frac{-3}{4\pi\phi_0^2}. \tag{3.13}$$

This is not Newton's constant as Cavendish measured, but instead a term which we identify to be analagous to the Newton gravitational constant! Observe that as we change scales, G_{eff} changes in inverse proportion.

Cosmological Constant Emerges

We can also identify the $\Lambda = \lambda\phi_0^4$ term as being a cosmological constant. Note that this term really is effectively a cosmological constant since it is a homogeneous and isotropic global scalar field. Observe that this cosmological constant is (1) scale dependent (that is, quartic in ϕ), (2) always positive (that is, we have de Sitter spacetime, so gravity is repulsive *but at this scale*). The notion that the cosmological constant is scale dependent is novel, but the important change is that the sign explains how gravity is repulsive instead of attractive. Due to the sign, there is no initial singularity in this model. Instead the universe expands from a finite minimum radius, and is not subject to the same problems that one encounters in the standard cosmological model.

Despite finding gravity being globally repulsive, it is locally attractive. This reconciles the use of the fourth order Poisson equation (1.3) which merely adds an extra term linear in r (radial distance) to the gravitational potential that would be negligible at *local* scales. It turns out that Mannheim et al [MK96] demonstrate the empirical strength of such a proposition at the galactic level, but that is beyond the scope of this article to review it too.

4. Conclusion

We introduced a different action which is based off of Weyl's attempt to unify gravity and electromagnetism. Instead of attempting such a unified field theory, we observed that it has interesting gravitational properties.

The vacuum satisfies the Schwarzschild solution for general relativity with a nonzero cosmological constant, plus some nonzero term and a term linear in r negligibly small at the "local" scale. Due to these extra terms, the scale invariance was spontaneously broken. This was purely accidental.

We also observed that when we solve the fourth order field equations for the isotropic and homogeneous case, we end up breaking symmetry again. But in doing so, we recover the standard cosmological model, and we explained why gravity is accelerating within the framework of the Conformal gravity model. Further, we have an effective gravitational constant that is scale dependent which allows gravity to be repulsive globally but (due to inhomogeneities in the scalar field) is locally attractive. This is consistent with the first investigation of spontaneous symmetry breaking in solving the static, spherically symmetric body's gravitational field as locally ("for small enough r") resembling Schwarzschild's solution.

Observe that this is really nothing surprising, since this is just another version of the Brans-Dicke theory. The Brans-Dicke action is

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\phi R - \omega\frac{\partial_\mu\phi\partial^\mu\phi}{\phi} + L_{matter}\right) \tag{4.1}$$

one can rearrange it by introducing $\Phi^2 = \phi$ to look like

$$I = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left(\Phi^2 R - 4\omega\partial_\mu\Phi\partial^\mu\Phi + L_{matter}\right) \tag{4.2}$$

which resembles the action in Eq (3.1). What the Brans-Dicke theory effectively does is replace $k = 16\pi G/c^4$ with a scalar field ϕ. We did something similar, except our scalar field spontaneously broke the scale invariance (so, analogously, we had a bare minimum value for k) which gave rise to a cosmological constant in addition to recovering the standard cosmological model. Further, we used covariant derivatives instead of partial derivatives, so we would need to include in the L_{matter} the extra terms, the Φ^4 term, and the coupling to matter. Nonetheless, the cosmological constant naturally emerges when we break symmetry.

Appendix A. Spontaneous Symmetry Breaking

Spontaneous symmetry breaking occurs whenever a given field in a given Lagrangian has a nonzero vacuum expectation value. Why exactly is this "breaking" the symmetry? Well, the Lagrangian appears symmetric under a symmetry group, but its vacuum state fails to be symmetric. The system no longer behaves symmetrically. So we went from symmetry to no symmetry due to a nonzero vacuum expectation value. It came about from condensed matter physics (see [Wen08] for applications of it in condensed matter physics) but has since been applied to quantum field theory and particle physics (see [PS95] for examples in particle physics).

Consider the scalar Lagrangian given by

$$\mathcal{L} = \underbrace{\frac{1}{2}(\partial_\mu \phi)^2}_{\text{"kinetic term"}} + \underbrace{\frac{1}{2}\mu^2\phi^2 - \frac{\lambda}{4!}\phi^4}_{\text{"potential term"}} \tag{A.1}$$

where ϕ is the scalar field, μ is a sort of "mass" parameter, and λ is the coupling. Observe that there is a symmetry of $\phi \to -\phi$ (a discrete symmetry). We can think of the potential as being

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{A.2}$$

which has extrema when its derivative is zero. There are two, given by

$$\phi_0 = \pm v = \pm\mu\sqrt{\frac{6}{\lambda}} \tag{A.3}$$

where the constant v is the "**vacuum expectation value**".

We can then write

$$\phi(x) = v + \sigma(x) \tag{A.4}$$

and then rewrite the Lagrangian as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 - \sqrt{\frac{\lambda}{6}}\mu\sigma^3 - \frac{\lambda}{4!}\sigma^4 \tag{A.5}$$

where we dropped the constant terms. We see that the symmetry $\phi \to -\phi$ is no longer identifiable.

Appendix B. Proof of Spontaneous Symmetry Breaking with extra Linear Term

The potential we are investigating has the form

$$V(\phi) = c\phi + \frac{\phi^2 R}{12} - \lambda\phi^4 \tag{B.1}$$

where c is some constant term, so its first derivative would be

$$V'(\phi) = c + \frac{\phi R}{6} - 4\lambda\phi^3. \tag{B.2}$$

By the fundamental theorem of algebra, it has exactly three roots at the values

$$\begin{aligned}
v_1 &= -\frac{\sqrt[3]{2}R\lambda + \left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{2/3}}{62^{2/3}\lambda\sqrt[3]{\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2}} \\
v_2 &= \frac{2\sqrt[3]{-2}R\lambda + \left(1 - i\sqrt{3}\right)\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{2/3}}{122^{2/3}\lambda\sqrt[3]{\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2}} \\
v_3 &= \frac{\left(1 + i\sqrt{3}\right)\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{2/3} - 2(-1)^{2/3}\sqrt[3]{2}R\lambda}{122^{2/3}\lambda\sqrt[3]{\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2}}.
\end{aligned}$$

We can plug each of these into the function with the extra linear term and find

$$\begin{aligned}
V(v_1) &= -\frac{\left(\sqrt[3]{2}R\lambda + \left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{2/3}\right)^4}{51842^{2/3}\lambda^3\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{4/3}} \\
&\quad + \frac{R\left(\sqrt[3]{2}R\lambda + \left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{2/3}\right)^2}{864\sqrt[3]{2}\lambda^2\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{2/3}} \\
&\quad - \frac{c\left(\sqrt[3]{2}R\lambda + \left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{2/3}\right)}{62^{2/3}\lambda\sqrt[3]{\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2}}
\end{aligned}$$

$$\begin{aligned}
V(v_2) &= -\frac{\left(2\sqrt[3]{-2}R\lambda + \left(1 - i\sqrt{3}\right)\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{2/3}\right)^4}{829442^{2/3}\lambda^3\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{4/3}} \\
&\quad + \frac{R\left(2\sqrt[3]{-2}R\lambda + \left(1 - i\sqrt{3}\right)\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{2/3}\right)^2}{3456\sqrt[3]{2}\lambda^2\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{2/3}} \\
&\quad + \frac{c\left(2\sqrt[3]{-2}R\lambda + \left(1 - i\sqrt{3}\right)\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2\right)^{2/3}\right)}{122^{2/3}\lambda\sqrt[3]{\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda - R^3\right)} - 108c\lambda^2}}
\end{aligned}$$

$$\begin{aligned} V(v_3) \quad = \quad & -\frac{\left(\left(1+i\sqrt{3}\right)\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda-R^3\right)}-108c\lambda^2\right)^{2/3}-2(-1)^{2/3}\sqrt[3]{2}R\lambda\right)^4}{829442^{2/3}\lambda^3\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda-R^3\right)}-108c\lambda^2\right)^{4/3}} \\ & +\frac{R\left(\left(1+i\sqrt{3}\right)\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda-R^3\right)}-108c\lambda^2\right)^{2/3}-2(-1)^{2/3}\sqrt[3]{2}R\lambda\right)^2}{3456\sqrt[3]{2}\lambda^2\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda-R^3\right)}-108c\lambda^2\right)^{2/3}} \\ & +\frac{c\left(\left(1+i\sqrt{3}\right)\left(\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda-R^3\right)}-108c\lambda^2\right)^{2/3}-2(-1)^{2/3}\sqrt[3]{2}R\lambda\right)}{122^{2/3}\lambda\sqrt[3]{\sqrt{2}\sqrt{\lambda^3\left(5832c^2\lambda-R^3\right)}-108c\lambda^2}}. \end{aligned}$$

Observe that these are nonzero quantities, just a whole lot messier than neglecting that linear term.

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