Application of Monte Carlo algorithms in the search of PATTERN-AVOIDING PERMUTATIONS

Under the supervision of Professor Neal Madras and NSERC sponsorship

ABSTRACT

Pattern-avoiding affine permutation represents an interface between combinatorics and group theory, which has been a topic of research interest for decades. Following the attainments made previously, this project, under the supervision of Professor Neal Madras, focuses on assessing various conjectures on the features of a permutation of a given size that avoids a certain pattern. Specifically, the 4231-avoiding permutation is at the center of this project.

The Markov chain Monte Carlo methods are implemented using MATLAB and Visual Studio. Well-defined algorithms are programmed into computer software with the aim of executing the long-run performance of Markov chains and graphical illustrations. The initial chain, preferably the one avoiding either 321, 4321, or 4231 patterns, is modified over time using deletion and insertion algorithms until it reaches the stationary state. The significant complexity of the work is to deal with the more intricate nature of affine permutations, which can be viewed as infinite periodic extensions of ordinary finite permutations.

Upon present completion, the inspection of typical shapes of pseudo-random permutations avoiding a given pattern, as well as the validity of coded algorithms, is being performed to eliminate systematic errors. Once accomplished, mathematical and other probabilistic methods are employed to examine the properties of a class of 4231-avoiding permutation. The simulation result is important since it not only verifies theoretical assumptions but also provides an intuitive sense and thorough guidance for further research in combinatorics and probability theory.

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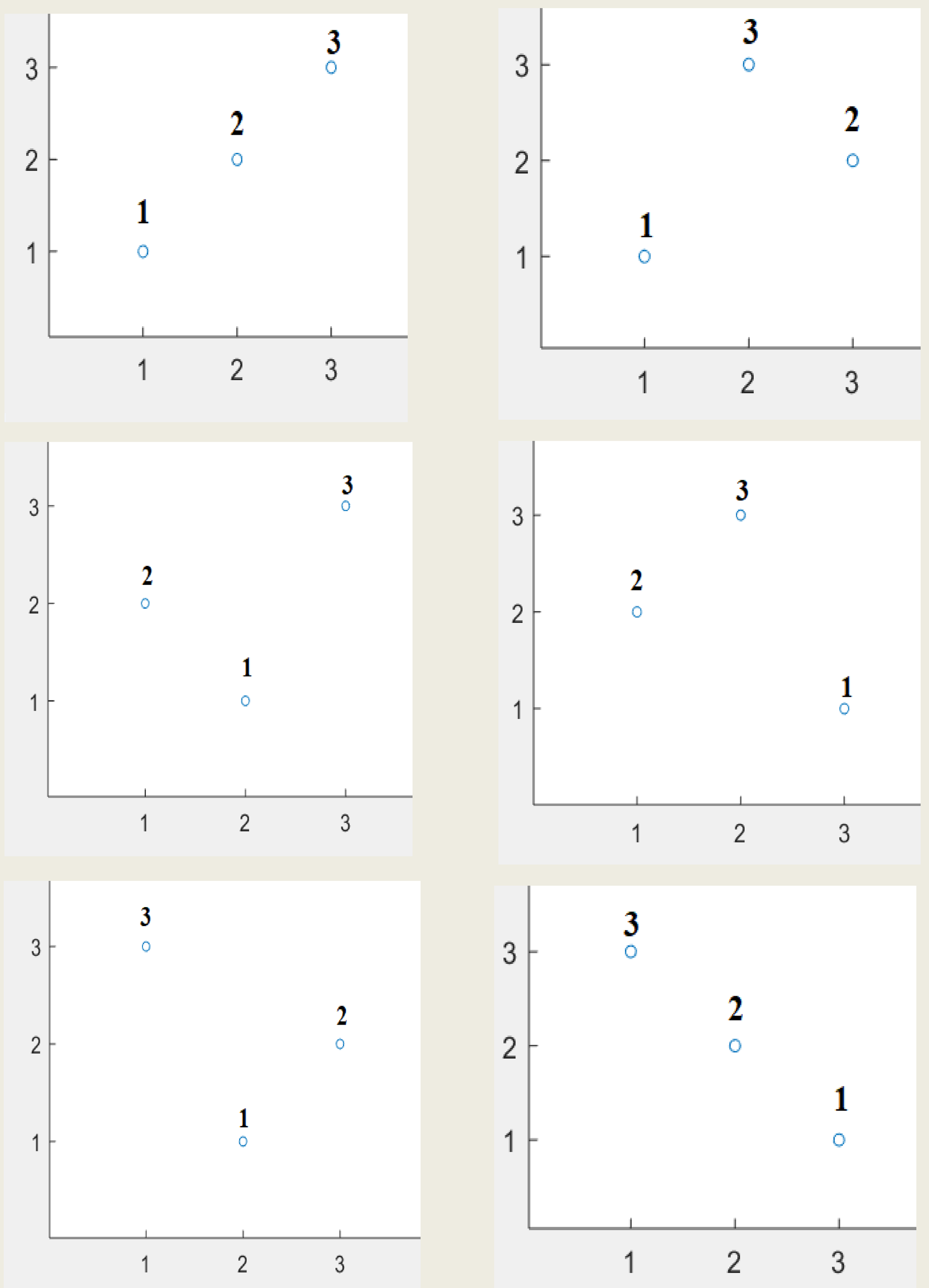
PROJECT'S OBJECTIVES

Use Markov Chain Monte Carlo methods (MCMC) to examine the typical shapes of affine permutations avoiding a given pattern in order to randomly generate these objects afterward.

MCMC methodologies draw samples from the target distribution by running a cleverly constructed Markov chain for a long time and then form sample averages to approximate expectations.

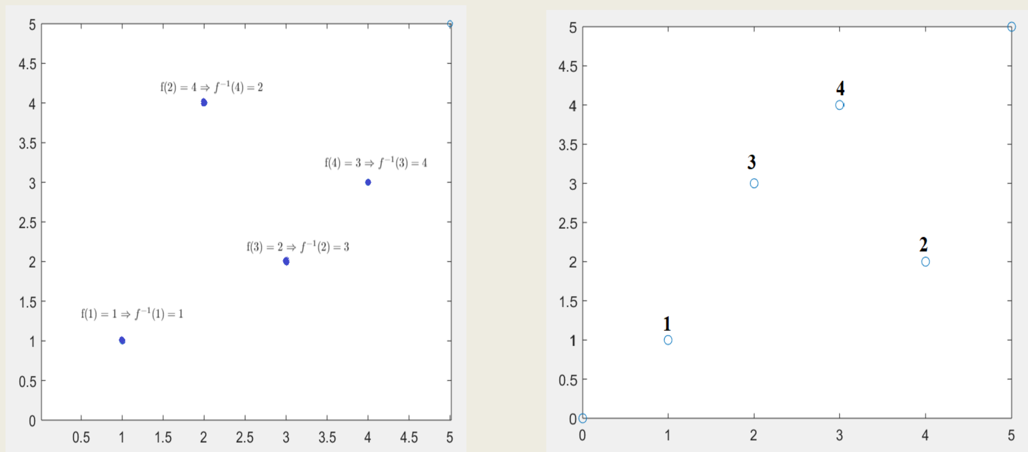
A permutation or a n-permutation is a linear ordering from the set [n] = {1, 2, 3, ..., n}. The number of n-permutations is n!

PATTERN OF LENGTH THREE

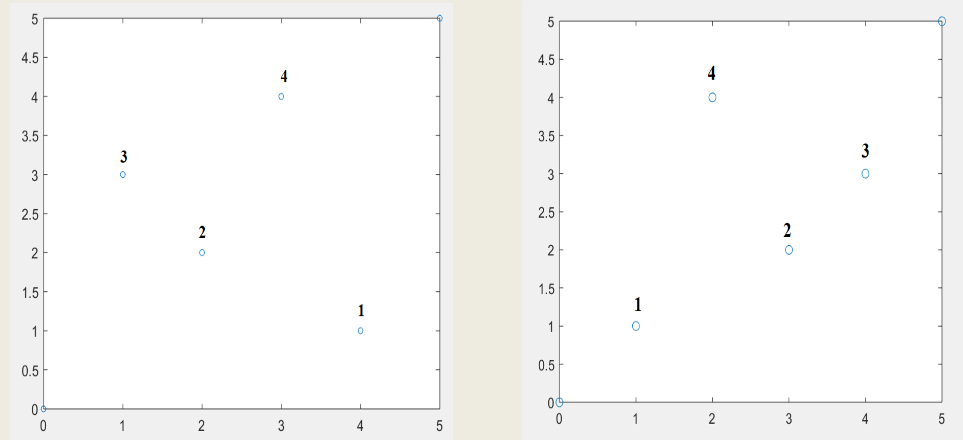


PATTERN OF LENGTH FOUR

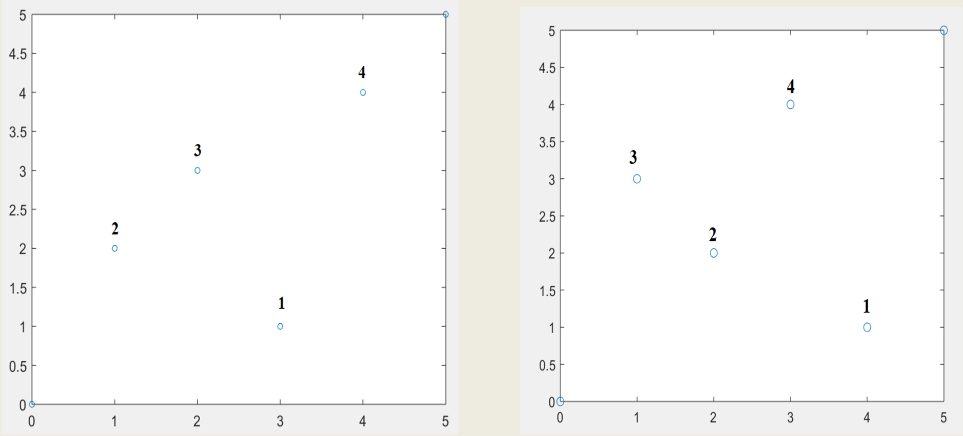
Due to the properties of inversion, reversion, and complement, an ordinary permutation is the rotation of the other. Hence, the scope of research is significantly reduced.



1342 is the inverse of 1423



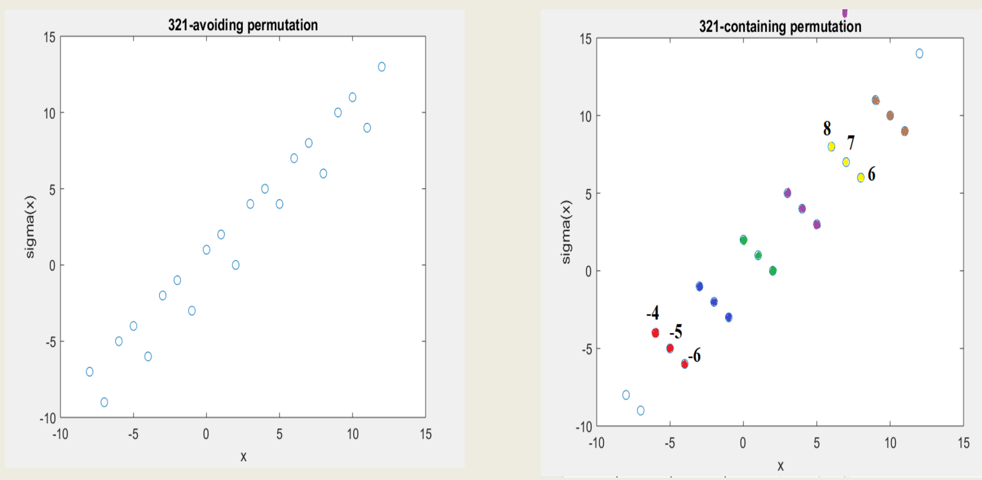
1423 is the reverse of 3241



3241 is the complement of 2314

PATTERN AVOIDANCE

A permutation w = [w(1), w(2), ..., w(N)] avoids pattern T if some k-element subsequences of w does not form pattern T.



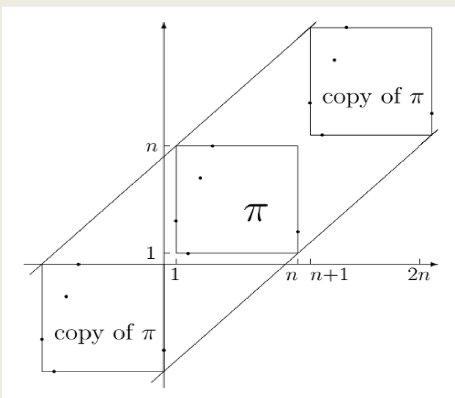
In comparison with 321-pattern, it clearly shows the permutation on the right-hand side does not avoid 321. In other words, the subsequences [-4, -5, -6] or [8, 7, 6] forms pattern 321.

AFFINE PERMUTATIONS

An ordinary permutation that has infinite periodic extensions. The use of affine permutations helps get rid of the critical issue when examining the ordinary permutations. That is, the typical shapes of pattern-avoiding permutations do not squash down at the corner.

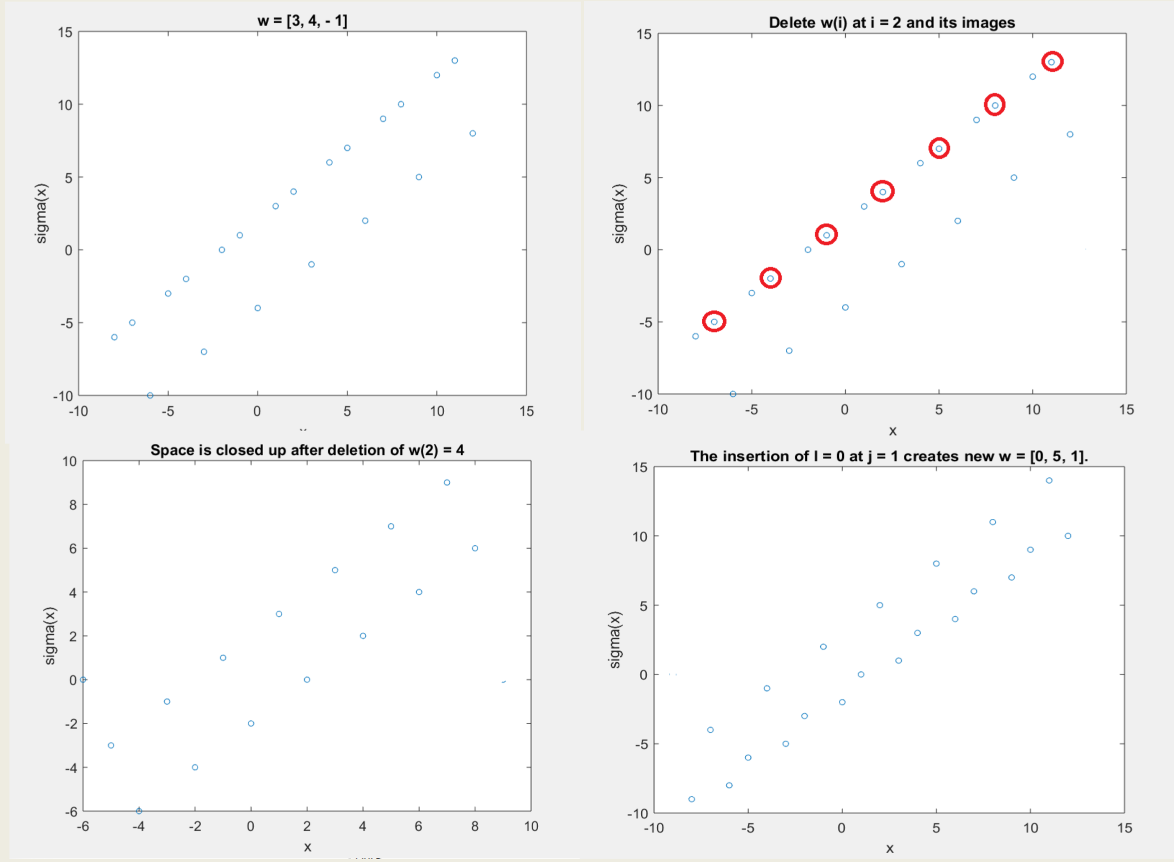
[w(1), w(2), ..., w(N)] is a bounded affine permutation if and only if

- w is a bijection
- $w(i+N) = w(i) + N \ \forall i \in \mathbb{Z}$ (periodic boundary condition)
- $w(1) + w(2) + \ldots + w(N) = \sum_{i=1}^{N} i = \frac{N(N+1)}{2}$ (centering condition)
- $|w(i) - i| < N$ (boundedness condition).



For an affine permutation of size n to be bounded, all points of the plot must lie on or between the two diagonal lines (Madras, 2020).

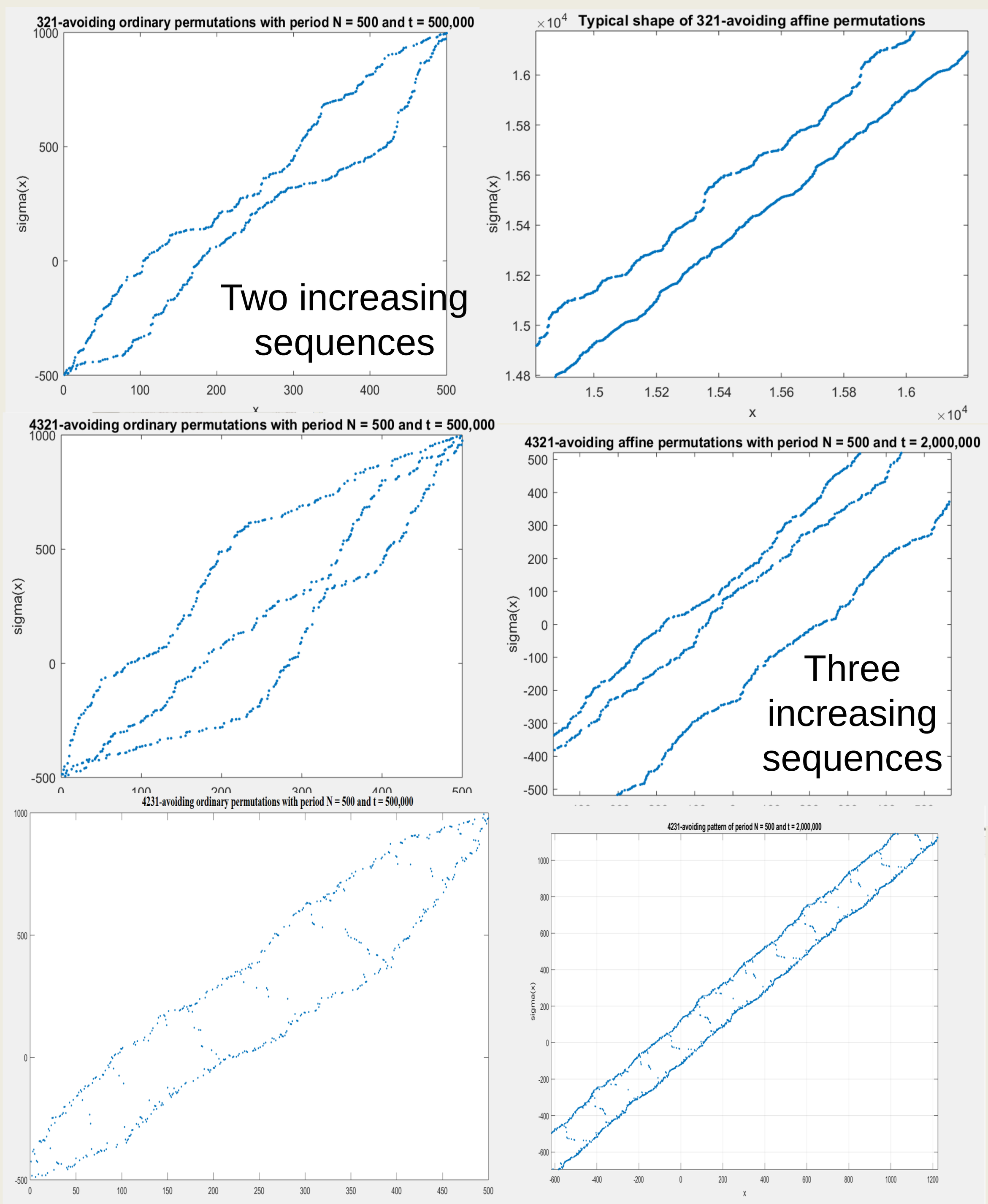
DELETION AND INSERTION



We start with a preferred Markov chain, and then randomly delete one point and insert another point. The process continues until the Markov chain reaches equilibrium.

RESULTS

The points of the initial permutation drift apart as the run proceeds (t will refer to the number of iterations of the Markov chain).



CONCLUSIONS

Affine permutation eliminates the "corner" problem existing in the ordinary one in the case of 321-, 4321-, and 4231-avoiding permutations.

REFERENCES

1. Madras, N.(2020). Bounded affine permutations.
2. Bona, M (2014). *In any way but this. Pattern avoidance. The basics.* In K.H.Rosen (Ed.). Combinatorics of permutations. Cambridge, Massachusetts: CRC Press.