Decomposition of Persistent Homology and Spectral Sequences

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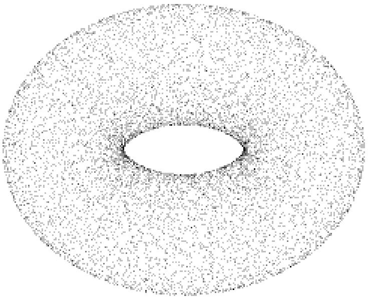
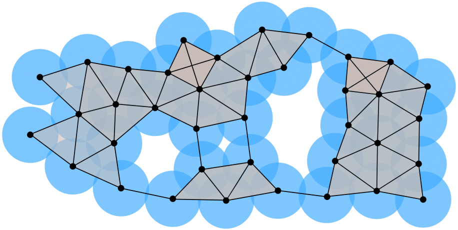
Persistent Homology - Point Cloud





Abstract Simplicial Complex

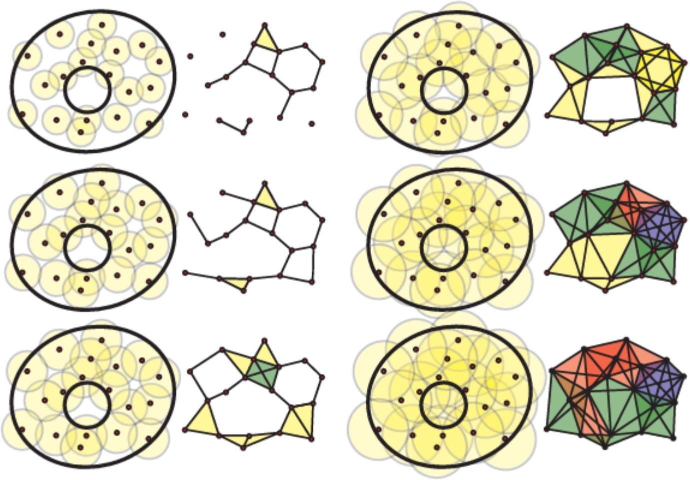
Definition

Let V be a finite set of vertices (points), 2^V is its power set. $\Delta \subseteq 2^V$ is an ***abstract simplicial complex*** *if, for all faces $f \in \Delta$ and every non-empty subset $f' \subseteq f$, f' is also a face in Δ.*

Definition (Vietoris–Rips complex)

$$\mathcal{R}(V, \epsilon) := \{f \subset 2^V : \ \forall u, v \in f, \ B(u, \epsilon) \cap B(v, \epsilon) \neq \emptyset\}.$$

Persistent Homology - Rips



Persistent Homology - Filtration

Definition

A ***filtered chain complex*** $(C, d, \mathcal{F})$ *over a field* $\mathbb{F}$ *is a chain complex of* $\mathbb{F}$*-spaces with an extra filtration structure*

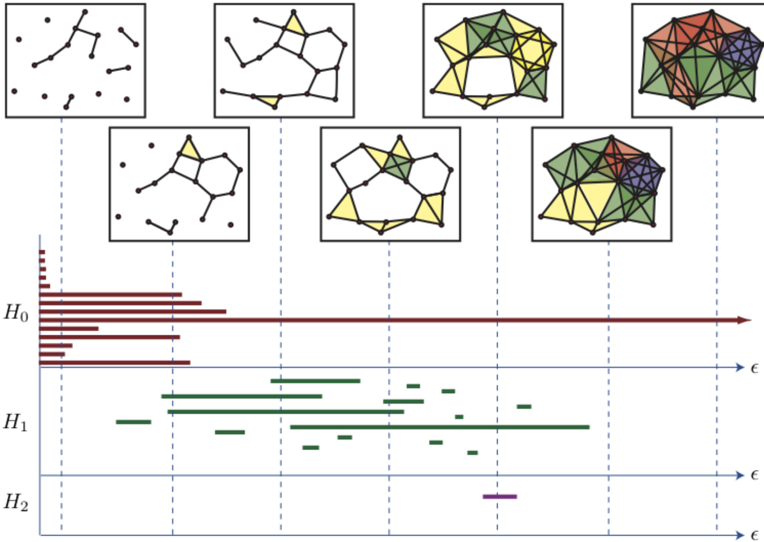
$$\emptyset = \mathcal{F}^0 C \subseteq \mathcal{F}^1 C \subseteq \cdots \subseteq \mathcal{F}^{p-1} C \subseteq \mathcal{F}^p C \subseteq \cdots \subseteq C$$

It is an ascending sequence of subchain complexes.

Remark

In the previous example, the entire chain complex C *is the ASC of all vertices, which is* 2^V *consisting of all subsets of* V*. Each filtration level in the example is a Vietoris–Rips complex, which is* $\mathcal{F}^p C = \mathcal{R}(V, p\epsilon)$ *for all* $p \geq 1$*.*

Persistent Homology - Barcodes



Persistent Homology

Definition

For $s \leq t$, the inclusion $\mathcal{F}^s C \hookrightarrow \mathcal{F}^t C$ induces a homomorphism $f_n^{s,t} : H_n(\mathcal{F}^s C) \to H_n(\mathcal{F}^t C)$ on homology groups. Then $H_n^{s,t} = \mathrm{Im}(f_n^{s,t})$ is defined to be the ***n-th persistent homology group*** *for all such $s \leq t$.*

Definition

The ***n-th persistent Betti number*** *is defined by $b_n^{s,t} = \dim_{\mathbb{F}} H_n^{s,t}$.*

Definition

The ***n-th persistent multiplicities*** *for all $s < t$ is defined by*

$$\mu_n^{s,t} = (b_n^{s,t-1} - b_n^{s,t}) - (b_n^{s-1,t-1} - b_n^{s-1,t}).$$

Persistent Homology and Spectral Sequence

Theorem (Basu and Parida, 2017)

Let $(C, d, \mathcal{F})$ be a filtered chain complex over $\mathbb{F}$ where each $\mathcal{F}^p C$ is the chain complex of a finite CW-complex. Then, for each $r > 0$, $n, s \geq 0$,

$$\dim_{\mathbb{F}} E_{n,s}^{(r)} = (b_n^{s,s+r-1} - b_n^{s-1,s+r-1}) - (b_{n-1}^{s-r,s-1} - b_{n-1}^{s-r,s}).$$

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Corollary (Spectral Sequence Theorem)

For every $r > 0$, and all $n \geq 0$,

$$\sum_s \dim_{\mathbb{F}} E^{(r)}_{n,s} = \sum_{j-i \geq r} (\mu_n^{i,j} + \mu_{n-1}^{i,j}) + b_n.$$

Lee-Gornik Spectral Sequence

The **equivariant $\mathfrak{sl}(N)$ Khovanov-Rozansky homology** of oriented link diagram D over $\mathbb{C}[a]$ is defined to be the homology

$$H_P(D) = H(H(C_P(D), d_{mf}), d_\chi).$$

Define $\hat{C}_P(D) = C_P(D)/(a-1)C_P(D)$. Then

$$\hat{H}_P(D) = H(H(\hat{C}_P(D), d_{mf}), d_\chi)$$

is called the **deformed $\mathfrak{sl}(N)$ Khovanov-Rozansky homology**.

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Theorem (Lee-Gornik)

Let D be a diagram of an oriented link L. Then the x-filtration $\mathcal{F}_x$ on the chain complex $(H(\hat{C}_P(D), d_{mf}), d_\chi)$ induces a spectral sequence $\{\hat{E}_r(L)\}$ converging to $\hat{H}_P(L)$ with $\hat{E}_1(L) \cong H_N(L)$.

Lobb's Decomposition

Lobb observed that $(H(C_P(D), d_{mf}), d_\chi)$, as a bounded chain complex of finitely generated graded free $\mathbb{C}[a]$-module, decomposes into a direct sum of simple graded chain complexes of the forms

$$\begin{aligned} F_{i,s} &= 0 \to \mathbb{C}[a]\|i\|\{s\} \to 0 \\ T_{i,m,s} &= 0 \to \mathbb{C}[a]\|i-1\|\{s+2km\} \xrightarrow{a^m} \mathbb{C}[a]\|i\|\{s\} \to 0 \end{aligned}$$

Lobb's Decomposition

Lemma (Wu, 2015)

For any $r \geq 0$,

$$E_r^{p,q}(\hat{F}_{i,s}) = \begin{cases} \mathbb{C} & \text{if } p = s \text{ and } q = i - s, \\ 0 & \text{otherwise}, \end{cases}$$

$$E_r^{p,q}(\hat{T}_{i,m,s}) = \begin{cases} \mathbb{C} & \text{if } p = s, q = i - s \text{ and } r \leq 2km, \\ \mathbb{C} & \text{if } p = s + 2km, q = i - 1 - s - 2km \\ & \text{and } r \leq 2km, \\ 0 & \text{otherwise}. \end{cases}$$

Remark

Here $\hat{F}_{i,s} = F_{i,s}/(a-1)F_{i,s}$ and $\hat{T}_{i,m,s} = T_{i,m,s}/(a-1)T_{i,m,s}$.

Persistent Chain Complex

Definition

*The **persistent chain complex** (PC, d_x) of filtered chain complex $(C, d, \mathcal{F})$ over $\mathbb{F}$ is defined to be the direct sum*

$$(PC, d_x) := \bigoplus_{p \in \mathbb{Z}} (\mathcal{F}^p C, d|_{\mathcal{F}^p C}).$$

Here x is a homogeneous indeterminate of degree 1 act on PC as the natural inclusion map $i^p : \mathcal{F}^p C \hookrightarrow \mathcal{F}^{p+1} C$ for each $p \in \mathbb{Z}$. This makes (PC, d_x) is a chain complex of graded $\mathbb{F}[x]$-modules, where the homogeneous component of degree p is $(\mathcal{F}^p C, d|_{\mathcal{F}^p C})$.

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Note that PC is a free $\mathbb{F}[x]$-module and d_x preserves the grading of PC. The homology $PH := H(PC, d_x)$ is a graded $\mathbb{F}[x]$-module and called the **persistent homology** of $(C, d, \mathcal{F})$.

Terminology

Definition

A filtered $\mathbb{F}$-space $(V, \mathcal{F})$ is ***locally finite dimensional*** *if $\mathcal{F}^p V$ is finite dimensional over $\mathbb{F}$ for every $p \in \mathbb{Z}$.*

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Definition

Let M be a graded $\mathbb{F}[x]$ module. Then $M\|n\|\{s\}$ is a bigraded $\mathbb{F}[x]$-module where

- *$\| \cdot \|$ indicates the homological grading,*
- *$\{\cdot\}$ indicates the polynomial degree shift, $(M\|n\|\{s\})^k = M^{k-s}$.*

Types of Factors

Definition

For $n, s, m \in \mathbb{Z}$ and $m > 0$,

$$U_{n,s,\infty} = 0 \to \mathbb{F}[x]\|n\|\{s\} \to 0 \quad (1)$$

$$U_{n,s,m} = 0 \to \mathbb{F}[x]\|n+1\|\{s+m\} \xrightarrow{x^m} \mathbb{F}[x]\|n\|\{s\} \to 0 \quad (2)$$

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Remark

$$H(U_{n,s,\infty}) \cong \mathbb{F}[x]\|n\|\{s\},$$
$$H(U_{n,s,m}) \cong \mathbb{F}[x]/(x^m)||n||\{s\}.$$

These factors are the generalization of Lobb's factors.

Decomposition of Persistent Chain Complex

Theorem (I)

Let (PC, d_x) be the persistent chain complex of a filtered chain complex $(C, d, \mathcal{F})$ over $\mathbb{F}$, such that:

- *For each n, the filtration on C_n is bounded below;*

Decomposition of Persistent Chain Complex

Theorem (I)

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- *For each n, the filtration on C_n is bounded below;*
- *For each n, $(C_n, \mathcal{F})$ is locally finite dimensional over $\mathbb{F}$.*

Then, up to chain homotopy and permutation of factors, (PC, d_x) can be uniquely decomposed as a direct sum of graded chain complexes of types (1) *and* (2).

Decomposition of Persistent Chain Complex

Theorem (I, Continued)

Define $\mathbb{Z}_{+,\infty} = \mathbb{Z}_+ \cup \{+\infty\}$. *Then for each* $n \in \mathbb{Z}$, *there exist* $K_n \in \mathbb{Z}_{+,\infty}$ *and a unique sequence* $\{(s_n(i), m_n(i))\}_{i=1}^{K_n} \subseteq \mathbb{Z} \times \mathbb{Z}_{+,\infty}$ *satisfying* $s_n(i) \leq s_n(i+1)$ *and* $m_n(i) \leq m_n(i+1)$ *if* $s_n(i) = s_n(i+1)$, *so that*

$$PC \simeq \bigoplus_{n=-\infty}^{\infty} \bigoplus_{i=1}^{K_n} U_{n,s_n(i),m_n(i)}.$$

Consequently, the persistent homology $PH = H(PC, d_x)$ *can be uniquely decomposed as a direct sum of*

$$PH \cong \bigoplus_{n=-\infty}^{\infty} \bigoplus_{i=1}^{K_n} H(U_{n,s_n(i),m_n(i)}).$$

Spectral Sequence

Definition

For a chain complex K of free $\mathbb{F}[x]$-modules, the short exact sequence $0 \to K\{1\} \xrightarrow{x} K \xrightarrow{\pi_x} K/xK \to 0$ induces an exact couple given by

$$\begin{tikzcd} H(K) \arrow[rr, "x"] & & H(K) \arrow[dl, "\pi_x"] \\ & H(K/xK) \arrow[ul, "\Delta"] & \end{tikzcd}$$

where π_x is the standard quotient map. Denote by

$$\mathcal{H}^{(r)}(K) = (A^{(r)}, E^{(r)}, f^{(r)}, g^{(r)}, \Delta^{(r)})$$

the above exact couple derived $(r-1)$ times. Then $\{E^{(r)}\}$ is called the ***spectral sequence*** *of K.*

Spectral Sequence

Remark

For a filtered chain complex $(C, d, \mathcal{F})$, $\{E^{(r)}\}$ in $\{\mathcal{H}^{(r)}(PC)\}$ is the standard spectral sequence of $(C, d, \mathcal{F})$ with some grading shifts.

Decomposition of Spectral Sequence

Theorem (II)

The decomposition of the persistent homology $PH = H(PC, d_x)$ *in Theorem (I) induces a direct sum decomposition of the spectral sequence* $\{E^{(r)}\}$ *of* (PC, d_x). *For* $r \in \mathbb{Z}_{+,\infty}$ *and the same sequence* $\{(s_n(i), m_n(i))\}_{i=1}^{K_n} \subseteq \mathbb{Z} \times \mathbb{Z}_{+,\infty}$ *defined in Theorem (I), we have*

$$E^{(r)} \cong \bigoplus_{n=-\infty}^{\infty} \bigoplus_{i=1}^{K_n} E^{(r)}(U_{n,s_n(i),m_n(i)}).$$

Decomposition of Spectral Sequence

More precisely, for $r, m \in \mathbb{Z}_+$,

$$E^{(r)}(U_{n,s,\infty}) \cong \mathbb{F}\|n\|\{s\} \quad \forall r \geq 1,$$
$$E^{(\infty)}(U_{n,s,\infty}) \cong \mathbb{F}\|n\|\{s\},$$

$$E^{(r)}(U_{n,s,m}) \cong \begin{cases} \mathbb{F}\|n+1\|\{s+m\} \oplus \mathbb{F}\|n\|\{s\} & \text{for } 1 \leq r \leq m, \\ 0 & \text{for } r > m, \end{cases}$$
$$E^{(\infty)}(U_{n,s,m}) \cong 0.$$

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$$E^{(\infty)}(U_{n,s,m}) \cong 0.$$

Corollary

$\{E^{(r)}\}$ collapses locally to $E^{(\infty)}$, which means $\{E_{n,s}^{(r)}\}$ collapses to $E_{n,s}^{(\infty)}$.

Relations of Persistent Homology and Spectral Sequence

Theorem (III)

Denote by $\nu_{n,s,\infty}$ the multiplicity of the factor $H(U_{n,s,\infty}) \cong \mathbb{F}[x]\|n\|\{s\}$ in the decomposition of the persistent homology $PH = H(PC, d_x)$ and $\nu_{n,s,m}$ the multiplicity of the factor $H(U_{n,s,m}) \cong \mathbb{F}[x]/(x^m)||n||\{s\}$.

- $\dim_{\mathbb{F}} E^{(r)}_{n,s} = \nu_{n,s,\infty} + \sum_{m \geq r}(\nu_{n,s,m} + \nu_{n-1,s-m,m})$, *which shows that the persistent homology determines the spectral sequence. Then:*
- *The spectral sequence $\{E^{(r)}\}$ collapses locally to $E^{(\infty)}$, and*
    - $\nu_{n,s,\infty} = \dim_{\mathbb{F}} E^{(\infty)}_{n,s}$,
    - $\nu_{n,s,m} = \dim_{\mathbb{F}} E^{(m)}_{n,s} - \dim_{\mathbb{F}} E^{(m+1)}_{n,s} - \nu_{n-1,s-m,m}$,

  which implies that the spectral sequence determines the persistent homology since $\mathcal{F}$ is bounded below and, therefore, $\nu_{n,s,m} = 0$ for $s \ll 0$.

Persistent Betti Number

Corollary

*Recall that $\mu_n^{i,j}$ is the **persistent multiplicities**, then*

- $\mu_n^{i,j} = \nu_{n,i,j-i}$,
- $\nu_{n,s,m} = \mu_n^{s,s+m}$.

*Also, the **persistent Betti number** $b_n^{i,j}$ can be counted with $\nu_{n,s,m}$ and $\nu_{n,s,\infty}$ by*

$$b_n^{i,j} = \sum_{j-m<s\leq i} \nu_{n,s,m}.$$

Future Work I (Peiqi Yang)

Definition (Wu, 2017)

The Khovanov-Rozansky chain complex $\mathscr{C}_*(G)$ *of digraph* G *is the graded Koszul chain complex* $C^{\mathbb{F}[E(G)]}(\Delta_G)$ *over* $\mathbb{F}[E(G)]$ *defined by the incidence set* Δ_G *of* G.

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There is a way to define an $\mathfrak{sl}(N)$ Koszul matrix factorization of digraphs, which induces a spectral sequence between the HOMFLYPT digraph homology and the $\mathfrak{sl}(N)$ digraph homology following our main result in this talk.

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The software packages for persistent homology is well developed nowadays. For another approach, there is a hope of using mature tools to compute some complicated examples in Khovanov-Rozansky graph homology.

Future Work II (Yingfeng Hu)

Jacob Rasmussen constructed a spectral sequence E whose first page is the HOMFLYPT homology that converges to the $\mathfrak{sl}(n)$ Khovanov-Rozansky homology.

More recently, Wu constructed a variant of the Khovanov-Rozansky homology for transverse links in the standard contact 3-sphere. Yingfeng plans to use the main results in this talk to give an alternative construction of the Rasmussen spectral sequence based on the transverse Khovanov-Rozansky homology.

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