Exercise 1.3.5 Prove Theorem 1.3.5 (1) (3) (4) (5) (6) (7) (8) (10) (11) (13) (14). Throughout the proofs, we let $x = [(x_0, x_1)]$, $y = [(y_0, y_1)]$ and $z = [(z_0, z_1)]$.

proof of (1).

$$\begin{aligned}(x+y)+z &= [(x_0+y_0, x_1+y_1)] + [(z_0, z_1)] \\ &= [((x_0+y_0)+z_0, (x_1+y_1)+z_1)] \\ &= [(x_0+(y_0+z_0), x_1+(y_1+z_1))] \\ &= [(x_0, x_1)] + [(y_0+z_0, y_1+z_1)] \\ &= [(x_0, x_1)] + ([(y_0, y_1)] + [(z_0, z_1)]) \\ &= x+(y+z).\end{aligned}$$

□

Proof of (3).

$$x + 0 = [(x_0, x_1)] + [(1,1)] = [(x_0+1, x_1+1)] = [(x_0, x_1)] = x.$$

One can justify $[(x_0+1, x_1+1)] = [(x_0, x_1)]$ by $(x_0+1)+x_1 = (x_1+1)+x_0$ (We've shown commutativity and associativity in $\mathbb{N}$). □

Proof of (4).

$$x + (-x) = [(x_0, x_1)] + [(x_1, x_0)] = [(x_0+x_1, x_1+x_0)] = [(1,1)] = 0.$$

$[(x_0+x_1, x_1+x_0)] = [(1,1)]$ because $x_0+x_1+1 = x_1+x_0+1$ (Apply commutativity, which we've proven on $+$ in $\mathbb{N}$). □

Proof of (5).

$$\begin{aligned}(xy)z &= [(x_0y_0+x_1y_1, x_0y_1+x_1y_0)] \cdot [(z_0, z_1)] \\ &= [(z_0(x_0y_0+x_1y_1)+z_1(x_0y_1+x_1y_0), z_1(x_0y_0+x_1y_1)+z_0(x_0y_1+x_1y_0))] \\ &= [(x_0y_0z_0+x_1y_1z_0+x_0y_1z_1+x_1y_0z_1, x_0y_0z_1+x_1y_1z_1+x_0y_1z_0+x_1y_0z_0)] \\ &= [(x_0(y_0z_0+y_1z_1)+x_1(y_1z_0+y_0z_1), x_0(y_0z_1+y_1z_0)+x_1(y_1z_1+y_0z_0))] \\ &= [(x_0, x_1)] \cdot [(y_0z_0+y_1z_1, y_1z_0+y_0z_1)] \\ &= x \cdot ([(y_0, y_1)] \cdot [(z_0, z_1)]) \\ &= x(yz).\end{aligned}$$

Note that we implicitly use properties we've proven for operations on the natural numbers (associativity and commutative for both addition and multiplication). □

Proof of (6).

$$\begin{aligned}xy &= [(x_0, x_1)] \cdot [(y_0, y_1)] \\ &= [(x_0y_0+x_1y_1, x_0y_1+x_1y_0)] \\ &= [(y_0x_0+y_1x_1, y_0x_1+y_1x_0)] \\ &= [(y_0, y_1)] \cdot [(x_0, x_1)] \\ &= yx\end{aligned}$$

□

Proof of (7).

$$\begin{aligned}x \cdot 1 &= [(x_0, x_1)] \cdot [(1+1, 1)] \\ &= [(x_0(1+1)+x_1 \cdot 1, x_0 \cdot 1 + x_1(1+1))] \\ &= [(x_0+x_0+x_1, x_0+x_1+x_1)] \\ &= [(x_0, x_1)].\end{aligned}$$

Since $(x_0+x_0+x_1)+x_1 = (x_0+x_1+x_1)+x_0$, the final step is justified. □

proof of (8).

$$
\begin{aligned}
x(y+z) &= [(x_0, x_1)] \cdot ([(y_0+z_0, y_1+z_1)]) \\
&= [(x_0(y_0+z_0) + x_1(y_1+z_1), x_0(y_1+z_1) + x_1(y_0+z_0))] \\
&= [(x_0y_0 + x_0z_0 + x_1y_1 + x_1z_1, x_0y_1 + x_0z_1 + x_1y_0 + x_1z_0)] \\
&= [((x_0y_0 + x_1y_1) + (x_0z_0 + x_1z_1), (x_0y_1 + x_1y_0) + (x_0z_1 + x_1z_0))] \\
&= [(x_0y_0 + x_1y_1, x_0y_1 + x_1y_0)] + [(x_0z_0 + x_1z_1, x_0z_1 + x_1z_0)] \\
&= [(x_0, x_1)] \cdot [(y_0, y_1)] + [(x_0, x_1)] \cdot [(z_0, z_1)] \\
&= xy + xz.
\end{aligned}
$$

□

proof of (10). We show that at least one of $x < y$, $x = y$ or $x > y$ holds. Suppose that $x \not> y$, $x \not< y$ and $x \neq y$. Hence, $x_0 + y_1 \not< x_1 + y_0$ and $y_0 + x_1 \not< y_1 + x_0$. Using the trichotomy of order in $\mathbb{N}$, we can then deduce that $x_0 + y_1 = x_1 + y_0$. However, this contradicts the assumption that $x \neq y$. Hence, at least one of the three statements holds. Now we show that no two statements can hold simultaneously. Suppose that $x < y$ and $x = y$. Then $x_0 + y_1 < x_1 + y_0$ but also $x_0 + y_1 = x_1 + y_0$ which clearly contradicts the trichotomy of order in $\mathbb{N}$. The other cases follow suit similarly. □

proof of (11). Suppose that $x < y$ and $y < z$. So $x_0 + y_1 < x_1 + y_0$ and $y_0 + z_1 < y_1 + z_0$. From the latter, we deduce that $y_1 + z_0 = y_0 + z_1 + q$ for some $q \in \mathbb{N}$. We can add this equality to both sides of the former to get $(x_0 + y_1) + (y_0 + z_1 + q) < (x_1 + y_0) + (y_1 + z_0)$. Then, using the cancellation law, we can simplify this to $x_0 + z_1 + q < x_1 + z_0$. So $x_1 + z_0 = x_0 + z_1 + q + r$ for some $r \in \mathbb{N}$. Since $q + r \in \mathbb{N}$, we get that $x_0 + z_1 < x_1 + z_0$. Hence, $x < z$. □

proof of (13). Suppose that $x < y$ and $z > \hat{0}$. We know, from Theorem 1.3.7 (2), that $z = [(a+1, 1)]$ for some $a \in \mathbb{N}$. Since $x_0 + y_1 < x_1 + y_0$, multiplying both sides by a yields $ax_0 + ay_1 < ax_1 + ay_0$. Hence,

$$
\begin{gathered}
[(ax_0, ax_1)] < [(ay_0, ay_1)] \\
[(ax_0 + x_0 + x_1, ax_1 + x_0 + x_1)] < [(ay_0 + y_0 + y_1, ay_1 + y_0 + y_1)] \\
[(x_0(a+1) + x_1, x_1(a+1) + x_0)] < [(y_0(a+1) + y_1, y_1(a+1) + y_0)] \\
[(x_0, x_1)][(a+1, 1)] < [(y_0, y_1)][(a+1, 1)] \\
xz < yz
\end{gathered}
$$

□

proof of (14). Suppose that $\hat{0} = \hat{1}$. So $[(1,1)] = [(1+1,1)]$. By definition, this means $1 + 1 = 1 + (1+1)$. Applying associativity with the law of cancellation for addition, we get $1 = 1 + 1$. This contradicts peano axioms. Hence, $\hat{1} \neq \hat{0}$. □

Exercise 1.3.6 Prove Theorem 1.3.7 (1) (3) (4b) (4c).

proof of (1). Suppose that $i(a) = i(b)$ for some $a, b \in \mathbb{N}$. So $[(a+1, 1)] = [(b+1, 1)]$ and $(a+1)+1 = 1+(b+1)$. Applying the cancellation law twice, we yield $a = b$. As desired. □

proof of (3). This holds true by definition of $\hat{1}$. □

proof of (4b). Let $a, b \in \mathbb{N}$ be arbitrary natural numbers. Then

$$\begin{aligned} i(a)i(b) &= [(a+1, 1)][(b+1, 1)] \\ &= [((a+1)(b+1)+1, a+1+b+1)] \\ &= [(ab+a+b+1+1, a+1+b+1)] \\ &= [(ab+1, 1)] \\ &= i(ab) \end{aligned}$$

□

proof of (4c). ($\Longrightarrow$) Suppose that $a < b$ for some $a, b \in \mathbb{N}$. Using properties of order, we see that $(a+1)+1 < (b+1)+1$. Therefore, $[(a+1, 1)] < [(b+1, 1)]$.

($\Longleftarrow$) Every step in $\Longrightarrow$ can be reversed to yield $\Longleftarrow$. □

Exercise 1.3.7 Let $x, y, z \in \mathbb{Z}$.

1. Prove that $x < y$ if and only if $-x > -y$.
2. Prove that if $z < 0$, then $x < y$ if and only if $xz > yz$.

proof for (1). Suppose that $x < y$. Adding $(-x) + (-y)$ to both sides, we yield $(-y) + ((-x) + x) < (-x) + ((-y) + y)$ which simplifies to $-y < -x$.

Now suppose that $-x > -y$. Adding $x + y$ to both sides gives $y + (x + (-x)) > x + (y + (-y))$ which gives $y > x$ after simplification using the law of additive inverses. □

proof of (2). Suppose that $z < 0$, and suppose that $x < y$. We know from (1) that $-z > -0 = 0$. And we know from Theorem 1.3.5 (13) that $-zx < -zy$. Applying (1) again gives $zx > zy$ which is the desired result.

We prove the converse using constraposition. Suppose that $x \geq y$. Either $x = y$ or $x > y$. In the former, we deduce that $xz = yz$, and therefore, $xz \leq yz$. In the latter, we apply the result we just proved to yield $xz < yz$ which is also equivalent to $xz \leq yz$. □

Exercise 1.3.8 Let $x \in \mathbb{Z}$. Prove that if $x > 0$ then $x \geq 1$. Prove that if $x < 0$ then $x \leq -1$.

Proof. Suppose that $x > 0$, and suppose that $x < 1$. So $0 < x < 1$ which contradicts Theorem 1.3.9. Now suppose that $x < 0$, and that $x > -1$. We deduce that $-1 < x < 0$ which also contadicts Theorem 1.3.9. □

Exercise 1.3.9

1. Prove that $1 < 2$.
2. Let $x \in \mathbb{Z}$. Prove that $2x \neq 1$.

proof of (1). We know $[(1+1+1, 1)] = 2$ and $[(1+1, 1)] = 1$. Since $(1+1)+1 < (1+1+1)+1$, we deduce that $1 < 2$. □

proof of (2). Let $x \in \mathbb{Z}$ be arbitrary. Suppose that $2x = 1$. Since $1 > 0$, we know that $2x > 0$ (Apply Lemma 1.3.8 (11)). And since $2 > 0$, $x > 0$. Since x is positive and $1 < 2$, it must be the case that $x < 2x = 1$. Therefore, $0 < x < 1$. However, we know from Theorem 1.3.9 that no such x can exist. Therefore, we have a contradiction. □

Exercise 1.3.10 Prove that the Well-Ordering Principle (Theorem 1.2.10), which was stated for $\mathbb{N}$ in Section 1.2, still holds when we think of $\mathbb{N}$ as the set of positive integers. That is, let $G \subseteq \{x \in \mathbb{Z} : x > 0\}$ be a non-empty set. Prove that there is some $m \in G$ such that $m \leq g$ for all $g \in G$. Use Theorem 1.3.7.

Proof. Let $R = \{x \in \mathbb{Z} : x > 0\}$. Let G be a non-empty subset of R with no least element. Let

$$A = \{a \in R : \text{if } r \in R \text{ and } r \leq a, \text{ then } r \notin G\}.$$

Clearly $A \cap G = \varnothing$. We show that $i^{-1}[A] = \mathbb{N}$, which implies that $A = R$ and lets us deduce that $G = \varnothing$ which is a contradiction. Let H denote $i^{-1}[A]$. Suppose that $1 \notin H$. By definition, $i(1) = \hat{1} \notin A$. So there exists $q \in R$ such that $q \leq \hat{1}$ and $q \in G$. Since $q \in R$, $\hat{0} < q \leq \hat{1}$. One can clearly see that $q = \hat{1}$. However, $\hat{1} \in G$ would contradict the assumption that G has no least element (since $\hat{0} < x < \hat{1}$ is false for all $x \in \mathbb{Z}$ as proven previously). Therefore, $1 \in H$.

Now suppose that $a \in H$ but $a + 1 \notin H$. By definition, that means $i(a) \in A$ but $i(a + 1) \notin A$. Then there must exist $p \in R$ such that $p \leq i(a + 1)$ and $p \in G$. Since $i(a) \in H$, showing that $p \leq i(a)$ would give us a contradiction. Hence, by the trichotomy of order in $\mathbb{Z}$, $i(a) < p$. And therefore, $i(a) < p \leq i(a + 1)$. It follows that $p = i(a+1)$. Therefore $i(a+1) \in G$. Now let $x \in G$ be arbitrary and suppose that $x < i(a+1)$. Then $x < i(a) + \hat{1}$ and $x \leq i(a)$. Because $i(a) \in A$, it follows that $x \notin G$, which is a contradiction. Hence, $a + 1 \leq x$ (trichotomy of order). We □

Exercise 1.3.11 Prove Theorem 1.3.8 (1) (3) (4) (5) (7) (10) (11).

proof of (1).

$$\begin{aligned} x + z &= y + z \\ x + z + (-z) &= y + z + (-z) \\ x + 0 &= y + 0 \\ x &= y. \end{aligned}$$

□

proof of (3). Consider $x + y + (-x) + (-y) = 0$.

$$\begin{aligned} x + y + (-x) + (-y) &= 0 \\ (x + y) + (-(x + y)) + (-x) + (-y) &= -(x + y) \\ 0 + (-x) + (-y) &= -(x + y) \\ (-x) + (-y) &= -(x + y) \end{aligned}$$

□

proof of (4).

$$\begin{aligned} x &= x \cdot 1 \\ &= x \cdot (1 + 0) \\ &= x \cdot 1 + x \cdot 0 \\ &= x + x \cdot 0 \end{aligned}$$

So $x = x + x \cdot 0$. Adding $-x$ to both sides yields the desired result. □

proof of (5). Suppose that $z \neq 0$ and $xz = yz$. Then $xz + (-(yz)) = 0$ and $xz + (-y)z = (x + (-y))z = 0$. Using Theorem 1.3.5 (9) (Which states that $\mathbb{Z}$ have no zero divisors), we deduce that $x + (-y) = 0$. Therefore, $x = y$. □

proof of (7). Suppose that □