Exercise 1.3.5 Prove Theorem 1.3.5 (1) (3) (4) (5) (6) (7) (8) (10) (11) (13) (14). Throughout the proofs, we let $x = [(x_0, x_1)]$, $y = [(y_0, y_1)]$ and $z = [(z_0, z_1)]$.

proof of (1).

$$\begin{aligned}(x+y)+z &= [(x_0+y_0, x_1+y_1)] + [(z_0, z_1)] \\ &= [((x_0+y_0)+z_0, (x_1+y_1)+z_1)] \\ &= [(x_0+(y_0+z_0), x_1+(y_1+z_1))] \\ &= [(x_0, x_1)] + [(y_0+z_0, y_1+z_1)] \\ &= [(x_0, x_1)] + ([(y_0, y_1)] + [(z_0, z_1)]) \\ &= x + (y+z).\end{aligned}$$

□

Proof of (3).

$$x + 0 = [(x_0, x_1)] + [(1,1)] = [(x_0+1, x_1+1)] = [(x_0, x_1)] = x.$$

One can justify $[(x_0+1, x_1+1)] = [(x_0, x_1)]$ by $(x_0+1)+x_1 = (x_1+1)+x_0$ (We've shown commutativity and associativity in $\mathbb{N}$). □

Proof of (4).

$$x + (-x) = [(x_0, x_1)] + [(x_1, x_0)] = [(x_0+x_1, x_1+x_0)] = [(1,1)] = 0.$$

$[(x_0+x_1, x_1+x_0)] = [(1,1)]$ because $x_0+x_1+1 = x_1+x_0+1$ (Apply commutativity, which we've proven on $+$ in $\mathbb{N}$). □

Proof of (5).

$$\begin{aligned}(xy)z &= [(x_0y_0+x_1y_1, x_0y_1+x_1y_0)] \cdot [(z_0, z_1)] \\ &= [(z_0(x_0y_0+x_1y_1)+z_1(x_0y_1+x_1y_0), z_1(x_0y_0+x_1y_1)+z_0(x_0y_1+x_1y_0))] \\ &= [(x_0y_0z_0+x_1y_1z_0+x_0y_1z_1+x_1y_0z_1, x_0y_0z_1+x_1y_1z_1+x_0y_1z_0+x_1y_0z_0)] \\ &= [(x_0(y_0z_0+y_1z_1)+x_1(y_1z_0+y_0z_1), x_0(y_0z_1+y_1z_0)+x_1(y_1z_1+y_0z_0))] \\ &= [(x_0, x_1)] \cdot [(y_0z_0+y_1z_1, y_1z_0+y_0z_1)] \\ &= x \cdot ([(y_0, y_1)] \cdot [(z_0, z_1)]) \\ &= x(yz).\end{aligned}$$

Note that we implicitly use properties we've proven for operations on the natural numbers (associativity and commutative for both addition and multiplication). □

Proof of (6).

$$\begin{aligned}xy &= [(x_0, x_1)] \cdot [(y_0, y_1)] \\ &= [(x_0y_0+x_1y_1, x_0y_1+x_1y_0)] \\ &= [(y_0x_0+y_1x_1, y_0x_1+y_1x_0)] \\ &= [(y_0, y_1)] \cdot [(x_0, x_1)] \\ &= yx\end{aligned}$$

□

Proof of (7).

$$\begin{aligned}x \cdot 1 &= [(x_0, x_1)] \cdot [(1+1, 1)] \\ &= [(x_0(1+1)+x_1 \cdot 1, x_0 \cdot 1 + x_1(1+1))] \\ &= [(x_0+x_0+x_1, x_0+x_1+x_1)] \\ &= [(x_0, x_1)].\end{aligned}$$

Since $(x_0+x_0+x_1)+x_1 = (x_0+x_1+x_1)+x_0$, the final step is justified. □

proof of (8).

$$\begin{aligned}
x(y+z) &= [(x_0,x_1)]\cdot([(y_0+z_0,y_1+z_1)]) \\
&= [(x_0(y_0+z_0)+x_1(y_1+z_1), x_0(y_1+z_1)+x_1(y_0+z_0))] \\
&= [(x_0y_0+x_0z_0+x_1y_1+x_1z_1, x_0y_1+x_0z_1+x_1y_0+x_1z_0)] \\
&= [((x_0y_0+x_1y_1)+(x_0z_0+x_1z_1),(x_0y_1+x_1y_0)+(x_0z_1+x_1z_0))] \\
&= [(x_0y_0+x_1y_1, x_0y_1+x_1y_0)]+[(x_0z_0+x_1z_1, x_0z_1+x_1z_0)] \\
&= [(x_0,x_1)]\cdot[(y_0,y_1)]+[(x_0,x_1)]\cdot[(z_0,z_1)] \\
&= xy+xz.
\end{aligned}$$

□

proof of (10). We show that at least one of $x<y$, $x=y$ or $x>y$ holds. Suppose that $x \not> y$, $x \not< y$ and $x \neq y$. Hence, $x_0+y_1 \not< x_1+y_0$ and $y_0+x_1 \not< y_1+x_0$. Using the trichotomy of order in $\mathbb{N}$, we can then deduce that $x_0+y_1 = x_1+y_0$. However, this contradicts the assumption that $x \neq y$. Hence, at least one of the three statements holds. Now we show that no two statements can hold simultaneously. Suppose that $x<y$ and $x=y$. Then $x_0+y_1<x_1+y_0$ but also $x_0+y_1=x_1+y_0$ which clearly contradicts the trichotomy of order in $\mathbb{N}$. The other cases follow suit similarly. □

proof of (11). Suppose that $x<y$ and $y<z$. So $x_0+y_1<x_1+y_0$ and $y_0+z_1<y_1+z_0$. From the latter, we deduce that $y_1+z_0=y_0+z_1+q$ for some $q\in\mathbb{N}$. We can add this equality to both sides of the former to get $(x_0+y_1)+(y_0+z_1+q)<(x_1+y_0)+(y_1+z_0)$. Then, using the cancellation law, we can simplify this to $x_0+z_1+q<x_1+z_0$. So $x_1+z_0=x_0+z_1+q+r$ for some $r\in\mathbb{N}$. Since $q+r\in\mathbb{N}$, we get that $x_0+z_1<x_1+z_0$. Hence, $x<z$. □

Proof. Suppose that $x<y$ and $z>\hat{0}$. We know, from Theorem 1.3.7 (2), that $z=[(a+1,1)]$ for some $a\in\mathbb{N}$. Since $x_0+y_1<x_1+y_0$, multiplying both sides by a yields $ax_0+ay_1<ax_1+ay_0$. Hence,

$$\begin{gathered}
[(ax_0,ax_1)]<[(ay_0,ay_1)] \\
[(ax_0+x_0+x_1, ax_1+x_0+x_1)]<[(ay_0+y_0+y_1, ay_1+y_0+y_1)] \\
[(x_0(a+1)+x_1, x_1(a+1)+x_0)]<[(y_0(a+1)+y_1, y_1(a+1)+y_0)] \\
[(x_0,x_1)][(a+1,1)]<[(y_0,y_1)][(a+1,1)] \\
xz<yz
\end{gathered}$$

□

proof of (14). Suppose that $\hat{0}=\hat{1}$. So $[(1,1)]=[(1+1,1)]$. By definition, this means $1+1=1+(1+1)$. Applying associativity with the law of cancellation for addition, we get $1=1+1$. This contradicts peano axioms. Hence, $\hat{1}\neq\hat{0}$. □