Title of the Document

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Contents

1 Exercises

1. Let (X, d) be a metric space and $S \subseteq X$. Show that $\overline{S}^{int} = \varnothing$.
2. Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $E = \{(x, y) \mid (x-a)^2 + (y-b)^2 \leq r^2\}$ is a bounded set in $\mathbb{R}^2$.
3. Let $f : X \to \mathbb{R}$ be a metric space and for $x, y \in X$. Show that if $d(x, y) < \varepsilon$ for every $\varepsilon > 0$, then $x = y$.

Exercise 2.1. (i) Assume $\overline{S}^{int} \neq \varnothing$. Then $\exists x \in \overline{S}^{int}, \exists r > 0$ such that $B_r(x) \subseteq S^{int}$. However, by $x \notin S$, this value of $r > 0$ implies $B_r(x) \cap S^c \neq \varnothing \Rightarrow B_r(x) \nsubseteq S$, which is a contradiction. Hence, $x \notin S \Rightarrow S^{int} = \varnothing$. □

Exercise 2.2. (ii) A set S is bounded iff $\exists M \in \mathbb{R}^+ \; \forall x, y \in S \; d(x, y) \leq M$. Let $a, b, r \in \mathbb{R}$. So $E = \{(x, y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 \leq r^2\}$.

$$\begin{aligned}
&\Rightarrow x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2 \\
&\Rightarrow x^2 - 2ax + y^2 - 2yb \leq r^2 - a^2 - b^2 \\
&\Rightarrow x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb
\end{aligned}$$

We need to show x^2 is bounded.

$$\begin{aligned}
&(x-a)^2 \leq r^2 \\
&\Rightarrow |x - a| \leq |r| \\
&\Rightarrow |x - a| + a \leq |r| + |a| \\
&\Rightarrow |x| \leq |x - a| + a \leq |x - a| + |a| \leq R + |a|
\end{aligned}$$

Thus x^2 is bounded. □

$\Rightarrow |y| \leq r + |a| \Rightarrow x^2 \leq (r + |a|)^2$ Same for y_i: $y_i^2 \leq (r + |b|)^2$ $\forall z = (x, y) \in D_{r,(a,b)}$ $\|z\| = \sqrt{x^2 + y^2} \leq \sqrt{(r + |a|)^2 + (r + |b|)^2}$ Thus if $M = \sqrt{(r + |a|)^2 + (r + |b|)^2}$, the bond holds.

15 named boundedness = distance boundedness.

Let $x = (x_1, x_2), y = (y_1, y_2) \in D_{r,a,b}$

$z_i \in \{x_i, y_i\}$ $(x_2 - a)^2 + (y_2 - b)^2 = r^2 \Rightarrow d(z_i, (a, b)) = \sqrt{(x_2 - a)^2 + (y_2 - b)^2} \leq r \Rightarrow d(x, y) \leq d(x_1, (a, b)) + d(y_1, (a, b)) = \sqrt{(x_1 - a)^2 + (x_1 - b)^2} + \sqrt{(y_1 - a)^2 + (y_1 - b)^2} \leq r + r = 2r$

(iii) Suppose that $x \neq y$. Then $d(x, y) \neq 0$. Thus if we choose $\epsilon = d(x, y) \implies \epsilon > 0$ but $d(x, y) \geq \epsilon$ (contradiction).

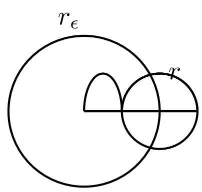
(contradiction) Suppose $x \neq y$ and so $d(x, y) \neq 0$.

Choose $\epsilon > 0$ so that $\epsilon = \frac{d(x,y)}{2}$. Then we must have $d(x, y) < \epsilon = \frac{d(x,y)}{2}$, which is a contradiction, as this implies if $d(x, y) \leq \frac{d(x,y)}{2} \implies d(x, y) = \epsilon = \frac{d(x,y)}{2}$

$\implies d(x, y) = \frac{d(x,y)}{2} \implies 2\epsilon \leq \epsilon$

Thus, $x = y$.

(iv) Let $(V, \|.\|)$ be a normed vsp. Then let $r > 0$ and $x \in V$. Then $B_r(x) = \{y \in V | d(x, y) < r\}$ $B_{\epsilon+r}(0) = \{y \in V | d(0, y) < r + \|x\|\}$



Let $y \in B_r(x)$. $d(0, y) \leq d(0, x) + d(x, y) \leq \|x\| + r \implies B_r(x) \leq B_{\epsilon+r}(0)$

(v) Suppose $\mathcal{S}$ is bounded. Then $\exists M \in \mathbb{R}$ such that $\forall x \in \mathcal{S} \|x\| \leq M$. (Equivalent to $\exists M \in \mathbb{R} : \forall x \in \mathcal{S} x \in B_M(0)$