Title of the Document

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1 Exercises

2.1. Let (X, d) be a metric space and $S \subset X$. Show that $\overline{S}^{\circ} = \varnothing$.

2.2. Show that for an arbitrary choice of $a, b, c \in \mathbb{R}$, the closed disk $(x-a)^2 + (y-b)^2 \leq r^2$ is a bounded set in $\mathbb{R}^2$.

2.3. Let (X, d) be a metric space and let $x, y \in X$. Show that if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

(2.1) Assume $S \neq \varnothing \Rightarrow \exists\, x \in S, \exists\, \epsilon > 0 \,.\, B_\epsilon(x) \subseteq S^\circ$.

Then by $x \in S^{**}$, $\exists\, \epsilon > 0 : B_{\frac{\epsilon}{2}}(x) \subseteq S_\eta$.

However, by $x \in \partial S$, this value of $\epsilon > 0$ implies

$B_{\frac{\epsilon}{2}}(x) \cap S^\nu \neq \varnothing \Rightarrow B_{\frac{\epsilon}{2}}(x) \nsubseteq S_\eta$

which is a contradiction, implying our assumption that $x \notin \overline{S}^s \cap S^{int}$ must be false and

$\overline{S}^\circ \cap S^{int} = \varnothing$.

(2.2) **A set** $\mathcal{S}$ **is bounded iff** $\exists\, M \in \mathbb{R}^+ : \forall x, y \in \mathcal{S}\, d(x,y) \leq M$.

Let $a, b, r \in \mathbb{R}$.

$\delta := \left\{(x,y) \in \mathbb{R} \,|\, (x-a)^2 + (y-b)^2 \leq r^2\right\}$

$$\Rightarrow x^2 - 2ax + a^2 + y^2 - 2by + b^2 \leq r^2$$

$$\Rightarrow x^2 - 2ax + y^2 - 2yb + r^2 - a^2 - b^2 \leq$$

$$\Rightarrow x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb$$

we need to show x^2 is bounded.

- $(x-a)^2 \leq r^2$
- $\Rightarrow |x-a| \leq |r|$
- $\Rightarrow |x-a| \leq |r+a| = |r| + |a|$
- $\Rightarrow |x| = |x-a+a| \leq |x-a| + |a| \leq r + |a|$

(diagrams of a circle with radius a and a square of side $2r$, situated around the circle)

$\Rightarrow |y| \leq r + |a|$
$\Rightarrow x^2 \leq (r + |a|)^2$

Same for y$_1$, $y_1^2 \leq (r + |b|)^2$

$\forall z = (x, y) \in D^2_{a,b}$

$\|z\| = \sqrt{x^2 + y_2}$

$\leq \sqrt{(r + |a|)^2 + (r + |b|)^2}$

Thus, if M $= \sqrt{(r + |a|)^2 + (r + |b|)^2}$, the bond holds.

** Normed boundedness = distance boundedness.

Let x = (x$_1, x_2$), $\quad y = (y_1, y_2) \in D_{a,b}$

z$_3 \in \{x, y\}$

(z$_1 - a)^2 + (z_2 - b)^2 \leq r^2$

$\Rightarrow d(z, (a, b)) = \sqrt{(z_1 - a)^2 + (z_2 - b)^2} \leq r$

$\Rightarrow d(x, y) \leq d(x, (a, b)) + d(y, (a, b))$

$= \sqrt{(x_1 - a)^2 + (x_2 - b)^2} + \sqrt{(y_1 - a)^2 + (y_2 - b)^2}$

$\leq r + r = 2r$.

(iii)

Suppose that $x \neq y$. Then $d(x, y) \neq 0$. Thus if we choose $\varepsilon = d(x, y) \Rightarrow \varepsilon > 0$ but $d(x, y) \notin \varepsilon$. (contradiction).

Suppose $x \neq y$ and so $d(x, y) \neq 0$. Choose $\varepsilon > 0$ so that $\varepsilon = d(x, y)$. Then we must have $d(x, y) < \varepsilon = \frac{d(x,y)}{2}$, which is a contradiction, as this implies $\frac{d(x,y)}{2} \leq d(x, y) = \varepsilon, \Rightarrow \varepsilon < \frac{\varepsilon}{2} \Rightarrow 2\varepsilon < \varepsilon$. Thus, $x = y$.

(iv)

Let $(V, \|\cdot\|)$ be a normed vsp. Then let $r > 0$ and $x \in V$. Then $B_r(x) = \{u \in V |\ d(x, u) < r\}$ $B_{r+\|x\|}(0) = \{u \in V |\ d(0, u) < r + \|x\|\}$

Let $y \in B_r(x)$. $d(0, y) \leq d(0, x) + d(x, y) \leq \|x\| + r \Rightarrow B_r(x) \subseteq B_{r+\|x\|}(0)$.

(v)

Suppose S is bounded. Then $\exists M \in \mathbb{R}_{>0} : \forall x \in S \|x\| \leq M$. (Equiv to $\exists M \in \mathbb{R} : \forall x \in V\ x \in B_M(0)$)