Title of the Document

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Contents

1 Exercises

2.1. Let (X, d) be a metric space and $S \subseteq X$. Show that $\partial S = \varnothing \implies S^o = \varnothing$.

2.2. Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $E = \{(x, y)|(x-a)^2 + (y-b)^2 \leq r^2\}$ is in a bounded set in $\mathbb{R}^2$.

2.3. Let (X, d) be a metric space and $x \in X, y \in X$. Show that if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

(1) Assume $\partial S = \varnothing \implies \exists n \in S^c, \exists l \in S^o, \exists x \in \partial S$".

Then, by $x \in S^{int} \forall \epsilon > 0$: $B_\epsilon(x) \subseteq S$.

However, by $x \in S^c$, this value of $\epsilon > 0$ implies $B_{\epsilon^+}(x) \cap S \subset S^c \implies B_{\epsilon^+}(x) \nsubseteq S$ which is a contradiction, implying our assumption that $x \in \partial S \cap S^{int}$ must be false and $\partial S \cap S^{int} = \varnothing$.

(2) A set S is bounded iff $\exists M \in \mathbb{R}^+ : \forall x, y \in S, d(x, y) \leq M$.

Let $a, b, r \in \mathbb{R}$.

$S = \{(x, y) \in \mathbb{R} \mid (x-a)^2 + (y-b)^2 \leq r^2\}$

$\Rightarrow x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2$

$\Rightarrow x^2 - 2ax + y^2 - 2yb \leq r^2 - a^2 - b^2$

$\Rightarrow x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb$

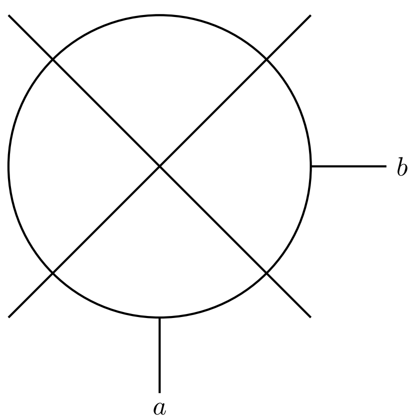
we need to show x^2 is bounded:

$(x-a)^2 \leq r^2$

$\Rightarrow |x-a| \leq |r|$

$\Rightarrow |x-a| \leq |r| = |r| + |a|$

$\Rightarrow |x| = |x-a+a| \leq |x-a| + |a| \leq r + |a|$



$\Rightarrow |y| \leq r + |a| \Rightarrow x^2 \leq (r + |a|)^2$

Same for y, $\quad y^2 \leq (r+|b|)^2$

$\forall z = (x,y) \in D_{r+\max(|a|,|b|)}$

$\|z\| = \sqrt{x^2+y^2} \leq \sqrt{(r+|a|)^2+(r+|b|)^2}$

Thus if $R = \sqrt{(r+|a|)^2+(r+|b|)^2}$, the bound holds.

#15 is normed boundness $\equiv$ distance boundness.

Let $x = (x_1, x_2), y = (y_1, y_2) \in D_{r \to xy}$

$z_0 = (x_1, x_2)$

$(x_2 - a)^2 + (y_2 - b)^2 = r^2$

$\Rightarrow d(x,(a,b)) = \sqrt{(x_1-a)^2+(x_2-b)^2} \leq r$

$\Rightarrow d(x,y) \leq d(x,(a,b)) + d(y,(a,b))$

$= \sqrt{(x_1-a)^2+(x_2-b)^2} + \sqrt{(y_1-a)^2+(y_2-b)^2}$

$\leq r + r = 2r.$

(iii)

Suppose that $x \neq y$. Then $d(x,y) \neq 0$. Thus if we choose $\varepsilon = d(x,y) \Rightarrow \varepsilon > 0$, but $d(x,y) \in \varepsilon$. (contradiction).

(contradiction) Suppose $x \neq y$ and so $d(x,y) \neq 0$. Choose $\varepsilon > 0$ so that $\varepsilon = d(y,x) = \frac{S}{2}$.

$d(x,y) < \varepsilon = d\left(\frac{S}{2}\right),$

which is a contradiction, as this implies if $d(x,y) = S > 0 \Rightarrow d(x,y) = S < \varepsilon = \frac{S}{2}$.

$\Rightarrow S < \frac{S}{2} \Rightarrow 2S < S.$

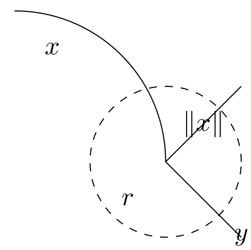
Thus $x = y$.

(iv)

Let $(V, \|\cdot\|)$ be a normed vector space. Then let $r > 0$ and $x \in V$. Then

$B_r'(x) = \{u \in V \mid d(x,u) < r\}$

$B_{r+\|x\|}(0) = \{y \in V \mid d(0,u) < r + \|x\|\}$



Let $y \in B_r(x)$.

$d(0,y) \leq d(0,x) + d(x,y)$

$\leq \|x\| + r$

$\Rightarrow B_r(x) \subseteq B_{r+\|x\|}(0)$.

(v)

Suppose S is bounded. Then $\exists M \in \mathbb{R} : \forall x \in S \; \|x\| \leq M$.

(Equal to $\exists M \in \mathbb{R} : \forall x \in S \; x \in B_{\mathbf{M}}(0)$)