1.3 Properties of Probability Models:

- $P(A) = 1 - P(A^c)$ $[A \cup A^c = \Omega$, $P(A) \cup P(A^c) = P(\Omega) = 1$.

Def'n (Partition): $\{A_i\}_{i=1}^{\infty}$ forms a partition of Ω iff:

- $\bigcup_{i=1}^{\infty} A_i = S$, and
- $\bigcap_{i=1}^{\infty} A_i = \emptyset$

THM 1.3.1 (Law of total probability, Unconditional Version)

Let $\{A_i\}_{i=1}^{\infty}$ form a partition over Ω, Let $B \in \Omega$, then: $P(B) = \sum_{i=1}^{\infty} P(A_i \cap B)$.

Proof: Let $A_i, A_j \in \{A_i\}_{i=1}^{\infty}$. Then $A_i \cap A_j = \emptyset$.

Consider $B \subseteq \Omega$. Then there exists a countable collection $\{A_i\}_{i=k}^{k'}$ so that $\bigcup_{i=k}^{k'} A_i \supseteq B$.

Moreover, $x \in A_i \cap B$ means that $x \notin A_j$, thus $(A_i \cap B) \cap (A_j \cap B) = \emptyset$.

Thus $B = \bigcup_{i=1}^{\infty} (A_i \cap B) \Rightarrow P(B) = \sum_{i=1}^{\infty} (A_i \cap B)$ #check

THM 1.3.2 Let A an $B \in \Omega : A \supseteq B$. Then

- $P(A) = P(B) + P(A \cap B^c)$

$\Rightarrow$ Let $x \in A$. Then $B \subseteq A$ implies $x \in A \cap B$ or $x \in A \cap B^c \overset{\text{disjoint}}{\Rightarrow} x \in B \cup (A \cap B)$ $[A \cap B = B]$.

$\Rightarrow P(A) = P(B) + P(A \cap B^c)$

$\Leftarrow$ Let $x \in B \cup (A \cap B^c)$. Then $x \in B$ or

$(x \in A$ and x is in $B^c)$ These are disjoint, and $x \in B \Rightarrow x \in B \cap A$ $(B \subseteq A)$. Obviously $(B \cap A) \cup (A \cap B^c) = A$, we can apply LTP(UC)

$$P(A) = P(A \cap B) + P(A \cap B^c) = P(B) + P(A \cap B^c).$$

Corollary 1.3.1 (Monotonicity) Let $A, B \subseteq \Omega$, $A \supseteq B$. Then $P(A) \geq P(B)$.

$$P(A) = P(A \cap B) + P(A \cap B^c).$$
$$= P(B) + P(A \cap B^c).$$

Since $P(A \cap B^c) \geq 0$, it holds.

Corollary 1.3.2 Let $A, B \subseteq \Omega \mid A \supseteq B$

$$P(A \cap B^c) = P(A) - P(B)$$

$$P(A) = P(A \cap B) + P(A \cap B^c)$$
$$= P(B) + P(A \cap B^c)$$
$$\Rightarrow P(A \cap B^c) = P(A) - P(B).$$

THM 1.3.3 (PIE, Two Events)

- $P(A \cup B) = P(A) + P(B) - P(A \cap B).$

$$A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$$

Notice that these are all disjoint

and, $(A \cap B) \cup (A \cap B^c) = A$,
$(A \cap B) \cup (A^c \cap B) = B$

Recall by Corollary 1.3.2, $P(A \cap B^c) = P(A) - P(B)$

* Replacing B w/ $A\cap B$ ($A\cap B \subseteq A$), we have
$$P(A\cap B^c) = P(A\cap (A\cap B)^c) = P(A) - P(A\cap B)$$
Similarly,
$$P(A^c\cap B) = P(B\cap (A\cap B)^c) = P(B) - P(A\cap B).$$
Thus, We can use LTP(UC):
$$P(A\cup B) = P(A\cap B) + P(A^c\cap B) + P(A\cap B^c)$$
$$= P(A\cap B) + (P(B) - P(A\cap B)) + (P(A) - P(A\cap B))$$
$$= P(A\cap B) + P(B) - P(A\cap B) + P(A) - P(A\cap B)$$
$$= P(A) + P(B) - P(A\cap B). \quad \square \quad \checkmark$$

THM 1.3.4 (Subadditivity). Let $A_1, A_2, \ldots$ be finite or countable, not necessarily disjoint. Then
$$P(A_1 \cup A_2 \cup \ldots) \le P(A_1) + P(A_2) + \ldots$$
Well if $A_1, A_2, \ldots$ are disjoint, then equality is trivial.
We now show that if there's a pair of non-disjoint events, that $\le$ occurs, and wlog can be applied to * $\bigcup_{i=1}^{\infty} A_i$ *.
Suppose $A_i \cap A_j \neq \emptyset$. Then $P(A_i\cap A_j) \ge 0$.
Thus $P(A_i \cup A_j) = P(A_i) + P(A_j) - P(A_i\cap A_j)$. By PIE
Hence, $P(A_1 \cup A_2 \cup \ldots \cup A_i \cup \ldots \cup A_j \cup \ldots)$
$$= P(A_1) + P(A_2) + \ldots + P(A_i \cup A_j) + \ldots \;{}^*$$
* Assuming rest are disjoint,
$$= P(A_1) + P(A_2) + \ldots + (P(A_i) + P(A_j) - P(A_i\cap A_j))$$
$$+ \ldots$$

* Replacing B w/ $A\cap B$ $(A\cap B \subseteq A)$, we have

$$P(A\cap B^c) = P(A\cap (A\cap B)^c) = P(A) - P(A\cap B)$$

Similarly,

$$P(A^c\cap B) = P(B\cap (A\cap B)^c) = P(B) - P(A\cap B).$$

Thus, We can use LTP(UC):

$$\begin{aligned} P(A\cup B) &= P(A\cap B) + P(A^c\cap B) + P(A\cap B^c) \\ &= P(A\cap B) + (P(B) - P(A\cap B)) + (P(A) - P(A\cap B)) \\ &= P(A\cap B) + P(B) - P(A\cap B) + P(A) - P(A\cap B) \\ &= P(A) + P(B) - P(A\cap B). \;\square \;\checkmark \end{aligned}$$

THM 1.3.4 (Subadditivity). Let $A_1, A_2, \ldots$ be finite or countable, not necessarily disjoint. Then

$$P(A_1 \cup A_2 \cup \ldots) \leq P(A_1) + P(A_2) + \ldots$$

Well if $A_1, A_2, \ldots$ are disjoint, then equality is trivial.

We now show that if there's a pair of non-disjoint events, that $\leq$ occurs, and wlog can be applied to * $\bigcup_{i=1}^{\infty} A_i$ *.

Suppose $A_i \cap A_j \neq \emptyset$. Then $P(A_i\cap A_j) \geq 0$.

Thus $P(A_i \cup A_j) = P(A_i) + P(A_j) - P(A_i\cap A_j)$. By PIE

Hence, $P(A_1 \cup A_2 \cup \cdots \cup A_i \cup \cdots \cup A_j \cup \cdots)$

$$= P(A_1) + P(A_2) + \ldots + P(A_i \cup A_j) + \ldots \; *$$

* Assuming rest are disjoint,

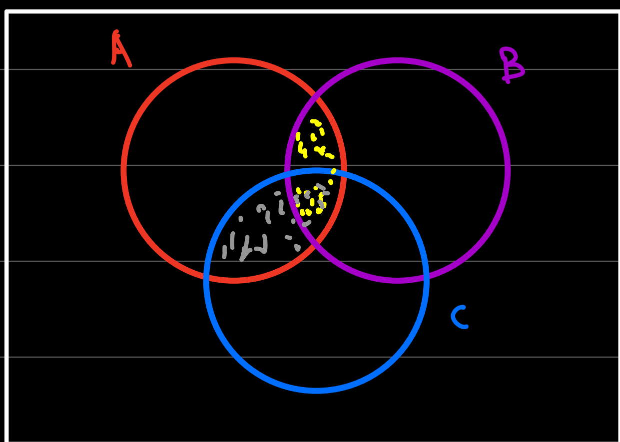
$$= P(A_1) + P(A_2) + \ldots + (P(A_i) + P(A_j) - P(A_i\cap A_j)) + \ldots$$

$\le \sum_{i=1}^{\infty} A_i$, as required. *need to check.

CHALLENGE: (PIE, 3-Events)

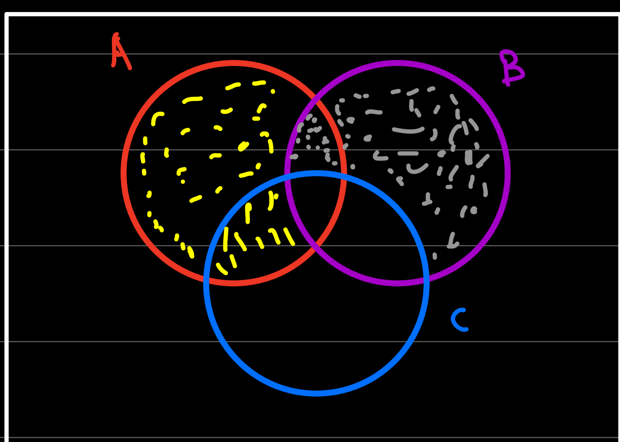
Let $A, B, C \subseteq \Omega$, not necessarily disjoint.

Then $A \cup B \cup C \overset{A}{=} \big[(A\cap B) \cup (A \cap B^c) \cup (A \cap C^c)$
$\cup (A \cap C)\big] \overset{B}{\cup} \big[(A \cap B) \cup (A^c \cap B) \cup (C^c \cap B)$
$\cup (B \cap C)\big] \overset{C}{\cup} \big[(B\cap C) \cup (B^c \cap C) \cup (A \cap C) \cup (A^c \cap C)\big]$



$A \cap B$

$A \cap C$



$A \cap B^c$

$A \cap C^c$

$P(A \cap (C \cap A \cup B \cap A)^c) = P(A) - P((C \cap A) \cup (B \cap A))$

By PIE for 2 events:

$$P\big((C \cap A) \cup (B \cap A)\big)$$
$$= P(A \cap C) + P(A \cap B) - P\big((A \cap C) \cap (A \cap B)\big)$$
$$= P(A \cap C) + P(A \cap B) - P(A \cap B \cap C).$$

Thus,