Title of the Document

Author Name

June 22, 2024

Contents

1 Exercises

1. Let (X, d) be a metric space and $S \subset X$. Show that $\partial S \subset \overline{S} \cap \overline{S^c}$.
2. Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $D = \{(x, y) \mid (x-a)^2 + (y-b)^2 \leq r^2\}$ is a bounded set in $\mathbb{R}^2$.
3. Let (X, d) be a metric space and for $x, y \in X$. Show that if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

(i)

Assume $\partial S \subseteq \overline{S} \cap \overline{S^c}$.

Then by $x \in \partial S \implies \forall \epsilon > 0 : B_\epsilon(x) \cap S^c \neq \emptyset$.

However, by $x \in \partial S$, this value of ϵ implies

$B_\epsilon(x) \cap S \neq \emptyset \Rightarrow B_\epsilon(x) \nsubseteq S$

which is a contradiction, implying our assumption that $x \in \partial S \cap S'$ must be false and

$\partial S \cap S' = \emptyset$

(ii)

A set S is bounded if and only if $\exists M \in \mathbb{R}^+ \forall x, y \in S$

$d(x, y) \leq M$

Let $a, b, r \in \mathbb{R}$.

$S = \{(x, y) \mid (x-a)^2 + (y-b)^2 \leq r^2\}$

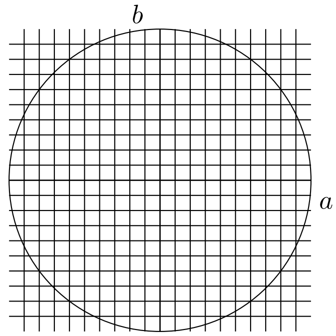
$$\begin{aligned}
&\implies x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2 \\
&\implies x^2 - 2ax + y^2 - 2yb + r^2 - a^2 - b^2 \\
&\implies x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb
\end{aligned}$$

We need to show x^2 is bounded:

$$\begin{aligned}
&(x-a)^2 \leq r^2 \\
&\implies |x-a| \leq |r| \\
&\implies |x-a| \leq |r| \implies |x| \leq |r| + |a| \\
&\implies |x| = |x + a - a| \leq |x-a| + |a| \\
&\implies |x| \leq |r| + |a|
\end{aligned}$$

(iii)

A set S is bounded if it is contained within some ball, i.e., $\exists M \in \mathbb{R}^+ \ \forall x, y \in S \ \ d(x, y) \leq M$



$\Rightarrow \quad |y| \leq r + |a| \quad \Rightarrow \quad x^2 \leq (r+|a|)^2$

Same for y, $\quad$ y$^2 \leq (r+|b|)^2$

$\forall z = (x,y) \in D_{r,a,b}, \quad \|z\| = \sqrt{x^2+y^2} \leq \sqrt{(r+|a|)^2+(r+|b|)^2}$

Thus, if $\mathcal{M} = \sqrt{(r+|a|)^2+(r+|b|)^2}$, the bound holds.

IS normed boundless = distance boundless.

Let $x = (x_1, x_2), y = (y_1, y_2) \in D_{r,a,b}$

$z_2 \in \{x, y\} \quad (x_2-a)^2 + (x_2-b)^2 = r^2$

$\Rightarrow d(z_1, (a,b)) = \sqrt{(x_2-a)^2+(x_2-b)^2} \leq r$

$\Rightarrow d(x,y) \leq d(x,(a,b)) + d(y,(a,b))$

$= \sqrt{(x_1-a)^2+(x_1-b)^2} + \sqrt{(y_1-a)^2+(y_1-b)^2}$

$\leq r + r = 2r.$

(iii) Suppose that $x \neq y$. Then $d(x,y) \neq 0$. Thus if we choose $\epsilon = d(x,y) \Rightarrow \epsilon > 0$ but $d(x,y) \notin \epsilon$. (contradiction).

(contradiction) Suppose $x \neq y$ and so $d(x,y) \neq 0$. Choose $\epsilon > 0$ so that $\epsilon = d(x,y)$. Then we must have $d(x,y) < \epsilon = d\left(\frac{d(x,y)}{2}\right) = \frac{d(x,y)}{2}$ which is a contradiction, as this implies $d(x,y) \leq \frac{d(x,y)}{2} \Rightarrow d(x,y) = s < \epsilon = \frac{s}{2} \Rightarrow s = s < \frac{s}{2} \Rightarrow 2s < s$. Thus $x = y$.

(iv) Let $(V, \|\cdot\|)$ be a normed vsp. Then let $r > 0$ and $x \in V$. Then $B_r(x) = \{u \in V \mid d(x,u) < r\}$ $B_{r+\|x\|}(0) = \{v \in V \mid d(0,v) < r + \|x\|\}$

Let $y \in B_r(x)$.
$d(0,y) \leq d(0,x) + d(x,y)$ $d(0,y) \leq \|x\| + r$
$\Rightarrow B_r(x) \subseteq B_{r+\|x\|}(0)$

(v) Suppose S is bounded. Then $\exists M \in \mathbb{R} : \forall x \in S \|x\| \leq M$.

(Equiv to $\exists M > 0 : \forall x \in V)$ $x \in B_M(0)$