Title of the Document

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Contents

1 Exercises

1. Let (X, d) be a metric space and $S \subseteq X$. Show that $\partial S = \emptyset$ if and only if S is both open and closed.
2. Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $(x-a)^2 + (y-b)^2 \leq r^2$ is in a bounded set in $\mathbb{R}^2$.
3. Let (X, d) be a metric space and for $x, y \in X$. Show that if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

(i) Assume $S \neq \emptyset$. Then $\exists x \in S$, such that $x \notin \partial S^*$. Then $x \in S^{int}$ and there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq S$.

However, by $x \notin \partial S$, this value of $\epsilon > 0$ implies $B_\epsilon(x) \cap S = \emptyset$, which is a contradiction, implying our assumption that $x \in S \cap S^{int}$ must be false and $\partial S \cap S^{int} = \emptyset$.

(iii) A set S is bounded iff $\exists M > 0$, such that $d(x, y) \leq M$, $\forall x, y \in S$

Let $a, b, r \in \mathbb{R}$.

$\delta = \{(x, y) \in \mathbb{R} | (x-a)^2 + (y-b)^2 \leq r^2\} \implies x^2 - 2ax + a^2 + y^2 - 2by + b^2 \leq r^2$

$\implies x^2 - 2ax + y^2 - 2by \leq r^2 - a^2 - b^2 \implies x^2 - 2ax + y^2 - 2by \leq r^2 - a^2 - b^2 + 2ax + 2by$

$\implies x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2by$

Need to show x^2 is bounded,

$(x-a)^2 \leq r^2 \implies |x-a| \leq |r| \implies |x-a| \leq |r| + |a|$

$\implies |x| = |x-a+a| \leq |x-a| + |a| \leq |r| + |a|$

$\implies |x| \leq |x-a+a| \leq |x-a| + |a| \leq |r| + |a|$

$$\Rightarrow |y| \leq r + |a|$$
$$\Rightarrow y^2 \leq (r + |a|)^2$$

Same for x, $x^2 \leq (r + |b|)^2$

$\forall z = (x, y) \in D_{r,(a,b)}$

$\|z\| = \sqrt{x^2 + y^2} \leq \sqrt{(r + |a|)^2 + (r + |b|)^2}$

Thus if $\sqrt{(r + |a|)^2 + (r + |b|)^2}$, the bound holds.

#IS named boundless = distance boundedness.

Let $x = (x_1, x_2), y = (y_1, y_2) \in D_{r,(a,b)}$

$z_i \in \{x_i, y_i\}$

$(x_i - a)^2 + (x_i - b)^2 = r^2$

$\Rightarrow d((x_i, b)) = \sqrt{(x_i - a)^2 + (x_i - b)^2} \leq r$

$\Rightarrow d((x, y)) \leq d((x, (a, b))) + d(y, (a, b))$

$= \sqrt{(x_i - a)^2 + (x_i - b)^2} + \sqrt{(y_i - a)^2 + (y_i - b)^2} \leq r + r = 2r$

(iii) **Suppose that $x \neq y$. Then $d(x, y) \neq 0$. Thus if we choose $\epsilon = d(x, y) \Rightarrow \epsilon > 0$ but $d(x, y) \notin \epsilon$. (contradiction).**

(contradiction) Suppose $x \neq y$ and so $d(x, y) \neq 0$.

Choose $\epsilon > 0$ so that $\epsilon = d(x, y)$. Then we must have $d(x, y) < \epsilon = \frac{d(x,y)}{2}$, which is a contradiction, as this implies $d(x, y) = \frac{d(x,y)}{2}$

If $d(x, y) < \frac{d(x,y)}{2}$

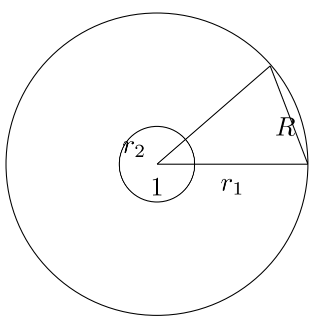
So $d(x, y) \leq s\epsilon = \frac{\epsilon}{2}$

Thus $\frac{\epsilon}{2} < \epsilon$

(iv)

Let $(V, || \cdot ||)$ be a normed vsp.

Then let $r > 0$ and $x \in V$. Then $B_r(x) = \{u \in V \mid d(x, u) < r\}$ $B_{r+||x||}(0) = \{v \in V \mid d(0, u) < r + ||x||\}$



Let $y \in B_r(x)$

$d(0, y) \leq d(0, x) + d(x, y)$

$\leq ||x|| + r$

$\Rightarrow \mathrm{B}_r(x) \subseteq B_{r+||x||}(0)$

(v) Suppose S is bounded. Then $\exists M \in \mathbb{R}$ such that $\forall x \in S ||x|| \leq M$

$(Equalto \exists M \in \mathbb{R} : \forall x \in V) \in S \subseteq B_M(0)$