Title of the Document

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Contents

1 Exercises

2.1

Let (X,d) be a metric space and $S \subset X$. Show that $\partial S = \emptyset$ if and only if $S = \emptyset$ or $S = X$.

2.2

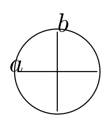
Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $\{(x,y) \mid (x-a)^2 + (y-b)^2 \leq r^2\}$ is in a bounded set in $\mathbb{R}^2$.

2.3

Let (X,d) be a metric space and $x, y \in X$. Show that if $d(x,y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

Proof of 2.1. Assume $S \neq \emptyset$. Then $\exists x \in S$ and $\exists \epsilon > 0 : B_\epsilon(x) \neq \emptyset$. Then by $x \in S^{\text{int}}, \exists \epsilon > 0 : B_\epsilon(x) \subseteq S$. However, by $x \in \partial S,$ this value of $\epsilon > 0$ implies $B_{\epsilon/2}(x) \cap S^c \neq \emptyset \implies B_{\epsilon/2}(x) \nsubseteq S$, which is a contradiction, implying our assumption that $x \in \partial S \cap S^{\text{int}}$ must be false and $\partial S \cap S^{\text{int}} = \emptyset$. $\square$

Proof of 2.2. A set S is bounded if and only if $\exists M \in \mathbb{R}^+ : \forall x, y \in S, d(x,y) \leq M$. Let $a, b, r \in \mathbb{R}$. $S := \{(x,y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 \leq r^2\} \implies x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2 \implies x^2 - 2ax + y^2 - 2yb \leq r^2 - a^2 - b^2 \implies x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb$ need to show x^2 is bounded $(x-a)^2 \leq r^2 \implies |x-a| \leq |r|$ $\implies |x-a|(|x+a| \leq |r| + |a| \implies |x| = |x-a+a| \leq |r| + |a|$ $\square$



$\Rightarrow |y| \leq r + |a|$

$\Rightarrow y^2 \leq (r + |a|)^2$

Same for y_2 , $y_2^2 \leq (r + |b|)^2$

$\forall z = (x,y) \in D_{r+|a|}$

$\|z\| = \sqrt{x^2 + y^2}$

$\leq \sqrt{(r+|a|)^2 + (r+|b|)^2}$

Thus if $\mathrm{M} = \sqrt{(r+|a|)^2 + (r+|b|)^2}$, the bound holds.

#15 Normed boundness = distance boundness.

Let $\mathrm{x} = (\mathrm{x}_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$

$\mathrm{z}_i \in [x, y]^i$

$(\mathrm{x}_2 - a)^2 + (x_2 - b)^2 = r^2$

$\Rightarrow d(x, (a,b)) = \sqrt{(x_1 - a)^2 + (x_2 - b)^2} \leq r$

$\Rightarrow d(x, y) \leq d(x, (a,b)) + d(y, (a,b))$

$= \sqrt{(x_1 - a)^2 + (x_2 - b)^2} + \sqrt{(y_1 - a)^2 + (y_2 - b)^2}$

$\leq r + r = 2r.$

(iii) Suppose that $x \neq y$. Then $d(x,y) \neq 0$. Thus if we choose $\epsilon = d(x,y)$, $\epsilon > 0$ but $d(x,y) \notin \epsilon$ (contradiction).

(contradiction) Suppose $x \neq y$ and so $d(x,y) = 0$.

Choose $\epsilon > 0$ such that $\epsilon = d(x,y)$. $d(x,y) < \epsilon = \frac{\epsilon}{2}$, which is a contradiction, as this implies if $d(x,y) \leq \frac{\epsilon}{2} \leq \epsilon = \frac{\epsilon}{2} \Rightarrow \epsilon > \frac{\epsilon}{2}, \Rightarrow 2\epsilon < \epsilon$. Thus $x = y$.

(iv) Let $(V, \|\cdot\|)$ be a normed vector space. Then let $r > 0$ and $x \in V$. Then $B_r^V(x) = \{u \in V | d(x,u) < r\}$ $B_{r+\|x\|}(0) = \{u \in V | d(0,u) < r + \|x\|\}$

Let $y \in B_r^V(x)$.

$d(0,y) \leq d(0,x) + d(x,y) \leq \|x\| + r$

$\Rightarrow B_r(x) \subseteq B_{r+\|x\|}(0)$.

(v) Suppose S is bounded. Then $\exists M : \forall x \in S, \|x\| \leq M$. (Equal to $\exists M \in \mathbb{R} : \forall x \in V, x \in B_M^V(0)$)