Title of the Document

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Contents

1 Exercises

1. Let (X, d) be a metric space and $S \subseteq X$. Show that $\overline{S}^0 \subseteq S^0 = \emptyset$.
2. Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $(x-a)^2 + (y-b)^2 \leq r^2$ is a bounded set in $\mathbb{R}^2$.
3. Let (X, d) be a metric space and let $x, y \in X$. Show that if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

Proof. (i) Assume $\exists \epsilon > 0, \exists x \in \overline{S}^0, x \notin S^0$.

Then by $x \in \overline{S}^0$, $\exists \epsilon > 0 : B_\epsilon(x) \subseteq S$.

However, by $x \notin S^0$, this value of $\epsilon > 0$ implies $B_{\epsilon/2}(x) \cap S^c \neq \emptyset \Rightarrow B_{\epsilon/2}(x) \nsubseteq S$, which is a contradiction.

This implies our assumption that $x \in \overline{S}^0 \cap S^0$ must be false and $\overline{S}^0 \cap S^0 = \emptyset$. □

Proof. (ii) A set S is bounded if and only if $\exists M \in \mathbb{R}^+ \ \forall x, y \in S$, $d(x, y) < M$.

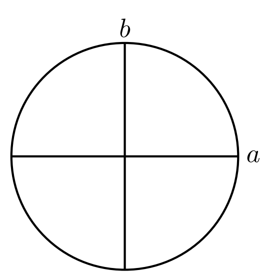
Let $a, b, r \in \mathbb{R}$. $S := \{(x, y) \in \mathbb{R}\}[(x-a)^2 + (y-b)^2 \leq r^2]$

$$\begin{aligned}
&\Rightarrow x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2 \\
&\Rightarrow x^2 - 2ax + y^2 - 2yb \leq r^2 - a^2 - b^2 \\
&\Rightarrow x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb
\end{aligned}$$

We need to show x^2 is bounded.

- $(x-a)^2 \leq r^2 \Rightarrow |x-a| \leq |r| \Rightarrow |x-a| \leq |r| + |a| \Rightarrow |x| = |x-a+a| \leq |x-a| + |a| \leq r + a$

□



$\Rightarrow |y| \leq r + |a|$

$\Rightarrow \ x^2 \leq (r + |a|)^2$

Same for y, $\quad y^2 \leq (r + |b|)^2$

$\forall z = (x, y) \in D^2_{a,b}$

$\|z\| = \sqrt{x^2+y^2} \leq \sqrt{(r+|a|)^2+(r+|b|)^2}$

Thus, if $\quad \mathrm{M} = \sqrt{(r+|a|)^2+(r+|b|)^2}, \quad$ the bound holds.

IS boundedness = distance boundedness:

Let $\quad \mathrm{x} = (\mathrm{x}_1, x_2), \quad y = (y_1, y_2) \in D^2_{a,b}, \quad z \in \{x, y\}$

$(\mathrm{x}_2 - a)^2 + (x_2 - b)^2 = r^2 \Rightarrow d(z,(a,b)) = \sqrt{(z_1-a)^2+(z_2-b)^2} \leq r$

$\Rightarrow d(x,y) \leq d(x,(a,b)) + d(y,(a,b)) = \sqrt{(x_1-a)^2+(x_2-b)^2} + \sqrt{(y_1-a)^2+(y_2-b)^2} \leq r + r = 2r.$

(iii) Suppose that $x \neq y$. Then $d(x,y) \neq 0$. Thus if we choose $\varepsilon = d(x,y) \implies \varepsilon > 0$ but $d(x,y) \notin \varepsilon$. (contradiction).

(contradiction) Suppose $x \neq y$ and so $d(x,y) \neq 0$. Choose $\varepsilon > 0$ so that $\varepsilon = d(x,y)$. Then we must have $d(x,y) \leq \varepsilon = d(\frac{x+y}{2})$, which is a contradiction, as this implies if $d(x,y) > 0$ then $d(x,y) = \epsilon = \frac{\epsilon}{2}$. Thus,
$\epsilon > 2\frac{\epsilon}{2} \implies 2\epsilon \leq \epsilon$
Thus, $x = y$.

(iv) Let $(V, \|\cdot\|)$ be a normed vector space. Then let $r > 0$ and $x \in V$. Then
$B_r(x) = \{u \in V | d(x,u) < r\}$
$B_{\epsilon+\|u\|}(0) = \{v \in V | d(0,u) < r + \|u\|\}$
Let $y \in B_r(x) \implies d(o,y) \leq d(o,x) + d(x,y)$
$\leq \|x\| + r \implies B_r(x) \subseteq B_{r+\|x\|}(0)$

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(v) Suppose S is bounded. Then $\exists M \in \mathbb{R} > 0 : \forall x \in S \|x\| \leq M$.
(Equivalent to $\exists M \in \mathbb{R} : \forall x \in V) x \in B_M(0)$