Title of the Document

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1 Exercises

2-1. Let (X, d) be a metric space and $S \subset X$. Show that $\overline{S} = S^*$ iff $S = \overline{S} \cap S^{int} = \emptyset$.
2-2. Show that for an arbitrary choice of $a, b, c \in \mathbb{R}$, the closed disk $(x-a)^2 + (y-b)^2 \leq r^2$ is a bounded set in $\mathbb{R}^2$.
2-3. Let (X, d) be a metric space and for $x, y \in X$. Show that if $d(x, y) < \varepsilon$ for every $\varepsilon > 0$, then $x = y$.

Solution 2-1. Assume $S \neq \overline{S} \cap S^{int}$.
Then $\exists x \in S^{int} : x \in \overline{S} \cap x \notin S^{int}$.
Then by $x \in S^{int} \Rightarrow \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq S^1$.
However, by $x \notin S^{int}$, this value of $\varepsilon > 0$ implies $B_{\frac{\varepsilon}{4}}(x) \cap S^1 = \emptyset \Rightarrow B_{\frac{\varepsilon}{4}}(x) \nsubseteq S$, which is a contradiction, implying our assumption that $x \in \overline{S} \cap S^{int}$ must be false and $\overline{S} \cap S^{int} = \emptyset$. □

Solution 2-2. A set S is bounded iff $\exists M \in \mathbb{R}^+ : \forall x, y \in S.\ d(x, y) \leq M$.

Let $a, b, r \in \mathbb{R}$. $S := \{(x, y) \in \mathbb{R}^2 | (x-a)^2 + (y-b)^2 \leq r^2\} \Rightarrow x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2$ $\Rightarrow x^2 - 2ax + y^2 - 2yb \leq r^2 - a^2 - b^2 \Rightarrow x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb$

Need to show x^2 is bounded: $(x-a)^2 \leq r^2 \Rightarrow |x-a| \leq |r| \Rightarrow |x-a| \leq |r| + |a| \Rightarrow |x| = |x-a+a| \leq |x-a| + |a| \leq r + |a|$.

□

$$\begin{aligned} &\implies |y| \leq r + |a| \\ &\implies x^2 \leq (r + |a|)^2 \end{aligned}$$

Same for y: $\quad$ y$^2 \leq (r + |b|)^2$

$$\begin{aligned} &\forall z = (x, y) \in D_{r,a,b}, \\ &\|z\| = \sqrt{x^2 + y^2} \\ &\leq \sqrt{(r + |a|)^2 + (r + |b|)^2} \end{aligned}$$

Thus, if $\mathcal{M} = \sqrt{(r + |a|)^2 + (r + |b|)^2}$, the bound holds.

#1S normed boundedness = distance boundedness.

Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{P}_{r,a,b}$

$$
\begin{aligned}
z_1 &= (x, y) \\
(z_1 - a)^2 + (z_2 - b)^2 &= r^2 \\
\Rightarrow d(z, (a, b)) &= \sqrt{(z_1 - a)^2 + (z_2 - b)^2} \leq r \\
\Rightarrow d(x, y) &\leq d(x, (a, b)) + d(y, (a, b)) \\
&= \sqrt{(x_1 - a)^2 + (x_2 - b)^2} + \sqrt{(y_1 - a)^2 + (y_2 - b)^2} \\
&\leq r + r = 2r.
\end{aligned}
$$

(iii)

Suppose that $x \neq y$. Then $d(x, y) \neq 0$. Thus if we choose $\varepsilon = d(x, y) \implies \varepsilon > 0$ but $d(x, y) \in \varepsilon$. (contradiction).

(contradiction) Suppose $x \neq y$ and so $d(x, y) \neq 0$.

Choose $\varepsilon > 0$ so that $\varepsilon = d(x, y)$. Then we must have $d(x, y) < \varepsilon = d\left(\frac{\varepsilon}{2}\right)$, which is a contradiction, as this implies $d(x, y) = \frac{\varepsilon}{2}$.

Thus $d(x, y) \leq 0 \implies d(x, y) = 0$ $\varepsilon = d(x, y) = \frac{\varepsilon}{2} \implies x = y$.

$\varepsilon > \frac{\varepsilon}{2} \quad \implies 2s < \varepsilon$

Thus $x = y$.

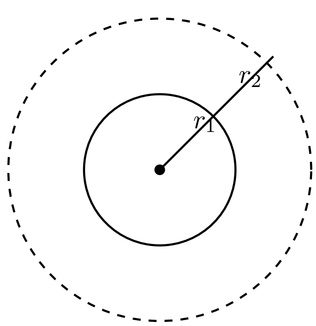
(iv)

Let $(V, \|\cdot\|)$ be a normed vsp.

Then let $r > 0$ and $x \in V$. Then

$B_r(x) = \{v \in V | d(x, v) < r\}$ $B_{\|x\|+r}(0) = \{v \in V | d(0, v) < r + \|x\|\}$

Diagram:



Let $y \in B_r(x)$.

$d(0,y) \leq d(0,x) + d(x,y) \leq \|x\| + r$

$\implies B_r(x) \subseteq B_{r+\|x\|}(0)$.

(v)

Suppose S is bounded. Then $\exists M \in \mathbb{R} : \forall x \in S \|x\| \leq M$.

(Equiv to $\exists M \in \mathbb{R}$:$\forall x \in S \subseteq V \ (x \in B_M(0))$)