2.6 Exercises

2-1. Let (X, d) be a metric space and $S \subseteq X$. Show that $\partial S \cap S^{\text{int}} = \emptyset$.

2-2. Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $(x-a)^2 + (y-b)^2 \leq r^2$ is a bounded set in $\mathbb{R}^2$.

2-3. Let (X, d) be a metric space and fix $\mathbf{x}, \mathbf{y} \in X$. Show that if $d(\mathbf{x}, \mathbf{y}) < \epsilon$ for every $\epsilon > 0$, then $\mathbf{x} = \mathbf{y}$.

(i) Assume f.s.o.c $\exists x \in S \,|\, x \in \partial S \cap S^{\text{int}}$.

Then by $x \in S^{\text{int}}$, $\exists \varepsilon > 0 : B_\varepsilon(x) \subseteq S$.

However, by $x \in \partial S$, this value of $\varepsilon > 0$ implies

$B_\varepsilon(x) \cap S^{\text{ext}} \neq \emptyset \Rightarrow B_\varepsilon(x) \not\subseteq S$ which is a contradiction,

implying our assumption that $x \in \partial S \cap S^{\text{int}}$ must be false and

$\partial S \cap S^{\text{int}} = \emptyset$.

(ii) A set S is bounded iff $\exists M \in \mathbb{R}^+ : \forall x, y \in S$

$d(x,y) < M$

Let $a, b, r \in \mathbb{R}$.

$S := \{ (x,y) \in \mathbb{R}^2 \,|\, (x-a)^2 + (y-b)^2 \leq r^2 \}$

$\Rightarrow x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2$

$\Rightarrow x^2 - 2ax + y^2 - 2yb \leq r^2 - a^2 - b^2$

$\Rightarrow x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb$

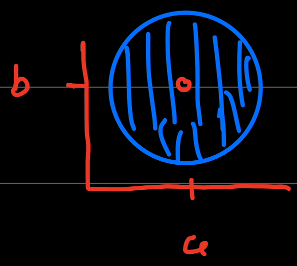
need to show x^2 is bounded

- $(x-a)^2 \leq r^2$

$\Rightarrow |x-a| \leq |r|$

$\Rightarrow |x-a| + |a| \leq |r| + |a|$

$\Rightarrow |x| = |x-a+a| \leq |x-a| + |a| \leq r + |a|$



$\Rightarrow |x| \leq r + |a|$

$\Rightarrow x^2 \leq (r + |a|)^2$

Same for y, $y^2 \leq (r + |b|)^2$

$\forall z = (x, y) \in D^2_{a,b}$

$$\|z\| = \sqrt{x^2 + y^2}$$

$$\leq \sqrt{(r+|a|)^2 + (r+|b|)^2}$$

Thus if $M = \sqrt{(r+|a|)^2 + (r+|b|)^2}$, the bound holds.

* IS normed boundedness = distance boundedness.

Let $x = (x_1, x_2)$, $y = (y_1, y_2) \in D_{[a,b]}$

$z \in \{x, y\}$

$$(z_1 - a)^2 + (z_2 - b)^2 \leq r^2$$

$$\Rightarrow d(z, (a,b)) = \sqrt{(z_1 - a) + (z_2 - b)^2} \leq r$$

$$\Rightarrow d(x, y) \leq d(x, (a,b)) + d(y, (a,b))$$

$$= \sqrt{(x_1 - a)^2 + (x_2 - b^2)} + \sqrt{(y_1 - a)^2 + (y_2 - b)^2}$$

$$\leq r + r = 2r.$$

(iii) Suppose that $x \neq y$. Then $d(x,y) \neq 0$. Thus if we choose $\varepsilon = d(x,y) \Rightarrow \varepsilon > 0$ but $d(x,y) \geq \varepsilon$. (contrapositive).

(contradiction) Suppose $x \neq y$ and so $d(x,y) \neq 0$.

choose $\varepsilon > 0$ so that $\varepsilon = \frac{d(x,y)}{2}$. Then we must have $d(x,y) < \varepsilon = \frac{d(x,y)}{2}$, which is a contradiction, as this implies if $d(x,y) = \delta > 0 \Rightarrow d(x,y) = \delta < \varepsilon = \frac{\delta}{2}$

$$\Rightarrow \delta < \frac{\delta}{2} \Rightarrow 2\delta < \delta.$$

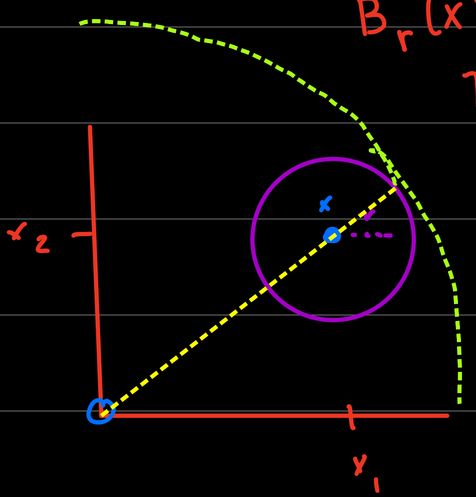
Thus $x = y$.

(iv) Let $(\bar{V}, \|\circ\|)$ be a normed vsp.

Then let $r > 0$ and $x \in \bar{V}$. Then

$$B_r(x) = \{v \in \bar{V} \mid d(x,v) < r\}$$

$$B_{r+\|x\|}(0) = \{v \in \bar{V} \mid d(0,v) < r + \|x\|\}$$



Let $y \in B_r(x)$.

$$d(0,y) \leq d(0,x) + d(x,y)$$
$$\leq \|x\| + r$$
$$\Rightarrow B_r(x) \subseteq B_{r+\|x\|}(0).$$

(v) Suppose S is bounded. Then $\exists M \in \mathbb{R} : \forall x \in S \ \|x\| \leq M$.

(Equiv to $\exists M \in \mathbb{R}^+ : \forall x \in \bar{V}$) $x \in B_M(0)$