

# Title of the Document

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# 1 Interior and Boundary Points

## Definition 2.1

Let  $(X, d)$  be a metric space. If  $x \in X$  and  $r > 0$ , we define the *open ball of radius  $r$  centered at  $x$*  as  $B_r(x) := \{y \in X : d(x, y) < r\}$ .

In  $\mathbb{R}^n$  with the Euclidean metric  $d(x, y) = \|x - y\|$ , the open ball  $B_r(x)$  is nothing more than the collection of points which are a distance at most  $r$  from  $x$ . This generalizes the interval, since in  $\mathbb{R}^1$  we have  $B_r(x) = \{y \in \mathbb{R} : |x - y| < r\} = (x - r, x + r)$ , or if we centre around 0,  $B_r(0) = (-r, r)$ . In  $\mathbb{R}^2$  we get a disk of radius  $r$ ,  $B_r(0) = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < r\}$ , which we recognize as being the same as  $x^2 + y^2 < r^2$ .

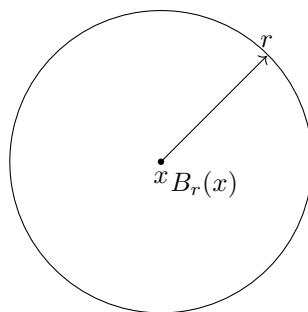


Figure 2.1: In  $\mathbb{R}^2$ , the open ball of radius  $r$  centered at  $x$  consists of all points which are a distance at most  $r$  from  $x$ .

## Definition 2.2

A subspace  $S$  of a metric space  $X$  is *bounded* if there exists an  $r > 0$  and an  $a \in X$  such that  $S \subseteq B_r(a)$ .

## Definition 2.3

Let  $(X, d)$  be a metric space, and  $S \subseteq \mathbb{R}^n$  be an arbitrary set.

1. We say that  $x \in S$  is an *interior point* of  $S$  if there exists an  $r > 0$  such that  $B_r(x) \subseteq S$ ; that is,  $x$  is an interior point if we can enclose it in an open ball which is entirely contained in  $S$ .
2. We say that  $x \in S$  is a *boundary point* of  $S$  if for every  $r > 0$ ,  $B_r(x) \cap S \neq \emptyset$  and  $B_r(x) \cap \partial S \neq \emptyset$ ; that is,  $x$  is a boundary point if no matter what ball we place around  $x$ , that ball lives both inside and outside of  $S$ .

The *interior* of  $S$  – denoted  $S^\circ$  – is the collection of interior points of  $S$ , while *boundary* of  $S$  – denoted  $\partial S$  – is the collection of boundary points of  $S$ .

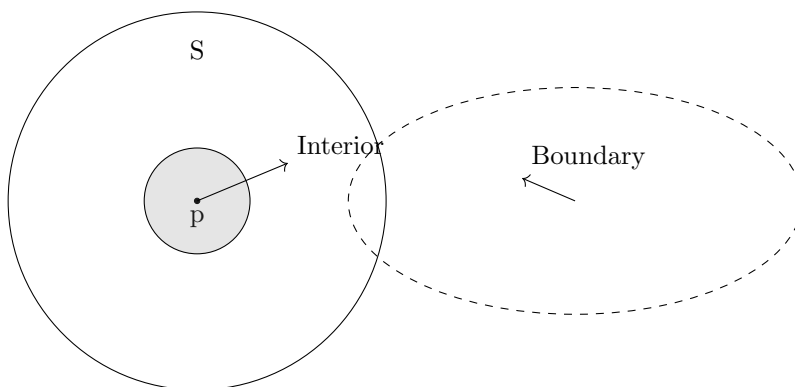


Figure 2.2: The point  $b$  is a boundary point. No matter what size ball we place around  $b$ , that ball will intersect both  $S$  and  $\partial S$ . On the other hand,  $p$  is an interior point, since we can place a ball around it which lives entirely within  $S$ .

We should take a moment and think about these definitions, and why they make sense. A boundary point is any point  $x$  which occurs at the very fringe of the set; that is, if I push a little further I will leave the set. An interior point should be a point inside of  $S$ , such that if I move in any direction a sufficiently small distance, I stay within the set. Note that if  $x$  is an interior point then we must have that  $x \in S$ ; however, boundary points do *not* need to be in the set. We start with a simple example.

#### Example 2.4

Let  $S = (-1, 1) \subseteq \mathbb{R}$  endowed with the Euclidean metric. What are the interior points and the boundary points of  $S$ ?

**Solution.** I claim that any point in  $(-1, 1)$  is an interior point. To see that this is the case, let  $p \in (-1, 1)$  be an arbitrary point. We need to place a ball around  $p$  which lies entirely within  $(-1, 1)$ . To do this, assume without loss of generality that  $p \geq 0$ . If  $p = 0$  then we can set  $r = 1/2$  and  $B_r(p) \subseteq (-1/2, 1/2) \subseteq (-1, 1)$ . Thus assume that  $p \neq 0$  and let  $r = (1 - p)/2$ , which represents half the distance from  $p$  to 1. I claim that  $B_r(p) \subseteq (-1, 1)$ . Indeed, let  $x \in B_r(p)$  be any point, so that  $|x - p| < r$  by definition. Then

$$\begin{aligned} |x| &= |x - p + p| \leq |x - p| + p \\ &\leq r + p = \frac{1-p}{2} + p \\ &= \frac{1+p}{2} < 1 \end{aligned}$$

where in the last inequality we have used the fact that  $p < 1$  so  $1 + p < 2$ . Thus  $x \in (-1, 1)$ , and since  $x$  was arbitrary,  $B_r(p) \subseteq (-1, 1)$ .

The boundary points are  $\pm 1$ , where we note that even though  $-1 \notin (-1, 1)$ , it is still a boundary point. To see that  $-1$  is a boundary point, let  $r > 0$  be arbitrary, so that  $B_r(p) = (-1 - r, 1 + r)$ . We then have  $B_r(p) \cap (-1, 1) = (-1 - r, 1) \neq \emptyset$ ,  $B_r(p) \cap (-1, 1)^c = (-1, 1 + r) \neq \emptyset$  as required. The proof for  $-1$  is analogous and left as an exercise.

#### Example 2.5

What is the boundary of  $\mathbb{Q}$  in  $\mathbb{R}$  with the Euclidean metric?

**Solution.** We claim that  $\partial\mathbb{Q} = \mathbb{R}$ . Since both the irrationals and rationals are dense in the real numbers, we know that every non-empty open interval in  $\mathbb{R}$  contains both a rational and irrational number. Thus let  $x \in \mathbb{R}$  be any real number, and  $r > 0$  be arbitrary. The set  $B_r(x)$  is an open interval around  $x$ , and

contains a rational number, showing that  $B_r(x) \cap \mathbb{Q} \neq \emptyset$ . Similarly,  $B_r(x)$  contains an irrational number demonstrating that  $B_r(x) \cap \mathbb{Q}^c \neq \emptyset$ , so  $x \in \partial\mathbb{Q}$ . Since  $x$  was arbitrary, we conclude that  $\partial\mathbb{Q} = \mathbb{R}$ .

## 2 Open and Closed Sets

### Definition 2.6

A set  $S$  in a metric space  $(X, d)$  is said to be open if every point of  $S$  is an interior point; that is,  $S$  is open if for every  $x \in S$  there exists an  $r > 0$  such that  $D_d(x) \subseteq S$ . The set  $S$  is closed if  $S^c$  is open. Given a point  $x \in X$ , an open neighborhood of  $x$  is some open set containing  $x$ .

### Example 2.7

The set  $S = \{(x, y) \in \mathbb{R}^2 : y > 0\} \subset \mathbb{R}^2$  is open in the Euclidean metric.

## 3 Open, Closed, and Everything in Between

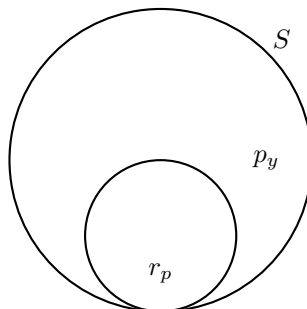


Figure 2.3: The upper half plane is open. For any point, look at its  $y$ -coordinate  $p_y$  and use the ball of radius  $p_y/2$ .

*Proof.* We need to show that around every point in  $S$  we can place an open ball that remains entirely within  $S$ . Choose a point  $p = (p_x, p_y) \in S$ , so that  $p_y > 0$ , and let  $r = p_y/2$ . Consider the ball  $B_r(p)$ , which we claim lies entirely within  $S$ . To see that this is the case, choose any other point  $q = (q_x, q_y) \in B_r(p)$ . Now

$$|p_y - q_y| \leq \|q - p\| < r = \frac{p_y}{2}$$

which implies that  $q_y > p_y - p_y/2 = p_y/2 > 0$ . Since  $q_y > 0$  this shows that  $q \in S$ , and since  $q$  was arbitrary,  $B_r(p) \subseteq S$  as required. ■ □