Title of the Document

Author Name

June 22, 2024

Contents

2.6 Exercises

2-1. Let (X, d) be a metric space and $S \subseteq X$. Show that $\partial S \subseteq S'^c$.

2-2. Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $(x-a)^2 + (y-b)^2 \leq r^2$ is a bounded set in $\mathbb{R}^2$.

2-3. Let (X, d) be a metric space and for $x, y \in X$. Show that if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

2-1. Assume $\partial S \subseteq S'^c$. Therefore, there exists $x \in S'$ such that $x \in S$ which is a contradiction.
Then by $x \in S'^c$, $\exists \epsilon > 0 : B_\epsilon(x) \subseteq S'^c$.
However, by $x \in S'$, this value of $\epsilon > 0$ implies $B_\epsilon(x) \cap S \neq \emptyset \Rightarrow B_\epsilon(x) \subseteq S$, which is a contradiction, implying our assumption that $x \in \partial S \cap S'$ must be false, and $S' \cap S^{\text{int}} = \emptyset \square$. $\square$

2-2. A set S is bounded iff $\exists M \in \mathbb{R}^+$ s.t. $\forall x, y \in S$, $d(x, y) \leq M$. Let $a, b, r \in \mathbb{R}$.
$\delta = \{(x, y) \in \mathbb{R}^2 | (x-a)^2 + (y-b)^2 \leq r^2\} \Rightarrow x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \;\leq\; r^2$
$\Rightarrow x^2 - 2ax + y^2 - 2yb \;\leq\; r^2 - a^2 - b^2 \Rightarrow x^2 + y^2 \;\leq\; r^2 - a^2 - b^2 + 2xa + 2yb$ *Need to show* x^2 is bounded.
$(x-a)^2 \leq r^2$
$\Rightarrow |x - a| \leq |r|$
$\Rightarrow |x - a| + |a| \leq |r| + |a|$
$\Rightarrow |x| = |x - a + a| \leq |x - a| + |a| \leq r + a|$. $\square$

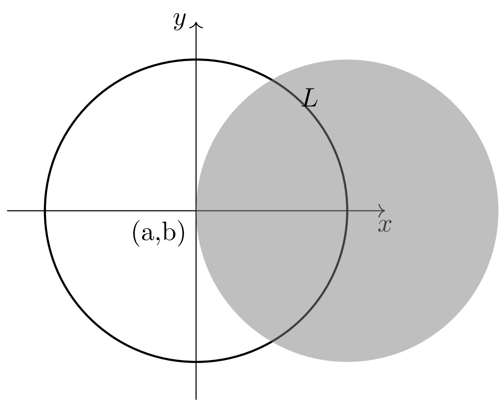

Figure 1: A circle with radius a and central point (a, b)

$\Rightarrow |y| \leq r + |a|$

$\Rightarrow y^2 \leq (r + |a|)^2$

Same for y, $\text{y}^2 \leq (r + |b|)^2$

$\forall z = (x, y) \in D^2_{a,b}$

$\|z\| = \sqrt{x^2 + y^2}$

$\leq \sqrt{(r + |a|)^2 + (r + |b|)^2}$

Thus if $\text{M} = \sqrt{(r + |a|)^2 + (r + |b|)^2}$, the bound holds.

\# 15 normed boundedness = distance boundedness.

Let x=(x_1, x_2), $y = (y_1, y_2) \in D_{a,b}$

$z_1 = (x_1, y_1)(z_1 - a)^2 + (z_2 - b)^2 = r^2$

$\Rightarrow d(z_1, (a, b)) = \sqrt{(x_1 - a)^2 + (x_2 - b)^2} \leq r$

$\Rightarrow d(x, y) \leq d(x, (a, b)) + d(y, (a, b))$

$= \sqrt{(x_1 - a)^2 + (x_2 - b)^2} + \sqrt{(y_1 - a)^2 + (y_2 - b)^2}$

$\leq r + r = 2r.$

(iii) **Suppose that** $x \neq y$. **Then** $d(x, y) \neq 0$. **Thus if we choose** $\epsilon = d(x, y) \Rightarrow \epsilon > 0$ **but** $d(x, y) \geq \epsilon$. *(contradiction).*

(contradiction) Suppose $x = y$ **and so** $d(x, y) = 0$.

Choose $\epsilon > 0$ **so that** $\epsilon = d(x, y)$. **Then we must have** $d(x, y) < \epsilon = \frac{d(0,0)}{2}$, **which is a contradiction, as this implies**

if $d(x, y) > 0$ **so** $d(x, y) = s < \epsilon = \frac{s}{2}$

$\Rightarrow s < \frac{s}{2} \Rightarrow 2s < s$

Thus, $x = y$.

(iv) **Let** $(V, ||\cdot||)$ **be a normed vsp.**

Then let $r > 0$ **and** $x \in V$.

$B_r(x) = \{y \in V \mid d(x, y) < r\}$ $B_{r+||x||}(0) = \{y \in V \mid d(0, y) < r + ||x||\}$

Let $y \in B_r(x)$.

$d(0, y) \leq d(0, x) + d(x, y) \leq ||x|| + r \Rightarrow B_r(x) \subseteq B_{r+||x||}(0)$.

(v) **Suppose** S **is bounded.Then** $\exists M \in \mathbb{R} : \forall x \in S \; ||x|| \leq M$.

(Equiv to $\exists M \in \mathbb{R} : \forall x \in S \; x \in B_M(0)$).