Title of the Document

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2.6 Exercises

2.1 Let (X, d) be a metric space and $S \subset X$. Show that $\overline{S}^0 \subset S^0 = \emptyset$.

2.2 Show that for an arbitrary choice of $a, b \in \mathbb{R}$, the closed disk $(x - a)^2 + (y - b)^2 \leq r^2$ is in a bounded set in $\mathbb{R}^2$.

2.3 Let (X, d) be a metric space and for $x, y \in X$. Show that if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

(i). Assume $\overline{S}^0 \subset S$.

$\exists x \in \overline{S}^0 \quad \exists \epsilon > 0 : \quad B_\epsilon(x) \subseteq S^0$.

Then by $x \in \overline{S}^0 \Rightarrow \forall \epsilon > 0 : \quad B_\epsilon(x) \subseteq S$.

However, by $x \notin S^0$, this value of $\epsilon > 0$ implies

$B_\epsilon(x) \cap S^0 = \emptyset \Rightarrow B_\epsilon(x) \nsubseteq S^0$,

which is a contradiction, implying our assumption that $x \in \overline{S}^0 \cap S^0$ must be false and $\overline{S}^0 \cap S^0 = \emptyset$.

(ii). A set S is bounded iff $\exists M \in \mathbb{R}^+ : \quad \forall x, y \in S \quad d(x, y) < M$.

Let $a, b, r \in \mathbb{R}$.

$S = \{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 \leq r^2\}$

$\Rightarrow x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2$

$\Rightarrow x^2 - 2ax + y^2 - 2yb \leq r^2 - a^2 - b^2$

$\Rightarrow x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb$

Need to show x^2 is bounded.

$(x - a)^2 \leq r^2 \Rightarrow |x - a| \leq |r| \Rightarrow |x - a| \leq |r| + |a|$

$\Rightarrow |x| = |x - a + a| \leq |x - a| + |a| \leq |r| + |a|$

Sure! Here is the LaTeX code for the given images:

$$\begin{matrix} \} & & \\ |) & b & \Rightarrow \quad |y| \leq r + |a| \\ \quad a & & \end{matrix}$$

$\Rightarrow x^2 \leq (r + |b|)^2$

Same for y, $\qquad y^2 \leq (r + |b|)^2$

$\forall z = (x, y) \in D_{r+|a|}$

$\|z\| = \sqrt{x^2 + y^2}$

$\leq \sqrt{(r+|a|)^2+(r+|b|)^2}$

Thus, if $M = \sqrt{(r+|a|)^2+(r+|b|)^2}$, the bound holds.

$*1S$ normed bounded set $=$ distance boundedness.

Let $x = (x_1, x_2), y = (y_1, y_2) \in D_{r,ab}$,

$z_i \in \{x_i, y_i\}$

$(z_i - a)^2 + (z_i - b)^2 = r^2$

$\Rightarrow d(z_i, [a,b]) = \sqrt{(z_i - a)^2 + (z_i - b)^2} \leq r$

$\therefore d(x,y) \leq d(x,[a,b]) + d(y,[a,b])$

$= \sqrt{(x_1 - a)^2 + (x_2 - b)^2} + \sqrt{(y_1 - a)^2 + (y_2 - b)^2}$

$\leq r + r = 2r.$

This code should produce a document that mirrors the math from the images provided.

(iii) Suppose that $x \neq y$. Then $d(x,y) \neq 0$. Thus if we choose $\epsilon = d(x,y) \Rightarrow \epsilon > 0$, but $d(x,y) \in \epsilon$ (contradiction).

(contradiction) Suppose $x = y$, and so $d(x,y) = 0$. Choose $\epsilon > 0$ such that $\epsilon = d(x,y)$. Then we must have $d(x,y) < \epsilon = \frac{d(0,0)}{2}$, which is a contradiction, as this implies: $d(x,y) = 0 \Rightarrow d(x,y) = 0 < \epsilon = \frac{\epsilon}{2}$ $\Rightarrow 0 < \frac{\epsilon}{2} \Rightarrow 2\epsilon < \epsilon$ Thus $x = y$.

(iv) Let $(V, \|\cdot\|)$ be a normed vector space. Then let $r > 0$ and $x \in V$. Then $B_r(x) = \{y \in V \mid d(x,y) < r\}$ $B_{\|\cdot\|+r}(0) = \{y \in V \mid d(0,y) < r + \|x\|\}$

`figure`.png

Let $y \in B_r(x)$. $d(0,y) \leq d(0,x) + d(x,y) \leq \|x\| + r \Rightarrow B_r(x) \subseteq B_{\|\cdot\|+r}(0)$

(v) Suppose S is bounded. Then $\exists M \in \mathbb{R} : \forall x \in S, \|x\| \leq M$. (Equivalent to $\exists M \geq 0 : \forall x \in S, x \in B_M(0)$)