Title of the Document

Author Name

June 22, 2024

Contents

1 Exercises

2.1. Let (X, d) be a metric space and $S \subset X$. Show that $\partial S \subset S^{int} = \emptyset$.

2.2. Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $(x-a)^2 + (y-b)^2 \leq r^2$ is in a bounded set in $\mathbb{R}^2$.

2.3. Let (X, d) be a metric space and let $x, y \in X$. Show that if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

2.1. Assume $\partial S \subset S^{int}$. Then $\exists x \in S^{int} \subset \partial S$. Then by $x \in S^{int} \implies \exists \epsilon > 0 : B_\epsilon(x) \subset S$.

However, by $x \in \partial S$, this value of $\epsilon > 0$ implies $B_{\frac{\epsilon}{2}}(x) \cap S^c \neq \emptyset \implies B_{\frac{\epsilon}{2}}(x) \not\subset S$, which is a contradiction, implying our assumption that $x \in \partial S \cap S^{int}$ must be false and $\partial S \cap S^{int} = \emptyset$. $\square$

2.2. A set S is bounded iff $\exists M \in \mathbb{R}^+ : \forall x, y \in S \ d(x, y) \leq M$.

Let $a, b, r \in \mathbb{R}$.

$S := \{(x, y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 \leq r^2\}$

$\implies x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2$

$\implies x^2 - 2ax + y^2 - 2yb \leq r^2 - a^2 - b^2$

$\implies x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb$

We need to show x^2 is bounded.

- $(x-a)^2 \leq r^2$
- $\implies |x-a| \leq |r|$
- $\implies |x-a| = |x| + |a| \leq |r| + |a|$
- $\implies |x| = |x-a+a| \leq |x-a| + |a| \leq r + |a|$

$\square$

$$\implies |y| \leq r + |a|$$
$$\implies y^2 \leq (r + |a|)^2$$

Same for x,

$x^2 \leq (r + |b|)^2$

For $z = (x, y) \in D^2_{a,b}$

$\|z\| = \sqrt{x^2 + y^2}$

$\leq \sqrt{(r+|a|)^2 + (r+|b|)^2}$

Thus if $M = \sqrt{(r+|a|)^2 + (r+|b|)^2}$ the bound holds.

IS named boundness = distance boundness.

Let $x = (x_1, x_2), y = (y_1, y_2) \in D_{a,b}$

$z_1, z_2 \in \{x, y\}$

$(z_2 - a)^2 + (z_2 - b)^2 = r^2$

$\implies d(z_i, (a,b)) = \sqrt{(z_1 - a)^2 + (z_2 - b)^2} \leq r$

$\implies d(x,y) \leq d(x,(a,b)) + d(y,(a,b))$

$= \sqrt{(x_1 - a)^2 + (x_2 - b)^2} + \sqrt{(y_1 - a)^2 + (y_2 - b)^2}$

$\leq r + r = 2r.$

(iii) Suppose that $x \neq y$. Then $d(x,y) \neq 0$. Thus if we choose $\epsilon = d(x,y)$ implies that $\epsilon > 0$ but $d(x,y) \notin \epsilon$. (contradiction).

Contradiction Suppose $x = y$ and so $d(x,y) = 0$. Choose $\epsilon > 0$ so that $\epsilon = d(x,y)$. Then we must have $d(x,y) < \epsilon = \frac{d(x,y)}{2}$, which is a contradiction, as this implies if $d(x,y) < \epsilon \implies d(x,y) = s < \epsilon = \frac{s}{2}$.

$-s < s/2 \implies 2s < s$.

Thus $x \neq y$.

(iv) Let $(V, ||\cdot||)$ be a normed vsp. Then let $r > 0$ and $x \in V$. Then $B_r(x) = \{u \in V | d(x,u) < r\}$ $B_{\epsilon+||x||}(0) = \{v \in V | d(0,v) < r + ||x||\}$ Let $y \in B_r(x)$. $d(0,y) \leq d(0,x) + d(x,y) \leq ||x|| + r \implies B_r(x) \subseteq B_{\epsilon+||x||}(0)$.

(v) Suppose S is bounded. Then $\exists M \epsilon \mathbb{R} : \forall x \epsilon S \ ||x|| \leq M$. $(Equivalentto \ \exists M \epsilon \mathbb{R} : \forall x \epsilon v) x \epsilon B_M(0)$