Title of the Document

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1 Interior and Boundary Points

Definition 2.1

Let (X, d) be a metric space. If $x \in X$ and $r > 0$, we define the **open ball of radius r centred at x** as $B_r(x) := \{y \in X : d(x, y) < r\}$.

In $\mathbb{R}^n$ with the Euclidean metric $d(x, y) = ||x - y||$, the open ball $B_r(x)$ is nothing more than the collection of points which are a distance at most r from x. This generalizes the interval, since in $\mathbb{R}^1$ we have $B_r(x) = \{y \in \mathbb{R} : |x - y| < r\} = (x - r, x + r)$, or if we centre around 0, $B_r(0) = (-r, r)$. In $\mathbb{R}^2$ we get a disk of radius r, $B_r(0) = \{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < r\}$, which we recognize as being the same as $x^2 + y^2 < r^2$.

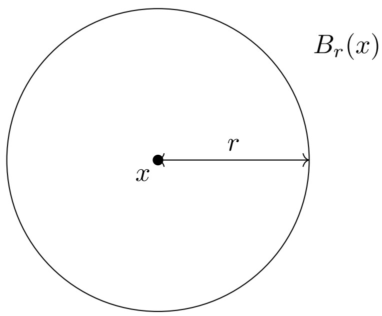

Figure 2.1: In $\mathbb{R}^2$, the open ball of radius r centred at x consists of all points which are a distance at most r from x.

Definition 2.2

A subspace S of a metric space X is **bounded** if there exists an $r > 0$ and an $a \in X$ such that $S \subseteq B_r(a)$.

Definition 2.3

Let (X, d) be a metric space, and $S \subseteq \mathbb{R}^n$ be an arbitrary set.

1. We say that $x \in S$ is an **interior point of S** if there exists an $r > 0$ such that $B_r(x) \subseteq S$; that is, x is an interior point if we can enclose it in an open ball which is entirely contained in S.

2. We say that $x \in S$ is a **boundary point of S** if for every $r > 0$, $B_r(x) \cap S \neq \emptyset$ and $B_r(x) \cap S^c \neq \emptyset$; that is, x is a boundary point if no matter what ball we place around x, that ball lives both inside and outside of S.

The **interior of** S - denoted $\mathring{S}$ - is the collection of interior points of S, while **boundary of** S - denoted ∂S - is the collection of boundary points of S.

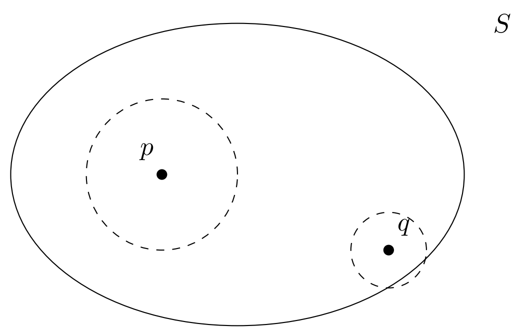

Figure 2.2: The point q is a boundary point. No matter what size ball we place around q, that ball will intersect both S and S^c. On the other hand, p is an interior point, since we can place a ball around it which lies entirely within S.

We should take a moment and think about these definitions, and why they make sense. A boundary point is any point at which occurs at the very fringe of the set; that is, if I push a little further I will leave the set. An interior point should be a point inside of S, such that if I move in any direction a sufficiently small distance, I stay within the set. Note that if x is an interior point then we must have that $x \in S$; however, boundary points do not need to be in the set. We start with a simple example.

Example 2.4 Let $S = (-1, 1) \subseteq \mathbb{R}$ endowed with the Euclidean metric. What are the interior points and the boundary points of S?

Solution. I claim that any point in $(-1, 1)$ is an interior point. To see that this is the case, let $p \in (-1, 1)$ be an arbitrary point. We need to place a ball around p which lies entirely within $(-1, 1)$.

To do this, assume without loss of generality that $p \geq 0$. If $p = 0$ then we can set $r = 1/2$ and $\mathbf{B}_r(p) = (-1/2, 1/2) \subseteq (-1, 1)$. Thus assume that $p \neq 0$ and let $r = (1-p)/2$, which represents half the distance from p to 1. I claim that $\mathbf{B}_r(p) \subseteq (-1, 1)$. Indeed, let $x \in \mathbf{B}_r(p)$ be any point, so that $|x - p| < r$ by definition. Then $|x| = |x - p + p| \leq |x - p| + |p| < r + p = \frac{1-p}{2} + p = \frac{1+p}{2} < 1$ where in the last inequality we have used the fact that $p < 1$ so $1 + p < 2$. Thus $x \in (-1, 1)$, and since x was arbitrary, $\mathbf{B}_r(p) \subseteq (-1, 1)$.

The boundary points are ± 1, where we note that even though $-1 \notin (-1, 1)$, it is still a boundary point. To see that -1 is a boundary point, let $r > 0$ be arbitrary, so that $\mathbf{B}_r(p) = (-1 - r, 1 + r)$. We then have $\mathbf{B}_r(p) \cap (-1, 1) = (-1 - r, 1) \neq \emptyset$,and $\mathbf{B}_r(p) \cap (-1, 1)^c = (-1 - r, 1) \neq \emptyset$, as required. The proof for -1 is analogous and left as an exercise. $\square$

Example 2.5 What is the boundary of $\mathbb{Q}$ in $\mathbb{R}$ with the Euclidean metric?

Solution. We claim that $\partial\mathbb{Q} = \mathbb{R}$. Since both the irrationals and the rationals are dense in the real numbers, we know that every non-empty open interval in $\mathbb{R}$ contains both a rational and irrational number. Thus let $x \in \mathbb{R}$ be any real number, and $r > 0$ be arbitrary. The set $\mathbf{B}_r(x)$ is an open interval around x, and contains a rational number, showing that $\mathbf{B}_r(x) \cap \mathbb{Q} \neq \emptyset$. Similarly, $\mathbf{B}_r(x)$ contains an irrational number, showing that $\mathbf{B}_r(x) \cap \mathbb{Q}^c \neq \emptyset$, so $x \in \partial\mathbb{Q}$. Since x was arbitrary, we conclude that $\partial\mathbb{Q} = \mathbb{R}$. $\square$

2 Open and Closed Sets

Definition 2.6

A set S in a metric space (X, d) is said to be **open** if every point of S is an interior point; that is, S is open if for every $x \in S$ there exists an $r > 0$ such that $B_d(x; r) \subseteq S$. The set S is **closed** if S is open. Given a point $x \in X$, an open neighbourhood of x is some open set containing x.

Example 2.7

The set $S = \{(x, y) \in \mathbb{R}^2 : y > 0\} \subseteq \mathbb{R}^2$ is open in the Euclidean metric.

3 Open, Closed, and Everything in Between

4 The Topology of $\mathbb{R}^n$

open_set_diagram.png

Figure 2.3: The upper half-plane is open. For any point, look at its y-coordinate p_y and use the ball of radius $p_y/2$.

Solution. We need to show that around every point in S we can place an open ball that remains entirely within S. Choose a point $p = (p_x, p_y) \in S$, so that $p_y > 0$, and let $r = p_y/2$. Consider the ball $B_r(p)$, which we claim lives entirely within S. To see that this is the case, choose any point $q = (q_x, q_y) \in B_r(p)$. Now

$p_y - q_y \leq \|q - p\| < r = \frac{p_y}{2}$

which implies that $q_y > p_y - p_y/2 = p_y/2 > 0$. Since $q_y > 0$, this shows that $q \in S$, and since q was arbitrary, $B_r(p) \subseteq S$ as required. □