### 2.1.1 Interior and Boundary Points

**Definition 2.1**

Let $(X, d)$ be a metric space. If $\mathbf{x} \in X$ and $r > 0$, we define the *open ball of radius $r$ centred at $\mathbf{x}$* as

$$B_r(\mathbf{x}) := \{\mathbf{y} \in X : d(\mathbf{x}, \mathbf{y}) < r\}.$$

In $\mathbb{R}^n$ with the Euclidean metric $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$, the open ball $B_r(\mathbf{x})$ is nothing more than the collection of points which are a distance at most $r$ from $\mathbf{x}$. This generalizes the interval, since in $\mathbb{R}^1$ we have

$$B_r(x) = \{y \in \mathbb{R} : |x - y| < r\} = (x - r, x + r),$$

or if we centre around 0, $B_r(0) = (-r, r)$. In $\mathbb{R}^2$ we get a disk of radius $r$,

$$B_r(\mathbf{0}) = \left\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} < r\right\},$$

which we recognize as being the same as $x^2 + y^2 < r^2$.

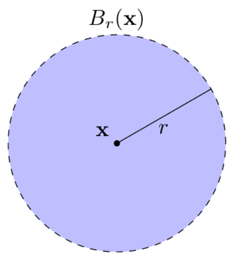


Figure 2.1: In $\mathbb{R}^2$, the open ball of radius $r$ centred at $\mathbf{x}$ consists of all points which are a distance at most $r$ from $\mathbf{x}$.

**Definition 2.2**

A subspace $S$ of a metric space $X$ is bounded if there exists an $r > 0$ and an $\mathbf{a} \in X$ such that $S \subseteq B_r(\mathbf{a})$.

**Definition 2.3**

Let $(X, d)$ be a metric space, and $S \subseteq \mathbb{R}^n$ be an arbitrary set.

1. We say that $\mathbf{x} \in S$ is an *interior point of $S$* if there exists an $r > 0$ such that $B_r(\mathbf{x}) \subseteq S$; that is, $\mathbf{x}$ is an interior point if we can enclose it in an open ball which is entirely contained in $S$.
2. We say that $\mathbf{x} \in S$ is a *boundary point of $S$* if for every $r > 0$, $B_r(\mathbf{x}) \cap S \neq \emptyset$ and $B_r(\mathbf{x}) \cap S^c \neq \emptyset$; that is, $\mathbf{x}$ is a boundary point if no matter what ball we place around $\mathbf{x}$, that ball lives both inside and outside of $S$.

The *interior of $S$* – denoted $S^{\text{int}}$ – is the collection of interior points of $S$, while *boundary of $S$* – denoted $\partial S$ – is the collection of boundary points of $S$.

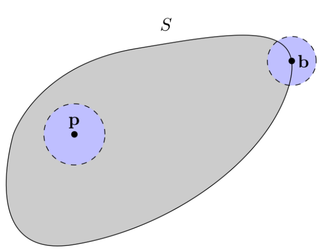


Figure 2.2: The point $\mathbf{b}$ is a boundary point. No matter what size ball we place around $\mathbf{b}$, that ball will intersect both $S$ and $S^c$. On the other hand, $\mathbf{p}$ is an interior point, since we can place a ball around it which lives entirely within $S$.

We should take a moment and think about these definitions, and why they make sense. A boundary point is any point which occurs at the very fringe of the set; that is, if I push a little further I will leave the set. An interior point should be a point inside of $S$, such that if I move in any direction a sufficiently small distance, I stay within the set. Note that if $\mathbf{x}$ is an interior point then we must have that $\mathbf{x} \in S$; however, boundary points *do not* need to be in the set. We start with a simple example.

**Example 2.4**

Let $S = (-1, 1] \subseteq \mathbb{R}$, endowed with the Euclidean metric. What are the interior points and the boundary points of $S$?

*Solution.* I claim that any point in $(-1, 1)$ is an interior point. To see that this is the case, let $p \in (-1, 1)$ be an arbitrary point. We need to place a ball around $p$ which lives entirely within







$(-1, 1)$. To do this, assume without loss of generality that $p \geq 0$. If $p = 0$ then we can set $r = 1/2$ and $B_{1/2}(0) = (-1/2, 1/2) \subseteq (-1, 1)$. Thus assume that $p \neq 0$ and let $r = (1 - p)/2$, which represents half the distance from $p$ to 1. I claim that $B_r(p) \subseteq (-1, 1)$. Indeed, let $x \in B_r(p)$ be any point, so that $|x - p| < r$ by definition. Then

$$\begin{aligned} |x| = |x - p + p| &\leq |x - p| + p \\ &\leq r + p = \frac{1 - p}{2} + p \\ &= \frac{1 + p}{2} < 1 \end{aligned}$$

where in the last inequality we have used the fact that $p < 1$ so $1 + p < 2$. Thus $x \in (-1, 1)$, and since $x$ was arbitrary, $B_r(p) \subseteq (-1, 1)$.

The boundary points are $\pm 1$, where we note that even though $-1 \notin (-1, 1]$, it is still a boundary point. To see that $+1$ is a boundary point, let $r > 0$ be arbitrary, so that $B_r(p) = (1 - r, 1 + r)$. We then have

$$B_r(p) \cap (-1, 1] = (1 - r, 1] \neq \emptyset, \qquad B_r(p) \cap (-1, 1)^c = (1, 1 + r) \neq \emptyset,$$

as required. The proof for $-1$ is analogous and left as an exercise. ■

**Example 2.5**

What is the boundary of $\mathbb{Q}$ in $\mathbb{R}$ with the Euclidean metric?

*Solution.* We claim that $\partial\mathbb{Q} = \mathbb{R}$. Since both the irrationals and rationals are dense in the real numbers, we know that every non-empty open interval in $\mathbb{R}$ contains both a rational and irrational number. Thus let $x \in \mathbb{R}$ be any real number, and $r > 0$ be arbitrary. The set $B_r(x)$ is an open interval around $x$, and contains a rational number, showing that $B_r(x) \cap \mathbb{Q} \neq \emptyset$. Similarly, $B_r(x)$ contains an irrational number, showing that $B_r(x) \cap \mathbb{Q}^c \neq \emptyset$, so $x \in \partial\mathbb{Q}$. Since $x$ was arbitrary, we conclude that $\partial\mathbb{Q} = \mathbb{R}$. ■

### 2.1.2 Open and Closed Sets

**Definition 2.6**

A set $S$ in a metric space $(X, d)$ is said to be *open* if every point of $S$ is an interior point; that is, $S$ is open if for every $\mathbf{x} \in S$ there exists an $r > 0$ such that $B_r(\mathbf{x}) \subseteq S$. The set $S$ is *closed* if $S^c$ is open. Given a point $\mathbf{x} \in X$, an *open neighbourhood* of $\mathbf{x}$ is some open set containing $\mathbf{x}$.

**Example 2.7**

The set $S = \{(x, y) \in \mathbb{R}^2 : y > 0\} \subseteq \mathbb{R}^2$ is open in the Euclidean metric.







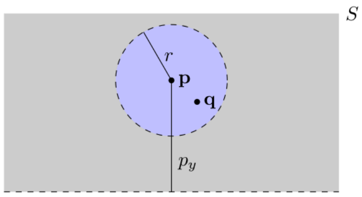


Figure 2.3: The upper half plane is open. For any point, look at its $y$-coordinate $p_y$ and use the ball of radius $p_y/2$.

*Solution.* We need to show that around every point in $S$ we can place an open ball that remains entirely within $S$. Choose a point $\mathbf{p} = (p_x, p_y) \in S$, so that $p_y > 0$, and let $r = p_y/2$. Consider the ball $B_r(\mathbf{p})$, which we claim lives entirely within $S$. To see that this is the case, choose any other point $\mathbf{q} = (q_x, q_y) \in B_r(\mathbf{p})$. Now

$$|q_y - p_y| \leq \|\mathbf{q} - \mathbf{p}\| < r = \frac{p_y}{2}$$

which implies that $q_y > p_y - p_y/2 = p_y/2 > 0$. Since $q_y > 0$ this shows that $\mathbf{q} \in S$, and since $\mathbf{q}$ was arbitrary, $B_r(\mathbf{p}) \subseteq S$ as required. ■