Title of the Document

Author Name

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Contents

1 Exercises

2.1. Let (x, d) be a metric space and $S \subseteq X$. Show that $0 < S^{int} = \emptyset$.

2.2. Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $\{x = a^2 + (y-b)^2 \leq r^2\}$ is a bounded set in $\mathbb{R}^2$.

2.3. Let (x, d) be a metric space and for $x, y, \epsilon \in X$, show that if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

(i) Assume $S \neq \emptyset$. Then $\exists x \in S, B_\epsilon(x) \subseteq S^{int}$.

Then by $x \in S^{int}$, $\exists \epsilon > 0, B_\epsilon(x) \subseteq S$.

However, by $x \in S^{int}$, this value of $\epsilon > 0$ implies $B_\epsilon(x) \cap S^C = \emptyset \implies B_\epsilon(x) \nsubseteq S$ which is a contradiction, implying our assumption that $x \in S \cap S^{int}$ must be false and $S^{int} = \emptyset$.

(ii) A set S is bounded iff $\exists M \in \mathbb{R}^+ 0 \forall x, y \in S d(x, y) \leq M$.

Let $a, b, r \in \mathbb{R}$.

$S := \{(x, y) \in \mathbb{R}^2 | (x-a)^2 + (y-b)^2 = r^2\}$

$\implies x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2$

$\implies x^2 - 2ax + y^2 - 2yb \leq r^2 - a^2 - b^2$

$\implies x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb$

need to show x^2 is bounded.

$(x-a)^2 \leq r^2$

$\implies |x-a| \leq |r|$

$\implies |x-a| \leq |r| + |a|$

$\implies |x| = |x-a+a| \leq |x-a| + |a| \leq r + |a|$

$$\Rightarrow |y| \leq r + |a|$$
$$\Rightarrow x^2 \leq (r + |a|)^2$$

Same for y : $\text{y}^2 \leq (r + |b|)^2$

$\forall z = (x, y) \in D^2_{a,b}$:
$\|z\| = \sqrt{x^2 + y^2}$
$\leq \sqrt{(r + |a|)^2 + (r + |b|)^2}$

Thus, if M $= \sqrt{(r + |a|)^2 + (r + |b|)^2}$, the band holds.

\# 1S normed boundless = distance boundless.

Let x $= (\text{x}_1, x_2)$, $y = (y_1, y_2) \in D_{a,b}$:
$z_i \in \{x, y\}^i$
$(x_i - a)^2 + (y_i - b)^2 = r^2$
$\Rightarrow d(z_i, (a, b)) = \sqrt{(x_i - a)^2 + (y_i - b)^2} \leq r$
$\Rightarrow d(x, y) \leq d(x, (a, b)) + d(y, (a, b))$
$= \sqrt{(x_1 - a)^2 + (x_2 - b)^2} + \sqrt{(y_1 - a)^2 + (y_2 - b)^2}$
$\leq r + r = 2r.$

(iii) Suppose that $x \neq y$. Then $d(x,y) \neq 0$. Thus if we choose $\varepsilon = d(x,y) \Rightarrow \varepsilon > 0$ but $d(x,y) \in \varepsilon$ (contradiction).

Proof. **(Contradiction)** Suppose $x = y$ and so $d(x,y) = 0$.

Choose $\varepsilon > 0$ such that $\varepsilon = d(x,y)$. Then we must have $d(x,y) < \varepsilon = \frac{d(x,y)}{2} = \frac{0}{2}$, which is a contradiction, as this implies if $d(x,y) = 0 \Rightarrow d(x,y) = 0 < \varepsilon = \frac{\varepsilon}{2} \Rightarrow 0 < \frac{\varepsilon}{2} \Rightarrow 2(0) < \varepsilon$.

Thus $x = y$. $\square$

(iv) Let $(V, \|\cdot\|)$ be a normed vsp.

Then let $r > 0$ and $x \in V$. Then $B_r(x) = \{u \in V \mid d(x,u) < r\}$ $B_{\|\cdot\|+r}(0) = \{v \in V \mid d(0,v) < r + \|x\|\}$

Let $y \in B_r(x)$. $d(0,y) \leq d(0,x) + d(x,y) \leq \|x\| + r \Rightarrow B_r(x) \subseteq B_{\|\cdot\|+r}(0)$.

(v) Suppose S is bounded. Then $\exists M \in \mathbb{R} > 0 \exists v \in S \|v\| \leq M$.

(Equiv to $\exists M > 0 : \forall x \in V$ and $x \in B_M(0)$).