Title of the Document

Author Name

June 22, 2024

Contents

1 Exercises

(2.1) Let (X, d) be a metric space and $S \subset X$. Show that $\partial S \subseteq \bar{S} \cap \overline{S^c} = \emptyset$.

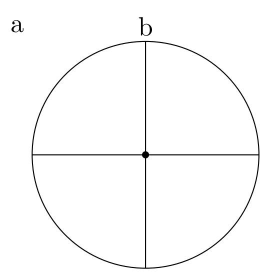
(2.2) Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $(x-a)^2 + (y-b)^2 \leq r^2$ is a bounded set in $\mathbb{R}$.

(2.3) Let (X, d) be a metric space and for $x, y \in X$, Show that if $d(x, y) < \varepsilon$ for every $\varepsilon > 0$, then $x = y$.

Proof for Exercise (2.1). Assume $\partial S \neq \emptyset$. Then there exists some $x \in \partial S$. Then $(x \in \overline{S} \cap \overline{S^c}) = B_\varepsilon(x) \subseteq S$. However, by $x \in \partial S$, this value of $\varepsilon > 0$ implies $B_\varepsilon(x) \cap S^c \neq \emptyset \Rightarrow B_\varepsilon(x) \nsubseteq S$ which is a contradiction, implying our assumption that $x \in \overline{S} \cap \partial S$ must be false and $\partial S \cap \partial S^c = \emptyset$. □

Proof for Exercise (2.2). A set S is bounded if and only if $\exists M \in \mathbb{R}^+ : \forall x, y \in S \; d(x, y) \leq M$.

Let $a, b, r \in \mathbb{R}$. $S = \{(x, y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 \leq r^2\} \Rightarrow x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2$ $\Rightarrow x^2 - 2ax + y^2 - 2yb + a^2 + b^2 - a^2 - b^2 \Rightarrow x^2 + y^2 \leq r^2 - a^2 - b^2 - 2ax - 2yb$ Need to show x^2 is bounded. $(x-a)^2 \leq r^2 \Rightarrow |x-a| \leq |r| \Rightarrow |x-a| \leq |r| + |a| \Rightarrow |x| = |x-a+a| \leq |x-a| + |a| \leq r + |a|$ □



$\Rightarrow |y| \leq r + |a|$

$\Rightarrow y^2 \leq (r + |a|)^2$

Same for y_i, $y_i^2 \leq (r + |b|)^2$

$\forall z = (x, y) \in D_{a,b}$

$\|z\| = \sqrt{x^2 + y^2}$

$\leq \sqrt{(r + |a|)^2 + (r + |b|)^2}$

Thus if $M = \sqrt{(r + |a|)^2 + (r + |b|)^2}$, the bound holds.

IS named boundness = distance boundedness.

Let $x = (x_1, x_2), y = (y_1, y_2) \in D_{a,b}$

$z_1 = \{x, y\}^2$

$(x_2 - a)^2 + (x_2 - b)^2 = r^2$

$\Rightarrow d(z_1, (a, b)) = \sqrt{(x_2 - a)^2 + (x_2 - b)^2} \leq r$

$\Rightarrow d(x,y) \leq d(x,(a,b)) + d(y,(a,b))$

$= \sqrt{(x_1-a)^2+(x_2-b)^2} + \sqrt{(y_1-a)^2+(y_2-b)^2}$

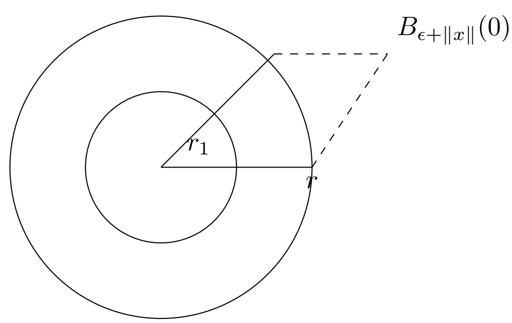
$\leq r + r = 2r.$

(iii) Suppose that $x \neq y$. Then $d(x,y) \neq 0$. Thus if we choose $\epsilon = d(x,y) \Rightarrow \epsilon > 0$ but $d(x,y) \geq \epsilon$. (contradiction).

(contradiction) Suppose $x \neq y$ and so $d(x,y) \neq 0$. Choose $\epsilon > 0$ such that $\epsilon = d(x,y)$. Then we must have $d(x,y) < \epsilon = \frac{d(x,y)}{2}$, which is a contradiction, as this implies if $d(x,y) > 0 \Rightarrow d(x,y) = \epsilon < \epsilon = \frac{\epsilon}{2}$ $\Rightarrow \epsilon > 0 \Rightarrow \frac{\epsilon}{2}$. Thus $x = y$.

(iv) Let $(V, \|\cdot\|)$ be a normed vector space.

Then let $r > 0$ and $x \in V$. Then $B_r(x) = \{u \in V \mid d(x,u) < r\}$ $B_{\epsilon+\|x\|}(0) = \{v \in V \mid d(0,v) < \epsilon + \|x\|\}$



Let $y \in B_r(x)$. $d(0,y) \leq d(0,x) + d(x,y) \leq \|x\| + r \Rightarrow B_r(x) \subseteq B_{\epsilon+\|x\|}(0)$.

(v) Suppose S is bounded. Then $\exists M \in \mathbb{R} : \forall x \in S$ $\|x\| \leq M$. (Equal to $\exists R \in \mathbb{R} : \forall x \in \mathbb{R}^n$) $x \in B_m(0)$