

Title of the Document

Author Name

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2.6 Exercises

2.1 Let (X, d) be a metric space and $S \subset X$. Show that $\overline{S}^0 \subset S^0 = \emptyset$.

2.2 Show that for an arbitrary choice of $a, b \in \mathbb{R}$, the closed disk $(x - a)^2 + (y - b)^2 \leq r^2$ is in a bounded set in \mathbb{R}^2 .

2.3 Let (X, d) be a metric space and for $x, y \in X$. Show that if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

(i). Assume $\overline{S}^0 \subset S$.

$$\exists x \in \overline{S}^0 \quad \exists \epsilon > 0 : \quad B_\epsilon(x) \subseteq S^0.$$

Then by $x \in \overline{S}^0 \Rightarrow \forall \epsilon > 0 : \quad B_\epsilon(x) \subseteq S$.

However, by $x \notin S^0$, this value of $\epsilon > 0$ implies

$$B_\epsilon(x) \cap S^0 = \emptyset \Rightarrow B_\epsilon(x) \not\subseteq S^0,$$

which is a contradiction, implying our assumption that $x \in \overline{S}^0 \cap S^0$ must be false and $\overline{S}^0 \cap S^0 = \emptyset$.

(ii). A set S is bounded iff $\exists M \in \mathbb{R}^+ : \quad \forall x, y \in S \quad d(x, y) < M$.

Let $a, b, r \in \mathbb{R}$.

$$S = \{(x, y) \in \mathbb{R}^2 \mid (x - a)^2 + (y - b)^2 \leq r^2\}$$

$$\Rightarrow x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2$$

$$\Rightarrow x^2 - 2ax + y^2 - 2yb \leq r^2 - a^2 - b^2$$

$$\Rightarrow x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb$$

Need to show x^2 is bounded.

$$(x - a)^2 \leq r^2 \Rightarrow |x - a| \leq |r| \Rightarrow |x - a| \leq |r| + |a|$$

$$\Rightarrow |x| = |x - a + a| \leq |x - a| + |a| \leq |r| + |a|$$

Sure! Here is the LaTeX code for the given images:

$$\left. \begin{array}{c} \} \\ | \end{array} \right) \begin{array}{c} b \\ a \end{array} \Rightarrow |y| \leq r + |a|$$

$$\Rightarrow x^2 \leq (r + |b|)^2$$

$$\text{Same for } y, \quad y^2 \leq (r + |b|)^2$$

$$\forall z = (x, y) \in D_{r+|a|}$$

$$\|z\| = \sqrt{x^2 + y^2}$$

$$\leq \sqrt{(r + |a|)^2 + (r + |b|)^2}$$

Thus, if $M = \sqrt{(r + |a|)^2 + (r + |b|)^2}$, the bound holds.

*1S normed bounded set = distance boundedness.

Let $x = (x_1, x_2), y = (y_1, y_2) \in D_{r,ab}$,

$$z_i \in \{x_i, y_i\}$$

$$(z_i - a)^2 + (z_i - b)^2 = r^2$$

$$\Rightarrow d(z_i, [a, b]) = \sqrt{(z_i - a)^2 + (z_i - b)^2} \leq r$$

$$\therefore d(x, y) \leq d(x, [a, b]) + d(y, [a, b])$$

$$= \sqrt{(x_1 - a)^2 + (x_2 - b)^2} + \sqrt{(y_1 - a)^2 + (y_2 - b)^2}$$

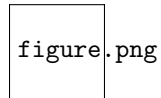
$$\leq r + r = 2r.$$

This code should produce a document that mirrors the math from the images provided.

(iii) Suppose that $x \neq y$. Then $d(x, y) \neq 0$. Thus if we choose $\epsilon = d(x, y) \Rightarrow \epsilon > 0$, but $d(x, y) \in \epsilon$ (contradiction).

(contradiction) Suppose $x = y$, and so $d(x, y) = 0$. Choose $\epsilon > 0$ such that $\epsilon = d(x, y)$. Then we must have $d(x, y) < \epsilon = \frac{d(0,0)}{2}$, which is a contradiction, as this implies: $d(x, y) = 0 \Rightarrow d(x, y) = 0 < \epsilon = \frac{\epsilon}{2} \Rightarrow 0 < \frac{\epsilon}{2} \Rightarrow 2\epsilon < \epsilon$ Thus $x = y$.

(iv) Let $(V, \|\cdot\|)$ be a normed vector space. Then let $r > 0$ and $x \in V$. Then $B_r(x) = \{y \in V \mid d(x, y) < r\}$
 $B_{\|\cdot\|+r}(0) = \{y \in V \mid d(0, y) < r + \|x\|\}$



Let $y \in B_r(x)$. $d(0, y) \leq d(0, x) + d(x, y) \leq \|x\| + r \Rightarrow B_r(x) \subseteq B_{\|\cdot\|+r}(0)$

(v) Suppose S is bounded. Then $\exists M \in \mathbb{R} : \forall x \in S, \|x\| \leq M$. (Equivalent to $\exists M \geq 0 : \forall x \in S, x \in B_M(0)$)