Title of the Document

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0.1 Exercises

2.1. Let (X, d) be a metric space and $S \subseteq X$. Show that $\partial S = \emptyset \implies S^0 = \bar{S}$.

2.2. Show that for an arbitrary choice of $a, b, r \in \mathbb{R}$, the closed disk $(x-a)^2 + (y-b)^2 \leq r^2$ is in a bounded set in $\mathbb{R}^2$.

2.3. Let (X, d) be a metric space and fix $x, y \in X$. Show that if $d(x, y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.

Solution to 2.1. Assume $\partial S = \varnothing$. Then $\forall x \in S^*, \exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq \bar{S}$. However, by $x \in \partial S$, this value of $\epsilon > 0$ implies $B_{\epsilon/2}(x) \cap S = \emptyset \implies B_{\epsilon/2}(x) \not\subset S$, which is a contradiction, implying our assumption that $x \in \partial S \cap S^0$ must be false and $\partial S \cap S^0 = \emptyset$. □

Solution to 2.2. A set S is bounded if $\exists M \in \mathbb{R}^+$ such that $\forall x, y \in S \to d(x, y) \leq M$.

Let $a, b, r \in \mathbb{R}$. $S = \{(x, y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 \leq r^2\} \implies x^2 - 2ax + a^2 + y^2 - 2yb + b^2 \leq r^2 \implies x^2 - 2ax + y^2 - 2yb \leq r^2 - a^2 - b^2 \implies x^2 + y^2 \leq r^2 - a^2 - b^2 + 2ax + 2yb$

Need to show x^2 is bounded: $(x-a)^2 \leq r^2 \implies |x-a| \leq |r| \implies |x-a| \leq |r| + |a|$ $|x| = |x - a + a| \leq |x-a| + |a| \leq |r| + |a|$ □

(drawing of a circle with diameter [a,b])

$\Rightarrow |y| \leq r + |a|$

$\Rightarrow x^2 \leq (r + |a|)^2$

Same for y, $\quad y^2 \leq (r + |b|)^2$

$\forall z = (x, y) \in D^2_{xy}$

$\|z\| = \sqrt{x^2 + y^2}$

$\leq \sqrt{(r + |a|)^2 + (r + |b|)^2}$

Thus, if $M = \sqrt{(r + |a|)^2 + (r + |b|)^2}$, the bound holds.

This is named boundedness = distance boundedness.

Let $x = (x_1, x_2), y = (y_1, y_2) \in D_{xy}$

$z_0 = \{x, y, z\}$

$(x_2 - a)^2 + (x_2 - b)^2 = r^2$

$\Rightarrow d(z_0, (a, b)) = \sqrt{(x_2 - a)^2 + (x_2 - b)^2} \leq r$

$\Rightarrow d(x, y) \leq d(x, (a, b)) + d(y, (a, b))$

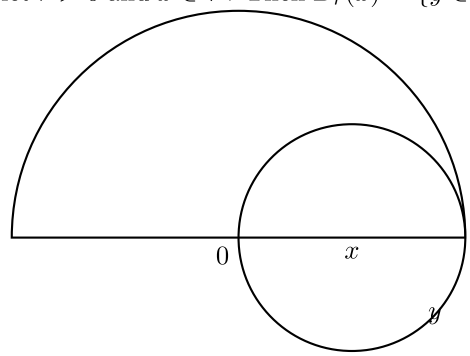
$= \sqrt{(x_1 - a)^2 + (x_1 - b)^2} + \sqrt{(y_1 - a)^2 + (y_1 - b)^2}$

$\leq r + r = 2r.$

(iii) Suppose that $x \neq y$. Then $d(x, y) \neq 0$. Thus if we choose $\epsilon = d(x, y) \implies \epsilon > 0$ but $d(x, y) \geq \epsilon$. (contradiction).

(contradiction) Suppose $x = y$ and so $d(x, y) = 0$. Choose $\epsilon > 0$ so that $\epsilon = d(x, y)$. Then we must have $d(x, y) < \epsilon = \frac{d(x,y)}{2} = \frac{0}{2}$, which is a contradiction, as this implies if $d(x, y) > 0 \implies d(x, y) = \epsilon < \epsilon = \frac{\epsilon}{2} \implies \epsilon \leq \frac{\epsilon}{2} \implies 2\epsilon \leq \epsilon$. Thus, $x = y$.

(iv) Let $(V, \|\cdot\|)$ be a normed vector space. Then let $r > 0$ and $x \in V$. Then $B_r(x) = \{y \in V \mid d(x, y) < r\}$,



$B_{r+\|x\|}(0) = \{y \in V \mid d(0, y) < r + \|x\|\}$ Let $y \in B_r(x)$. $d(0, y) \leq d(0, x) + d(x, y) \leq \|x\| + r \implies B_r(x) \subseteq B_{r+\|x\|}(0)$.

(v) Suppose S is bounded. Then $\exists M \in \mathbb{R} : \forall x \in V\ \|x\| \leq M$ (Equiv to $\exists M \in \mathbb{R} : \forall x \in V)x \in B_M(0)$.