Statistics

ISSN: 0233-1888 (Print) 1029-4910 (Online) Journal homepage: https://www.tandfonline.com/loi/gsta20

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To cite this article: John T. Chen & Arjun K. Gupta (2005) Matrix variate skew normal distributions, Statistics, 39:3, 247-253, DOI: 10.1080/02331880500108593

To link to this article: https://doi.org/10.1080/02331880500108593

Published online: 15 Aug 2006.

Statistics, Vol. 39, No. 3, June 2005, 247–253

Matrix variate skew normal distributions

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(Received 29 October 2003; in final form 1 March 2005)

Consider an experiment in which m measurements are observed in each of n treatments (for example, different dosages of a drug). When the experimenter is interested in the differences of the treatment effects, statistical testing of the corresponding mean vectors for skew populations necessitates a new model, which is called the matrix variate of skew normal distributions. In this note, we propose the model and derive the moment generating function and distribution of the quadratic forms of such random matrices. We show that the quadratic form of a skew normal matrix variate follows a Wishart distribution.

Keywords: Skew normal distribution; Multivariate skew normal distribution; Quadratic form; Moment generating function; Normal matrix variate; Wishart distribution

AMS Subject Classification: Primary 62H10; Secondary 62H05

1. Introduction

Consider the statistical problem of comparing the treatment effects of n dosages of a drug, with K patients in each treatment, and in practice, each patient normally has m measurements of interest (for example, blood pressure, cholesterol level, weight, height, abdomen measurement, body fat percentage, etc.). Let X_{ij} represent the result of the ith measurement ($i = 1, \ldots, m$) from the jth treatment ($j = 1, \ldots, n$). When the treatments are correlated, the study of the matrix variate model is required. In the literature, the model of normal matrix variate distribution refers to the distribution family in which $\mathbf{X} = (X_{ij}) \sim N_{m,n}(\mathbf{M}, \Sigma \otimes \mathbf{\Psi})$ where $\mathbf{M} \in R^{m\times n}$ is an $m \times n$ mean matrix, Σ is an $m \times m$ positive definite matrix (describing the dependence structure among the measurements) and $\mathbf{\Psi}$ is an $n \times n$ positive definite matrix (describing the dependence structure among treatments). The density of a normal matrix variate, $\mathbf{X} \sim N_{m,n}(\mathbf{M}, \Sigma \otimes \mathbf{\Psi})$, takes the following form:

$$f(\mathbf{X}) = (2\pi)^{-mn/2}\det(\Sigma)^{-n/2}\det(\mathbf{\Psi})^{-m/2}\mathrm{etr}\left[-\frac{1}{2}\Sigma^{-1}(\mathbf{X}-\mathbf{M})\mathbf{\Psi}^{-1}(\mathbf{X}-\mathbf{M})'\right] \tag{1}$$

where $\mathbf{M}$ is the $m \times n$ mean matrix.

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Statistics
ISSN 0233-1888 print/ISSN 1029-4910 online © 2005 Taylor & Francis Group Ltd
http://www.tandf.co.uk/journals
DOI: 10.1080/02331880500108593

Recently there have been many studies on matrix variate distributions. For example, Neudecker [1] discussed the multinormal matrix variate, Fang [2] studied the asymptotic properties of symmetric matrix variates, Mathew *et al.* [3] addressed the problem of confidence regions for matrix variate and Nagar and Gupta [4] discussed the matrix variate Kummer-beta distribution, to list just a few.

As elaborated in the articles by Nguyen [5], Gupta and Varga [6] as well as the book by Gupta and Nagar [7], the application of matrix variate distributions extends to the areas of agriculture, econometrics, education, medicine as well as psychology in scenarios where the observed random phenomena can be described by random matrices with dependence structures of random vectors. Cohen and Sackrowitz [8] proposed directional tests for multivariate models.

Although cases of matrix variate normal distributions are well addressed in the literature, so far as we know, there is no study concerning the model for matrix variate observations when the underlying distribution is skew. This motivates the investigation in this article. As a prelude, it is relative at this point to give a brief account on the concepts of univariate and multivariate skew normal distributions.

The skew normal random variable refers to a random variable X that has the probability density

$$f(x) = 2\phi(x)\mathbf{\Phi}(\lambda x) \quad \lambda \in R, \quad x \in R$$

where $\phi(x)$ and $\mathbf{\Phi}(x)$ are the probability density function and the cumulative distribution function of the standard normal distribution, respectively. Here λ is the skewness parameter [9].

The multivariate skew normal distribution is defined as follows. Random vector $\mathbf{y}$ is said to follow a multivariate skew normal distribution if the joint density of $\mathbf{y}$ takes the following form:

$$f(\mathbf{y}) = 2\phi_k(\mathbf{y}, \Omega)\mathbf{\Phi}(\mathbf{t}'\mathbf{y}) \quad \mathbf{y} \in R^k \tag{2}$$

where the skewness vector $\mathbf{t} \in R^k$ is the vector of k real numbers. $\phi_k(\mathbf{y}, \Omega)$ is the density of k-dimensional normal density with the covariance matrix Ω. Note that $\mathbf{t}'\mathbf{y} \in R$ and $\mathbf{\Phi}(x)$ is the CDF of the standard normal distribution. The studies of multivariate skew normal distribution can be seen in Azzalini and Dalla Valle [10], Gupta and Chen [11] as well as Gupta and Chen [12]. For the application of the multivariate skew normal distributions, Azzalini and Capitanio [13] used the model to analyse a set of sport data, Gupta and Chen [14] applied the univariate model to study a set of epidemiology data on disease occurrence rates in twins.

The material in this article is arranged as follows. Section 2 sets the scene and defines the skew normal matrix variate model, which is followed by section 3 discussing the moment generating function. Section 4 proves the result that the quadratic forms of a skew normal matrix variate follows a Wishart distribution, which indicates that there exists a non-normal random matrix $\mathbf{Y}$ such that $\mathbf{YY}'$ follows a Wishart distribution.

2. Matrix variate skew normal distributions

Prior to the discussion of statistical properties, we provide a definition of the matrix variate skew normal distribution. For notational convenience, we use a bold upper case letter to represent a random matrix or its realization, a bold lower case letter to indicate a random vector or its realization and an upper case letter to denote a random variable.

Definition 1 *An $m \times n$ random matrix X is said to follow a matrix variate skew normal distribution if the density of X takes the following form,*

$$f(\mathbf{X}) = c^* \phi_{m,n}(\mathbf{X}, \Sigma \otimes \mathbf{\Psi})\mathbf{\Phi}_n(\mathbf{X}'\mathbf{b}, \Psi) \quad \mathbf{X} \in R^{m\times n} \tag{3}$$

where $\phi_{m,n}(\mathbf{X}, \Sigma \otimes \mathbf{\Psi})$ is the density of the standard normal matrix variate given in equation (1) with $\mathbf{M} = \mathbf{0}_{m\times n}$ and covariance matrix $\Sigma \otimes \mathbf{\Psi}$; $\mathbf{b} \in R^m$, $\mathbf{\Phi}_n(\mathbf{X}'\mathbf{b}, \mathbf{\Psi})$ is the cumulative distribution function of an n-dimensional normal random vector with covariance matrix $\mathbf{\Psi}$, and $c^* = (\mathbf{\Phi}_n(\mathbf{0}, (1 + \mathbf{b}'\Sigma\mathbf{b})\mathbf{\Psi}))^{-1}$. In the sequel, the skew normal matrix variate X is denoted as $\mathbf{X} \sim SN_{m,n}(\mathbf{b}, \Sigma, \mathbf{\Psi})$.

Note: When $n = 1$ and $m = k$, the density in equation (3) becomes the density of the multivariate skew normal introduced in equation (2). The following theorem clarifies the eligibility for the function defined in equation (3) to be a density.

Theorem 1 *Let $f(\mathbf{X}) = c^* \phi_{m,n}(\mathbf{X}, \Sigma \otimes \mathbf{\Psi})\mathbf{\Phi}_n(\mathbf{X}'\mathbf{b}, \mathbf{\Psi})$, then*

$$\int f(\mathbf{X})\mathrm{d}\mathbf{X} = 1. \tag{4}$$

We need the following lemma for the proof of Theorem 1.

Lemma 1 *Let $U_{m\times n} \sim N_{m,n}(\mathbf{0}_{m\times n}, \Sigma_1 \otimes \mathbf{\Psi}_1)$ where $\mathbf{\Psi}_1$ is an $n \times n$ positive definite matrix and Σ_1 is an $m \times m$ positive definite matrix. For any constant vector, $\mathbf{a} \in R^n$ and $\mathbf{q} \in R^m$,*

$$E_{\mathbf{U}}[\mathbf{\Phi}_n(\mathbf{a} + \mathbf{U}'\mathbf{q}, \mathbf{\Psi}_1)] = \mathbf{\Phi}_n(\mathbf{a}, c_0\mathbf{\Psi}_1), \tag{5}$$

where $c_0 = (1 + \mathbf{q}'\Sigma_1\mathbf{q})$.

Proof The left-hand side of equation (5) reads

$$\begin{aligned} E_{\mathbf{U}}[\mathbf{\Phi}_n(\mathbf{a} + \mathbf{U}'\mathbf{q}, \mathbf{\Psi}_1)] &= E_{\mathbf{U}}[P(\mathbf{z} \le \mathbf{a} + \mathbf{U}'\mathbf{q}|\mathbf{U})] \quad \text{where } \mathbf{z} \sim N_n(\mathbf{0}_{n\times 1}, \mathbf{\Psi}_1) \\ &= E_{\mathbf{U}}[P(\mathbf{z} - \mathbf{U}'\mathbf{q} \le \mathbf{a}|\mathbf{U})] \\ &= P(\mathbf{z} - \mathbf{U}'\mathbf{q} \le \mathbf{a}), \end{aligned} \tag{6}$$

since $\mathbf{z} \sim N_n(\mathbf{0}_{n\times 1}, \mathbf{\Psi}_1)$, $\mathbf{U} \sim N_{m,n}(\mathbf{0}_{m\times n}, \Sigma_1 \otimes \mathbf{\Psi}_1)$, and $\mathbf{z}$ and $\mathbf{U}$ are independent. Thus $\mathbf{U}'\mathbf{q} \sim N_n(\mathbf{0}_{n\times 1}, \mathbf{q}'\Sigma_1\mathbf{q}\mathbf{\Psi}_1)$ and $\mathbf{v} = \mathbf{z} - \mathbf{U}'\mathbf{q} \sim N_n(\mathbf{0}_{n\times 1}, (1 + \mathbf{q}'\Sigma_1\mathbf{q})\mathbf{\Psi}_1)$. From equation (6) we get

$$\begin{aligned} E_{\mathbf{U}}[\mathbf{\Phi}_n(\mathbf{a} + \mathbf{U}'\mathbf{q}, \mathbf{\Psi}_1)] &= P(\mathbf{z} - \mathbf{U}'\mathbf{q} \le \mathbf{a}) \\ &= P(\mathbf{v} \le \mathbf{a}) \\ &= \mathbf{\Phi}_n(\mathbf{a}, c_0\mathbf{\Psi}_1), \end{aligned}$$

which is the right-hand side of equation (5). ■

Proof of Theorem 1 With Lemma 1, we can readily prove Theorem 1. Notice that the left-hand side of equation (4) can be expressed as follows:

$$\begin{aligned} \int f(\mathbf{X})\mathrm{d}\mathbf{X} &= c^* E_{\mathbf{V}}[\mathbf{\Phi}_n(\mathbf{V}'\mathbf{b}, \mathbf{\Psi})] \quad \text{with } \mathbf{V} \sim N_{m,n}(\mathbf{0}_{m,n}, \Sigma \otimes \mathbf{\Psi}) \\ &= c^* \mathbf{\Phi}_n(\mathbf{0}_{n\times 1}, (1 + \mathbf{b}'\Sigma\mathbf{b})\mathbf{\Psi}) \quad \text{by Lemma 1 with } \mathbf{a} = \mathbf{0}_{n\times 1} \\ &= 1, \end{aligned}$$

which completes the proof of Theorem 1. ■

Remark 1 When the underlying population is a normal matrix variate, if the treatments are independent, we have $\mathbf{\Psi} = \mathbf{I}_{n\times n}$; if the measurements are not correlated, we get $\Sigma = \mathbf{I}_{m\times m}$. However, for the skew normal random matrix variate defined in this paper, $\mathbf{\Psi} = \mathbf{I}_{n\times n}$ does not imply that the treatments are independent; $\Sigma = \mathbf{I}_{m\times m}$ does not imply that the measurements are independent, due to the skewness factor $\mathbf{b}$ in the density of the skew normal random matrix.

Remark 2 For the skew normal matrix variate, when $\mathbf{\Psi} = \mathbf{I}_{n\times n}$, $c^* = 2^n$, the density in equation (3) becomes

$$f(\mathbf{X}) = 2^n \phi_{m,n}(\mathbf{X}, \Sigma \otimes \mathbf{I})\mathbf{\Phi}_n(\mathbf{X}'\mathbf{b}, \mathbf{I}_{n\times n}), \quad \mathbf{b} \in R^n. \tag{7}$$

When $\Sigma = \mathbf{I}_{m\times m}$, the density in (3) becomes

$$f(\mathbf{X}) = c^* \phi_{m,n}(\mathbf{X}, \mathbf{I} \otimes \mathbf{\Psi})\mathbf{\Phi}_n(\mathbf{X}'\mathbf{b}, \mathbf{\Psi}), \quad \mathbf{b} \in R^n. \tag{8}$$

The densities in equations (7) and (8) are two commonly applied skew normal matrix variate models.

In the following sections, we discuss statistical properties of the matrix variate skew normal model.

3. Moment generation function

We now derive the moment generating function of the skew normal matrix variate defined in the preceding section.

THEOREM 2 *Let $m \times n$ random matrix $\mathbf{X} \sim SN_{m,n}(\mathbf{b}, \Sigma \otimes \mathbf{\Psi})$, the moment generating function of $\mathbf{X}$, $M(\mathbf{T})$, reads*

$$M(\mathbf{T}) = c^* \mathrm{etr}\left[\left(\frac{1}{2}\Sigma\mathbf{T}\mathbf{\Psi}\mathbf{T}'\right)\right] \mathbf{\Phi}_n[\mathbf{\Psi}\mathbf{T}'\Sigma\mathbf{b}, (1 + \mathbf{b}'\Sigma\mathbf{b})\mathbf{\Psi}] \tag{9}$$

where $\mathbf{T} \in R^{m\times n}$.

Proof By the definition of the moment generating function for matrix variates in conjunction with the density of $\mathbf{X}$, for $\mathbf{T} \in R^{m\times n}$, $M(\mathbf{T}) = E(\mathrm{etr}(\mathbf{X}\mathbf{T}'))$ reads

$$M(\mathbf{T}) = c^* \int (2\pi)^{-(mn/2)} \det(\Sigma)^{-n/2} \det(\mathbf{\Psi})^{-m/2} \times \mathrm{etr}\left(-\frac{1}{2}\Sigma^{-1}\mathbf{X}\mathbf{\Psi}^{-1}\mathbf{X}' + \mathbf{X}\mathbf{T}'\right) \mathbf{\Phi}_n(\mathbf{X}'\mathbf{b}, \mathbf{\Psi})\mathrm{d}\mathbf{X}.$$

Since

$$\mathrm{etr}\left(-\frac{1}{2}\Sigma^{-1}\mathbf{X}\mathbf{\Psi}^{-1}\mathbf{X}'\right) = \mathrm{etr}\left(-\frac{1}{2}\Sigma^{-1/2}\mathbf{X}\mathbf{\Psi}^{-1}(\Sigma^{-1/2}\mathbf{X})'\right),$$

letting $\mathbf{Y}_{m\times n} = \Sigma^{-1/2}\mathbf{X}$ yields

$$M(\mathbf{T}) = c^* \int (2\pi)^{-(mn/2} \det(\mathbf{\Psi})^{-m/2} \mathrm{etr}\left(-\frac{1}{2}\mathbf{Y}\mathbf{\Psi}^{-1}\mathbf{Y}' + \Sigma^{1/2}\mathbf{Y}\mathbf{T}'\right) \mathbf{\Phi}_n(\mathbf{Y}'\Sigma^{1/2}\mathbf{b}, \mathbf{\Psi})\mathrm{d}\mathbf{Y}. \tag{10}$$

Now, notice that $\mathrm{tr}(\Sigma^{1/2}\mathbf{Y}\mathbf{T}') = \mathrm{tr}(\mathbf{Y}\mathbf{T}'\Sigma^{1/2})$ and

$$\mathrm{tr}\left(-\frac{1}{2}\mathbf{Y}\boldsymbol{\Psi}^{-1}\mathbf{Y}'\right) = \mathrm{tr}\left[-\frac{1}{2}(\mathbf{Y} - \Sigma^{1/2}\mathbf{T}\boldsymbol{\Psi})\boldsymbol{\Psi}^{-1}(\mathbf{Y} - \Sigma^{1/2}\mathbf{T}\boldsymbol{\Psi})'\right] - \mathrm{tr}(\mathbf{Y}\mathbf{T}'\Sigma^{1/2}) + \mathrm{tr}\left(\frac{1}{2}\Sigma^{1/2}\mathbf{T}\boldsymbol{\Psi}\mathbf{T}'\Sigma^{1/2}\right).$$

From equation (10) we have

$$\begin{aligned} M(\mathbf{T}) &= c^*\mathrm{etr}\left(\frac{1}{2}\Sigma\mathbf{T}\boldsymbol{\Psi}\mathbf{T}'\right)\int (2\pi)^{-(mn/2}\det(\boldsymbol{\Psi})^{-m/2} \\ &\quad\times \mathrm{etr}\left[-\frac{1}{2}(\mathbf{Y} - \Sigma^{1/2}\mathbf{T}\boldsymbol{\Psi})\boldsymbol{\Psi}^{-1}(\mathbf{Y} - \Sigma^{1/2}\mathbf{T}\boldsymbol{\Psi})'\right]\boldsymbol{\Phi}_n(\mathbf{Y}'\Sigma^{1/2}\mathbf{b},\,\boldsymbol{\Psi})\mathrm{d}\mathbf{Y} \\ &= c^*\mathrm{etr}\left(\frac{1}{2}\Sigma\mathbf{T}\boldsymbol{\Psi}\mathbf{T}'\right)E_{\mathbf{U}}[\boldsymbol{\Phi}_n((\mathbf{U} + \Sigma^{1/2}\mathbf{T}\boldsymbol{\Psi})'\Sigma^{1/2}\mathbf{b},\,\boldsymbol{\Psi})] \end{aligned} \tag{11}$$

where $\mathbf{U} \sim N_{m,n}(\mathbf{0}_{m\times n}, \mathbf{I}_{m\times m} \otimes \boldsymbol{\Psi})$. Thus equation (11) can be written as

$$\begin{aligned} M(\mathbf{T}) &= c^*\mathrm{etr}\left(\frac{1}{2}\Sigma\mathbf{T}\boldsymbol{\Psi}\mathbf{T}'\right)E_{\mathbf{U}}[\boldsymbol{\Phi}_n(\mathbf{U}'\Sigma^{1/2}\mathbf{b} + \boldsymbol{\Psi}\mathbf{T}'\Sigma\mathbf{b},\,\boldsymbol{\Psi})] \\ &= c^*\mathrm{etr}\left(\frac{1}{2}\Sigma\mathbf{T}\boldsymbol{\Psi}\mathbf{T}'\right)\boldsymbol{\Phi}_n(\boldsymbol{\Psi}\mathbf{T}'\Sigma\mathbf{b},\,(1 + \mathbf{b}'\Sigma\mathbf{b})\boldsymbol{\Psi}), \end{aligned}$$

with $\mathbf{a} = \boldsymbol{\Psi}\mathbf{T}'\Sigma\mathbf{b}$, $\mathbf{q} = \Sigma^{1/2}\mathbf{b}$ and $\Sigma_1 = \mathbf{I}_{m\times m}$ in Lemma 1. This completes the proof of Theorem 2. ■

Another important property of the newly proposed model is the distribution of its quadratic form.

4. Distribution of quadratic forms

In this section, we discuss the quadratic form of the skew normal matrix variates. For univariate skew normal random variables, as mentioned in Azzalini [9], the quadratic form of a skew normal random variable X is $X^2 \sim \chi_1^2$, which is the same as the distribution of Y^2 when Y follows the standard normal distribution. For the multivariate skew normal random vectors, if $\mathbf{v} \sim SN_k(\mathbf{0}_{k\times 1}, \mathbf{I}_{k\times k})$, $\mathbf{v}'\mathbf{v} \sim \chi_k^2$, which is the same as the distribution of the quadratic form $\mathbf{z}'\mathbf{z}$ with $\mathbf{z} \sim N_k(\mathbf{0}_{k\times 1}, \mathbf{I}_{k\times k})$, the standard k-dimensional normal distribution. For the matrix variates, the quadratic form $\mathbf{X}\mathbf{X}'$ of the normal matrix variate $\mathbf{X}$ follows the Wishart distribution [7], and we observe that, for the skew normal matrix variate defined in this paper, the distribution of its quadratic form follows a Wishart distribution.

THEOREM 3 *If an $m \times n$ matrix variate $\mathbf{X} \sim SN_{m,n}(\mathbf{b}, \Sigma \otimes \boldsymbol{\Psi})$ with $\Psi = \mathbf{I}_{n\times n}$, then $\mathbf{X}\mathbf{X}' \sim W_m(n, \Sigma)$.*

Proof The moment generating function of the quadratic form of $\mathbf{X}$, $\mathbf{XX}'$, can be obtained as follows, for any $\mathbf{T} \in R^{m\times m}$, with $\Sigma^{-1} - 2\mathbf{T}$ being a positive definite matrix,

$$
\begin{aligned}
E[\mathrm{etr}(\mathbf{XX}'\mathbf{T})] &= \int \mathrm{etr}(\mathbf{XX}'\mathbf{T}) f(\mathbf{X})\mathrm{d}\mathbf{X} \\
&= \int 2^n (2\pi)^{-mn/2} \det(\Sigma)^{-n/2} \mathrm{etr}\left(-\frac{1}{2}\Sigma^{-1/2}\mathbf{XX}'\Sigma^{-1/2} + \mathbf{XX}'\mathbf{T}\right) \mathbf{\Phi}_n(\mathbf{X}'\mathbf{b})\mathrm{d}\mathbf{X} \\
&\quad \text{by the density of } \mathbf{X} \text{ given in equations (1) and (7) with } \mathbf{\Psi} = \mathbf{I}_{n\times n} \\
&= \int 2^n (2\pi)^{-mn/2} \det(\Sigma)^{-n/2} \mathrm{etr}\left(-\frac{1}{2}\mathbf{X}'(\Sigma^{-1} - 2\mathbf{T})\mathbf{X}\right) \mathbf{\Phi}_n(\mathbf{X}'\mathbf{b})\mathrm{d}\mathbf{X} \\
&= 2^n [\det(\Sigma^{-1} - 2\mathbf{T})]^{-n/2} [\det(\Sigma)]^{-n/2} E_{\mathbf{W}}(\mathbf{\Phi}_n(\mathbf{W}'\mathbf{b})) \\
&\quad \text{where } \mathbf{W} \sim N_{m,n}(\mathbf{0}_{m\times n}, \Sigma \otimes \mathbf{I}_{n\times n}) \\
&\quad \text{Using Lemma 1 with } \Sigma_1 = \Sigma^{-1} - 2\mathbf{T},\ \mathbf{a} = \mathbf{0}_{n\times 1},\ \mathbf{q} = \mathbf{b} \text{ and } \Psi_1 = \mathbf{I}_{n\times n} \\
&= 2^n [\det(\Sigma^{-1} - 2\mathbf{T})]^{-n/2} [\det(\Sigma)]^{-n/2} \mathbf{\Phi}_n(\mathbf{0}_{n\times 1}) \\
&= \det(\mathbf{I}_{m\times m} - 2\mathbf{T}\Sigma)^{-n/2},
\end{aligned}
$$

which is the moment generating function of the Wishart distribution $W_m(n, \Sigma)$. ■

Theorem 3 implies that the decomposition for a Wishart matrix variate is not unique. The decomposition theorem states as follows [7]:

$$
\text{If} \quad \mathbf{S} \sim W_p(n, \Sigma) \quad \text{then} \quad \mathbf{S} = \mathbf{XX}' \quad \text{where} \quad \mathbf{X} \sim N_{p,n}(\mathbf{0}_{p\times n}, \Sigma \otimes \mathbf{I}_{n\times n}). \tag{12}
$$

With Theorem 3, we show that the quadratic form of a skew normal matrix variate is also a Wishart variate. Thus there exists a random matrix $\mathbf{Y} \sim SN_{p,n}(\mathbf{b}, \Sigma \otimes \mathbf{I}_{n\times n})$ with any $\mathbf{b} \in R^p$ such that $\mathbf{YY}' \sim W_p(n, \Sigma)$. In this sense, the decomposition in equation (12) is not unique.

Theorem 3 provides the distribution of a simple quadratic form of the skew normal matrix variate to show the non-uniqueness of the decomposition theorem. The general expression for the distribution of the general quadratic form $\mathbf{XAX}'$, when $\mathbf{X} \sim SN_{m,n}(\mathbf{b}, \Sigma \otimes \mathbf{\Psi})$, is tedious and it does not provide any new insight on the matrix variate distribution theory. We therefore do not include it in this paper. Interested readers are referred to chapter 7 of Gupta and Nagar [7] for similar arguments on the matrix variate normal distributions.

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