Full Conditionals for B & Σ based on Matrix Normal Multivariate Linear Regression

Model: $\underset{N\times k}{Y} \mid \underset{p\times k}{B}, \underset{k\times k}{\Sigma} = \underset{N\times p}{X}\,\underset{p\times k}{B} + \underset{N\times k}{E}$ $\quad E \sim \text{MatNorm}(\underset{\sim}{0}, I_N, \Sigma)$

I_N = N-Dim. Identity Mat.

Priors: $B \mid \Sigma \sim \text{MatNorm}(B_0, T_0^{-1}, \Sigma)$ $\quad T_0$ = Prior Precision

$\Sigma \sim IW(\nu, \underset{k\times k}{C_0})$

NOTE: Prior for B depends on Σ!

So (B, Σ) Forms Conjugate joint MatNorm-IW Prior

(B Not $\perp\!\!\!\perp$ of Σ a priori)

Full Cond. for B:

$$B \mid \Sigma, Y \propto e^{-\frac{1}{2}\left\{tr[\Sigma^{-1}(Y-XB)'(Y-XB)] + tr[\Sigma^{-1}(B-B_0)'T_0(B-B_0)]\right\}}$$

$$\propto e^{-\frac{1}{2} tr\left\{\Sigma^{-1}\left[\underbrace{(Y-XB)'(Y-XB) + (B-B_0)'T_0(B-B_0)}_{B'(X'X+T_0)B \,-\, 2(T_0B_0+X'Y)B}\right]\right\}}$$

$$\propto e^{-\frac{1}{2} tr[\Sigma^{-1}(B-M)'V^{-1}(B-M)]}$$

Where: $\underset{p\times p}{V} = (T_0 + X'X)^{-1}$

$\underset{p\times k}{M} = V(T_0B_0 + X'Y)$

$\therefore B \mid Y, \Sigma \sim \text{MatNorm}(M, V, \Sigma)$

Equivalently, $\underset{pk\times 1}{Vec(B)} \sim N_{pk}(Vec(M), \Sigma \otimes V)$

Key Observation: Can Update V & M using Low-Dimensional MatNorm Formulas, Then Draw $Vec(B)$ using Low-Dim (pk) Multivariate Normal.

$$\Sigma \mid Y, B \propto \underbrace{|\Sigma|^{-n/2} e^{-\frac{1}{2} tr[\Sigma^{-1}(Y-XB)'(Y-XB)]}}_{f(Y \mid B, \Sigma)} \cdot \underbrace{|\Sigma|^{-p/2} e^{-\frac{1}{2} tr[\Sigma^{-1}(B-B_0)' T_0 (B-B_0)]}}_{\pi(B \mid \Sigma)} \cdot \underbrace{|\Sigma|^{-\frac{\nu+k+1}{2}} e^{-\frac{1}{2} tr(\Sigma^{-1} C_0)}}_{\pi(\Sigma)}$$

$$\propto |\Sigma|^{-\frac{(n+\nu+p)+k+1}{2}} e^{-\frac{1}{2} tr\{\Sigma^{-1}[C_0 + (Y-XB)'(Y-XB) + (B-B_0)' T_0 (B-B_0)]\}}$$

$$\therefore \quad \Sigma \mid Y, B \sim IG(\nu^*, \underset{k \times k}{C^*})$$

Where: $\nu^* = \nu + n + p$

$$C^* = \underset{k \times k}{C_0} + \underset{k \times N}{(Y - X\beta)'} \underset{N \times k}{(Y - X\beta)} + \underset{k \times p}{(B - \beta_0)'} \underset{p \times p}{T_0} \underset{p \times k}{(B - \beta_0)}$$