Lecture 9: Two-sample problem I

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Recall

- One sample sign test, Wilcoxon signed rank test, large-sample approximation, median, Hodges-Lehman estimator, distribution-free confidence interval.
- Jackknife for bias and standard error of an estimator.
- Bootstrap samples, bootstrap replicates.
- Bootstrap standard error of an estimator.
- Bootstrap percentile confidence interval.
- Hypothesis testing with the bootstrap (one-sample problem.)
- Assessing the error in bootstrap estimates.
- Example: inference on ratio of heart attack rates in the aspirin-intake group to the placebo group.
- The exhaustive bootstrap distribution
- Discrete data problems (one-sample, two-sample proportion tests, test of homogeneity, test of independence)

Two-sample location problem

- Data: two samples from two populations. $X_1, \cdots X_m$ from population 1 and corresponding to distribution function $F(.)$ and $Y_1, \cdots Y_n$ and corresponding to distribution function $G(.)$.
- Assumptions:
  - X and Y are mutually independent.
  - $F(.)$ and $G(.)$ are continuous.
  - Populations differ only by a location shift.
- Hypothesis
  - $\mathrm{H}_0 : F(t) = G(t)$ for each t versus $\mathrm{H}_A : \Delta \neq 0$, where $G(t) = F(t - \Delta)$.
  - Other alternatives $\mathrm{H}_A : \Delta > 0$, $\mathrm{H}_A : \Delta < 0$ (stochastic ordering of alternatives).
  - For example, Δ can be expected treatment effect compared to control.

Two-sample location problem

- Wilcoxon two-sample rank sum test
  - Combined sample of $N = m + n$.
  - Let S_n denote the rank of Y_n in this joint ordering.
  - Wilcoxon two-sample rank sum statistic $W = \sum_{j=1}^{n} S_j$.
- R command: `wilcox.test(x, y, paired = FALSE, alternative = "two.sided")`.
- Under H_0, all $\binom{N}{n}$ possible assignments for the Y-ranks are equally likely, each having probability $\frac{1}{\binom{N}{n}}$.

Two-sample location problem

- Mann–Whitney statistic
  - $U = \sum_{i=1}^{m} \sum_{j=1}^{n} \phi(X_i, Y_j)$, where $\phi(X_i, Y_j) = 1 \quad if \quad X_i < Y_j$.
- Tests based on W and U are equivalent.
  - $W = U + \dfrac{n(n+1)}{2}$.

Two-sample location problem (large-sample approximation)

- Test statistic: the standardized version of W.
  - $Z = \dfrac{W - \{m\,(m+n+1)\,/2\}}{\{mn\,(m+n+1)\,/12\}^{1/2}}$.
  - $Z \sim \mathrm{N}\,(0, 1)$.
- If there are ties
  - Use the average of the ranks.
  - not an exact test.

Example (Two-sample location problem)

- **HWC** Example 4.1
- Whether there is a difference in the transfer of tritiated water (water containing tritium, a radioactive isotope of hydrogen) across the tissue layers in the term human chorioamnion (a placental membrane) and in the human chorioamnion between 3- and 6-months' gestational age.
- Data: measurd the permeability constant Pd of the human chorioamnion to water.
  - collected data after 5 min of delivery from the placentas of healthy subjects, between 12 and 26 weeks following termination of pregnancy and
  - from term, uncomplicated vaginal deliveries.
- $\mathrm{H}_0 : \Delta = 0$ versus $\mathrm{H}_A : \Delta \neq 0$.

Example (Two-sample location problem)

```r
at.term = c(.80, .83, 1.89, 1.04, 1.45,
  1.38, 1.91, 1.64, .73, 1.46)
gest.age = c(1.15, .88, .90, .74, 1.21)
wilcox.test(at.term, gest.age,
  alternative="greater", conf.int=T)
```

```
##
##  Wilcoxon rank sum test
##
## data:  at.term and gest.age
## W = 35, p-value = 0.1272
## alternative hypothesis: true location shift is greater t
## 95 percent confidence interval:
##  -0.08   Inf
## sample estimates:
## difference in location
##                  0.305
```

Two-sample location problem (Hodges–Lehmann estimator)

- Compute mn differences $Y_j - X_i; i = 1, \cdots, m, j = 1, \cdots, n$. $\hat{\Delta} = \text{median}\{Y_j - X_i; i = 1, \cdots, m, j = 1, \cdots, n\}$.
- `wilcox.test` gives a point estimator: `difference in location`
- In our example $\hat{\Delta} = .305$.
- There is no need to perform the large-sample approximation because we have the result for the exact test.

Comments on Wilcoxon rank sum test

- The confidence interval for Δ is found directly from the R command wilcox.test.
- The significance level of the rank sum test is not preserved if the two populations differ in dispersion or shape.
- If $\mathrm{H}_0 : \Delta = \Delta_0$, form a pseudo-sample $Y_1' = Y_1 - \Delta_0, \cdots, Y_n' = Y_n - \Delta_0$.
- Run Wilcoxon rank sum test on $X_1, \cdots, X_m, Y_1', \cdots, Y_n'$.
- In a two-sample problem, we sometimes wish to estimate $\delta = P(X < Y)$. Read **HWC** page 138 - 141, comment 18.
  - For example in quantifying stress-strength reliability.
  - X stress (random) applied to a system.
  - Y strength of the system.

References for this lecture

HWC Chapter 4.1-4.3