Derivation of Skew Normal Density

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In the 2010 paper by Sylvia Frühwirth-Schnatter and Saumyadipta Pyne titled *Bayesian inference for finite mixtures of univariate and multivariate skew-normal and skew-t distributions*, the authors claim that a standard skew-normal random variable X as defined by Azzalini (1985) can be expressed as as a convex combination of a truncated normal and standard normal random variable. Specifically, if $Z \sim \mathcal{TN}_{[0,\infty)}(0,1)$ and $\epsilon \sim \mathcal{N}(0,1)$, then X defined as

$$X = \delta Z + \sqrt{1-\delta^2}\epsilon$$

is a standard skew-normal random variable with parameter $\alpha = \frac{\delta}{\sqrt{1-\delta^2}}$. The authors claim that X has density function $2\phi(x)\Phi(\alpha X)$, which is Azzalini's original definition of the standard skew-normal distribution. Below is a proof of this claim, which will start with finding an expression for the distribution function of X. First though, note the following re-parameterization.

$$X = \delta Z + \sqrt{1-\delta^2}\epsilon = \sqrt{\frac{\alpha^2}{1+\alpha^2}}Z + \sqrt{\frac{1}{1+\alpha^2}}\epsilon$$

$$= \frac{\alpha}{\sqrt{1+\alpha^2}}Z + \frac{1}{\sqrt{1+\alpha^2}}\epsilon = aZ + b\epsilon$$

$$P(X \leq x) = P(aZ + b\epsilon \leq x) = \int_0^\infty P(b\epsilon \leq x - az | Z = z)P(Z = z)dz$$

$$\int_0^\infty P(\epsilon \leq \frac{x-az}{b} | Z = z)P(Z = z)dz = \int_0^\infty \Phi(\frac{x-az}{b})2\phi(z)dz$$

Note that the pdf of Z, a truncated standard normal random variable, is $2\phi(z)$. We wish to differentiate this last expression with respect to x in order to obtain the density function of X. Note that the order of integration and differentiation can be switched since $\Phi(\frac{x-az}{b})2\phi(z)$ is continuously differentiable on $[0,\infty)$.

$$\frac{d}{dx}\int_0^\infty \Phi(\frac{x-az}{b})2\phi(z)dz = \int_0^\infty \frac{d}{dx}\Phi(\frac{x-az}{b})2\phi(z)dz$$

$$= 2\int_0^\infty \phi(\frac{x-az}{b})\frac{1}{b}\phi(z)dz$$

$$= 2\int_0^\infty \frac{1}{\sqrt{2\pi b^2}}e^{-(\frac{x-az}{b})^2/2}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz$$

$$= 2\int_0^\infty \frac{1}{\sqrt{2\pi b^2}}e^{-\frac{x^2-2xaz+a^2z^2}{2b^2}}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz$$

$$= 2\int_0^\infty \frac{1}{\sqrt{2\pi b^2}}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2+2xaz-a^2z^2-b^2z^2}{2b^2}}dz$$

$$= 2\int_0^\infty \frac{1}{\sqrt{2\pi b^2}}\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2+2xaz-z^2(a^2+b^2)}{2b^2}}dz$$

The fact that $a^2 + b^2 = 1$ can be used now.

$$
\begin{aligned}
&= 2\int_0^\infty \frac{1}{\sqrt{2\pi b^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2-2xaz+z^2}{2b^2}}dz \\
&= 2\int_0^\infty \frac{1}{\sqrt{2\pi b^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2-2xaz}{2b^2}}e^{-\frac{x^2}{2b^2}}dz \\
&= 2\int_0^\infty \frac{1}{\sqrt{2\pi b^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-ax)^2-a^2x^2}{2b^2}}e^{-\frac{x^2}{2b^2}}dz \\
&= 2\int_0^\infty \frac{1}{\sqrt{2\pi b^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-ax)^2}{2b^2}}e^{-\frac{x^2-a^2x^2}{2b^2}}dz \\
&= 2\int_0^\infty \frac{1}{\sqrt{2\pi b^2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{(z-ax)^2}{2b^2}}e^{-\frac{x^2}{2}}dz \\
&= 2\phi(x)\int_0^\infty \frac{1}{\sqrt{2\pi b^2}}e^{-\frac{(z-ax)^2}{2b^2}}dz \\
&= 2\phi(x)[1-\Phi(-\frac{a}{b}x)] = 2\phi(x)\Phi(\alpha x)
\end{aligned}
$$