Homework - 02

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Instructions: Completed assignments are due on Monday, January 7, 2019. They are to be uploaded to the Virtual Campus.

Solutions may be submitted by groups of two or three students, possibly different from those in Homework 1. Only a single copy of each group's work should be uploaded (by any member), clearly stating the names of all contributors.

Exercise 1 (Estimating a log-odds with a normal prior - End-of-chapter Exercise 1 from Jim Albert (2009), Bayesian computations with R, 2nd ed., chapter 5 - See the Docs folder in the *Laplace approximation* chapter - Virtual Campus.)**.** *Suppose y has a binomial distribution with parameters n and p, and we are interested in the log-odds value $\theta = \log(p/(1-p))$. Our prior for θ is that $\theta \sim \mathrm{N}(\mu, \sigma)$. It follows that the posterior density of θ is given, up to a proportionality constant, by*

$$g(\theta|y) \propto \frac{exp(y\,\theta)}{(1+exp(\theta))^n}\,\exp\left[-\frac{(\theta-\mu)^2}{2\,\sigma^2}\right].$$

More concretely, suppose we are interested in learning about the probability that a special coin lands heads when tossed. A priori we believe that the coin is fair, so we assign θ a $\mathrm{N}(0, 0.25)$ prior. We toss the coin $n = 5$ times and obtain $y = 5$ heads.

a) Using a normal approximation to the posterior density, compute the probability that the coin is biased toward heads (i.e., that θ is positive).

b) Using the prior density as a proposal density, design a rejection algorithm for sampling from the posterior distribution. Using simulated draws from your algorithm, approximate the probability that the coin is biased toward heads.

c) Using the prior density as a proposal density, simulate values from the posterior distribution using the SIR algorithm. Approximate the probability that the coin is biased toward heads.

Exercise 2 (Robert, Casella (2010), *Introducing Monte Carlo Methods with R*, Exercise 6.13, Chapter 6 and Example 7.11, Chapter 7, and Robert, Casella (2004), *Monte Carlo statistical methods,* Example 7.11. See the Docs folder - Virtual Campus.)**.** *In 1986, the space shuttle* Challenger *exploded during takeoff, killing the seven astronauts aboard. The explosion was the result of an O-ring failure, a splitting of a ring of rubber that seals the parts of the ship together. The accident was believed to have been caused by the unusually cold weather (31^oF or 0^oC) at the time of launch, as there is reason to believe that the O-ring failure probabilities increase as temperature decreases. Data on previous space shuttle launches and O-ring failures are given in the dataset* `challenger` *provided with the* `mcsm` *package. The first column corresponds to the failure indicators y_i and the second column to the corresponding temperature x_i ($1 \le i \le 23$).*

1. *Fit this dataset with a classic-frequentist logistic regression, where:*

$$\mathrm{P}(Y_i = 1|x_i) \equiv p(x_i) = \mathrm{Logistic}(\alpha + \beta\,x_i) = \frac{\exp(\alpha+\beta\,x_i)}{1+\exp(\alpha+\beta\,x_i)},$$

using the `glm` *function, obtaining the classic MLE's for α and β, along with standard errors. Estimate the probabilities of failure at $60^o F$, $50^o F$, and $40^o F$.*

2. *Bayesian version: As suggested in Robert, Casella (2004), pag. 282, take a flat prior for β and, for α, consider it the logarithm of a new parameter:*

$$\alpha = \log(w), \quad \textit{where } w \sim \operatorname{Exp}\left(\frac{1}{s}\right), \quad s > 0,$$

then, $w = e^{\alpha}$ and the α prior pdf is:

$$h(\alpha|s) = \frac{1}{s}\, e^{\alpha} \exp\left(-\frac{1}{s}\, e^{\alpha}\right), \quad \alpha \in \mathbb{R}, \quad s > 0. \tag{1}$$

Warning! *In page 282, it reads: "an exponential prior on* $\log \alpha$*", instead of the other way round, which is the correct pdf. The expectation of α with this distribution, is:*

$$E(\alpha) = \log s - \gamma, \quad \textit{where } \gamma = 0.577216 = -\int_0^{\infty} e^{-t} \log t \, dt$$

is Euler's constant. Set up, run and discuss the output of a Metropolis-Hastings algorithm with an independent candidate:

$$g(\alpha, \beta) = h(\alpha|\hat{s}) \cdot \phi(\hat{b}),$$

where ϕ is either a normal pdf, as suggested in R.C. (2004) or a Laplace pdf, as suggested in R.C. (2010), where $\hat{s}$ and $\hat{b}$ are chosen as described there. Study from the simulation the probabilities of failure at $60^o F$, $50^o F$, and $40^o F$.

3. *(Bivariate product slice sampler) Write the likelihood of this logistic regression model as a product of n functions and follow the derivations in R.C. (2010), pag. 220. Function* `challenge` *in the* `mcsm` *package implements a bivariate product slice sampler for this example. List* (i.e., understand the underlying algebra & document/improve the code) *the function (*Warning: *Robert-Cassella switch notations $(\alpha, \beta) \to (a, b)$ in mid-example).* Hint: *To generate a new (α, β), uniform on the intersection region, consider each Bernoulli factor in the likelihood function, treating separately those i for which $y_i = 1$ and those for which $y_i = 0$. In the first case the function:*

$$h_i(\alpha, \beta) = p_i(\alpha, \beta) = \operatorname{Logistic}(\alpha + \beta\, x_i),$$

and in the second one,

$$h_i(\alpha, \beta) = 1 - p_i(\alpha, \beta) = 1 - \operatorname{Logistic}(\alpha + \beta\, x_i).$$

Run the simulation. Compare the resulting posterior pdf's and the inferences about failure probabilities with those in the Metropolis-Hastings procedure.

Exercise 3. *Load the* `Credit` *dataset in the* `ISLR` *package.*

(1) Fit the following two classic, frequentist linear models. Describe the results, such as significance of predictors and overall quality of predictions:

```
Credit.lm1<-lm(Balance ~ Limit + Income, data=Credit)
Credit.lm2<-lm(Balance ~ Student + Limit, data=Credit)
```

(2) Using JAGS, prepare Bayesian versions for `Credit.lm1` *and* `Credit.lm2`*, with a discussion of the possible priors.*

(3) Same as above with Stan.
