Evaluating an integral using Feynman's technique, Beta and Gamma function, and Euler's reflection formula

The integral I required to evaluate is given as-

$$I = \int_0^\infty \frac{lnx}{(1+x^2)^2} dx$$

Introducing a new integral parameterization -

$$I(a) = \int_0^\infty \frac{x^a}{(1+x^2)^2} dx$$

Substituting $x = \sqrt{t}$ gives-

$$I(a) = \frac{1}{2} \int_0^\infty \frac{t^{\frac{(a-1)}{2}}}{(1+t)^2} dt$$

Which can be written as-

$$I(a) = \frac{1}{2} \int_0^\infty \frac{t^{\frac{(a+1)}{2}-1}}{(1+t)^{\frac{(a+1)}{2}+\frac{(3-a)}{2}}} dt$$

The Beta function is defined as-

$$\beta\left(m,n\right) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Hence our integral $I(a)$ can be represented as

$$I(a) = \frac{1}{2}\beta\left(\frac{a+1}{2}, \frac{3-a}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{3-a}{2}\right)}{\Gamma\left(\frac{a+1}{2}+\frac{3-a}{2}\right)}$$

$$I(a) = \frac{1}{2}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{3-a}{2}\right)$$

Where $\Gamma(z)$ is the Gamma function and using the property that $\Gamma(z+1) = z\Gamma(z)$ and simplifying we have-

$$I(a) = \frac{(1-a)}{4}\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{1-a}{2}\right)$$
$$I(a) = \frac{(1-a)}{4}\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(1-\frac{1+a}{2}\right)$$

Using Euler's refelction formula, $\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin(\pi z)}$, we have -

$$I(a) = \frac{\pi(1-a)}{4\sin\left(\pi\left(\frac{1+a}{2}\right)\right)}$$

$$I(a) = \frac{\pi(1-a)}{4\cos\left(\frac{\pi a}{2}\right)}$$

The required integral I is the derivative of $I(a)$ evaluated at $a = 0$

$$I = I'(a)\Big|_{a=0}$$

$$I'(a) = \frac{\pi}{4\cos\left(\frac{\pi a}{2}\right)}\left[\frac{\pi}{2}(1-a)\,tan\left(\frac{\pi a}{2}\right) - 1\right]$$

$$I'(0) = -\frac{\pi}{4}$$

Hence,

$$\mathbf{I = \int_0^\infty \frac{lnx}{(1+x^2)^2}dx = -\frac{\pi}{4}}$$

NOTE: The Euler's reflection formula is valid when $|z| < 1$. $I(a)$ will then converge only for $-3 < a < 1$.