A sum involving the Riemann-Zeta function

The required sum S to be evaluated, is given as-

$$S = \sum_{n=2}^{\infty} \frac{\zeta(n)}{\pi^n}$$

Where $\zeta(z)$ is theRiemannn-Zeta function given by-

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

which converges for $z > 1$.

The Bose-Einstein integral relating the Gamma and Zeta function is given as follows -

$$\Gamma(s)\zeta(s) = \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx$$

Using this defination of $\zeta(s)$ in our sum-

$$S = \sum_{n=2}^{\infty} \frac{\pi}{\Gamma(n)} \int_0^{\infty} \frac{\frac{x}{\pi}^{n-1}}{e^x - 1} dx$$

$$S = \pi \int_0^{\infty} \frac{1}{(e^x - 1)} \sum_{n=2}^{\infty} \frac{\frac{x}{\pi}^{n-1}}{\Gamma(n)}$$

Using the series defination o e^x, the inner sum reduces to $e^{\frac{x}{\pi}} - 1$,

$$S = \pi \int_0^\infty \frac{e^{\frac{x}{\pi}} - 1}{e^x - 1} dx$$

Multiplying and dividing by e^{-x}

$$S = \pi \int_0^\infty \frac{e^{-(1-\frac{1}{\pi})x} - e^{-x}}{1 - e^{-x}} dx$$

Since $e^{-x} < 1$ for the given limits of integration, we can expand it using the power series of $\frac{1}{1-x}$ as-

$$S = \pi \int_0^\infty \left(e^{-(1-\frac{1}{\pi})x} - e^{-x}\right) \sum_{n=0}^\infty e^{-nx}$$

$$S = \pi \int_0^\infty \sum_{n=0}^\infty \left(e^{-(n+1-\frac{1}{\pi})x} - e^{-(n+1)x}\right) dx$$

Since

$$\int_0^\infty e^{-ax} dx = \frac{1}{a}$$

for $a > 0$

$$S = \pi \sum_{n=0}^\infty \left(\frac{1}{n+1-\frac{1}{\pi}} - \frac{1}{n+1}\right)$$

$$S = \pi \sum_{n=1}^\infty \left(\frac{1}{n-\frac{1}{\pi}} - \frac{1}{n}\right)$$

The series representation of the digamma function, $\psi(z+1)$ for all not negativw integer values of z, we have-

$$\psi(z+1) = -\gamma + \sum_{n=0}^\infty \left(\frac{1}{n} - \frac{1}{n+z}\right)$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{n+z}-\frac{1}{n}\right)=-\gamma-\psi(z+1)$$

γ is the Euler-Mascheroni constant.

Thus, our required sum is -

$$S=\pi\left(-\gamma-\psi(z+1)\right)\Big|_{z=-\frac{1}{\pi}}$$

$$S=-\gamma\pi-\pi\psi(1-\frac{1}{\pi})$$

$$\boxed{\boldsymbol{\sum_{n=2}^{\infty}\frac{\zeta(n)}{\pi^{n}}=\frac{\zeta(2)}{\pi^{2}}+\frac{\zeta(3)}{\pi^{3}}+\frac{\zeta(4)}{\pi^{4}}+\ldots=-\gamma\pi-\pi\psi(1-\frac{1}{\pi})}}$$

$$\sum_{n=2}^{\infty}\frac{\zeta(n)}{\pi^{n}}\approx 0.6958$$