Multi-Server System Analysis

Prabal Gupta

November 7, 2017

1 Problem Summary

- Analyze a multi-server system receiving periodic requests at rate λ, and having processing times distributed exponentially with mean $\frac{1}{\mu}$.
- Find $B(t)$, which defines the number of busy servers at any given time t.
- Calculate $\mathrm{E}[B(t)]$ as a function of t, and take a limit as $t \to \infty$.

2 Proof

2.1 Formal Statement

Let,

$$\begin{aligned} X_k &\triangleq \text{Arrival time of request } k \\ S_k &\triangleq \text{Service duration for request } k \end{aligned}$$

Now let,

$$X_k \triangleq \sum_{i=1}^{k-1} \Delta X_i \sim f_{X_k}(x) = \mathop{\star}_{i=1}^{k-1} f_{\Delta X_i}(x) \tag{2.1}$$

Since,

$$\Delta X \triangleq X_{k+1} - X_k = \frac{1}{\lambda}$$

We can rewrite X_k as $X_k = (k-1) \cdot \frac{1}{\lambda}$, where $\frac{1}{\lambda} \in \mathbb{R}^+$ is a constant. Thus,

$$\frac{1}{\lambda} \sim f_{\frac{1}{\lambda}}(x) = \delta(x - \frac{1}{\lambda}) \tag{2.2}$$

$$X_k \sim f_{X_k}(x) = \delta(x - (k-1) \cdot \frac{1}{\lambda}) \tag{2.3}$$

where $\delta(x)$ represents the *Dirac Delta Impulse*. Additionally, S_k, the exponential distributed random variable satisfies:

$$S_k \sim f_{S_k}(x) = \mu e^{-\mu x} \cdot u(x) \tag{2.4}$$

where the unit-step function $u(x)$ is defined as:

$$u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Morevover, note that we can rewrite $B(t)$ as:

$$B(t) \triangleq \sum_{i=1}^{k} \left[u(t - X_k) - u(t - (X_k + S_k))\right] \tag{2.5}$$

Now, since X_k and S_k are random variables, so are $u(t-X_k)$ and $u(t-(X_k+S_k))$. Thus, it is easy to observe that:

$$\begin{aligned} u(t - X_k) \sim f_{u(t-X_k)}(x) &= \mathrm{P}\{t < X_k\} \cdot \delta(x) + \mathrm{P}\{t \geq X_k\} \cdot \delta(x-1) \\ &= \overline{F_{X_k}(x)} \cdot \delta(x) + F_{X_k}(x) \cdot \delta(x-1) \end{aligned} \tag{2.6}$$

and similarly,

$$u(t - (X_k + S_k)) \sim f_{u(t-(X_k+S_k))}(x)$$

where,

$$\begin{aligned} f_{u(t-(X_k+S_k))}(x) &= \mathrm{P}\{t < (X_k + S_k)\} \cdot \delta(x) + \mathrm{P}\{t \geq (X_k + S_k)\} \cdot \delta(x-1) \\ &= \overline{F_{(X_k+S_k)}(x)} \cdot \delta(x) + F_{(X_k+S_k)}(x) \cdot \delta(x-1) \end{aligned} \tag{2.7}$$

where $F_G(x)$ represents the *Cumulative Distribution Function* of some random variable G.
Now let $Z_k \triangleq u(t - X_k) - u(t - (X_k + S_k))$ be a random variable. Since,

$$B(t) \triangleq \sum_{i=1}^{k} Z_k \tag{2.8}$$

We can rewrite $B(t)$ as a random variable using **(2.5)** such that,

$$\boxed{B(t) \sim \mathop{\star}_{i=1}^{k} f_{Z_k}(x)} \tag{2.9}$$

2.2 Caculating $\mathbf{E}[B(t)]$ in terms of t

We know from **(2.8)** that $Z_k \triangleq u(t-X_k)-u(t-(X_k+S_k))$ and $B(t) \sim \underset{i=1}{\overset{k}{*}} f_{Z_k}(x)$ from **(2.9)**. Thus, to calculate $\mathrm{E}[B(t)]$, we need to calculate $\mathrm{E}[Z_k]$. We do so by using the *Probability Density Functions* for random variables $u(t-X_k)$, $u(t-(X_k+S_k))$, and $-u(t-(X_k+S_k))$. We know that,

$$
\begin{array}{rcl}
X & \rightarrow & -X \\
f_X(x) & \rightarrow & f_{-X}(x) = f_X(-x)
\end{array}
$$

Thus using **(2.7)**,

$$
\begin{aligned}
f_{-u(t-(X_k+S_k))}(x) &= f_{u(t-(X_k+S_k))}(-x) \qquad (2.10)\\
&= \overline{F_{(X_k+S_k)}(-x)} \cdot \delta(-x) + F_{(X_k+S_k)}(-x) \cdot \delta(-(x+1))
\end{aligned}
$$

We know using **(2.3)** that $f_{X_k}(x) = \delta(x-(k-1)\cdot\frac{1}{\lambda})$, thus:

$$
F_{X_k}(x) = \int_0^x f_{X_k}(k) \cdot dk = u(x-(k-1)\cdot\frac{1}{\lambda}) \qquad (2.11)
$$

Now, using **(2.11)** and **(2.6)** we can express $f_{u(t-X_k)}(x)$ as:

$$
f_{u(t-X_k)}(x) = u(t-X_k)\cdot\delta(x-1) + (1-u(t-X_k))\cdot\delta(x) \qquad (2.12)
$$

We now use convolution to find $f_{(X_k+S_k)}(x)$ and $F_{(X_k+S_k)}(x)$,

$$
\begin{aligned}
X_k + S_k \sim f_{(X_k+S_k)}(x) &= f_{X_k}(x) \star f_{S_k}(x)\\
&= f_{S_k}(x-X_k)\\
&= \mu e^{-\mu(x-X_k)} \cdot u(x-X_k) \qquad (2.13)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
F_{(X_k+S_k)}(x) &= \int_0^x f_{(X_k+S_k)}(k)\cdot dk\\
&= F_{S_k}(x-X_k)\\
&= (1-\mu e^{-\mu(x-X_k)}) \cdot u(x-X_k) \qquad (2.14)
\end{aligned}
$$

We now use **(2.14)** and **(2.10)** to obtain $f_{-u(t-(X_k+S_k))}(x)$:

$$
\begin{aligned}
f_{-u(t-(X_k+S_k))}(x) &= [1-(1-e^{-\mu(t-X_k)})\cdot u(t-X_k)]\cdot\delta(-x) \\
&+ (1-e^{-\mu(t-X_k)})\cdot u(t-X_k)\cdot\delta(-(x+1)) \quad (2.15)
\end{aligned}
$$

Using **(2.12)** and **(2.15)**, we calculate $f_{Z_k}(x)$ as follows:

$$
\begin{aligned}
f_{Z_k}(x) &= f_{u(t-X_k)}(x)\star f_{-u(t-(X_k+S_k))}(x) \\
&= e^{-\mu(t-X_k)}\cdot u(t-X_k)\cdot\delta(x-1) \\
&+ (1-e^{-\mu(t-X_k)}\cdot u(t-X_k))\cdot\delta(x) \quad (2.16)
\end{aligned}
$$

Thus, $\mathrm{E}[Z_k]$ can be calculated as follows:

$$
\mathrm{E}[Z_k] = \int_{-\infty}^{\infty} x\cdot f_{Z_k}(x)\cdot dx = e^{-\mu(t-X_k)}\cdot u(t-X_k) \quad (2.17)
$$

We now calculate $\mathrm{E}[B(t)]$ as $\sum_{i=1}^{n}\mathrm{E}[Z_i]$:

$$
\boxed{\mathrm{E}[B(t)] = \sum_{i=1}^{n}\mathrm{E}[Z_i] = \sum_{i=1}^{n} e^{-\mu(t-X_i)}\cdot u(t-X_i)} \quad (2.18)
$$

Where, n is the total number of requests we want to find $\mathrm{E}[B(t)]$ over. We take the limit as $t\to\infty$ and,

$$
\lim_{t\to\infty}\mathrm{E}[B(t)] = \lim_{t\to\infty}\sum_{i=1}^{n}\mathrm{E}[Z_i] = \lim_{t\to\infty}\sum_{i=1}^{\lambda t} e^{-\mu(t-X_i)} \quad (2.19)
$$

Using geometric series summation, we can re-write the above result as:

$$
\begin{aligned}
\sum_{i=1}^{\lambda t} e^{-\mu(t-X_i)} &= \sum_{i=0}^{\lambda t-1} e^{-\mu(t-i\cdot\frac{1}{\lambda})} \\
&= \frac{(1-e^{\lambda t\cdot(\frac{\mu}{\lambda})})}{(1-e^{\frac{\mu}{\lambda}})}\cdot e^{-\mu t} \\
&= \frac{(e^{\frac{-\mu}{\lambda}}-e^{(\lambda t-1)\cdot(\frac{\mu}{\lambda})})}{(e^{\frac{-\mu}{\lambda}}-1)}\cdot e^{-\mu t} \\
&= \frac{(e^{\frac{-\mu}{\lambda}}\cdot e^{-\mu t}-e^{\frac{-\mu}{\lambda}})}{(e^{\frac{-\mu}{\lambda}}-1)} \quad (2.20)
\end{aligned}
$$

Now we can find the value of $\mathrm{E}[B(t)]$ as $t\to\infty$ using the above expression:

$$
\begin{aligned}
\lim_{t\to\infty}\mathrm{E}[B(t)] &= \lim_{t\to\infty}\frac{(e^{\frac{-\mu}{\lambda}}\cdot e^{-\mu t}-e^{\frac{-\mu}{\lambda}})}{(e^{\frac{-\mu}{\lambda}}-1)} \\
&= \frac{e^{\frac{-\mu}{\lambda}}}{1-e^{\frac{-\mu}{\lambda}}}
\end{aligned}
$$

Thus, we can conclude that:

$$\boxed{\lim_{t\to\infty} \mathrm{E}[B(t)] = \frac{e^{\frac{-\mu}{\lambda}}}{1-e^{\frac{-\mu}{\lambda}}} = \frac{e^{\frac{-1}{\rho}}}{1-e^{\frac{-1}{\rho}}}} \tag{2.21}$$

$\blacksquare$