Exercise 1

1pt 1. $cov(y_1, y_2) = 1$, $var(y_1) = 2$

0.5pt 2. Since $cov(y_1, y_3) = 0$ and y is a Gaussian vector y_1 and y_3 are independent.

1pt 3. $cor(y_1, y_2) = \frac{cov(y_1, y_2)}{\sigma(y_1)\sigma(y_2)} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2} = 0,5$

4. $E[y_1 - 2y_2 + y_3] = 1 - 2 \times 2 - 1 = -4$

1pt $V[y_1 - 2y_2 + y_3] = (1 \ -2 \ 1) \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$

$$(1 \ -2 \ 1) \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} = 4$$

Donc $y_1 - 2y_2 + y_3 \sim \mathcal{N}(-4, 4)$

5. $E[y_1 + y_2] = 3$, $V[y_1 + y_2] = V[y_1] + V[y_2] + 2\,cov(y_1, y_2)$
$= 2 + 2 + 2 \times 1 = 6$

1pt

$cov(z_1, z_2) = (1 \ -2 \ 1) \underbrace{\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}}$

$$(1 \ -2 \ 1) \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = -2$$

D'où $\underline{z} \sim \mathcal{N}\left(\begin{pmatrix} -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix} \right)$

6. 1pts $f(y) = \dfrac{1}{(2\pi)^{3/2} \, |\Sigma|^{1/2}} \exp\left(-\frac{1}{2} (y-\mu)' \Sigma^{-1} (y-\mu) \right)$

where $|\Sigma|$ and Σ^{-1} must be computed

Exercise 2

1. $x_1^2 + x_2^2 + x_3^2 \sim \chi^2_{(3)}$

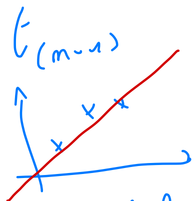
2. $\dfrac{x_1}{\sqrt{\dfrac{x_1^2 + x_2^2 + x_3^2}{3}}} \sim t_{(3)}$

3. $\dfrac{\dfrac{x_1^2 + x_2^2}{2}}{\dfrac{x_1^2 + x_2^2 + x_3^2}{3}} \sim F_{2,3}$

Exercise 3

1pt 1. $(n-1)\dfrac{s^2}{\sigma^2} \sim \chi^2_{(n-1)}$

1pt 2. $t = \dfrac{(\bar{y} - \mu)}{(s/\sqrt{n})} \sim t_{(n-1)}$



Exercise 4

$y_i = \beta x_i + \epsilon_i$

0,5 pt 1. It a simple linear model without intercept

2. Let $S(\beta) = \sum_{i=1}^{n} (y_i - \beta x_i)^2$

1pt

$S'(\beta) = -2 \sum_{i=1}^{n} x_i (y_i - \beta x_i)$

$S''(\beta) = 2 \sum_{i=1}^{n} x_i^2 > 0$ if at least one x_i is different from 0.

Thus the unique minimum is obtained by canceling $S'(\beta)$

$$S'(\beta) = 0 \iff -2\sum_{i=1}^{m} x_i (y_i - \beta x_i)$$

$$\iff \beta \sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i y_i$$

$$\iff \boxed{\beta = \frac{\sum_{i=1}^{m} x_i y_i}{\sum_{i=1}^{m} x_i^2}}$$

Thus $\hat{\beta} = \dfrac{\sum_{i=1}^{m} x_i y_i}{\sum_{i=1}^{m} x_i^2}$

3. $E[\hat{\beta}] = \dfrac{1}{\sum_{i=1}^{m} x_i^2} \sum_{i=1}^{m} x_i E[y_i]$ By linearity of the expectation

1pt

$$= \frac{1}{\sum_{i=1}^{m} x_i^2} \sum_{i=1}^{m} x_i \cdot \beta x_i$$

since $E[y_i] = E[x_i \beta + \epsilon_i]$
$= x_i \beta + \underbrace{E[\epsilon_i]}_{0}$
$= \beta x_i$

$= \beta$, thus $\hat{\beta}$ is an unbiased estimator of β.

4. $V(\hat{\beta}) = \dfrac{1}{\left(\sum_{i=1}^{m} x_i^2\right)^2} \sum_{i=1}^{m} x_i^2 V[y_i]$

1pt

$$= \frac{\sigma^2}{\sum_{i=1}^{m} x_i^2}$$

5. $\hat{\beta} \sim \mathcal{N}\left(\beta, \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}\right)$

1 pt

6. $\sigma^2 = Var[\epsilon_i]$

1 pt Thus $\hat{\epsilon}_i = y_i - \hat{y}_i = y_i - \hat{\beta} x_i$, $E[\hat{\epsilon}_i] = 0$

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (y_i - \hat{\beta} x_i)^2$$

Another possible estimator (which would be unbiased)

is $\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (y_i - \hat{\beta} x_i)^2$

Exercice 5

1pt 1. $\hat{\beta}_1 = 2{,}992544$

1pt 2. $\hat{\sigma}(\hat{\beta}_1) = 0{,}020010$

1pt 3. $R^2 = 0{,}998$ (cf multiple R-squared)

1pt 4. $\hat{\sigma}^2 = 0{,}1596^2$, since $\hat{\sigma}^2 = \frac{1}{n-k-1} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$

- Juliana 4.5/20
- Prabal 13.5/20
- Arslan 6.25/20
- Ishfaaq 19/20
- Hammed Habeeb 11/2
- MD ABDUL MAZED 9/20
- Ludwig Hagen 18.5/20