Statistical inference
Practice 6

Exercises are extracted from the book : Rencher, A. C., & Schaalje, G. B. (2008). *Linear models in statistics*. John Wiley & Sons.

Available on the website :
https://www.utstat.toronto.edu/~brunner/books/LinearModelsInStatistics.pdf

Exercise from chapter 3 (page 83)

3.10 Show that $E[(\mathbf{y}-\boldsymbol{\mu})(\mathbf{y}-\boldsymbol{\mu})'] = E(\mathbf{y}\mathbf{y}') - \boldsymbol{\mu}\boldsymbol{\mu}'$ as in (3.25).

3.20 Let $\mathbf{y} = (y_1, y_2, y_3)'$ be a random vector with mean vector and covariance matrix

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 10 \end{pmatrix}.$$

(a) Let $z = 2y_1 - 3y_2 + y_3$. Find $E(z)$ and $\mathrm{var}(z)$.

(b) Let $z_1 = y_1 + y_2 + y_3$ and $z_2 = 3y_1 + y_2 - 2y_3$. Find $E(\mathbf{z})$ and $\mathrm{cov}(\mathbf{z})$, where $\mathbf{z} = (z_1, z_2)'$.

Exercise from chapter 4 (page 101)

4.2 Obtain (4.8) from (4.7); that is, show that $|\boldsymbol{\Sigma}^{-1/2}| = |\boldsymbol{\Sigma}|^{-1/2}$.

4.9 Assuming that $\mathbf{y}$ is $N_p(\boldsymbol{\mu}, \sigma^2\mathbf{I})$ and $\mathbf{C}$ is an orthogonal matrix, show that $\mathbf{C}\mathbf{y}$ is $N_p(\mathbf{C}\boldsymbol{\mu}, \sigma^2\mathbf{I})$.

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4.16 Suppose that $\mathbf{y}$ is $N_4(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ -2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{pmatrix} 4 & 2 & -1 & 2 \\ 2 & 6 & 3 & -2 \\ -1 & 3 & 5 & -4 \\ 2 & -2 & -4 & 4 \end{pmatrix}.$$

(a) The joint marginal distribution of y_1 and y_3
(b) The marginal distribution of y_2
(c) The distribution of $z = y_1 + 2y_2 - y_3 + 3y_4$
(d) The joint distribution of $z_1 = y_1 + y_2 - y_3 - y_4$ and $z_2 = -3y_1 + y_2 + 2y_3 - 2y_4$

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4.18 If $\mathbf{y}$ is $N_3(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\Sigma} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 3 \end{pmatrix},$$

which variables are independent? (See Corollary 1 to Theorem 4.4a)

4.19 If $\mathbf{y}$ is $N_4(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & -4 & 6 \end{pmatrix},$$

which variables are independent?

Exercises from chapter 5 (page 122)

5.16 **(a)** Show that if $t = z/\sqrt{u/p}$ is $t(p)$ as in (5.33), then t^2 is $F(1, p)$.

with $z \sim \mathcal{N}(0,1)$ and $u \sim \chi^2_{(p)}$
and z and u are independent

$\mathbf{y}$ is $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

5.17 Show that $\boldsymbol{\Sigma}^{-1/2}(\mathbf{y} - \boldsymbol{\mu})$ is $N_n(\mathbf{0}, \mathbf{I})$, as used in the illustration at the beginning of Section 5.5.

5.19 If $\mathbf{y}$ is $N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, verify that $(\mathbf{y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu})$ is $\chi^2(n)$,

As a reminder $\Sigma^{-\frac{1}{2}} \Sigma^{-\frac{1}{2}} = \Sigma^{-1}$ and $\Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} = \Sigma$

$j = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is the column vector of dimension 1

5.24 Suppose that $y_1, y_2, \ldots, y_n$ is a random sample from $N(\mu, \sigma^2)$ so that $\mathbf{y} = (y_1, y_2, \ldots, y_n)'$ is $N_n(\mu\mathbf{j}, \sigma^2\mathbf{I})$. It was shown in Example 5.5 that $(n-1)s^2/\sigma^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2/\sigma^2$ is $\chi^2(n-1)$. In Example 5.6a, it was demonstrated that $\bar{y}$ and $s^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2/(n-1)$ are independent.

(a) Show that $\bar{y}$ is $N(\mu, \sigma^2/n)$.

(b) Show that $t = (\bar{y} - \mu)/(s/\sqrt{n})$ is distributed as $t\ (n-1)$.

Hint : $t = \dfrac{\sqrt{n}(\bar{y}-\mu)}{\sigma} \Big/ \left(\dfrac{s}{\sigma}\right)$. Let denote $w = (n-1)\dfrac{s^2}{\sigma^2} \sim \chi^2_{(n-1)}$, $\dfrac{s}{\sigma} = \sqrt{\dfrac{w}{n-1}}$
thus $\frac{s}{\sigma}$ is the square root of a chi-square divided by its degrees of freedom

5.32 Suppose that $\mathbf{y}$ is $N_n(\boldsymbol{\mu}, \sigma^2\mathbf{I})$ and that $\mathbf{X}$ is an $n \times p$ matrix of constants with rank $p < n$.

(a) Show that $\mathbf{H} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ and $\mathbf{I} - \mathbf{H} = \mathbf{I} - \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'$ are idempotent, and find the rank of each.

(b) Assuming $\boldsymbol{\mu}$ is a linear combination of the columns of $\mathbf{X}$, that is $\boldsymbol{\mu} = \mathbf{Xb}$ for some $\mathbf{b}$ [see (2.37)], find $E(\mathbf{y}'\mathbf{H}\mathbf{y})$ and $E[\mathbf{y}'(\mathbf{I} - \mathbf{H})\mathbf{y}]$, where $\mathbf{H}$ is as defined in part (a) .

(c) Find the distributions of $\mathbf{y}'\mathbf{H}\mathbf{y}/\sigma^2$ and $\mathbf{y}'(\mathbf{I} - \mathbf{H})\mathbf{y}/\sigma^2$.

(d) Show that $\mathbf{y}'\mathbf{H}\mathbf{y}$ and $\mathbf{y}'(\mathbf{I} - \mathbf{H})\mathbf{y}$ are independent.

(e) Find the distribution of

$$\frac{\mathbf{y}'\mathbf{H}\mathbf{y}/p}{\mathbf{y}'(\mathbf{I} - \mathbf{H})\mathbf{y}/(n-p)}.$$