Lecture 2: Convex smooth optimisation, Tikhonov regularisation

Luca Calatroni
CR CNRS, Laboratoire I3S
CNRS, UniCA, Inria SAM, France

Inverse problems in biological imaging

MSc Data Science and Artificial Intelligence

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Table of contents

Introduction

Motivation

Goal: providing theoretical & practical tools (i.e. algorithms) for solving

$$\min_{x \in \mathbb{R}^n} F(x), \quad x \in \mathbb{R}^n \text{ is a vectorised image of size } n_1 \times n_2 = n$$

for a functional $F : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ with suitable properties.

- F is smooth $\to$ gradient descent algorithms
- $F := f + g$, f smooth & g non-smooth $\to$ proximal-gradient algorithms
- $F := f + \|x\|_0$ with f smooth and

$$\|x\|_0 := \# \{i : x_i \neq 0\} .$$

Such problems often appear in:

- **Inverse problems** in signal/image processing: image reconstruction, variable/parameter selection, compressed sensing. . . .
- **Statistical/machine learning**: empirical risk minimisation, regression. . .
- **Optimisation per se**: analysis/implementation of fast algorithms for solving large-scale problems. . .

References

Some standard reference books/surveys:

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- S. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.
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- A. Beck, *First-order methods in optimization*, Volume 25, MOS-SIAM series on Optimization, 2017.
- A. Chambolle, T. Pock, *An introduction to continuous optimization for imaging*, Acta Numerica, 2016
- S. Salzo, S. Villa, *Proximal Gradient Methods for Machine Learning and Imaging*, Handbook on Harmonic and Applied Analysis, Applied and Numerical Harmonic Analysis, 2021.

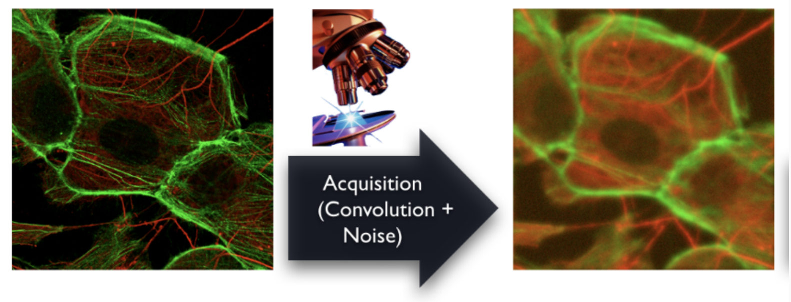
Optimisation for inverse problems in imaging

$$\text{Given } y \in \mathbb{R}^m, A \in \mathbb{R}^{m \times n} \quad \text{find} \quad x \in \mathbb{R}^n \quad \text{s.t.} \quad y = \mathcal{T}(Ax)$$

where $m \leq n$ and $\mathcal{T} : \mathbb{R}^m \to \mathbb{R}^m$ models noise degradation.

- **Image restoration** (denoising, deconvolution, super-resolution)

$$A \text{ is a convolution matrix} \quad Ax \Leftrightarrow h * X$$



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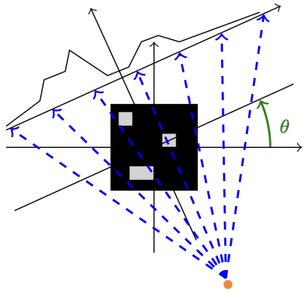

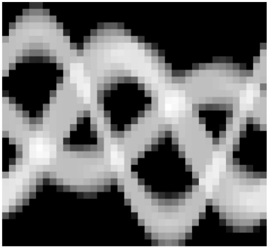

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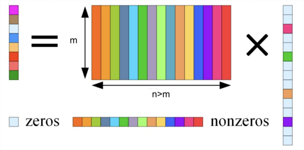
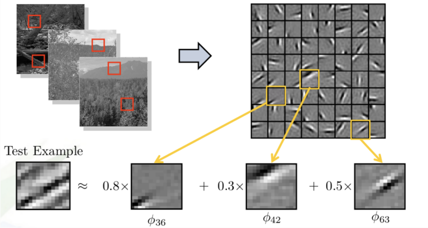
- **Dictionary representation** (data analysis, vision): $x = Dw$





Bad positioning of inverse filtering, Max. Likelihood approach

$$y = Ax + n$$

Naive approach: inverse filtering approach:

$$A^{-1}y = A^{-1}(Ax + n) = x + A^{-1}n$$

Amplification of the noise if A^{-1} is bad conditioned!

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Maximum-likelihood approach: find estimate $\mathbb{R}^n \ni x^* \approx x$ by solving

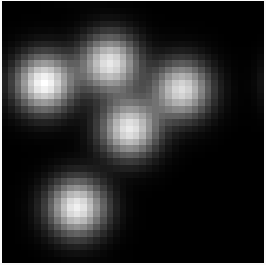
$$\boxed{x^* \in \arg\min_{x \in \mathbb{R}^n} \; F(x) = \frac{1}{2}\|Ax - y\|^2}$$

Regularisation idea

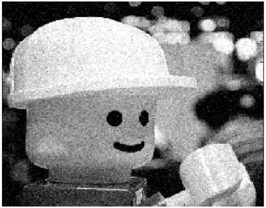
Consider instead:

$$\boxed{x^* \in \arg\min_{x \in \mathbb{R}^n} \; F(x) = f(x) + g(x)}$$

- where, e.g., $f(x) = \frac{1}{2}\|Ax - y\|^2$ is the data fidelity term whose form relates to noise statistics (Gaussian, Poisson...)
- g is a regularisation term which encodes *a priori* information on the desired solution...



few non-zeros



piecewise constant

piecewise linear

Regularisation: Bayesian motivation

Following a Bayesian/MAP approach consider:

$$\pi_Y(y|Ax;\theta_f) \quad \text{(likelihood)}, \qquad \pi_X(x;\theta_g) \quad \text{(prior)}$$

with $\theta_f, \theta_g > 0$ hyperparameters of the distributions. By Bayes' theorem:

$$x^* \in \arg\max_x \ \pi_{X|Y}(x|y) = \arg\max_x \ \frac{\pi_{Y|X}(y|Ax;\theta_f)\pi_X(x;\theta_g)}{\pi_Y(y)}$$

$$\Leftrightarrow x^* \in \arg\min_x - \ln(\pi_{X|Y}(x|y)) = \arg\min_x - \ln(\pi_{Y|X}(y|Ax;\theta_f)) - \ln(\pi_X(x;\theta_g)) + \cancel{\ln(\pi_Y(y))}$$

Now, if $\pi_X(x;\theta_g) = e^{-\theta_g g(x)}, \theta_g > 0$ and $\pi_{Y|X}(y|Ax;\theta_f) = e^{-\theta_f f(x)}, \theta_f > 0$, then:

$$\boxed{x^* \in \arg\min_{x\in\mathbb{R}^n} \ f(x) + \lambda g(x), \qquad \lambda := \theta_g/\theta_f}$$

$\lambda > 0$ is the **regularisation parameter**: it weights the amount of regularisation against the trust in the data.

Some examples of regularisation

- **Modelling sparsity**: the prior is "the image has *few* non-zero entries". Natural choice is $g(x) = \|x\|_0$ (complex problem), so an alternative is:

$$g(x) = \|x\|_1 = \sum_{i=1}^{n} |x_i|$$

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- **Modelling sparsity in some basis**: let $W \in \mathbb{R}^{n \times p}$ a dictionary. Assume that $x = Wa$, with $a \in \mathbb{R}^p$ (synthesis view point) and consider:

$$g(x) = g(a) = \|Wa\|_1 \to \min_{a \in \mathbb{R}^p} \frac{1}{2}\|AWa - y\|^2 + \lambda \|Wa\|_1$$

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- **Modelling piece-wise constancy**: constant regions in images = regions with little variations = regions with small gradients. Hence natural choice is:

$$g(x) = \frac{1}{2}\|Dx\|_{2,2}^2 = \frac{1}{2}\sum_{i=1}^{n}\left((D_h x)_i^2 + (D_v x)_i^2\right), \quad g(x) = \|Dx\|_{2,1} = \sum_{i=1}^{n}\sqrt{(D_h x)_i^2 + (D_v x)_i^2}$$

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- **Modelling piece-wise linearity**: piece-wise linear regions in images = regions with small Hessian...

$$g(x) = \frac{1}{2}\|D^2 x\|^2$$

...how to choose a good prior? Open question!

A smooth example: Tikhonov regularisation

Idea: smooth regularisation of the image in some basis.

$$g(x) = \frac{1}{2}\|Bx\|_2^2, \quad B \in \mathbb{R}^{N \times n}$$

- **Ridge regularization**: $N = n$ and $B = \mathrm{Id}$:

$$x^* \in \operatorname*{arg\,min}_{x \in \mathbb{R}^n} \; \frac{1}{2}\|Ax - y\|^2 + \lambda\|x\|^2$$

Reduces high-values of the image x.

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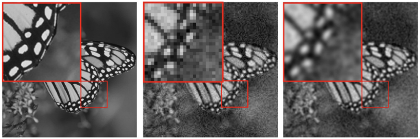
  Reduces high-values of the image x.

- **Sobolev regularization** (Tikhonov, Arsenin, '83): $N = 2n$ and $B = D = \begin{pmatrix} D_h \\ D_v \end{pmatrix}$:

$$\boxed{x^* \in \arg\min_{x\in\mathbb{R}^n} \; \frac{1}{2}\|Ax-y\|^2 + \lambda\|Dx\|_{2,2}^2}$$

  Reduces high-values of the finite-difference image **gradient** Dx (hence oscillations, but also edge sharpness).



Notation, preliminaries & basic notions

Notation and basic tools

- $(X, \langle v, w \rangle) = (\mathbb{R}^n, v^T w)$ with Euclidean norm $\| \cdot \|$
- $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, $\mathbb{R}_+ := \{\alpha \in \mathbb{R} : \alpha \geq 0\}$, $\mathbb{R}_{++} := \{\alpha \in \mathbb{R} : \alpha > 0\}$
- Closed ball of radius $\delta > 0$ in $x \in X$:

$$B_\delta(x) = \{y \in X : \|y - x\| \leq \delta\}$$

- Convex set $C \subset X$

$$(\forall x, y \in C) \quad \forall \alpha \in [0, 1] \quad \alpha x + (1 - \alpha) y \in C$$

Proper functions

Minimal property to have well-defined minimisation problems.

Definition (proper function)

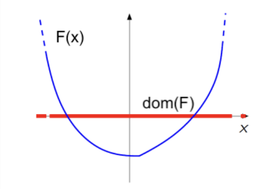
A function $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ is said *proper* iff

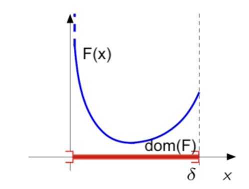
$$\exists x \in \mathbb{R}^n \quad \text{such that} \quad F(x) \neq +\infty.$$

We define

$$\mathrm{dom}(F) := \{x \in \mathbb{R}^n : F(x) < +\infty\}$$

Clearly, F is proper $\Leftrightarrow \mathrm{dom}(F) \neq \emptyset$.

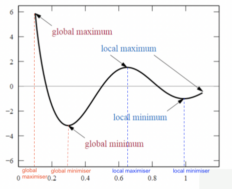




Global/local minimisers

Given a proper function:

- **global minimiser**: $x^* \in \mathbb{R}^n$: $F(x^*) \leq F(x)$ for $\underline{\text{every } x \in \mathbb{R}^n}$.
- **local minimiser**: $x^* \in \mathbb{R}^n$: there exists $\delta > 0$ and a neighbourhood $B_\delta(x^*)$ such that $F(x^*) \leq F(x)$ for $\underline{\text{every } x \in B_\delta(x^*)}$.



$$\min_{x \in \mathbb{R}^n} F(x) \quad \text{VS} \quad \arg\min_{x \in \mathbb{R}^n} F(x)$$

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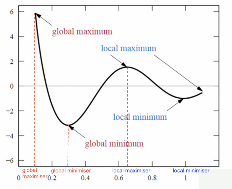


$$\min_{x \in \mathbb{R}^n} F(x) \quad \text{VS} \quad \arg\min_{x \in \mathbb{R}^n} F(x)$$

Definition (set of minimisers)

The **set** of global minimisers of F is denoted by:

$$\arg\min F = \{x^* \in \mathbb{R}^n : x^* \text{ is a minimiser of } F\} \subset \mathbb{R}^n$$

Empty? Singleton? (it depends on F)

Notation, preliminaries & basic notions

Convexity

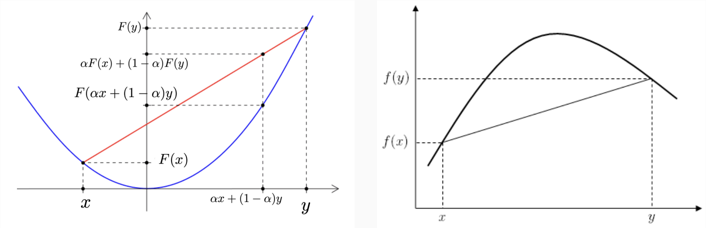
Convex functions

Definition (convex function)

A proper function $F : \mathbb{R}^n \to \overline{\mathbb{R}}$ is *convex* if:

$$(\forall x, y \in \mathbb{R}^n), \quad (\forall \alpha \in [0,1]), \quad F(\alpha x + (1-\alpha)y) \leq \alpha F(x) + (1-\alpha)F(y).$$

Moreover, F is *strictly convex* if the inequality holds when $x, y \in \mathrm{dom}(F)$, $x \neq y$ and $\alpha \in (0,1)$. We say that $G : \mathbb{R}^n \to [-\infty, +\infty)$ is *concave* is $F = -G$ is convex. If a function is not convex nor concave we say that is *non-convex*.



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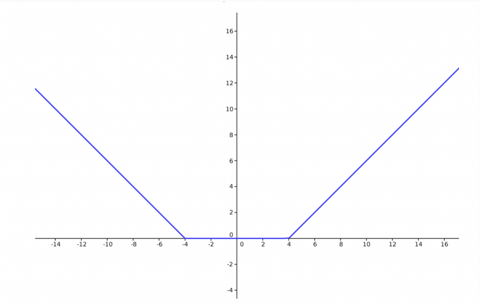

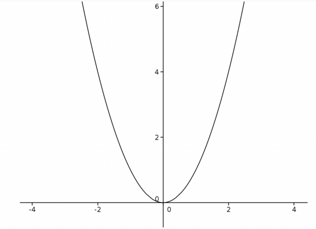

Convex VS. strictly convex functions

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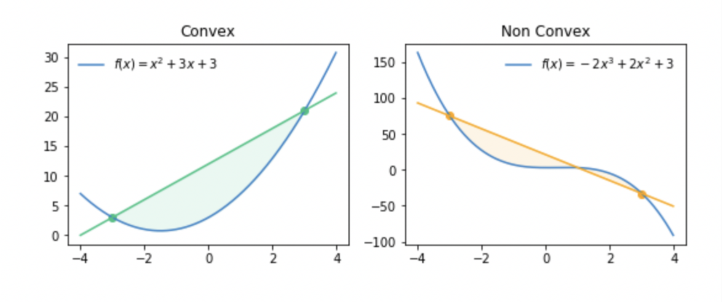

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Examples:

- $F(x) = \|x\|$ is convex

$$\|\alpha x + (1-\alpha)y\| \leq \|\alpha x\| + \|(1-\alpha)y\| = \alpha\|x\| + (1-\alpha)\|y\| \qquad \forall x, y \in \mathbb{R}^n$$

- $F(x) = \|x\|^2$ is strictly convex
- $F(x) = \|x\|_p$, $p \in [1, +\infty)$ are convex

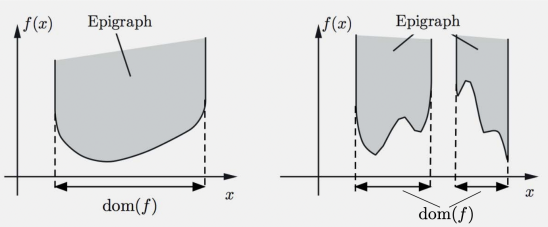
Useful properties

Proposition (epigraph of convex functions is convex set)

Let F be proper. Then F is convex if and only if

$$\mathrm{epi}(f) = \{(x,t) \in X \times \mathbb{R} : f(x) \leq t\}$$

is convex.



Proposition (operations with convex functions)

Let f and g be two convex functions and let $\beta \in \mathbb{R}_{++}$. Then, the sum $f + g$ is a convex function and the function βf is a convex function.

Notation, preliminaries & basic notions

Lower semi-continuity & coercivity

Lower semi-continuity

Definition (lower semi-continuity)

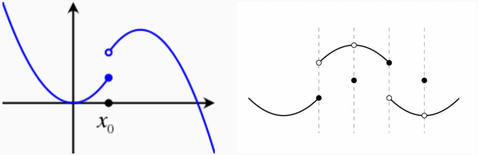
Let F be a proper function. F is *lower semi-continuous (l.s.c.) at* $x \in \mathbb{R}^n$ iff

$$F(x) \leq \liminf_{y \to x} F(y).$$

Equivalently, for every sequence $(x_k)_{k \in \mathbb{N}}$ with $x_k \to x$:

$$F(x) \leq \liminf_{k \to +\infty} F(x_k) \left(= \lim_{k \to +\infty} \inf \left\{ F(x_j) : j \geq k \right\} \right).$$

If F is l.s.c. at every $x \in \mathbb{R}^n$, we say that the function is l.s.c.



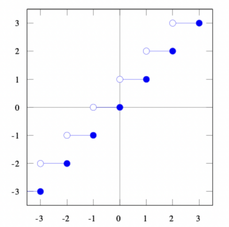
Left: lower l.s.c. **Right**: where the function is lower l.s.c.?

Examples of l.s.c. functions

- The functions $F : \mathbb{R} \to \mathbb{R}$

$$F(x) = |x|_0 = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}, \qquad F(x) = \lceil x \rceil = \min \{k \in \mathbb{Z} : x \leq k\}$$

are l.s.c. (but not continuous).



$F(x) = \lceil x \rceil$

- All continuous functions (l.s.c + u.s.c.).

Coercivity

How to ensure that the minimum is not attained at “extreme points” of the domain?

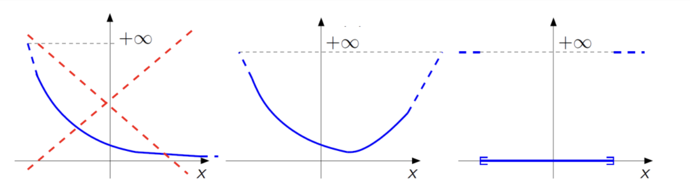
Definition (coercivity)

Let F be proper. We say that F is *coercive* iff

$$\lim_{\|x\| \to +\infty} F(x) = +\infty$$

Examples:

- $F : \mathbb{R} \to \mathbb{R}_+$, $F(x) = e^x$ is **not** coercive, but $F : \mathbb{R} \to \mathbb{R}_+$, $F(x) = e^{|x|}$ is.
- $F : \mathbb{R}^2 \to \mathbb{R}_+$, $F(x_1, x_1) = x_1^2 + x_2^2$ is coercive.
- $F : \mathbb{R}^2 \to \mathbb{R}_+$, $F(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$ is **not** coercive. Why?



Existence of minimisers

Theorem (existence of minimisers)

If F is proper, l.s.c. and coercive, then $\operatorname{argmin} F \neq \emptyset$.

Note: generalises the Bolzano-Weirestrass theorem holding for problems

$$\boxed{\min_{x \in C} F(x)}$$

for compact $C \subset \mathbb{R}^n$ s.t. $C \cap \operatorname{dom}(F) \neq \emptyset$ and *continuous* F.

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Theorem (convex case)

If F is proper, coercive and convex, then every local minimiser is a global minimiser.

Definition ($\Gamma_0(\mathbb{R}^n)$)

$$\Gamma_0(X) := \left\{ F : X \to \overline{\mathbb{R}} : F \text{ is proper, convex and l.s.c.} \right\}$$

Remark (importance of coercivity): $F \in \Gamma_0(X) \not\Rightarrow F$ admits a minimiser.

Take e.g. $F(x) = -\log x, x > 0$ and $F(x) = +\infty, x \leq 0 \ldots$ no coercivity guaranteed!

Uniqueness of minimisers

How to guarantee uniqueness?

Theorem (existence+uniqueness of minimisers)

If F is proper, l.s.c., coercive and strictly convex, then F admits a unique minimiser.

Equivalently, $\arg\min F = \{x^*\}$, a singleton.

Notation, preliminaries & basic notions

Differentiability and L-smoothness

Gâteaux differentiability

How to provide a characterisation of the minimisers of a function f in terms of a suitable notion of "∇f"?

How to provide a characterisation of the minimisers of a function f in terms of a suitable notion of "∇f"?

Definition (Gâteaux differentiability)

Let $f \in \mathcal{P}$ and let $x \in \mathrm{dom}(f)$. For $v \in \mathbb{R}^n$, we denote the *directional derivative* in x along the direction v as the limit

$$f'(x; v) = f'(x)[v] := \lim_{t \to 0^+} \frac{f(x + tv) - f(x)}{t},$$

when it exists. If there exists $w \in \mathbb{R}^n$ such that:

$$(\forall v \in \mathbb{R}^n) \quad f'(x)[v] = \langle w, v \rangle,$$

then we say that f is *Gâteaux differentiable* (in short, *differentiable*) in x and denote by $\nabla f(x) = w$ the *gradient* of f at x.

Optimality conditions and relations with convexity

Theorem (Fermat's rule)

Let $f \in \Gamma_0(\mathbb{R}^n)$ be differentiable at point x^*. Then:

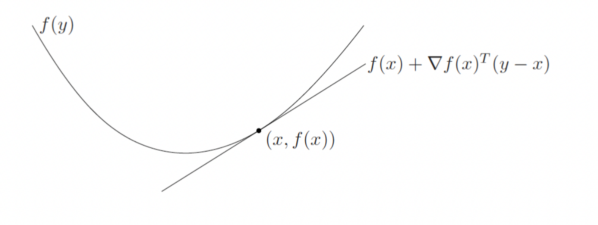
$$x^* \text{ is a minimiser of } f(\cdot) \quad \Longleftrightarrow \quad \nabla f(x^*) = 0.$$

Proposition (Differentiability and convexity)

Let $f \in \Gamma_0(\mathbb{R}^n)$. Suppose that f is differentiable on $\mathrm{dom}(f)$. Then the following statements are equivalent:

1. f is convex;
2. $\forall x, y \in \mathrm{dom}(f)$, $f(y) \geq \overbrace{f(x) + \langle \nabla f(x), y - x \rangle}^{\phi(y;x):=}$;
3. $\forall x, y \in \mathrm{dom}(f)$, $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$.

- the function $\phi(y; x)$ is an affine lower bound/estimator of f
- the tangent to f is below f at all points.



Lipschitz smoothness (L-smoothness)

In the framework of first-order optimisation methods, it's important to provide conditions on the growth of functions considered.

Definition (L-smoothness)

Let $f \in \Gamma_0(\mathbb{R}^n)$ be differentiable. We say that f is an *L-smooth function* with constant $L \geq 0$ iff ∇f is L-Lipschitz continuous:

$$\exists L \geq 0 : \quad \forall w, z \in \mathbb{R}^n \quad \|\nabla f(w) - \nabla f(z)\| \leq L\|w - z\|.$$

Remark: For $f(x) = \frac{1}{2}\|Ax - y\|_2^2$, you can check $L = \|A^T A\| \leq \|A\|^2$.

Smoothness VS strong convexity

- f is L-smooth if and only if:

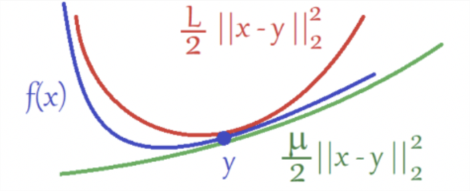
$$(\forall w, z \in \mathrm{dom}(f)) \quad f(z) \leq f(w) + \langle \nabla f(w), z - w \rangle + \frac{L}{2}\|w - z\|^2$$

- f is μ-strongly convex if and only if:

$$(\forall w, z \in \mathrm{dom}(f)) \quad f(z) \geq f(w) + \langle \nabla f(w), z - w \rangle + \frac{\mu}{2}\|w - z\|^2$$

It can be proved that if f is a C^2 function there holds:

$$\mu \mathrm{Id} \preceq \nabla^2 f(x) \preceq L \mathrm{Id}, \qquad \text{for all } x$$



Strong convexity entails better convergence properties.

Smooth optimisation algorithms

Smooth optimisation algorithms

Gradient descent

Gradient descent

Gradient descent (GD) algorithm: ubiquitous in many applications for minimising (non-)convex, differentiable and proper functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$

Algorithm: Gradient Descent (GD) algorithm

Input: $\tau \in \left(0, \frac{2}{L}\right), x^0 \in \mathbb{R}^n$.
for $k \geq 0$ **do**

$$x_{k+1} = x_k - \tau \nabla f(x_k)$$

end for

- Choice of τ: important to guarantee convergence (need to be sufficiently small), it relates to L ($\sim$growth of f).

  Example: minimise $f(x) = x^2/2$. GD iteration: $x_{k+1} = (1-\tau)x_k$, convergence for...?
- Convexity assumption: no dependence on x_0.
- Stopping criterion: relative error $\|x_{k+1} - x_k\| \leq$ `tol` or gradient check $\|\nabla f(x_{k+1})\| \leq$ `tol` (approaching 0).

Understanding the step-size upper bound

Lemma

For all $k \geq 0$, there holds:

$$\tau\left(1-\frac{\tau L}{2}\right)\|f(x_k)\|^2 \leq f(x_k)-f(x_{k+1}).$$

Thus, if $\tau < \frac{2}{L}$, then $f(x_{k+1}) < f(x_k)$, i.e. the GD algorithm is descending.

Proof. Since $x_{k+1} - x_k = -\tau\nabla f(x_k)$, then by the characterisation of L-smoothness we have:

$$\begin{aligned} f(x_{k+1}) &\leq f(x_k) - \tau\langle\nabla f(x_k), \nabla f(x_k)\rangle + \frac{L}{2}\tau^2\|\nabla f(x_k)\|^2, \\ &= f(x_k) + \left(\frac{L\tau}{2}-1\right)\tau\|\nabla f(x_k)\|^2 \end{aligned}$$

so the thesis follows.

Convergence of GD algorithm

Theorem (convergence of GD)

Let $(x_k)_k$ the GD sequence and $x^* \in \arg\min f$. Then, if $\tau \in (0, 2/L)$, there holds:

$$f(x_k) - f(x^*) \leq \underbrace{\frac{\|x^0 - x^*\|^2}{2\tau}}_{C(x^0,x^*,\tau)} \frac{1}{k} = O\left(\frac{1}{k}\right)$$

Remarks:

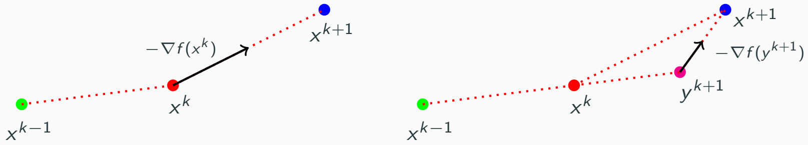
- Convergence in function values with speed $O(1/k)$.
- Note that the constant is unknown, as x^* is, but it is finite.

Smooth optimisation algorithms

Accelerated GD

Accelerated gradient descent

Idea: add inertia to "shift" the sequence of iterates.



Algorithm: Accelerated Gradient Descent (AGD) algorithm [1]

Input: $x_0 = x_{-1} \in \mathbb{R}^n$, $\tau \in \left(0, \frac{1}{L}\right]$, $t_0 = 1$.

for $k \geq 0$ **do**

$$t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2}$$

$$y_{k+1} = x_k + \frac{t_k - 1}{t_{k+1}}(x_k - x_{k-1})$$

$$x_{k+1} = y_{k+1} - \tau \nabla f(y_{k+1})$$

end for

Note: For the use of AGD in inverse problems, the sequence (y_k) does not have to be related with the data y, which typically appears in the expression of ∇f.

[1] Nesterov, 1983

Lemma (behaviour of the sequence (t_k))

Let t_0 and the sequence t_k be defined by:

$$t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2}.$$

Then $t_k \geq \frac{k+2}{2}$ for all $k \geq 0$. In particular, $t_k \to +\infty$.

Proof: by induction. For $k = 0$ we have $t_0 \geq 1$. Suppose that the claim holds for some k, meaning that $t_k \geq \frac{k+2}{2}$. Want to show:

$$t_{k+1} \geq \frac{k+1+2}{2} = \frac{k+3}{2}.$$

Using recursion and $2t_k \geq k+2$ (induction)

$$t_{k+1} = \frac{1 + \sqrt{1 + 4t_k^2}}{2} \geq \frac{1 + \sqrt{1 + (k+2)^2}}{2} \geq \frac{1 + \sqrt{(k+2)^2}}{2} = \frac{k+3}{2}.$$

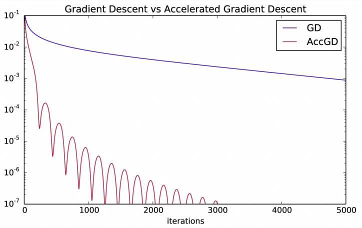
Accelerated convergence result

Theorem (convergence of AGD)[2]

Let $(x_k)_k$ be the AGD sequence and $x^* \in \arg\min f$. Then, there holds:

$$f(x_k) - f(x^*) \leq \underbrace{\frac{2\|x^0 - x^*\|^2}{\tau}}_{C(x^0, x^*, \tau)} \frac{1}{(k+1)^2}$$

Get *faster* to a reasonably accurate approximation of x^*.



[2] Nesterov, 2004, Chambolle-Pock, 2016

Accuracy viewpoint

How many iterations are needed for such algorithms to achieve ε-accuracy, i.e.

$$\boxed{f(x_k) - f(x^*) \leq \varepsilon}$$

- GD: all $k \geq 0$ such that $k \geq \lceil C/\varepsilon \rceil$
- AGD: all $k \geq 0$ such that $k \geq \lceil C/\sqrt{\varepsilon} - 1 \rceil$

First order optimisation or more?

This quadratic rate matches the *worst-case* lower bound for first-order optimisation methods (i.e., methods using only gradient information for optimisation the function).

Other possibility: **Newton's method**, but more involved:

$$x_{k+1} = x_k - (H_f(x_k))^{-1}(\nabla f(x_k))$$

where $H_f(x_k)$ is the Hessian of f evaluated in x_k (could be hard to invert).

Application to inverse problems

Maximum-likelihood approach

For $A \in \mathbb{R}^{m \times n}$ and $n \sim \mathcal{N}(0, \sigma^2 \mathrm{Id})$, observe noisy/blurred image y through:

$$\boxed{y = Ax + n}$$

Consider maximum-likelihood functional:

$$\boxed{x^* \in \operatorname*{arg\,min}_{x \in \mathbb{R}^n} \frac{1}{2}\|Ax - y\|^2 = f(x)}$$

How to set-up GD iteration?

- Compute the (Gateaux) gradient $\nabla f(x)$
- Compute its Lipschitz constant L
- Choose $\tau < 2/L$ (as big as possible) and $x_0 \in \mathbb{R}^n$ (often, $x_0 = y$ in convex problems) to launch the algorithm
- Choose stopping criterion:

$$\|x_{k+1} - x_k\| \leq \epsilon, \quad \|f(x_{k+1}) - f(x_k)\| \leq \epsilon, \quad \|\nabla f(x_k)\| \leq \epsilon$$

Tikhonov regularisation

$$x^* \in \underset{x\in\mathbb{R}^n}{\arg\min}\ \frac{1}{2}\|Ax-y\|^2 + \lambda\|Bx\|^2 = f(x)$$

- Compute gradient $\nabla f = \nabla f_1 + \nabla f_2$
- Estimate L observing that

$$\|\nabla f(w) - \nabla f(z)\| \leq \|\nabla f_1(w) - \nabla f_1(z)\| + \|\nabla f_2(w) - \nabla f_2(z)\| \leq \underbrace{(L_1 + L_2)}_{L}\|w - z\|$$

- Choose $\tau < 2/L$ (as big as possible) and $x_0 \in \mathbb{R}^n$ to launch the algorithm

Conclusions

We focused on convex, smooth optimisation problems arising in imaging inverse problems.

- We revised basic notions for having well-posedness of the underlying problem and basic optimisation tools
- We considered GD as a reference first-order algorithm
- We discussed Nesterov acceleration for improving convergence speed

How to explore analogous ideas in the structured smooth+non-smooth setting?

Questions?

calatroni@i3s.unice.fr