Sparse $\ell_2 - \ell_0$ image reconstruction

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Projet MORPHEME - UCA, CNRS, INRIA -

M2 MScDAI

Outline of the talk

1. Introduction and examples
2. ℓ_1 promotes sparsity
3. Algorithms for $\ell_2 - \ell_1$ optimization
4. Algorithms for $\ell_2 - \ell_0$ optimization
5. Results on super-resolution Microscopy by Single Molecule Localization.

1.Introduction

Many signal processing areas are concerned with

- Linear observation : $A\mathrm{x} = d$
  - d : observed data, vector in $\mathbb{R}^M$
  - x unknown data to be estimated in $\mathbb{R}^N$
  - A observation matrix, $M \times N$ matrix.

  where we have **few observations** for a **large explicative unknown variables** x $M << N$
  The system is undertermined, A is ill-conditioned, observations are noisy
- Least square solution $\hat{\mathrm{x}} = \arg \min_{\mathrm{x} \in \mathbb{R}^N} \|A\mathrm{x} - d\|_2^2$
  $(\|\mathrm{x}\|_2^2 = \|\mathrm{x}\|^2 = \sum_{i=1}^N x_i^2)$
- Regularization: **sparse** signal hypothesis modeled by considering ℓ_1-norm or ℓ_0 semi-norm constraints:

$$\|\mathrm{x}\|_1 \leq K \quad \text{where} \quad \|\mathrm{x}\|_1 = \sum_{i=1}^N |\mathrm{x}_i|$$

$$\|\mathrm{x}\|_0 \leq K \quad \text{where} \quad \|\mathrm{x}\|_0 = \#\{\mathrm{x}_i,\ i = 1, \ldots, N : \mathrm{x}_i \neq 0\}$$

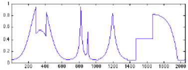
NB: ℓ_0-norm **is NOT** a norm as $\|\lambda \mathrm{x}\|_0 = \|\mathrm{x}\|_0 \neq \lambda \|\mathrm{x}\|_0$.

1.0 Dictionary representation in image processing

- Image are non-stationary, they exhibit smooth areas, oscillations, edges, textures,...

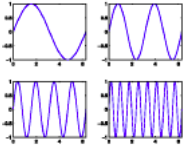
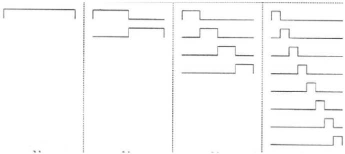
Let's $d \in \mathbb{R}^M$ be a patch of an image or a signal:



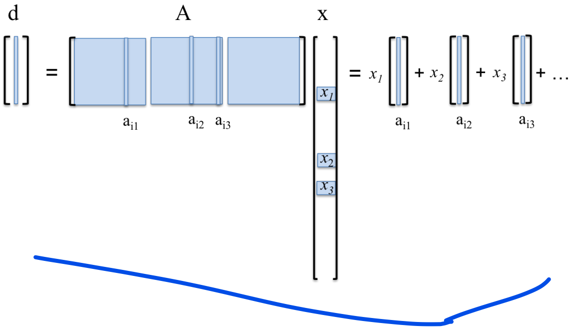


- Each part is represented by given waveforms which best match the image structure, for example Basis B_i as Haar, smooth wavelets, sine/cosine transform,...

Let's $A = [\mathrm{a}_1, ..., \mathrm{a}_N] \in \mathbb{R}^{M \times N}$ be a set of basis vectors, or normalized vectors

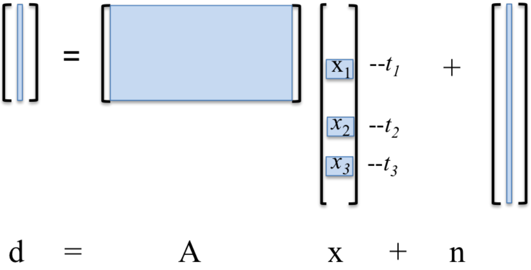
- Such A is a redundant **dictionary** (succession of representative waveforms, possibly a succession of bases)
- The dictionary A is adapted to the signal d if d can be represented by a **few** number of vectors of the dictionary A, that is $d \approx A\mathrm{x}$ with x is a **sparse** vector, that is $\|\mathrm{x}\|_0 \leq K$, where $K << N$.



1.1 Examples in Signal/image Processing

- signal is a sum of pulses, spikes, modeled by a sum of Dirac $\sum_{r=1}^{K} x_r \delta_{t_r}$.
- acquisition system, channel, is modeled as a linear system, e.g. convolution by a Gaussian function: $d(.) = h * \sum_{r=1}^{K} x_r \delta_{t_r} = \sum_{r=1}^{K} x_r h(. - t_r)$.

By assuming the Dirac locations t_r are on a regular grid indexed by $i = 1, ...N$

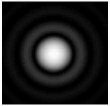


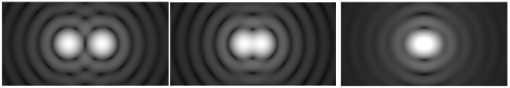
- 1D example: Channel estimation in communications, ...
- 2D example: Single Molecule Localization in super-resolution microscopy , ...

2D example in Super-resolution microscopy: SMLM (continued)

Conventional fluorescence microscopy limits

- physical diffraction limit of optical systems
- Airy patch = impulse response of the microscope (PSF: *Point Spread Function*)
- overlapping patches limit at ≈200nm the distance between two molecules to be resolved (Rayleigh limit)





Super-resolution by single molecule localization

- **Photo-activable molecules**: PALM *Photo Activated Localisation Microscopy* ([Betzig & al 06, Hess & al, 2006]) et STORM *STochastic Optical Reconstruction Microscopy* ([Rust & al, 2006])
- Sequentially activate and image a small random set of fluorescent molecules.

2D example in Super-resolution microscopy: SMLM (continued)

- activation
- imaging
- localization
- assembling

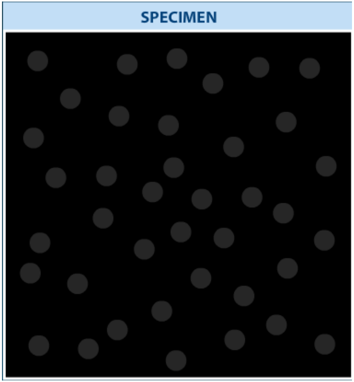


Figure: PALM microscopy principle. From Zeiss tutorials [http://zeiss-campus.magnet.fsu.edu/tutorials/index.html]

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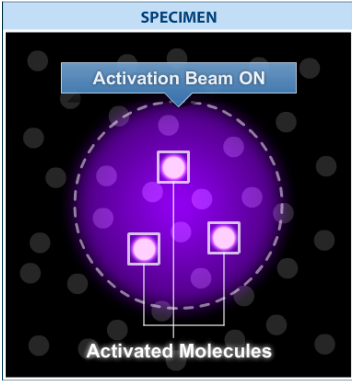

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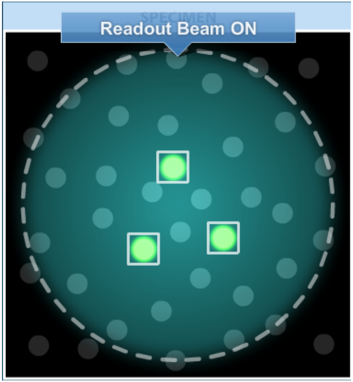

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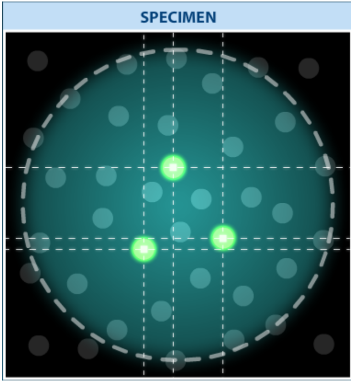

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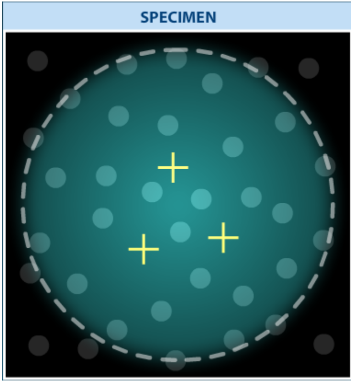

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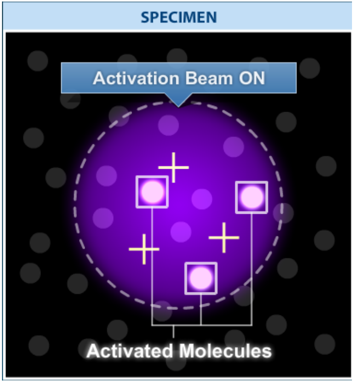

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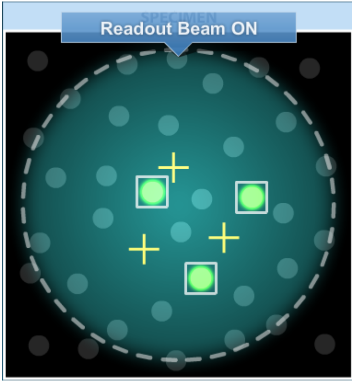

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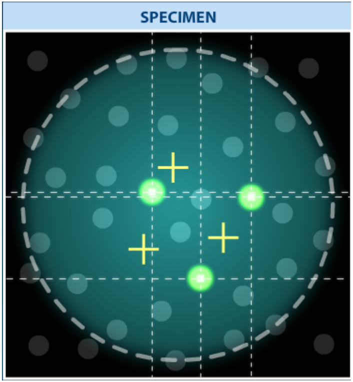

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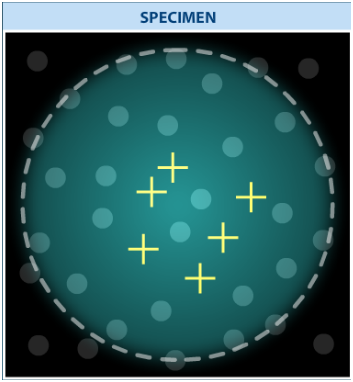

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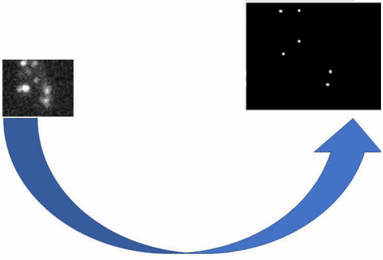

$$\hat{x} \in \arg\min_{x} \frac{1}{2}\|Ax - d\|_2^2 + \lambda\|x\|_0$$

2D example in Super-resolution microscopy: SMLM (continued)

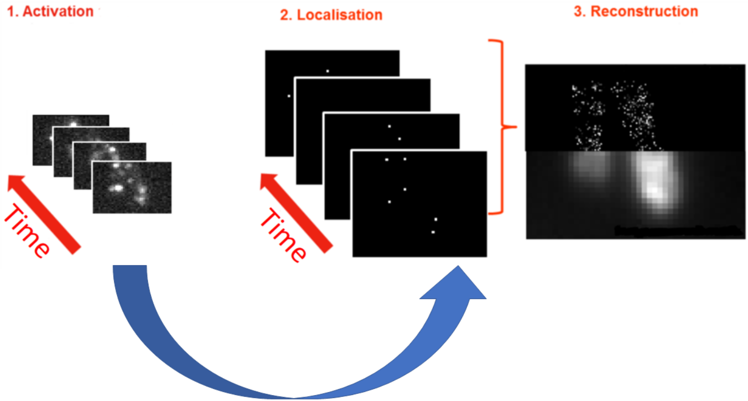

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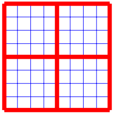
Image formation model PALM / STORM

$\mathbf{d} \in \mathbb{R}^{M \times M}$ one acquisition.
$\mathbf{X} \in \mathbb{R}^{ML \times ML}$ an image where each pixel of $\mathbf{d}$ is divided in $\mathbf{L} \times \mathbf{L}$ pixels.



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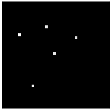

$$* \quad \text{PSF} \xrightarrow{H(\cdot)} H(X) \xrightarrow{M_L(\cdot)} M_L(H(X)) \xrightarrow{+\eta} d$$

X

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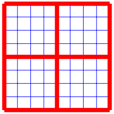

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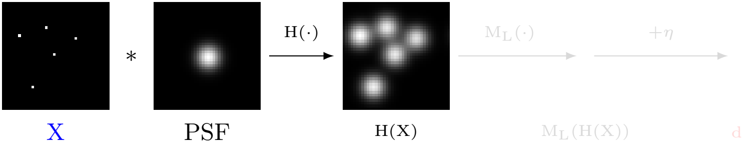

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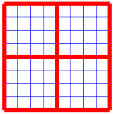

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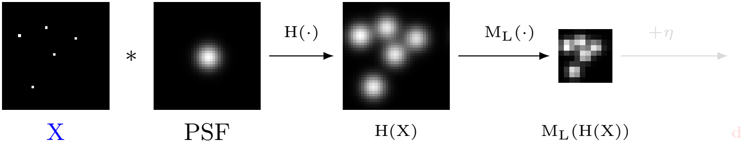

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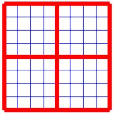

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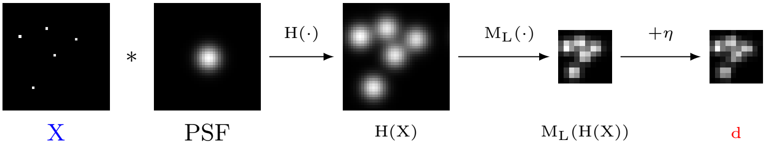

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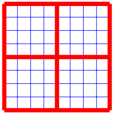

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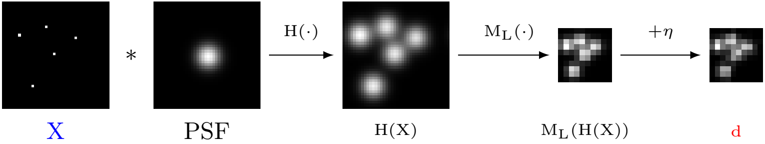

Model

$$\mathbf{d} = \mathrm{M_L}(\mathrm{H}(\mathrm{X})) + \eta,$$

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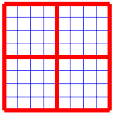

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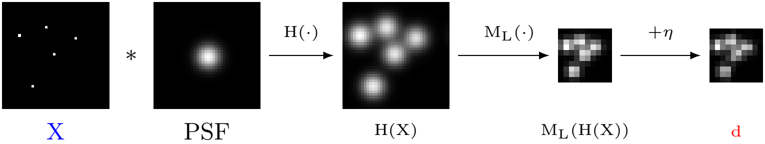


Problem $\ell_2 - \ell_0$

$$\hat{X} \in \arg\min_{X} \frac{1}{2}\|d - M_L(H(X))\|_2^2 + \lambda\|X\|_0$$

$\|X\|_0 = \#\{X_i / X_i \neq 0\}$ is the number of non zero components of X.

1.3 ℓ_2-ℓ_0 optimization problems

Noisy problem: two constrained forms ($\epsilon > 0$, $K > 0$)

$$\hat{\mathrm{x}} = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \frac{1}{2}\|A\mathrm{x} - d\|_2^2 \text{ subject to } \|\mathrm{x}\|_0 \leq K$$

$$\hat{\mathrm{x}} = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \|\mathrm{x}\|_0 \text{ subject to } \|A\mathrm{x} - d\|_2^2 \leq \epsilon$$

Noisy problem : penalized form ($\lambda > 0$)

$$\hat{\mathrm{x}} = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \mathrm{G}_{\ell_0}(\mathrm{x}) := \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \lambda\|\mathrm{x}\|_0$$

$$A \in \mathbb{R}^{M\times N} \text{ with } M \ll N$$

- Non equivalent formulations
- Existence of an optimal solution and relationships between optimal solutions in [Nikolova 16]
- Intensive work in signal and image processing, and in statistics.
- **non-continuous**, **non-convex** and **NP-hard** optimization problem. [Natarajan 95] [Davis & al 97]. Rouhgly speaking, *a solution cannot be verified in polynomial time w.r.t the dimension of the problem*

ℓ_2-ℓ_0 Optimization

1. Iterative Hard Thresholding,
2. Continuous relaxation,
3. Greedy algorithms,
4. Exact reformulation.

FBS = IHT Algorithm

Penalized form

$$\hat{x} = \arg\min_{x \in \mathbb{R}^N} \frac{1}{2}\|Ax - d\|_2^2 + \lambda\|x\|_0$$

- $\frac{1}{2}\|Ax - d\|_2^2$ is L-gradient Lipschitz ($L = \|A\|^2$)
- Proximal of $\|.\|_0$ has explicit expression, this is the Hard Threshold

Iterative Hard Thresholding

(IHT): Forward-Backward Splitting (FBS) algorithm

$$x^{k+1} = \mathrm{prox}_{\gamma\lambda\|.\|_0}\left(x^k - \gamma A^t\left(Ax^k - d\right)\right)$$

$\gamma < \frac{1}{L}$ is the gradient step.

Computation of $\mathrm{prox}_{\gamma\lambda\|.\|_0}$:

$$\mathrm{prox}_{\gamma\lambda\|.\|_0}(\mathrm{y}) = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \left\{ \frac{1}{2}\|\mathrm{x}-\mathrm{y}\|^2 + \gamma\lambda\|\mathrm{x}|\|_0 \right\}$$

$$\frac{1}{2}(\mathrm{x}-\mathrm{y})^2 + \gamma\lambda\|\mathrm{x}|\|_0 = \sum_{i=1}^{N}(x_i - y_i)^2 + \gamma\lambda|x_i|_0$$

where $|u|_0 = 1$ if $u \neq 0$, 0 elsewhere.

Then it is sufficient to compute in 1D
$\arg\min_{u\in\mathbb{R}} \left\{ g(u) := \frac{1}{2}(u-y)^2 + \gamma\lambda|u|_0 \right\}$

Computation of $\arg\min_{u\in\mathbb{R}} \left\{ g(u) := \frac{1}{2}(u-y)^2 + \gamma\lambda|u|_0 \right\}$

- if $u = 0$ then $g(0) = \frac{1}{2}(y)^2$
- The minimum could be reached at $\hat{u} = 0$, the value is $g(\hat{u}) = \frac{1}{2}(y)^2$

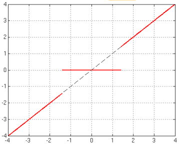
- if $u \neq 0$ then $g(u) = \frac{1}{2}(u-y)^2 + \lambda$
- The minimum is reached at $\hat{u} = y$ and the value is $g(\hat{u}) = \lambda$

if $|y| \leq \sqrt{2\lambda}$ then $\hat{u} = 0$

if $|y| \geq \sqrt{2\lambda}$ then $\hat{u} = y$

The solution is given by the Hard Threshold function

$$\boxed{\hat{u} = \begin{cases} y & \text{if } |y| > \sqrt{2\lambda}\,, \\ 0 & \text{if } |y| \leq \sqrt{2\lambda}\,. \end{cases}} \Rightarrow$$



non convex

Find the solution of the optimal problem

$$\hat{x} = \arg\min_{x \in \mathbb{R}^N} \frac{1}{2}\|Ax - d\|_2^2 + \lambda\|x\|_0$$

by Forward Backward Splitting algorithm (Iterative Hard Thresholding)

$$x^{k+1} = \mathrm{prox}_{\gamma\lambda\|.\|_0}\left(x^k - \gamma A^t\left(Ax^k - d\right)\right)$$

- IHT algorithm converges to a critical point [Blumensath and Davies 08, Attouch et al 13].
- Initialization point is important, for example initialize with the solution with the ℓ_1-norm problem: $\arg\min_{x \in \mathbb{R}^N} \left\{\frac{1}{2}\|Ax - y\|^2 + \gamma\lambda\|x\|_1\right\}$. It is not guaranty that this solution is sparse.

Continuous separable relaxation (convex and non-convex)

$$\frac{1}{2}\|Ax-d\|_2^2+\lambda\|x\|_0 \quad \rightarrow \quad \frac{1}{2}\|Ax-d\|_2^2+\lambda\sum_{i\in\mathbb{I}_N}\phi(x_i)$$

Continuous approximation of the ℓ_0-norm function:

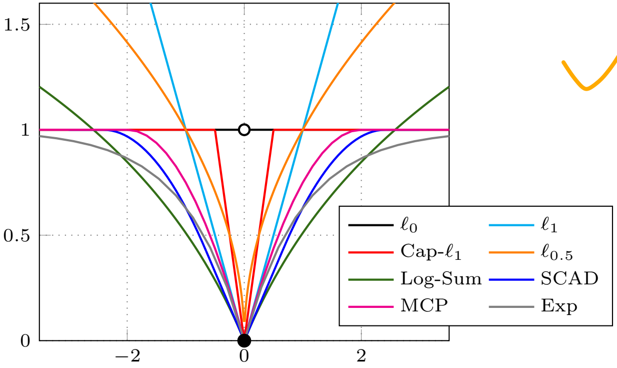
- ℓ_1-norm: Lasso [Tibshirani 96] ; Basic Pursuit [Chen et al 98] ; Compressed Sensing [Donoho 06, Candes et al 06])
- Adaptive Lasso [Zou 06] ;
- Nonnegative Garrote [Breiman 95] ;
- Exponential approximation [Mangasarian 96] ;
- Log-Sum Penalty [Candes et al 08] ;
- Smoothly Clipped Absolute Deviation (SCAD) [Fan and Li 01] ;
- Minimax Concave Penalty (MCP) [Zhang 10] ;
- ℓ_p-norms $0 < p < 1$ [Chartrand 07, Foucart and Lai 09] ;
- Smoothed ℓ_0-norm Penalty (SL0) [Mohimani et al 09] ;

Are they *good* approximations?
Which one to use?

Continuous separable relaxation (convex and non-convex)

$$\frac{1}{2}\|Ax - d\|_2^2 + \lambda\|x\|_0 \quad \rightarrow \quad \frac{1}{2}\|Ax - d\|_2^2 + \lambda\sum_{i\in\mathbb{I}_N} \phi(x_i)$$

Continuous approximation of the ℓ_0-norm function:



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Example of continuous approximation functions of the ℓ_0-norm:

- ℓ_1-norm: $\phi(t) = |t|$
- Log-Sum Penalty [Candes et al 08] $\phi_{Log}(\theta; t) := \log(1 + |t|\theta)$, with $\theta \in \mathbb{R}_+^\star$.
- Minimax Concave Penalty (MCP) [Zhang 10]
  $\phi_{MCP}(\gamma, \lambda; t) = \lambda \left(\frac{\gamma\lambda}{2} \mathbb{1}_{\{|t| > \gamma\lambda\}} + \left(|t| - \frac{t^2}{2\gamma\lambda} \right) \mathbb{1}_{\{|t| \leq \gamma\lambda\}} \right)$
  with $\mathbb{1}_{\{x \in C\}} = 1$ if $x \in C$ and 0 otherwise.

$$\mathrm{G}_{\ell_0}(x) := \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \lambda\|\mathrm{x}\|_0 \quad \rightarrow \quad \tilde{\mathrm{G}}(x) := \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \sum_{i=1}^{N} \phi(\mathrm{x}_i)$$

Definition of a *good* continuous approximation

- $\mathrm{G}_{\ell_0}(x)$ and $\tilde{\mathrm{G}}(x)$ have **same global** minimizers

$$\arg\min_{\mathrm{x}\in\mathbb{R}^N} \tilde{\mathrm{G}}(\mathrm{x}) = \arg\min_{\mathrm{x}\in\mathbb{R}^N} \mathrm{G}_{\ell_0}(\mathrm{x}) \tag{P1}$$

- $\tilde{\mathrm{G}}(x)$ has **less local** minimizers than $\mathrm{G}_{\ell_0}(x)$

$$\hat{\mathrm{x}} \text{ minimiseur de } \tilde{\mathrm{G}} \implies \hat{\mathrm{x}} \text{ minimiseur de } \mathrm{G}_{\ell_0} \tag{P2}$$

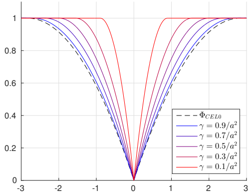
ϕ **depends on** $\|a_i\|$ and λ when applied on x_i:

$$\tilde{\mathrm{G}}(x) := \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \sum_{i \in \mathbb{I}_N} \phi(\|a_i\|, \lambda, \mathrm{x}_i)$$

The one which removes the most of local minimizers is $\phi_{MCP}(\frac{1}{\|a_i\|}, \lambda, t)$ that we call $\phi_{\texttt{CEL0}}$:

$$\phi_{\texttt{CEL0}}(\|a_i\|, \lambda, x) = \lambda - \frac{\|a_i\|^2}{2}\left(|x| - \frac{\sqrt{2\lambda}}{\|a_i\|}\right)^2 \mathbb{1}_{\left\{|x| \leq \frac{\sqrt{2\lambda}}{\|a_i\|}\right\}}$$

where $\mathbb{1}_{\{\mathrm{x} \in D\}} = 1$ if $\mathrm{x} \in D$; 0 otherwise.



$\phi_{MCP}(\gamma, \lambda, t)$

continuous exact local optimization

The $\ell_2 - \ell_0$ and ℓ_2- CEL0 functionals :

$$\mathrm{G}_{\ell_0}(\mathrm{x}) := \frac{1}{2}\|A\mathrm{x} - d\|^2 + \lambda\|\mathrm{x}\|_0$$

$$\mathrm{G}_{\text{CEL0}}(\mathrm{x}) = \frac{1}{2}\|A\mathrm{x} - d\|^2 + \sum_{i\in\mathbb{I}_N} \phi_{\text{CEL0}}(\|a_i\|, \lambda, \mathrm{x}_i)$$

where $\phi_{\text{CEL0}}(\|a_i\|, \lambda, x) = \lambda - \frac{\|a_i\|^2}{2}\left(|x| - \frac{\sqrt{2\lambda}}{\|a_i\|}\right)^2 \mathbb{1}_{\left\{|x|\leq \frac{\sqrt{2\lambda}}{\|a_i\|}\right\}}$

Properties of $\mathrm{G}_{\text{CEL0}}(\mathrm{x})$

- **Limit inf** of the functions satisfying (P1) and (P2) ; the one which removes the most of local minimizers
- **Continuity**
- **Non convex** in the general case (for any A)
- but **convexity** with respect to each **component**

Nonsmooth nonconvex algorithms

The **continuity of G_{CEL0}** allows to use recent *nonsmooth nonconvex* algorithms to minimize (indirectly) G_{ℓ_0},

- *Difference of Convex* (DC) functions programming [Gasso et al 09]
- *Majorization-Minimization*(MM) algorithms (*e.g.* Iteratively Reweighted ℓ_1 (IRL1) [Ochs et al 2015])
- *Forward-Backward splitting* (GIST [Gong et al 13], [Attouch et al 13])

Forward-Backward Splitting Algorithm

$$x^{k+1} \in \mathrm{prox}_{\gamma \Phi_{\mathrm{CEL0}}(\cdot)} \left(x^k - \gamma^k A^T (A x^k - d) \right),$$

where $0 < \gamma < \frac{1}{\|A\|^2}$ and

$$\mathrm{prox}_{\gamma \phi_{\mathrm{CEL0}}(a,\lambda;\cdot)}(u) = \begin{cases} \mathrm{sign}(u) \min \left(|u|, (|u| - \sqrt{2\lambda}\gamma a)_+ / (1 - a^2\gamma) \right) & \text{if } a^2\gamma < 1 \\ u \mathbb{1}_{\{|u| > \sqrt{2\gamma\lambda}\}} + \{0, u\} \mathbb{1}_{\{|u| = \sqrt{2\gamma\lambda}\}} & \text{if } a^2\gamma \geq 1 \end{cases}$$

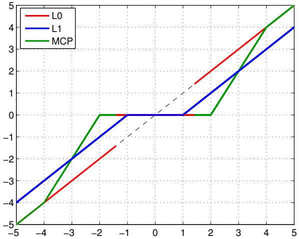


Figure: Proximal operators. Red: ℓ_0, Blue: ℓ_1, Green: Φ_{CEL0} (depends on $a = \|a_i\|$ at component $u = x_i$).

Forward-Backward Splitting Algorithm

$$\mathrm{x}^{k+1} \in \mathrm{prox}_{\gamma \Phi_{\mathtt{CEL0}}(\cdot)} \left(\mathrm{x}^k - \gamma^k A^T (A\mathrm{x}^k - d) \right),$$

where $0 < \gamma < \frac{1}{\|A\|^2}$ and

$$\mathrm{prox}_{\gamma \phi_{\mathtt{CEL0}}(a,\lambda;\cdot)}(u) = \begin{cases} \operatorname{sign}(u) \min \left(|u|, (|u| - \sqrt{2\lambda}\gamma a)_+ / (1 - a^2\gamma) \right) & \text{if } \ a^2\gamma < 1 \\ u \mathbb{1}_{\{|u| > \sqrt{2\gamma\lambda}\}} + \{0, u\} \mathbb{1}_{\{|u| = \sqrt{2\gamma\lambda}\}} & \text{if } \ a^2\gamma \geq 1 \end{cases}$$

- Convergence to a critical point under Kurdyka-Lojaseiwicz (KL) property [Attouch et al 13].
- Accelerated algorithm in the non convex case [Li Lin 15]

5.1 Results, ISBI challenge 2013, simulated dataset

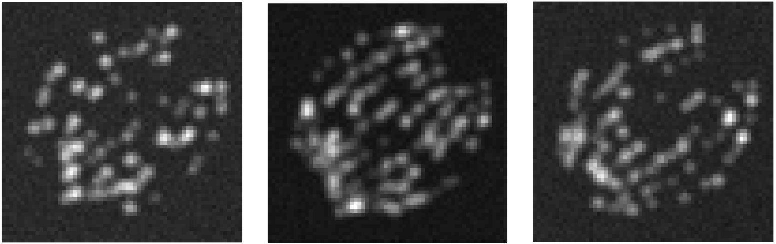


Figure: Simulated images (among the 361 simulated high density images for this sample). Data from IEEE ISBI Challenge 2013.
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8 simulated tubes of 30nm diameter
Camera of 64×64 pixels of size 100nm.
Gaussian PSF, FWHM = 258.21 nm (full width at half maximum)
80932 molecules activated on 361 frames.

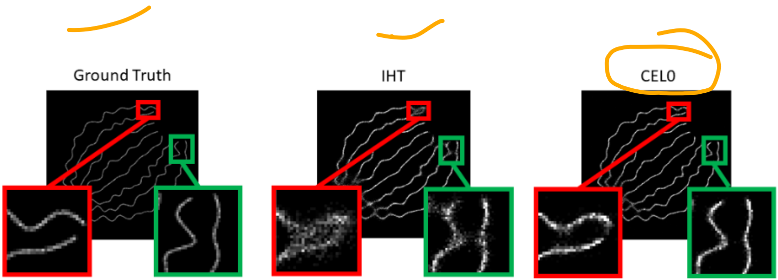


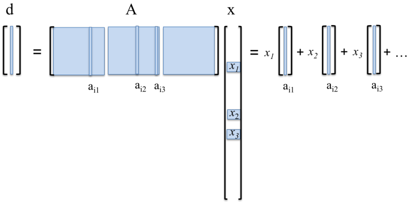
Figure: Reconstruction from simulated data set, reduction ratio $L = 4$.

Greedy algorithms

Greedy algorithms, *Matching Pursuit (MP)* [Mallat et al 93], *Orthogonal MP* [Pati et al 93], *Orthogonal Least Squares (OLS)* [Chen et al 89], *Bayesian OMP* [Herzet et al 10], *Single Best Replacement* [Soussen et al 11] and further variants.

Matching Pursuit:

d is the signal we want to represent with the a limited number $K << N$ of waveforms or atoms of dictionary A, one atom is one column of A, *i.e.* $A_{.,i} = \mathrm{a}_i$, $i = 1, ..N$.



For that we have to solve

$$\hat{\mathrm{x}} = \arg\min_{\mathrm{x} \in \mathbb{R}^N} \|A\mathrm{x} - d\|_2^2 \ \text{ subject to } \ \|\mathrm{x}\|_0 \leq K.$$

$$\left(\text{or } \hat{\mathrm{x}} = \arg\min_{\mathrm{x} \in \mathbb{R}^N} \|\mathrm{x}\|_0 \ \text{ subject to } \ \|A\mathrm{x} - d\|_2^2 \leq \epsilon \right)$$

Matching Pursuit principle

It is assumed without loss of generality that A has unit norm columns, $\|A_{.,i}\| = \|a_i\| = 1$.

The **first component** $i^1 \in \{1, ..., N\}$ will be such that the **correlation** between d and atom i is maximum: $i^1 = \arg \max_{j \in \{1,..,N\}} |\langle a_j, d\rangle|$.

Then the **optimal solution** is $\mathrm{x}^1 = (0, 0, .., \langle a_{i^1}, d\rangle, 0, .., 0)$, where the non null component is at index i^1, which is written as $\mathrm{x}^1 = \langle a_{i^1}, d\rangle . e_{i^1}$, $e_i \in \mathbb{R}^N$, $i \in \{1, .., N\}$ is the canonical basis in $\mathbb{R}^N$.

The criterion is $\|A.\mathrm{x}^1 - d\|^2 = \|d\|^2 - (\langle a_{i^1}, d\rangle)^2$.

The **residual** is $r = d - A.\mathrm{x}^1 = d - \langle a_{i^1}, d\rangle a_{i^1}$, and the process is repeated.

Matching Pursuit Algorithm
Input: A (with unit norm column), d, K.

Initialize: $r^0 = d, \sigma^0 = \varnothing, (x^0 = 0)$.

Repeat, while $\#\sigma^k \leq K$: (or while $\|r^k\| > \epsilon$)

$$\begin{array}{rcl} i^k & = & \arg\max_{j\in\{1,..,N\}} |\langle r^k, a_j\rangle| \\ \sigma^{k+1} & = & \sigma^k \cup \{i^k\} \\ r^{k+1} & = & r^k - \langle r^k, a_{i^k}\rangle . a_{i^k} \end{array} \tag{1}$$

σ^k is the support of the current solution x^k, that is the indexes of the non-zero components. $\#\sigma^k$ is the cardinal of σ^k. The initial value of $\#\sigma^0$ is 0 and it increases by 1 at each iteration.
The optimal solution at current iteration is $x^{k+1} = x^k + \langle r^k, a_{i^k}\rangle . e_{i^k}$.

- The residual $\|r^k\|$ converges exponentially to 0 [Mallat et al 93].
- Sub-optimal solution: retro-project the residual onto $Span\{(a_i)_{i\in\sigma^K}\}$ reduce the approximation error $(\|A.\mathrm{x}^K - d\|^2)$.

Orthogonal Matching Pursuit [Pati et al 93, Tropp 04]: at each iteration, optimally estimate the intensities with the current support of the solution fixed, by $\mathrm{x}^{k+1} = \arg \min_{\{\mathrm{x}/\sigma_{\mathrm{x}} \subset \sigma^{k+1}\}} \|A\mathrm{x} - d\|^2$.

Orthogonal Matching Pursuit (OMP) Algorithm Input: A (with unit norm column), d, K.

i = intensity

Initialize: $r^0 = d, \sigma^0 = \varnothing$

Repeat, while $\#\sigma^k \leq K$**:**

$$
\begin{aligned}
i^k &= \arg\max_{j \notin \sigma^k} |\langle r^k, a_j \rangle| \\
\sigma^{k+1} &= \sigma^k \cup \{i^k\} \\
\mathrm{x}^{k+1} &= \arg \min_{\{\mathrm{x}/\sigma_{\mathrm{x}} \subset \sigma^{k+1}\}} \|A\mathrm{x} - d\|^2 \\
r^{k+1} &= d - A\mathrm{x}^{k+1}
\end{aligned}
$$

- Convergence in N iterations at most (at each iteration a **new** component is selected),
- Exact sparse recovery results (under conditions on A) [Tropp 04].

Further algorithms:

At each iteration, several strategies for one component to be

- added,
- removed,
- replaced.

Orthogonal Least Squares (OLS) [Chen et al 89], *Bayesian OMP* [Herzet et al 10], *Single Best Replacement* [Soussen et al 11] and further variants [Jain & al 11, Soussen et al 15]...

The more complex is the strategy, the best is the solution and the longest is the computing time.

Exact reformulation

- *Class of continuous nonconvex penalties* → asymptotic connections with the ℓ_2-ℓ_0 criteria [Chouzenoux et al 13]
- *Reformulation using Difference of Convex functions* → asymptotic or local minimizer results [Le Thi et al 14, Le Thi et al 15]
- *Equivalence of ℓ_0- and ℓ_p-norm $(0 < p \leq 1)$ minimization under linear equalities or inequalities (e.g. exact reconstruction problem)* [Fung and Mangasarian 11]
- *Reformulation and optimization through Mixed-Integer Programs (MIPs)* → global optimum for problems of reasonable size (a few hundred variables) [Bourguignon et al 15]
- **Exact reformulation** ([Bi et al 14, Yuan & Ghanem 16, Liu et al 18], ,...)

Exact reformulation of ℓ_0: Penalized reformulation

Lemma 1 [Liu et al 18, Yuan & Ghanem 16]

$$\|x\|_0 = \min_{-1\leq u\leq 1} \|u\|_1 \text{ s.t } \|x\|_1 = < u, x >$$

Exact reformulation for the $\ell_2 - \ell_0$ penalized problem

Initial problem:

$$\min_x \frac{1}{2}\|Ax - d\|_2^2 + \lambda\|x\|_0$$

Penalized reformulation:

$$\min_{x,u} G_\rho(x, u) := \frac{1}{2}\|Ax - d\|^2 + \iota_{\{-1\leq\cdot\leq 1\}}(u) + \lambda\|u\|_1 + \rho(\|x\|_1 - < x, u >)$$

with $\iota_{\{x\in D\}}(x) = 0$ if $x \in D$, $+\infty$ otherwise.

Theorem [Bechensteen,et al.]

If $\rho > \sigma_{max}(A)\|d\|_2$, and A is of full rank. Then:

1. If (x_ρ, u_ρ) is a local (respectively global) minimizer of G_ρ, then x_ρ is a local (respectively global) minimizer of the initial problem.
2. If $\hat{x}$ is a global minimizer of the initial problem, then $(\hat{x}, \hat{u})$ is a global minimizer of G_ρ with $\hat{u}$ associated with Lemma 1.

Exact reformulation of ℓ_0: Constrained reformulation

Lemma 1 [Liu et al 18, Yuan & Ghanem 16]

$$\|\mathrm{x}\|_0 = \min_{-1\leq \mathrm{u} \leq 1} \|\mathrm{u}\|_1 \text{ s.t } \|\mathrm{x}\|_1 = <\mathrm{u}, \mathrm{x}>$$

Exact reformulation for the $\ell_2 - \ell_0$ constrained problem

Initial problem:

$$\min_{\mathrm{x}} \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \iota_{\{\|\cdot\|_0 \leq K\}}(\mathrm{x})$$

Constrained reformulation:

$$\min_{\mathrm{x},\mathrm{u}} G_\rho(\mathrm{x}, \mathrm{u}) := \frac{1}{2}\|A\mathrm{x} - d\|^2 + \iota_{\{\cdot \geq 0\}}(\mathrm{x}) + \iota_{\{-1\leq \cdot \leq 1\}}(\mathrm{u}) + \iota_{\{\|\cdot\|_1 \leq K\}}(u) + \rho(\|\mathrm{x}\|_1 - <\mathrm{x}, \mathrm{u}$$

Theorem [Bechensteen,et al.]

If $\rho > \sigma_{max}(A)\|d\|_2$, and A is of full rank. Then:

1. If (x_ρ, u_ρ) is a local (respectively global) minimizer of G_ρ, then x_ρ is a local (respectively global) minimizer of the initial problem.
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Why minimize the constrained or penalized reformulation instead of their initial formulation?

Constrained reformulation:

$$\min_{\mathrm{x},\mathrm{u}} \frac{1}{2}\|A\mathrm{x}-d\|^2+\iota_{\{\cdot\geq 0\}}(\mathrm{x})+\iota_{\{-1\leq\cdot\leq 1\}}(u)+\iota_{\{\|\cdot\|_1\leq K\}}(u)+\rho(\|\mathrm{x}\|_1-<\mathrm{x},\mathrm{u}>)$$

Penalized reformulation:

$$\min_{\mathrm{x},\mathrm{u}} \frac{1}{2}\|A\mathrm{x}-d\|^2+\iota_{\{\cdot\geq 0\}}(\mathrm{x})+\iota_{\{-1\leq\cdot\leq 1\}}(\mathrm{u})+\lambda\|u\|_1+\rho(\|\mathrm{x}\|_1-<\mathrm{x},\mathrm{u}>)$$

- Biconvex
- Non-convexity linked to the coupling term $<\mathrm{x},\mathrm{u}>$
- Minimizing the reformulation is equivalent to minimize the initial problem regarding local and global minimizers

We add a positivity constraint on x and we finally define

$$G_\rho(\mathrm{x},\mathrm{u}) = \frac{1}{2}\|A\mathrm{x}-d\|^2 + \iota_{\{\cdot\geq 0\}}(\mathrm{x}) + \rho\|\mathrm{x}\|_1 + \iota_{\{\|\cdot\|_1\leq K\}}(\mathrm{u}) + \iota_{\{-1\leq\cdot\leq 1\}}(\mathrm{u}) - \rho < \mathrm{x},\mathrm{u}$$

The global optimization scheme is (continuation method)

Initialize: $\rho^0 > 0, n = 0$

Repeat: Solve the problem G_{ρ^n}:

$$\{\mathrm{x}^{n+1},\mathrm{u}^{n+1}\} = \arg\min_{\mathrm{x},\mathrm{u}} G_{\rho^n}(\mathrm{x},\mathrm{u})$$

Update: $\rho^{n+1} = \alpha\rho^n$, $\alpha > 1$

Until: $\rho^{n+1} > \sigma_{max}(A)\|d\|_2$

$$G_{\rho^n}(\mathrm{x},\mathrm{u}) = \frac{1}{2}\|A\mathrm{x}-d\|^2 + \iota_{\{\cdot\geq 0\}}(\mathrm{x}) + \rho^n\|\mathrm{x}\|_1 + \iota_{\{\|\cdot\|_1\leq K\}}(\mathrm{u}) + \iota_{\{-1\leq\cdot\leq 1\}}(\mathrm{u}) - \rho^n < \mathrm{x}$$

At fixed ρ^n we apply the Proximal Alternate Minimization (PAM) algorithm [Attouch & al 10]

Initialize: $\mathrm{u}^0 = 0 \in \mathbb{R}^M$

Repeat: $\arg\min G_{\rho^n}$ using alternate minimizations

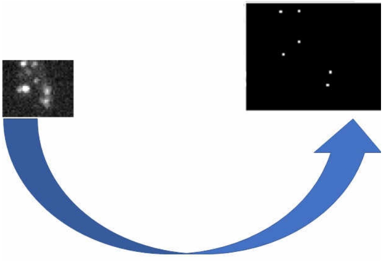
- $\{\mathrm{x}^{n+1}\} = \arg\min_{\mathrm{x}} G_{\rho^n}(\mathrm{x},\mathrm{u}^n) + \frac{1}{2c^n}\|\mathrm{x}-\mathrm{x}^n\|^2$
  $\rightarrow$ FISTA Algorithm [Beck et al 09]

- $\{\mathrm{u}^{n+1}\} = \arg\min_{\mathrm{u}} G_{\rho^n}(\mathrm{x}^{n+1},\mathrm{u}) + \frac{1}{2d^n}\|\mathrm{u}-\mathrm{u}^n\|^2$
  $\rightarrow$ Algorithm [Stefanov, 2004]

Until: convergence

Convergence of the algorithm towards a critical point of G_{ρ^n} for c^n and d^n such that $0 < r_- < c^n, d^n < r_+$ and under KL condition on G_{ρ^n} and assuming that x_n and u_n are bounded [Attouch & al 10].

Results: Single-Molecule Localization Microscopy



$$\hat{x} \in \arg\min_{x} \frac{1}{2}\|Ax - d\|_2^2 + \iota_{\{\cdot \geq 0\}}(x) + R(x)$$

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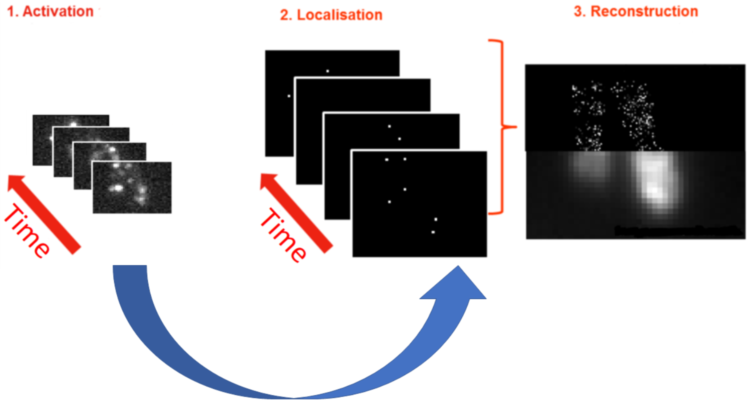


$$\hat{\mathrm{x}} \in \arg\min_{\mathrm{x}} \frac{1}{2}\|A\mathrm{x} - d\|_2^2 + \iota_{\{\cdot \geq 0\}}(\mathrm{x}) + R(\mathrm{x})$$

Results, ISBI challenge 2013, simulated dataset

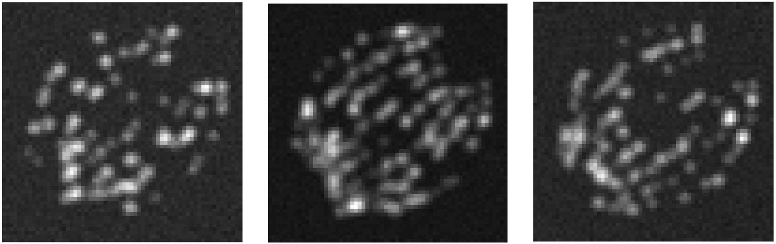


Figure: Simulated images (among the 361 simulated high density images for this sample). Data from IEEE ISBI Challenge 2013.
http://bigwww.epfl.ch/smlm/datasets/index.html

8 simulated tubes of 30nm diameter
Camera of 64×64 pixels of size 100nm.
Gaussian PSF, FWHM = 258.21 nm (full width at half maximum)
80932 molecules activated on 361 frames.

Results, ISBI challenge 2013, simulated dataset

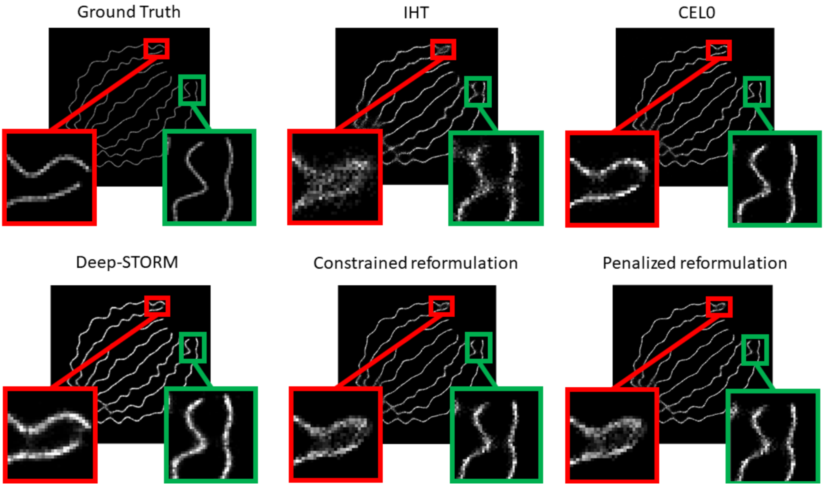
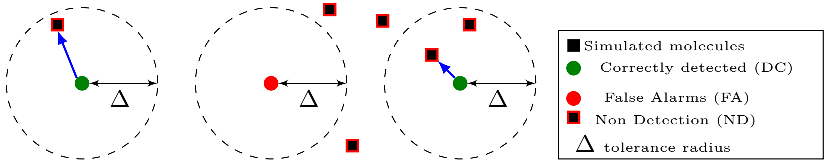


Figure: Reconstruction from simulated data set, reduction ratio $L = 4$.

Results, ISBI challenge 2013, simulated dataset

Jaccard index calculus



$$\text{Jaccard index} = \frac{\text{DC}}{\text{DC} + \text{FA} + \text{ND}}$$

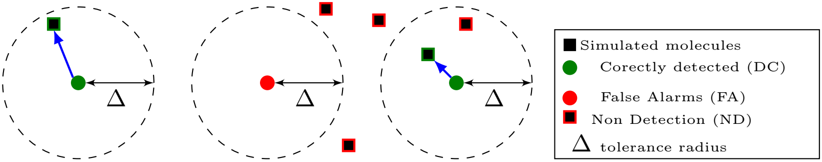
Jaccard index results

	Jaccard index (%)			
Method - Tolerance (nm)	50	100	150	200
IHT	20.1	35.9	40.4	41.3
CEL0	**29.3**	**41.3**	42.4	42.6
Constrained reformulation	25.2	40.0	**43.2**	**43.9**
Penalized reformulation	25.0	39.3	42.2	42.8
Deep-STORM	×	×	×	×

Table: The Jaccard index obtained and the tolerance

Results, ISBI challenge 2013, simulated dataset

Jaccard index calculus



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Deep-STORM	×	×	×	×

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Results, ISBI challenge 2013, Real dataset

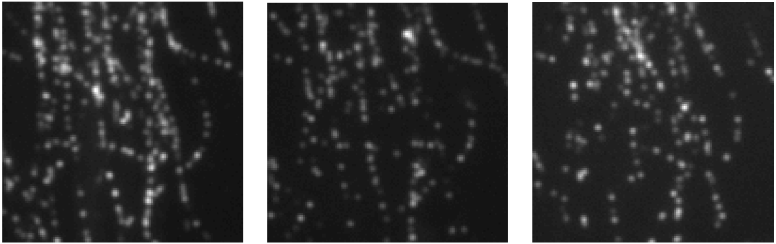

Figure: Real images (among the 500 real high density images for this sample). Data from IEEE ISBI Challenge 2013.
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Camera of 128×128 pixels of size 100nm.
Gaussian PSF, FWHM = 358.1 nm (full width at half maximum)

Results, ISBI challenge 2013, Real dataset

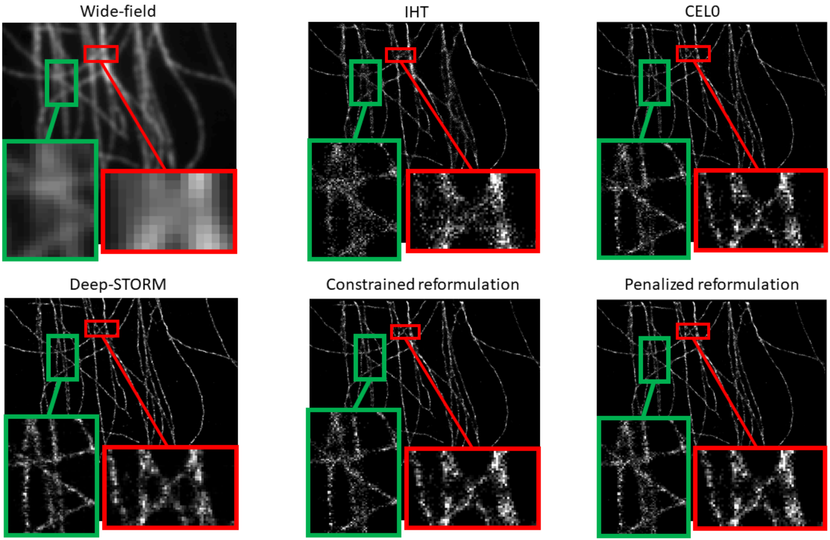

Figure: Reconstruction from the real data set, reduction ratio $L = 4$.

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