MSc Data Science & Artificial Intelligence - AIEDA323

Biomedical Signal Processing

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Introduction

Course organization

- 4 lectures
- 4 computer lab sessions
- Course material sur Moodle: **Biomedical Signal Processing** - **AIEDA323**
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Resources

- Reference books
  - [RAN02] Rangayyan, *Biomedical Signal Analysis*, IEEE Press, 2002
  - [OPP89] Oppenheim, Schafer, *Discrete-time Signal Processing*, Prentice-Hall, 1989
  - [HAY96] Hayes, *Statistical Digital Signal Processing and Modeling*, John Wiley, 1996
- Course slides
- Lab guide

Evaluation

- Written reports of computer labs

Goals and organization

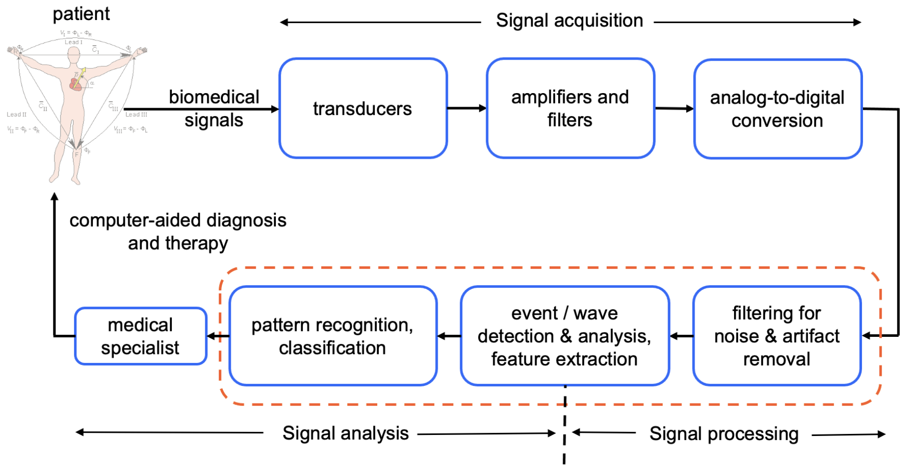
Course objectives

- Understand the genesis of **biomedical signals** — a.k.a. *biosignals* — issued from
  - different organs (heart, brain, muscles, nerves) and pathophysiological processes
  - different modalities (electric / magnetic / acoustic, invasive / noninvasive, etc.)
- Recognize the need for **biomedical signal processing (BSP)** to
  - remove noise and artifacts from observed records
  - extract useful information for monitoring, diagnosis, prognosis, therapy selection
  - model physiological phenomena
- Get acquainted with basic and advanced **BSP techniques** for
  - spectral (frequency-domain) characterization and filtering
  - optimal spatial filtering
  - blind source separation
- **Implement and apply** BSP techniques on real signals using scientific computing software
- Acknowledge that BSP is a key ingredient of **explainable AI for health**

Syllabus

1. Introduction to biosignals and BSP (O. Meste, 1 lecture + 1 lab)
2. Spectral analysis (4 lectures / labs)
3. Optimal spatial filtering (1 lecture / lab)
4. Blind source separation (1 lecture / lab)

Biomedical signal processing and analysis workflow



Chapter 1: Spectral analysis and filtering

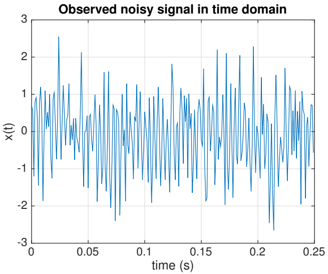
Objectives

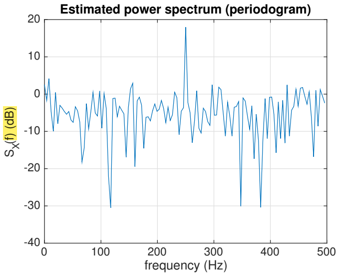
- Review the **fundamentals of digital signal processing**
- Realize the need for representing signals in the **frequency domain**
- Get acquainted with the main **tools for spectral analysis** of deterministic signals
  - discrete-time Fourier transform (DTFT)
  - discrete Fourier transform (DFT), fast Fourier transform (FFT)
- Study **power spectral density estimators** of random signals
  - periodogram and variants (Bartlett, Welch)
- Understand the concept of **frequency filtering**
- **Design digital filters** with the desired frequency response
- **Implement** spectral analysis and filtering techniques using **MATLAB**
- Illustrate their performance in **biomedical signal processing problems** [lab assignment]

Spectral analysis — motivating examples (1/2)

Finding harmonic structure

Spectral analysis can often reveal repetitive or periodic components that are otherwise hidden in the time-domain representation of noisy signals.





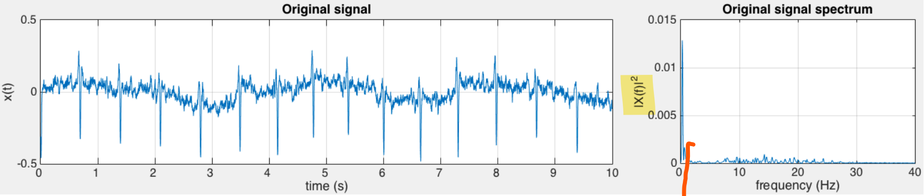
- [Left plot] Noisy data in time domain seem to lack ‘interesting’ components.
- [Right plot] Spectral analysis reveals periodic component at $f_0 = 250$ Hz.

Spectral analysis — motivating examples (2/2)

Artifact cancellation in biomedical data

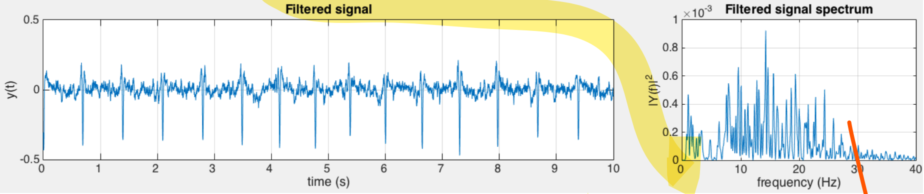
- Electrocardiogram (ECG) records are often corrupted by noise and artifacts.
- Spectral estimation allows the identification of corrupted frequency bands.
- Optimal frequency filters can be designed for artifact cancellation and signal enhancement.

Original signal: ECG record corrupted by baseline wandering and high-frequency noise

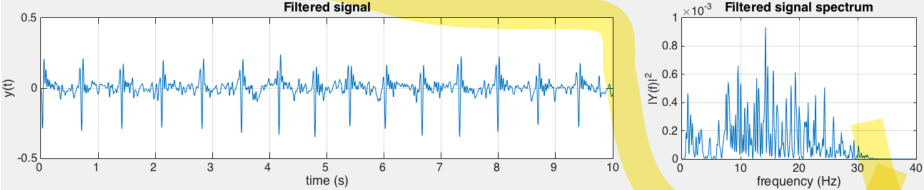


Spectral analysis — motivating examples (2/2, cont'd)

Highpass filtering ($f_c = 0.5$ Hz) $\rightarrow$ baseline wandering removal



Lowpass filtering ($f_c = 30$ Hz) $\rightarrow$ high-frequency noise suppression



Spectral analysis (or estimation)

Definition

From a finite record of a stationary data sequence, estimate how the signal power is distributed over frequency — i.e., find its **power spectral density (PSD)**.

Useful in many application domains

- **Medicine**: physiological data analysis (electrocardiogram, electroencephalogram, ...).
- **Mechanics**: vibration monitoring, fault detection.
- **Astronomy, finance**: hidden periodicity finding.
- **Speech and audio processing**: speech recognition, audio compression, music recognition.
- **Seismology**: earthquake analysis, focus localization, tremor prediction.
- **Control systems**: dynamic behavior analysis, controller synthesis.

Spectral analysis is a fundamental tool

for the electrical engineer and the data scientist.

Continuous-time (analog) signals

Most signals — including biosignals — can be mathematically described as a function of continuous time, i.e., they are **analog signals**:

$$x(t), \quad t \in \mathbb{R}.$$

Continuous-time Fourier transform (FT)

$$X(F) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi Ft} dt$$

$$x(t) = \mathcal{F}^{-1}\{X(F)\} = \int_{-\infty}^{+\infty} X(f) e^{j2\pi Ft} dF$$

where j represents the imaginary unit: $j^2 = -1$.

- $x(t)$ and $X(F)$ are called *FT pairs*:

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(F)$$

- The FT is a key tool in signal processing. It provides an alternative representation of the signal, emphasizing its periodic (harmonic) components.

From continuous time to discrete time

To be processed by digital means, analog signals must be transformed into **digital signals** through time sampling and quantization.

- **Time sampling**:

$$x[n] = x(nT_s), \quad n \in \mathbb{Z}$$

T_s: sampling period (seconds); $F_s \stackrel{\text{def}}{=} \frac{1}{T_s}$: sampling frequency (samples/second or Hz)

- **Quantization**: signal values are restricted to a discrete set.

A discrete-time signal $x[n]$, $n \in \mathbb{Z}$, is also called a *sequence*.

Example 1: Regular sampling of $x(t) = \cos(2\pi F_0 t)$ at sampling rate $F_s = 1/T_s$.

$$x[n] = x(nT_s) \underset{\substack{\uparrow \\ t=nT_s}}{=} \cos(2\pi F_0 n T_s) = \cos(\omega_0 n), \quad \omega_0 \stackrel{\text{def}}{=} \frac{2\pi F_0}{F_s}$$

ω_0: radians/sample

Some usual sequences

- **Unit sample sequence**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Any sequence $x[n]$ can be expressed as the *convolution*

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n] * \delta[n] \tag{1}$$

- **Unit step sequence**

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- **Exponential sequence**

$$x[n] = Ae^{\alpha n}$$

Noting $A = |A|e^{j\phi}$ and $\alpha = |\alpha|e^{j\omega_0}$ $\Rightarrow$ $x[n] = |A||\alpha|^n e^{j(\omega_0 n + \phi)}$

- If $|\alpha| = 1$: **complex exponential sequence**

$$x[n] = |A|e^{j(\omega_0 n + \phi)} = |A|\cos(\omega_0 n + \phi) + j|A|\sin(\omega_0 n + \phi)$$

$|A|$: amplitude $\qquad \omega_0$: frequency $\qquad \phi$: phase

- $\mathrm{Re}\{x[n]\} = |A|\cos(\omega_0 n + \phi)$: **discrete sinusoid**

Some usual sequences (cont'd)

→ *Exercise 1:* Prove that discrete-time sinusoids are periodic in frequency, i.e., sinusoids with frequency $\omega_0 + 2\pi k$ are identical sequences for any integer $k \in \mathbb{Z}$.

→ *Exercise 2:* Using MATLAB `stem` command, plot in the same figure the discrete sinusoids $\cos(\omega_0 n)$ and $\cos((\omega_0 + 2\pi)n)$, with $\omega_0 = \pi/5$ rad/sample, for $0 \leq n \leq 20$.

Frequency-domain representation of discrete-time signals

Discrete-time Fourier transform (DTFT)

$$X(e^{j\omega}) = \mathcal{F}\{x[n]\} = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}, \quad \omega \in \mathbb{R}$$

$$x[n] = \mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega, \quad n \in \mathbb{Z}$$

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

- $X(e^{j\omega})$ is a complex-valued function of real-valued argument ω:

$$X(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}$$

$|X(e^{j\omega})|$: magnitude $\quad\quad$ $\sphericalangle X(e^{j\omega})$: phase

- $X(e^{j\omega})$ is 2π-periodic: $X(e^{j(\omega+2\pi k)}) = X(e^{j\omega})$, $k \in \mathbb{Z}$ (corollary of [Exercise 1])

  $\rightarrow$ it suffices to specify the DTFT over an interval of length 2π rad/sample, typically $[-\pi, \pi]$

- *Sufficient condition for existence*: $x[n]$ absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]| < +\infty \qquad (2)$$

DTFT properties

- *Hermitian symmetry*:

$$\text{If } x[n] \in \mathbb{R} \quad \Rightarrow \quad X(e^{j\omega}) = X^*(e^{-j\omega}) \quad \begin{cases} |X(e^{j\omega}) = |X(e^{-j\omega})|| \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$$

- *Linearity*:

$$\mathcal{F}\left\{\sum_{p=1}^{P} a_p x_p[n]\right\} = \sum_{p=1}^{P} a_p X_p(e^{j\omega}), \quad \text{with } X_p(e^{j\omega}) = \mathcal{F}\{x_p[n]\}$$

- *Time shift*:

$$\mathcal{F}\{x[n-n_0]\} = X(e^{j\omega})e^{-j\omega n_0}$$

- *Modulation*:

$$\mathcal{F}\{x[n]e^{j\omega_0 n}\} = X(e^{j(\omega-\omega_0)})$$

DTFT properties (cont'd)

- *Convolution*:

$$\mathcal{F}\{x[n] * y[n]\} = X(e^{j\omega})Y(e^{j\omega})$$

- *Product*:

$$\mathcal{F}\{x[n]y[n]\} = X(e^{j\omega}) * Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\xi})Y(e^{j(\omega-\xi)})d\xi \qquad (3)$$

- *Parseval's theorem*:

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \int_{-1/2}^{1/2} |X(e^{j2\pi f})|^2 df \qquad (4)$$

  - E_x: **energy** of sequence $x[n]$.
  - $|X(e^{j\omega})|^2 d\omega$: infinitesimal signal energy in the frequency band $\left[\omega - \frac{d\omega}{2}, \omega + \frac{d\omega}{2}\right]$ (rad/sample).
  - Hence, $|X(e^{j\omega})|^2$ is the **energy spectral density** of $x[n]$.

Usual Fourier transform pairs

$x[n]$	$X(\omega), \quad \lvert\omega\rvert < \pi$	$X(f), \quad \lvert f\rvert < 1/2$
1	$2\pi\delta(\omega)$	$\delta(f)$
$e^{j\omega_0 n}$	$2\pi\delta(\omega - \omega_0)$	$\delta(f - f_0)$
$\cos(\omega_0 n)$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$	$\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)$
$\delta[n]$	1	1
$\delta[n - n_0]$	$e^{-j\omega n_0}$	$e^{-j2\pi f n_0}$
$a^n u[n], \ \lvert a\rvert < 1$	$\frac{1}{1-ae^{-j\omega}}$	$\frac{1}{1-ae^{-j2\pi f}}$

Remark

- $\omega = 2\pi f \ \Rightarrow \ \delta(\omega) = \frac{1}{2\pi}\delta(f)$

$\rightarrow$ *Exercise 3:* Prove the last row of the above table.

Usual Fourier transform pairs (cont'd)

→ *Exercise 4:* Compute the DTFT of the length-N *rectangular window*

$$w_N[n] \stackrel{\text{def}}{=} \begin{cases} 1, & 0 \leq n \leq (N-1) \\ 0, & \text{elsewhere} \end{cases}$$

Hint: $1 - e^{-a} = e^{-a/2}(e^{a/2} - e^{-a/2})$, for any $a \in \mathbb{C}$.

→ *Exercise 5:* Using MATLAB commands `linspace`, `abs` and `plot`, plot the DTFT magnitude (absolute value) of the 64-point rectangular window $w_{64}[n]$ in 1000 equispaced points of the interval $[-\pi, \pi]$ rad/sample. Compare the theoretical expression found in the previous exercise and the output of the `freqz` command. Label the plot axes with `xlabel` and `ylabel`.

Link with continuous-time Fourier transform

Nyquist-Shannon sampling theorem

Let $x(t)$ a continuous-time signal with bandwidth B, i.e., $X(f) = 0$, $|f| > B$.

Signal $x(t)$ can be recovered *exactly* from its time samples $x[n] = x(nT_s)$ if

$$F_s > 2B.$$

- The Nyquist-Shannon theorem is a fundamental result in signal processing.
- It provides the sufficient condition for perfect recovery of an analog signal from its digital counterpart.
- It underlies today's digital world, as it enables the *lossless* storage, processing and transmission of analog signals by digital means.

Link with continuous-time Fourier transform (cont'd)

Link between FT and DTFT

If $x[n] = x(nT_s)$ and the conditions of the Nyquist-Shannon theorem are fulfilled, then:

$$X(F) = \frac{1}{F_s} X\left(e^{j2\pi \frac{F}{F_s}}\right), \quad |F| < \frac{F_s}{2}$$

- In $|F| < F_s/2$, $|\omega| < \pi$, frequency axes of $\mathcal{F}\{x(t)\}$ and $\mathcal{F}\{x[n]\}$ follow the linear maps

$$F \text{ (Hz)} \mapsto \omega = 2\pi \frac{F}{F_s} \text{ (rad/sample)}$$

$$\omega \text{ (rad/sample)} \mapsto F = \frac{\omega}{2\pi} F_s \text{ (Hz)}$$

- Discrete-time frequency ω = *normalized* continuous-time frequency F.
- Continuous-time frequency F = *denormalized* discrete-time frequency ω.
- Normalization factor: sampling frequency F_s.

DTFT of pure sinusoidal components

The DTFT of a **pure sinusoidal component** or **tone** of frequency ω_0:

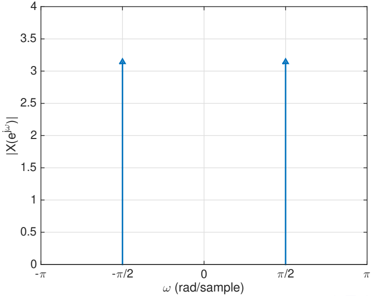
$$x[n] = \cos(\omega_0 n)$$

consists of two Dirac's impulses at $\omega = \pm\omega_0$:

$$X(e^{j\omega}) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0), \quad |\omega| < \pi$$

Hence, the position of Dirac's impulses allows us to identify the periodicity of the signal.

Example 2: If $x[n] = \cos\left(\frac{\pi}{2}n\right) \quad \Rightarrow \quad X(e^{j\omega}) = \pi\delta\left(\omega - \frac{\pi}{2}\right) + \pi\delta\left(\omega + \frac{\pi}{2}\right), \quad |\omega| < \pi$



Finite observation length effects

In practice, only a finite number of samples N are observed:

$$x_N[n] = \begin{cases} x[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

We can express the observed sequence:

$$x_N[n] = w_N[n]x[n]$$

where $w_N[n]$ is the length-N rectangular window. By the product property (3):

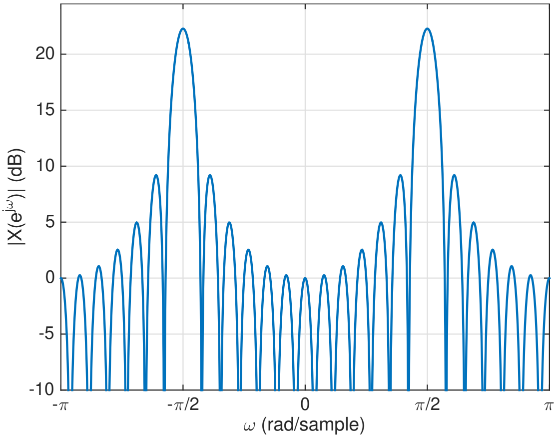
$$X_N(e^{j\omega}) = \mathcal{F}\{w_N[n]x[n]\} = W_N(e^{j\omega}) * X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W_N(e^{j\xi})X(e^{j(\omega-\xi)})d\xi$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} W_N(e^{j\xi})\big[\delta(\omega-\omega_0-\xi)+\delta(\omega+\omega_0-\xi)\big]d\xi \underset{\substack{\uparrow \\ \int f(x)\delta(x-x_0)dx=f(x_0)}}{=} \frac{1}{2}W_N(e^{j(\omega-\omega_0)})+\frac{1}{2}W_N(e^{j(\omega+\omega_0)})$$

In the frequency spectrum of the finite length observation, Dirac's impulses $\delta(\omega \pm \omega_0)$ are replaced by the DTFT of the rectangular window, $W_N(e^{j(\omega\pm\omega_0)})$.

Finite observation length effects (cont'd)

Example 3: $x[n] = \cos\left(\frac{\pi}{2}n\right)$, $0 \leq n \leq 24$ $(N = 25)$



Finite observation length effects (cont'd)

Finite sample size N causes **two effects** on the frequency spectrum:

Spectral smearing or smoothing

Caused by the **main lobe** of $W_N(e^{j\omega})$:

Frequency components spaced by less the **main lobe width (MLW)** of $W_N(e^{j\omega})$ cannot be resolved in the DTFT.

Power leakage

Caused by **sidelobes** of $W_N(e^{j\omega})$:

Frequency components with power below the **sidelobe level (SLL)** of $W_N(e^{j\omega})$ may be masked in the DTFT.

→ *Exercise 6:* Using MATLAB, plot in each case the DTFT of length-N sequence:

$$x[n] = \cos(0.4\pi n) + A\cos(0.45\pi n), \quad 0 \leq n \leq (N-1)$$

	case 1	case 2	case 3	case 4
A	1	1	1	0.2
N	64	32	40	40

Is it always possible to identify the two harmonic components? Why?

Usual windows

Name	*Definition*	*MLW*	*SLL*
rectangular	$w[n] = \begin{cases} 1, & 0 \leq n \leq (N-1) \\ 0, & \text{otherwise} \end{cases}$	$4\pi/N$	-13 dB
triangular (Bartlett)	$w[n] = \begin{cases} \frac{2n}{N-1}, & 0 \leq n \leq (N-1)/2 \\ 2 - \frac{2n}{N-1}, & (N-1)/2 < n \leq N-1 \end{cases}$	$8\pi/N$	-26 dB
Hanning	$w[n] = \frac{1}{2}\left(1 - \cos(\frac{2\pi n}{N-1})\right),\ 0 \leq n \leq N-1$	$8\pi/N$	-31 dB
Hamming	$w[n] = 0.54 - 0.46\cos(\frac{2\pi n}{N-1}),\ 0 \leq n \leq N-1$	$8\pi/N$	-41 dB
Blackman	$w[n] = 0.42 - 0.5\cos(\frac{2\pi n}{N-1}) + 0.08\cos(\frac{4\pi n}{N-1}),\ 0 \leq n \leq N-1$	$12\pi/N$	-57 dB

Table 1: Typical windows used in digital signal processing. MLW: main lobe width (rad/sample); SLL: sidelobe level.

→ *Exercise 7:* Using MATLAB `window` and `freqz` commands, compute the DTFT of the above windows with length $N = 25$ and plot their normalized magnitude $|W(e^{j\omega})|/|W(e^{j0})|$ (in dB) in the interval $[-\pi, \pi]$ rad/sample. Check the MLW and SLL values given in the table.

Discrete Fourier transform (DFT)

The DTFT $X(e^{j\omega})$ is a function of a real variable, $\omega \in \mathbb{R}$.

Q: How to compute and store the DTFT by digital means? **A**: By sampling ω

Discrete Fourier series (DFS)

Let $\tilde{x}[n]$ be an N-periodic sequence: $\tilde{x}[n] = \tilde{x}[n + rN]$, $\forall r \in \mathbb{Z}$.

Because of its periodicity, sequence $\tilde{x}[n]$ admits the **discrete Fourier series (DFS)** representation:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j\frac{2\pi kn}{N}}$$

with

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\frac{2\pi kn}{N}}$$

$$\tilde{x}[n] \overset{\text{DFS}}{\longleftrightarrow} \tilde{X}[k]$$

Discrete Fourier transform (DFT), cont'd

Let $x[n]$ be a non-periodic sequence with DTFT $X(e^{j\omega})$.

Uniformly sample $X(e^{j\omega})$ on N equispaced points in the interval $[0, 2\pi]$ rad/sample to form the DFS coefficients:

$$\tilde{X}[k] = X(e^{j\omega_k}), \quad \omega_k = \frac{2\pi k}{N}, \quad 0 \leq k \leq N-1$$

Then:

$$\tilde{x}[n] = \mathrm{DFS}^{-1}\{\tilde{X}[k]\} = \sum_{r=-\infty}^{\infty} x[n - rN] \tag{5}$$

i.e., $\tilde{x}[n]$ is the N-periodic extension of $x[n]$.

→ *Exercise 8:* Can you prove equation (5)?

Two cases:

- length$\{x[n]\} \leq N$: no overlap in the repetitions → one period of $\tilde{x}[n] = x[n]$
- length$\{x[n]\} > N$: overlap in the repetitions → $x[n]$ cannot be recovered from $\tilde{x}[n]$.

Discrete Fourier transform (DFT), cont'd

Discrete Fourier transform (DFT)

The DFT of sequence $x[n]$ of length N is defined as the DFS coefficients of its N-periodic extension:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}}, \qquad 0 \leq k \leq N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}}, \qquad 0 \leq n \leq N-1$$

$$x[n] \overset{\text{DFT}}{\longleftrightarrow} X[k]$$

- DFT complexity: $O(N^2)$ multiplications.

Fast Fourier transform (FFT)

When $N = 2^p$, $p \in \mathbb{Z}^+$, the DFT admits an equivalent fast implementation called the **fast Fourier transform (FFT)**, with complexity $O(N \log N)$.

The FFT was invented by Carl Friedrich Gauss in 1805, and rediscovered by James Cooley (IBM) and John Tukey (Princeton) in 1965. This tool boosted the development of DSP as a discipline.

Discrete Fourier transform (DFT), cont'd

→ *Exercise 9:* Using the MATLAB commands `fft` and `plot`:

1. Compute the 64-point DFT of the 64-point rectangular window, $w_{64}[n]$, defined in [Exercise 4].
2. Plot the absolute values $|X[k]|$ of the 64-point DFT of $w_{64}[n]$ using the actual discrete frequency values on the horizontal axis: $\omega_k = 2\pi k/64$, $0 \le k \le 63$.
3. By superimposing the curves in the same interval, compare with the theoretical DTFT found in [Exercise 4], plotted in [Exercise 5]. If plotting in the interval $[-\pi, \pi]$ rad/sample, use `fftshift` to bring the FFT values to the desired frequency interval.

→ *Exercise 10:* Consider the 64-sample Bartlett window (cf. [Table 1]).

1. Compute the 64-point DFT, $X[k]$, $0 \le k \le 63$. Plot its absolute values and compare with those of the DTFT, as done in the previous exercise.
2. Downsample the DTFT to generate the 32-point DFT:

$$\tilde{X}[k] = X\left(e^{j\frac{2\pi k}{32}}\right), \quad 0 \le k \le 31.$$

   Using command `ifft`, compute the inverse DFT, $\tilde{x}[n] = \mathrm{DFT}^{-1}\{\tilde{X}[k]\}$. What do you obtain? Why?

→ *Exercise 11:* Repeat [Exercise 9] by padding 64 zeros to $w_{64}[n]$ and then computing the 128-point DFT of the zero-padded sequence. What can you conclude?

Discrete Fourier transform (DFT), cont'd

Zero padding

Let $x[n]$ be a length-N sequence and consider the length-M zero-padded sequence, $M > N$:

$$y[n] \stackrel{\text{def}}{=} \begin{cases} x[n], & 0 \leq n \leq N-1 \\ 0, & N \leq n \leq M-1 \end{cases}$$

Then:

- The DTFT of $y[n]$ is the same as that of $x[n]$:

$$Y(e^{j\omega}) = X(e^{j\omega})$$

- The M-point DFT of $y[n]$ yields the DTFT of the original sequence $x[n]$ sampled at frequencies $\omega_k = \frac{2\pi k}{M}$:

$$Y[k] = X(e^{j\frac{2\pi k}{M}}), \quad 0 \leq k \leq M-1$$

→ *Exercise 12:* Prove the above results.

Corollary: Zero-padding has no other effect than sampling the DTFT with finer detail.

In the sequel, zero-padding will be implicitly used when computing the M-point DFT with $M > N$.

Random processes

- The DTFT / DFT / FFT characterize **deterministic signals** in the frequency domain.
- Pathophysiological phenomena present a certain degree of randomness or uncertainty, for which deterministic modeling is no longer suitable.
- Biosignals are better modeled as **random** or **stochastic processes**, and call for a statistical characterization.

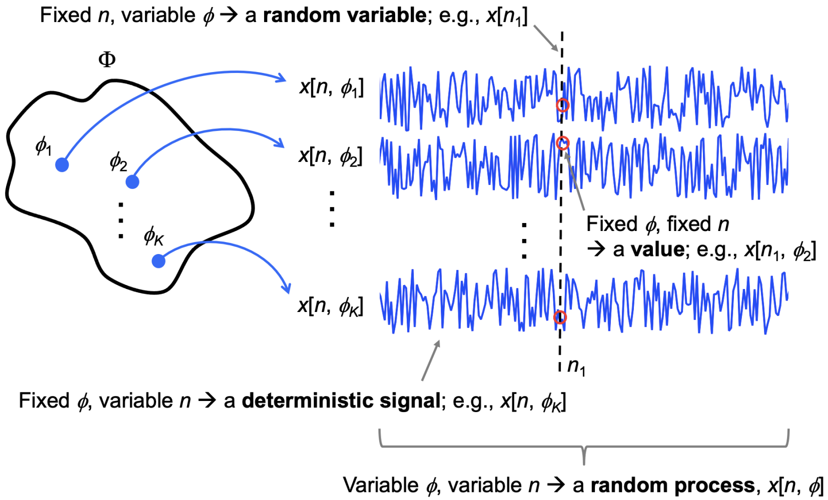
Random process

A **random process** is a function of the elements of a sample space Φ, as well as another independent variable, n, typically a discrete-time index.

Given an experiment E, with sample space Φ, the random process $x[n]$ maps each possible outcome $\phi \in \Phi$ to a function of n, denoted $x[n, \phi]$.

The function may be real-valued, $x[n, \phi] \in \mathbb{R}$, or complex-valued, $x[n, \phi] \in \mathbb{C}$.

Random processes (cont'd)

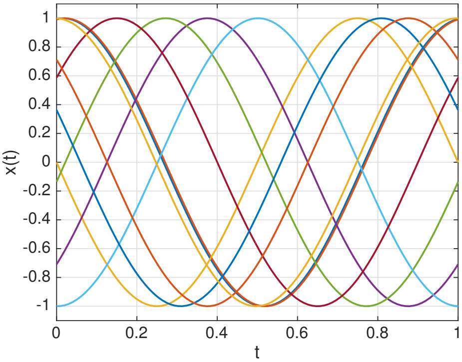


Random processes (cont'd)

Example 4: Harmonic process with random initial phase.

Let $x(t, \phi) = A\cos(\omega_0 t + \phi)$ be a continuous-time random process, where

- $A, \omega_0 \in \mathbb{R}$ are real-valed constants
- $\phi \equiv U(-\pi, \pi)$ is a random variable uniformly distributed in the interval $[-\pi, \pi[$.



Ten realizations of the sinusoidal r.p. with uniform initial phase, for $A = 1$ and $\omega_0 = 2\pi$. Each plot (color) represents $x(t, \phi)$ as a function of time index t for a particular realization of random variable ϕ.

Statistics

Let $x[n]$ be a discrete-time random process.
Its probability density function (pdf) at time instant n is noted $f_x(x; n)$.

Mean, variance, power

$$\mu_x[n] = \mathrm{E}\{x[n]\} = \int_{-\infty}^{+\infty} x\, f_x(x; n) dx$$

$$\sigma_x^2[n] = \mathrm{E}\{|x[n] - \mu_x[n]|^2\} = \int_{-\infty}^{+\infty} |x - \mu_x|^2 f_x(x; n) dx$$

$$P_x[n] = \mathrm{E}\{|x[n]|^2\} = \int_{-\infty}^{+\infty} |x|^2 f_x(x; n) dx = |\mu_x[n]|^2 + \sigma_x^2[n]$$

Autocorrelation function

$$r_x[n_1, n_2] = \mathrm{E}\{x[n_1]x^*[n_2]\} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1 x_2^* f_x(x_1, x_2; n_1, n_2) dx_1 dx_2$$

Through the change of variable $n_1 = n$ and $n_2 = n - k$, we can also write

$$r_x[n, n-k] = \mathrm{E}\{x[n]x^*[n-k]\}$$

$\rightarrow$ *Exercise 13:* Compute the mean, variance, power, and autocorrelation function of the harmonic process of [Example 4].

Stationarity

A r.p. is said to be **stationary** when its statistical properties are invariant to a shift in time origin.

Strict sense stationary random process (up to order p)

Random process $x[n]$ is strict sense stationary if and only if (iff) for any time shift k

$$f_x(x_1, \ldots, x_p; n_1, \ldots, n_p) = f_x(x_1, \ldots, x_p; n_1 + k, \ldots, n_p + k)$$

By limiting stationarity to orders $p = 1$ and $p = 2$, we obtain a

Wide sense stationary random process (WSS)

Random process $x[n]$ is wide sense stationary if:

$$\mu_x[n] = \mu_x = \text{constant}$$

and

$$r_x[n, n-k] = r_x[k] \qquad \text{i.e., it only depends on time lag } k.$$

→ *Exercise 14:* Prove that the variance and power of a WSS process are independent of n.

→ *Exercise 15:* Show that the harmonic process of [Example 4] is WSS.

Power spectral density

Let $x[n]$ be a **discrete-time wide sense stationary** random process (random sequence).

Let $r_x[k]$ denote its **autocorrelation sequence**.

Wiener-Khinchin Theorem (discrete time)

$$S_x(\omega) = \mathcal{F}\{r_x[k]\} = \sum_{k=-\infty}^{+\infty} r_x[k] e^{-j\omega k} \tag{6}$$

$$r_x[k] = \mathcal{F}^{-1}\{S_x(\omega)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_x(\omega) e^{j\omega k} d\omega. \tag{7}$$

- According to eqn. (7), the power of $x[n]$ (assuming a finite energy signal) is given by

$$P_x = r_x[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_x(\omega) d\omega.$$

  (stochastic equivalent of Parseval's theorem (4))

- $S_x(\omega)d\omega$: infinitesimal signal power in the frequency band $\left[\omega - \frac{d\omega}{2}, \omega + \frac{d\omega}{2}\right]$ (rad/sample).
- Hence, $S_x(\omega)$ is the **power spectral density (PSD)** of $x[n]$.

Power spectral density (cont'd)

Spectral estimation problem

From a finite length record of a WSS random sequence $x[n]$

$$\{x[n]\}_{n=0}^{N-1}$$

determine an estimate $\hat{S}_x(\omega)$ of its power spectral density $S_x(\omega)$.

We are looking for *good* estimates in terms of different **performance criteria**:

- Frequency resolution, bias, variance, computational cost, ...

Two main **approaches**: non-parametric and parametric.

We shall focus on popular non-parametric estimators:

- periodogram
- improved periodogram-based methods:
  - Bartlett's averaged periodogram
  - Welch's averaged modified periodogram

Periodogram

Given a length-N realization of a WSS process $x[n]$, its **periodogram** spectral estimator is defined as

$$\hat{S}_{\mathrm{P}}(\omega) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n]e^{-j\omega n}\right|^2 = \frac{1}{N}|X_N(\omega)|^2 \qquad (8)$$

where $X_N(\omega) = \mathcal{F}\{x_N[n]\}$ is the DTFT of the zero-padded data sequence obtained from the N-point observation:

$$x_N[n] \stackrel{\text{def}}{=} w_N[n]x[n] = \begin{cases} x[n] & 0 \leq n \leq (N-1) \\ 0 & \text{otherwise} \end{cases}$$

and $w_N[n]$ is the length-N rectangular window defined in [Exercise 4].

Spectral estimators are random processes

Each N-sample realization of $x[n]$ will provide a (generally) different spectral estimate.

Hence, $\hat{S}_{\mathrm{P}}(\omega)$ and in general any spectral estimator can be considered as random processes that are functions of frequency ω.

Consequence: spectral estimation quality can be measured by means of performance indices (statistics) such as bias, variance and MSE, recalled next.

Periodogram (cont'd)

→ *Exercise 16:*

1. Generate a signal $x[n]$ composed of two sinusoids with the same amplitude, having frequencies $F_1 = 300$ Hz and $F_2 = 400$ Hz and initial phase randomly drawn from a uniform distribution $U(-\pi, \pi)$, sampled at $F_s = 2$ kHz for $T = 2$ s. Listen to the signal (`sound`).

2. Generate a noisy signal
$$y[n] = x[n] + b[n]$$
by adding Gaussian noise $b[n]$ with a signal-to-noise ratio (SNR) = 10 dB, where
$$\text{SNR} = \frac{P_x}{P_b}$$
P_x: signal power; P_b: noise power. Listen to the noisy signal.

3. Compute the periodogram of $y[n]$ using the nearest power of 2 for the FFT size. Plot the periodogram in the interval $[0, \pi]$ rad/sample. Denormalize the horizontal axis to represent frequency in Hz. Does the spectral analysis reflect the signal content?

4. Repeat the above steps a few times for different random realizations of the sinusoids and the noise. Superimpose each new periodogram to the first plot. What can you observe?

Estimation performance analysis

Let $\hat{\theta}$ be an estimator of an unknown quantity $\theta \in \mathbb{R}$.

Estimation performance can be characterized by several statistical indices:

Bias

$$\text{bias}\{\hat{\theta}\} \stackrel{\text{def}}{=} \mathsf{E}\{\hat{\theta}\} - \theta = \mu_{\hat{\theta}} - \theta.$$

How close the average estimate $\mu_{\hat{\theta}}$ is from the actual value θ.

Variance

$$\text{var}\{\hat{\theta}\} = \sigma^2_{\hat{\theta}} \stackrel{\text{def}}{=} \mathsf{E}\{(\hat{\theta} - \mu_{\hat{\theta}})^2\} = \mathsf{E}\{\hat{\theta}^2\} - \mu^2_{\hat{\theta}}.$$

How spread the estimates $\hat{\theta}$ lie around their mean value $\mu_{\hat{\theta}}$.

Mean square error (MSE)

$$\text{MSE}\{\hat{\theta}\} \stackrel{\text{def}}{=} \mathsf{E}\{(\hat{\theta} - \theta)^2\} = (\mathsf{E}\{\hat{\theta}\} - \theta)^2 + \mathsf{E}\{(\hat{\theta} - \mathsf{E}\{\hat{\theta}\})^2\} = \text{bias}^2\{\hat{\theta}\} + \text{var}\{\hat{\theta}\}.$$

Average square error between estimates $\hat{\theta}$ and actual value θ.

Extension to complex-valued case is straightforward by replacing $|\cdot|^2$ for $(\cdot)^2$.

Estimation performance analysis (cont'd)

Let $\hat{\theta}_N$ be an estimator of a quantity θ computed from a set of N samples of available data.

Several **statistical properties** of the estimator can be defined:

Unbiasedness

The estimator is said to be **unbiased** if $\mathrm{E}\{\hat{\theta}_N\} = \theta$, i.e., if $\mathrm{bias}\{\hat{\theta}_N\} = 0$.

The estimator is **asymptotically unbiased** if $\lim_{N\to\infty} \mathrm{E}\{\hat{\theta}_N\} = \theta$, i.e., if $\lim_{N\to\infty} \mathrm{bias}\{\hat{\theta}_N\} = 0$.

Consistency

The estimator is said to be **consistent** if $\lim_{N\to\infty} \mathrm{var}\{\hat{\theta}_N\} = 0$.

The estimator is **MSE consistent** if $\lim_{N\to\infty} \mathrm{MSE}\{\hat{\theta}_N\} = 0$.

MSE consistency implies variance consistency and asymptotic unbiasedness.

Performance analysis — bias of the periodogram

Periodogram mean

The **mean of the periodogram** is the convolution of the actual PSD $S(\omega)$ and the Bartlett kernel:

$$\mathrm{E}\{\hat{S}_{\mathrm{P}}(\omega)\} = S(\omega) * W_{\mathrm{B}}(\omega) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S(\xi) W_{\mathrm{B}}(\omega - \xi) d\xi.$$

$$W_{\mathrm{B}}(\omega) = \mathcal{F}\{w_{\mathrm{B}}[n]\} = \frac{1}{N}\left[\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right]^2$$

$$w_{\mathrm{B}}[k] = \begin{cases} 1 - \frac{|k|}{N} & 0 \le |k| \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

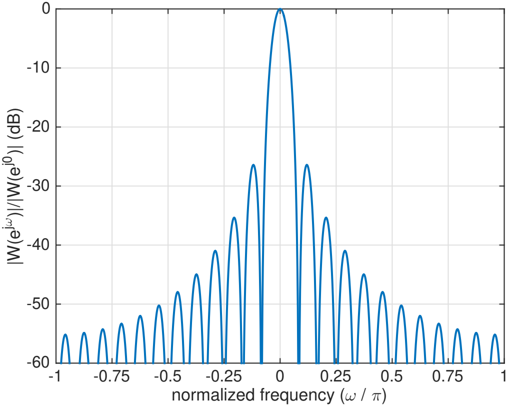
Since $\mathrm{E}\{\hat{S}_{\mathrm{P}}(\omega)\} \neq S(\omega)$, **the periodogram is a biased estimator of the PSD**

Remark:
Sequence $w_{\mathrm{B}}[n]$ is the $(2N-1)$-point **Bartlett** or **triangular window**. It can be expressed as:

$$w_{\mathrm{B}}[n] = \frac{1}{N} w_N[n] * w_N[-n] \quad \Rightarrow \quad W_{\mathrm{B}}(\omega) = \frac{1}{N}|W_N(\omega)|^2$$

where $w_N[n]$ is the N-point rectangular window and $W_N(\omega)$ its DTFT, defined in [Exercise 4].

Performance analysis — bias of the periodogram (cont'd)



Normalized DTFT of the Bartlett window, $|W_\mathrm{B}(e^{j\omega})|/|W_\mathrm{B}(e^{j0})|$, for $N = 25$ samples (window length: $2N - 1 = 49$).

Performance analysis — bias of the periodogram (cont'd)

Asymptotically, as observation length N tends to infinity:

$$\lim_{N\to\infty} w_B[n] = 1,\ \forall n \quad \Rightarrow \quad \lim_{N\to\infty} W_\mathrm{B}(\omega) = \delta(\omega) \quad \Rightarrow \quad \lim_{N\to\infty} \mathrm{E}\{\hat{S}_\mathrm{P}(\omega)\} = S(\omega).$$

The periodogram is an asymptotically unbiased estimator of the PSD

In practice, for finite sample size N, **two side effects**:

Spectral smearing or smoothing: caused by main lobe of $W_\mathrm{B}(\omega)$

The half-power bandwidth of $W_\mathrm{B}(\omega)$ can be shown to be approximately $f_{3\mathrm{dB}} \simeq 1/N$.

This is the **fundamental resolution of the periodogram**: $\hat{S}_\mathrm{P}(\omega)$ cannot resolve frequency components spaced by less than $\Delta f_\mathrm{min} \simeq 1/N$ sample^{-1} (or $\Delta\omega_\mathrm{min} \simeq 2\pi/N$ rad/sample).

Power leakage: caused by sidelobes of $W_\mathrm{B}(\omega)$

A frequency component present at f_0 will leak at frequencies $f_0 \pm (2p+1)/(2N)$, $p \in \mathbb{Z}^+$.

Performance analysis — variance of the periodogram

Periodogram variance

For N sufficiently large, we have:

$$\mathrm{var}\{\hat{S}_{\mathrm{P}}(\omega)\} \approx S^2(\omega).$$

Hence, **the periodogram is an inconsistent estimator of the PSD**.

Consequences:

- Variance cannot be improved by increasing the observation length N.
- Inconsistency — as spectral smearing — has an adverse effect on resolvability properties.

These results will be illustrated in the computer lab and highlight the need for better estimators.

Performance analysis (cont'd)

→ *Exercise 17:*

1. Generate 100 random realizations of the noisy sum of sinusoids considered in [Exercise 16], and compute the corresponding periodogram for each realization.

2. By averaging the spectral realizations, compute and plot the mean and variance of the periodogram at each frequency value.

3. Repeat the above experiment by doubling the observation length. What changes do you observe in terms of resolution and variance?

Periodogram averaging

The sample mean: averaging reduces variance (Law of Large Numbers)

Let θ be a random variable with mean μ_θ and variance σ_θ^2. Given K uncorrelated observations $\{\theta_k\}_{k=1}^K$, the **sample mean** is defined as

$$\hat{\mu}_\theta = \frac{1}{K}\sum_{k=1}^{K}\theta_k.$$

Its **mean** and **variance** are given by:

$$\mathrm{E}\{\hat{\mu}_\theta\} = \frac{1}{K}\sum_{k=1}^{K}\mathrm{E}\{\theta_k\} = \mu_\theta$$

$$\mathrm{var}\{\hat{\mu}_\theta\} = \frac{1}{K^2}\sum_{k=1}^{K}\mathrm{var}\{\theta_k\} = \frac{1}{K}\sigma_\theta^2$$

$\rightarrow$ the sample mean is an **unbiased, consistent estimator** of the ensemble mean.

Idea to reduce variance of spectral estimators:

- compute the PSD estimates of several data segments
- compute the sample mean of the PSD estimates.

Bartlett method: periodogram averaging

Bartlett method

1. **Divide** the observed data sequence in K segments of L samples each:

$$x_j[n] = x[(j-1)L + n], \quad j = 1, 2, \ldots, K, \quad n = 0, 1, \ldots, L-1, \quad \text{with } K \stackrel{\text{def}}{=} \lfloor N/L \rfloor.$$

2. Compute the **periodogram** of each segment:

$$\hat{S}_j(\omega) = \frac{1}{L} \left| \sum_{n=0}^{L-1} x_j[n] e^{-j\omega n} \right|^2.$$

3. **Average** periodograms to produce the Bartlett spectral estimate:

$$\hat{S}_{\mathrm{B}}(\omega) = \frac{1}{K} \sum_{j=1}^{K} \hat{S}_j(\omega).$$

→ *Exercise 18:* For the noisy sum of sinusoids generated in [Exercise 16], compute Bartlett spectral estimator with $K = 4$ segments, and perform its performance analysis following the procedure of [Exercise 17]. Compare with the periodogram.

Bartlett method — properties

Bartlett's mean and variance

$$\mathrm{E}\{\hat{S}_\mathrm{B}(\omega)\} = \frac{1}{K}\sum_{j=1}^{K}\mathrm{E}\{\hat{S}_\mathrm{P}(\omega)\} = \mathrm{E}\{\hat{S}_\mathrm{P}(\omega)\} = S(\omega) * W_\mathrm{B}(\omega) \quad \Rightarrow \quad L\text{-sample periodogram}$$

$$\mathrm{var}\{\hat{S}_\mathrm{B}(\omega)\} \approx \frac{1}{K^2}\sum_{j=1}^{K}\mathrm{var}\{\hat{S}_\mathrm{P}(\omega)\} = \frac{1}{K}S^2(\omega) \quad \Rightarrow \quad \text{variance reduction by factor of } K$$

Bartlett method limitations:
The link between the number of segments K and the size of each segment L is given by:

$$K \cdot L \leq N.$$

- Increasing the number of segments K improves variance, but...
- ... worsens resolution, as L decreases in the same proportion.

$\rightarrow$ variance is decreased at the expense of increased bias.

Q: How can this tradeoff be relaxed?

A: Allow overlapping between consecutive segments.

Welch method: averaged, overlapped, windowed periodograms

Welch method

1. Choose **window size** L and **overlap factor** $\Delta \in [0, 1[$ between consecutive segments.
2. **Divide** the data in K segments of L samples each:

$$x_j[n] = x[(j-1)D + n], \quad j = 1, \ldots, K, \quad n = 0, \ldots, L-1$$

with $D = \text{round}\big(L(1-\Delta)\big)$ and $K = \lceil (N-L)/D \rceil + 1 \simeq N/D$ if $N \gg L$.
   - no overlap: $\Delta = 0 \quad \Rightarrow \quad D = L \quad \Rightarrow \quad K \simeq N/L$ data segments (as in Bartlett)
   - 50% overlap (typical): $\Delta = 0.5 \quad \Rightarrow \quad D = L/2 \quad \Rightarrow \quad K \simeq 2N/L$ data segments
3. Choose a suitable **window** sequence $w(n)$ of length L.
4. For each segment, compute the **periodogram** of the weighted segment:

$$\hat{S}_j(\omega) = \frac{1}{LU} \left| \sum_{n=0}^{L-1} w[n] x_j[n] e^{-j\omega n} \right|^2$$

where $U \stackrel{\text{def}}{=} \frac{1}{L} \sum_{n=0}^{L-1} w[n]^2$ is the power of window $w[n]$.
5. **Average** the computed periodograms to produce the Welch PSD estimate:

$$\hat{S}_{\text{W}}(\omega) = \frac{1}{K} \sum_{j=1}^{K} \hat{S}_j(\omega)$$

Welch method — properties

Welch's mean and variance

$$\mathrm{E}\{\hat{S}_{\mathrm{W}}(\omega)\} = S(\omega) * \frac{1}{LU}|W(\omega)|^2$$

$$\mathrm{var}\{\hat{S}_{\mathrm{W}}(\omega)\} \approx \frac{1}{K}\mathrm{var}\{\hat{S}_j(\omega)\}$$

- Decreasing window length L yields more segments and thus reduced variance
  - but if L becomes too small $\rightarrow$ loss of spectral resolution caused by broader main lobe of $W(\omega)$.
- **Overlap** (higher Δ) provides more segments and thus increased averaging and reduced variance, without decreasing window length L
  - but it requires sufficient decorrelation between consecutive segments.
- **Data window**
  - provides mainlobe-sidelobe tradeoff capability to tune compromise between smearing and leakage
  - puts less weight on data samples at the ends of segments $\rightarrow$ reduced crosscorrelation between segments $\rightarrow$ improved variance reduction.
- **Overlap for optimal variance reduction** $\approx 50\%$ of window length ($\Delta = 0.5$).

Welch method (cont'd)

→ *Exercise 19:* For the noisy sum of sinusoids generated and analyzed in [Exercise 16] to [Exercise 18], compute Welch spectral estimator with $L = 1000$ samples, $\Delta = 50\%$ and

1. rectangular window
2. Hamming window.

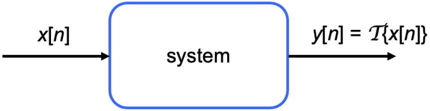
Compare with the periodogram and Bartlett estimators.

Lineal time-invariant (LTI) systems

LTI systems define a fundamental class of signal transformations including frequency filters.

Frequency filters are a key preprocessing step to suppress noise and interference from biosignals before further processing and analysis.

Let $y[n] = \mathcal{T}\{x[n]\}$ be the **output (or response)** of a system $\mathcal{T}\{\cdot\}$ excited by **input (or excitation)** $x[n]$.



The system is called **linear time-invariant (LTI)** if it fulfils:

- *Linearity*

$$\mathcal{T}\{a_1 x_1[n] + a_2 x_2[n]\} = a_1 \mathcal{T}\{x_1[n]\} + a_2 \mathcal{T}\{x_2[n]\}, \quad \text{for any } a_1, a_2 \in \mathbb{C}$$

- *Time invariance*

$$\mathcal{T}\{x[n - n_0]\} = y[n - n_0], \quad \text{for any } n_0 \in \mathbb{Z}$$

Lineal time-invariant (LTI) systems (cont'd)

$$y[n] = \mathcal{T}\{x[n]\} \underset{\substack{\uparrow \\ \text{eqn. (1)}}}{=} \mathcal{T}\{x[n] * \delta[n]\} = \mathcal{T}\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}$$

$$\underset{\substack{\uparrow \\ \text{linearity}}}{=} \sum_{k=-\infty}^{\infty} x[k]\mathcal{T}\{\delta[n-k]\} \underset{\substack{\uparrow \\ \text{time invariance}}}{=} \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

Output sequence of an LTI system

Output of an LTI system = convolution of the input $x[n]$ and the system's **impulse response** $h[n]$:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{9}$$

$h[n] \stackrel{\text{def}}{=} \mathcal{T}\{\delta[n]\}$: response (or output) of the system to the unit sample input $x[n] = \delta[n]$.

Frequency response of LTI systems

Transforming both sides of eqn. (9):

$$y[n] = h[n] * x[n] \quad \Rightarrow \quad Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \quad H(e^{j\omega}) \stackrel{\text{def}}{=} \mathcal{F}\{h[n]\}$$

Frequency response

$$H(e^{j\omega}) = \mathcal{F}\{h[n]\} = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\sphericalangle H(e^{j\omega})} \in \mathbb{C}$$

$|H(e^{j\omega})|$: **magnitude response** $\quad \sphericalangle H(e^{j\omega})$: **phase response**

The frequency response determines how the system responds to sinusoidal input signals — why?

Frequency response of LTI systems (cont'd)

The response to input $x[n] = e^{j\omega n}$ is:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)}$$

$$= e^{j\omega n} \underbrace{\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}}_{\mathcal{F}\{h[n]\}=H(e^{j\omega})} = H(e^{j\omega})e^{j\omega n}$$

- Complex exponentials $e^{j\omega n}$ are the *eigenfunctions* of LTI systems, with eigenvalues $H(e^{j\omega})$:

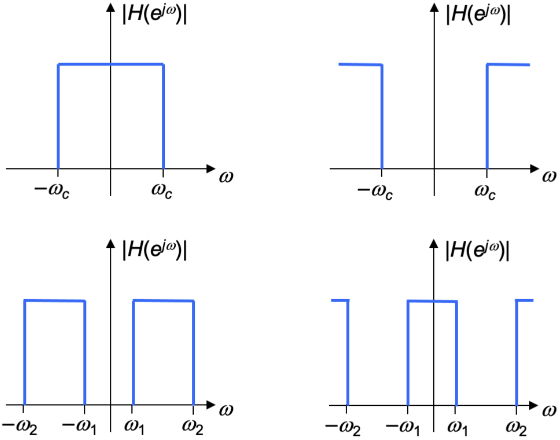
  The response to a complex exponential is another complex exponential with the same frequency, but possibly different amplitude and phase, as determined by the magnitude and phase response of the system.

LTI system response to a sinusoidal input

$$x[n] = A\cos(\omega n + \phi) \quad \Rightarrow \quad y[n] = A|H(e^{j\omega})|\cos\big(\omega n + \phi + \sphericalangle H(e^{j\omega})\big)$$

Frequency response of LTI systems (cont'd)

→ *Exercise 20:* Identify the type of frequency filter (lowpass, highpass, bandpass, bandstop) that can be associated with the following magnitude responses:



LTI systems defined by difference equations

Difference equations

A linear **difference equation** with constant coefficients can be expressed as:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{10}$$

Under zero initial conditions, i.e.,

$$x[n] = 0,\ n \leq n_0 \quad \Rightarrow \quad y[n] = 0,\ n \leq n_0$$

eqn. (10) defines an LTI system.

Two types of filters according to their impulse response length:

- *Finite impulse response (FIR)*:

$$a_k = 0,\ \forall k \geq 1 \quad \Rightarrow \quad y[n] = \sum_{k=0}^{M} b_k x[n-k] \quad \Rightarrow \quad h[k] = \begin{cases} b_k, & 0 \leq k \leq M \\ 0, & \text{otherwise} \end{cases}$$

- *Infinite impulse response (IIR)*:

$$\exists k \geq 1 : a_k \neq 0 \quad \Rightarrow \quad h[k] \text{ is of infinite length}$$

LTI systems defined by difference equations (cont'd)

→ *Exercise 21:* Consider the LTI system characterized by the difference equation:

$$y[n] - 0.5y[n-1] = x[n] \tag{11}$$

1. Determine the system's impulse response $h[n]$. Is it an FIR or an IIR filter?
2. Determine the system's frequency response $H(e^{j\omega})$ by two equivalent methods:
   - Computing the DTFT of $h[n]$.
   - Computing the DTFT on both sides of the difference equation.
3. Compute the system's output to the input signal $x[n] = \cos(\pi/2n)$.

→ *Exercise 22:* Using MATLAB:

1. Compute (`filter`) and plot (`stem`) the first 20 samples of the impulse response of system (11). Superimpose the theoretical values found in [Exercise 21] (question 1).
2. Compute and plot the first 20 samples of the response to input $x[n] = \cos(\pi/2n)$. Superimpose the theoretical values found in [Exercise 21] (question 3).

System properties

Causality

A system is **causal** if its output $y[n_0]$ only depends on $x[n]$, $n \le n_0$.

In terms of the impulse response:

$$h[n] = 0, \quad n < 0$$

Stability

A system is **stable** if a bounded input always yields a bounded output:

$$\text{If } |x[n]| \le B_x,\ B_x < +\infty,\ \forall n \quad \Rightarrow \quad |y[n]| \le B_y,\ B_y < +\infty,\ \forall n, \text{ with } y[n] = \mathcal{T}\{x[n]\}$$

In terms of the impulse response:

$$\sum_{n=-\infty}^{\infty} |h[n]| < +\infty$$

Corollaries:

- A non-causal system is not physically realizable, as it anticipates ('divines') the input.
- FIR filters are always stable.
- The frequency response of an FIR filter is always defined (cf. DTFT existence condition (2)).

Filter design

Goal: given a desired, ideal frequency response, find the LTI system fulfilling the response.

3 steps:

1. *Specify* the desired system properties
2. *Approximate* the specifications by a causal discrete system
3. *Realize* the system

IIR vs FIR filter design

Type	*Advantages*	*Drawbacks*
FIR	– always stable: $\sum_n \lvert h[n] \rvert < +\infty$	– high order
	– linear phase if symmetric $h[n]$	– computational cost
	– flexible design for generic approximations	– numerical design
IIR	– low order	– stability problems
	– low computational cost	– nonlinear phase
	– analytic design with analog tools (Butterworth, Chebyshev, ...)	– lack of flexibility

FIR filter design — window method

Idea: truncate the ideal impulse response over its most significant interval.

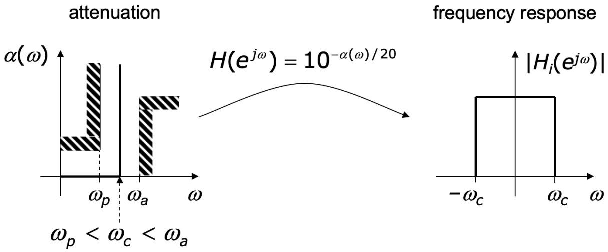
Steps

1. Specify the ideal frequency response
2. Determine the ideal impulse response
3. Select the truncation length
4. Delay the ideal impulse response
5. Truncate the delayed impulse response

FIR filter design — window method (cont'd)

1) Specify the ideal frequency response

Example 5: Lowpass filter specification



Hypothesis: ideal frequency response is zero-phase

$$\sphericalangle H_i(e^{j\omega}) = 0$$

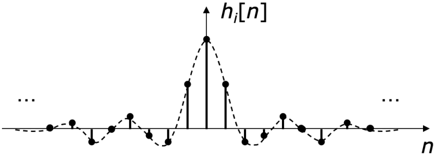
$$H_i(e^{j\omega}) = |H_i(e^{j\omega})| = \Pi\left(\frac{\omega}{2\omega_c}\right) \stackrel{\text{def}}{=} \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

FIR filter design — window method (cont'd)

2) Determine the ideal impulse response

$$h_i[n] = \mathcal{F}^{-1}\{H_i(e^{j\omega})\}$$

→ *Exercise 23:* Compute the impulse response of the frequency response of [Example 5].



Ideal impulse response

- Infinite duration: not FIR but IIR filter
- Non-causal:

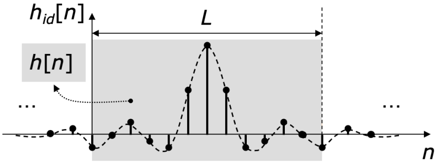
$$\exists n < 0, \quad h_i[n] \neq 0$$

FIR filter design — window method (cont'd)

3) Select the truncation length: odd value L

4) Delay the ideal impulse response

$$h_{id} = h_i[n - n_d], \quad n_d \stackrel{\text{def}}{=} (L-1)/2$$



Remark: Delay doesn't affect the magnitude response; it only introduces a linear phase term.

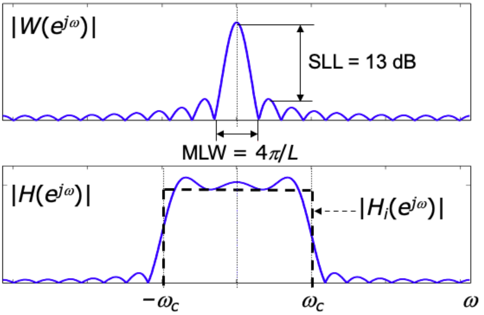
5) Truncate the delayed impulse response

$$h[n] = w[n]h_{id}[n]$$

for a suitable length-L window $w[n]$ (rectangular, Bartlett, Hamming, etc. — see [Table 1]).

FIR filter design — windowing effects

$$H(e^{j\omega}) = \mathcal{F}\{w[n]h_{id}[n]\} = W(e^{j\omega}) * H_{id}(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{j\xi})H_{id}(e^{j(\omega-\xi)})d\xi$$



Two effects caused by windowing:

- Main lobe smoothens transition band → loss in frequency selectivity.
- Sidelobes create ripples in passband and stopband.

FIR filter design — windowing effects (cont'd)

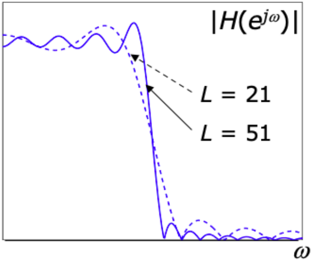
Increasing L

- decreases MLW

  $\rightarrow$ narrower transition band $\rightarrow$ improved selectivity

- increases number of sidelobes, but
  - keeps SLL and sidelobe power

  $\rightarrow$ more ripples, with constant amplitude



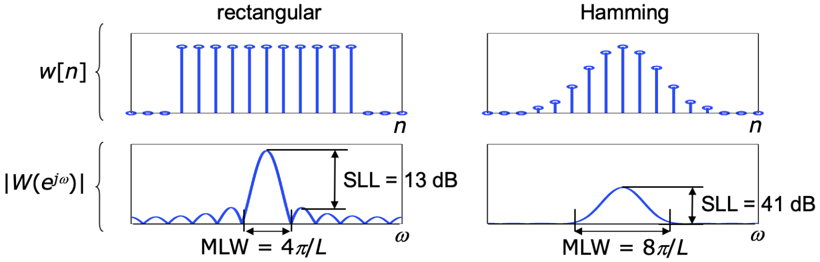
This is the so-called **Gibbs phenomenon**.

Ripples are caused by abrupt truncation of the ideal impulse response. They can be alleviated by using smoother windows.

FIR filter design — windowing effects (cont'd)

Using tapering windows $w[n]$

- increases SLL $\rightarrow$ decreases ripple amplitude
- decreases window effective length (for same L) $\rightarrow$ increases MLW $\rightarrow$ widens transition band



Trade-off: ripple amplitude $\longleftrightarrow$ transition bandwidth

IIR filter design

- Based on analog filter design

- **Steps**

  1. Define digital filter specifications
  2. Obtain analog filter specifications through a suitable transformation:
     - invariant impulse response
     - bilineal transformation
     - frequency transformation
  3. Apply analog filter approximation theory
  4. Inverse-transform to determine digital filter approximation

- **Analog filter approximations**

Filter type	*passband*	*stopband*
Butterworth	maximally flat	monotonically increasing
Cheyshev, type I	constant amplitude ripple	monotonically increasing
Chebyshev, type II	maximally flat	constant amplitude ripple
Cauer (elliptical)	constant amplitude ripple	constant amplitude ripple

Forward-backward zero-phase filtering

A nonlinear phase term $\sphericalangle H(e^{j\omega})$ may distort the output signal in IIR filters.

Zero-phase filtering

Let $h[n]$ be the impulse response of an LTI system.

Then, the system characterized by impulse response

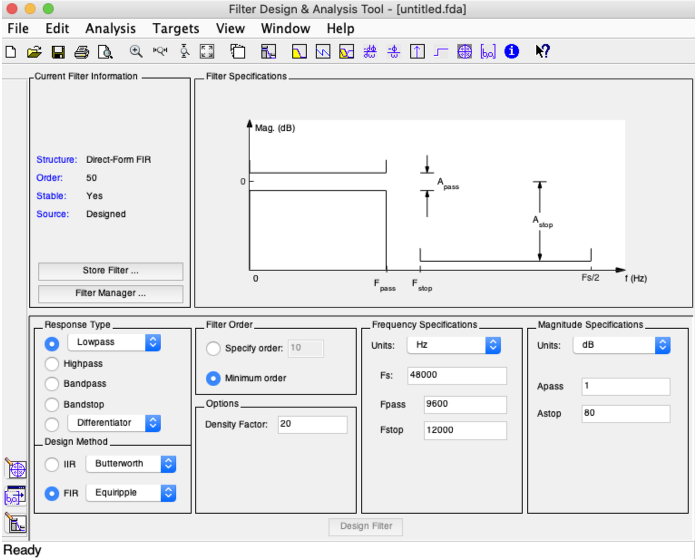
$$g[n] = h[n] * h[-n]$$

has a zero phase response.

$\rightarrow$ *Exercise 24:* Prove the above result.

Filter design in practice

MATLAB's filter design and analysis tool: `fdatool`.



Filter design in practice (cont'd)

→ *Exercise 25:* For the noisy sum of sinusoids generated in [Exercise 16]:

1. Using MATLAB's `fdatool`, design a Chebyshev type-II lowpass filter to attenuate the sinusoid with frequency F_2 and the high-frequency noise by at least 40 dB.
2. Plot the magnitude response, impulse response and step response of the designed filter.
3. Export the designed filter as an object in a MAT-file.
4. Load the filter and use it to process the noisy signal (`load`, `filter`).
5. Listen to the original and filtered signals. Can you notice any difference?
6. Using the spectral estimator of your choice, compute, plot and compare the magnitude spectrum of the original and filtered signals.
7. With the aid of the the data cursors, verify whether the filter fulfils the design requirements.
8. Use the same filter to perform zero-phase filtering (`filtfilt`), and verify its effects on the power spectrum.

Summary

- Digital signal processing fundamentals
  - time sampling, basic sequences, FT, PSD, windowing, zero-padding, LTI systems, impulse response, frequency response, difference equations, causality, stability, FIR and IIR filters
- Signals can be represented in the frequency domain by Fourier transforms and variants
  - deterministic signals: DTFT, DFT, FFT
  - random processes: periodogram, averaged periodogram (Bartlett, Welch)
- Finite observation length impacts spectral resolution
  - spectral smearing, power leakage
- Tapering windows alleviate power leakage at the expense of spectral smearing
- Digital filter design
  - advantages and limitations of FIR and IIR filters
  - window method for FIR filter synthesis