Lecture 3:
Convex optimization
(Part 2)

Optimization for data sciences

Rémy Sun
remy.sun@inria.fr

Course organization

What can we optimize?

- Reduce the complexity/overhead of a problem
  - E.g. Network quantization
  - E.g. Computational optimization

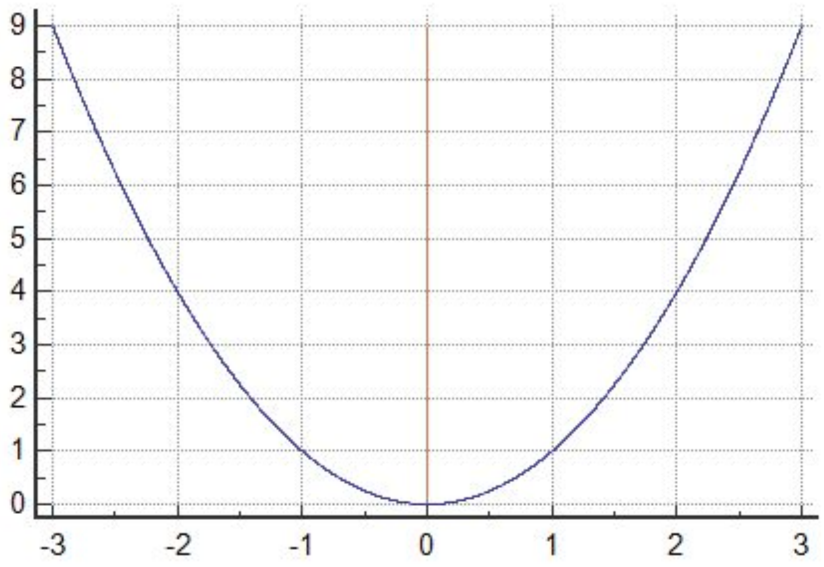
- **Find the best solution to a problem**
  - **Numerical optimization**
  - **Evaluate solutions according to a criterion**
  - **Look at solutions from some given space of possible solutions to consider**

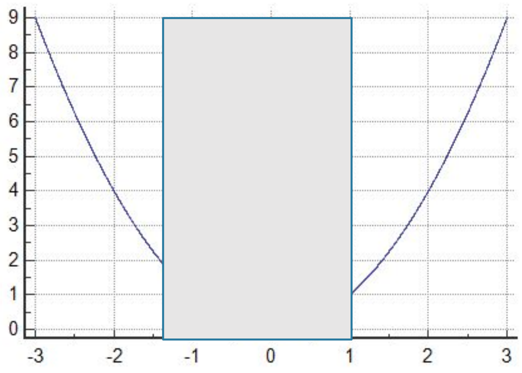
Defining an optimization problem

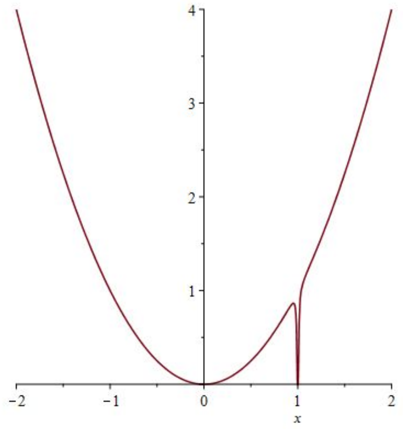
- Minimize a quantity $f_0(x)$
  - Under inequality and equality constraints
  - Constraints define a domain D
  - Could have no constraint except x in D

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& g_i(x) = 0, \quad i = 1, \ldots, p
\end{array}
$$

Can you formalize these problems?







Course organization

UNIVERSITÉ CÔTE D'AZUR

- Introduction to optimization
  - A few problems of interest
  - Quick mathematical refresher

- **Convex problems (Following Stephen Boyd)**
  - **Quick refresher on last week**
  - **Convex sets**
  - **Convex functions**
  - **Convex problems**
  - **Simplex algorithm for Linear Programming**

Course organization

UNIVERSITÉ CÔTE D'AZUR

- Duality (for convex problems)
- Newton's Descent and Barrier methods for convex case
- (First order) descent methods for the general case
- Backpropagation
- Some more properties on stochastic gradient descent

Evaluation

- Reports on lab sessions
  - Labs on jupyter notebooks
    - Not every session
  - Explain the code done in the session
  - Summarize what is done in the practical

- Written Exam
  - Theoretical questions
  - We will do exercises in class

Refresher on last week

Convex problem

convex optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

- variable $x \in \mathbf{R}^n$
- equality constraints are linear
- $f_0, \dots, f_m$ are **convex**: for $\theta \in [0, 1]$,

$$f_i(\theta x + (1 - \theta)y) \leq \theta f_i(x) + (1 - \theta) f_i(y)$$

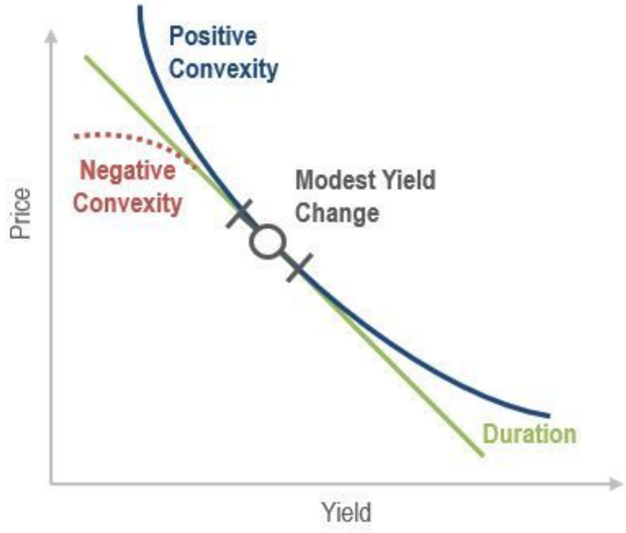
i.e., f_i have nonnegative (upward) curvature

Easy problem

- classical view:
  - linear (zero curvature) is easy
  - nonlinear (nonzero curvature) is hard

- the classical view is **wrong**

- the correct view:
  - convex (nonnegative curvature) is easy
  - nonconvex (negative curvature) is hard



Easy to solve!

- many different algorithms (that run on many platforms)
  - interior-point methods for up to 10000s of variables
  - first-order methods for larger problems
  - do not require initial point, babysitting, or tuning
- can develop and deploy quickly using modeling languages such as CVXPY
- solvers are reliable, so can be embedded
- code generation yields real-time solvers that execute in milliseconds (*e.g.*, on Falcon 9 and Heavy for landing)

Convex sets overview

- Convex sets
  - Affine sets, norm balls, norm cones
  - Convex combination, convex hull and Convex cones
  - Hyperplanes, halfspaces and polyhedron
  - Positive Semidefinite Cone
- Convexity preserving operations
  - Intersection
  - Affine mapping
  - Perspective and Linear fractional mappings
- Proper cones and generalized inequalities
- Separating and supporting hyperplanes

Convex sets overview

- **Convex sets (definition!)**
  - **Affine sets, norm balls, norm cones**
  - **Convex combination, convex hull and Convex cones**
  - **Hyperplanes, halfspaces and polyhedron**
  - Positive Semidefinite Cone
- **Showing a set is convex (with operations!)**
  - **Intersection**
  - *Affine mapping*
  - *Perspective and Linear fractional mappings*
- Proper cones and generalized inequalities
- Separating and supporting hyperplanes

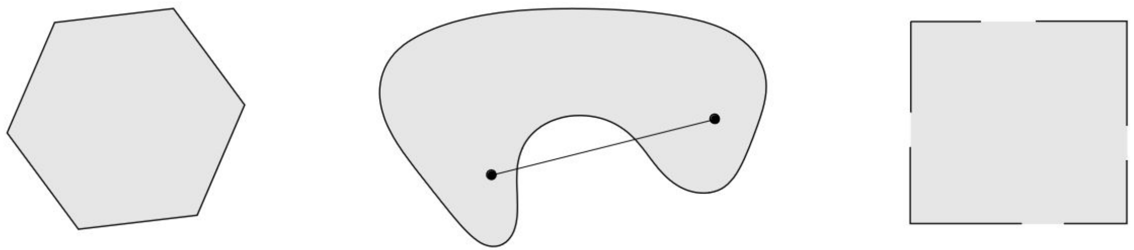
Convex sets

line segment between x_1 and x_2: all points of form $x = \theta x_1 + (1-\theta)x_2$, with $0 \le \theta \le 1$

convex set: contains line segment between any two points in the set

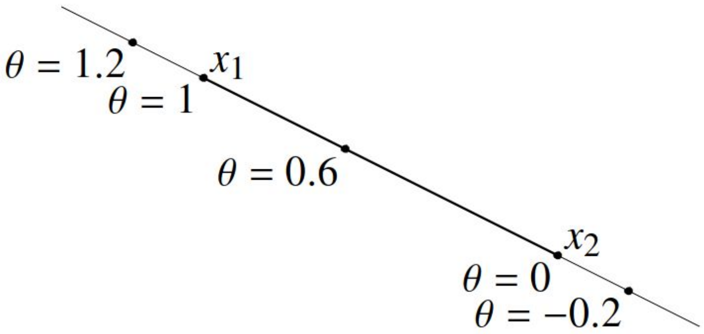
$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

examples (one convex, two nonconvex sets)



Affine sets

line through x_1, x_2: all points of form $x = \theta x_1 + (1 - \theta)x_2$, with $\theta \in \mathbf{R}$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$
(conversely, every affine set can be expressed as solution set of system of linear equations)

Balls

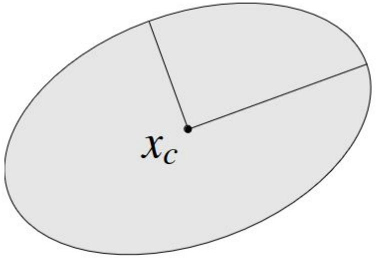
(Euclidean) ball with center x_c and radius r:

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

ellipsoid: set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)



another representation: $\{x_c + Au \mid \|u\|_2 \leq 1\}$ with A square and nonsingular

Norm balls and cones

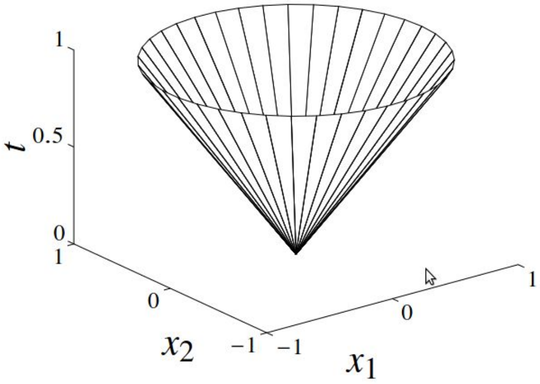
Norm balls and norm cones

- **norm:** a function $\|\cdot\|$ that satisfies
  - $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
  - $\|tx\| = |t|\,\|x\|$ for $t \in \mathbf{R}$
  - $\|x + y\| \leq \|x\| + \|y\|$
- notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm
- **norm ball** with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$
- **norm cone:** $\{(x, t) \mid \|x\| \leq t\}$
- norm balls and cones are convex

Euclidean norm cone

$$\{(x, t) \mid \|x\|_2 \leq t\} \subset \mathbf{R}^{n+1}$$

is called **second-order cone**



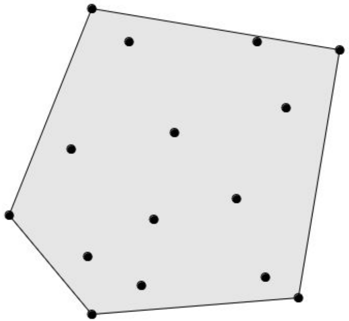
Convex combination and hull

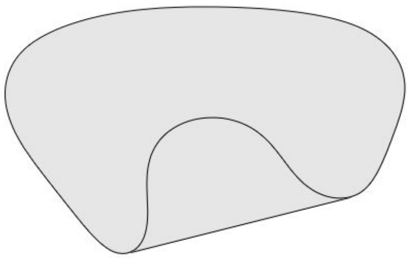
convex combination of $x_1, \ldots, x_k$: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

convex hull $\mathbf{conv}\, S$: set of all convex combinations of points in S



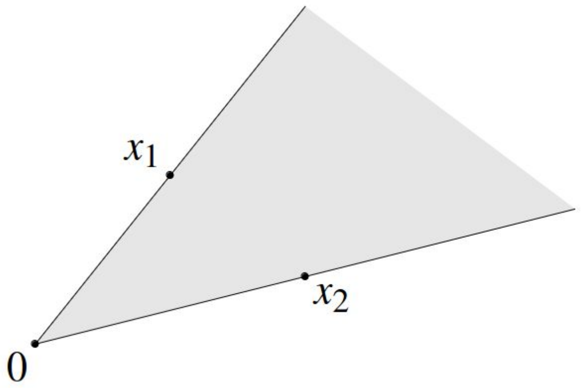


Convex cone

conic (nonnegative) combination of x_1 and x_2: any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

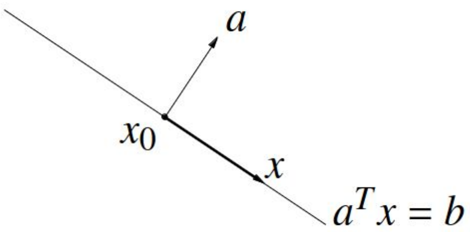
with $\theta_1 \geq 0$, $\theta_2 \geq 0$



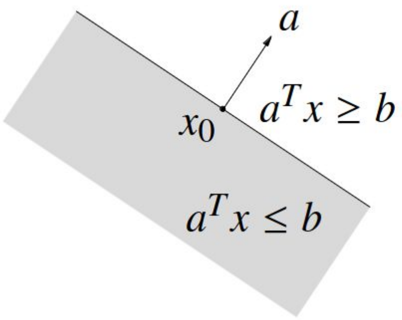
convex cone: set that contains all conic combinations of points in the set

Hyperplanes and halfspaces

hyperplane: set of the form $\{x \mid a^T x = b\}$, with $a \neq 0$



halfspace: set of the form $\{x \mid a^T x \leq b\}$, with $a \neq 0$



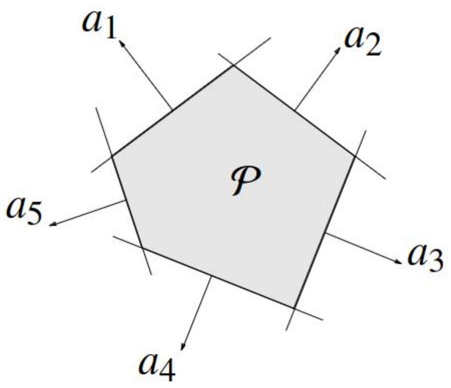
- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex

Polyhedron

- **polyhedron** is solution set of finitely many linear inequalities and equalities

$$\{x \mid Ax \preceq b,\ Cx = d\}$$

  ($A \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{p\times n}$, $\preceq$ is componentwise inequality)
- intersection of finite number of halfspaces and hyperplanes
- example with no equality constraints; a_i^T are rows of A



Showing a set is convex

methods for establishing convexity of a set C

1. apply definition: show $x_1, x_2 \in C, 0 \leq \theta \leq 1 \implies \theta x_1 + (1-\theta)x_2 \in C$
   - recommended only for **very simple** sets

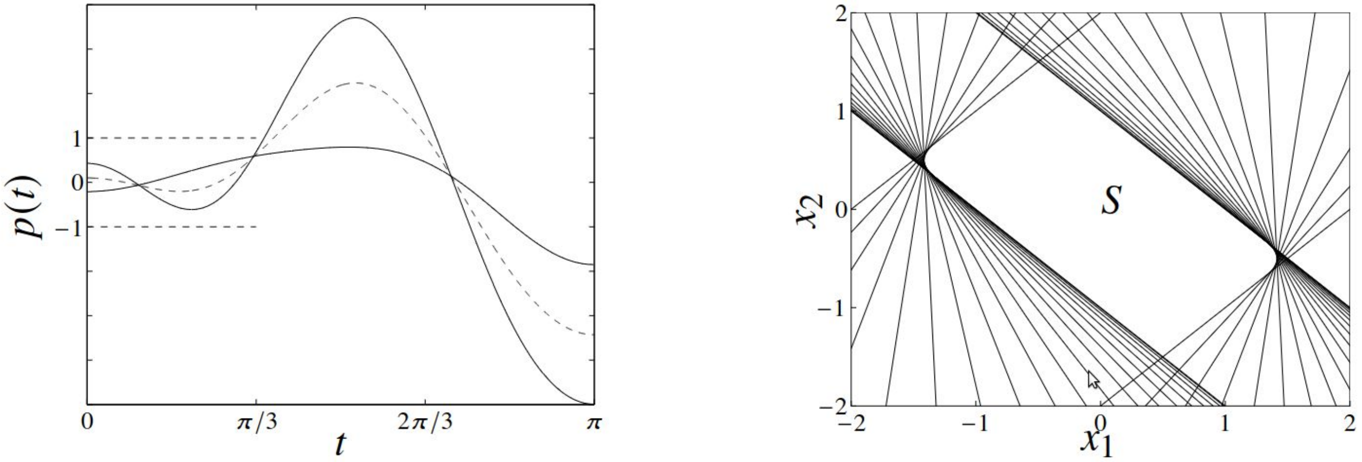
2. use convex functions (next lecture)

3. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity
   - intersection
   - affine mapping
   - perspective mapping
   - linear-fractional mapping

you'll mostly use methods 2 and 3

Showing a set is convex

- the intersection of (any number of) convex sets is convex
- **example:**
  - $S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$, with $p(t) = x_1 \cos t + \cdots + x_m \cos mt$
  - write $S = \bigcap_{|t| \leq \pi/3} \{x \mid |p(t)| \leq 1\}$, *i.e.*, an intersection of (convex) slabs
- picture for $m = 2$:



Showing a set is convex

- suppose $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine, *i.e.*, $f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$

- the **image** of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the **inverse image** $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

Showing a set is convex

- **perspective function** $P : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$:

$$P(x, t) = x/t, \qquad \mathbf{dom}\, P = \{(x, t) \mid t > 0\}$$

- images and inverse images of convex sets under perspective are convex

- **linear-fractional function** $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$:

$$f(x) = \frac{Ax + b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

- images and inverse images of convex sets under linear-fractional functions are convex

Takeaway

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta) x_2 \in C$$

- Classic convex sets
  - Affine sets, hyperplanes, cones, balls, polyhedrons

- Convexity preserving operations
  - Intersection
  - Affine mapping
  - Perspective
  - Linear Fractional mapping

Convex function overview

UNIVERSITÉ CÔTE D'AZUR

- Convex functions (with definition!)
  - Examples of classic convex functions
  - Extended value function
  - Line restriction
  - First and second order conditions
  - Epigraph and sublevel sets
- Showing a function is convex with operations
  - Non-negative weighted sum and affine composition
  - Pointwise maximum
  - Composition rules
  - Partial minimization and perspective
- Conjugate function

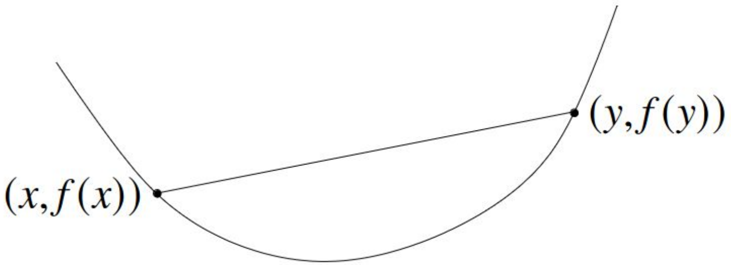
Convex function overview

- **Convex functions (with definition!)**
  - **Examples of classic convex functions**
  - *Extended value function*
  - *Line restriction*
  - **First and second order conditions**
  - *Epigraph and sublevel sets*
- **Showing a function is convex with operations**
  - **Non-negative weighted sum and affine composition**
  - **Pointwise maximum**
  - **Composition rules**
  - *Partial minimization and perspective*
- *Conjugate function*

Convex functions

- $f : \mathbf{R}^n \to \mathbf{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and for all $x, y \in \mathbf{dom}\, f$, $0 \le \theta \le 1$,

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$$



- f is concave if $-f$ is convex
- f is strictly convex if $\mathbf{dom}\, f$ is convex and for $x, y \in \mathbf{dom}\, f$, $x \neq y$, $0 < \theta < 1$,

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$$

Convex functions (Examples)

convex functions:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- exponential: e^{ax}, for any $a \in \mathbf{R}$
- powers: x^α on $\mathbf{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbf{R}$, for $p \geq 1$
- positive part (relu): $\max\{0, x\}$

concave functions:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- powers: x^α on $\mathbf{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$
- entropy: $-x \log x$ on $\mathbf{R}_{++}$
- negative part: $\min\{0, x\}$

Extended value

- suppose f is convex on $\mathbf{R}^n$, with domain $\mathbf{dom}\, f$
- its extended-value extension $\tilde{f}$ is function $\tilde{f} : \mathbf{R}^n \to \mathbf{R} \cup \{\infty\}$

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbf{dom}\, f \\ \infty & x \notin \mathbf{dom}\, f \end{cases}$$

- often simplifies notation; for example, the condition

$$0 \le \theta \le 1 \quad \Longrightarrow \quad \tilde{f}(\theta x + (1-\theta)y) \le \theta \tilde{f}(x) + (1-\theta)\tilde{f}(y)$$

(as an inequality in $\mathbf{R} \cup \{\infty\}$), means the same as the two conditions

  - $\mathbf{dom}\, f$ is convex
  - $x, y \in \mathbf{dom}\, f,\ 0 \le \theta \le 1 \quad \Longrightarrow \quad f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$

Line restriction

- $f : \mathbf{R}^n \to \mathbf{R}$ is convex if and only if the function $g : \mathbf{R} \to \mathbf{R}$,

$$g(t) = f(x + tv), \qquad \mathbf{dom}\, g = \{t \mid x + tv \in \mathbf{dom}\, f\}$$

  is convex (in t) for any $x \in \mathbf{dom}\, f$, $v \in \mathbf{R}^n$

- can check convexity of f by checking convexity of functions of one variable

First order condition

- f is **differentiable** if $\mathbf{dom}\, f$ is open and the gradient

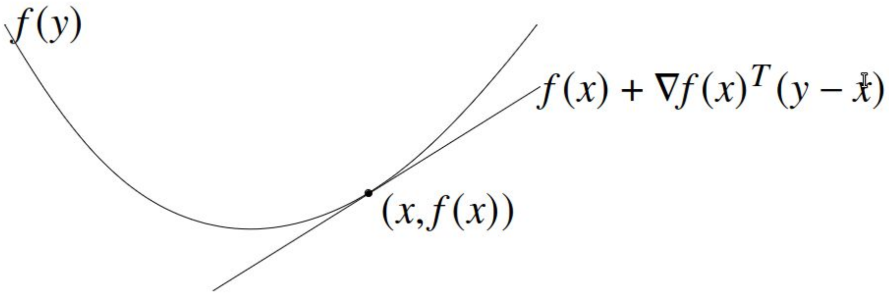
$$\nabla f(x) = \left(\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \ldots, \frac{\partial f(x)}{\partial x_n} \right) \in \mathbf{R}^n$$

  exists at each $x \in \mathbf{dom}\, f$

- **1st-order condition:** differentiable f with convex domain is convex if and only if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \mathbf{dom}\, f$$

- first order Taylor approximation of convex f is a **global underestimator** of f



Second order condition

UNIVERSITÉ CÔTE D'AZUR

- f is **twice differentiable** if $\mathbf{dom}\, f$ is open and the Hessian $\nabla^2 f(x) \in \mathbf{S}^n$,

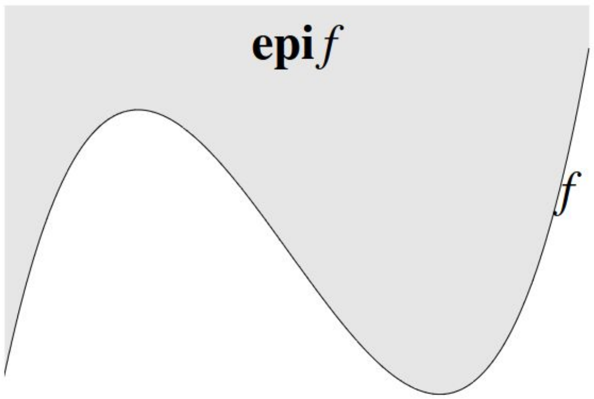
$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, n,$$

  exists at each $x \in \mathbf{dom}\, f$

- **2nd-order conditions:** for twice differentiable f with convex domain
  - f is convex if and only if $\nabla^2 f(x) \succeq 0$ for all $x \in \mathbf{dom}\, f$
  - if $\nabla^2 f(x) \succ 0$ for all $x \in \mathbf{dom}\, f$, then f is strictly convex

Epigraph and sublevel set

- α**-sublevel set** of $f : \mathbf{R}^n \to \mathbf{R}$ is $C_\alpha = \{x \in \mathbf{dom}\, f \mid f(x) \leq \alpha\}$
- sublevel sets of convex functions are convex sets (but converse is false)
- **epigraph** of $f : \mathbf{R}^n \to \mathbf{R}$ is $\mathbf{epi}\, f = \{(x, t) \in \mathbf{R}^{n+1} \mid x \in \mathbf{dom}\, f,\ f(x) \leq t\}$



- f is convex if and only if $\mathbf{epi}\, f$ is a convex set

Showing a function is convex

methods for establishing convexity of a function f

1. verify definition (often simplified by restricting to a line)
2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$
   - recommended only for **very simple** functions
3. show that f is obtained from simple convex functions by operations that preserve convexity
   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum and supremum
   - composition
   - minimization
   - perspective

you'll mostly use methods 2 and 3

Showing a function is convex

- **nonnegative multiple:** αf is convex if f is convex, $\alpha \geq 0$
- **sum:** $f_1 + f_2$ convex if f_1, f_2 convex
- **infinite sum:** if $f_1, f_2, \ldots$ are convex functions, infinite sum $\sum_{i=1}^{\infty} f_i$ is convex
- **integral:** if $f(x, \alpha)$ is convex in x for each $\alpha \in \mathcal{A}$, then $\int_{\alpha \in \mathcal{A}} f(x, \alpha)\, d\alpha$ is convex
- there are analogous rules for concave functions

Showing a function is convex

UNIVERSITÉ CÔTE D'AZUR

(pre-)composition with affine function: $f(Ax+b)$ is convex if f is convex

examples

- log barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \qquad \mathbf{dom}\, f = \{x \mid a_i^T x < b_i, i = 1, \ldots, m\}$$

- norm approximation error: $f(x) = ||Ax - b||$ (any norm)

Showing a function is convex

if $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex

examples

- piecewise-linear function: $f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$
- sum of r largest components of $x \in \mathbf{R}^n$:

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

($x_{[i]}$ is ith largest component of x)

proof: $f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$

Showing a function is convex

if $f(x, y)$ is convex in x for each $y \in \mathcal{A}$, then $g(x) = \sup_{y \in \mathcal{A}} f(x, y)$ is convex

examples

- distance to farthest point in a set C: $f(x) = \sup_{y \in C} \|x - y\|$
- maximum eigenvalue of symmetric matrix: for $X \in \mathbf{S}^n$, $\lambda_{\max}(X) = \sup_{\|y\|_2=1} y^T X y$ is convex
- support function of a set C: $S_C(x) = \sup_{y \in C} y^T x$ is convex

Showing a function is convex

- the function $g(x) = \inf_{y \in C} f(x, y)$ is called the **partial minimization** of f (w.r.t. y)
- if $f(x, y)$ is convex in (x, y) and C is a convex set, then partial minimization g is convex

examples

- $f(x, y) = x^T A x + 2x^T B y + y^T C y$ with

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0, \qquad C \succ 0$$

  minimizing over y gives $g(x) = \inf_y f(x, y) = x^T (A - BC^{-1}B^T) x$
  g is convex, hence Schur complement $A - BC^{-1}B^T \succeq 0$

- distance to a set: $\mathbf{dist}(x, S) = \inf_{y \in S} \|x - y\|$ is convex if S is convex

Showing a function is convex

- composition of $g : \mathbf{R}^n \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$ is $f(x) = h(g(x))$ (written as $f = h \circ g$)
- composition f is convex if
  - g convex, h convex, $\tilde{h}$ nondecreasing
  - or g concave, h convex, $\tilde{h}$ nonincreasing

  (monotonicity must hold for extended-value extension $\tilde{h}$)
- proof (for $n = 1$, differentiable g, h)

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

examples

- $f(x) = \exp g(x)$ is convex if g is convex
- $f(x) = 1/g(x)$ is convex if g is concave and positive

Showing a function is convex

- composition of $g : \mathbf{R}^n \to \mathbf{R}^k$ and $h : \mathbf{R}^k \to \mathbf{R}$ is $f(x) = h(g(x)) = h(g_1(x), g_2(x), \ldots, g_k(x))$
- f is convex if h is convex and for each i one of the following holds
  - g_i convex, $\tilde{h}$ nondecreasing in its ith argument
  - g_i concave, $\tilde{h}$ nonincreasing in its ith argument
  - g_i affine

- you will use this composition rule **constantly** throughout this course
- you need to commit this rule to memory

Perspective

- the **perspective** of a function $f : \mathbf{R}^n \to \mathbf{R}$ is the function $g : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$,

$$g(x,t) = tf(x/t), \qquad \mathbf{dom}\, g = \{(x,t) \mid x/t \in \mathbf{dom}\, f,\ t > 0\}$$

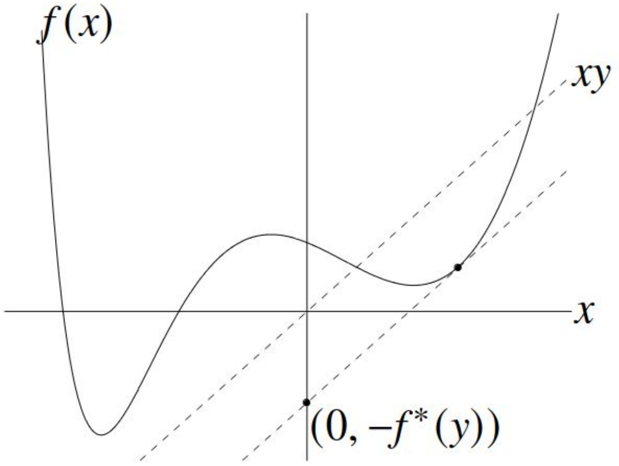
- g is convex if f is convex

examples

- $f(x) = x^T x$ is convex; so $g(x,t) = x^T x/t$ is convex for $t > 0$
- $f(x) = -\log x$ is convex; so relative entropy $g(x,t) = t\log t - t\log x$ is convex on $\mathbf{R}^2_{++}$

Conjugate

- the **conjugate** of a function f is $f^*(y) = \sup_{x \in \mathbf{dom} f}(y^T x - f(x))$



- f^* is convex (even if f is not)
- will be useful in chapter 5

Takeaway

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

- Classic convex functions
  - Affine, exponential, norms, max, …
- Convexity preserving operations
  - Non negative weighted sum, composition with affine
  - Pointwise maximum and supremum
  - Composition
  - Minimization
  - Perspective

1. Convex problems

Standard form optimization problem

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & i = 1, \ldots, p \end{array}$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

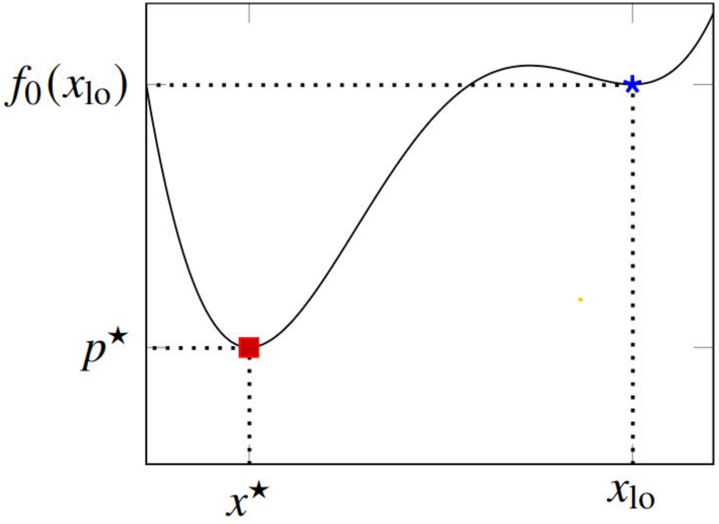
Feasible and optimal points

- $x \in \mathbf{R}^n$ is **feasible** if $x \in \mathbf{dom} f_0$ and it satisfies the constraints
- **optimal value** is $p^\star = \inf\{f_0(x) \mid f_i(x) \le 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}$
- $p^\star = \infty$ if problem is infeasible
- $p^\star = -\infty$ if problem is **unbounded below**
- a feasible x is **optimal** if $f_0(x) = p^\star$
- X_{opt} is the set of optimal points

Optimal points

x is **locally optimal** if there is an $R > 0$ such that x is optimal for

$$
\begin{array}{lll}
\text{minimize (over } z) & f_0(z) & \\
\text{subject to} & f_i(z) \leq 0, \quad i = 1, \dots, m, & h_i(z) = 0, \quad i = 1, \dots, p \\
& \|z - x\|_2 \leq R &
\end{array}
$$



Implicit and explicit constraints

standard form optimization problem has **implicit constraint**

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\, f_i \;\cap\; \bigcap_{i=1}^{p} \mathbf{dom}\, h_i,$$

- we call $\mathcal{D}$ the **domain** of the problem
- the constraints $f_i(x) \leq 0$, $h_i(x) = 0$ are the **explicit constraints**
- a problem is **unconstrained** if it has no explicit constraints ($m = p = 0$)

example:

$$\text{minimize} \quad f_0(x) = -\textstyle\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

Feasibility problem

$$
\begin{array}{lll}
\text{find} & x & \\
\text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\
 & h_i(x) = 0, & i = 1, \ldots, p
\end{array}
$$

can be considered a special case of the general problem with $f_0(x) = 0$:

$$
\begin{array}{lll}
\text{minimize} & 0 & \\
\text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\
 & h_i(x) = 0, & i = 1, \ldots, p
\end{array}
$$

- $p^\star = 0$ if constraints are feasible; any feasible x is optimal
- $p^\star = \infty$ if constraints are infeasible

Standard form convex problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & a_i^T x = b_i, \quad i = 1, \dots, p \end{array}$$

- objective and inequality constraints $f_0, f_1, \dots, f_m$ are convex
- equality constraints are affine, often written as $Ax = b$
- feasible and optimal sets of a convex optimization problem are convex

- problem is **quasiconvex** if f_0 is quasiconvex, $f_1, \dots, f_m$ are convex, $h_1, \dots, h_p$ are affine

Optimum in a convex set

any locally optimal point of a convex problem is (globally) optimal

proof:

- suppose x is locally optimal, but there exists a feasible y with $f_0(y) < f_0(x)$
- x locally optimal means there is an $R > 0$ such that

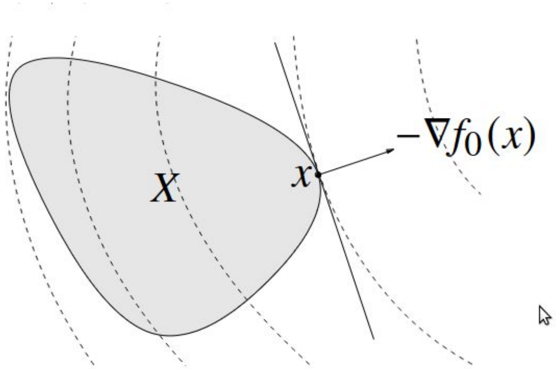
$$z \text{ feasible}, \quad \|z - x\|_2 \leq R \quad \Longrightarrow \quad f_0(z) \geq f_0(x)$$

- consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$
- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- z is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$ and $f_0(z) \leq \theta f_0(y) + (1 - \theta)f_0(x) < f_0(x)$, which contradicts our assumption that x is locally optimal

First order criterion

- x is optimal for a convex problem if and only if it is feasible and

$$\nabla f_0(x)^T(y-x) \geq 0 \quad \text{for all feasible } y$$



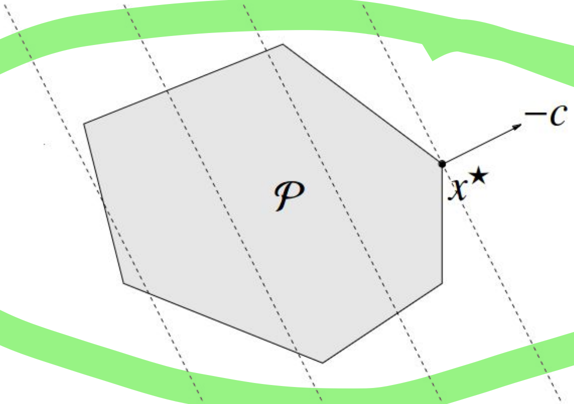
- if nonzero, $\nabla f_0(x)$ defines a supporting hyperplane to feasible set X at x

Linear programming

$$
\begin{array}{ll}
\text{minimize} & c^T x + d \\
\text{subject to} & Gx \preceq h \\
& Ax = b
\end{array}
$$

Often written as a maximization

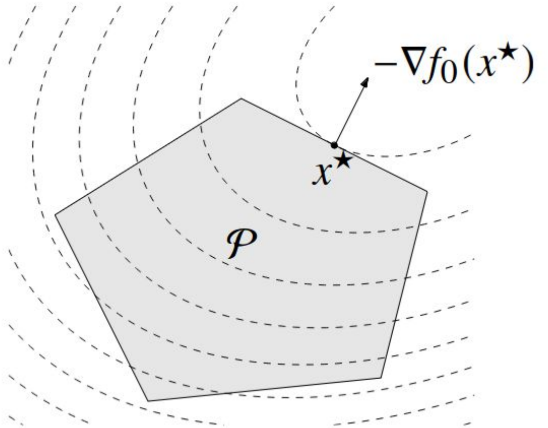
- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



Quadratic programming

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- $P \in \mathbf{S}^n_+$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Quadratically constrained Quadratic programming (QCQP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- $P_i \in \mathbf{S}^n_+$; objective and constraints are convex quadratic
- if $P_1, \ldots, P_m \in \mathbf{S}^n_{++}$, feasible region is intersection of m ellipsoids and an affine set

Change of variable

- $\phi : \mathbf{R}^n \to \mathbf{R}^n$ is one-to-one with $\phi(\mathbf{dom}\,\phi) \supseteq \mathcal{D}$
- consider (possibly non-convex) problem

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \dots, m \\ & h_i(x) = 0, & i = 1, \dots, p \end{array}$$

- change variables to z with $x = \phi(z)$
- can solve equivalent problem

$$\begin{array}{lll} \text{minimize} & \tilde{f}_0(z) & \\ \text{subject to} & \tilde{f}_i(z) \leq 0, & i = 1, \dots, m \\ & \tilde{h}_i(z) = 0, & i = 1, \dots, p \end{array}$$

where $\tilde{f}_i(z) = f_i(\phi(z))$ and $\tilde{h}_i(z) = h_i(\phi(z))$

- recover original optimal point as $x^\star = \phi(z^\star)$

Transformation

UNIVERSITÉ CÔTE D'AZUR

suppose

- ϕ_0 is monotone increasing
- $\psi_i(u) \leq 0$ if and only if $u \leq 0$, $i = 1, \ldots, m$
- $\varphi_i(u) = 0$ if and only if $u = 0$, $i = 1, \ldots, p$

standard form optimization problem is equivalent to

$$
\begin{array}{lll}
\text{minimize} & \phi_0(f_0(x)) & \\
\text{subject to} & \psi_i(f_i(x)) \leq 0, & i = 1, \ldots, m \\
 & \varphi_i(h_i(x)) = 0, & i = 1, \ldots, p
\end{array}
$$

example: minimizing $\|Ax - b\|$ is equivalent to minimizing $\|Ax - b\|^2$

Maximization and minimization

- suppose ϕ_0 is monotone decreasing
- the maximization problem

$$\begin{array}{lll} \text{maximize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & i = 1, \ldots, p \end{array}$$

is equivalent to the minimization problem

$$\begin{array}{lll} \text{minimize} & \phi_0(f_0(x)) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & i = 1, \ldots, p \end{array}$$

- **examples**:
  - $\phi_0(u) = -u$ transforms maximizing a concave function to minimizing a convex function
  - $\phi_0(u) = 1/u$ transforms maximizing a concave positive function to minimizing a convex function

Eliminating equality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } z) & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \leq 0, \quad i = 1, \ldots, m \end{array}$$

where F and x_0 are such that $Ax = b \iff x = Fz + x_0$ for some z

Introducing equality constraints

$$\begin{array}{ll} \text{minimize} & f_0(A_0x + b_0) \\ \text{subject to} & f_i(A_ix + b_i) \leq 0, \quad i = 1, \dots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } x, y_i) & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leq 0, \quad i = 1, \dots, m \\ & y_i = A_ix + b_i, \quad i = 0, 1, \dots, m \end{array}$$

Slack variables for linear equalities

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

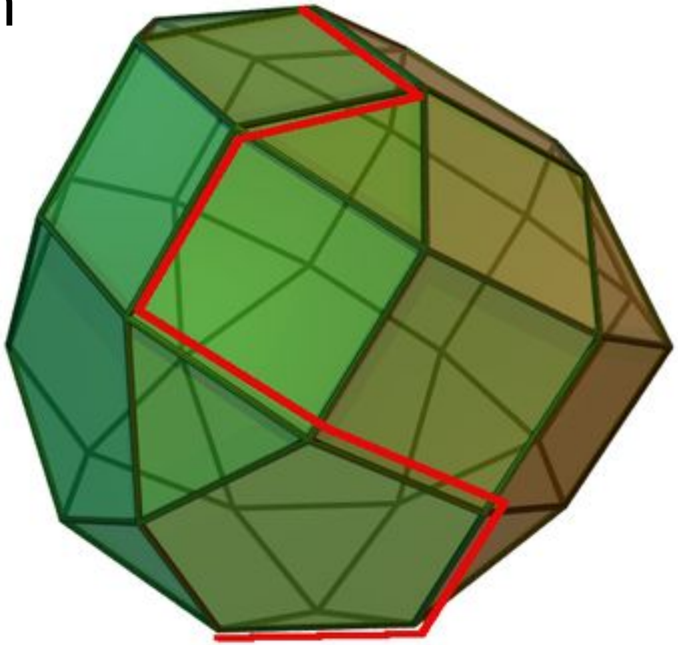
is equivalent to

$$\begin{array}{lll} \text{minimize (over } x, s) & f_0(x) & \\ \text{subject to} & a_i^T x + s_i = b_i, & i = 1, \ldots, m \\ & s_i \geq 0, & i = 1, \ldots m \end{array}$$

2. Simplex algorithm (for LP)

Introduction

- Historical algorithm of optimization
  - Proposed by Dantzig
- Widely used even nowadays
  - “Usually” efficient
  - Bad worst case
- Move along constraint edges



Standard form LP

UNIVERSITÉ CÔTE D'AZUR

$$
\begin{array}{lrcrcrcr}
\text{Maximize} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{Subject to :} & & & & & & & \\
 & 2x_1 & + & 3x_2 & - & x_3 & \leq & 5 \\
 & 4x_1 & + & x_2 & - & 2x_3 & \leq & 11 \\
 & 3x_1 & + & 4x_2 & - & 2x_3 & \leq & 8 \\
 & & & & & x_1, x_2, x_3 & \geq & 0.
\end{array}
$$

Standard form LP

UNIVERSITÉ CÔTE D'AZUR

$$
\begin{array}{lrcrcrcr}
\boxed{\text{Maximize}} & 5x_1 & + & 4x_2 & + & 3x_3 & & \\
\text{Subject to :} & & & & & & & \\
 & 2x_1 & + & 3x_2 & - & x_3 & \leq & 5 \\
 & 4x_1 & + & x_2 & - & 2x_3 & \leq & 11 \\
 & 3x_1 & + & 4x_2 & - & 2x_3 & \leq & 8 \\
 & & & & & x_1, x_2, x_3 & \geq & 0.
\end{array}
$$

Add slack variables

$$2x_1 + 3x_2 + x_3 \leq 5$$

Add slack variables

$$2x_1 + 3x_2 + x_3 \leq 5$$

$$x_4 = 5 - 2x_1 - 3x_2 - x_3.$$

$$x_4 \geq 0.$$

Add slack variables

$$2x_1 + 3x_2 + x_3 \leq 5$$

$$\begin{array}{rcrcrcr} 4x_1 & + & x_2 & - & 2x_3 & \leq & 11 \\ 3x_1 & + & 4x_2 & - & 2x_3 & \leq & 8 \end{array}$$

$$x_4 = 5 - 2x_1 - 3x_2 - x_3.$$

$$x_4 \geq 0.$$

Add slack variables

$$2x_1 + 3x_2 + x_3 \leq 5$$

$$\begin{array}{rcrcrcr} 4x_1 & + & x_2 & - & 2x_3 & \leq & 11 \\ 3x_1 & + & 4x_2 & - & 2x_3 & \leq & 8 \end{array}$$

$$x_4 = 5 - 2x_1 - 3x_2 - x_3.$$

$$\begin{array}{rcrcrcrcr} x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \end{array}$$

$$x_4 \geq 0.$$

$$x_5 \geq 0, x_6 \geq 0.$$

Canonical form LP

Maximize z subject to $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$.

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 .
\end{array}
$$

Canonical form LP

Maximize z subject to $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$.

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 .
\end{array}
$$

Find initial feasible solution

$$x_1 = 0, x_2 = 0, x_3 = 0$$

Find initial feasible solution

$$x_1 = 0, x_2 = 0, x_3 = 0$$

$$x_4 \quad = \quad 5 \quad - \quad 2x_1 \quad - \quad 3x_2 \quad - \quad x_3 = 5$$

Find initial feasible solution

$$x_1 = 0, x_2 = 0, x_3 = 0$$

$$x_4 \quad = \quad 5 \quad - \quad 2x_1 \quad - \quad 3x_2 \quad - \quad x_3 = 5$$

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 5, x_5 = 11, x_6 = 8$$

$$z = 0$$

Solving the simplex dictionary

Let's borrow an unconventional presentation from F. Giroire!

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 .
\end{array}
$$

Basic variables: x_4, x_5, x_6, variables on the left.
Non-basic variable: x_1, x_2, x_3, variables on the right.

A dictionary is feasible if a feasible solution is obtained by setting all non-basic variables to 0.

Find the most influential variable

Simplex strategy: find an optimal solution by successive improvements.

Rule: we increase the value of the variable of largest positive coefficient in z.

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & \boxed{5}\, x_1 & + & 4x_2 & + & 3x_3 .
\end{array}
$$

Here, we try to increase x_1.

How far can we go?

How much can we increase x_1?

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 .
\end{array}
$$

We have $\boxed{x_4 \geq 0}$.

It implies $5 - 2x_1 \geq 0$, that is $\boxed{x_1 \leq \frac{5}{2}}$.

How far can we go?

How much can we increase x_1?

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ \hline z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 . \end{array}$$

We have $\boxed{x_4 \geq 0}$.

It implies $5 - 2x_1 \geq 0$, that is $\boxed{x_1 \leq 5/2}$.

Similarly,

$x_5 \geq 0$ gives $x_1 \leq 11/4$.

$x_6 \geq 0$ gives $x_1 \leq 8/3$.

How far can we go?

How much can we increase x_1?

$$\begin{array}{rcrcrcrcr}
\color{red}{x_4} & \color{red}{=} & \color{red}{5} & \color{red}{-} & \color{red}{2x_1} & \color{red}{-} & \color{red}{3x_2} & \color{red}{-} & \color{red}{x_3} \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & 5x_1 & + & 4x_2 & + & 3x_3.
\end{array}$$

We have $\boxed{x_4 \geq 0}$.

It implies $5 - 2x_1 \geq 0$, that is $\boxed{x_1 \leq 5/2}$ Strongest constraint

Similarly,

$x_5 \geq 0$ gives $x_1 \leq 11/4$.

$x_6 \geq 0$ gives $x_1 \leq 8/3$.

How far can we go?

How much can we increase x_1?

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 .
\end{array}
$$

We have $\boxed{x_4 \geq 0}$.

It implies $5 - 2x_1 \geq 0$, that is $\boxed{x_1 \leq 5/2}$ Strongest constraint

We get a new solution: $x_1 = 5/2$, $x_4 = 0$
with better value $z = 5 \cdot 5/2 = 25/2$.
We still have $x_2 = x_3 = 0$ and now $x_5 = 11 - 4 \cdot 5/2 = 1$,
$x_6 = 8 - 3 \cdot 5/2 = 1/2$

Pivot around the chosen variable

We build a new feasible dictionary.

$$
\begin{array}{rcrcrcrcr}
x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\
x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & 5x_1 & + & 4x_2 & + & 3x_3.
\end{array}
$$

x_1 enters the bases and x_4 leaves it:

$$
x_1 = 5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4
$$

Pivot around the chosen variable

We replace x_1 by its expression in function of x_2, x_3, x_4.

$$
\begin{array}{rclcrcrcr}
x_1 & = & 5/2 & - & 1/2x_4 & - & 3/2x_2 & - & 1/2x_3 \\
x_5 & = & 11 & - & 4(5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4) & - & x_2 & - & 2x_3 \\
x_6 & = & 8 & - & 3(5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4) & - & 4x_2 & - & 2x_3 \\
\hline
z & = & & & 5(5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4) & + & 4x_2 & + & 3x_3 .
\end{array}
$$

Pivot around the chosen variable

Finally, we get the new dictionary:

$$
\begin{array}{rcrcrcrcrcrcrc}
x_1 & = & \frac{5}{2} & - & \frac{3}{2} & x_2 & - & \frac{1}{2} & x_3 & - & \frac{1}{2} & x_4 \\
x_5 & = & 1 & + & 5 & x_2 & & & & + & 2 & x_4 \\
x_6 & = & \frac{1}{2} & + & \frac{1}{2} & x_2 & - & \frac{1}{2} & x_3 & + & \frac{3}{2} & x_4 \\
\hline
z & = & \frac{25}{2} & - & \frac{7}{2} & x_2 & + & \frac{1}{2} & x_3 & - & \frac{5}{2} & x_4 .
\end{array}
$$

New system

Finally, we get the new dictionary:

$$\begin{array}{rclclclcl}
x_1 & = & \boxed{5/2} & - & \frac{3}{2} x_2 & - & \frac{1}{2} x_3 & - & \frac{1}{2} x_4 \\
x_5 & = & \boxed{1} & + & 5 x_2 & & & + & 2 x_4 \\
x_6 & = & \boxed{1/2} & + & \frac{1}{2} x_2 & - & \frac{1}{2} x_3 & + & \frac{3}{2} x_4 \\
\hline
z & = & \boxed{25/2} & - & 7/2 x_2 & + & 1/2 x_3 & - & 5/2 x_4 .
\end{array}$$

We can read the solution directly from the dictionary:

Non basic variables: $x_2 = x_3 = x_4 = 0$.

Basic variables: $x_1 = 5/2$, $x_5 = 1$, $x_6 = 1/2$.

Value of the solution: $z = 25/2$.

New step!

$$
\begin{array}{rcrcrcrcrcr}
x_1 & = & \frac{5}{2} & - & \frac{3}{2} & x_2 & - & \frac{1}{2} & x_3 & - & \frac{1}{2} & x_4 \\
x_5 & = & 1 & + & 5 & x_2 & & & & + & 2 & x_4 \\
x_6 & = & \frac{1}{2} & + & \frac{1}{2} & x_2 & - & \frac{1}{2} & x_3 & + & \frac{3}{2} & x_4 \\
\hline
z & = & \frac{25}{2} & - & \frac{7}{2} & x_2 & + & \frac{1}{2} & x_3 & - & \frac{5}{2} & x_4 .
\end{array}
$$

New step of the simplex:

- x_3 enters the basis (variable with largest positive coefficient).
- 3^d equation is the strictest constaint $x_3 \leq 1$.
- x_6 leaves the basis.

And we’re done!

New feasible dictionary:

$$
\begin{array}{rcrcrcrcr}
x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\
x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\
x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\
\hline
z & = & 13 & - & 3x_2 & - & x_4 & - & x_6 .
\end{array}
$$

With new solution:

$$x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0$$

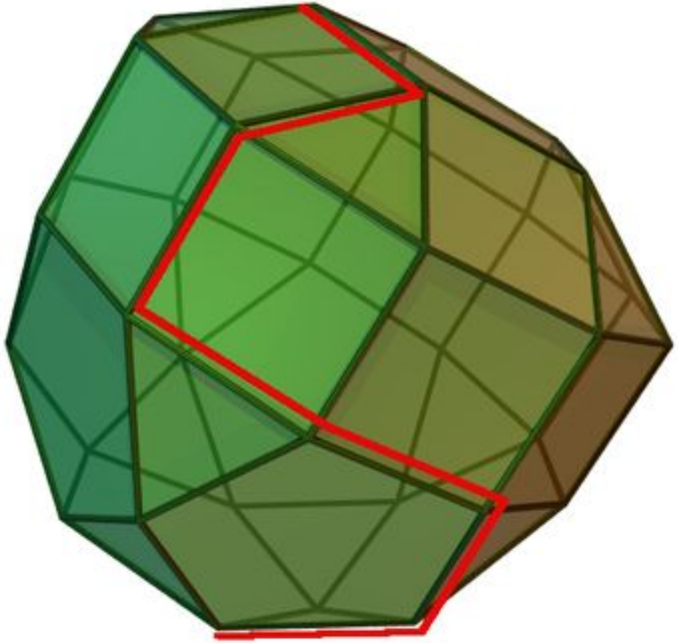
of value $z = 13$.

This solution is optimal.

All coefficients in z are negative and $x_2 \geq 0, x_4 \geq 0, x_6 \geq 0$, *so* $z \leq 13$.

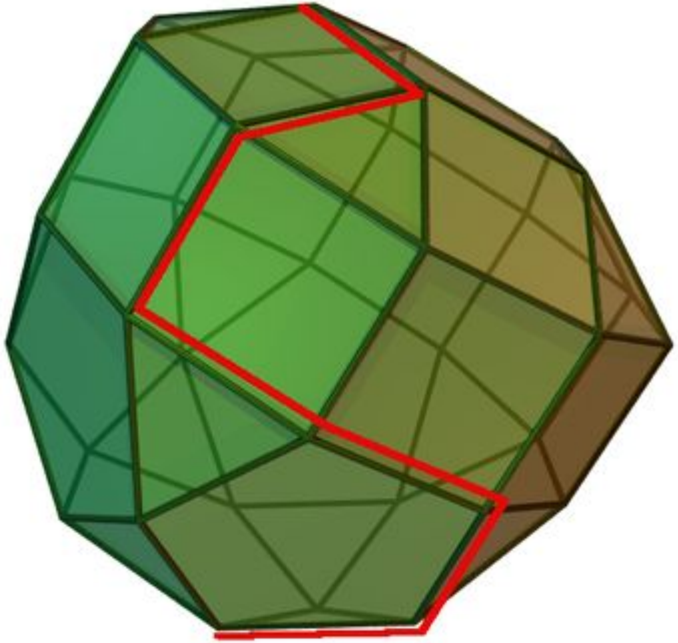
Let’s recap

- We convert to canonical form to work along equalities
- We start along some corner
- We move as far as we can on the best edge
  - Until we are done



Let’s recap

- We convert to canonical form to work along equalities
- We start along some corner
- We move as far as we can on the best edge
  - Until we are done
- **Usually solved with a simplex tableau**



Simplex tableau

UNIVERSITÉ CÔTE D'AZUR

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ \hline z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 . \end{array}$$

x1	x2	x3	x4	x5	x6	Z	C
2	3	1	1	0	0	0	5
4	1	2	0	1	0	0	11
3	4	2	0	0	1	0	8
-5	-4	-3	0	0	0	1	0

Simplex tableau

x1	x2	x3	x4	x5	x6	Z	C
2	3	1	1	0	0	0	5 (5/2)
4	1	2	0	1	0	0	11 (11/4)
3	4	2	0	0	1	0	8 (8/3)
-5	-4	-3	0	0	0	1	0

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
4	1	2	0	1	0	0	11
3	4	2	0	0	1	0	8
-5	-4	-3	0	0	0	1	0

Simplex tableau

UNIVERSITÉ CÔTE D'AZUR

x1	x2	x3	x4	x5	x6	Z	C
2	3	1	1	0	0	0	5 (5/2)
4	1	2	0	1	0	0	11 (11/4)
3	4	2	0	0	1	0	8 (8/3)
-5	-4	-3	0	0	0	1	0

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
0	-5	0	-2	1	0	0	1
0	-0.5	0.5	-1.5	0	1	0	0.5
0	3.5	-0.5	2.5	0	0	1	12.5

Simplex tableau

UNIVERSITÉ CÔTE D'AZUR

$$
\begin{array}{rcl}
x_1 & = & \frac{5}{2} - \frac{3}{2} x_2 - \frac{1}{2} x_3 - \frac{1}{2} x_4 \\
x_5 & = & 1 + 5 x_2 + 2 x_4 \\
x_6 & = & \frac{1}{2} + \frac{1}{2} x_2 - \frac{1}{2} x_3 + \frac{3}{2} x_4 \\
\hline
z & = & \frac{25}{2} - \frac{7}{2} x_2 + \frac{1}{2} x_3 - \frac{5}{2} x_4 .
\end{array}
$$

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
0	-5	0	-2	1	0	0	1
0	-0.5	0.5	-1.5	0	1	0	0.5
0	3.5	-0.5	2.5	0	0	1	12.5

Simplex tableau

UNIVERSITÉ CÔTE D'AZUR

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5 (5)
0	-5	0	-2	1	0	0	1 (inf)
0	-0.5	0.5	-1.5	0	1	0	0.5 (1)
0	3.5	-0.5	2.5	0	0	1	12.5

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
0	-5	0	-2	1	0	0	1
0	-1	1	-3	0	2	0	1
0	3.5	-0.5	2.5	0	0	1	12.5

Simplex tableau

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5 (5)
0	-5	0	-2	1	0	0	1 (inf)
0	-0.5	0.5	-1.5	0	1	0	0.5 (1)
0	3.5	-0.5	2.5	0	0	1	12.5

x1	x2	x3	x4	x5	x6	Z	C
1	2	0	2	0	-1	0	2
0	-5	0	-2	1	0	0	1
0	-1	1	-3	0	2	0	1
0	3	0	1	0	1	1	13

Simplex tableau

$$\begin{array}{rcrcrcrcr}
x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\
x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\
x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\
\hline
z & = & 13 & - & 3x_2 & - & x_4 & - & x_6 .
\end{array}$$

x1	x2	x3	x4	x5	x6	Z	C
1	2	0	2	0	-1	0	2
0	-5	0	-2	1	0	0	1
0	-1	1	-3	0	2	0	1
0	3	0	1	0	1	1	13

Algorithm

- Build tableau from canonical
- Check we have feasible solution
- Do a pivot step if negative coef
  - Pick column c w/ most negative coefficient
  - Pick row r w/ smallest ratio
  - Pivot! (set c to 1 in r and 0 in other rows)