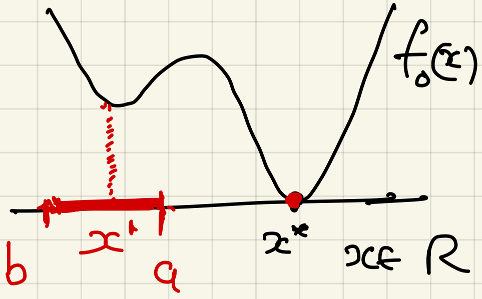
Optimization problem

Convex

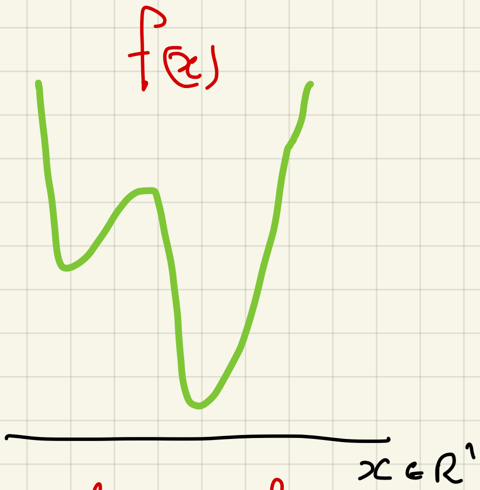
Non convex

minimize $f_0(x)$

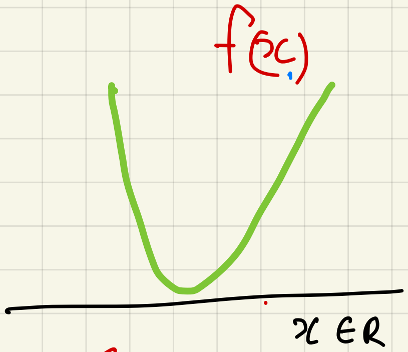
s.t. $f_i(x) \le b_i$, $i = 1, 2, \dots m$



objective function



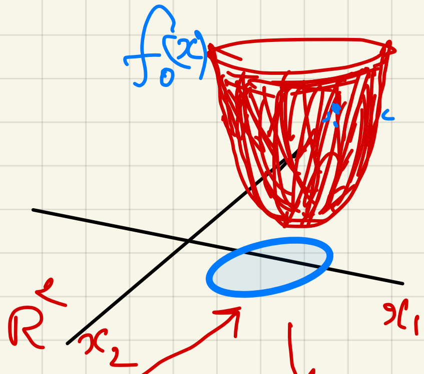
Non confex



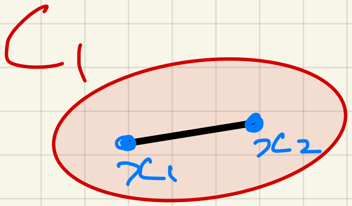
Convex

Constraints

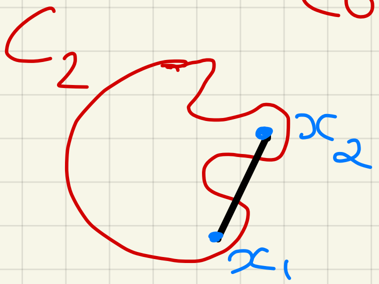
forms a set



convex?



Convex



Non-convex.

2. Convex sets

- affine and convex sets
- some important examples
- operations that preserve convexity
- generalized inequalities
- separating and supporting hyperplanes
- dual cones and generalized inequalities

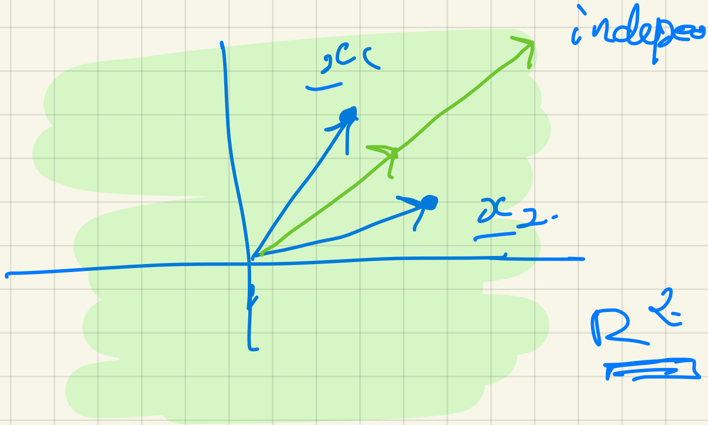
Different Combinations

① Linear combination: $\underline{x}_1, \underline{x}_2 \dots$

$$\sum_i \theta_i x_i, \quad \theta_i \in R$$

linear span $(\underline{x}_1, \underline{x}_2) = R^2$.

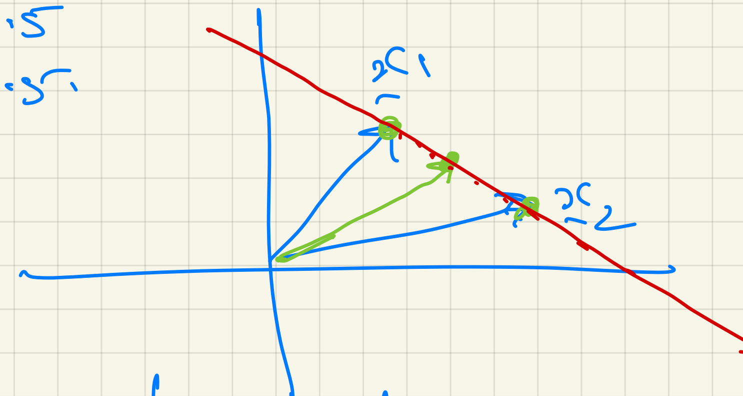
x_1 and x_2 are linearly indep.



② Affine combination:

$$\sum_i \theta_i x_i, \quad \theta_i \in R, \quad \sum_i \theta_i = 1$$

$\theta_1 = 0.5$
$\theta_2 = 0.5$

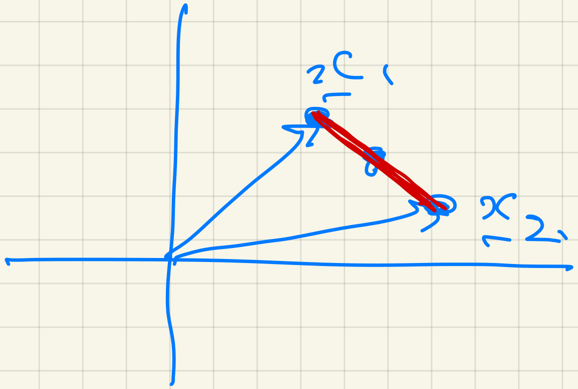


affine span (x_1, x_2) is the line passing through $\underline{x}_1$ and $\underline{x}_2$.

③ Convex combination:

$$\sum_i \theta_i x_i, \quad \theta_i \in R, \quad \sum_i \theta_i = 1, \quad \theta_i \in [0, 1]$$

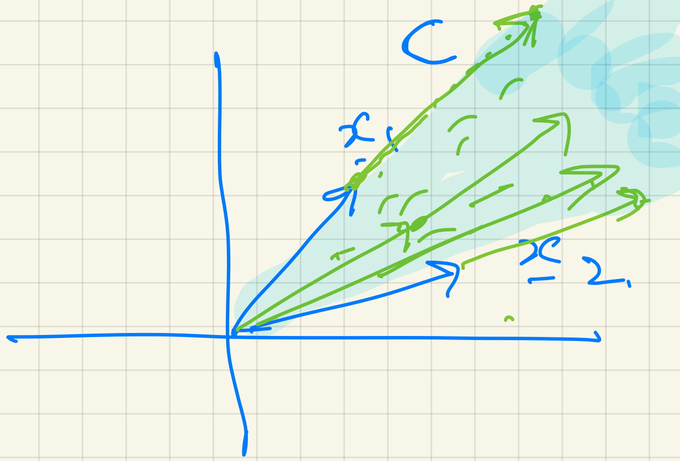
'convex span' (x_1, x_2) is the line segment between $\underline{x}_1$ and $\underline{x}_2$



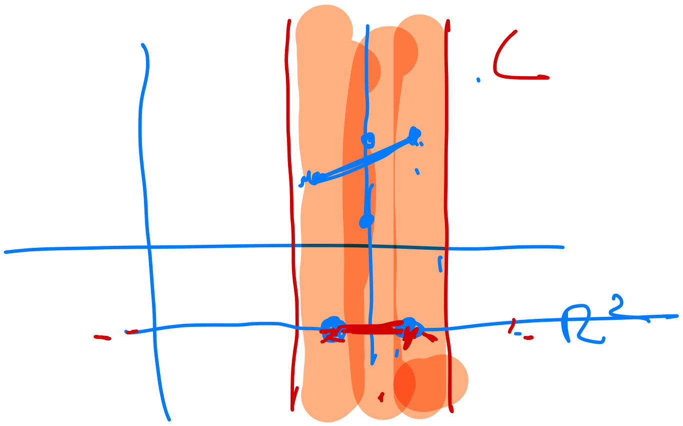
④ conic combination:

$$\sum_i a_i x_i, \quad a_i \in \mathbb{R}, \quad a_i \geq 0$$

conic span (x_1, x_2) is the convex cone C



Affine set



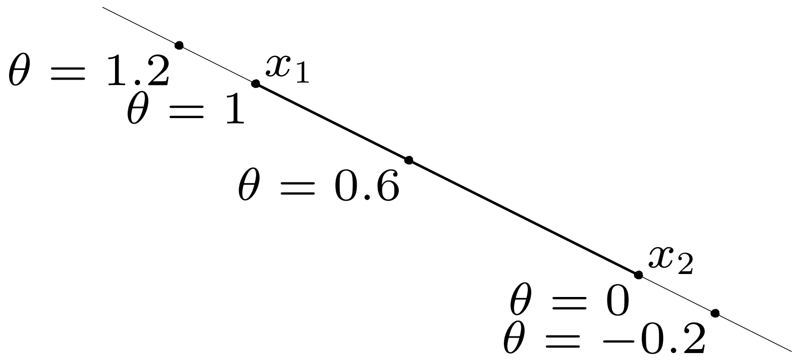
not an affine set.

$\sum_i \theta_i = 1$

$\sum \theta + 1 - \theta = 1$

line through x_1, x_2: all points

$$x = \theta x_1 + (1-\theta)x_2 \qquad (\theta \in \mathbf{R})$$



affine set: contains the line through any two distinct points in the set

example: solution set of linear equations $\{x \mid Ax = b\}$ ✓

(conversely, every affine set can be expressed as solution set of system of linear equations)

$C = \{\underline{x} \mid A\underline{x} = \underline{b}\}$

$\underline{x}_1 \in C \Rightarrow A\underline{x}_1 = \underline{b}$

$\underline{x}_2 \in C \Rightarrow A\underline{x}_2 = \underline{b}$

$\underline{y} = \theta x_1 + (1-\theta) x_2$

$A\underline{y} = \underline{b}$?

$$A\underline{y} = A(\theta \underline{x}_1 + (1-\theta) x_2) = \theta \underline{b} + \underline{b} - \theta \underline{b}$$

$$= \underline{\underline{b}}$$

$\Rightarrow y \in \underline{\underline{C}}$

C is affine.

Convex set

line segment between x_1 and x_2: all points

$$x = \theta x_1 + (1 - \theta) x_2$$

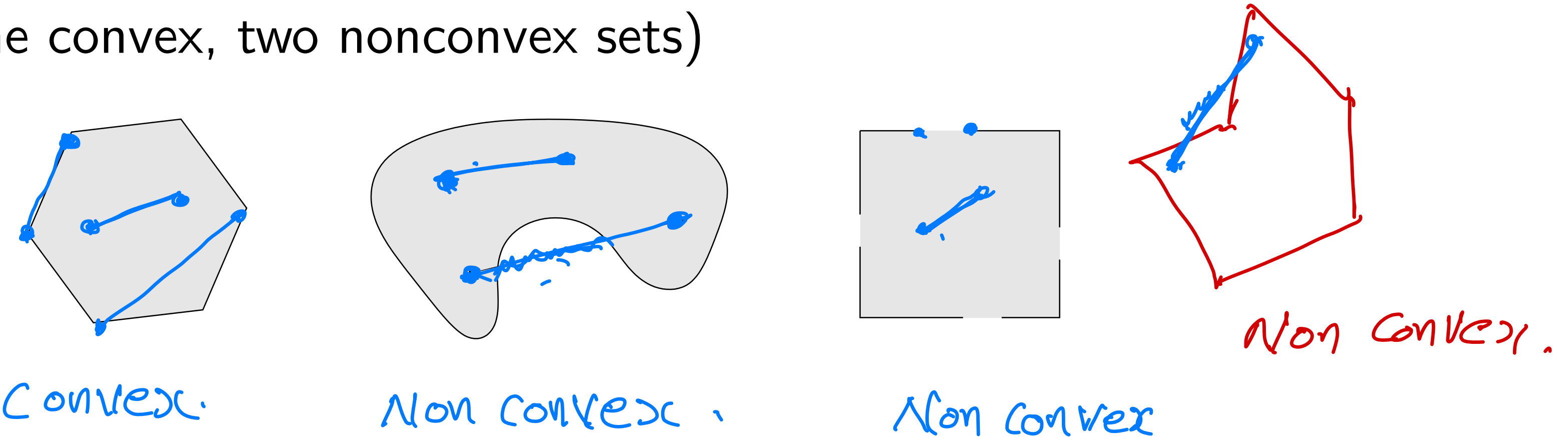
$\theta + (1 - \theta) = 1$

with $0 \leq \theta \leq 1$

convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1 - \theta) x_2 \in C$$

examples (one convex, two nonconvex sets)



Convex combination and convex hull

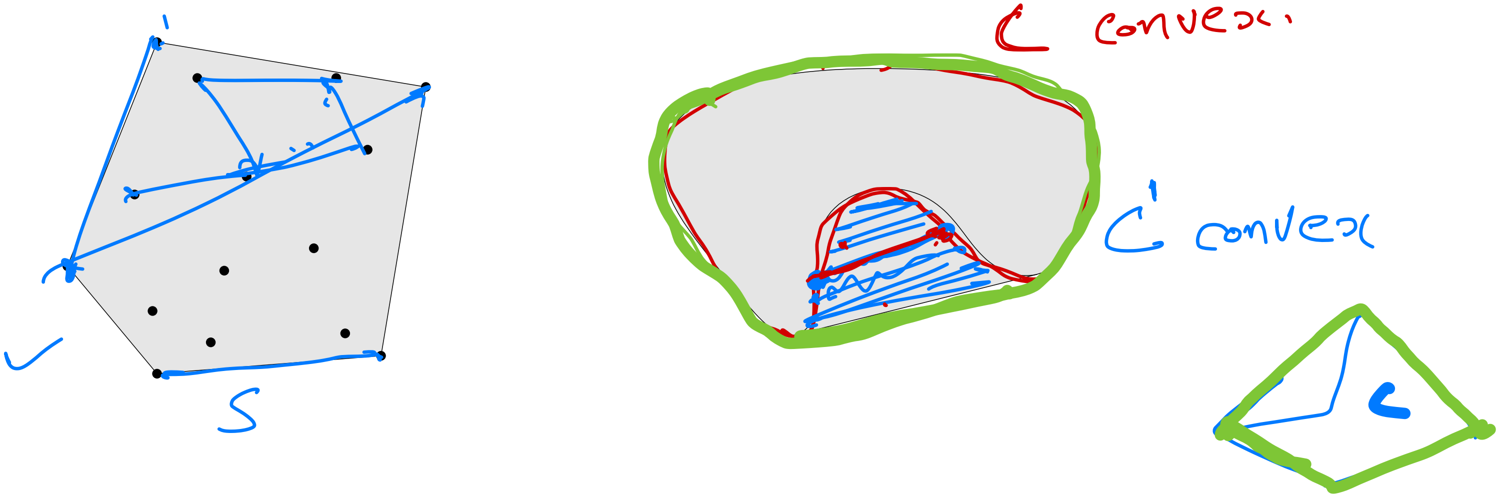
convex combination of x_1,. . . , x_k: any point x of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$, $\theta_i \geq 0$

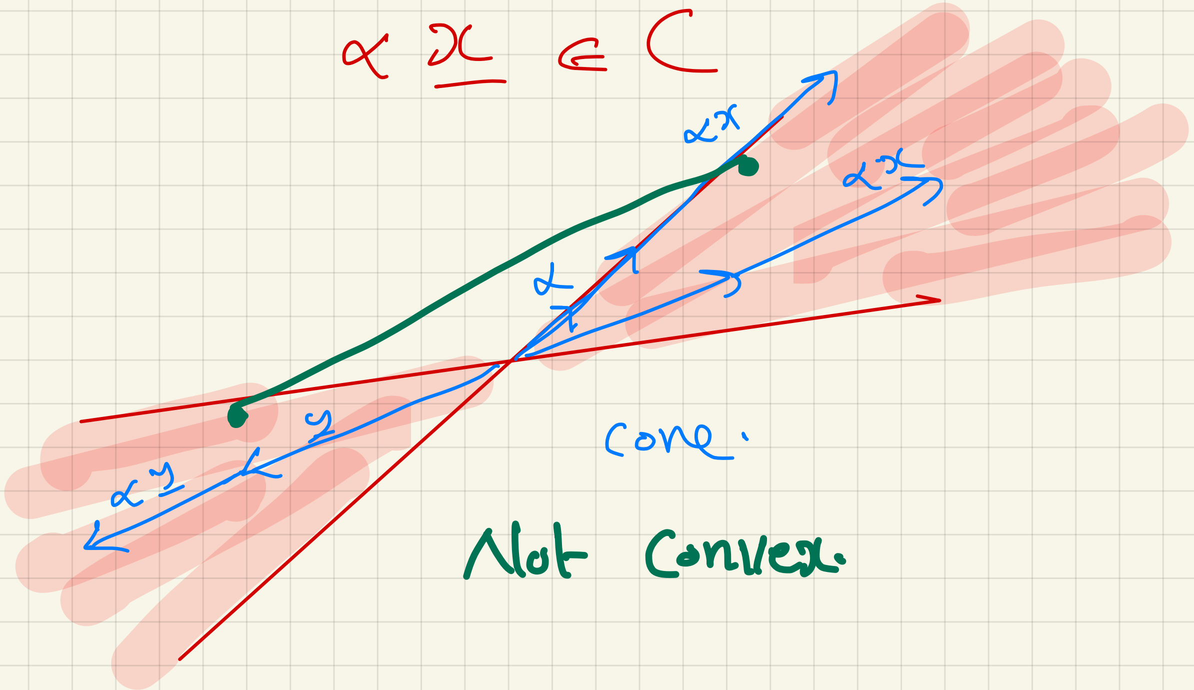
$\theta_i \in [0,1]$

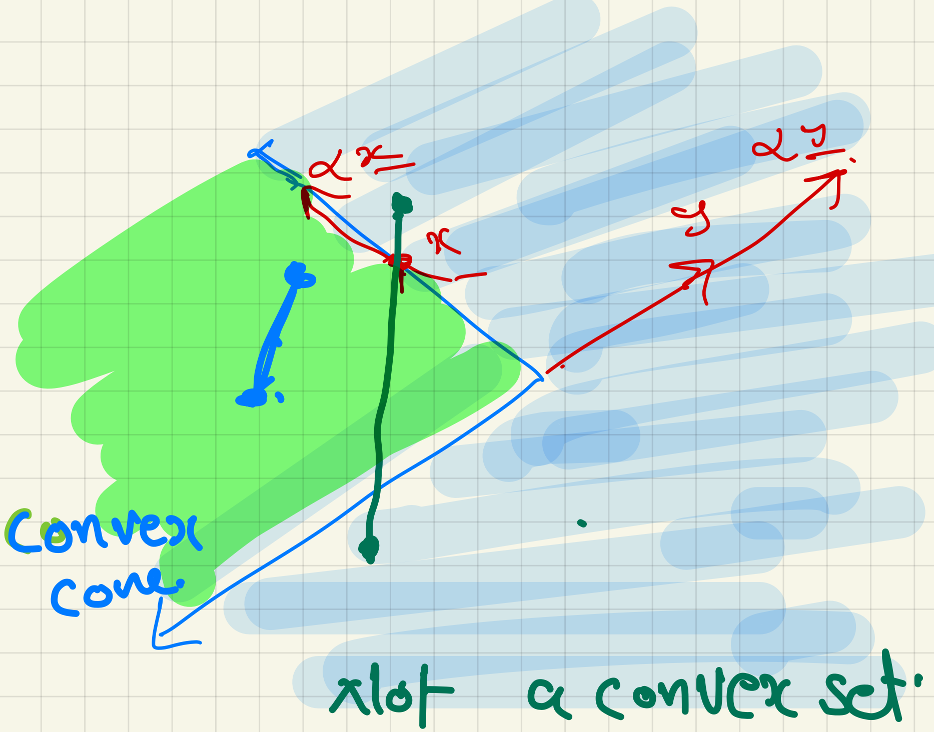
convex hull $\mathbf{conv}\, S$: set of all convex combinations of points in S

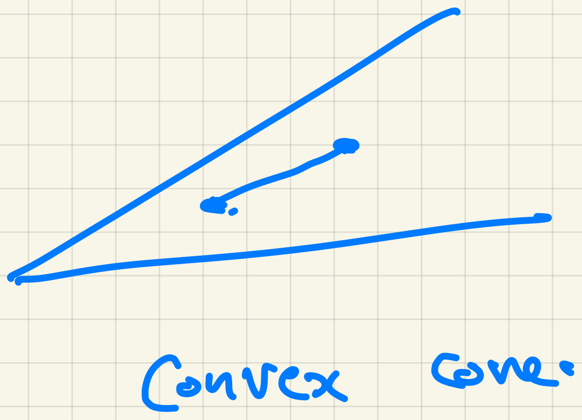


Cone: A set C is a cone if $\forall\ \underline{x} \in C$ and $\alpha \geq 0$

$\alpha \underline{x} \in C$







Convex cone

conic (nonnegative) combination of x_1 and x_2: any point of the form

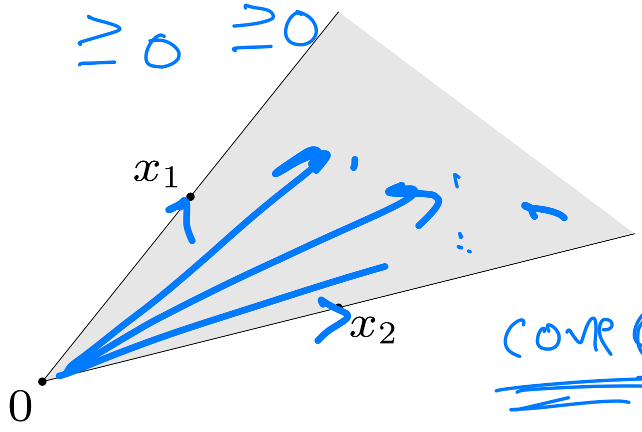
$$x = \theta_1 x_1 + \theta_2 x_2$$

with $\theta_1 \geq 0$, $\theta_2 \geq 0$

$(1-\theta)$ $\theta \in [0,1]$

$[0,1]$ $[0,1]$ $\theta x_1 + (1-\theta) x_2 \in C$

≥ 0 ≥ 0



conv C → convex

convex cone: set that contains all conic combinations of points in the set

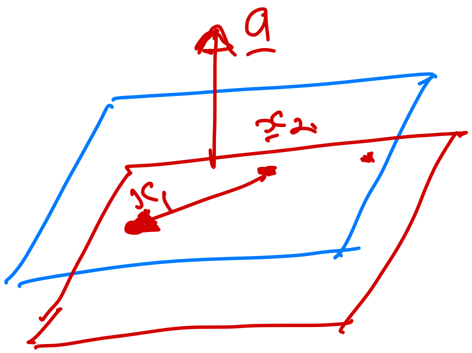
Hyperplanes and halfspaces

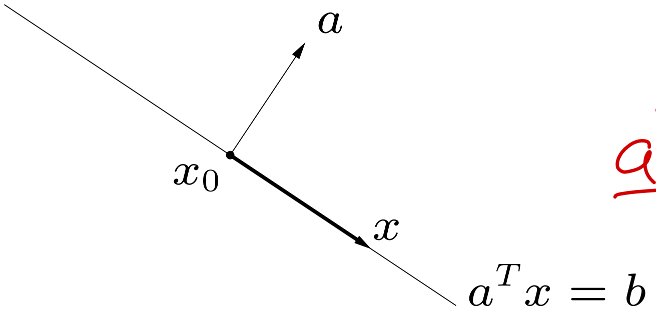
hyperplane: set of the form $\{x \mid a^T x = b\}$ $(a \neq 0)$

$a^T x_1 = b$ — ①

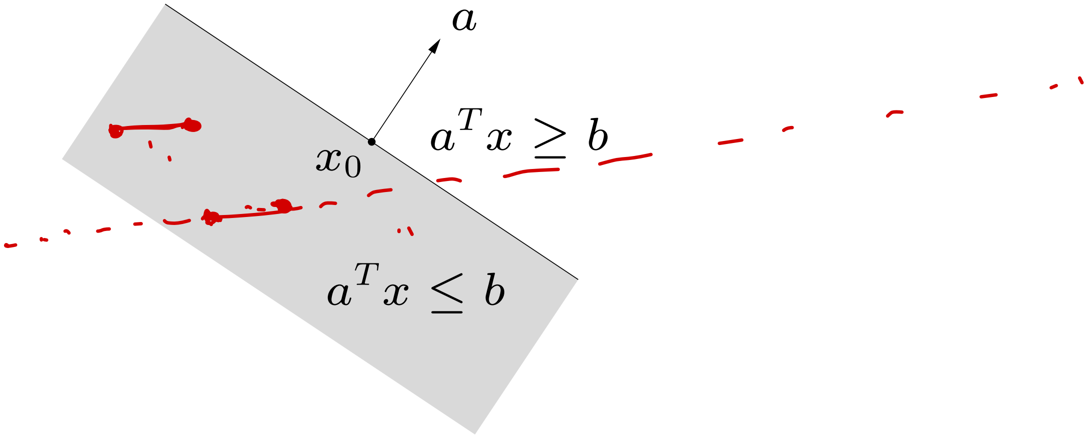
$a^T x_2 = b$ — ②

$a^T(x_1 - x_2) = 0$





halfspace: set of the form $\{x \mid a^T x \leq b\}$ $(a \neq 0)$



- a is the normal vector
- hyperplanes are affine and convex; halfspaces are convex Not affine.

$$C = \{ \underline{x} \mid \underline{a}^T \underline{x} = \underline{b} \} \quad (a \neq 0)$$

$$x_1 \in C \Rightarrow \underline{a}^T \underline{x}_1 = \underline{b}$$

$$x_2 \in C \Rightarrow \underline{a}^T \underline{x}_2 = \underline{b}$$

$$\underline{y} = \underline{\theta}\, \underline{x}_1 + (\underline{1-\theta})\, \underline{x}_2 \Leftarrow$$

$$\left.\begin{aligned} \underline{a}^T(\theta \underline{x}_1 + (1-\theta)\underline{x}_2) &= \theta\, \underline{a}^T \underline{x} + (1-\theta)\, \underline{a}^T \underline{x}_2 \\ &= \theta\, \underline{b} + (1-\theta)\, \underline{b} = \underline{b} \end{aligned}\right\}$$

$$\underline{\theta} \Rightarrow$$

$$\boxed{\theta \in [0,1]} \leftarrow$$

$$\underline{C} = \{ \underline{x} \mid \underline{a}^T \underline{x} \leq \underline{b} \}$$

$$x_1 \in C \Rightarrow \underline{a}^T \underline{x}_1 \leq b \ ✓$$

$$x_2 \in C \Rightarrow \underline{a}^T \underline{x}_2 \leq b \ ✓$$

$$\Rightarrow$$

$$y = \underline{\theta}\, x_1 + (1-\theta)\, \underline{x}_2 \ ✓$$

$$\underline{a}^T(\theta \underline{x}_1 + (1-\theta)\underline{x}_2)$$

$$= \underbrace{\theta}_{\geq 0}\, \underline{a}^T x_1 + \underbrace{(1-\theta)}_{\geq 0}\, \underline{a}^T \underline{x}_2$$

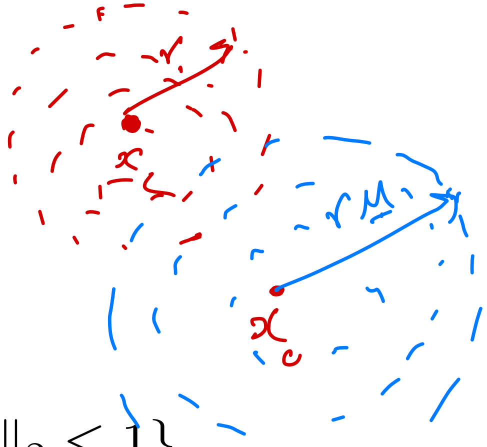
$$\leq \theta b + (1-\theta) b$$

$$= b$$

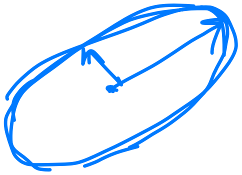
$$\underline{\theta \in [0,1]} \Rightarrow \theta \geq 0, \quad (1-\theta) \geq 0$$

$$a^T y \leq b \Rightarrow y \in C$$

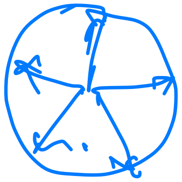
Euclidean balls and ellipsoids



(Euclidean) ball with center x_c and radius r:



$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \le r\} = \{x_c + ru \mid \|u\|_2 \le 1\}$$



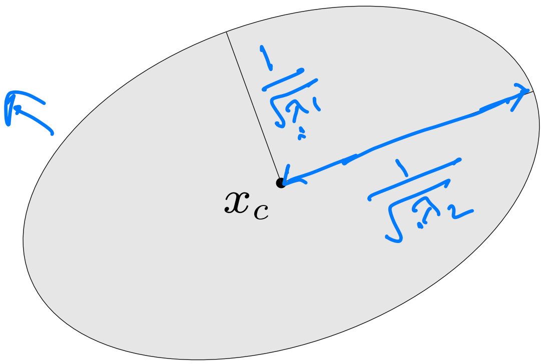
$(x - x_c)^T(x - x_c)$

ellipsoid: set of the form

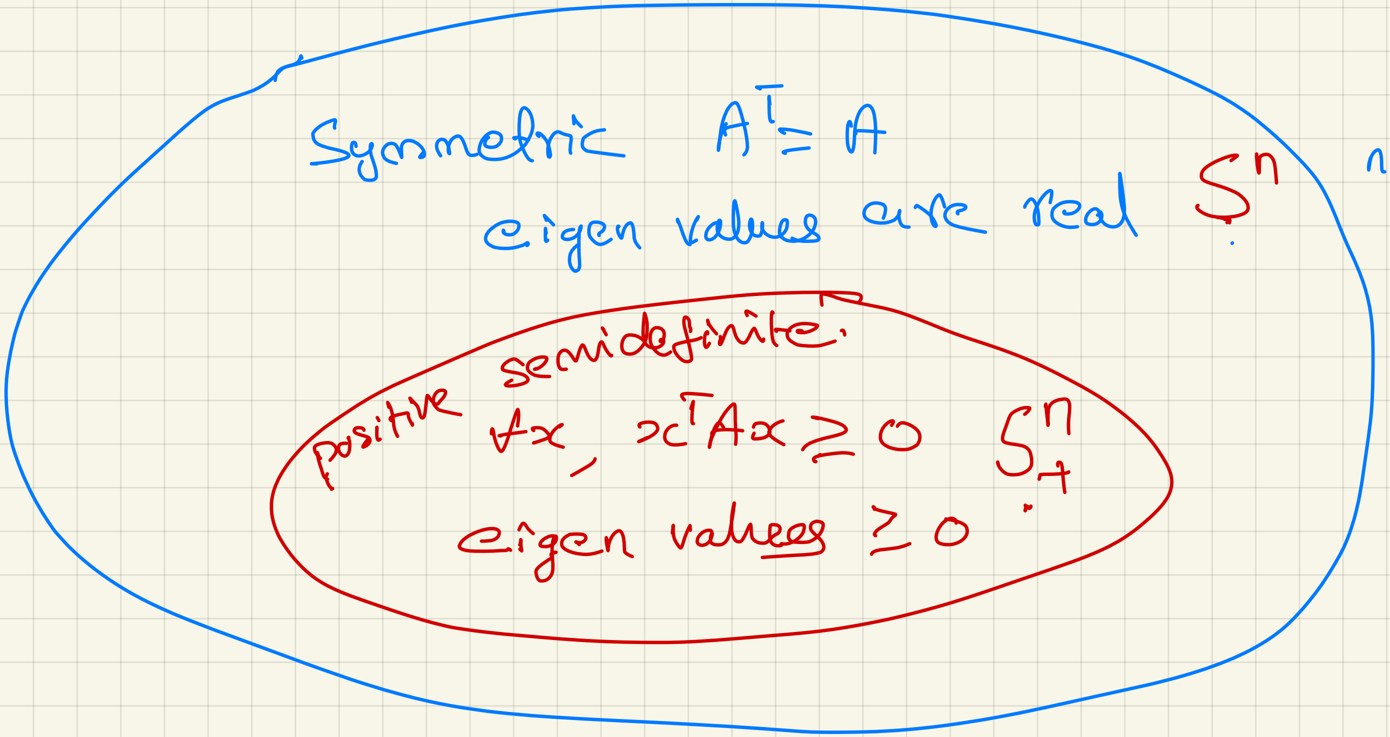
$$\{x \mid (x - x_c)^T P^{-1}(x - x_c) \le 1\}$$

with $P \in \mathbf{S}^n_{++}$ (*i.e.*, P symmetric positive definite)

λ_1 and λ_2 are the eigen values of P



other representation: $\{x_c + Au \mid \|u\|_2 \le 1\}$ with A square and nonsingular


Symmetric $A^T = A$

eigen values are real S^n

n×n

positive semidefinite

$\forall x,\ x^TAx \geq 0$ S^n_+

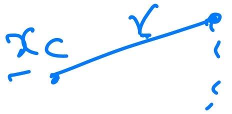
eigen values ≥ 0

S^n_{++}

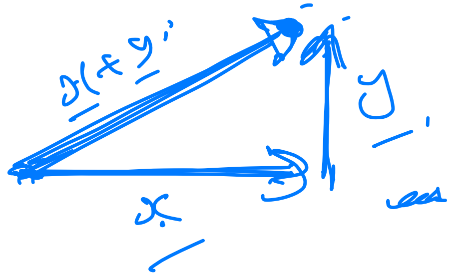
posite definite

eigen values >0

Norm balls and norm cones



norm: a function $\|\cdot\|$ that satisfies



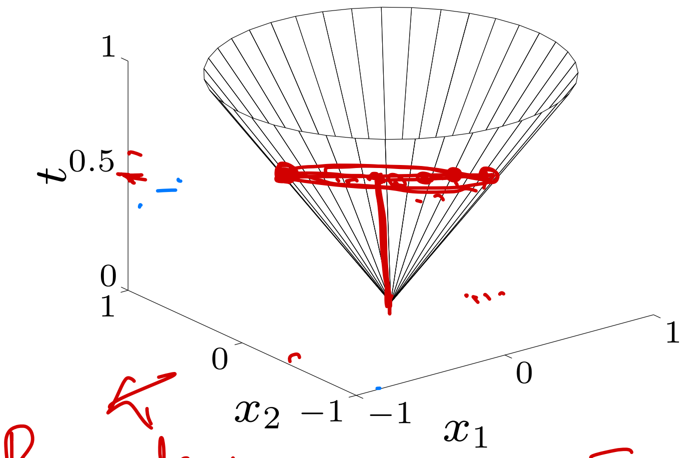
- $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- $\|tx\| = |t| \, \|x\|$ for $t \in \mathbf{R}$ homogenity.
- $\|x + y\| \leq \|x\| + \|y\|$ triangular inequality $\|x\|_2$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

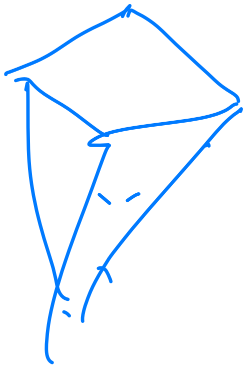
norm ball with center x_c and radius r: $\{x \mid \|x - x_c\| \leq r\}$

norm cone: $\{\underset{R^{n+1}}{(\overset{R^n}{x}, t)} \mid \|x\| \leq t\}$ $t \geq 0$

Euclidean norm cone is called second-order cone



Boundary of the Euclidean norm cone in R^2



norm balls and cones are convex

Norm balls are convex ?. ✓

$\underline{x_1}, \underline{x_2} \in \underline{B}(x_c, r) \Rightarrow \|x_1 - x_c\| \leq r$ — ①

$\|\underline{x_2} - x_c\| \leq r$ — ②

$\underline{y} = \theta x_1 + (1-\theta) x_2, \quad \theta \in [0,1]$

$+\theta x_c - \theta x_c$

$\underline{\|y - x_c\|} = \|\theta \underline{x_1} + (1-\theta) \underline{x_2} - x_c\| = \|\theta (x_1 - x_c) + (1-\theta)(x_2 - x_c)\|$

triangular ineq. $\leq \|\theta(x_1 - x_c)\| + \|(1-\theta)(x_2 - x_c\|$

homogenity. $\leq \theta \|x_1 - x_c\| + (1-\theta)\|x_2 - x_c\|$

$\leq r\theta + (1-\theta) r$

$= r\theta + r - r\theta$

$= r$

$\|y - x_c\| \leq r$ //

Unit Norm Circle in 2D

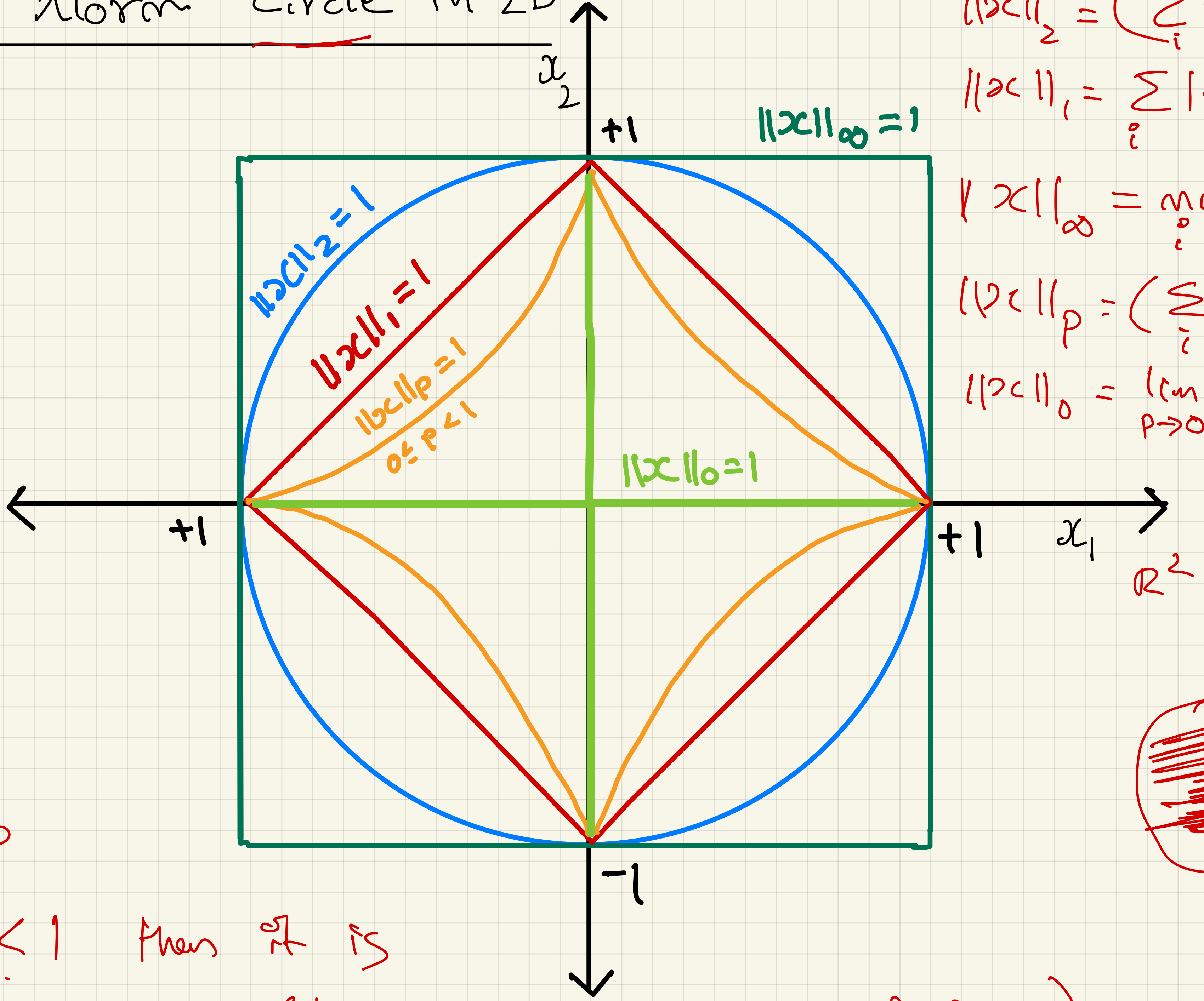
$$\|x\|_2 = \left(\sum_i x_i^2\right)^{1/2}$$

$$\|x\|_1 = \sum_i |x_i|$$

$$\|x\|_\infty = \max_i \{|x_i|\}$$

$$\|x\|_p = \left(\sum_i x_i^p\right)^{1/p}$$

$$\|x\|_0 = \lim_{p \to 0} \sum |x_i|^p$$



$\|x\|_p$

$P < 1$ then it is not a norm. (triang ineq is not satisfied.)

Norm Cone Convex ? ✓

$$B = \{(\underline{x}, t) \mid \|\underline{x}\| \le t\}$$

① ← $(\underline{x}_1, t_1) \in B \Rightarrow \|\underline{x}_1\| < t_1$

② ← $(\underline{x}_2, t_2) \in B \Rightarrow \|\underline{x}_2\| \le t_2$

$$(\underline{x}, t) = \theta(\underline{x}_1, t_1) + (1-\theta)(\underline{x}_2, t_2) \qquad \theta \in [0,1]$$

$$= \Big(\underbrace{\theta \underline{x}_1 + (1-\theta)\underline{x}_2}_{\underline{x}\ \uparrow}, + \underbrace{\theta t_1 + (1-\theta) t_2}_{t}\Big)$$

$$\|x\| = \|\theta \underline{x}_1 + (1-\theta)\underline{x}_2\|$$

$$\le \|\theta \underline{x}_1\| + \|(1-\theta)\underline{x}_2\| \quad \text{(triang. ineq.)}$$

$$\le \theta \|x_1\| + (1-\theta)\|x_2\| \quad \text{(homogenity)}$$

$$\le \theta t_1 + (1-\theta) t_2$$

$$= t \ /\!/$$

$$\underline{\underline{\|\underline{x}\| \le t}}$$

$2 \leq 3$

$\underline{a} \preceq \underline{b}$

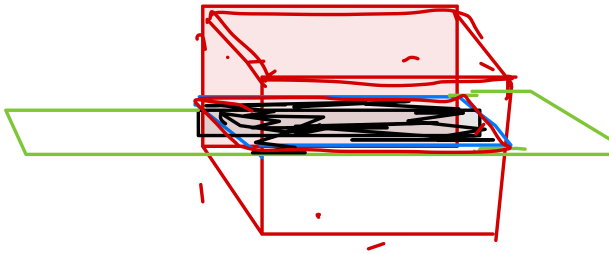
Polyhedra $= \{\underline{x}, \underline{Ax \leq b}, \underline{Cx = d}\}$

solution set of finitely many linear inequalities and equalities

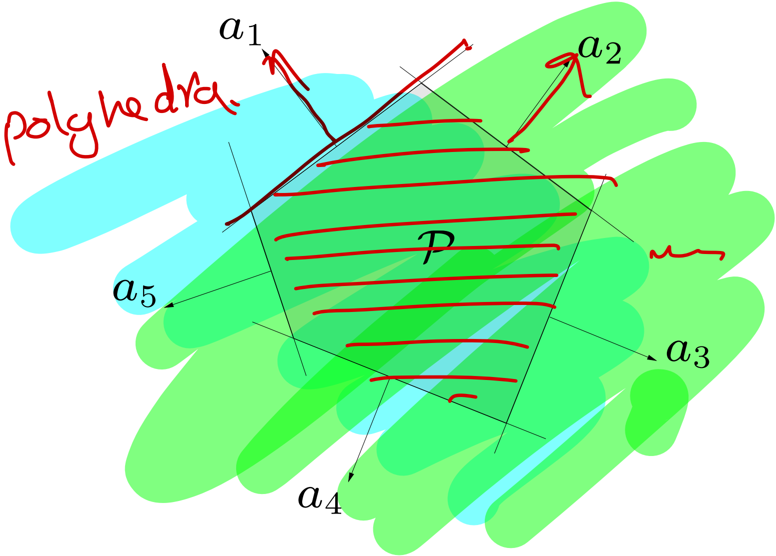
$$Ax \preceq b, \qquad Cx = d$$

componentwise inequality.

$(A \in \mathbf{R}^{m\times n}$, $C \in \mathbf{R}^{p\times n}$, $\preceq$ is componentwise inequality)



hyperplane,
lines
rays
line segment } → polyhedra.



polyhedron is intersection of finite number of halfspaces and hyperplanes

$a \preceq b$ → componentwise.

Positive semidefinite cone

$X \preceq Y$

$\Rightarrow Y - X \succeq 0$

$Y - X \in S_+^n$

notation:

- $\mathbf{S}^n$ is set of symmetric $n \times n$ matrices $A^T = A$
- $\mathbf{S}_+^n = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}_+^n \quad \Longleftrightarrow \quad z^T X z \geq 0 \text{ for all } z$$

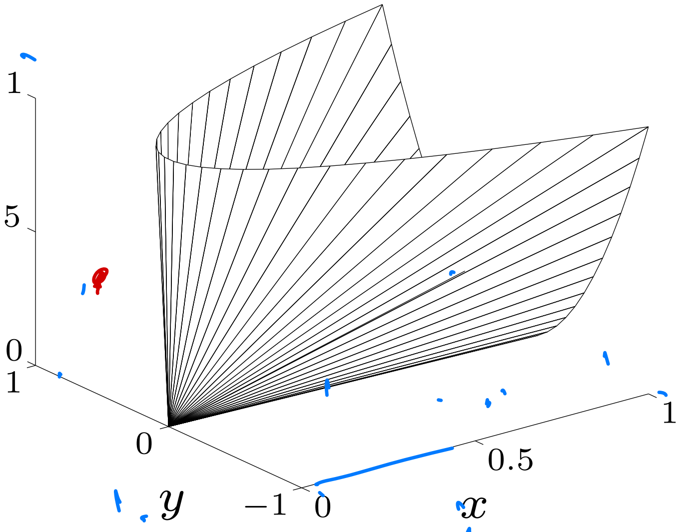
  $\mathbf{S}_+^n$ is a convex cone

- $\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2$

$n = 2$

$\dfrac{n(n+1)}{2} = \dfrac{2 \times 3}{2} = 3$



set of

Symmetric matrix (S_n) ✓

$X_1, X_2 \in S_n$

$\Downarrow \quad \{ X_1^T = X_1, \quad X_2^T = X_2 \}$

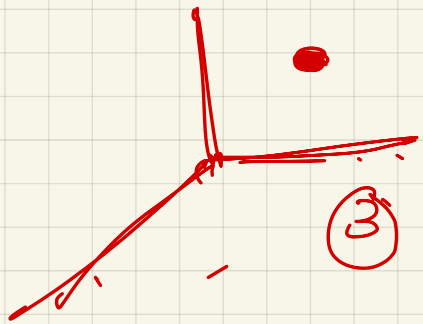
$Y = \theta X_1 + (1-\alpha) X_2 \qquad \theta \in [0,1]$

$Y^T = \theta X_1^T + (1-\alpha) X_2^T$

$= \theta X_1 + (1-\alpha) X_2$

$= Y$

$\boxed{Y^T = Y}$



$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_2 & x_5 & x_6 \end{bmatrix} \begin{matrix} \rightarrow 3 \\ \rightarrow 2 \\ \rightarrow 1 \end{matrix}$$

x_3 (not x_2)

$\Rightarrow$

$3 + 2 + 1.$

$n + (n-1) + (n-2) - \cdots - 1$

$\Rightarrow \dfrac{n(n+1)}{2}.$

S^n is a vector space with dimension $\boxed{\dfrac{n(n+1)}{2}}$

positive semidefinite matrix → convex cone.

$$S_n^+ = \{ X \in S^n \mid X \succeq 0 \}$$

$$X_1, X_2 \in S_n^+ \Rightarrow \underbrace{z^T X_1 z \geq 0}, \ \underbrace{z^T X_2 z \geq 0},$$

$$\theta_1 X_1 + \theta_2 X_2, \quad \boxed{\theta_1, \theta_2 \geq 0} \ \checkmark$$

$$z^T(\underbrace{\theta_1 X_1 + \theta_2 X_2}_{\in S_+^n}) z = \underbrace{\underbrace{\theta_1}_{\geq 0} \underbrace{z^T X_1 z}_{\geq 0} + \underbrace{\theta_2}_{\geq 0} \underbrace{z^T X_2 z}_{\geq 0}}_{\geq 0}$$

Plotting S_+^2 in MATLAB

```
close all
x=0:0.01:3;
z=0:0.01:3;

[x_axis z_axis]=meshgrid(x,z);
y=sqrt(x_axis.*z_axis);
surf(x_axis, z_axis, y)
hold on
surf(x_axis, z_axis, -y)
xlabel('x axis')
ylabel('z axis')
zlabel('y axis')
```

$$\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in S_+^2$$

$$\left\{ \begin{array}{l} x \geq 0 \\ z \geq 0 \\ xz \geq y^2. \end{array} \right.$$

(principle minors)

boundary $xz = y^2.$

$$\Rightarrow \quad y = \sqrt{xz}$$

$$y = -\sqrt{xz}.$$

Operations that preserve convexity

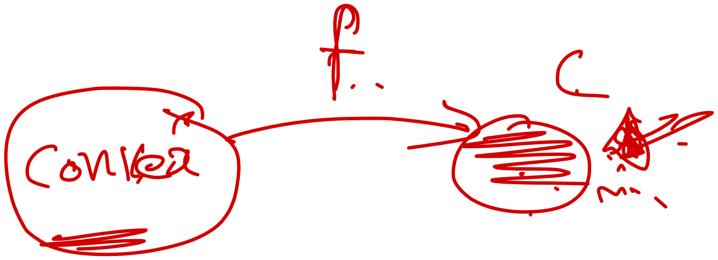
practical methods for establishing convexity of a set C

1. apply definition

$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \underline{\theta x_1 + (1-\theta) x_2 \in C}$$

2. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, . . .) by operations that preserve convexity

   - intersection
   - affine functions
   - perspective function
   - linear-fractional functions



intersection preserve convexity. (prove).

Intersection

$p(t) = \underline{a_t}^T \underline{x}$

$-1 \leq \underline{a_t}^T \underline{x} \leq 1$

the intersection of (any number of) convex sets is convex

$|\underline{a_t}^T \underline{x}| \leq 1$

example:

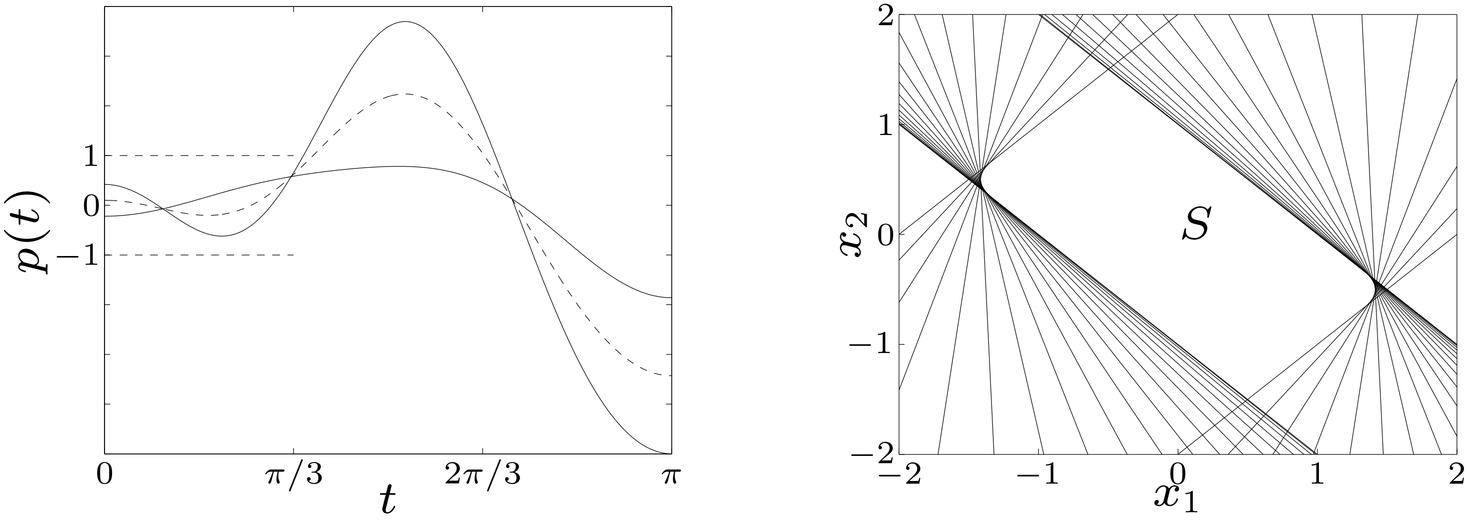
$$S = \{x \in \mathbf{R}^m \mid \underline{|p(t)|} \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \underline{\cos 2t} + \cdots + x_m \cos mt$

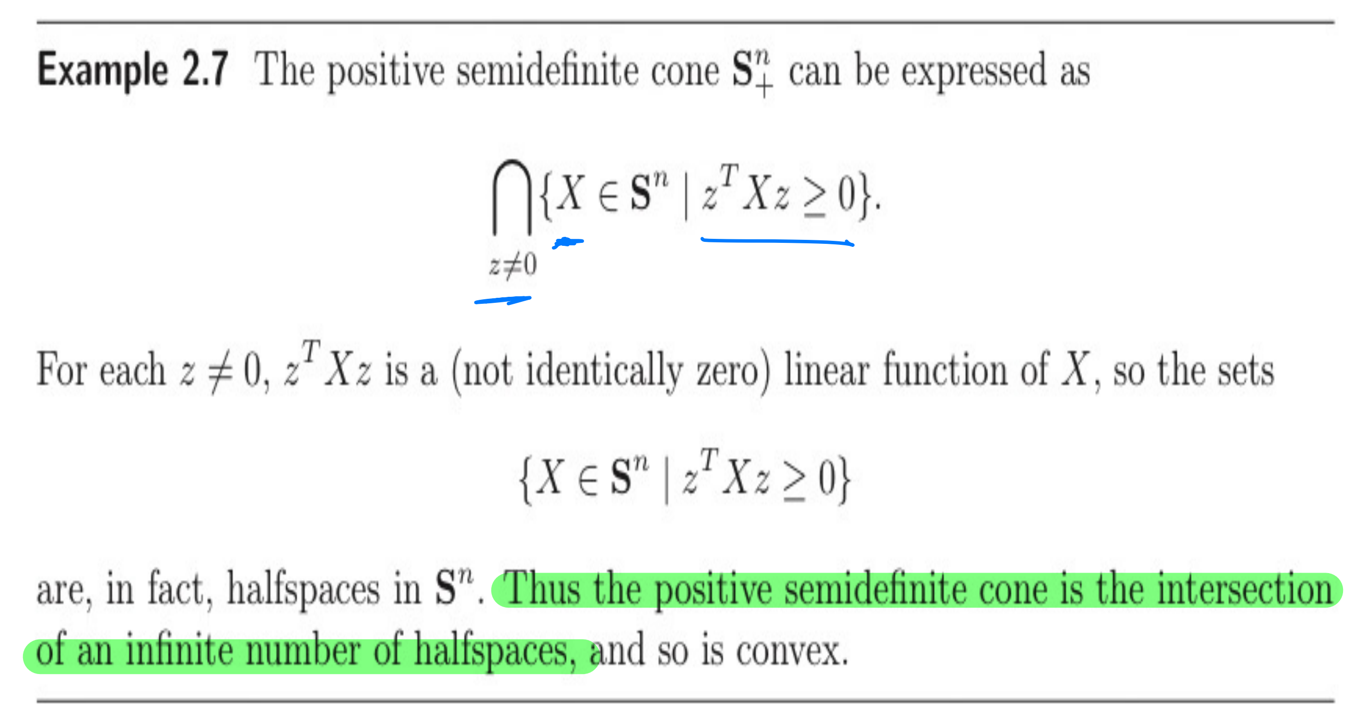
$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

$$\underline{a_t} = \begin{bmatrix} \cos t \\ \cos 2t \\ \vdots \\ \cos mt \end{bmatrix}$$

for $m = 2$:



$$X = \begin{bmatrix} x & z \\ z & y \end{bmatrix}$$



Example 2.7 The positive semidefinite cone $\mathbf{S}_+^n$ can be expressed as

$$\bigcap_{z \neq 0} \{X \in \mathbf{S}^n \mid \underline{z^T X z \geq 0}\}.$$

For each $z \neq 0$, $z^T X z$ is a (not identically zero) linear function of X, so the sets

$$\{X \in \mathbf{S}^n \mid z^T X z \geq 0\}$$

are, in fact, halfspaces in $\mathbf{S}^n$. Thus the positive semidefinite cone is the intersection of an infinite number of halfspaces, and so is convex.

$\underline{z}_1$

$$\underline{z}_1^T X \underline{z}_1 \geq 0$$

$\Downarrow$

half space

linear function $f(x) = Ax$

$\rightarrow f(0) = 0$

Affine function

suppose $f : \mathbf{R}^n \to \mathbf{R}^m$ is affine ($\underbrace{f(x) = Ax + b}_{\text{affine function}}$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$)

- the image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- the inverse image $f^{-1}(C)$ of a convex set under f is convex

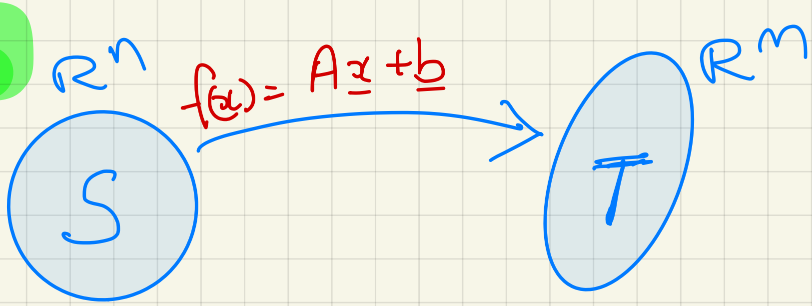
$$C \subseteq \mathbf{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$ (with $A_i, B \in \mathbf{S}^p$)
- hyperbolic cone $\{x \mid x^T P x \leq (c^T x)^2,\ c^T x \geq 0\}$ (with $P \in \mathbf{S}^n_+$)

Image of Convex set under $f(\underline{x}) = A\underline{x}+\underline{b}$ is Convex (proof)

Assume S is convex



$\underline{x}_1, \underline{x}_2 \in S$

$\Rightarrow \underline{y}_1 = A\underline{x}_1 + \underline{b} \in T$

$\underline{y}_2 = A\underline{x}_2 + \underline{b} \in T$

Also since since S is cvx

$\theta \underline{x}_1 + (1-\theta)\underline{x}_2 \in S$

for $\theta \in [0,1]$

T is convex ?

For $\theta \in [0,1]$

$$\theta \underline{y}_1 + (1-\theta)\underline{y}_2 = \theta(A\underline{x}_1 + b) + (1-\theta)(A\underline{x}_2 + b)$$

$$= \underbrace{A\underbrace{(\theta x_1 + (1-\theta)x_2)}_{\in S} + b}_{\in T}$$

$$\Downarrow$$

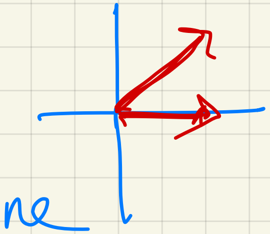
T is Convex

Scaling : $\alpha S = \{\alpha \underline{x} \mid \underline{x} \in S\}, \quad \alpha \in R$

translation : $\underline{a} + S = \{\underline{a} + \underline{x} \mid \underline{x} \in S\}, \quad a \in R^n.$

projection : $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1$

} Affine function.

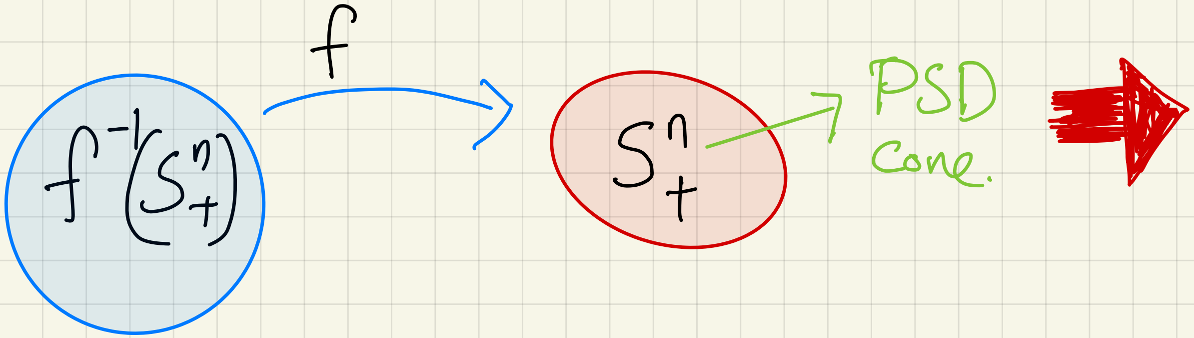


Solution set of Linear Matric Inequality (LMI)

$$A(x) = x_1 A_1 + x_2 A_2 + \dots \dots x_m A_m \preceq B$$

Solution set $\{\underline{x} \in R^n \mid A(x) \preceq B\}$

* Consider a function $f(x) = B - A(\underline{x})$



$$f^{-1}(S^n_+) = \{\underline{x} \in R^m \mid \underbrace{B - A(\underline{x})}_{} \in S^n_+\}$$

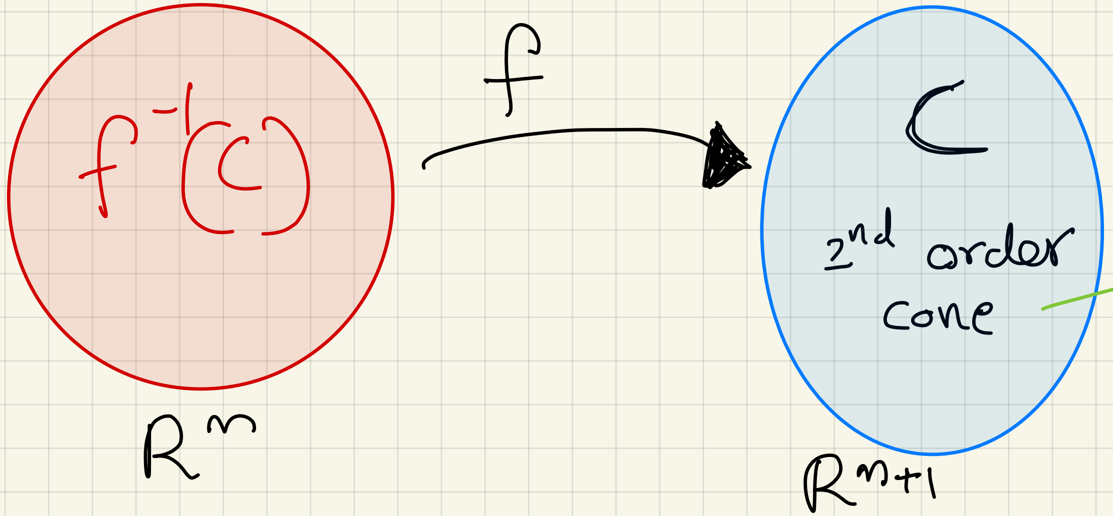
$$B - A(\underline{x}) \succeq 0$$

$$A(\underline{x}) \preceq B$$

Conclusion : Solution set of LMI is the inverse image of PSD cone under the affine function $f(x)$

Hyperbolic Cone

$\{ \underline{x} \mid \underline{x}^T P x \le (c^T x)^2, \; c^T \underline{x} \ge 0, P \in S_+^n \}$



$C = \{ (z,t) \mid z^T z \le t^2, \; t > 0 \}$

$$f(x) = \left(P^{1/2} \underline{x}, \; \underline{c}^T \underline{x} \right)$$

Hyperbolic Cone.

$$f^{-1}(C) = \{ \underline{x} \in R^n \mid \underbrace{(P^{1/2} \underline{x})^T (P^{1/2} \underline{x})}_{x^T P \underline{x}} \le (\underline{c}^T \underline{x})^2, \; \underline{c}^T \underline{x} > 0 \}$$

Conclusion: Hyperbolic cone is the inverse image of a 2nd order cone under affine mapping f(x)

Perspective and linear-fractional function

perspective function $P : \mathbf{R}^{n+1} \to \mathbf{R}^n$:

$$\underline{P}(\underbrace{\underline{x}, t}_{\mathbf{R}^{n+1}}) = \underbrace{x/t}_{\mathbf{R}^{n}}, \qquad \mathbf{dom}\, P = \{(x,t) \mid t > 0\}$$

images and inverse images of convex sets under perspective are convex

linear-fractional function $f : \mathbf{R}^n \to \mathbf{R}^m$:

$$f(x) = \frac{Ax + b}{c^T x + d}, \qquad \mathbf{dom}\, f = \{x \mid c^T x + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

Perspective function

$$f : \mathbb{R}^{n+1} \to \mathbb{R}^{n}$$

$$f(\underline{x}, t) = \underline{x}/t.$$

$$\text{dom } f = \{(\underline{x}, t) \mid t > 0\}$$

$$f\left(\begin{bmatrix} \overset{\underline{x}}{x_1} \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix}\right) = \begin{bmatrix} \overset{\underline{x}/x_{n+1}}{x_1/x_{n+1}} \\ x_2/x_{n+1} \\ \vdots \\ x_n/x_{n+1} \end{bmatrix}$$

$$\begin{tikzcd} \mathbb{R}^{n+1} \arrow[r] & \mathbb{R}^{n}. \\ S \arrow[r] & f(S) \\ (\underline{x}, t) \arrow[r] & \underline{x}/t. \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[r, "f"] & f(S) \\ (\mathbb{R}^{n+1}) & (\mathbb{R}^{n}) \end{tikzcd}$$

convex ⟷ Convex.

Proof (S is convex → f(S) is convex.)

Assume S is a convex set.

consider two points in S

$\underline{y}_1 = (\underline{x}_1, t_1) \in S$ | $\underline{y}_2 = (\underline{x}_2, t_2) \in S$

take the convex combination of these 2 points

$$\underline{y}_3 = \left(\overbrace{\theta \underline{x}_1 + (1-\theta)\underline{x}_2}^{\underline{x}}, \overbrace{\theta t_1 + (1-\theta) t_2}^{t}\right) ; \quad \theta \in [0, 1].$$

Since S is convex $\underline{y}_3 \in S$

Now

$$f(\underline{y}_1) = \frac{\underline{x}_1}{t_1} \in f(S)$$

$$f(\underline{y}_2) = \frac{\underline{x}_2}{t_2} \in f(S)$$

We have to check whether for $\alpha \in [0, 1]$

$$\alpha \frac{\underline{x}_1}{t_1} + (1-\alpha) \frac{\underline{x}_2}{t_2}$$ belongs to $f(S)$

$$\alpha \frac{\underline{x}_1}{t_1} + (1-\alpha) \frac{\underline{x}_2}{t_2} = \frac{\overbrace{\theta \underline{x}_1 + (1-\theta)\underline{x}_2}^{\underline{x}}}{\underbrace{\theta t_1 + (1-\theta) t_2}_{t}} = f(\underline{y}_3) \in f(S)$$ —— ①

① holds when $\theta = \dfrac{\alpha t_2}{(1-\alpha)t_1 + \alpha t_2}$ (verify) ——— ②

Note: Since $\alpha \in [0,1]$, $\theta \in [0,1]$ in ②

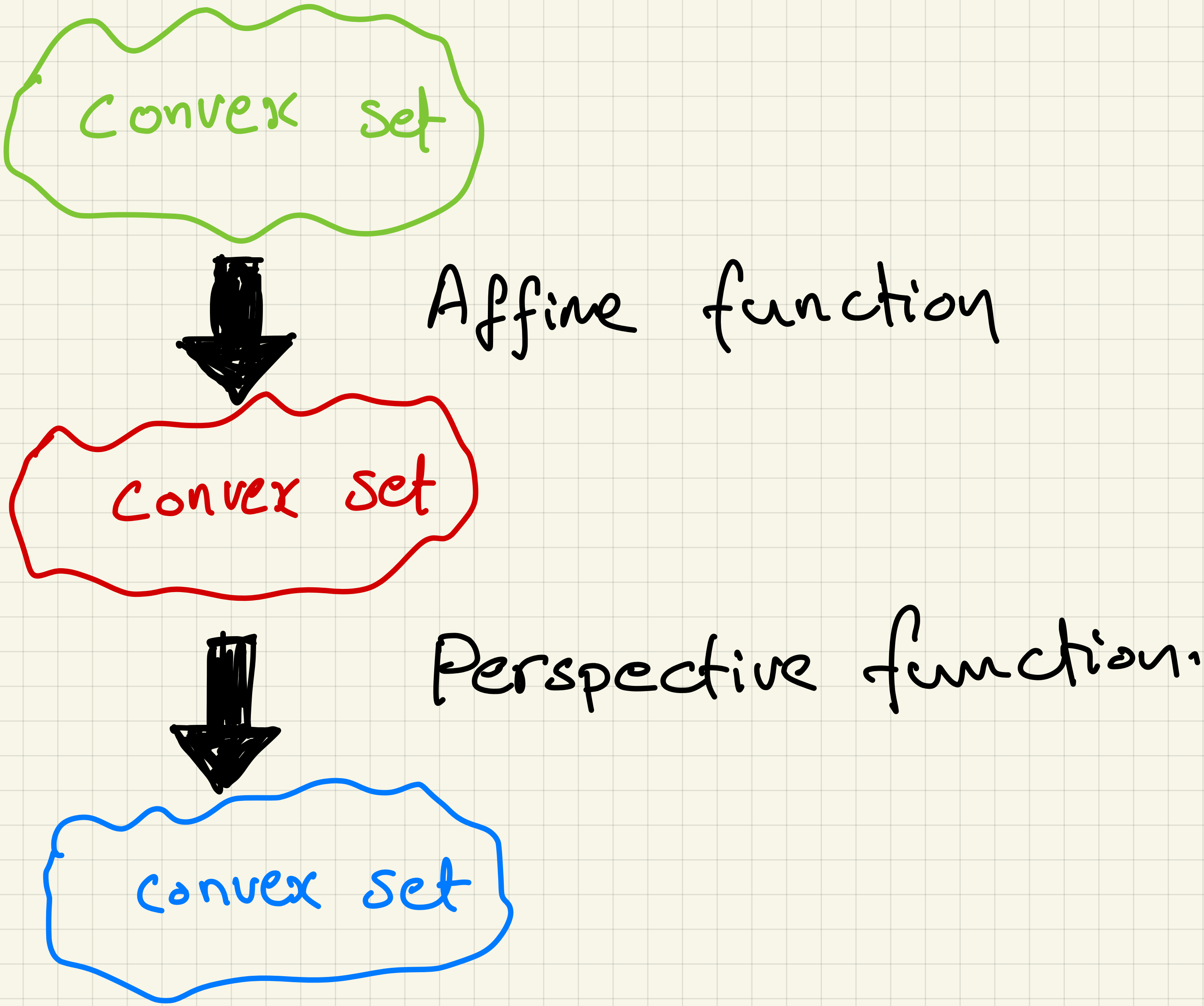
Linear fractional function → pespective transform of an affine function

$$f(\underline{x}) = \frac{A\underline{x}+\underline{b}}{\underline{c}^T\underline{x}+u}$$

$$= \text{perspective}\left(\overbrace{\begin{pmatrix} A \\ \underline{c}^T \end{pmatrix}\underline{x} + \begin{pmatrix} \underline{b} \\ u \end{pmatrix}}^{\text{Affine function}}\right)$$

$$= \text{perspective}\begin{pmatrix} A\underline{x}+\underline{b} \\ \underline{c}^T\underline{x}+u \end{pmatrix}\begin{matrix} \to \in \mathbb{R}^m \\ \to \in \mathbb{R} \end{matrix}$$

Linear fraction function preseves Convexity.



$\underline{\underline{\mathbb{R}^2}}$

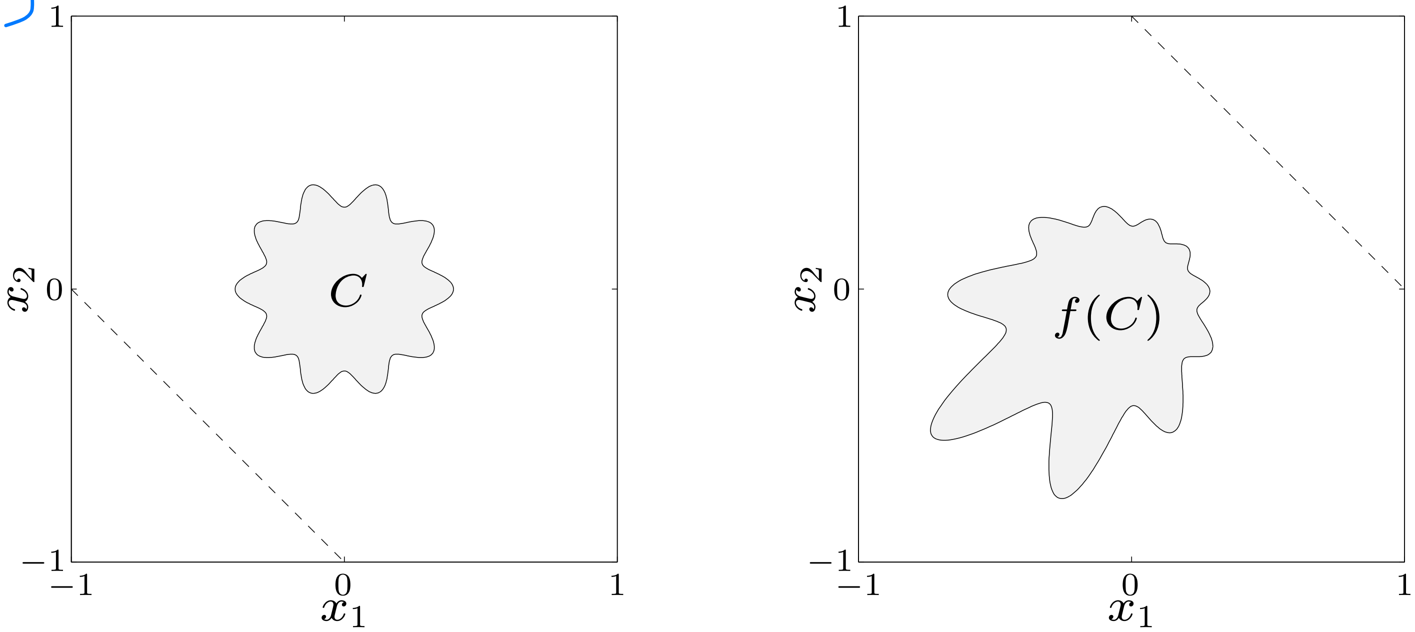
example of a linear-fractional function

I.

$$f(x) = \frac{1}{x_1 + x_2 + 1} x$$

$$\frac{\underline{A}\,\underline{x} + \underline{b}}{c^T x + d}$$

$$\frac{I\,\underline{x} + 0}{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1}$$



Generalized inequalities

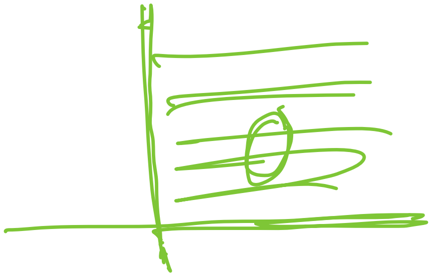
$2 < 3$

$\underline{x} \underline{\preceq} \underline{y}$

$X \underline{\preceq} Y$

a convex cone $K \subseteq \mathbf{R}^n$ is a **proper cone** if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

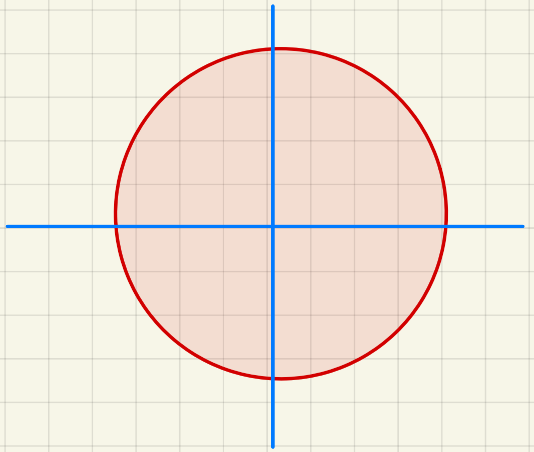


examples

- nonnegative orthant $K = \mathbf{R}^n_+ = \{x \in \mathbf{R}^n \mid x_i \geq 0, \ i = 1, \ldots, n\}$
- positive semidefinite cone $K = \mathbf{S}^n_+$
- nonnegative polynomials on $[0, 1]$:

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

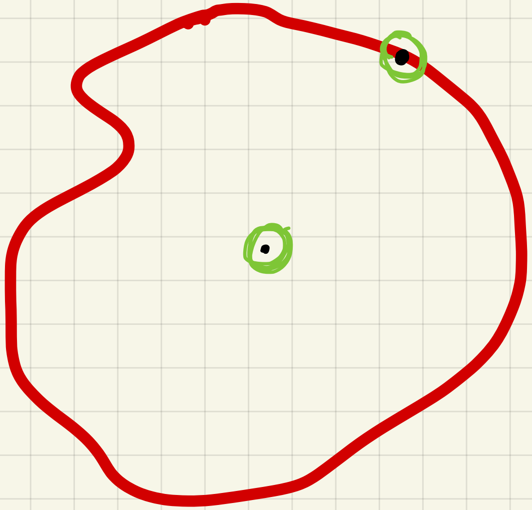
Closed sets : "Sets containing the boundary points"



$\{(x,y) \mid x^2+y^2 < 1\} \Rightarrow$ Open $\Leftarrow (-5, 8)$

$\{(x,y) \mid x^2+y^2 \leq 1\} \Rightarrow$ Closed $\Leftarrow [-5, 8]$

Open set (formal definition hint)

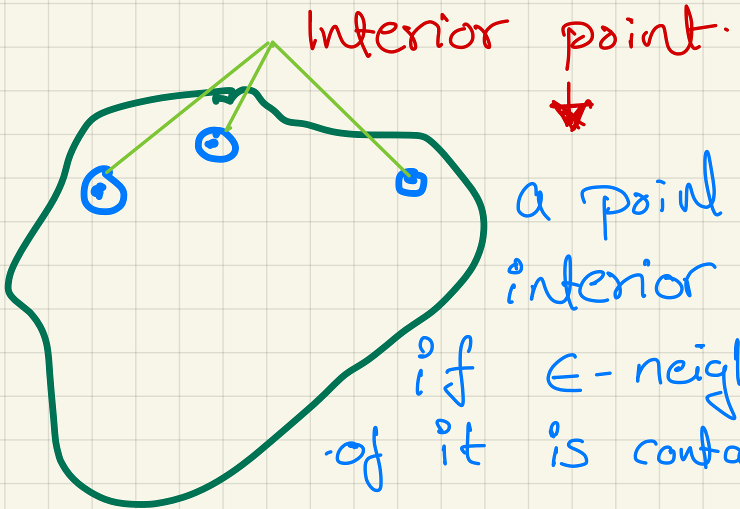


C is open set if for any $\underline{x} \in C$ there is an ϵ-neighbourhood fully contained in C

Closed set : A set whose compliment is an open set.

Solid Set

S is solid $\text{int}(S) \neq \phi$

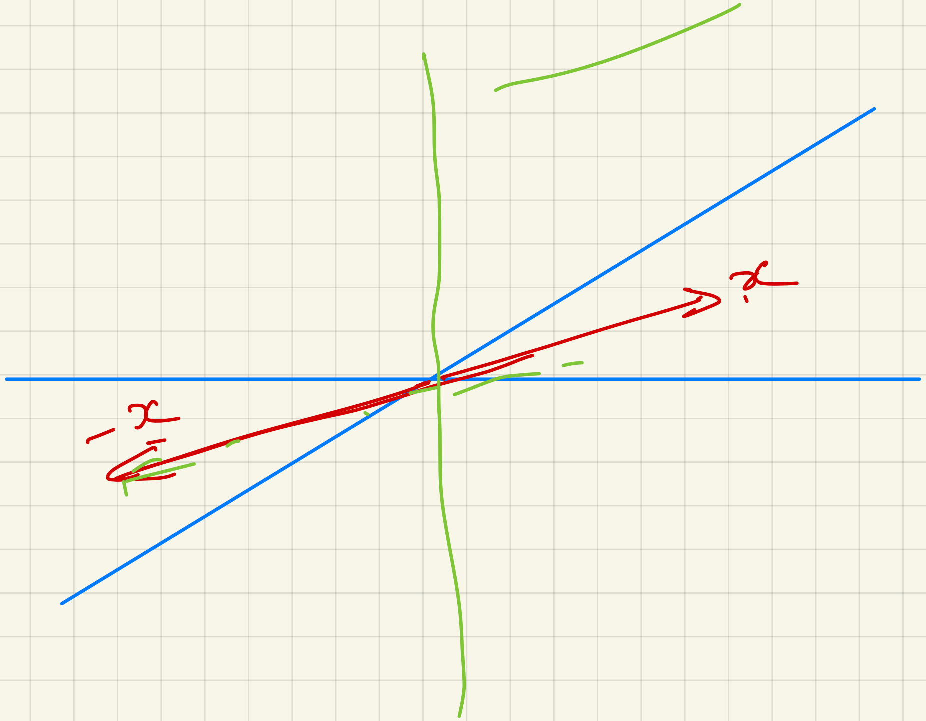


a point is an interior point if ε-neighbourhood of it is contained in the set

Pointed Set

$$\underline{x} \in S, \; -\underline{x} \in S \Rightarrow \underline{x} = 0$$

'S contains ∞ line'



generalized inequality defined by a proper cone K:

$$x \preceq_K y \quad \Longleftrightarrow \quad y - x \in K, \qquad\qquad x \prec_K y \quad \Longleftrightarrow \quad y - x \in \textbf{int}\, K$$

examples

- componentwise inequality ($K = \mathbf{R}^n_+$)

$$x \preceq_{\mathbf{R}^n_+} y \quad \Longleftrightarrow \quad x_i \leq y_i, \quad i = 1, \ldots, n$$

- matrix inequality ($K = \mathbf{S}^n_+$)

$$X \preceq_{\mathbf{S}^n_+} Y \quad \Longleftrightarrow \quad Y - X \text{ positive semidefinite}$$

these two types are so common that we drop the subscript in $\preceq_K$

properties: many properties of $\preceq_K$ are similar to $\leq$ on $\mathbf{R}$, *e.g.*,

$$x \preceq_K y, \quad u \preceq_K v \quad \Longrightarrow \quad x + u \preceq_K y + v$$

Read properties of Gen. Ineq. from text book.

Generalized Inequality (w.r.t. a proper cone K)

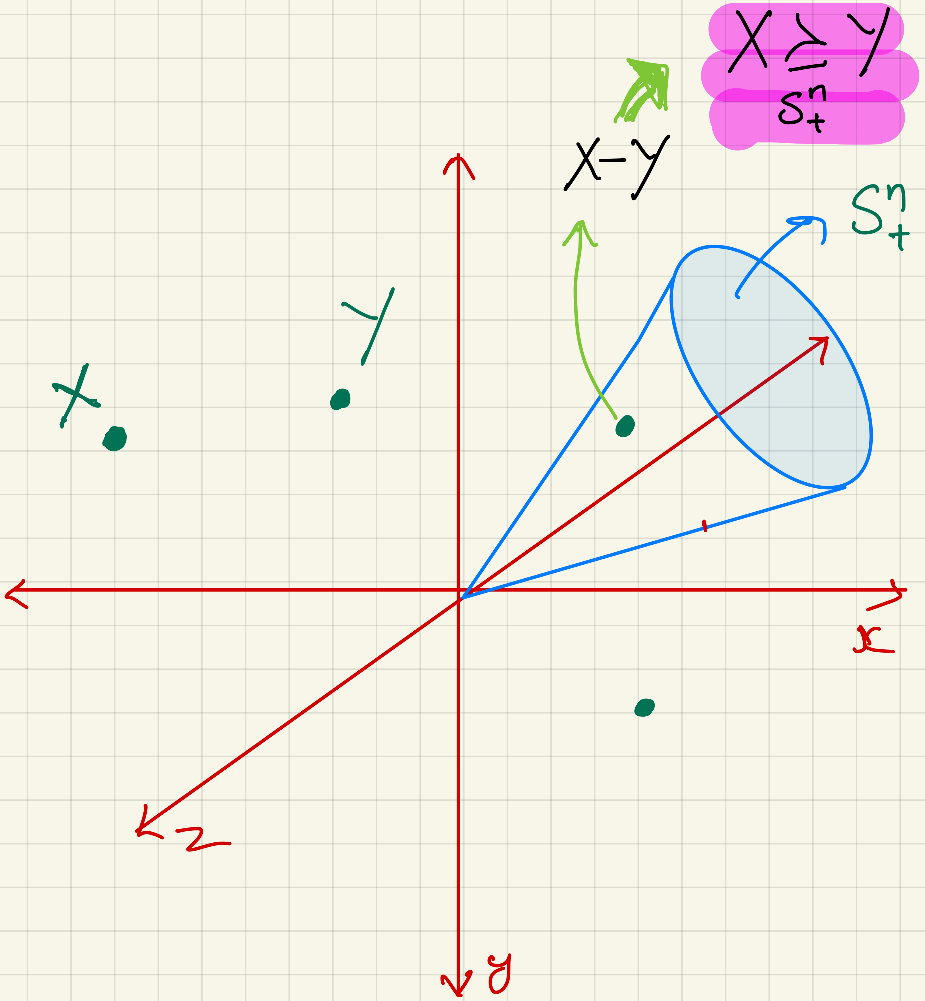
2 Commonly used generalized Inequalities

$K = \mathbb{R}^n_+$ $\swarrow$ $\searrow$ $K = S^n_+$

vectors. Point wise inequality	Matrix inequality (PSD Inequality)
$K = \mathbb{R}^n_+$	$K = S^n_+$
$\underline{x} \preceq_{\mathbb{R}^n_+} \underline{y} \iff \underline{y}-\underline{x} \in \mathbb{R}^n_+$	$X \succeq_{S^n_+} Y \Rightarrow X-Y \in S^n_+$
$\Rightarrow y_i \geq x_i \ \forall i$	$\Rightarrow X-Y$ is PSD

$$x = \begin{bmatrix} -1 \\ 5 \end{bmatrix} \qquad y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$x - y = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \notin \mathbb{R}^2_+$$



Generalized inequality is not a linear order'

$1 \leq 3 \leq 7$

$\underline{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\qquad$ $\cancel{\underline{x} \preceq_{\mathbb{R}^2_+} \underline{y}}$ $\qquad$ $\underline{y} - \underline{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\underline{y} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\qquad$ $\cancel{\underline{y} \preceq_{\mathbb{R}^2_+} \underline{x}}$ $\qquad$ $\underline{x} - \underline{y} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\left.\begin{array}{ll} X \preceq_K Y & \text{False} \\ Y \preceq_K X & \text{False.} \end{array}\right\} \text{This can happen.}$$

$\mathbb{R}^n \to \mathbb{R}^q$

Minimum and minimal elements

$\preceq_K$ is not in general a *linear ordering*: we can have $x \not\preceq_K y$ and $y \not\preceq_K x$

$x \in S$ is **the minimum element** of S with respect to $\preceq_K$ if

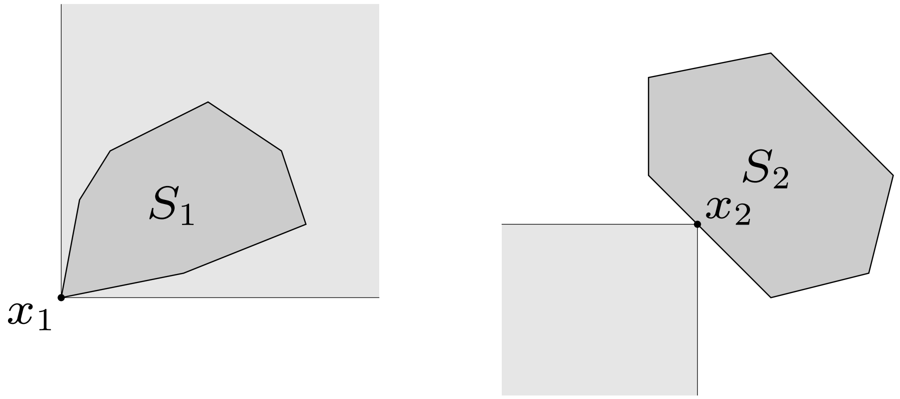
$$y \in S \quad \Longrightarrow \quad x \preceq_K y$$

$x \in S$ is **a minimal element** of S with respect to $\preceq_K$ if

$$y \in S, \quad y \preceq_K x \quad \Longrightarrow \quad y = x$$

example $(K = \mathbf{R}^2_+)$

x_1 is the minimum element of S_1
x_2 is a minimal element of S_2

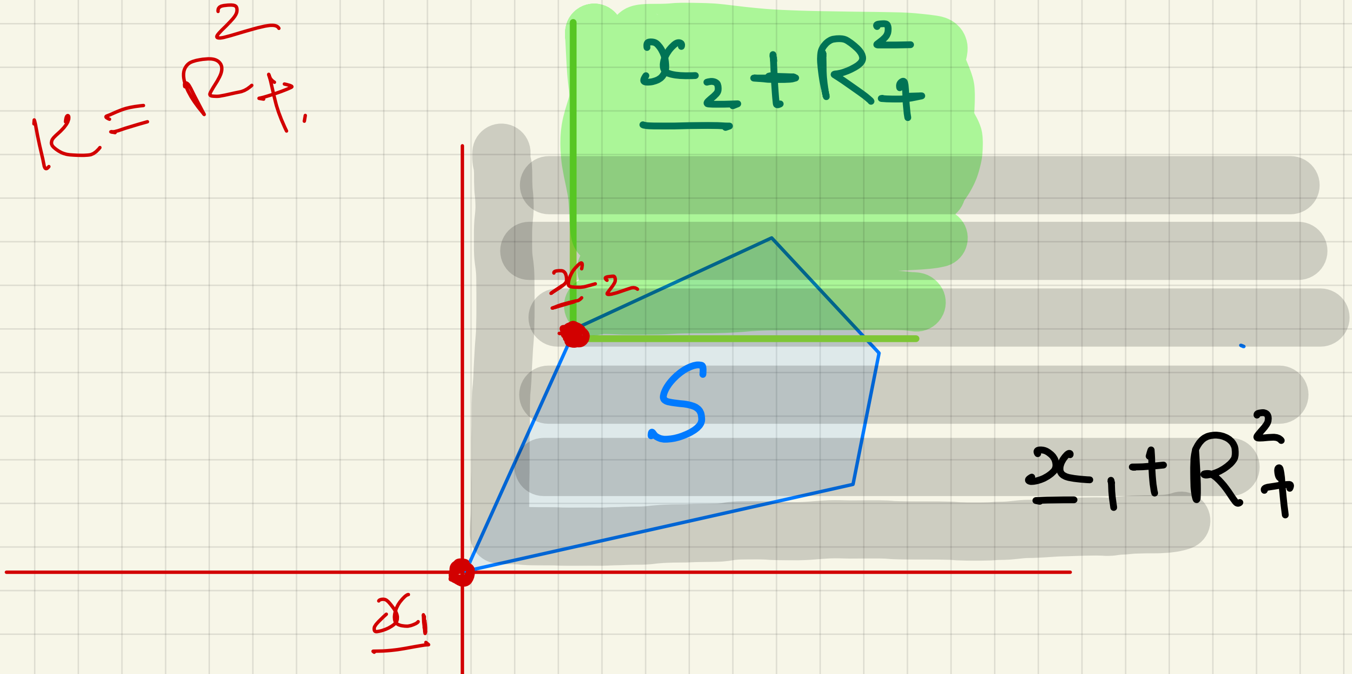


$\boxed{\text{Minimum}}$ → Unique

- Set S
- $\underline{x}$ is a minimum element of S w.r.t $\preceq_K$ iff

$$\boxed{S \subseteq x + K.}$$

$K = R^2_+$.

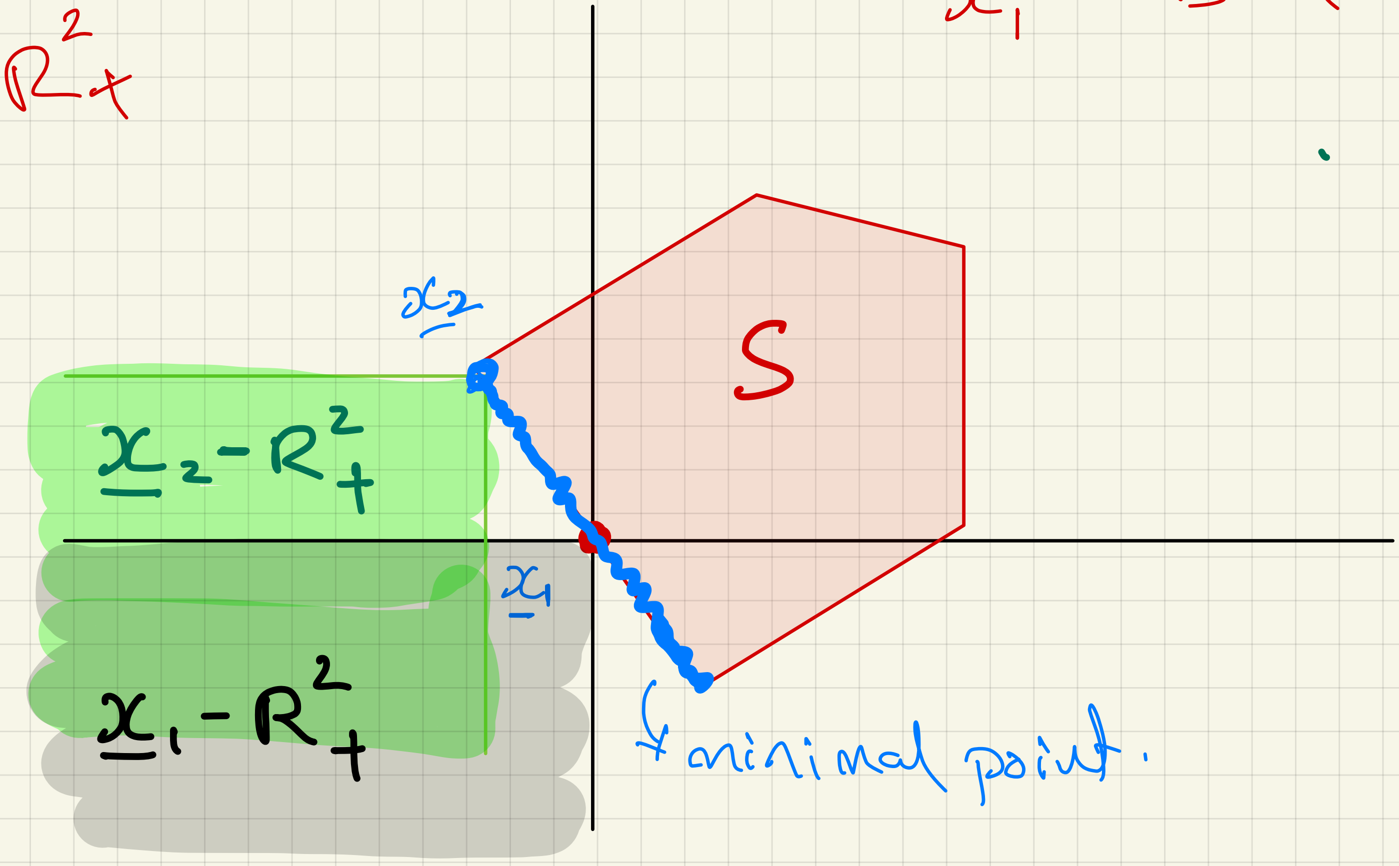


$\boxed{\text{Minimal}}$ → Not Unique

$\underline{x}$ is a minimal element of S $\overset{\text{w.r.t. } \preceq_K}{\wedge}$ iff $\boxed{x - K \cap S = \{x\}}$

x_1 is a minimal element.

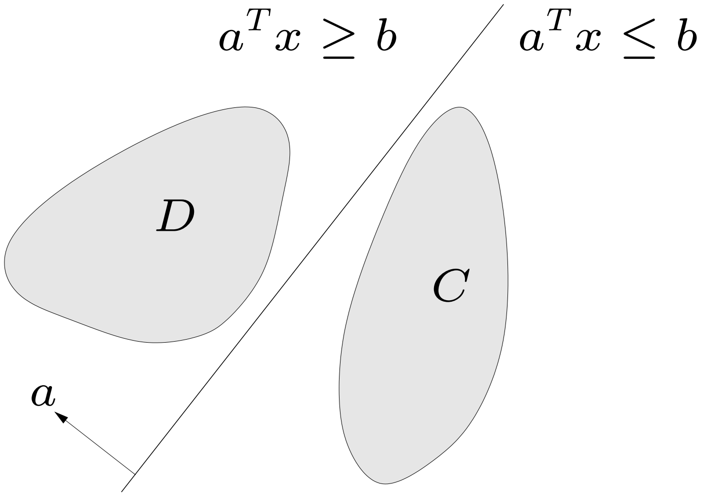
$K = \mathbb{R}^2_+$

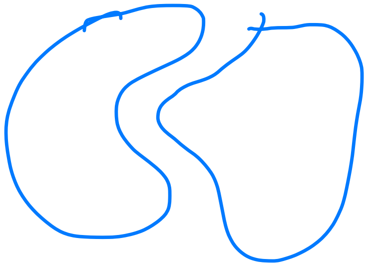


Separating hyperplane theorem

if C and D are nonempty disjoint convex sets, there exist $a \neq 0$, b s.t.

$$a^T x \leq b \text{ for } x \in C, \qquad a^T x \geq b \text{ for } x \in D$$





the hyperplane $\{x \mid a^T x = b\}$ separates C and D

strict separation requires additional assumptions ($e.g.$, C is closed, D is a singleton)

Supporting hyperplane theorem

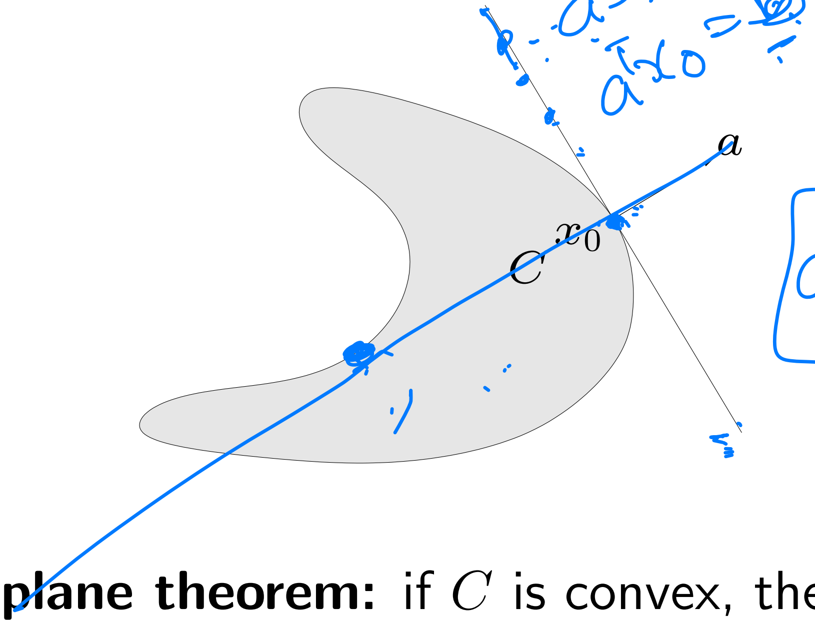
supporting hyperplane to set C at boundary point x_0:

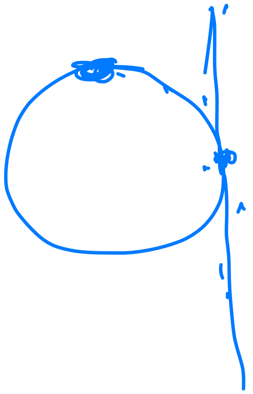
$$\{x \mid a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$

$a^Tx = b$

$a^Tx_0 = b$





$$\boxed{a^Tx = a^Tx_0}$$

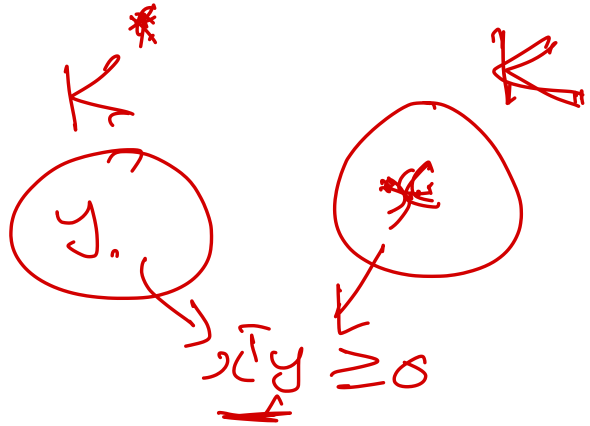
supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

Dual cones and generalized inequalities

need not be convex

dual cone of a cone K:

$$K^* = \{y \mid y^Tx \geq 0 \text{ for all } x \in K\}$$



examples

- $K = \mathbf{R}^n_+$: $K^* = \mathbf{R}^n_+$
- $K = \mathbf{S}^n_+$: $K^* = \mathbf{S}^n_+$
- $K = \{(x,t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_2 \leq t\}$
- $K = \{(x,t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x,t) \mid \|x\|_\infty \leq t\}$

$$\underbrace{[-1, 2]}_{y}\underbrace{\begin{bmatrix}1\\3\end{bmatrix}}_{x} = +5 \geq 0$$

first three examples are **self-dual** cones

Dual of a norm.

dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \iff y^Tx \geq 0 \text{ for all } x \succeq_K 0$$

$$[-1, 2]\begin{bmatrix}1\\0\end{bmatrix} = -1$$

x_2

Dual cone of S^n_+ is S^n_+

Proof: ① $Y \notin S^n_+ \Rightarrow Y \notin (S^n_+)^*$

② $Y \in S^n_+ \Rightarrow Y \in (S^n_+)^*$

$\underline{x^T y}$

$tr(\underline{X}^T \underline{Y})$..

$X(:)^T Y(:)$

① Assume $Y \notin S^n_+$ ⇒ there exists $q \in \mathbb{R}^n$ such that

$$q^T Y q \le 0$$

$$tr(q^T Y q) \le 0$$

$$tr(q q^T Y) \le 0$$

(cyclic property of trace)

we know that

$$X = qq^T \in S^n_+$$

$$tr(X^T Y) \le 0$$

$$\Rightarrow Y \notin (S^n_+)^*$$

② Assume $Y \in S_+^n$

Let $X \in S_+^n$

eigen values

eigen vector .

$$X = \sum_{i=1}^{N} \lambda_i q_i q^T \quad \text{(eigen value decomposition)}$$

$$tr(YX) = tr\left(Y \sum_{i=1}^{N} \lambda_i q_i q_i^T\right)$$

$$= \sum_{i=1}^{N} \lambda_i \, tr(Y q_i q_i^T) \quad \text{(trace is linear)}$$

(since $X \in S_+^n$)

$$= \sum_{i=1}^{N} \lambda_i \, tr(q_i^T Y q_i) \quad \text{(cyclic property)}$$

(since $Y \in S_+^n$)

$$= \sum_{i=1}^{N} \underbrace{\lambda_i}_{\geq 0} \underbrace{q_i^T Y q_i}_{\geq 0} \geq 0 \quad \text{(dropping trace for the scalar } q_i^T Y q_i\text{)}$$

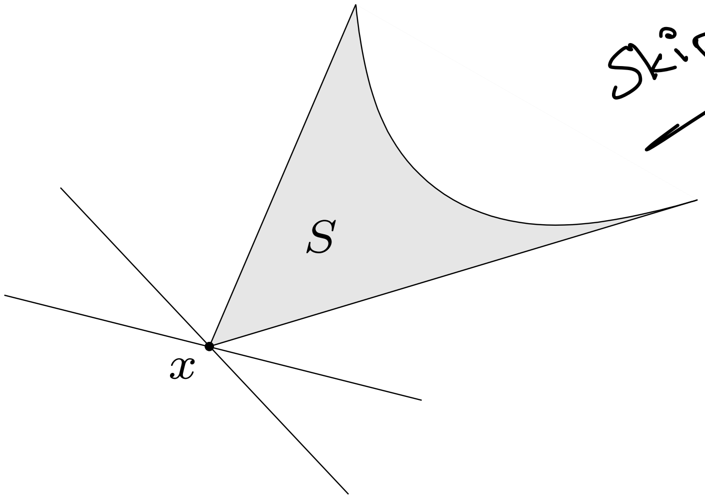
$$\Rightarrow Y \in (S_+^n)^*$$

Minimum and minimal elements via dual inequalities

Skipping

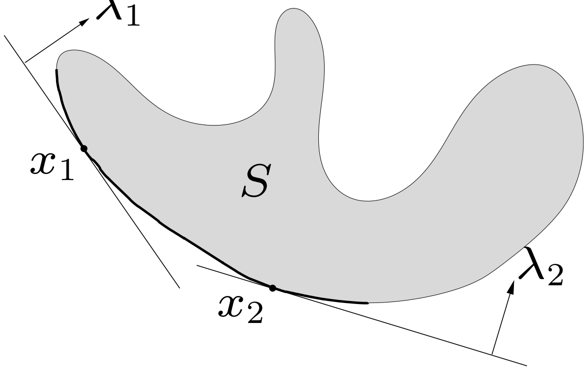
minimum element w.r.t. $\preceq_K$

x is minimum element of S iff for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over S



minimal element w.r.t. $\preceq_K$

- if x minimizes $\lambda^T z$ over S for some $\lambda \succ_{K^*} 0$, then x is minimal



- if x is a minimal element of a *convex* set S, then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that x minimizes $\lambda^T z$ over S

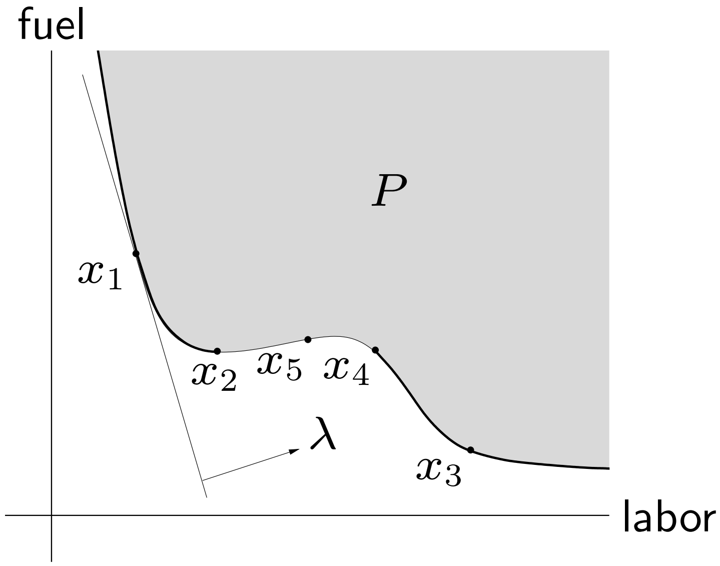
optimal production frontier

Skipping

- different production methods use different amounts of resources $x \in \mathbf{R}^n$
- production set P: resource vectors x for all possible production methods
- efficient (Pareto optimal) methods correspond to resource vectors x that are minimal w.r.t. $\mathbf{R}^n_+$

example $(n = 2)$

x_1, x_2, x_3 are efficient; x_4, x_5 are not



optimal production frontier

- different production methods use different amounts of resources $x \in \mathbf{R}^n$
- production set P: resource vectors x for all possible production methods
- efficient (Pareto optimal) methods correspond to resource vectors x that are minimal w.r.t. $\mathbf{R}^n_+$

example $(n = 2)$

x_1, x_2, x_3 are efficient; x_4, x_5 are not