Lecture 5:
Duality and convex solvers

Optimization for data sciences

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Course organization

Course organization

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- Introduction to optimization
    - A few problems of interest
    - Quick mathematical refresher

- Convex problems (Following Stephen Boyd)
    - Quick refresher on last week
    - Convex sets
    - Convex functions
    - Convex problems
    - Simplex algorithm for Linear Programming

Course organization

- **Duality (for convex problems)**
  - **Lagrangian and dual function**
  - **Dual problem**
  - **Qualification constraints**
  - **KKT conditions**

- **Newton's Descent and Barrier methods for convex case**
  - **Descent for the unconstrained problems**
  - **Equality constrained problems**
  - **Interior point methods**

Course organization

- (First order) descent methods for the general case
- Backpropagation
- Some more properties on stochastic gradient descent

Evaluation

- Reports on lab sessions
  - Labs on jupyter notebooks
    - Not every session
  - Explain the code done in the session
  - Summarize what is done in the practical

- Written Exam
  - Theoretical questions
  - We will do exercises in class

Refresher on last weeks

Convex problem

convex optimization problem:

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

- variable $x \in \mathbf{R}^n$
- equality constraints are linear
- $f_0, \dots, f_m$ are **convex**: for $\theta \in [0, 1]$,

$$f_i(\theta x + (1-\theta)y) \leq \theta f_i(x) + (1-\theta) f_i(y)$$

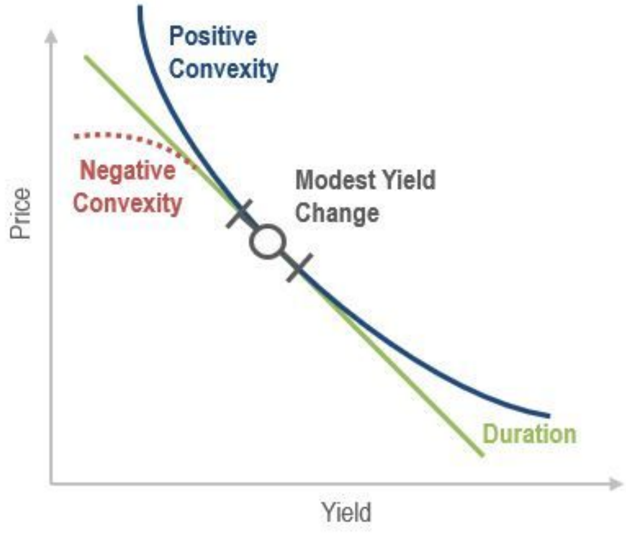
i.e., f_i have nonnegative (upward) curvature

Easy problem

- classical view:
  - linear (zero curvature) is easy
  - nonlinear (nonzero curvature) is hard

- the classical view is **wrong**

- the correct view:
  - convex (nonnegative curvature) is easy
  - nonconvex (negative curvature) is hard



Easy to solve!

- many different algorithms (that run on many platforms)
  - interior-point methods for up to 10000s of variables
  - first-order methods for larger problems
  - do not require initial point, babysitting, or tuning
- can develop and deploy quickly using modeling languages such as CVXPY
- solvers are reliable, so can be embedded
- code generation yields real-time solvers that execute in milliseconds (*e.g.*, on Falcon 9 and Heavy for landing)

Convex sets overview

- **Convex sets (definition!)**
  - **Affine sets, norm balls, norm cones**
  - **Convex combination, convex hull and Convex cones**
  - **Hyperplanes, halfspaces and polyhedron**
  - Positive Semidefinite Cone
- **Showing a set is convex (with operations!)**
  - **Intersection**
  - *Affine mapping*
  - *Perspective and Linear fractional mappings*
- Proper cones and generalized inequalities
- Separating and supporting hyperplanes

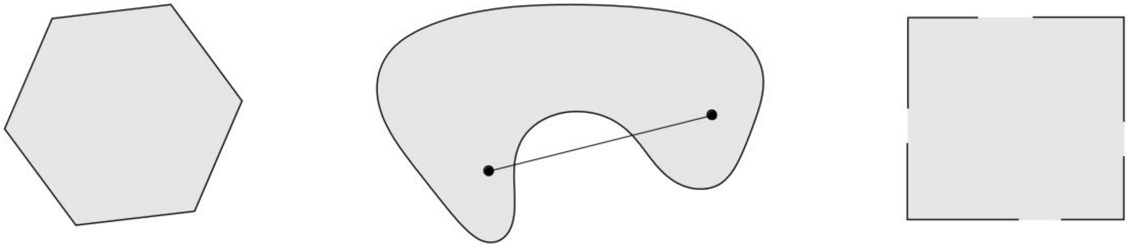
Convex sets

line segment between x_1 and x_2: all points of form $x = \theta x_1 + (1-\theta)x_2$, with $0 \le \theta \le 1$

convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad 0 \le \theta \le 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

examples (one convex, two nonconvex sets)



Showing a set is convex

methods for establishing convexity of a set C

1. apply definition: show $x_1, x_2 \in C, 0 \leq \theta \leq 1 \implies \theta x_1 + (1-\theta)x_2 \in C$
   - recommended only for **very simple** sets

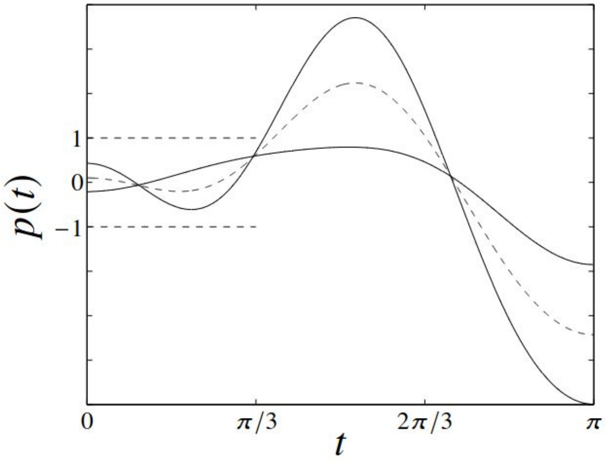
2. use convex functions (next lecture)

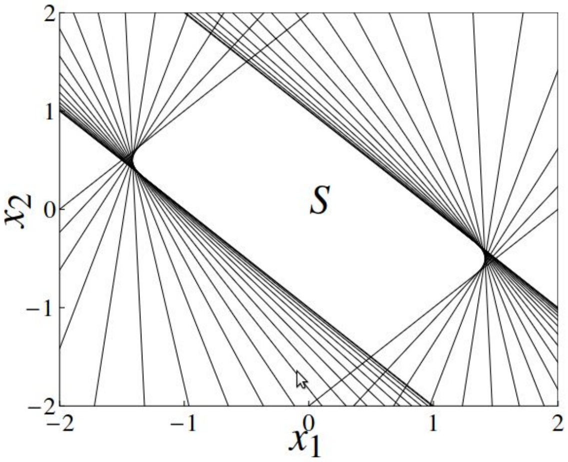
3. show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity
   - intersection
   - affine mapping
   - perspective mapping
   - linear-fractional mapping

you'll mostly use methods 2 and 3

Showing a set is convex

- the intersection of (any number of) convex sets is convex
- **example:**
  - $S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$, with $p(t) = x_1 \cos t + \cdots + x_m \cos mt$
  - write $S = \bigcap_{|t| \leq \pi/3} \{x \mid |p(t)| \leq 1\}$, *i.e.*, an intersection of (convex) slabs
- picture for $m = 2$:





Takeaway

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta x_1 + (1-\theta)x_2 \in C$$

- Classic convex sets
  - Affine sets, hyperplanes, cones, balls, polyhedrons

- Convexity preserving operations
  - Intersection
  - Affine mapping
  - Perspective
  - Linear Fractional mapping

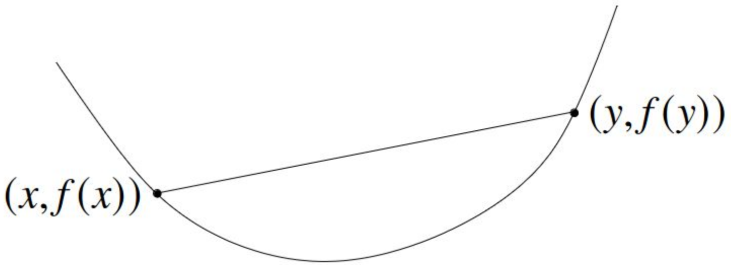
Convex function overview

- **Convex functions (with definition!)**
  - **Examples of classic convex functions**
  - *Extended value function*
  - *Line restriction*
  - **First and second order conditions**
  - *Epigraph and sublevel sets*
- **Showing a function is convex with operations**
  - **Non-negative weighted sum and affine composition**
  - **Pointwise maximum**
  - **Composition rules**
  - *Partial minimization and perspective*
- *Conjugate function*

Convex functions

- $f : \mathbf{R}^n \to \mathbf{R}$ is convex if $\mathbf{dom}\, f$ is a convex set and for all $x, y \in \mathbf{dom}\, f$, $0 \le \theta \le 1$,

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y)$$



- f is concave if $-f$ is convex
- f is strictly convex if $\mathbf{dom}\, f$ is convex and for $x, y \in \mathbf{dom}\, f$, $x \ne y$, $0 < \theta < 1$,

$$f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta)f(y)$$

Showing a function is convex

methods for establishing convexity of a function f

1. verify definition (often simplified by restricting to a line)
2. for twice differentiable functions, show $\nabla^2 f(x) \succeq 0$
   - recommended only for **very simple** functions
3. show that f is obtained from simple convex functions by operations that preserve convexity
   - nonnegative weighted sum
   - composition with affine function
   - pointwise maximum and supremum
   - composition
   - minimization
   - perspective

you'll mostly use methods 2 and 3

Showing a function is convex

- **nonnegative multiple:** αf is convex if f is convex, $\alpha \geq 0$
- **sum:** $f_1 + f_2$ convex if f_1, f_2 convex
- **infinite sum:** if $f_1, f_2, \ldots$ are convex functions, infinite sum $\sum_{i=1}^{\infty} f_i$ is convex
- **integral:** if $f(x, \alpha)$ is convex in x for each $\alpha \in \mathcal{A}$, then $\displaystyle\int_{\alpha \in \mathcal{A}} f(x, \alpha)\, d\alpha$ is convex

- there are analogous rules for concave functions

Showing a function is convex

(pre-)composition with affine function: $f(Ax+b)$ is convex if f is convex

examples

- log barrier for linear inequalities

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \qquad \mathbf{dom}\, f = \{x \mid a_i^T x < b_i, i = 1, \dots, m\}$$

- norm approximation error: $f(x) = ||Ax - b||$ (any norm)

Showing a function is convex

if $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex

examples

- piecewise-linear function: $f(x) = \max_{i=1,\ldots,m}(a_i^T x + b_i)$
- sum of r largest components of $x \in \mathbf{R}^n$:

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

($x_{[i]}$ is ith largest component of x)

proof: $f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$

Showing a function is convex

- composition of $g : \mathbf{R}^n \to \mathbf{R}$ and $h : \mathbf{R} \to \mathbf{R}$ is $f(x) = h(g(x))$ (written as $f = h \circ g$)
- composition f is convex if
  - g convex, h convex, $\tilde{h}$ nondecreasing
  - or g concave, h convex, $\tilde{h}$ nonincreasing

  (monotonicity must hold for extended-value extension $\tilde{h}$)
- proof (for $n = 1$, differentiable g, h)

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

examples

- $f(x) = \exp g(x)$ is convex if g is convex
- $f(x) = 1/g(x)$ is convex if g is concave and positive

Takeaway

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$

- Classic convex functions
  - Affine, exponential, norms, max, …
- Convexity preserving operations
  - Non negative weighted sum, composition with affine
  - Pointwise maximum and supremum
  - Composition
  - Minimization
  - Perspective

Standard form optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array}$$

- $x \in \mathbf{R}^n$ is the optimization variable
- $f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
- $f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \dots, m$, are the inequality constraint functions
- $h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

Feasible and optimal points

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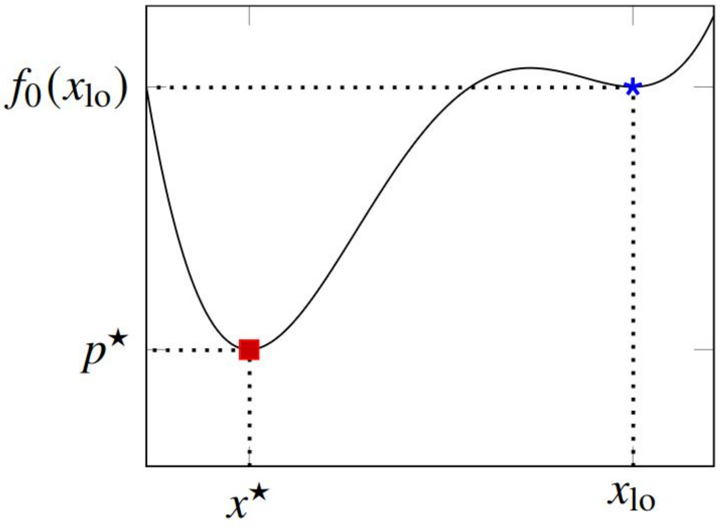
- $x \in \mathbf{R}^n$ is **feasible** if $x \in \mathbf{dom}\, f_0$ and it satisfies the constraints
- **optimal value** is $p^\star = \inf\{f_0(x) \mid f_i(x) \le 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, p\}$
- $p^\star = \infty$ if problem is infeasible
- $p^\star = -\infty$ if problem is **unbounded below**
- a feasible x is **optimal** if $f_0(x) = p^\star$
- X_{opt} is the set of optimal points

Optimal points

x is **locally optimal** if there is an $R > 0$ such that x is optimal for

$$\begin{array}{ll} \text{minimize (over } z) & f_0(z) \\ \text{subject to} & f_i(z) \leq 0, \quad i = 1, \ldots, m, \quad h_i(z) = 0, \quad i = 1, \ldots, p \\ & \|z - x\|_2 \leq R \end{array}$$



Standard form convex problem

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& a_i^T x = b_i, \quad i = 1, \ldots, p
\end{array}
$$

- objective and inequality constraints $f_0, f_1, \ldots, f_m$ are convex
- equality constraints are affine, often written as $Ax = b$
- feasible and optimal sets of a convex optimization problem are convex

- problem is **quasiconvex** if f_0 is quasiconvex, $f_1, \ldots, f_m$ are convex, $h_1, \ldots, h_p$ are affine

Optimum in a convex set

any locally optimal point of a convex problem is (globally) optimal

proof:

- suppose x is locally optimal, but there exists a feasible y with $f_0(y) < f_0(x)$
- x locally optimal means there is an $R > 0$ such that

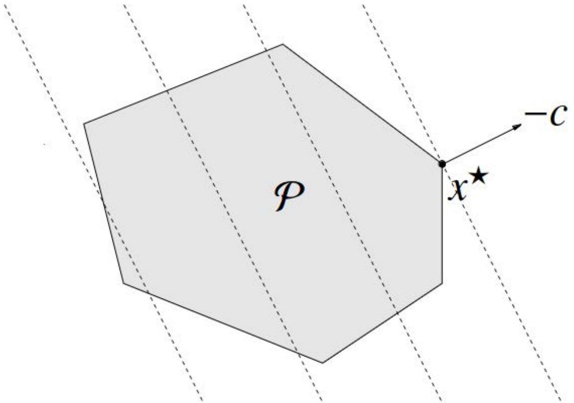
$$z \text{ feasible}, \quad \|z - x\|_2 \leq R \quad \Longrightarrow \quad f_0(z) \geq f_0(x)$$

- consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$
- $\|y - x\|_2 > R$, so $0 < \theta < 1/2$
- z is a convex combination of two feasible points, hence also feasible
- $\|z - x\|_2 = R/2$ and $f_0(z) \leq \theta f_0(y) + (1 - \theta)f_0(x) < f_0(x)$, which contradicts our assumption that x is locally optimal

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Often written as a maximization

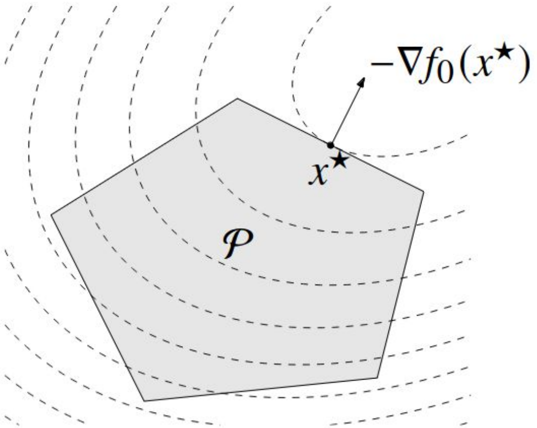
- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



Quadratic programming

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- $P \in \mathbf{S}_{+}^{n}$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



Quadratically constrained Quadratic programming (QCQP)

$$
\begin{array}{ll}
\text{minimize} & (1/2)x^T P_0 x + q_0^T x + r_0 \\
\text{subject to} & (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- $P_i \in \mathbf{S}_+^n$; objective and constraints are convex quadratic
- if $P_1, \ldots, P_m \in \mathbf{S}_{++}^n$, feasible region is intersection of m ellipsoids and an affine set

Change of variable

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- $\phi : \mathbf{R}^n \to \mathbf{R}^n$ is one-to-one with $\phi(\mathbf{dom}\,\phi) \supseteq \mathcal{D}$
- consider (possibly non-convex) problem

$$\begin{array}{lll} \text{minimize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & i = 1, \ldots, p \end{array}$$

- change variables to z with $x = \phi(z)$
- can solve equivalent problem

$$\begin{array}{lll} \text{minimize} & \tilde{f}_0(z) & \\ \text{subject to} & \tilde{f}_i(z) \leq 0, & i = 1, \ldots, m \\ & \tilde{h}_i(z) = 0, & i = 1, \ldots, p \end{array}$$

where $\tilde{f}_i(z) = f_i(\phi(z))$ and $\tilde{h}_i(z) = h_i(\phi(z))$

- recover original optimal point as $x^\star = \phi(z^\star)$

Transformation

suppose

- ϕ_0 is monotone increasing
- $\psi_i(u) \leq 0$ if and only if $u \leq 0$, $i = 1, \ldots, m$
- $\varphi_i(u) = 0$ if and only if $u = 0$, $i = 1, \ldots, p$

standard form optimization problem is equivalent to

$$
\begin{array}{lll}
\text{minimize} & \phi_0(f_0(x)) & \\
\text{subject to} & \psi_i(f_i(x)) \leq 0, & i = 1, \ldots, m \\
 & \varphi_i(h_i(x)) = 0, & i = 1, \ldots, p
\end{array}
$$

example: minimizing $\|Ax - b\|$ is equivalent to minimizing $\|Ax - b\|^2$

Maximization and minimization

- suppose ϕ_0 is monotone decreasing
- the maximization problem

$$\begin{array}{lll} \text{maximize} & f_0(x) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \dots, m \\ & h_i(x) = 0, & i = 1, \dots, p \end{array}$$

  is equivalent to the minimization problem

$$\begin{array}{lll} \text{minimize} & \phi_0(f_0(x)) & \\ \text{subject to} & f_i(x) \leq 0, & i = 1, \dots, m \\ & h_i(x) = 0, & i = 1, \dots, p \end{array}$$

- **examples**:
  - $\phi_0(u) = -u$ transforms maximizing a concave function to minimizing a convex function
  - $\phi_0(u) = 1/u$ transforms maximizing a concave positive function to minimizing a convex function

Eliminating equality constraints

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } z) & f_0(Fz + x_0) \\ \text{subject to} & f_i(Fz + x_0) \le 0, \quad i = 1, \ldots, m \end{array}$$

where F and x_0 are such that $Ax = b \iff x = Fz + x_0$ for some z

Introducing equality constraints

$$\begin{array}{ll} \text{minimize} & f_0(A_0x + b_0) \\ \text{subject to} & f_i(A_ix + b_i) \leq 0, \quad i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{minimize (over } x, y_i) & f_0(y_0) \\ \text{subject to} & f_i(y_i) \leq 0, \quad i = 1, \ldots, m \\ & y_i = A_ix + b_i, \quad i = 0, 1, \ldots, m \end{array}$$

Slack variables for linear equalities

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\text{minimize (over } x, s) & f_0(x) \\
\text{subject to} & a_i^T x + s_i = b_i, \quad i = 1, \ldots, m \\
& s_i \geq 0, \quad i = 1, \ldots m
\end{array}
$$

Takeaway

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p \end{array}$$

- Convex f and linear h
  - X feasible: satisfies implicit and explicit constraints
- Quite a few classical convex problems(linear, quadratic, …)
- Easy to change variables between equivalent problems

Simplex tableau

$$\begin{array}{rcrcrcrcr} x_4 & = & 5 & - & 2x_1 & - & 3x_2 & - & x_3 \\ x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \\ \hline z & = & & & 5x_1 & + & 4x_2 & + & 3x_3 . \end{array}$$

x1	x2	x3	x4	x5	x6	Z	C
2	3	1	1	0	0	0	5
4	1	2	0	1	0	0	11
3	4	2	0	0	1	0	8
-5	-4	-3	0	0	0	1	0

Simplex tableau

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x1	x2	x3	x4	x5	x6	Z	C
2	3	1	1	0	0	0	5 (5/2)
4	1	2	0	1	0	0	11 (11/4)
3	4	2	0	0	1	0	8 (8/3)
-5	-4	-3	0	0	0	1	0

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
4	1	2	0	1	0	0	11
3	4	2	0	0	1	0	8
-5	-4	-3	0	0	0	1	0

Simplex tableau

x1	x2	x3	x4	x5	x6	Z	C
2	3	1	1	0	0	0	5 (5/2)
4	1	2	0	1	0	0	11 (11/4)
3	4	2	0	0	1	0	8 (8/3)
-5	-4	-3	0	0	0	1	0

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
0	-5	0	-2	1	0	0	1
0	-0.5	0.5	-1.5	0	1	0	0.5
0	3.5	-0.5	2.5	0	0	1	12.5

Simplex tableau

$$
\begin{array}{rcl}
x_1 & = & \frac{5}{2} - \frac{3}{2} x_2 - \frac{1}{2} x_3 - \frac{1}{2} x_4 \\
x_5 & = & 1 + 5 x_2 + 2 x_4 \\
x_6 & = & \frac{1}{2} + \frac{1}{2} x_2 - \frac{1}{2} x_3 + \frac{3}{2} x_4 \\
\hline
z & = & \frac{25}{2} - \frac{7}{2} x_2 + \frac{1}{2} x_3 - \frac{5}{2} x_4 .
\end{array}
$$

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
0	-5	0	-2	1	0	0	1
0	-0.5	0.5	-1.5	0	1	0	0.5
0	3.5	-0.5	2.5	0	0	1	12.5

Simplex tableau

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x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5 (5)
0	-5	0	-2	1	0	0	1 (inf)
0	-0.5	0.5	-1.5	0	1	0	0.5 (1)
0	3.5	-0.5	2.5	0	0	1	12.5

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
0	-5	0	-2	1	0	0	1
0	-1	1	-3	0	2	0	1
0	3.5	-0.5	2.5	0	0	1	12.5

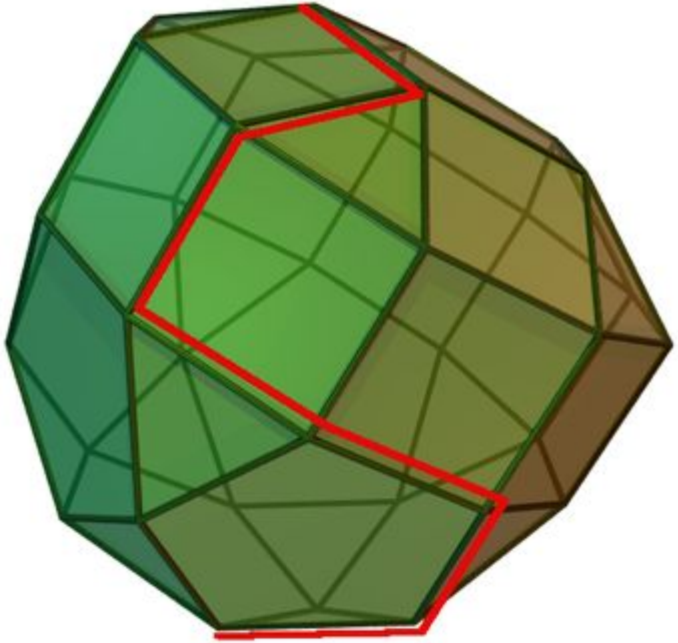
Simplex tableau

$$
\begin{array}{rcrcrcrcr}
x_3 & = & 1 & + & x_2 & + & 3x_4 & - & 2x_6 \\
x_1 & = & 2 & - & 2x_2 & - & 2x_4 & + & x_6 \\
x_5 & = & 1 & + & 5x_2 & + & 2x_4 & & \\
\hline
z & = & 13 & - & 3x_2 & - & x_4 & - & x_6 .
\end{array}
$$

x1	x2	x3	x4	x5	x6	Z	C
1	2	0	2	0	-1	0	2
0	-5	0	-2	1	0	0	1
0	-1	1	-3	0	2	0	1
0	3	0	1	0	1	1	13

Algorithm

- Build tableau from canonical
- Check we have feasible solution
- Do a pivot step if negative coef
  - Pick column c w/ most negative coefficient
  - Pick row r w/ smallest ratio
  - Pivot! (set c to 1 in r and 0 in other rows)



Unconstrained problem version

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\
& h_i(x) = 0, \quad i = 1, \dots, p,
\end{array}
$$

Unconstrained problem version

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & h_i(x) = 0, \quad i = 1, \ldots, p, \end{array}$$

$$I_-(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0, \end{cases}$$

$$\text{minimize} \quad f_0(x) + \sum_{i=1}^{m} I_-(f_i(x)) + \sum_{i=1}^{p} I_0(h_i(x)),$$

Relaxing the indicator function

$$I_{-}(u) = \begin{cases} 0 & u \leq 0 \\ \infty & u > 0, \end{cases}$$

$$\text{minimize} \quad f_0(x) + \textstyle\sum_{i=1}^{m} I_{-}(f_i(x)) + \textstyle\sum_{i=1}^{p} I_0(h_i(x)),$$

$$\text{minimize} \quad L(x, \lambda, \nu) = f_0(x) + \textstyle\sum_{i=1}^{m} \lambda_i f_i(x) + \textstyle\sum_{i=1}^{p} \nu_i h_i(x).$$

Lagrangian

- **standard form problem** (not necessarily convex)

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p \end{array}$$

  variable $x \in \mathbf{R}^n$, domain $\mathcal{D}$, optimal value $p^\star$

- **Lagrangian:** $L : \mathbf{R}^n \times \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$, with $\mathbf{dom}\, L = \mathcal{D} \times \mathbf{R}^m \times \mathbf{R}^p$,

$$L(x, \lambda, \nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$$

  - weighted sum of objective and constraint functions
  - λ_i is **Lagrange multiplier** associated with $f_i(x) \le 0$
  - ν_i is Lagrange multiplier associated with $h_i(x) = 0$

Lagrange dual

- **Lagrange dual function:** $g : \mathbf{R}^m \times \mathbf{R}^p \to \mathbf{R}$,

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

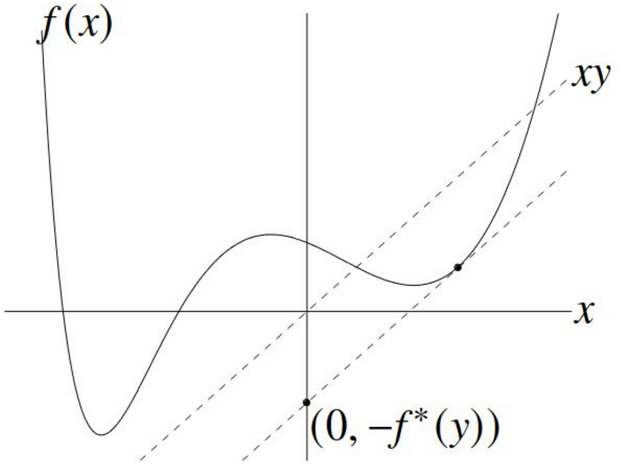
- g is concave, can be $-\infty$ for some λ, ν
- **lower bound property:** if $\lambda \succeq 0$, then $g(\lambda, \nu) \leq p^\star$
- proof: if $\tilde{x}$ is feasible and $\lambda \succeq 0$, then

$$f_0(\tilde{x}) \geq L(\tilde{x}, \lambda, \nu) \geq \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = g(\lambda, \nu)$$

minimizing over all feasible $\tilde{x}$ gives $p^\star \geq g(\lambda, \nu)$

Conjugate

- the **conjugate** of a function f is $f^*(y) = \sup_{x \in \mathbf{dom} f} (y^T x - f(x))$



- f^* is convex (even if f is not)
- will be useful in chapter 5

Lagrange dual and conjugate

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & Ax \preceq b, \quad Cx = d \end{array}$$

- dual function

$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathbf{dom} f_0} \left(f_0(x) + (A^T\lambda + C^T\nu)^T x - b^T\lambda - d^T\nu \right) \\ &= -f_0^*(-A^T\lambda - C^T\nu) - b^T\lambda - d^T\nu \end{aligned}$$

where $f^*(y) = \sup_{x \in \mathbf{dom} f}(y^T x - f(x))$ is conjugate of f_0

- simplifies derivation of dual if conjugate of f_0 is known
- **example: entropy maximization**

$$f_0(x) = \sum_{i=1}^n x_i \log x_i, \qquad f_0^*(y) = \sum_{i=1}^n e^{y_i - 1}$$

Dual problem

(Lagrange) **dual problem**

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

- finds best lower bound on $p^\star$, obtained from Lagrange dual function
- a convex optimization problem, even if original **primal** problem is not
- dual optimal value denoted $d^\star$
- λ, ν are dual feasible if $\lambda \succeq 0$, $(\lambda, \nu) \in \mathbf{dom}\, g$
- often simplified by making implicit constraint $(\lambda, \nu) \in \mathbf{dom}\, g$ explicit

Weak and strong duality

weak duality: $d^\star \leq p^\star$

- always holds (for convex and nonconvex problems)
- can be used to find nontrivial lower bounds for difficult problems, *e.g.*, solving the SDP

$$\begin{array}{ll} \text{maximize} & -\mathbf{1}^T \nu \\ \text{subject to} & W + \mathbf{diag}(\nu) \succeq 0 \end{array}$$

  gives a lower bound for the two-way partitioning problem on page 5.7

strong duality: $d^\star = p^\star$

- does not hold in general
- (usually) holds for convex problems
- conditions that guarantee strong duality in convex problems are called **constraint qualifications**

Slater's constraint qualification

strong duality holds for a convex problem

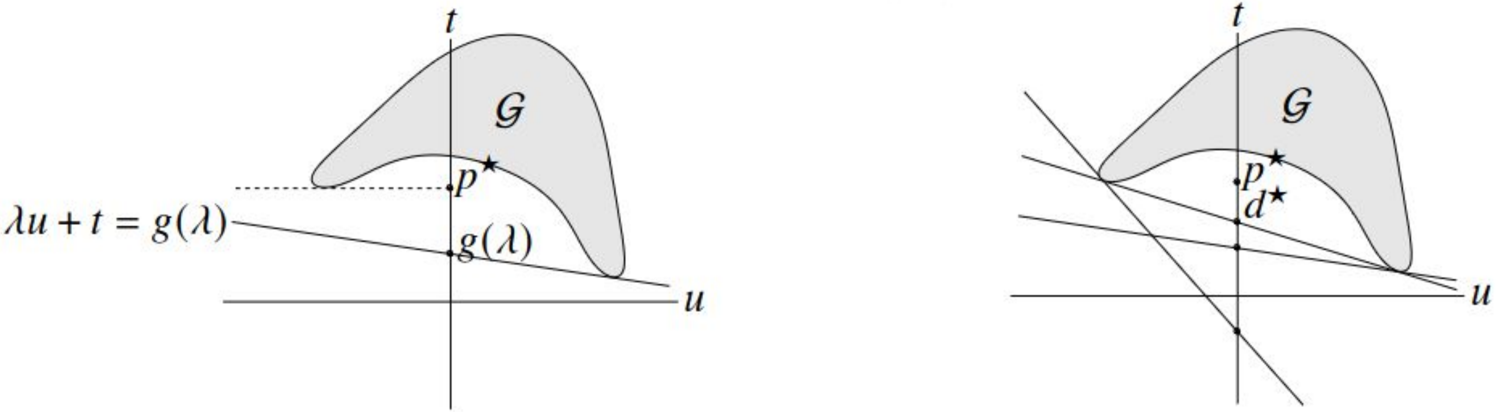
$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

if it is **strictly feasible**, *i.e.*, there is an $x \in \mathbf{int}\,\mathcal{D}$ with $f_i(x) < 0$, $i = 1, \ldots, m$, $Ax = b$

- also guarantees that the dual optimum is attained (if $p^\star > -\infty$)
- can be sharpened: *e.g.*,
  - can replace $\mathbf{int}\,\mathcal{D}$ with $\mathbf{relint}\,\mathcal{D}$ (interior relative to affine hull)
  - affine inequalities do not need to hold with strict inequality
- there are many other types of constraint qualifications

Geometric interpretation

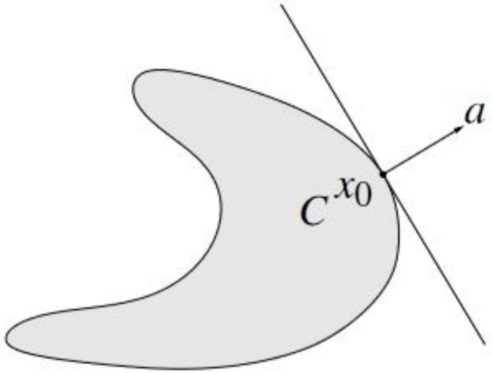
- for simplicity, consider problem with one constraint $f_1(x) \leq 0$
- $\mathcal{G} = \{(f_1(x), f_0(x)) \mid x \in \mathcal{D}\}$ is set of achievable (constraint, objective) values
- **interpretation of dual function:** $g(\lambda) = \inf_{(u,t)\in\mathcal{G}}(t + \lambda u)$



- $\lambda u + t = g(\lambda)$ is (non-vertical) supporting hyperplane to $\mathcal{G}$
- hyperplane intersects t-axis at $t = g(\lambda)$

Supporting hyperplanes

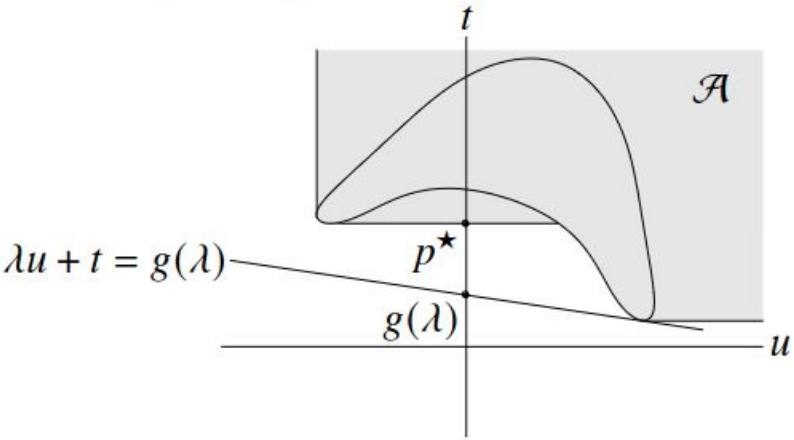
- suppose x_0 is a boundary point of set $C \subset \mathbf{R}^n$
- **supporting hyperplane** to C at x_0 has form $\{x \mid a^T x = a^T x_0\}$, where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$



- **supporting hyperplane theorem:** if C is convex, then there exists a supporting hyperplane at every boundary point of C

Geometric interpretation

- same with $\mathcal{G}$ replaced with $\mathcal{A} = \{(u,t) \mid f_1(x) \le u,\ f_0(x) \le t \text{ for some } x \in \mathcal{D}\}$



- strong duality holds if there is a non-vertical supporting hyperplane to $\mathcal{A}$ at $(0, p^\star)$
- for convex problem, $\mathcal{A}$ is convex, hence has supporting hyperplane at $(0, p^\star)$
- Slater's condition: if there exist $(\tilde{u}, \tilde{t}) \in \mathcal{A}$ with $\tilde{u} < 0$, then supporting hyperplane at $(0, p^\star)$ must be non-vertical

Complementary slackness

- assume strong duality holds, $x^\star$ is primal optimal, $(\lambda^\star, \nu^\star)$ is dual optimal

$$
\begin{aligned}
f_0(x^\star) = g(\lambda^\star, \nu^\star) &= \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^\star f_i(x) + \sum_{i=1}^{p} \nu_i^\star h_i(x) \right) \\
&\leq f_0(x^\star) + \sum_{i=1}^{m} \lambda_i^\star f_i(x^\star) + \sum_{i=1}^{p} \nu_i^\star h_i(x^\star) \\
&\leq f_0(x^\star)
\end{aligned}
$$

- hence, the two inequalities hold with equality
- $x^\star$ minimizes $L(x, \lambda^\star, \nu^\star)$
- $\lambda_i^\star f_i(x^\star) = 0$ for $i = 1, \ldots, m$ (known as **complementary slackness**):

$$
\lambda_i^\star > 0 \implies f_i(x^\star) = 0, \qquad f_i(x^\star) < 0 \implies \lambda_i^\star = 0
$$

Karush-Kuhn-Tucker (KKT) conditions

the **KKT conditions** (for a problem with differentiable f_i, h_i) are

1. primal constraints: $f_i(x) \leq 0$, $i = 1, \ldots, m$, $h_i(x) = 0$, $i = 1, \ldots, p$
2. dual constraints: $\lambda \succeq 0$
3. complementary slackness: $\lambda_i f_i(x) = 0$, $i = 1, \ldots, m$
4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{i=1}^{p} \nu_i \nabla h_i(x) = 0$$

if strong duality holds and x, λ, ν are optimal, they satisfy the KKT conditions

KKT for convex problems

if $\tilde{x}$, $\tilde{\lambda}$, $\tilde{\nu}$ satisfy KKT for a convex problem, then they are optimal:

- from complementary slackness: $f_0(\tilde{x}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$
- from 4th condition (and convexity): $g(\tilde{\lambda}, \tilde{\nu}) = L(\tilde{x}, \tilde{\lambda}, \tilde{\nu})$

hence, $f_0(\tilde{x}) = g(\tilde{\lambda}, \tilde{\nu})$

if Slater's condition is satisfied, then

x is optimal if and only if there exist λ, ν that satisfy KKT conditions

- recall that Slater implies strong duality, and dual optimum is attained
- generalizes optimality condition $\nabla f_0(x) = 0$ for unconstrained problem

Takeaway

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} L(x, \lambda, \nu) = \inf_{x \in \mathcal{D}} \left(f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right)$$

- Mirror problem that is always convex!
- Gives a lower bound on solution (weak duality)
- Can give the exact solution
  - Under qualifications on constraints for convex problems
- KKT conditions can help reverse engineer a solution

1. Unconstrained problems

General problem statement

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- unconstrained minimization problem

$$\text{minimize} \quad f(x)$$

- we assume
  - f convex, twice continuously differentiable (hence $\mathbf{dom}\, f$ open)
  - optimal value $p^\star = \inf_x f(x)$ is attained at $x^\star$ (not necessarily unique)
- optimality condition is $\nabla f(x) = 0$
- minimizing f is the same as solving $\nabla f(x) = 0$
- a set of n equations with n unknowns

Closed form solution

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- convex quadratic: $f(x) = (1/2)x^T Px + q^T x + r,\ P \succeq 0$
- we can solve exactly via linear equations

$$\nabla f(x) = Px + q = 0$$

- much more on this special case later

Iterative methods for reality

- for most non-quadratic functions, we use **iterative methods**
- these produce a sequence of points $x^{(k)} \in \mathbf{dom} f$, $k = 0, 1, \ldots$
- $x^{(0)}$ is the **initial point** or **starting point**
- $x^{(k)}$ is the kth **iterate**
- we hope that the method **converges**, *i.e.*,

$$f(x^{(k)}) \to p^\star, \qquad \nabla f(x^{(k)}) \to 0$$

Starting point and sublevel sets

- algorithms in this chapter require a starting point $x^{(0)}$ such that
  - $x^{(0)} \in \mathbf{dom} f$
  - sublevel set $S = \{x \mid f(x) \le f(x^{(0)})\}$ is closed
- 2nd condition is hard to verify, except when **all** sublevel sets are closed
  - equivalent to condition that $\mathbf{epi} f$ is closed
  - true if $\mathbf{dom} f = \mathbf{R}^n$
  - true if $f(x) \to \infty$ as $x \to \mathbf{bd}\,\mathbf{dom} f$
- examples of differentiable functions with closed sublevel sets:

$$f(x) = \log\left(\sum_{i=1}^{m} \exp(a_i^T x + b_i)\right), \qquad f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x)$$

Descent methods

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- **descent methods** generate iterates as

$$x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)}$$

  with $f(x^{(k+1)}) < f(x^{(k)})$ (hence the name)
- other notations: $x^+ = x + t\Delta x$, $x := x + t\Delta x$
- $\Delta x^{(k)}$ is the **step**, or **search direction**
- $t^{(k)} > 0$ is the **step size**, or **step length**
- from convexity, $f(x^+) < f(x)$ implies $\nabla f(x)^T \Delta x < 0$
- this means Δx is a **descent direction**

Descent methods

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General descent method.

given a starting point $x \in \mathbf{dom}\, f$.

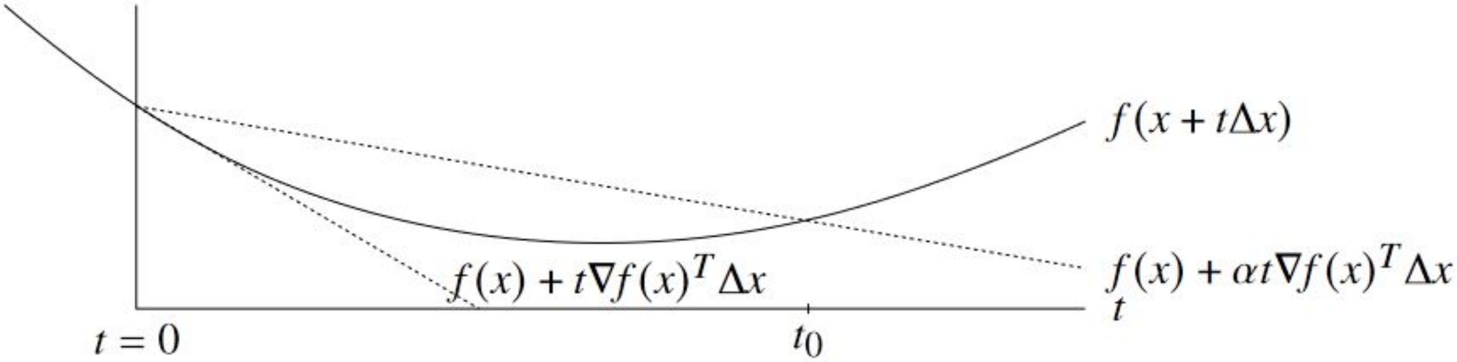
repeat

1. Determine a descent direction Δx.
2. **Line search.** Choose a step size $t > 0$.
3. **Update.** $x := x + t\Delta x$.

until stopping criterion is satisfied.

Line search

- **exact line search:** $t = \text{argmin}_{t>0} f(x + t\Delta x)$
- **backtracking line search** (with parameters $\alpha \in (0, 1/2)$, $\beta \in (0, 1)$)
  - starting at $t = 1$, repeat $t := \beta t$ until $f(x + t\Delta x) < f(x) + \alpha t \nabla f(x)^T \Delta x$
- graphical interpretation: reduce t (*i.e.*, backtrack) until $t \leq t_0$



Gradient descent

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- general descent method with $\Delta x = -\nabla f(x)$

---

given a starting point $x \in \textbf{dom} f$.
repeat
1. $\Delta x := -\nabla f(x)$.
2. **Line search.** Choose step size t via exact or backtracking line search.
3. **Update.** $x := x + t\Delta x$.

until stopping criterion is satisfied.

- stopping criterion usually of the form $\|\nabla f(x)\|_2 \le \epsilon$
- convergence result: for strongly convex f,

$$f(x^{(k)}) - p^\star \le c^k (f(x^{(0)}) - p^\star)$$

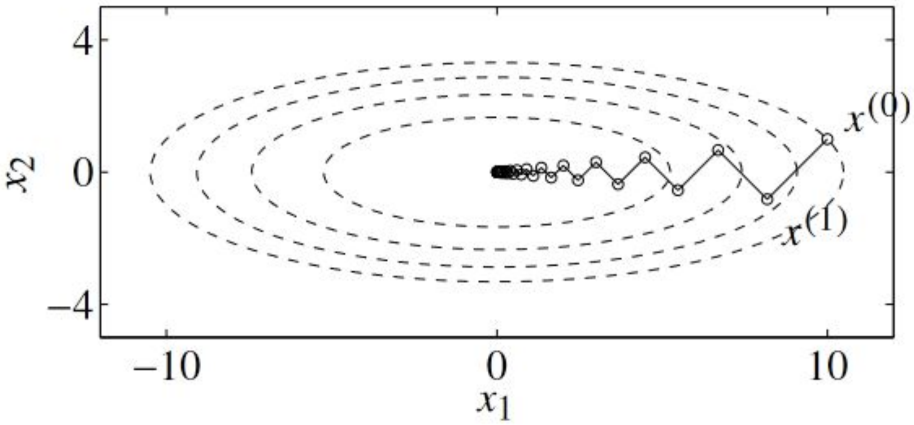
  $c \in (0,1)$ depends on m, $x^{(0)}$, line search type
- very simple, but can be very slow

Example of gradient descent

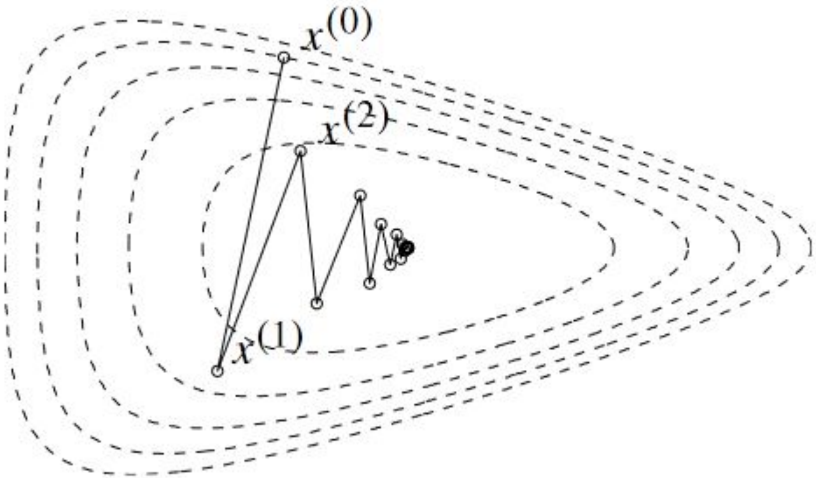
- take $f(x) = (1/2)(x_1^2 + \gamma x_2^2)$, with $\gamma > 0$
- with exact line search, starting at $x^{(0)} = (\gamma, 1)$:

$$x_1^{(k)} = \gamma \left(\frac{\gamma - 1}{\gamma + 1}\right)^k, \qquad x_2^{(k)} = \left(-\frac{\gamma - 1}{\gamma + 1}\right)^k$$

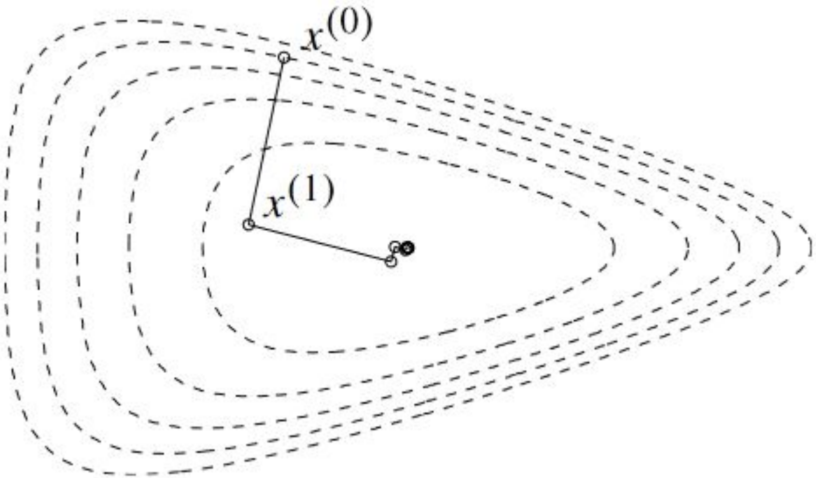
  - very slow if $\gamma \gg 1$ or $\gamma \ll 1$
  - example for $\gamma = 10$ at right
  - called **zig-zagging**



Example of gradient descent

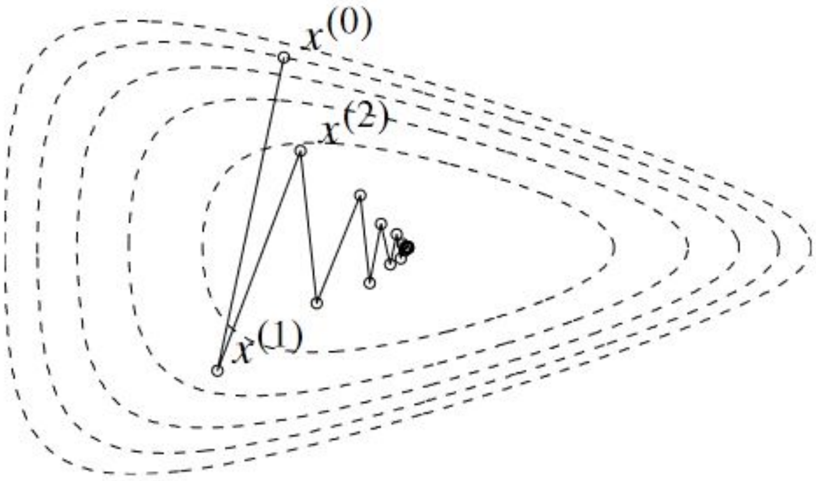


backtracking line search

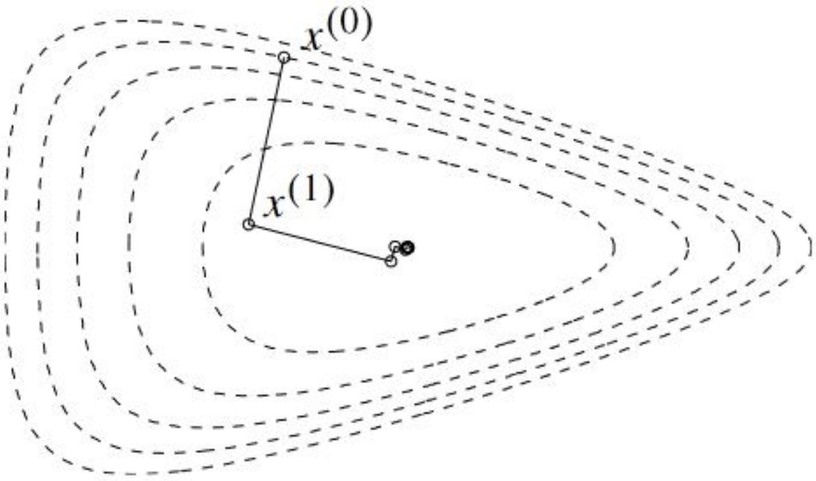


exact line search

Example of gradient descent



backtracking line search



exact line search

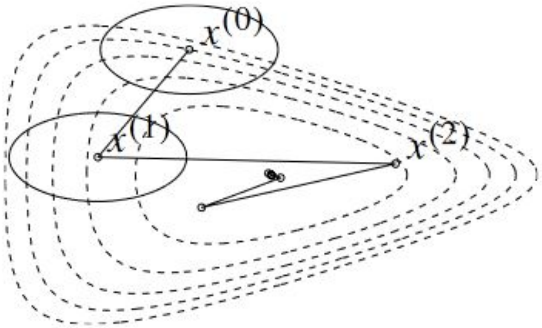
Steepest descent

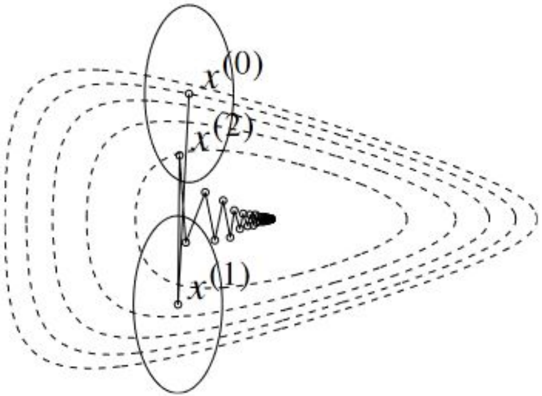
- **normalized steepest descent direction** (at x, for norm $\|\cdot\|$):

$$\Delta x_{\mathrm{nsd}} = \operatorname{argmin}\{\nabla f(x)^T v \mid \|v\| = 1\}$$

- interpretation: for small v, $f(x+v) \approx f(x) + \nabla f(x)^T v$;
- direction Δx_{nsd} is unit-norm step with most negative directional derivative
- **(unnormalized) steepest descent direction**: $\Delta x_{\mathrm{sd}} = \|\nabla f(x)\|_* \Delta x_{\mathrm{nsd}}$
- satisfies $\nabla f(x)^T \Delta x_{\mathrm{sd}} = -\|\nabla f(x)\|_*^2$
- **steepest descent method**
  - general descent method with $\Delta x = \Delta x_{\mathrm{sd}}$
  - convergence properties similar to gradient descent

Steepest descent norm

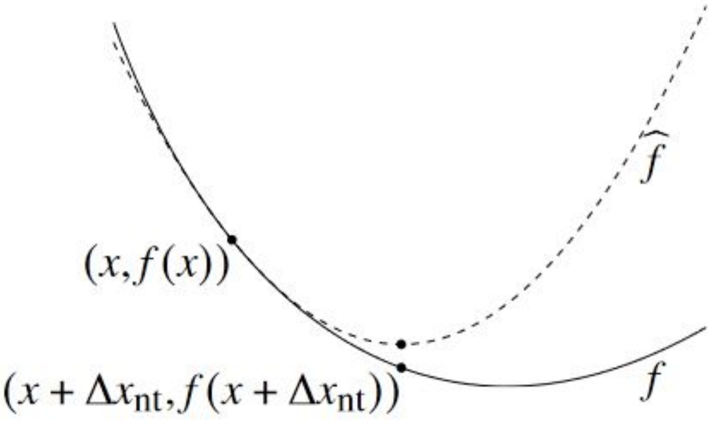




- steepest descent with backtracking line search for two quadratic norms
- ellipses show $\{x \mid \|x - x^{(k)}\|_P = 1\}$
- interpretation of steepest descent with quadratic norm $\|\cdot\|_P$: gradient descent after change of variables $\bar{x} = P^{1/2}x$
- shows choice of P has strong effect on speed of convergence

- **Newton step** is $\boxed{\Delta x_{\mathrm{nt}} = -\nabla^2 f(x)^{-1} \nabla f(x)}$
- **interpretation**: $x + \Delta x_{\mathrm{nt}}$ minimizes second order approximation

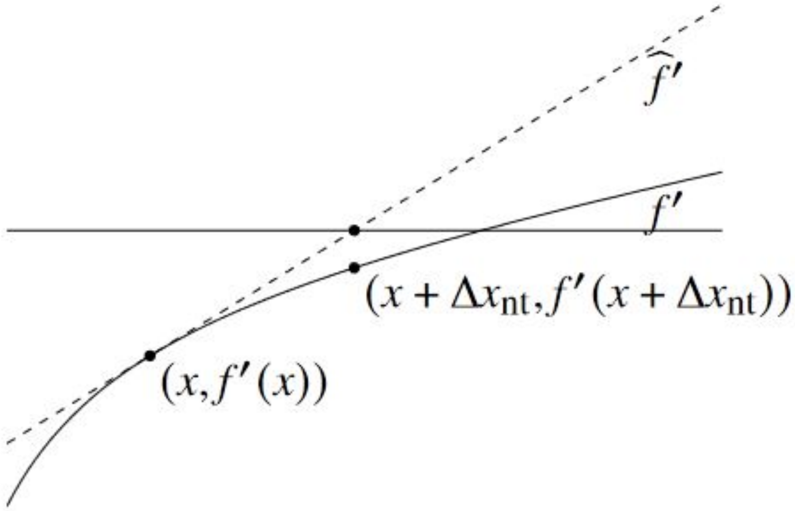
$$\widehat{f}(x+v) = f(x) + \nabla f(x)^T v + \frac{1}{2} v^T \nabla^2 f(x) v$$



Linearized interpretation

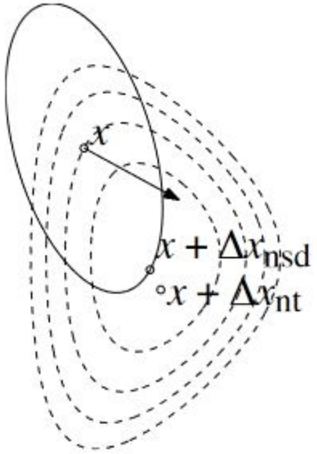
- $x + \Delta x_{\text{nt}}$ solves linearized optimality condition

$$\nabla f(x+v) \approx \nabla \widehat{f}(x+v) = \nabla f(x) + \nabla^2 f(x) v = 0$$



Hessian norm interpretation

- Δx_{nt} is steepest descent direction at x in local Hessian norm $\|u\|_{\nabla^2 f(x)} = (u^T \nabla^2 f(x) u)^{1/2}$



- dashed lines are contour lines of f; ellipse is $\{x + v \mid v^T \nabla^2 f(x) v = 1\}$
- arrow shows $-\nabla f(x)$

Newton decrement

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- **Newton decrement** is $\lambda(x) = \left(\nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)\right)^{1/2}$
- a measure of the proximity of x to $x^\star$
- gives an estimate of $f(x) - p^\star$, using quadratic approximation $\widehat{f}$:

$$f(x) - \inf_y \widehat{f}(y) = \frac{1}{2}\lambda(x)^2$$

- equal to the norm of the Newton step in the quadratic Hessian norm

$$\lambda(x) = \left(\Delta x_{\mathrm{nt}}^T \nabla^2 f(x) \Delta x_{\mathrm{nt}}\right)^{1/2}$$

- directional derivative in the Newton direction: $\nabla f(x)^T \Delta x_{\mathrm{nt}} = -\lambda(x)^2$
- affine invariant (unlike $\|\nabla f(x)\|_2$)

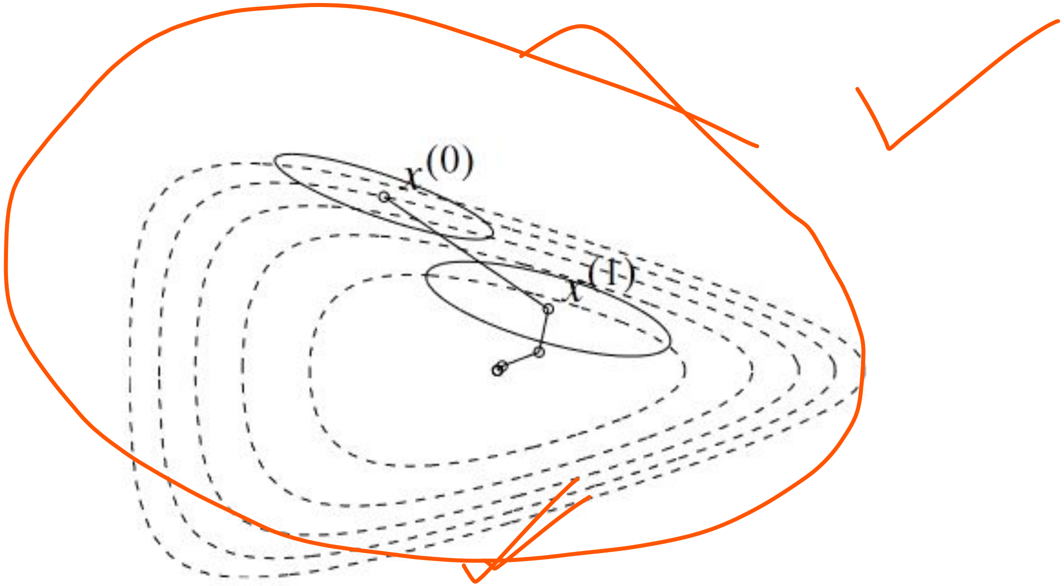
Algorithm

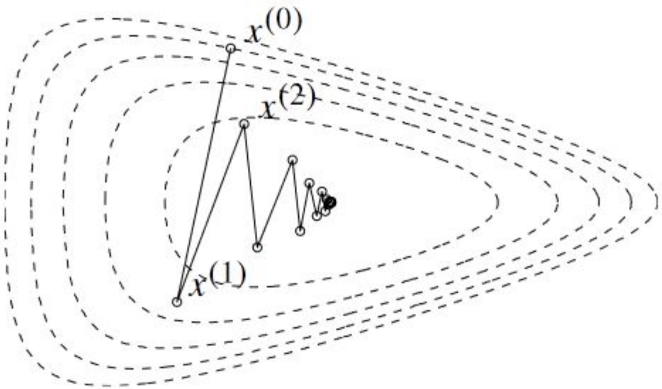
given a starting point $x \in \mathbf{dom} f$, tolerance $\epsilon > 0$.
repeat

1. **Compute the Newton step and decrement.**
   $\Delta x_{\rm nt} := -\nabla^2 f(x)^{-1} \nabla f(x); \quad \lambda^2 := \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x).$
2. **Stopping criterion. quit** if $\lambda^2/2 \le \epsilon$.
3. **Line search.** Choose step size t by backtracking line search.
4. **Update.** $x := x + t\Delta x_{\rm nt}$.

- **affine invariant**, *i.e.*, independent of linear changes of coordinates
- Newton iterates for $\tilde{f}(y) = f(Ty)$ with starting point $y^{(0)} = T^{-1}x^{(0)}$ are $y^{(k)} = T^{-1}x^{(k)}$

Backtrack analysis

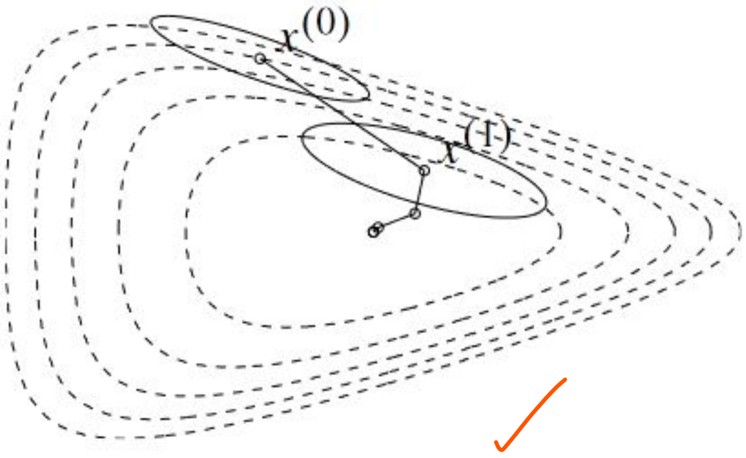


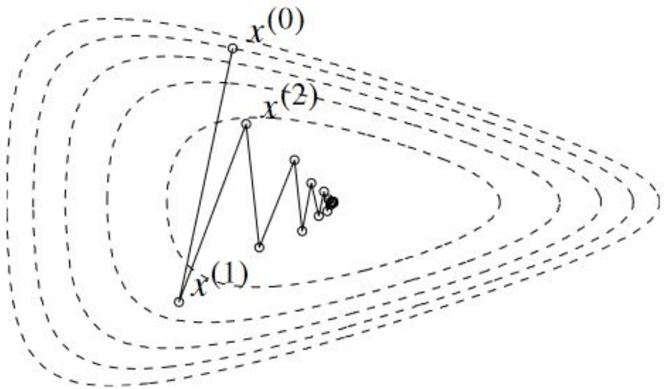


backtracking line search

- backtracking parameters $\alpha = 0.1$, $\beta = 0.7$
- converges in only 5 steps
- quadratic local convergence

Backtrack analysis





backtracking line search

- backtracking parameters $\alpha = 0.1$, $\beta = 0.7$
- converges in only 5 steps
- quadratic local convergence

2. Equality constrained problems

x

Equality constrained problems

- equality constrained smooth minimization problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & Ax = b \end{array}$$

- we assume
  - f convex, twice continuously differentiable
  - $A \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
  - $p^\star$ is finite and attained

- **optimality conditions:** $x^\star$ is optimal if and only if there exists a $\nu^\star$ such that

$$\nabla f(x^\star) + A^T \nu^\star = 0, \qquad Ax^\star = b$$

KKT to the rescue

- $f(x) = (1/2)x^T Px + q^T x + r,\ P \in \mathbf{S}^n_+$
- $\nabla f(x) = Px + q$
- optimality conditions are a **system of linear equations**

$$\begin{bmatrix} P & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x^\star \\ \nu^\star \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix}$$

- coefficient matrix is called KKT matrix
- KKT matrix is nonsingular if and only if

$$Ax = 0, \quad x \neq 0 \qquad \Longrightarrow \qquad x^T Px > 0$$

- equivalent condition for nonsingularity: $P + A^T A \succ 0$

Just change the variable (for the most part)

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- represent feasible set $\{x \mid Ax = b\}$ as $\{Fz + \hat{x} \mid z \in \mathbf{R}^{n-p}\}$
  - $\hat{x}$ is (any) **particular solution** of $Ax = b$
  - range of $F \in \mathbf{R}^{n\times(n-p)}$ is nullspace of A ($\mathbf{rank}\, F = n - p$ and $AF = 0$)
- **reduced or eliminated problem**: minimize $f(Fz + \hat{x})$
- an unconstrained problem with variable $z \in \mathbf{R}^{n-p}$
- from solution $z^\star$, obtain $x^\star$ and $\nu^\star$ as

$$x^\star = Fz^\star + \hat{x}, \qquad \nu^\star = -(AA^T)^{-1}A\nabla f(x^\star)$$

3. Inequality constrained problems (and interior point methods)

Inequality constrained problems

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

we assume

- f_i convex, twice continuously differentiable
- $A \in \mathbf{R}^{p \times n}$ with $\mathbf{rank}\, A = p$
- $p^\star$ is finite and attained
- problem is strictly feasible: there exists $\tilde{x}$ with

$$
\tilde{x} \in \mathbf{dom}\, f_0, \qquad f_i(\tilde{x}) < 0, \quad i = 1, \ldots, m, \qquad A\tilde{x} = b
$$

hence, strong duality holds and dual optimum is attained

Remember this?

- reformulation via **indicator function**:

$$\begin{array}{ll} \text{minimize} & f_0(x) + \sum_{i=1}^m I_-(f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

  where $I_-(u) = 0$ if $u \leq 0$, $I_-(u) = \infty$ otherwise

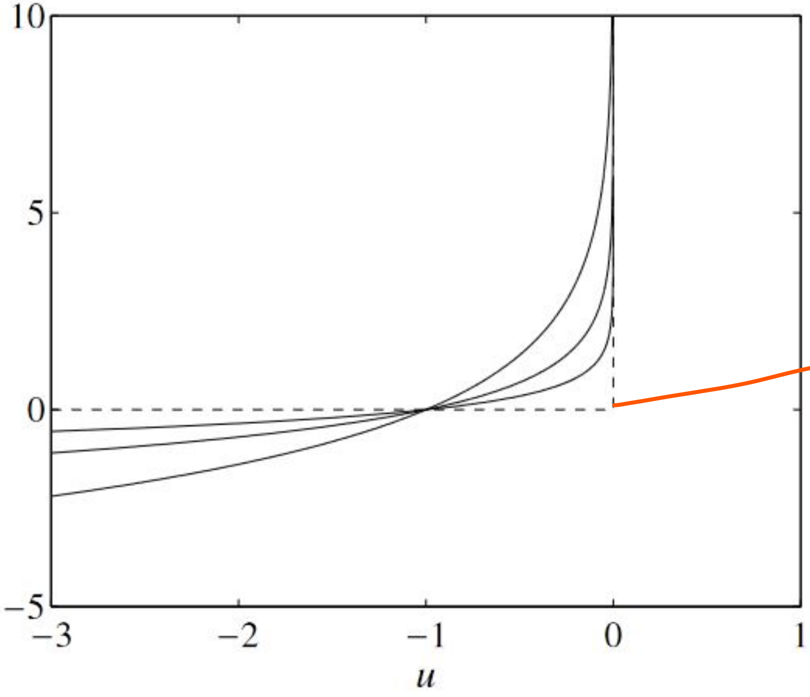
- **approximation via logarithmic barrier**:

$$\begin{array}{ll} \text{minimize} & f_0(x) - (1/t) \sum_{i=1}^m \log(-f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

- an equality constrained problem
- for $t > 0$, $-(1/t)\log(-u)$ is a smooth approximation of I_-
- approximation improves as $t \to \infty$

Logarithmic barrier

- $-(1/t)\log u$ for three values of t, and $I_-(u)$



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Logarithmic barrier

- log barrier function for constraints $f_1(x) \leq 0, \ldots, f_m(x) \leq 0$

$$\phi(x) = -\sum_{i=1}^{m} \log(-f_i(x)), \quad \mathbf{dom}\,\phi = \{x \mid f_1(x) < 0, \ldots, f_m(x) < 0\}$$

- convex (from composition rules)
- twice continuously differentiable, with derivatives

$$\begin{aligned}
\nabla\phi(x) &= \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) \\
\nabla^2\phi(x) &= \sum_{i=1}^{m} \frac{1}{f_i(x)^2} \nabla f_i(x) \nabla f_i(x)^T + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla^2 f_i(x)
\end{aligned}$$

Central path

- for $t > 0$, define $x^\star(t)$ as the solution of

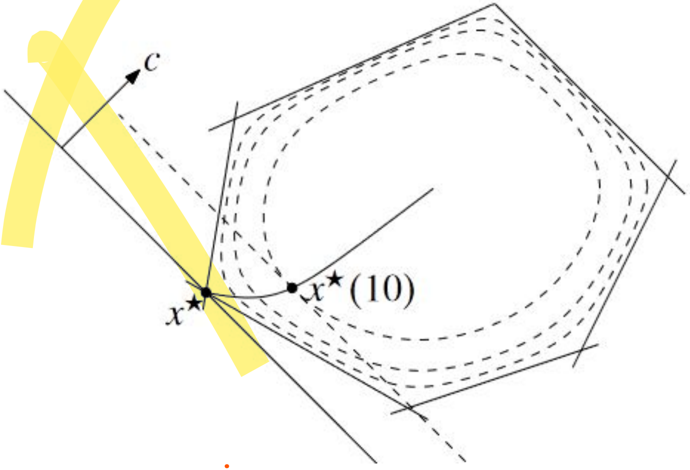
$$\begin{array}{ll} \text{minimize} & tf_0(x) + \phi(x) \\ \text{subject to} & Ax = b \end{array}$$

(for now, assume $x^\star(t)$ exists and is unique for each $t > 0$)

- central path is $\{x^\star(t) \mid t > 0\}$

example: central path for an LP

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, 6 \end{array}$$



hyperplane $c^T x = c^T x^\star(t)$ is tangent to level curve of ϕ through $x^\star(t)$

Duality and the central path

- $x = x^\star(t)$ if there exists a w such that

$$t\nabla f_0(x) + \sum_{i=1}^{m} \frac{1}{-f_i(x)} \nabla f_i(x) + A^T w = 0, \qquad Ax = b$$

- therefore, $x^\star(t)$ minimizes the Lagrangian

$$L(x, \lambda^\star(t), \nu^\star(t)) = f_0(x) + \sum_{i=1}^{m} \lambda_i^\star(t) f_i(x) + \nu^\star(t)^T (Ax - b)$$

where we define $\lambda_i^\star(t) = 1/(-tf_i(x^\star(t))$ and $\nu^\star(t) = w/t$

- this confirms the intuitive idea that $f_0(x^\star(t)) \to p^\star$ if $t \to \infty$:

$$p^\star \geq g(\lambda^\star(t), \nu^\star(t)) = L(x^\star(t), \lambda^\star(t), \nu^\star(t)) = f_0(x^\star(t)) - m/t$$

Approximated KKT

$x = x^\star(t)$, $\lambda = \lambda^\star(t)$, $\nu = \nu^\star(t)$ satisfy

1. primal constraints: $f_i(x) \leq 0$, $i = 1, \ldots, m$, $Ax = b$
2. dual constraints: $\lambda \succeq 0$
3. approximate complementary slackness: $-\lambda_i f_i(x) = 1/t$, $i = 1, \ldots, m$
4. gradient of Lagrangian with respect to x vanishes:

$$\nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + A^T \nu = 0$$

difference with KKT is that condition 3 replaces $\lambda_i f_i(x) = 0$

Force field interpretation

- **centering problem** (for problem with no equality constraints)

$$\text{minimize} \quad tf_0(x) - \textstyle\sum_{i=1}^m \log(-f_i(x))$$

- **force field interpretation**
  - $tf_0(x)$ is potential of force field $F_0(x) = -t\nabla f_0(x)$
  - $-\log(-f_i(x))$ is potential of force field $F_i(x) = (1/f_i(x))\nabla f_i(x)$
- forces balance at $x^\star(t)$:

$$F_0(x^\star(t)) + \sum_{i=1}^m F_i(x^\star(t)) = 0$$

Barrier method

given strictly feasible x, $t := t^{(0)} > 0$, $\mu > 1$, tolerance $\epsilon > 0$.

repeat

1. *Centering step.* Compute $x^\star(t)$ by minimizing $tf_0 + \phi$, subject to $Ax = b$.
2. *Update.* $x := x^\star(t)$.
3. *Stopping criterion.* **quit** if $m/t < \epsilon$.
4. *Increase* t. $t := \mu t$.

- terminates with $f_0(x) - p^\star \leq \epsilon$ (stopping criterion follows from $f_0(x^\star(t)) - p^\star \leq m/t$)
- centering usually done using Newton's method, starting at current x
- choice of μ involves a trade-off: large μ means fewer outer iterations, more inner (Newton) iterations; typical values: $\mu = 10$ or 20
- several heuristics for choice of $t^{(0)}$

Phase I: feasibility

- barrier method needs strictly feasible starting point, *i.e.*, x with

$$f_i(x) < 0, \quad i = 1, \ldots, m, \qquad Ax = b$$

- (like the infeasible start Newton method, more sophisticated interior-point methods do not require a feasible starting point)
- **phase I** method forms an optimization problem that
  - is itself strictly feasible
  - finds a strictly feasible point for original problem, if one exists
  - certifies original problem as infeasible otherwise
- **phase II** uses barrier method starting from strictly feasible point found in phase I

Phase I: feasibility

- introduce slack variable s in **phase I problem**

$$\begin{array}{ll} \text{minimize (over } x,\ s) & s \\ \text{subject to} & f_i(x) \leq s, \quad i = 1, \ldots, m \\ & Ax = b \end{array}$$

  with optimal value $\bar{p}^\star$

  - if $\bar{p}^\star < 0$, original inequalities are strictly feasible
  - if $\bar{p}^\star > 0$, original inequalities are infeasible
  - $\bar{p}^\star = 0$ is an ambiguous case

- start phase I problem with
  - any $\tilde{x}$ in problem domain with $A\tilde{x} = b$
  - $s = 1 + \max_i f_i(\tilde{x})$

Phase I: feasibility

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- minimize **sum** of slacks, not max:

$$
\begin{array}{ll}
\text{minimize} & \mathbf{1}^T s \\
\text{subject to} & s \succeq 0, \quad f_i(x) \leq s_i, \quad i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

- will find a strictly feasible point if one exists
- for infeasible problems, produces a solution that satisfies many (but not all) inequalities
- can weight slacks to set **priorities** (in satifying constraints)