

# Lecture 3: Convex optimization (Part 2)

Optimization for data sciences



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# Course organization

# What can we optimize?

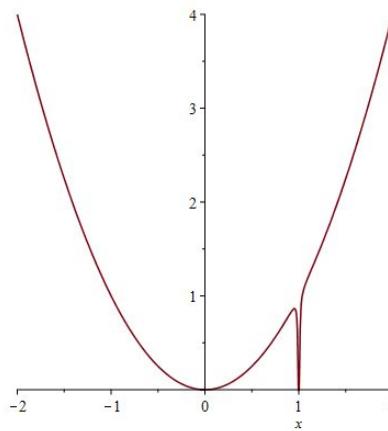
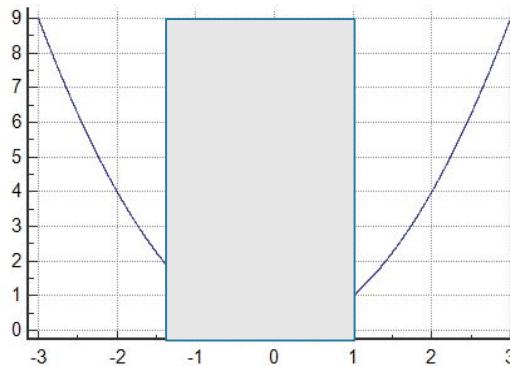
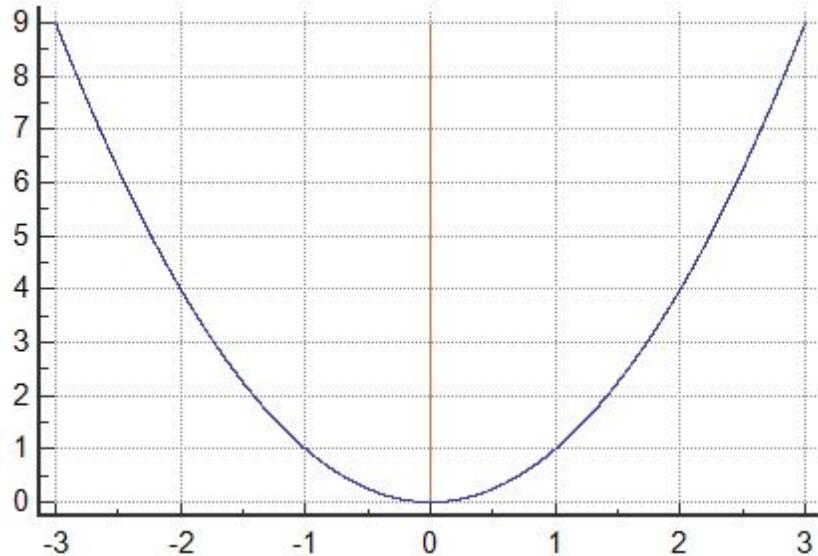
- Reduce the complexity/overhead of a problem
  - E.g. Network quantization
  - E.g. Computational optimization
- **Find the best solution to a problem**
  - **Numerical optimization**
  - **Evaluate solutions according to a criterion**
  - **Look at solutions from some given space of possible solutions to consider**

# Defining an optimization problem

- Minimize a quantity  $f_0(x)$ 
  - Under inequality and equality constraints
  - Constraints define a domain D
  - Could have no constraint except  $x \in D$

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && g_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

# Can you formalize these problems?



# Course organization

- Introduction to optimization
  - A few problems of interest
  - Quick mathematical refresher
- **Convex problems (Following Stephen Boyd)**
  - **Quick refresher on last week**
  - **Convex sets**
  - **Convex functions**
  - **Convex problems**
  - **Simplex algorithm for Linear Programming**

# Course organization

- Duality (for convex problems)
- Newton's Descent and Barrier methods for convex case
- (First order) descent methods for the general case
- Backpropagation
- Some more properties on stochastic gradient descent

- Reports on lab sessions
  - Labs on jupyter notebooks
    - Not every session
  - Explain the code done in the session
  - Summarize what is done in the practical
- Written Exam
  - Theoretical questions
  - We will do exercises in class

# Refresher on last week

# Convex problem

convex optimization problem:

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && Ax = b \end{aligned}$$

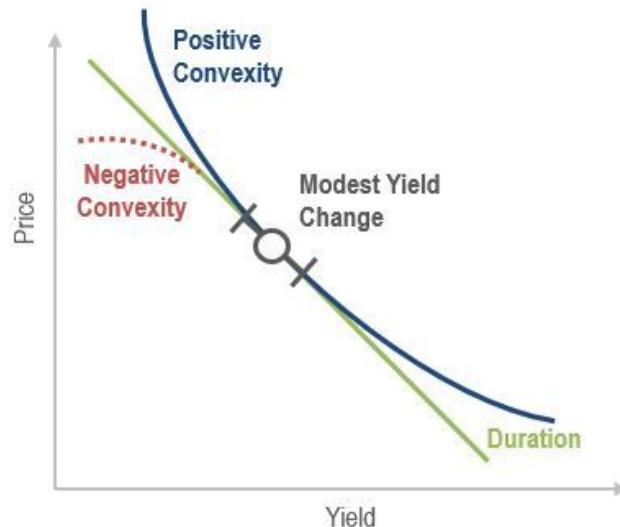
- ▶ variable  $x \in \mathbf{R}^n$
- ▶ equality constraints are linear
- ▶  $f_0, \dots, f_m$  are **convex**: for  $\theta \in [0, 1]$ ,

$$f_i(\theta x + (1 - \theta)y) \leq \theta f_i(x) + (1 - \theta)f_i(y)$$

i.e.,  $f_i$  have nonnegative (upward) curvature

# Easy problem

- ▶ classical view:
  - linear (zero curvature) is easy
  - nonlinear (nonzero curvature) is hard
- ▶ the classical view is **wrong**
- ▶ the correct view:
  - convex (nonnegative curvature) is easy
  - nonconvex (negative curvature) is hard



# Easy to solve!

- ▶ many different algorithms (that run on many platforms)
  - interior-point methods for up to 10000s of variables
  - first-order methods for larger problems
  - do not require initial point, babysitting, or tuning
- ▶ can develop and deploy quickly using modeling languages such as CVXPY
- ▶ solvers are reliable, so can be embedded
- ▶ code generation yields real-time solvers that execute in milliseconds (e.g., on Falcon 9 and Heavy for landing)

- Convex sets
  - Affine sets, norm balls, norm cones
  - Convex combination, convex hull and Convex cones
  - Hyperplanes, halfspaces and polyhedron
  - Positive Semidefinite Cone
- Convexity preserving operations
  - Intersection
  - Affine mapping
  - Perspective and Linear fractional mappings
- Proper cones and generalized inequalities
- Separating and supporting hyperplanes

- **Convex sets (definition!)**
  - **Affine sets, norm balls, norm cones**
  - **Convex combination, convex hull and Convex cones**
  - **Hyperplanes, halfspaces and polyhedron**
  - Positive Semidefinite Cone
- **Showing a set is convex (with operations!)**
  - **Intersection**
  - **Affine mapping**
  - **Perspective and Linear fractional mappings**
- Proper cones and generalized inequalities
- Separating and supporting hyperplanes

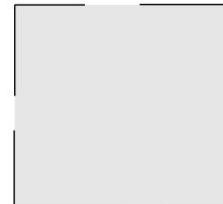
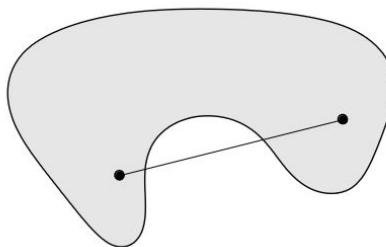
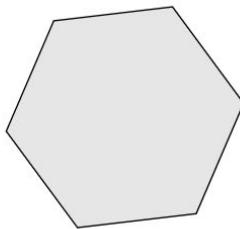
# Convex sets

**line segment** between  $x_1$  and  $x_2$ : all points of form  $x = \theta x_1 + (1 - \theta)x_2$ , with  $0 \leq \theta \leq 1$

**convex set**: contains line segment between any two points in the set

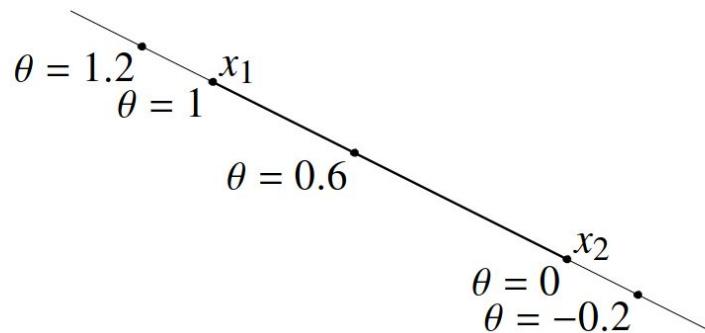
$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \implies \quad \theta x_1 + (1 - \theta)x_2 \in C$$

**examples** (one convex, two nonconvex sets)



# Affine sets

**line** through  $x_1, x_2$ : all points of form  $x = \theta x_1 + (1 - \theta)x_2$ , with  $\theta \in \mathbf{R}$



**affine set**: contains the line through any two distinct points in the set

**example**: solution set of linear equations  $\{x \mid Ax = b\}$

(conversely, every affine set can be expressed as solution set of system of linear equations)

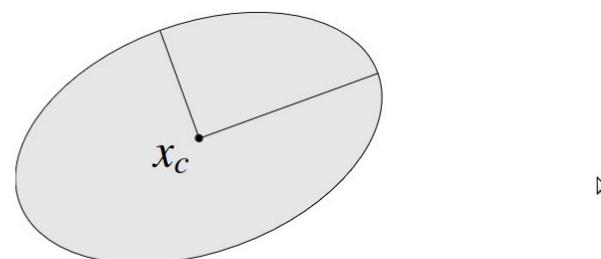
**(Euclidean) ball** with center  $x_c$  and radius  $r$ :

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

**ellipsoid:** set of the form

$$\{x \mid (x - x_c)^T P^{-1} (x - x_c) \leq 1\}$$

with  $P \in \mathbf{S}_{++}^n$  (i.e.,  $P$  symmetric positive definite)



another representation:  $\{x_c + Au \mid \|u\|_2 \leq 1\}$  with  $A$  square and nonsingular

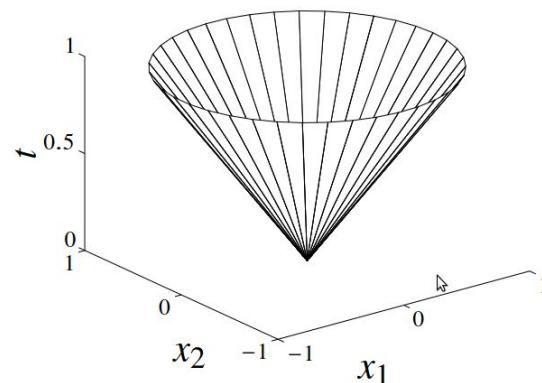
## Norm balls and norm cones

- ▶ **norm:** a function  $\|\cdot\|$  that satisfies
  - $\|x\| \geq 0$ ;  $\|x\| = 0$  if and only if  $x = 0$
  - $\|tx\| = |t| \|x\|$  for  $t \in \mathbf{R}$
  - $\|x + y\| \leq \|x\| + \|y\|$
- ▶ notation:  $\|\cdot\|$  is general (unspecified) norm;  $\|\cdot\|_{\text{symb}}$  is particular norm
- ▶ **norm ball** with center  $x_c$  and radius  $r$ :  $\{x \mid \|x - x_c\| \leq r\}$
- ▶ **norm cone:**  $\{(x, t) \mid \|x\| \leq t\}$
- ▶ norm balls and cones are convex

Euclidean norm cone

$$\{(x, t) \mid \|x\|_2 \leq t\} \subset \mathbf{R}^{n+1}$$

is called **second-order cone**



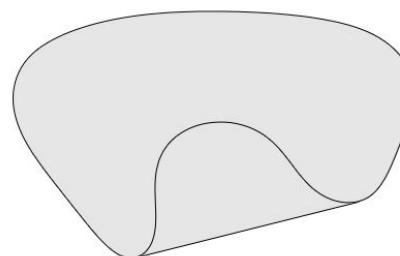
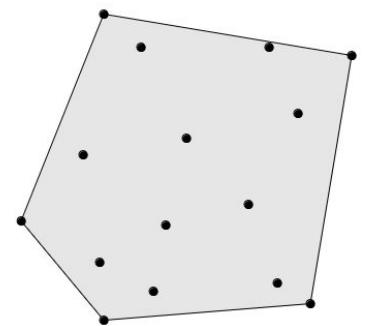
# Convex combination and hull

**convex combination** of  $x_1, \dots, x_k$ : any point  $x$  of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k$$

with  $\theta_1 + \cdots + \theta_k = 1$ ,  $\theta_i \geq 0$

**convex hull**  $\text{conv } S$ : set of all convex combinations of points in  $S$

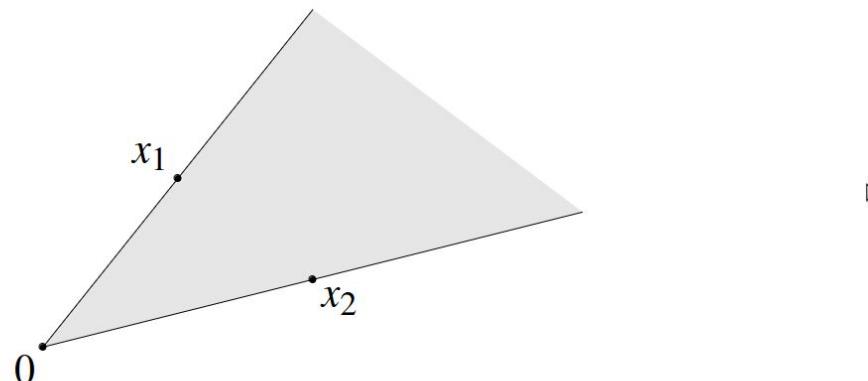


# Convex cone

**conic (nonnegative) combination** of  $x_1$  and  $x_2$ : any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2$$

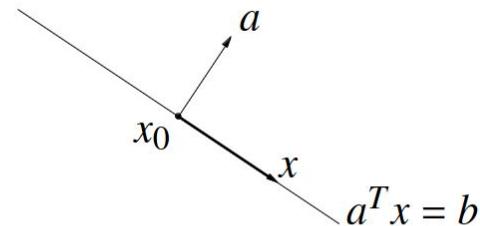
with  $\theta_1 \geq 0, \theta_2 \geq 0$



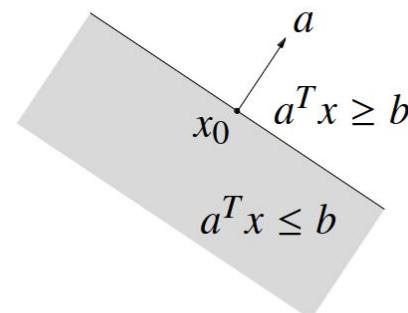
**convex cone**: set that contains all conic combinations of points in the set

# Hyperplanes and halfspaces

**hyperplane:** set of the form  $\{x \mid a^T x = b\}$ , with  $a \neq 0$



**halfspace:** set of the form  $\{x \mid a^T x \leq b\}$ , with  $a \neq 0$



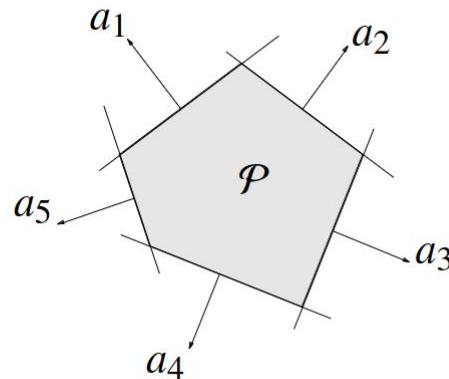
- ▶  $a$  is the normal vector
- ▶ hyperplanes are affine and convex; halfspaces are convex

- ▶ **Polyhedron** is solution set of finitely many linear inequalities and equalities

$$\{x \mid Ax \leq b, Cx = d\}$$

( $A \in \mathbf{R}^{m \times n}$ ,  $C \in \mathbf{R}^{p \times n}$ ,  $\leq$  is componentwise inequality)

- ▶ intersection of finite number of halfspaces and hyperplanes
- ▶ example with no equality constraints;  $a_i^T$  are rows of  $A$



# Showing a set is convex

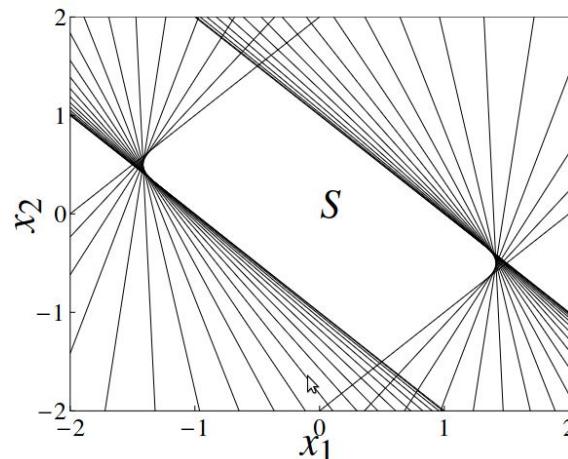
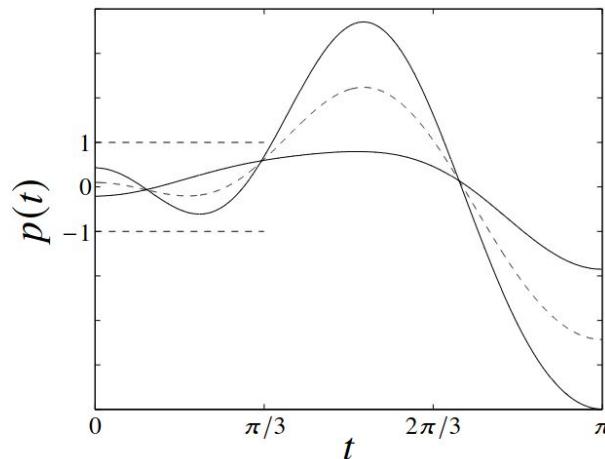
methods for establishing convexity of a set  $C$

1. apply definition: show  $x_1, x_2 \in C, 0 \leq \theta \leq 1 \implies \theta x_1 + (1 - \theta)x_2 \in C$ 
  - recommended only for **very simple** sets
2. use convex functions (next lecture)
3. show that  $C$  is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity
  - intersection
  - affine mapping
  - perspective mapping
  - linear-fractional mapping

you'll mostly use methods 2 and 3

# Showing a set is convex

- ▶ the intersection of (any number of) convex sets is convex
- ▶ **example:**
  - $S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$ , with  $p(t) = x_1 \cos t + \cdots + x_m \cos mt$
  - write  $S = \bigcap_{|t| \leq \pi/3} \{x \mid |p(t)| \leq 1\}$ , i.e., an intersection of (convex) slabs
- ▶ picture for  $m = 2$ :



# Showing a set is convex

- ▶ suppose  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$  is affine, i.e.,  $f(x) = Ax + b$  with  $A \in \mathbf{R}^{m \times n}$ ,  $b \in \mathbf{R}^m$
- ▶ the **image** of a convex set under  $f$  is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \implies f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

- ▶ the **inverse image**  $f^{-1}(C)$  of a convex set under  $f$  is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \implies f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

# Showing a set is convex

- ▶ **perspective function**  $P : \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$ :

$$P(x, t) = x/t, \quad \text{dom } P = \{(x, t) \mid t > 0\}$$

- ▶ images and inverse images of convex sets under perspective are convex
  - ▶ **linear-fractional function**  $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ :
- $$f(x) = \frac{Ax + b}{c^T x + d}, \quad \text{dom } f = \{x \mid c^T x + d > 0\}$$
- ▶ images and inverse images of convex sets under linear-fractional functions are convex

# Takeaway

$$x_1, x_2 \in C, \quad 0 \leq \theta \leq 1 \quad \implies \quad \theta x_1 + (1 - \theta)x_2 \in C$$

- Classic convex sets
  - Affine sets, hyperplanes, cones, balls, polyhedrons
- Convexity preserving operations
  - Intersection
  - Affine mapping
  - Perspective
  - Linear Fractional mapping

# Convex function overview

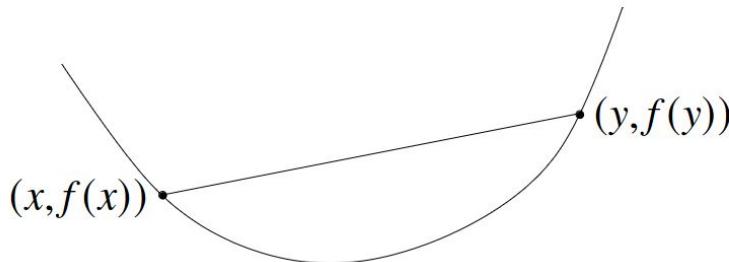
- Convex functions (with definition!)
  - Examples of classic convex functions
  - Extended value function
  - Line restriction
  - First and second order conditions
  - Epigraph and sublevel sets
- Showing a function is convex with operations
  - Non-negative weighted sum and affine composition
  - Pointwise maximum
  - Composition rules
  - Partial minimization and perspective
- Conjugate function

- **Convex functions (with definition!)**
  - **Examples of classic convex functions**
  - *Extended value function*
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# Convex functions

- $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is convex if  $\mathbf{dom}f$  is a convex set and for all  $x, y \in \mathbf{dom}f$ ,  $0 \leq \theta \leq 1$ ,

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$



- $f$  is concave if  $-f$  is convex
- $f$  is strictly convex if  $\mathbf{dom}f$  is convex and for  $x, y \in \mathbf{dom}f$ ,  $x \neq y$ ,  $0 < \theta < 1$ ,

$$f(\theta x + (1 - \theta)y) < \theta f(x) + (1 - \theta)f(y)$$

↳

# Convex functions (Examples)

convex functions:

- ▶ affine:  $ax + b$  on  $\mathbf{R}$ , for any  $a, b \in \mathbf{R}$
- ▶ exponential:  $e^{ax}$ , for any  $a \in \mathbf{R}$
- ▶ powers:  $x^\alpha$  on  $\mathbf{R}_{++}$ , for  $\alpha \geq 1$  or  $\alpha \leq 0$
- ▶ powers of absolute value:  $|x|^p$  on  $\mathbf{R}$ , for  $p \geq 1$
- ▶ positive part (relu):  $\max\{0, x\}$

concave functions:

- ▶ affine:  $ax + b$  on  $\mathbf{R}$ , for any  $a, b \in \mathbf{R}$
- ▶ powers:  $x^\alpha$  on  $\mathbf{R}_{++}$ , for  $0 \leq \alpha \leq 1$
- ▶ logarithm:  $\log x$  on  $\mathbf{R}_{++}$
- ▶ entropy:  $-x \log x$  on  $\mathbf{R}_{++}$
- ▶ negative part:  $\min\{0, x\}$

# Extended value

- ▶ suppose  $f$  is convex on  $\mathbf{R}^n$ , with domain  $\mathbf{dom} f$
- ▶ its extended-value extension  $\tilde{f}$  is function  $\tilde{f} : \mathbf{R}^n \rightarrow \mathbf{R} \cup \{\infty\}$

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \mathbf{dom} f \\ \infty & x \notin \mathbf{dom} f \end{cases}$$

- ▶ often simplifies notation; for example, the condition

$$0 \leq \theta \leq 1 \implies \tilde{f}(\theta x + (1 - \theta)y) \leq \theta \tilde{f}(x) + (1 - \theta) \tilde{f}(y)$$

(as an inequality in  $\mathbf{R} \cup \{\infty\}$ ), means the same as the two conditions

- $\mathbf{dom} f$  is convex
- $x, y \in \mathbf{dom} f, 0 \leq \theta \leq 1 \implies f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta) f(y)$

- ▶  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is convex if and only if the function  $g : \mathbf{R} \rightarrow \mathbf{R}$ ,

$$g(t) = f(x + tv), \quad \text{dom } g = \{t \mid x + tv \in \text{dom } f\}$$



- is convex (in  $t$ ) for any  $x \in \text{dom } f$ ,  $v \in \mathbf{R}^n$
- ▶ can check convexity of  $f$  by checking convexity of functions of one variable

# First order condition

- $f$  is **differentiable** if  $\text{dom}f$  is open and the gradient

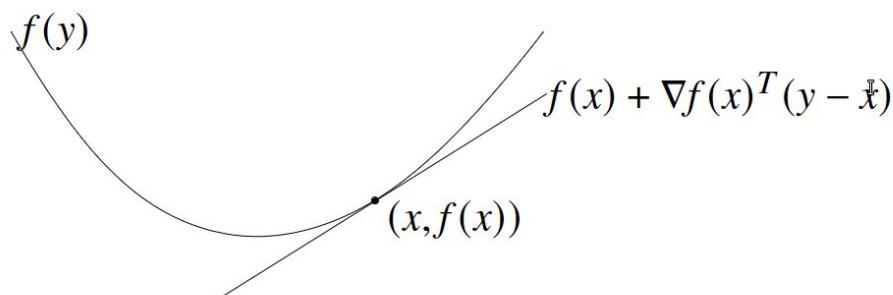
$$\nabla f(x) = \left( \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}, \dots, \frac{\partial f(x)}{\partial x_n} \right) \in \mathbf{R}^n$$

exists at each  $x \in \text{dom}f$

- **1st-order condition:** differentiable  $f$  with convex domain is convex if and only if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \text{for all } x, y \in \text{dom}f$$

- first order Taylor approximation of convex  $f$  is a **global underestimator** of  $f$



- ▶  $f$  is **twice differentiable** if  $\text{dom } f$  is open and the Hessian  $\nabla^2 f(x) \in \mathbf{S}^n$ ,

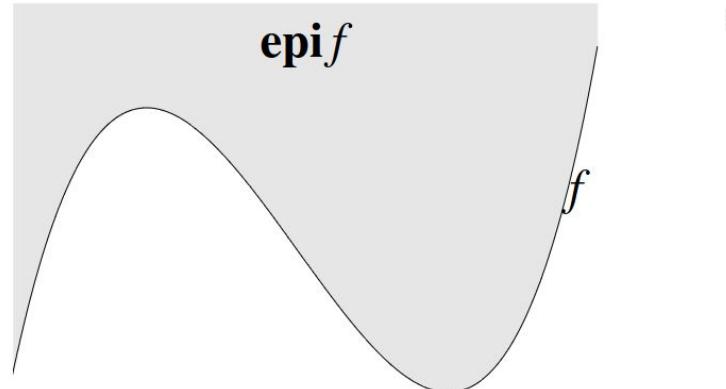
$$\nabla^2 f(x)_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}, \quad i, j = 1, \dots, n,$$

exists at each  $x \in \text{dom } f$

- ▶ **2nd-order conditions:** for twice differentiable  $f$  with convex domain
  - $f$  is convex if and only if  $\nabla^2 f(x) \succeq 0$  for all  $x \in \text{dom } f$
  - if  $\nabla^2 f(x) > 0$  for all  $x \in \text{dom } f$ , then  $f$  is strictly convex

# Epigraph and sublevel set

- ▶  **$\alpha$ -sublevel set** of  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is  $C_\alpha = \{x \in \mathbf{dom}f \mid f(x) \leq \alpha\}$
- ▶ sublevel sets of convex functions are convex sets (but converse is false)
- ▶ **epigraph** of  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is  $\text{epi } f = \{(x, t) \in \mathbf{R}^{n+1} \mid x \in \mathbf{dom}f, f(x) \leq t\}$



- ▶  $f$  is convex if and only if  $\text{epi } f$  is a convex set

# Showing a function is convex

methods for establishing convexity of a function  $f$

1. verify definition (often simplified by restricting to a line)
2. for twice differentiable functions, show  $\nabla^2 f(x) \geq 0$ 
  - recommended only for **very simple** functions
3. show that  $f$  is obtained from simple convex functions by operations that preserve convexity
  - nonnegative weighted sum
  - composition with affine function
  - pointwise maximum and supremum
  - composition
  - minimization
  - perspective

you'll mostly use methods 2 and 3



# Showing a function is convex

- ▶ **nonnegative multiple:**  $\alpha f$  is convex if  $f$  is convex,  $\alpha \geq 0$
  - ▶ **sum:**  $f_1 + f_2$  convex if  $f_1, f_2$  convex
  - ▶ **infinite sum:** if  $f_1, f_2, \dots$  are convex functions, infinite sum  $\sum_{i=1}^{\infty} f_i$  is convex
  - ▶ **integral:** if  $f(x, \alpha)$  is convex in  $x$  for each  $\alpha \in \mathcal{A}$ , then  $\int_{\alpha \in \mathcal{A}} f(x, \alpha) d\alpha$  is convex
- ↗
- ▶ there are analogous rules for concave functions

# Showing a function is convex

**(pre-)composition with affine function:**  $f(Ax + b)$  is convex if  $f$  is convex

## examples

- ▶ log barrier for linear inequalities

$$f(x) = - \sum_{i=1}^m \log(b_i - a_i^T x), \quad \text{dom } f = \{x \mid a_i^T x < b_i, i = 1, \dots, m\}$$

- ▶ norm approximation error:  $f(x) = \|Ax - b\|$  (any norm)

# Showing a function is convex

if  $f_1, \dots, f_m$  are convex, then  $f(x) = \max\{f_1(x), \dots, f_m(x)\}$  is convex

## examples

- ▶ piecewise-linear function:  $f(x) = \max_{i=1,\dots,m}(a_i^T x + b_i)$
- ▶ sum of  $r$  largest components of  $x \in \mathbf{R}^n$ :

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

( $x_{[i]}$  is  $i$ th largest component of  $x$ )

□

proof:  $f(x) = \max\{x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$

# Showing a function is convex

if  $f(x, y)$  is convex in  $x$  for each  $y \in \mathcal{A}$ , then  $g(x) = \sup_{y \in \mathcal{A}} f(x, y)$  is convex

## examples

- ▶ distance to farthest point in a set  $C$ :  $f(x) = \sup_{y \in C} \|x - y\|$
- ▶ maximum eigenvalue of symmetric matrix: for  $X \in \mathbf{S}^n$ ,  $\lambda_{\max}(X) = \sup_{\|y\|_2=1} y^T X y$  is convex
- ▶ support function of a set  $C$ :  $S_C(x) = \sup_{y \in C} y^T x$  is convex

# Showing a function is convex

- ▶ the function  $g(x) = \inf_{y \in C} f(x, y)$  is called the **partial minimization** of  $f$  (w.r.t.  $y$ )
- ▶ if  $f(x, y)$  is convex in  $(x, y)$  and  $C$  is a convex set, then partial minimization  $g$  is convex

## examples

- ▶  $f(x, y) = x^T A x + 2x^T B y + y^T C y$  with

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0, \quad C > 0$$



minimizing over  $y$  gives  $g(x) = \inf_y f(x, y) = x^T (A - BC^{-1}B^T)x$   
 $g$  is convex, hence Schur complement  $A - BC^{-1}B^T \succeq 0$

- ▶ distance to a set:  $\text{dist}(x, S) = \inf_{y \in S} \|x - y\|$  is convex if  $S$  is convex

# Showing a function is convex

- ▶ composition of  $g : \mathbf{R}^n \rightarrow \mathbf{R}$  and  $h : \mathbf{R} \rightarrow \mathbf{R}$  is  $f(x) = h(g(x))$  (written as  $f = h \circ g$ )
- ▶ composition  $f$  is convex if
  - $g$  convex,  $h$  convex,  $\tilde{h}$  nondecreasing
  - or  $g$  concave,  $h$  convex,  $\tilde{h}$  nonincreasing(monotonicity must hold for extended-value extension  $\tilde{h}$ )
- ▶ proof (for  $n = 1$ , differentiable  $g, h$ )

$$f''(x) = h''(g(x))g'(x)^2 + h'(g(x))g''(x)$$

## examples

- ▶  $f(x) = \exp g(x)$  is convex if  $g$  is convex
- ▶  $f(x) = 1/g(x)$  is convex if  $g$  is concave and positive



# Showing a function is convex

- ▶ composition of  $g : \mathbf{R}^n \rightarrow \mathbf{R}^k$  and  $h : \mathbf{R}^k \rightarrow \mathbf{R}$  is  $f(x) = h(g(x)) = h(g_1(x), g_2(x), \dots, g_k(x))$
- ▶  $f$  is convex if  $h$  is convex and for each  $i$  one of the following holds
  - $g_i$  convex,  $\tilde{h}$  nondecreasing in its  $i$ th argument
  - $g_i$  concave,  $\tilde{h}$  nonincreasing in its  $i$ th argument
  - $g_i$  affine
- ▶ you will use this composition rule **constantly** throughout this course
- ▶ you need to commit this rule to memory



- ▶ the **perspective** of a function  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  is the function  $g : \mathbf{R}^n \times \mathbf{R} \rightarrow \mathbf{R}$ ,

$$g(x, t) = tf(x/t), \quad \text{dom } g = \{(x, t) \mid x/t \in \text{dom } f, t > 0\}$$

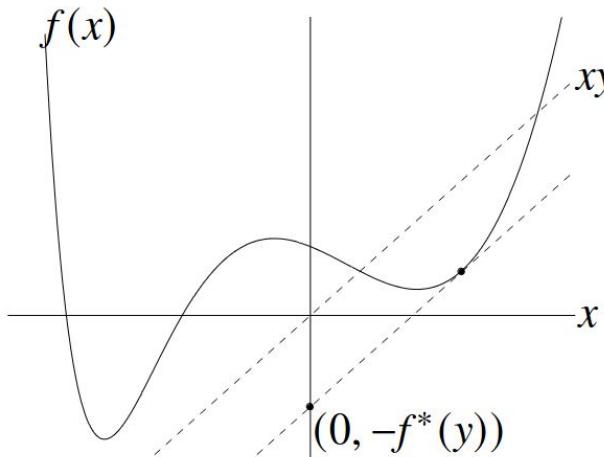
- ▶  $g$  is convex if  $f$  is convex

## examples

- ▶  $f(x) = x^T x$  is convex; so  $g(x, t) = x^T x/t$  is convex for  $t > 0$
- ▶  $f(x) = -\log x$  is convex; so relative entropy  $g(x, t) = t \log t - t \log x$  is convex on  $\mathbf{R}_{++}^2$

# Conjugate

- ▶ the **conjugate** of a function  $f$  is  $f^*(y) = \sup_{x \in \text{dom } f} (y^T x - f(x))$



↳

- ▶  $f^*$  is convex (even if  $f$  is not)
- ▶ will be useful in chapter 5

# Takeaway

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

- Classic convex functions
  - Affine, exponential, norms, max, ...
- Convexity preserving operations
  - Non negative weighted sum, composition with affine
  - Pointwise maximum and supremum
  - Composition
  - Minimization
  - Perspective

# 1. Convex problems

# Standard form optimization problem

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

- ▶  $x \in \mathbf{R}^n$  is the optimization variable
- ▶  $f_0 : \mathbf{R}^n \rightarrow \mathbf{R}$  is the objective or cost function
- ▶  $f_i : \mathbf{R}^n \rightarrow \mathbf{R}, i = 1, \dots, m$ , are the inequality constraint functions
- ▶  $h_i : \mathbf{R}^n \rightarrow \mathbf{R}$  are the equality constraint functions



# Feasible and optimal points

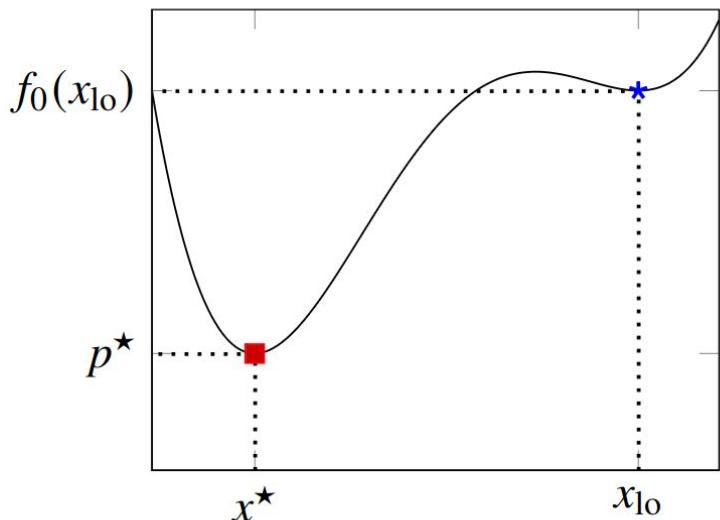
- ▶  $x \in \mathbf{R}^n$  is **feasible** if  $x \in \text{dom } f_0$  and it satisfies the constraints
- ▶ **optimal value** is  $p^\star = \inf\{f_0(x) \mid f_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, p\}$
- ▶  $p^\star = \infty$  if problem is infeasible
- ▶  $p^\star = -\infty$  if problem is **unbounded below**
- ▶ a feasible  $x$  is **optimal** if  $f_0(x) = p^\star$
- ▶  $X_{\text{opt}}$  is the set of optimal points



# Optimal points

$x$  is **locally optimal** if there is an  $R > 0$  such that  $x$  is optimal for

$$\begin{array}{ll}\text{minimize (over } z) & f_0(z) \\ \text{subject to} & f_i(z) \leq 0, \quad i = 1, \dots, m, \quad h_i(z) = 0, \quad i = 1, \dots, p \\ & \|z - x\|_2 \leq R\end{array}$$



standard form optimization problem has **implicit constraint**

$$x \in \mathcal{D} = \bigcap_{i=0}^m \mathbf{dom} f_i \cap \bigcap_{i=1}^p \mathbf{dom} h_i,$$

- ▶ we call  $\mathcal{D}$  the **domain** of the problem
- ▶ the constraints  $f_i(x) \leq 0, h_i(x) = 0$  are the **explicit constraints**
- ▶ a problem is **unconstrained** if it has no explicit constraints ( $m = p = 0$ )

**example:**

$$\text{minimize } f_0(x) = -\sum_{i=1}^k \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints  $a_i^T x < b_i$

# Feasibility problem

$$\begin{array}{ll}\text{find} & x \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$

can be considered a special case of the general problem with  $f_0(x) = 0$ :

$$\begin{array}{ll}\text{minimize} & 0 \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p\end{array}$$



- ▶  $p^\star = 0$  if constraints are feasible; any feasible  $x$  is optimal
- ▶  $p^\star = \infty$  if constraints are infeasible

# Standard form convex problem

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && a_i^T x = b_i, \quad i = 1, \dots, p \end{aligned}$$

- ▶ objective and inequality constraints  $f_0, f_1, \dots, f_m$  are convex
  - ▶ equality constraints are affine, often written as  $Ax = b$
  - ▶ feasible and optimal sets of a convex optimization problem are convex
- ◀
- 
- ▶ problem is **quasiconvex** if  $f_0$  is quasiconvex,  $f_1, \dots, f_m$  are convex,  $h_1, \dots, h_p$  are affine

# Optimum in a convex set

any locally optimal point of a convex problem is (globally) optimal

**proof:**

- ▶ suppose  $x$  is locally optimal, but there exists a feasible  $y$  with  $f_0(y) < f_0(x)$
- ▶  $x$  locally optimal means there is an  $R > 0$  such that

$$z \text{ feasible}, \quad \|z - x\|_2 \leq R \quad \implies \quad f_0(z) \geq f_0(x)$$

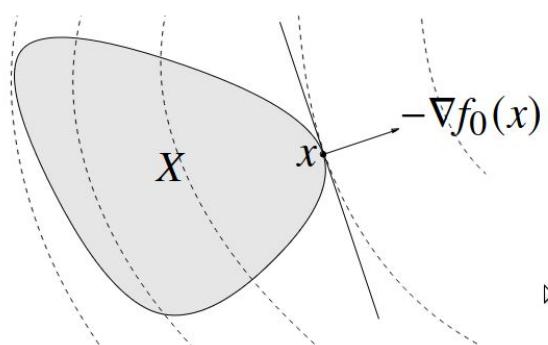
- ▶ consider  $z = \theta y + (1 - \theta)x$  with  $\theta = R/(2\|y - x\|_2)$
- ▶  $\|y - x\|_2 > R$ , so  $0 < \theta < 1/2$
- ▶  $z$  is a convex combination of two feasible points, hence also feasible
- ▶  $\|z - x\|_2 = R/2$  and  $f_0(z) \leq \theta f_0(y) + (1 - \theta)f_0(x) < f_0(x)$ , which contradicts our assumption that  $x$  is locally optimal



# First order criterion

- $x$  is optimal for a convex problem if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0 \text{ for all feasible } y$$

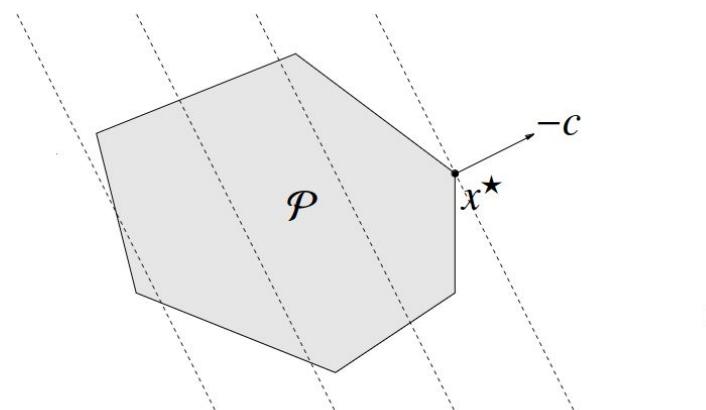


- if nonzero,  $\nabla f_0(x)$  defines a supporting hyperplane to feasible set  $X$  at  $x$

$$\begin{array}{ll}\text{minimize} & c^T x + d \\ \text{subject to} & Gx \leq h \\ & Ax = b\end{array}$$

**Often written as**  
**a maximization**

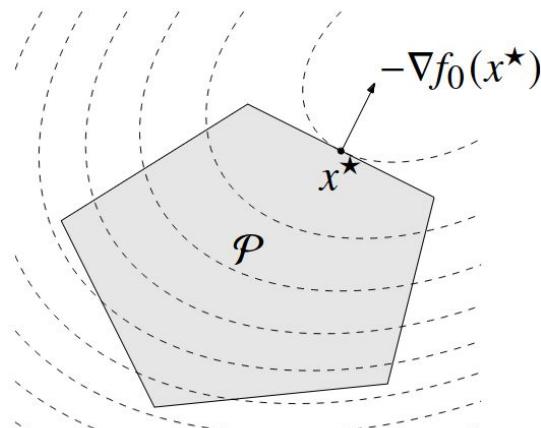
- ▶ convex problem with affine objective and constraint functions
- ▶ feasible set is a polyhedron



# Quadratic programming

$$\begin{array}{ll}\text{minimize} & (1/2)x^T Px + q^T x + r \\ \text{subject to} & Gx \leq h \\ & Ax = b\end{array}$$

- ▶  $P \in \mathbf{S}_+^n$ , so objective is convex quadratic
- ▶ minimize a convex quadratic function over a polyhedron



# Quadratically constrained Quadratic programming (QCQP)

$$\begin{aligned} & \text{minimize} && (1/2)x^T P_0 x + q_0^T x + r_0 \\ & \text{subject to} && (1/2)x^T P_i x + q_i^T x + r_i \leq 0, \quad i = 1, \dots, m \\ & && Ax = b \end{aligned}$$

- ▶  $P_i \in \mathbf{S}_+^n$ ; objective and constraints are convex quadratic
- ▶ if  $P_1, \dots, P_m \in \mathbf{S}_{++}^n$ , feasible region is intersection of  $m$  ellipsoids and an affine set

# Change of variable

- ▶  $\phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$  is one-to-one with  $\phi(\mathbf{dom} \phi) \supseteq \mathcal{D}$
- ▶ consider (possibly non-convex) problem

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

- ▶ change variables to  $z$  with  $x = \phi(z)$
- ▶ can solve equivalent problem

$$\begin{aligned} & \text{minimize} && \tilde{f}_0(z) \\ & \text{subject to} && \tilde{f}_i(z) \leq 0, \quad i = 1, \dots, \overset{\mathbb{I}}{m} \\ & && \tilde{h}_i(z) = 0, \quad i = 1, \dots, p \end{aligned}$$

where  $\tilde{f}_i(z) = f_i(\phi(z))$  and  $\tilde{h}_i(z) = h_i(\phi(z))$

- ▶ recover original optimal point as  $x^\star = \phi(z^\star)$

# Transformation

suppose

- ▶  $\phi_0$  is monotone increasing
- ▶  $\psi_i(u) \leq 0$  if and only if  $u \leq 0$ ,  $i = 1, \dots, m$
- ▶  $\varphi_i(u) = 0$  if and only if  $u = 0$ ,  $i = 1, \dots, p$

standard form optimization problem is equivalent to

$$\begin{aligned} & \text{minimize} && \phi_0(f_0(x)) \\ & \text{subject to} && \psi_i(f_i(x)) \leq 0, \quad i = 1, \dots, m \\ & && \varphi_i(h_i(x)) = 0, \quad i = 1, \dots, p \end{aligned}$$



example: minimizing  $\|Ax - b\|$  is equivalent to minimizing  $\|Ax - b\|^2$

# Maximization and minimization

- ▶ suppose  $\phi_0$  is monotone decreasing
- ▶ the maximization problem

$$\begin{aligned} & \text{maximize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

is equivalent to the minimization problem

$$\begin{aligned} & \text{minimize} && \phi_0(f_0(x)) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

- ▶ **examples:**
  - $\phi_0(u) = -u$  transforms maximizing a concave function to minimizing a convex function
  - $\phi_0(u) = 1/u$  transforms maximizing a concave positive function to minimizing a convex function

# Eliminating equality constraints

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && Ax = b \end{aligned}$$

is equivalent to

$$\begin{aligned} & \text{minimize (over } z) && f_0(Fz + x_0) \\ & \text{subject to} && f_i(Fz + x_0) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

where  $F$  and  $x_0$  are such that  $Ax = b \iff x = Fz + x_0$  for some  $z$

# Introducing equality constraints

$$\begin{aligned} & \text{minimize} && f_0(A_0x + b_0) \\ & \text{subject to} && f_i(A_i x + b_i) \leq 0, \quad i = 1, \dots, m \end{aligned}$$

is equivalent to

$$\begin{aligned} & \text{minimize (over } x, y_i\text{)} && f_0(y_0) \\ & \text{subject to} && f_i(y_i) \leq 0, \quad i = 1, \dots, m \\ & && y_i = A_i x + b_i, \quad i = 0, 1, \dots, m \end{aligned}$$



# Slack variables for linear equalities

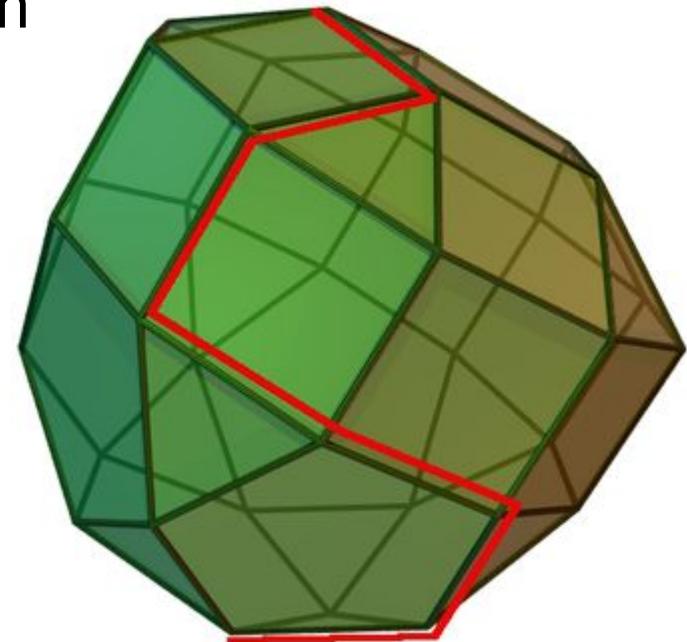
$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && a_i^T x \leq b_i, \quad i = 1, \dots, m \end{aligned}$$

is equivalent to

$$\begin{aligned} & \text{minimize (over } x, s) && f_0(x) \\ & \text{subject to} && a_i^T x + s_i = b_i, \quad i = 1, \dots, m \\ & && s_i \geq 0, \quad i = 1, \dots, m \end{aligned}$$

## 2. Simplex algorithm (for LP)

- Historical algorithm of optimization
  - Proposed by Dantzig
- Widely used even nowadays
  - “Usually” efficient
  - Bad worst case
- Move along constraint edges



## Standard form LP

Maximize  $5x_1 + 4x_2 + 3x_3$

Subject to :

$$2x_1 + 3x_2 - x_3 \leq 5$$

$$4x_1 + x_2 - 2x_3 \leq 11$$

$$3x_1 + 4x_2 - 2x_3 \leq 8$$

$$x_1, x_2, x_3 \geq 0.$$

Maximize  $5x_1 + 4x_2 + 3x_3$

Subject to :

$$2x_1 + 3x_2 - x_3 \leq 5$$

$$4x_1 + x_2 - 2x_3 \leq 11$$

$$3x_1 + 4x_2 - 2x_3 \leq 8$$

$$x_1, x_2, x_3 \geq 0.$$

# Add slack variables

$$2x_1 + 3x_2 + x_3 \leq 5$$

# Add slack variables

$$2x_1 + 3x_2 + x_3 \leq 5$$

$$x_4 = 5 - 2x_1 - 3x_2 - x_3.$$

$$x_4 \geq 0.$$

# Add slack variables

$$2x_1 + 3x_2 + x_3 \leq 5$$

$$\begin{array}{rclcl} 4x_1 & + & x_2 & - & 2x_3 \leq 11 \\ 3x_1 & + & 4x_2 & - & 2x_3 \leq 8 \end{array}$$

$$x_4 = 5 - 2x_1 - 3x_2 - x_3.$$

$$x_4 \geq 0.$$

# Add slack variables

$$2x_1 + 3x_2 + x_3 \leq 5$$

$$\begin{array}{rclclclcl} 4x_1 & + & x_2 & - & 2x_3 & \leq & 11 \\ 3x_1 & + & 4x_2 & - & 2x_3 & \leq & 8 \end{array}$$

$$x_4 = 5 - 2x_1 - 3x_2 - x_3.$$

$$\begin{array}{rclclclcl} x_5 & = & 11 & - & 4x_1 & - & x_2 & - & 2x_3 \\ x_6 & = & 8 & - & 3x_1 & - & 4x_2 & - & 2x_3 \end{array}$$

$$x_4 \geq 0.$$

$$x_5 \geq 0, x_6 \geq 0.$$

Maximize  $z$  subject to  $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$ .

$$x_4 = 5 - 2x_1 - 3x_2 - x_3$$

$$x_5 = 11 - 4x_1 - x_2 - 2x_3$$

$$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$$

$$z = 5x_1 + 4x_2 + 3x_3.$$

Maximize  $z$  subject to  $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$ .

$$x_4 = 5 - 2x_1 - 3x_2 - x_3$$

$$x_5 = 11 - 4x_1 - x_2 - 2x_3$$

$$x_6 = 8 - 3x_1 - 4x_2 - 2x_3$$

$$z = 5x_1 + 4x_2 + 3x_3.$$

# Find initial feasible solution

$$x_1 = 0, x_2 = 0, x_3 = 0$$

# Find initial feasible solution

$$x_1 = 0, x_2 = 0, x_3 = 0$$

$$x_4 = 5 - 2x_1 - 3x_2 - x_3 = 5$$

# Find initial feasible solution

$$x_1 = 0, x_2 = 0, x_3 = 0$$

$$x_4 = 5 - 2x_1 - 3x_2 - x_3 = 5$$

$$x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 5, x_5 = 11, x_6 = 8$$

$$z = 0$$

# Solving the simplex dictionary

Let's borrow an unconventional presentation from F. Giroire!

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & 5x_1 + 4x_2 + 3x_3. \end{array}$$

Basic variables:  $x_4, x_5, x_6$ , variables on the left.

Non-basic variable:  $x_1, x_2, x_3$ , variables on the right.

A dictionary is **feasible** if a feasible solution is obtained by setting all non-basic variables to 0.

# Find the most influential variable

**Simplex strategy:** find an optimal solution by successive improvements.

**Rule:** we increase the value of the variable of **largest positive coefficient** in  $z$ .

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & \boxed{5}x_1 + 4x_2 + 3x_3. \end{array}$$

Here, we try to increase  $x_1$ .

# How far can we go?

How much can we increase  $x_1$ ?

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & 5x_1 + 4x_2 + 3x_3. \end{array}$$

We have  $x_4 \geq 0$ .

It implies  $5 - 2x_1 \geq 0$ ,

that is  $x_1 \leq \frac{5}{2}$ .

# How far can we go?

How much can we increase  $x_1$ ?

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & 5x_1 + 4x_2 + 3x_3. \end{array}$$

We have  $x_4 \geq 0$ .

It implies  $5 - 2x_1 \geq 0$ , that is  $x_1 \leq 5/2$ .

Similarly,

$x_5 \geq 0$  gives  $x_1 \leq 11/4$ .

$x_6 \geq 0$  gives  $x_1 \leq 8/3$ .

# How far can we go?

How much can we increase  $x_1$ ?

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & 5x_1 + 4x_2 + 3x_3. \end{array}$$

We have  $x_4 \geq 0$ .

It implies  $5 - 2x_1 \geq 0$ ,

that is  $x_1 \leq 5/2$

Strongest constraint

Similarly,

$x_5 \geq 0$  gives  $x_1 \leq 11/4$ .

$x_6 \geq 0$  gives  $x_1 \leq 8/3$ .

# How far can we go?

How much can we increase  $x_1$ ?

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & 5x_1 + 4x_2 + 3x_3. \end{array}$$

We have  $x_4 \geq 0$ .

It implies  $5 - 2x_1 \geq 0$ , that is  $x_1 \leq 5/2$  Strongest constraint

We get a new solution:  $x_1 = 5/2$ ,  $x_4 = 0$

with better value  $z = 5 \cdot 5/2 = 25/2$ .

We still have  $x_2 = x_3 = 0$  and now  $x_5 = 11 - 4 \cdot 5/2 = 1$ ,  
 $x_6 = 8 - 3 \cdot 5/2 = 1/2$

# Pivot around the chosen variable

We build a new feasible dictionary.

$$\begin{array}{rcl} x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\ x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\ x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\ \hline z & = & 5x_1 + 4x_2 + 3x_3. \end{array}$$

$x_1$  enters the bases and  $x_4$  leaves it:

$$x_1 = 5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4$$

# Pivot around the chosen variable

We replace  $x_1$  by its expression in function of  $x_2, x_3, x_4$ .

$$\begin{array}{rcl} x_1 & = & 5/2 - 1/2x_4 - 3/2x_2 - 1/2x_3 \\ x_5 & = & 11 - 4(5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4) - x_2 - 2x_3 \\ x_6 & = & 8 - 3(5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4) - 4x_2 - 2x_3 \\ \hline z & = & 5(5/2 - 3/2x_2 - 1/2x_3 - 1/2x_4) + 4x_2 + 3x_3. \end{array}$$

# Pivot around the chosen variable

Finally, we get the new dictionary:

$$\begin{array}{rcl} x_1 & = & \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 & = & 1 + 5x_2 \\ x_6 & = & \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\ \hline z & = & \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4. \end{array}$$

# New system

Finally, we get the new dictionary:

$$\begin{array}{rcl} x_1 & = & \boxed{\frac{5}{2}} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 & = & \boxed{1} + 5x_2 \quad \quad \quad + 2x_4 \\ x_6 & = & \boxed{\frac{1}{2}} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\ \hline z & = & \boxed{\frac{25}{2}} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4. \end{array}$$

We can read the solution directly from the dictionary:

Non basic variables:  $x_2 = x_3 = x_4 = 0$ .

Basic variables:  $x_1 = 5/2$ ,  $x_5 = 1$ ,  $x_6 = 1/2$ .

Value of the solution:  $z = 25/2$ .

# New step!

$$\begin{array}{rcl} x_1 & = & \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ x_5 & = & 1 + 5x_2 + 2x_4 \\ x_6 & = & \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\ \hline z & = & \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4. \end{array}$$

New step of the simplex:

- $x_3$  enters the basis (variable with largest positive coefficient).
- 3<sup>d</sup> equation is the strictest constraint  $x_3 \leq 1$ .
- $x_6$  leaves the basis.

# And we're done!

New feasible dictionary:

$$\begin{array}{rcl} x_3 & = & 1 + x_2 + 3x_4 - 2x_6 \\ x_1 & = & 2 - 2x_2 - 2x_4 + x_6 \\ \hline x_5 & = & 1 + 5x_2 + 2x_4 \\ \hline z & = & 13 - 3x_2 - x_4 - x_6. \end{array}$$

With new solution:

$$x_1 = 2, x_2 = 0, x_3 = 1, x_4 = 0, x_5 = 1, x_6 = 0$$

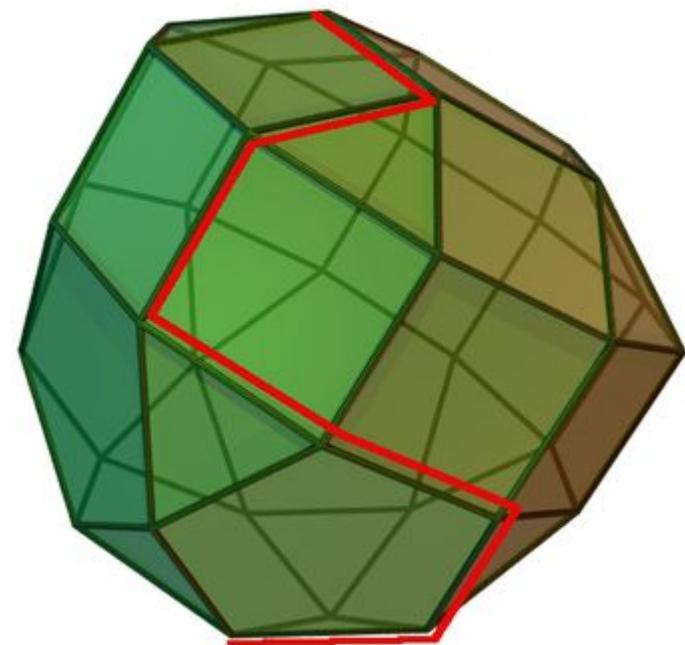
of value  $z = 13$ .

This solution is optimal.

All coefficients in  $z$  are negative and  $x_2 \geq 0, x_4 \geq 0, x_6 \geq 0$ , so  $z \leq 13$ .

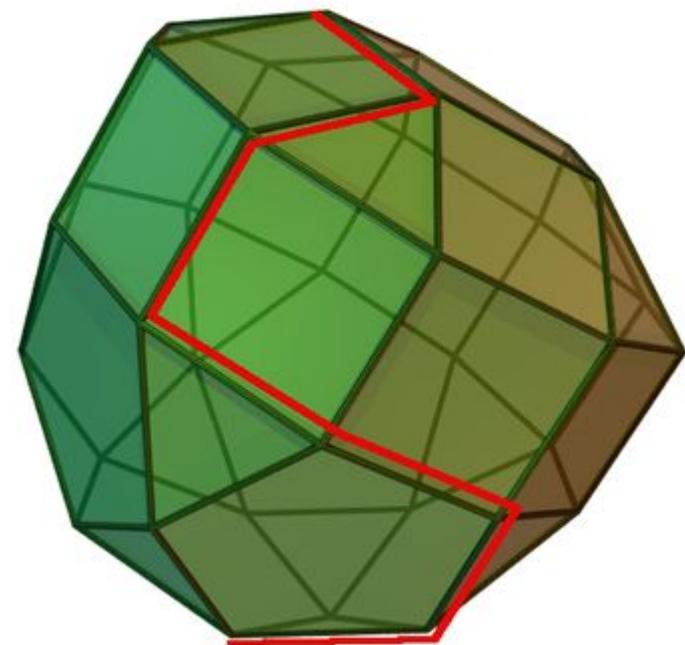
# Let's recap

- We convert to canonical form to work along equalities
- We start along some corner
- We move as far as we can on the best edge
  - Until we are done



# Let's recap

- We convert to canonical form to work along equalities
- We start along some corner
- We move as far as we can on the best edge
  - Until we are done
- **Usually solved with a simplex tableau**



# Simplex tableau

$$\begin{array}{rcl}
 x_4 & = & 5 - 2x_1 - 3x_2 - x_3 \\
 x_5 & = & 11 - 4x_1 - x_2 - 2x_3 \\
 x_6 & = & 8 - 3x_1 - 4x_2 - 2x_3 \\
 \hline
 z & = & 5x_1 + 4x_2 + 3x_3.
 \end{array}$$

x1	x2	x3	x4	x5	x6	z	c
2	3	1	1	0	0	0	5
4	1	2	0	1	0	0	11
3	4	2	0	0	1	0	8
-5	-4	-3	0	0	0	1	0

# Simplex tableau

x1	x2	x3	x4	x5	x6	Z	C
2	3	1	1	0	0	0	5 (5/2)
4	1	2	0	1	0	0	11 (11/4)
3	4	2	0	0	1	0	8 (8/3)
-5	-4	-3	0	0	0	1	0

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
4	1	2	0	1	0	0	11
3	4	2	0	0	1	0	8
-5	-4	-3	0	0	0	1	0

# Simplex tableau

x1	x2	x3	x4	x5	x6	Z	C
2	3	1	1	0	0	0	5 (5/2)
4	1	2	0	1	0	0	11 (11/4)
3	4	2	0	0	1	0	8 (8/3)
-5	-4	-3	0	0	0	1	0

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
0	-5	0	-2	1	0	0	1
0	-0.5	0.5	-1.5	0	1	0	0.5
0	3.5	-0.5	2.5	0	0	1	12.5

# Simplex tableau

$$\begin{array}{rcl}
 x_1 & = & \frac{5}{2} - \frac{3}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \\
 x_5 & = & 1 + 5x_2 + 2x_4 \\
 x_6 & = & \frac{1}{2} + \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\
 \hline
 z & = & \frac{25}{2} - \frac{7}{2}x_2 + \frac{1}{2}x_3 - \frac{5}{2}x_4.
 \end{array}$$

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
0	-5	0	-2	1	0	0	1
0	-0.5	0.5	-1.5	0	1	0	0.5
0	3.5	-0.5	2.5	0	0	1	12.5

# Simplex tableau

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5 (5)
0	-5	0	-2	1	0	0	1 (inf)
0	-0.5	0.5	-1.5	0	1	0	0.5 (1)
0	3.5	-0.5	2.5	0	0	1	12.5

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5
0	-5	0	-2	1	0	0	1
0	-1	1	-3	0	2	0	1
0	3.5	-0.5	2.5	0	0	1	12.5

# Simplex tableau

x1	x2	x3	x4	x5	x6	Z	C
1	1.5	0.5	0.5	0	0	0	2.5 (5)
0	-5	0	-2	1	0	0	1 (inf)
0	-0.5	0.5	-1.5	0	1	0	0.5 (1)
0	3.5	-0.5	2.5	0	0	1	12.5

x1	x2	x3	x4	x5	x6	Z	C
1	2	0	2	0	-1	0	2
0	-5	0	-2	1	0	0	1
0	-1	1	-3	0	2	0	1
0	3	0	1	0	1	1	13

# Simplex tableau

$$\begin{array}{rcl}
 x_3 & = & 1 + x_2 + 3x_4 - 2x_6 \\
 x_1 & = & 2 - 2x_2 - 2x_4 + x_6 \\
 x_5 & = & 1 + 5x_2 + 2x_4 \\
 \hline
 z & = & 13 - 3x_2 - x_4 - x_6.
 \end{array}$$

x1	x2	x3	x4	x5	x6	Z	C
1	2	0	2	0	-1	0	2
0	-5	0	-2	1	0	0	1
0	-1	1	-3	0	2	0	1
0	3	0	1	0	1	1	13

# Algorithm

- Build tableau from canonical
- Check we have feasible solution
- Do a pivot step if negative coef
  - Pick column  $c$  w/ most negative coefficient
  - Pick row  $r$  w/ smallest ratio
  - Pivot! (set  $c$  to 1 in  $r$  and 0 in other rows)

