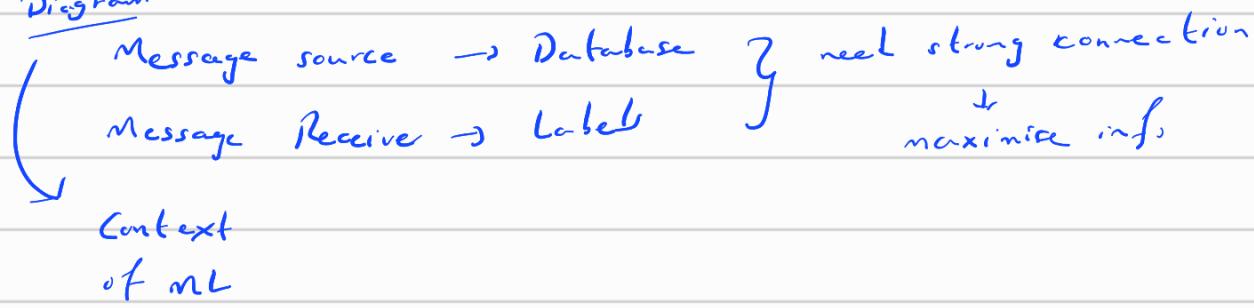


LI

Information Theory & Coding

Diagram



Self-Information

slide 2: when A occurs then, \Rightarrow lot of information
 (if $P(A)$ is low)

$I(A)$: qty of info

$$f(p^n) = f(p^{n-1} \cdot p) = n f(p)$$

$$f(p) = \left(f(p^{\frac{1}{n}}) \right)^n \approx n f(p^{\frac{1}{n}})$$

$$f(p^{\frac{m}{n}}) = f\left(\left(p^{\frac{1}{n}}\right)^m\right) = m f(p^{\frac{1}{n}})$$

$$\Rightarrow f(p^{\frac{1}{n}}) = \underbrace{f(p^{\frac{m}{n}})}_m$$

$$f(p^{\frac{1}{n}}) = \underline{\underline{f(p)}}$$

Exercise 1

$$\begin{aligned} h(B|A) &= -\log_2 P(A, B) \\ &= -\log_2 P(A) - \log_2 P(B|A) \end{aligned}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

when A and B are independent

$$P(B|A) = \frac{P(A) P(B)}{P(A)} = P(B)$$

$$\Rightarrow h(A, B) = -\log_2 P(A) - \log_2 P(B) \\ = h(A) + h(B)$$

Exercise 2

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$h(B|A) = -\log_2 P(A, B) \\ = -\log_2 P(A) - \log_2 P(B|A)$$

$$B \subset A \Leftrightarrow P(A|B) = 1$$

$$\therefore h(A, B) = h(A) + h(B|A)$$

$$P(A|B) = \frac{P(B|A) P(A)}{P(B)} \quad \text{Bayes Theorem}$$

$$\Rightarrow P(B|A) = \frac{P(B)}{P(A)}$$

$$\Rightarrow h(A, B) = h(A) + h(B) = h(A) \\ = h(B)$$

$$h(A, B) = h(A) + h(B)$$

$(h(A, B) \rightarrow \text{whole diagram})$

$$i(A, B) = h(A) - h(A|B) = h(B) - h(B|A)$$

$$A \subset B \Rightarrow i(B|A) = 1$$

$$\Rightarrow h(B|A) = -\log_2(\beta/A) = 0$$

$$i(A, B) = h(B) - h(B|A)$$

$$= h(B)$$



$$(i) \quad i(A, B) = h(A, B) - h(B|A) - h(A|B) \\ = h(A) + h(B) - h(B) - h(A) \\ = 0$$

$$(ii) \quad i(A, B) = h(A, B) - h(B|A) - h(A|B) \\ = h(B) - h(B|A) -$$

$$\begin{aligned} \cdot h(A) &= -\log P(A) & p(A, B) = p(A) p(B|A) \\ \cdot h(A, B) &= -\log p(A, B) = -\log p(A) - \log p(B|A) \\ \cdot h(B|A) &= -\log p(B|A) \\ \cdot h(B|A) &= h(A, B) - h(A) \quad | \quad h(A, B) = h(A) + h(B|A) \end{aligned}$$

$$\begin{aligned} \cdot B \subset A \Rightarrow p(A|B) &= 1 \\ \cdot A = B \Rightarrow \frac{A \subset B \text{ & } B \subset A}{h(A, B) = h(A)} \end{aligned}$$

$$\cdot i(A, B) = h(A) - h(A|B) = h(B) - h(B|A)$$

$$\cdot H(S) = E[h(S)] = - \sum_{i=1}^n p_i \log p_i \Rightarrow \begin{matrix} \text{richness} \\ \text{of source} \\ (\text{entropy}) \end{matrix}$$

$$\cdot H(X) = - \sum_{i=1}^n p(X=x_i) \log p(X=x_i)$$

$$\cdot H_2(p) = -p \log p - (1-p) \log (1-p)$$

$$\cdot \lim_{p \rightarrow 0} p \log p = 0$$

$$\cdot \text{Gibbs' inequality: } \sum_{i=1}^n p_i \log \frac{q_i}{p_i} \leq 0 \quad \left(\begin{matrix} p_i \text{ & } q_i \text{ are} \\ \text{prob} \end{matrix} \right)$$

$$\cdot H_n(p_1, \dots, p_n) \leq \log n \quad \left(\begin{matrix} \text{entropy of } n \text{ states} \\ \text{is bound by log of} \\ n \text{ of states} \end{matrix} \right)$$

↑
maximum entropy obtained when $p_i = 1/n$ (same prob for all states)

$$\cdot H(X, Y) = - \sum_{i=1}^n \sum_{j=1}^m p(X=x_i, Y=y_j) \log p(X=x_i, Y=y_j)$$

↓
→ $m n$ states \Rightarrow bounded by $\log mn$
→ symmetric

- $H(X, Y) = H(X) + H(Y)$ if X and Y are ind.
- $H(X|Y=y_i) = -\sum_{i=1}^n P(X=x_i|Y=y_i) \log P(X=x_i|Y=y_i)$
(y_i is fixed)
- $H(X|Y) = \sum_{j=1}^n P(Y=y_j) H(X|Y=y_j)$
- $H(X, Y) = H(X) + H(Y|X)$
 $= H(Y) + H(X|Y)$
- $H(x_1, \dots, x_n) = \sum_{i=1}^n H(x_i|x_1, \dots, x_{i-1})$
- $H(X) \leq H(X, Y)$ (since entropy is always +ve)
 $H(Y) \leq H(X, Y)$
- $H(X|Y) \leq H(X)$ (not proved)
- $I(X, Y) = H(X) - H(X|Y)$
 $= \sum \sum P(X=x_i, Y=y_j) \log \frac{P(X=x_i, Y=y_j)}{P(X=x_i) P(Y=y_j)}$

Slide 23: Say when each one becomes equality
 \downarrow
 s inequalities

X, Y independent

or $X = f(Y)$ with f one-to-one application
 (f^{-1})

depending on the inequality

If X & Y independent:

$$\begin{aligned} H(X, Y) &= - \sum_{i=1}^n \sum_{j=1}^m p(X=x_i) p(Y=y_j) \log [p(X=x_i) p(Y=y_j)] \\ &= - \sum_{i=1}^n \sum_{j=1}^m p(X=x_i) p(Y=y_j) [\log p(X=x_i) \\ &\quad + \log p(Y=y_j)] \\ &= - \sum_{i=1}^n \sum_{j=1}^m p(X=x_i) \log p(X=x_i) p(Y=y_j) \\ &\quad + p(Y=y_j) \log p(Y=y_j) p(X=x_i) \\ &= - \sum_{i=1}^n p(X=x_i) \log p(X=x_i) \sum_{j=1}^m p(Y=y_j) \\ &\quad - \sum_{j=1}^m p(Y=y_j) \log p(Y=y_j) \sum_{i=1}^n p(X=x_i) \\ &= - \sum_{i=1}^n p(X=x_i) \log [p(X=x_i)] + \\ &\quad - \sum_{j=1}^m p(Y=y_j) \log [p(Y=y_j)] \\ &= H(X) + H(Y) \end{aligned}$$

L2 : Part 1 Exercises (Quantitative measure of info)

Exercise 1

Event B : { Today is my birthday }

$$(i) P(B) = 1/365$$

$$H(B) = -\log_2 P(B) = -\log_2 (1/365) = \log_2 (365) = 8.5$$

$$(ii) H(s) = -\sum_{i=1}^2 p_i \log p_i = \frac{1}{365} \log_2 365 + \frac{364}{365} \log_2 \left(\frac{365}{364} \right)$$

$$= 0.028$$

Exercise 2

$$64 \underbrace{32}_{\text{}} \underbrace{16}_{\text{}} \underbrace{8}_{\text{}} \underbrace{4}_{\text{}} \underbrace{2}_{\text{}} \underbrace{1}_{\text{}}$$

$$P(x_i) = 1/64$$

$$H(x_i) = -\log_2 1/64 = \log_2 (64) = 6$$

$$H[x_i] = -\sum_{i=1}^{64} \frac{1}{64} \log_2 \left(\frac{1}{64} \right) = 6$$

$$H(s) = \log_2 (64) = 6 \text{ sh/state}$$

= number of question

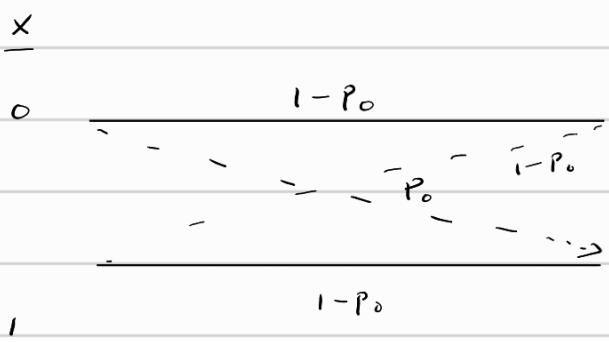
Interpretation: Each question \rightarrow 1 sh (2 responses per each question since same prob for each response)

\Rightarrow We require 6 sh of info to find the piece

\Rightarrow 6 questions needed

\Rightarrow That's why we cut board in half \rightarrow max info each time
(2 equal parts) (1 sh)

Ex 6



$$\begin{aligned}
 & \underline{x} & \underline{y} & P(X=0) = 1/4 \\
 & 0 & 0 & P(X=1) = 3/4 \\
 & \quad \quad \quad p_0 = P(Y=0|X=1) \\
 & \quad \quad \quad = P(Y=1|X=0) = 0.1 \\
 & 1 & 1 & 1-p_0 = P(Y=0|X=0) \\
 & & & = P(Y=1|X=1)
 \end{aligned}$$

$$H(X) = - \sum_{i=0}^1 p(X=i) \log_2 p(X=i) = 0.9$$

$$= - \left[\frac{1}{4} \log_2 \left(\frac{1}{4} \right) + \frac{3}{4} \log_2 \left(\frac{3}{4} \right) \right]$$

$$= - \left[\frac{1}{4} (-2) + \frac{3}{4} (-0.415) \right]$$

= 0.811 sh/state of X

$$(Y=0) = (Y=0 \text{ & } X=1) + (Y=0 \text{ & } X=0)$$

$$\begin{aligned}
 P(Y=0) &= P(X=0)(1-p_0) + P(X=1)p_0 \\
 &= 0.25(0.1) + 0.75(0.1) \\
 &= \frac{3}{10}
 \end{aligned}$$

$$\begin{aligned}
 P(Y=1) &= P(X=0)p_0 + P(X=1)(1-p_0) \\
 &= (0.25)(0.1) + 0.75(0.1) \\
 &= 0.025 + 0.675 \\
 &= 0.7 (1 - 3/10)
 \end{aligned}$$

$$H[Y] = - \left[\frac{3}{10} \log_2 \frac{3}{10} + 0.7 \log_2 (0.7) \right]$$

-

$$H(X, Y) = - \sum_{i=0}^1 \sum_{j=0}^1 P(X=x_i, Y=y_j) \log_2 P(X=x_i, Y=y_j)$$

$$P(X=0, Y=0) = 9/40$$

$$\frac{P(X=0)}{P(X=1)} \frac{P(Y/X=0)}{P(Y/X=1)}$$

$$P(X=0, Y=1) = 0.025$$

$$P(X=0)$$

$$P(X=1, Y=1) = 0.675$$

$$H(X|Y=y_j) = - \sum_{i=1}^n P(X=x_i | Y=y_j) \log P(X=x_i | Y=y_j)$$

(y_j is fixed)

$$H(X|Y) = \sum_{j=1}^n P(Y=y_j) H(X|Y=y_j)$$

$$H(Y|X) = \sum_{i=1}^n P(X=x_i) H(Y|X=x_i)$$

$$\rightarrow H(Y|X=0) = - P(Y=0 | X=0) \log P(Y=0 | X=0)$$

$$- P(Y=1 | X=0) \log P(Y=1 | X=0)$$

$$0.332^2$$

$$0.1368$$

$$- (0.9 \log_2(0.9)) - (0.1 \log(0.1))$$

$$\approx 0.469$$

$$H(Y|X=1) = - P(Y=0 | X=1) \log P(Y=0 | X=1)$$

$$- P(Y=1 | X=1) \log P(Y=1 | X=1)$$

$$- 0.1 \log 0.1 + 0.9 \log 0.9$$

$$\approx 0.469$$

$$H(Y/X) = 0.25(0.469) + 0.75(0.469)$$

$$= 0.469$$

$$H(X,Y) = H(X/Y) + H(Y)$$

$$= H(Y/X) + H(X)$$

$$\Rightarrow H(Y/X) = 0.4$$

$$H(X/Y) = 0.47$$

$$I(X,Y) = H(X,Y) - H(X/Y) - H(Y/X)$$

Ex 3

$$X \sim \text{Geo}(1/2)$$

$$P(X=k) = 0.5^{k-1} \cdot 0.5 = 0.5^k$$

$$H(X) = - \sum_{n=1}^{\infty} 0.5^n \log(0.5)^n \stackrel{\log 0.5}{=} -\log(2)^{-1}$$

$$= - \sum_{n=1}^{\infty} n \cdot 0.5^n \cdot \log(1/2) = -n$$

$$= \frac{0.5}{(1-0.5)^2} = \frac{0.5}{0.25} = 2$$

Problem 2

$$H(S) = \log_2 18 \text{ sh/scenario}$$

2) 3 possible outcomes : Equilibrium (P_E)
Left heavier (P_L)
Right heavier (P_R)

$$H_{\max}(P) = \log_2 3 \text{ sh/weight}$$

experiment

$$\text{Min no. of operations} = \frac{\log_2 18}{\log_2 3} \approx 2.63 \text{ experiments}$$

$$3) \quad P_{eq} = \frac{\binom{2n}{8}}{\binom{2n}{9}} = \frac{\frac{8!}{(8-2n)! \cdot 2^n!}}{\frac{9!}{(9-2n)! \cdot 2^n!}} = \frac{1 - \frac{2n}{9}}{1 - \frac{8}{9}}$$

$$P_L = P_r = \frac{1}{2} \left(1 - \left(1 - \frac{2n}{9} \right) \right)$$

4) Cond⁺ to get max of info

$$P_e = P_r = P_c \Leftrightarrow 1 - \frac{z^n}{q} = \frac{n}{q} \Rightarrow n = 3$$

Part 2 : Exerciser

$$1) H_{\max}(X) = \log_2(m)$$

When X is uniformly distributed

$$2) H(Y) = \log_2(n)$$

$$3) 0 \leq H(X|Y) \leq [H(X) \leq H(Y)] \leq H(X, Y) \leq H(X) + H(Y)$$

$$4) \begin{array}{l} \text{Equality: } X \text{ & } Y \\ \text{are independent} \end{array} \quad \begin{array}{l} \text{Equality if} \\ X \text{ is uniformly} \\ \text{dist} \end{array} \quad \begin{array}{l} \text{Equality} \\ \text{if } X \subset Y \end{array} \quad \begin{array}{l} \text{Equality} \\ \text{if } X \\ \text{& } Y \text{ are} \\ \text{independent} \end{array}$$

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$$5) P(Y=y_1) = \frac{1}{6} + \frac{1}{24} + \frac{1}{24} = \frac{1}{4}$$

$$P(Y=y_2) = \frac{1}{12} + \frac{1}{8} + \frac{1}{24} = \frac{1}{4}$$

$$P(Y=y_3) = \frac{1}{4}$$

$$P(Y=y_4) = \frac{1}{4}$$

$$6) I(X, Y) = H(Y) - H(Y|X)$$

$$= 2 -$$

$$H(Y|X) = \sum_{i=1}^3 H(Y|X=x_i) p(X=x_i)$$

To calculate $H(Y|X=x_i)$

We need the dist of $P(Y|X=x_i)$

$$\text{Use } P(Y=y_i | X=x_i) = \frac{P(X=x_i, Y=y_i)}{P(X=x_i)}$$

$P(Y x)$	y_1	y_2	y_3	y_4
x_1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$
x_2	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{3}$
x_3	y_4	y_1	y_3	y_2

we have
 the cond
 in table by
 $P(X=x_i) = \frac{1}{3}$

$$H(Y|X=x_1) = H\left(\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{1}{8}\right)$$

$$H(Y|X=x_2) = \dots$$

$$H(Y|X=x_3) = \dots$$

$$\therefore I(X,Y) = 2 - \left[\frac{1}{3} H\left(\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{1}{8}\right) + \frac{1}{2} H(\dots) + \frac{1}{6} H(\dots) \right]$$

$$2 - \left(\frac{1.75}{3} + \frac{1.85}{2} + \frac{2}{6} \right)$$

$$= 2 - 0.58 - 0.925 + 0.33$$

$$= 0.15$$

Problem

$$1) P(m=0) = \frac{3}{16} + \frac{1}{8} = \frac{5}{16}$$

$$P(m=1) = \frac{5}{8} + \frac{1}{16} = \frac{11}{16}$$

$$P(T=1) = \frac{5}{8} + \frac{3}{16} = \frac{13}{16}$$

$$P(T=0) = \frac{3}{16}$$

$P_{\text{err}}(m) =$

$$2) P(m=1, T=0) + P(m=0, T=1)$$

$$= \frac{1}{16} + \frac{3}{16} = \frac{4}{16} = \frac{1}{4}$$

$$3) P_{\text{err}}(s) = P(s=1, T=0) + P(s=0, T=1)$$

$$= \frac{3}{16} < \frac{4}{16}$$

Student is better

$$4) I(E, T) = H(E) - H(E|T)$$

$$= 0$$

$$5) I(T, m) = H(T) + H(m) - H(T, m) \approx 0.091$$

$$7) P(E=1) = 496/512 = \frac{31}{32}$$

$$P(\bar{E}=0) = 16/512 = \underline{\frac{1}{32}}$$

