Optimization

Assignment → HomeWork Solution

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① Convex Sets →

Exercise ① Is it Convex?

ⓐ $A := \{x \in R^n \mid Ax = b\}$ with $A \in R^{n \times n}$, $b \in R^n$

⇒ Let, $x_1, x_2 \in A$

$\therefore Ax_1 = b$ — ①

$\therefore Ax_2 = b$ — ②

Now take, y is a convex set of x_1 & x_2.

$\therefore y = \theta x_1 + (1-\theta) x_2$, for $\theta \in [0,1]$ & $\sum \theta_i = 1$

So, we need to prove that, $Ay = b$.

Now, $Ay = A[\theta x_1 + (1-\theta) x_2] = \theta A x_1 + (1-\theta) A x_2$

$= \theta b + (1-\theta) b$ {From ① & ②}

$= b$

$\Rightarrow \boxed{Ay = b}$ $\Rightarrow A := \{x \in R^n \mid Ax = b\}$ is convex set.

ⓑ $S = \{x \in R^n \mid \|x - x_c\|_2 = r\}$ with $x_c \in R^n$, $r > 0$

⇒ Let, $x_1, x_2 \in S$

$\therefore \|x_1 - x_c\|_2 = r$ — ①

$\therefore \|x_2 - x_c\|_2 = r$ — ②

Now take y is a convex set of x_1 & x_2

$\therefore y = \theta x_1 + (1-\theta)x_2$, For $\theta \in [0,1]$, & $\sum_{i=1}^{n} \theta = 1$

So, we have to prove, $\|y - x_c\|_2 = r$

Now, $\|y - x_c\|_2 = \|\theta x_1 + (1-\theta)x_2 - x_c\|_2$

$= \|\theta x_1 + (1-\theta)x_2 - x_c + \theta x_c - \theta x_c\|_2$

$= \|\theta(x_1 - x_c) + (1-\theta)(x_2 - x_c)\|_2$

$\leq \|\theta(x_1 - x_c)\|_2 + \|(1-\theta)(x_2 - x_c)\|_2$ {By Applying Triangular inequality}

$\leq \theta\|x_1 - x_c\|_2 + (1-\theta)\|x_2 - x_c\|_2$ {By Homogeneity}

$\leq \theta r + (1-\theta)r$

$= \cancel{\theta r} + r - \cancel{\theta r}$

$\Rightarrow \|y - x_c\|_2 \leq r$

$\Rightarrow$ So, the Condition fails

$\Rightarrow$ It is Not Convex (Answer)

(c) $H = \{x \in R^n \mid \|x - x_1\|_2 \leq \|x - x_2\|_2\}$ with $x_1, x_2 \in R^n$

$\Rightarrow \|x - x_1\|_2^2 - \|x - x_2\|_2^2 = \|x\|_2^2 - 2x^T x_1 + \|x_1\|_2^2 - \|x\|_2^2 + 2x^T x_2 - \|x_2\|_2^2$

$= 2x^T(x_2 - x_1) - (\|x_2\|_2^2 - \|x_1\|_2^2) \leq 0$

$\Rightarrow x^T(x_2 - x_1) \leq \frac{1}{2}\left(\|x_2\|_2^2 - \|x_1\|_2^2\right)$

Let, $u, v \in H$ and $\theta \in [0,1]$ $\Rightarrow \begin{cases} u^T(x_2 - x_1) \leq \frac{1}{2}\left(\|x_2\|_2^2 - \|x_1\|_2^2\right) \\ v^T(x_2 - x_1) \leq \frac{1}{2}\left(\|x_2\|_2^2 - \|x_1\|_2^2\right) \end{cases}$

$\therefore$ Now, take y is a convex set of u & v.

$\therefore y = u\theta + (1-\theta)v$, for, $\theta \in [0,1]$, & $\sum \theta = 1$

$$\therefore y^T(x_2 - x_1) = \left[\theta u + (1-\theta)v\right]^T (x_2 - x_1)$$

$$= \left\{\theta u^T + (1-\theta)v^T\right\}(x_2 - x_1)$$

$$= \theta u^T (x_2 - x_1) + (1-\theta) v^T (x_2 - x_1)$$

$$\leq \theta \cdot \frac{1}{2}\left(\|x_2\|_2^2 - \|x_1\|_2^2\right) + (1-\theta)\cdot\frac{1}{2}\cdot\left(\|x_2\|_2^2 - \|x_1\|_2^2\right)$$

$$= \frac{1}{2}\left(\|x_2\|_2^2 + \|x_1\|_2^2\right)$$

$\Rightarrow$ ~~The~~ It is Convex Set. (Answer)

Exercise (2) Set of polynomial Coefficients.

$S = \{x \in R^n \mid |P_x(t)| \leq 1, \text{ for } |t| \leq 1\}$ with $P_x(t) = x_1 t + \dots + x_n t^n$ is Convex

$\Rightarrow P_x(t) = \sum_{i=1}^{n} x_i t^i = x^T \underline{t}$ where, $t = (t^1 t^2 t^3 \dots t^n)^T$

$\therefore S = \{|P_x(t)| \leq 1, \text{ for } |t| \leq 1\}$

$$= \bigcap_{t \in [-1,1]} \{P_x(t) \leq 1\}$$

$$= \bigcap_{t \in [-1,1]} \left(\{x \in R^n : P_x(t) \leq 1\} \cap \{x \in R^n : P_x(t) \geq 1\}\right)$$

$$= \bigcap_{t \in [-1,1]} \left(\{ x \in \mathbb{R}^n : x^T t \le 1 \} \cap \{ x \in \mathbb{R}^n : x^T t \ge 1 \} \right)$$

We know that $x^T t \le 1$ & $x^T t \ge 1$ are two half spaces.

$\Rightarrow$ We know that Half spaces are Convex.

So, By using this property that Half space is Convex and Also we know that intersection of Convex set is also Convex, we prove that it is Convex set.

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(2) Convex Functions $\Rightarrow$

Exercise (3) Is it Convex?

(a) $f : \mathbb{R}^n \to \mathbb{R}, \; x \longrightarrow \|x\|_2$

$\mathbb{R}^n \to$ Dom of f

$\Rightarrow$ For, $x, y \in \mathbb{R}^n$ and $\theta \in [0,1]$, $\Sigma\theta = 1$

$$\Rightarrow f(\theta x + (1-\theta)y) = \|\theta x + (1-\theta)y\|_2$$

$$\le \theta \|x\|_2 + (1-\theta)\|y\|_2$$

$$= \theta f(x) + (1-\theta) f(y) \quad \{ \because \text{Triangular inequality \& Homogeneity} \}$$

$$\Rightarrow f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y)$$

$\Rightarrow f$ is Convex function.

(b) $f : \mathbb{R} \to \mathbb{R}$, $x \to x^3$

$\Rightarrow$ We know that, x^a on $\mathbb{R}_{++}$ is Convex fun^c, for $a \geq 1$ (or) $a \leq 0$

But here ~~B~~ the Domain is whole $\mathbb{R}$

$\Rightarrow$ x^3 is Non Convex.

If It's Convex then, $f(\theta x + (1-\theta) y) \leq \theta f(x) + (1-\theta) f(y)$

$\therefore f(0.3 \times (-1) + 0.7 \times 0) = f(-0.3) = -0.027$

$\therefore 0.3 f(-1) + 0.7 f(0) = \cancel{-0.027} \; 0.3 \times (-1)^3 = -0.3$

$\Rightarrow$ But here $-0.027 \not\leq -0.3$

$\Rightarrow$ Not Convex function (Answer)

EX.– (4) Sum of Biggest ~~of r ord~~ absolute value Coordinate.

Let, ~~R~~ $f : \mathbb{R}^n \to \mathbb{R}$, $x \to \sum_{i=1}^{r} |x|_{[i]}$

$\Rightarrow$ Sum of largest Components of $x \in \mathbb{R}^n$

$$f(x) = x_{[1]} + x_{[2]} + \cdots + x_{[r]}$$

$$\Rightarrow f(x) = \max \{ x_{i_1} + x_{i_2} + \cdots + x_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n \}$$

An index of a vector entry goes from 1 to n

There are n choose r sets of r different indices.

$\Rightarrow f(x) = \sum_{i=1}^{} \max \; a_i^T \underline{x}$ $\Rightarrow$ f is Point Wise Max of Affine Fun^c

Chose ^nC_r different a_i & ~~..~~ $\Rightarrow$ It's Convex (Answer)

Ex ⑤ Epigraph is Convex

Epigraph of a func $f: R^n \rightarrow R$ is defined as

$$epi(f) = \{(x,t) \mid x \in dom(f), t \in R, f(x) \leq t\}$$

Proof →

We have to prove: epi(f) is Convex if $f(x)$ is Convex.

We prove this by Contradiction. Assume $f(x)$ is Convex but epi(f) is not.

Then, $\exists$ ~~the~~ two points (x,t), $(y,t') \in epi(f)$ such that, $\forall \theta \in [0,1]$.

$$\theta(x,t) + (1-\theta)(y,t') \notin epi(f)$$

Therefore,

$$f(\theta x + (1-\theta)y) > \theta t + (1-\theta)t' \geq \theta f(x) + (1-\theta) f(y).$$

which is Contradictory to the fact that $f(x)$ is Convex.

Hence epi(f) is Convex

Conversely, assume that epi(f) is Convex. We verify the Convexity of f on it's domain.

Let, $x, y \in f$ & $\theta \in [0,1]$. Then the points $(x, f(x))$ and $(y, f(y))$ are in the epigraph of f.

Since, the epigraph is Convex. ~~The point~~

The point $\theta(x, f(x)) + (1-\theta)(y, f(y))$ is also in the epigraph of f.

Therefore, $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$.

Hence, f is convex.

$\Rightarrow$ We have thus proved that epi(f) is convex if and only if f is convex. (Answer)

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③ Convex Problem

Ex ⑥

ⓐ $\min \; x_1^2 + x_2^2 \longrightarrow$ Objective function $\Rightarrow$ Convex function

such that $\dfrac{x_1}{1+x_2^2} \leq 0 \longrightarrow$ inequality Constraint

$(x_1 + x_2)^2 = 0 \longrightarrow$ equality Constraint

$\Rightarrow$ It is Not Convex Problem. (As f_i & h_i is Non Convex)

as, $\dfrac{x_1}{1+x_2^2} \leq 0 \rightarrow$ Non Convex and $(x_1+x_2)^2 = 0$

$\Rightarrow$ Non convex

$\Rightarrow \; 1+x_2^2 > 0$ $\qquad \Rightarrow (x_1 + x_2) = 0$

$\Rightarrow$ the equivalent Convex Problem

$$\left.\begin{aligned} &\min \; x_1^2 + x_2^2 \\ &\text{such that } x_1 \leq 0 \\ &x_1 + x_2 = 0 \end{aligned}\right\}$$

(b) $\min \ \frac{x_1}{x_2} + \frac{x_3}{x_1} \longrightarrow$ Non Convex

s.t. $\frac{x_2}{x_3} + x_1 \leq 1$

$(x_1+x_2)^2 = 0$

$\Rightarrow$ Non Convex as, $\frac{x_2}{x_3} + x_1 \leq 1 \Rightarrow$ Non Convex

And $(x_1+x_2)^2 = 0$ is Not Linear

Here, $\frac{x_1}{x_2} + \frac{x_3}{x_1} \rightarrow$ Objective fun^{c} is Not Convex.

For Equality Constraint $(x_1+x_2)^2 = 0$

$\Rightarrow x_1 + x_2 = 0$

But ~~Eq~~ the equivalent Convex Problem is Not valid

Ex (7) Solve the Linear Problems

(a) $\min \ -3x_1 + x_2$

s.t. $x_1 + x_2 \leq 5$

$2x_1 + x_2 \leq 8$

$x_1, x_2 \geq 0$

By applying Simplex Algo,

x_1	x_2	x_3	x_4	z	C
1	1	1	0	0	5
2	1	0	1	0	8
3	1	0	0	1	0
0	0.5	1	-0.5	0	1
1	0.5	0	0.5	0	4
0	-2.5	0	-1.5	1	-12

$\Rightarrow \quad x_1 = 4, \; x_2 = 0, \quad \Rightarrow -3x_1 + x_2 = -12$

~~$\Rightarrow -3x_1 + x_2 = -12$~~

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④ Duality

Ex ⑧ We consider the following Optimization Problem over $x \in \mathbb{R}$.

$$\min \; x^2 + 1$$
$$\text{s.t.} \; (x-2)(x-4) \le 0$$

ⓐ The Ans is $x = 2$; and value $= 5$ (Answer)

ⓑ

Lagrangian, $L(x, \lambda) = x^2 + 1 + \lambda (x-2)(x-4)$

$$= (1+\lambda)x^2 - 6\lambda x + 1 + 8\lambda$$

For Minimum, $\dfrac{\partial L(x,\lambda)}{\partial x} = 2(1+\lambda)x - 6\lambda = 0$

$$\Rightarrow x = \frac{3\lambda}{(1+\lambda)}$$

which is feasible for $\lambda \geq 2$.

$$g(\lambda) = \inf L(x, \lambda) = (1+\lambda) \cdot \frac{9\lambda^2}{(1+\lambda)^2} - \lambda \cdot 6 \cdot \frac{3\lambda}{(1+\lambda)} + 1 + 8\lambda$$

$$= -9 \cdot \frac{\lambda^2}{(1+\lambda)} + 8\lambda + 1$$

$\Rightarrow$ dual Problem,
$$\text{Max} \; -9\frac{\lambda^2}{(1+\lambda)} + 8\lambda + 1$$
$$\text{such that } \lambda \geq 2$$
(Answer)

(c)

$$g(\lambda) = \frac{-9\lambda^2}{1+\lambda} + 8\lambda + 1$$

$$\Rightarrow \frac{\partial g(\lambda)}{\partial \lambda} = 8 - 9 \cdot \frac{2\lambda+\lambda^2}{1+2\lambda+\lambda^2} = 0$$

$$\Rightarrow 8 + 16\lambda + 8\lambda^2 - 18\lambda - 9\lambda^2 = 0$$

$$\Rightarrow \lambda^2 + 2\lambda - 8 = 0$$

$$\Rightarrow (\lambda+4)(\lambda-2) = 0$$

$$\Rightarrow \lambda = -4, +2$$

$\Rightarrow \lambda = +2$ is feasible.

$\Rightarrow$ Dual optimal value $= 5$

(d) Strong duality confirmation via Slater's constraint qualification hings on the presence of a strictly feasible solution, where the inequality constraints are satisfied with strict inequality.

For instance, at $x=3$, this condition is met as

$$(x-2)(x-4) = (3-2)(3-4) = -1 < 0$$

Ex – (9)