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Quantitative Measure of Information
Part I

Exercise 1

One person says: "Today is my birthday". Calculate the amount of self-information conveyed by this statement. Calculate the average amount of information conveyed by this source over one year.

Exercise 2

The 64 squares of a chessboard are assumed to be equiprobable. Determine the average amount of information contained in a communication indicating the position of a given chess piece. Propose a dichotomous strategy, based on questions of the form "Is the chess piece on that part of the chessboard?", that would allow to guess the position of this chess piece in a minimum average number of questions. Compare this average number of questions to the entropy calculated at the beginning of the exercise.

Exercise 3

A perfectly balanced coin is tossed until the first head appears. Calculate the entropy $H(X)$ in Shannon, where the random variable X denotes the number of flips required to get the first head. Propose a dichotomous strategy, based on questions with binary response of the form "Is X is smaller or greater than (...)", making it possible to guess the value of X in a minimum average number of questions. Compare this number of questions to $H(X)$.
In order to resolve this exercise, the following equality can be used $\sum_{n=1}^{\infty} n\, a^n = \frac{a}{(1-a)^2}$.

Exercise 5

Consider a tank that consists of two compartments of identical volumes. Compartment I is filled with two inert gases with respective proportions $(\frac{2}{5}, \frac{3}{5})$. The same gases fill compartment II with respective proportions $(\frac{1}{3}, \frac{2}{3})$. Assuming the pressure and temperature in both compartments are the same, calculate the tank entropy before and after the two compartments communicate. Interpret the result.

Exercise 6

A source emits symbols 0 and 1 with probabilities $P(0) = \frac{1}{4}$ and $P(1) = \frac{3}{4}$. These symbols are transmitted to a receiver through an imperfect symmetric channel illustrated by Figure 1, with $p_0 = 10^{-1}$. Denoting by X and Y the transmitted and received symbols, calculate the following quantities: $H(X)$, $H(Y)$, $H(X,Y)$, $H(Y|X)$, $H(X|Y)$ and $I(X,Y)$.

Problem 1

Let $\{\mathcal{E}_k\}_{k=1}^{n}$ be a partition of $\mathcal{E}$. We denote by N and N_k the numbers of elements in sets $\mathcal{E}$ and $\mathcal{E}_k$, respectively. Assume that the elements of $\mathcal{E}$ are equiprobable. We set $p_k = N_k/N$.

1. Determine the self-information of any element of $\mathcal{E}_k$. Calculate the average amount of information needed to determine any element in $\mathcal{E}_k$.

2. Calculate the average amount of information needed to characterize any element of $\mathcal{E}$. By noticing that we can split the identification procedure of an element of $\mathcal{E}$ in 2 steps, (a) identification of the set $\mathcal{E}_k$, and then (b) identification of the element in $\mathcal{E}_k$, estimate the average amount of information needed to identify $\mathcal{E}_k$.

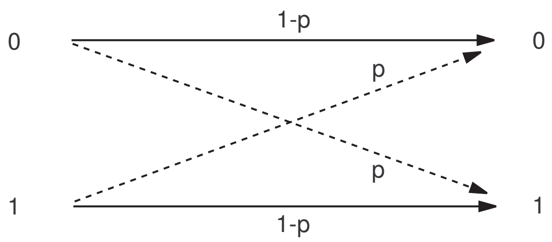

Figure 1: Imperfect channel.

Problem 2

Consider a twin-pan balance and 9 coins. We know that one of these coins is fake. The problem is to find the fake coin given that it only differs from the other 8 coins by its weight.

1. Determine the number of possible cases, considering that the fake coin may be heavier or lighter than the others. Calculate the average amount of information necessary to identify the fake coin.

2. To identify the fake coin, the weights of two sets of n coins each are compared using the twin-pan balance. Enumerate the possible outcomes of each weighting operation. Assuming these outcomes are equiprobable, determine in that case the amount of information provided by every weighing operation. Determine the average number of weighting operations to plan.

3. One wants to determine n in order to maximize the amount of information provided by each weighting operation. Let P_ℓ, resp. P_r, be the probability that the set of coins in the left pan, resp. right pan, is heavier. Let P_e be the probability that an equilibrium is achieved. Calculate P_ℓ, P_r and P_e.

4. Calculate n to maximize the entropy of each weighting operation.

5. Calculate the minimum average number of weighting operations required to identify the fake coin.

6. Propose a strategy to identify the fake coin.