FA'25

HW 7

① $L_1 = \{a^r b^s c^t \mid r>s,\ s>t,\ t>0\}$

1. Assume L_1 is context free.
2. Let p be the pumping length
3. Choose a string $w \in L_1$ and $|w| \geq p$

$a^{p+2} b^{p+1} c^p$ in the language ✓
$|w| \geq p$ ✓

4. Decompose: $uvxyz$ such that $|vxy| \leq p$, $|v|+|y|>0$

Cases:
1. v, y in a's only
2. v, y in b's only
3. v, y in c's only
4. v, y overlap a's and b's
   a. v in a's, y in b's
   b. v in a's & b's, y in b's
   c. v in a's, y in a's & b's
5. v, y overlap in b's and c's
   a. v in b's, y in c's
   b. v in b's & c's, y in c's
   c. v in b's, y in b's and c's
6. v, y overlap a's, b's, and c's
   - this one is not possible because for v and y to overlap the a's b's and c's There must be one or more a's all the b's, and one or more c's. That means $|vxy| \not\leq p$

Case 1: v, y in a's only

$u = a^m$ $\qquad j+k>0$
$v = a^j$
$x = a^l$ $\qquad j+l+k \leq p$
$y = a^k$
$z = a^{p+2-j-k-l-m} b^{p+1} c^p$

Options are to pump up or down. Pumping up will lead to more a's which will not be a problem for the string. So pump down!

$$uxz = a^m a^l a^{p+2-j-k-l-m} b^{p+1} c^p$$
$$= a^{p+2-j-k} b^{p+1} c^p$$

After pumping, the string should still be in the language. However, since $j + k > 0$, it cannot be the case that $p+2-j-k > p+1$.
Case 1, then, leads to a contradiction.

Case 2: v, y in b's only

$u = a^{p+2} b^{p+1-j-k-l-m}$
$v = b^j$ $\qquad j+k>0$
$x = b^l$
$y = b^k$ $\qquad j+l+k \leq p$
$z = b^m c^p$

Both pumping up and down will lead to a contradiction. I'll pump up.

$uvvxyyz$
$\longrightarrow a^{p+2} b^{p+1-j-k-l-m} b^j b^j b^l b^k b^k b^m c^p$

$= a^{p+2} b^{p+1+j+k} c^p$ $\quad$ Since $j+k>0$, $p+2 \not> p+1+j+k$. Case 2 leads to a contradiction.

Case 3: v, y in c's only

$u = a^{p+2} b^{p+1} c^{p-j-k-l-m}$

$v = c^j$ $\quad\quad$ $j+k > 0$

$x = c^l$ $\quad\quad$ $j+k+l \leq p$

$y = c^k$

$z = c^m$

Pump up $\rightarrow$ $uvvxyyz = a^{p+2} b^{p+1} c^{p-j-k-l-m} c^j c^j c^l c^k c^k c^m$

$= a^{p+2} b^{p+1} c^{p+j+k}$

Since $j+k > 0$, $p+1 \not> p+j+k$. Therefore Case 3 leads to a contradiction.

Case 4: v, y overlap a's & b's

There are three subcases to consider.

a) v is in a's only, y is in b's only

$u = a^{p+2-j-l}$

$v = a^j$ $\quad\quad$ $j+l+m+k \leq p$

$x = a^l b^m$ $\quad\quad$ $j+k > 0$

$y = b^k$

$z = b^{p+1-m-k} c^p$

Pump down

$uxz = a^{p+2-j-l} a^l b^m b^{p+1-m-k} c^p$

$= a^{p+2-j} b^{p+1-k} c^p$

Since $j+k > 0$, there are three possibilities to consider:

$j > 0, k = 0$ in which case there would be too few a's compared to b's

$j = 0, k > 0$ in which case there would be too few b's compared to c's

$j > 0, k > 0$ in which case there would at the minimum be too few b's compared to c's.

All possibilities for subcase 4a lead to contradictions.

b) v is in a's and b's, y is in b's only

$$u = a^{p+2-j}$$
$$v = a^j b^k \qquad j+k+l > 0$$
$$x = b^m \qquad j+k+m+l \le p$$
$$y = b^l$$
$$z = b^{p+1-k-l-m} c^p$$

Pump down

$$a^{p+2-j} b^m b^{p+1-k-l-m} c^p$$
$$= a^{p+2-j} b^{p+1-k-l} c^p$$

There are seven subsubcases ~~cases~~ to consider here based on the values of j, k, and l.

$j>0, k=0, l=0 \Rightarrow a^{p+2-j} b^{p+1} c^p$ too few a's

$j=0, k>0, l=0 \Rightarrow a^{p+2} b^{p+1-k} c^p$ too few b's

$j=0, k=0, l>0 \Rightarrow a^{p+2} b^{p+1-l} c^p$ too few b's

$j>0, k>0, l=0 \Rightarrow a^{p+2-j} b^{p+1-k} c^p$ too few b's

$j>0, k=0, l>0 \Rightarrow a^{p+2-j} b^{p+1-l} c^p$ too few b's

$j=0, k>0, l>0 \Rightarrow a^{p+2} b^{p+1-k-l} c^p$ too few b's

$j>0, k>0, l>0 \Rightarrow a^{p+2-j} b^{p+1-k-l} c^p$ too few b's

Every possible decomposition for subcase 4b leads to a contradiction.

c) v in a's only, y in a's and b's

$u = a^{p+2-j-k-m}$

$v = a^{j}$ $\qquad j+k+l > 0$

$x = a^{m}$

$y = a^{k}b^{l}$ $\qquad j+m+k+l \leq p$

$z = b^{p+1-l}c^{p}$

Pump down... $uxz = a^{p+2-j-k-m}a^{m}b^{p+1-l}c^{p}$

$= a^{p+2-j-k}b^{p+1-l}c^{p}$

There are seven possible value combinations for j, k, and l:

i $j > 0$, $k = 0$, $l = 0$

ii $j = 0$, $k > 0$, $l = 0$

iii $j = 0$, $k = 0$, $l > 0$

iv $j > 0$, $k > 0$, $l = 0$

v $j > 0$, $k = 0$, $l > 0$

vi $j = 0$, $k > 0$, $l > 0$

vii $j > 0$, $k > 0$, $l > 0$

subsubcases iii, v, vi, and vii all have too many b's compared to c's since $l > 0$ in those cases. In subsubcases i, ii, iv either j or k (or both) are greater than zero meaning there are too few a's compared to b's.

All subsubcases for subcase 4c lead to contradictions.

Case 4 conclusion: all subcases for case 4 lead to contradictions.

Case 5: v, y overlap b's & c's

There are three subcases to cover for case 5:

a) v in b's only, y in c's only

b) v in b's and c's, y in c's only

c) v in b's only, y in b's and c's

a) v in b's only, y in c's only

$u = a^{p+2} b^{p+1-j-l}$

$v = b^j$ $\quad$ $j + k > 0$

$x = b^l c^m$ $\quad$ $j + l + m + k \leq p$

$y = c^k$

$z = c^{p-k-m}$

Pump up... $uvvxyyz$

$= a^{p+2} b^{p+1-j-l} b^j b^j b^l c^m c^k c^k c^{p-k-m}$

$= a^{p+2} b^{p+1+j} c^{p+k}$

$j+k>0$ ~~which means~~ If j is greater than zero, regardless of k, there will be too few a's compared to b's. If $j=0$ and $k>0$, there will be too few b's compared to the number of c's. All possible values of j and k in subcase 5a lead to contradictions.

Case 5, subcase b: v in b's and c's, y in c's only

$u = a^{p+2} b^{p+1-j}$

$v = b^j c^k$ $\qquad j+k+l > 0$

$x = c^m$ $\qquad j+k+m+l \leq p$

$y = c^l$

$z = c^{p-k-l-m}$

Pump up ... $uvvxyyz$

$= a^{p+2} b^{p+1-j} b^j c^k b^j c^k c^m c^l c^l c^{p-k-l-m}$

$= a^{p+2} b^{p+1} c^k b^j c^{p+l}$

There are seven possible combinations of values for j, k, and l:

$j > 0,\ k = 0,\ l = 0 \rightarrow a^{p+2} b^{p+1+j} c^p$ ← too many b's vs. a's

$j = 0,\ k > 0,\ l = 0 \rightarrow a^{p+2} b^{p+1} c^{p+k}$ ← Too many c's to b's

$j = 0,\ k = 0,\ l > 0 \rightarrow a^{p+2} b^{p+1} c^{p+l}$ ← too many c's to b's

$j > 0,\ k > 0,\ l = 0 \rightarrow a^{p+2} b^{p+1} c^k b^j c^p$ ← wrong order

$j > 0,\ k = 0,\ l > 0 \rightarrow a^{p+2} b^{p+1+j} c^{p+l}$ ← too many b's to a's

$j = 0,\ k > 0,\ l > 0 \rightarrow a^{p+2} b^{p+1} c^{p+k+l}$ ← too many c's to b's

$j > 0,\ k > 0,\ l > 0 \rightarrow a^{p+2} b^{p+1} c^k b^j c^{p+l}$ ← wrong order

All combinations of values for j, k, and l lead to contradictions. So all decompositions covered by subcase b of case 5 lead to contradictions.

Subcase c (case 5): v in b's only, y in b's & c's

$u = a^{p+2} b^{p+1-j-k-m}$
$v = b^j$
$x = b^m$
$y = b^k c^l$
$z = c^{p-l}$

Pump up... $u\,v\,v\,x\,y\,y\,z$

$= a^{p+2} b^{p+1-j-k-m} b^j b^j b^m b^k c^l b^k c^l c^{p-l}$

$= a^{p+2} b^{p+1+j} c^l b^k c^p$

We consider the seven possible combinations for j, k, and l.

$j > 0,\ k = 0,\ l = 0 \Rightarrow a^{p+2} b^{p+1+j} c^p$ too many b's to a's

$j = 0,\ k > 0,\ l = 0 \Rightarrow a^{p+2} b^{p+1+k} c^p$ too many b's to a's

$j = 0,\ k = 0,\ l > 0 \Rightarrow a^{p+2} b^{p+1} c^{p+l}$ too many c's to b's

$j > 0,\ k > 0,\ l = 0 \Rightarrow a^{p+2} b^{p+1+j+k} c^p$ too many b's to a's

$j > 0,\ k = 0,\ l > 0 \Rightarrow a^{p+2} b^{p+1+j} c^{p+l}$ too many b's to a's

$j = 0,\ k > 0,\ l > 0 \Rightarrow a^{p+2} b^{p+1} c^l b^k c^p$ wrong order

$j > 0,\ k > 0,\ l > 0 \Rightarrow a^{p+2} b^{p+1+j} c^l b^k c^p$ wrong order

All possibilities for subcase c lead to contradictions.

All possibilities for subcase c lead to contradictions.

Conclusion: It has been shown that all decompositions lead to contradictions. The original assumption must be false. ΔL_1 is not context free.

(2')

$$L_{2a} = \{a^r b^s c^t \mid r>s,\ s>t,\ t>0,\ r<4\}$$

$$L_{2b} = \{a^r b^s c^t d^q \mid r=t,\ s<2q;\ q,r,s,t>0\}$$

$$L_{2c} = \{a^r b^s c^t d^q \mid r=q,\ s<t;\ q,r,s,t>0\}$$

L_{2a} and L_{2c} are context free. L_{2b} is not CF.

L_{2a} only has one string "aaabbc".

PDA

$$\to q_0 \xrightarrow{a\,\varepsilon\,\varepsilon} q_1 \xrightarrow{a\,\varepsilon\,\varepsilon} q_2 \xrightarrow{a\,\varepsilon\,\varepsilon} q_3 \xrightarrow{b\,\varepsilon\,\varepsilon} q_4 \xrightarrow{b\,\varepsilon\,\varepsilon} q_5 \xrightarrow{c\,\varepsilon\,\varepsilon} (q_6)$$

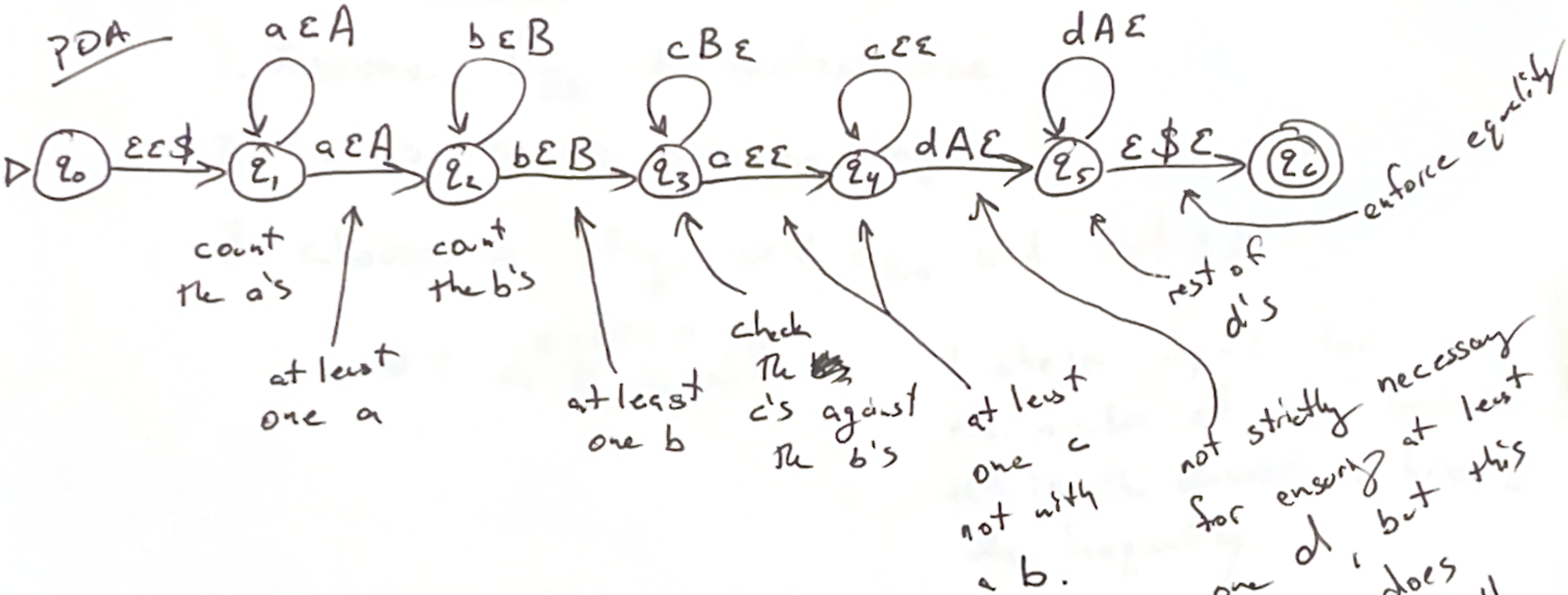
CFG

$S \to aaabbc$

$L_{2c} \quad a^r b^s c^t d^q \mid r=q,\ s<t;\ q,r,s,t>0$

$= a^r b^s c^s c^t d^r \quad ,\ r,s,t>0$

PDA



not strictly necessary for ensuring at least one d, but this one transition does gaurantee that all B's have been popped from stack.

CFG

$S \rightarrow aSd \mid aAd$

$A \rightarrow BD$

$B \rightarrow bBc \mid bc$

$D \rightarrow cD \mid c$

$$L_{2b} = \{a^r b^s c^t d^q \mid r=t,\ s<2q;\ q,r,s,t>0\}$$

1. Assume L_{2b} is context free
2. Let p be the pumping length
3. choose a string $w \in L_{2b}$ and $|w| \geq p$

$$w = a^p b^{2p-1} c^p d^p$$

I chose $2p-1$ for the number of b's because that is the closest to breaking the inequality.

4. Decompose into $uvxyz$

$$|vxy| \leq p$$
$$|v|+|y| > 0$$

1. v,y in a's only
2. v,y in b's only
3. v,y in c's only
4. v,y in d's only
5. v,y overlap a's and b's
6. v,y overlap b's and c's
7. v,y overlap c's and d's

Case 5 subcases

a. v in a's, y in b's
b. v in a's & b's, y in b's
c. v in a's, y in a's & b's

Case 6 subcases

a. v in b's, y in c's
b. v in b's & c's, y in c's
c. v in b's, y in b's & c's

Case 7 subcases

a. v in c's, y in d's
b. v in c's & d's, y in d's
c. v in c's, y in c's & d's