

# **Study of color chaos model and Time frequency analysis of S&P 500 and NASDAQ indexes**

by

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Dedicated to my wife Raji Praba, my sons, Deepak Praba and Darshan Praba

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## **LIST OF ABBREVIATIONS**

## **ABSTRACT**

Study of color chaos model and Time frequency analysis of S&P 500 and NASDAQ indexes

by

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Change the abstract based on the finding of the report.. as of now, it is replication of the Chen's abstract Time frequency model and random walk model are two polar models in linear systems. Color chaos is a model that is in between these models which generates irregular oscillation with a narrow frequency band. The deterministic component from noisy data can be recovered by time variant filter in Gabor space. The characteristic frequency is calculated by Wigner decomposed distribution series. It is found that 7% of the detrended by HP filter can be explained by the deterministic color chaos. The existence of persistent chaotic cycle reveals a new perspective of market resilience and new sources of economic uncertainties. The nonlinear pattern in the stock market may not be wiped out by the market competition under non-equilibrium situations with trend evolution and frequency shifts.

# CHAPTER I

## Introduction

### 1.0.1 Objective of the project

The object of the project is to study the color chaos model that was paper published by Prof. Ping Chen in the paper [? ]. The approach of this paper is to understand the mathematical concepts of color chaos, time frequency analysis - Wigner distribution and Gabor transformation. In the paper [? ], the S&P 500 composite monthly index price was taken for the analysis, and in this project the color chaos model is studied for both the S&P 500 and NASDAQ indices.

### 1.0.2 Organization of Project

This report has been organized into five chapters. Chapter 1 outlines the entire project giving an introduction to time series forecasting using stochastic models. Chapter 2 introduces color-chaos models and their applications. Chapter 3 provides an overview of time-frequency analysis. Chapter 4 **Finish up after the chapters are finalized**

### 1.0.3 Data Source:

The Color chaos model is evaluated for two indices in this study and the sources of the data for the indices are given below in table 1.1.

Symbol	Description	Source	Frequency	Duration
FSPCOM	S&P 500 Price Composite index	Citibase	Monthly	1942-1992
NASDAQ	NASDAQ Composite index	Yahoo Finance	Monthly	1970-2010

Table 1.1: Meta data on FSPCOM(S&P 500) & NASDAQ

#### 1.0.4 Time series forecasting using stochastic models

In general, models for time series data can have many forms and represent different stochastic processes. The most widely used linear time series models are Auto Regressive (AR) and Moving Average (MA) models. Combining these two, the Autoregressive and Moving Average (ARMA) and Autoregressive Integrated Moving Average (ARIMA) are also used. The Autoregressive Fractionally Integrated Moving Average (ARFIMA) model generalized ARMA and ARIMA models. For seasonal time series forecasting, a variation of ARIMA, the Seasonal Autoregressive Integrated Moving Average (SARIMA) model is used.

Linear models have drawn much attention due to their relative simplicity in understanding and implementation. Many practical time series show non-linear patterns. Non-linear models are appropriate for predicting the volatility changes in economic and financial time series. Considering these facts, various non linear models have been proposed over the years. A few widely used non-linear models are Autoregressive Conditional Heteroskedasticity (ARCH) model, its variations like Generalized ARCH (GARCH), Exponential GARCH (EGARCH), the Threshold Autoregressive (TAR), the Non-linear AutoRegressive (NAR), the Non-linear Moving Average (NMA) model and others.

The Autoregressive Moving Average (ARMA) models: An ARMA( $p,q$ ) model is a combination of AR( $p$ ) and MA( $q$ ) models and is suitable for univariate time series modeling. In an AR( $p$ ) model the future value of a variable is assumed to be a linear combination of  $p$  past observations and a random error together with a constant term.

Mathematically the AR(p) model can be expressed as:

$$y_t = c + \sum_{i=1}^p \varphi_i y_{t-i} + \epsilon_t \quad (1.1)$$

Here  $y_t$  and  $\epsilon_t$  are the actual value and random error respectively at time period  $t$ ,  $\varphi_i (i = 1, 2, 3, \dots, p)$  are model parameters and  $c$  is a constant. The integer constant  $p$  is known as the order of the model. Sometimes the constant term is omitted for simplicity. A AR( $p$ ) model regresses against past values of the series; an MA( $q$ ) model uses past errors as the explanatory variables. The MA( $q$ ) model is given by

$$y_t = \mu + \sum_{j=1}^q \theta_j \epsilon_{t-j} + \epsilon_t \quad (1.2)$$

Here  $\mu$  is the mean of the series,  $\theta_j (j = 1, 2, 3, \dots, q)$  are the model parameters and  $q$  is the order of the model. The random shocks are assumed to be a white noise process, i.e. a sequence of independent and identically distributed (i.i.d) random variables with zero mean and a constant variance  $\sigma^2$ . Generally, the random shocks are assumed to follow a normal distribution. Conceptually a moving average model is a linear regression of the current observation of the time series against the random shocks of one or more prior observations. Fitting an MA model to a time series is more complicated than fitting an AR model because in the former one the random error terms are not fore-seeable. Autoregressive (AR) and moving average (MA) models can be effectively combined together to form a general and useful class of time series models, known as ARMA models. Mathematically an ARMA( $p, q$ ) model is represented as:

$$y_t = c + \epsilon_t + \sum_{i=1}^p \varphi_i y_{t-i} + \sum_{j=1}^q \theta_j \epsilon_{t-j} \quad (1.3)$$

Here the model orders  $p, q$  refer to  $p$  autoregressive and  $q$  moving average terms.

### 1.0.5 Difference Stationary

Loosely speaking a stationary process is one whose statistical properties do not change over time. More formally, a strictly stationary stochastic process is one where given  $t_1, \dots, t_l$  the joint statistical distribution of  $X_{t_1}, \dots, X_{t_l}$  is the same as the joint statistical distribution of  $X_{t_l+\tau}$  for all  $l$  and  $\tau$ . It means that all moments of all degrees (expectations, variances, third order and higher) of the process anywhere are the same. It also means that the joint distribution of  $(X_t, X_s)$  is the same as  $(X_{t+\tau}, X_{s+\tau})$ .

A stochastic process is said to be stationary if its mean and variance are constant over time. i.e. time invariant. A stationary process will not drift too far away from its mean value because of the finite variance.

**Difference Stationary:** Sometimes even de-trending is not sufficient to make a time series stationary, in such case, it may be necessary to transform it into a series of period-to-period and/or season-to-season differences to make the series stationary. Such a series is said to be difference-stationary.

If the trend in a time series is a deterministic function of time, such as  $t$  or  $t^2$ , we call it a deterministic (predictable) trend. If it is not predictable, we have a stochastic trend.

Consider the following model.

$$Y_t = \alpha + \beta_1 t + \beta_2 Y_{t-1} + u_t \quad (1.4)$$

where  $u_t$  is white noise.

**Pure Random Walk:**  $\alpha = 0$ ,  $\beta_1=0$ , and  $\beta_2=1$ . This is non stationary as we get  $Y_t = Y_{t-1} + u_t$ . The mean is constant over the time, but the variance is increasing linearly with time. If we find the difference, we get  $\Delta Y_t = u_t$ . Note that differenced series is stationary (DS) because  $E(\Delta Y_t) = E(u_t) = 0$  and  $Var(\Delta Y_t) = Var(u_t) =$

$\sigma^2$ . Both are time invariant. Hence, a random walk without a drift is difference-stationary(DS).

**Random Walk with a drift:**  $\alpha \neq 0, \beta_1 = 0$ , and  $\beta_2=1$ . This is non stationary as we get  $Y_t = Y_{t-1} + u_t + \alpha$ . The mean and variance are both increasing linearly with time. If we find the difference, we get  $\Delta Y_t = \alpha + u_t$ . Note that differenced series is stationary (DS) because  $E(\Delta Y_t) = E(\alpha + u_t) = \alpha$  and  $Var(\Delta Y_t) = Var(u_t) = \sigma^2$ . Both are time invariant. Hence, a random walk with a drift is also difference-stationary(DS).  $Y_t$  is trending upward or downward depending on the sign of the drift ( $\alpha$ ).

**Stationary Test:** There are several tests of stationary available and we used Dickey Fuller test (DF test) to test stationary of the first difference for both indices. The  $p$ -value of DF test for both indices is 0.001 suggests that both indices are difference stationary. **Deterministic Trend:**  $\alpha \neq 0, \beta_1 \neq 0$ , and  $\beta_2 = 0$ .  $Y_t = \alpha + \beta_1 t + \mu_t$ . Note that the mean of the series,  $E(Y_t) = E(\alpha + \beta_1 t) = \alpha + \beta_1 t$ , which is time-varying but its variance,  $Var(\Delta Y_t) = Var(\alpha + \beta_1 + u_t) = \sigma^2$  which is time-invariant. Still, the series with a deterministic trend is non-stationary. Once we know the values of  $\alpha$  and  $\beta_1$ , we can subtract the mean from the series (detrending) and create a detrended series which is stationary.

**Random walk with drift and deterministic trend:**  $\alpha \neq 0, \beta_1 \neq 0$ , and  $\beta_2 = 1$ . We get  $Y_t = \alpha + \beta_1 t + Y_{t-1} + u_t$ . Note that the difference series,  $\Delta Y_t = \alpha + \beta_1 t + u_t$  is still time varying and hence, the mean of the differenced series is nonstationary. Detrending is still necessary on the differenced series to make it stationary.

### 1.0.6 Seasonal Trend Decomposition Procedure based on LOESS (STL)

STL is a filtering procedure for decomposing a time series into trend, seasonal, and remainder components. STL has a simple design that consists of a sequence of applications of the LOESS (LOcal regrESSion) smoother; the simplicity allows

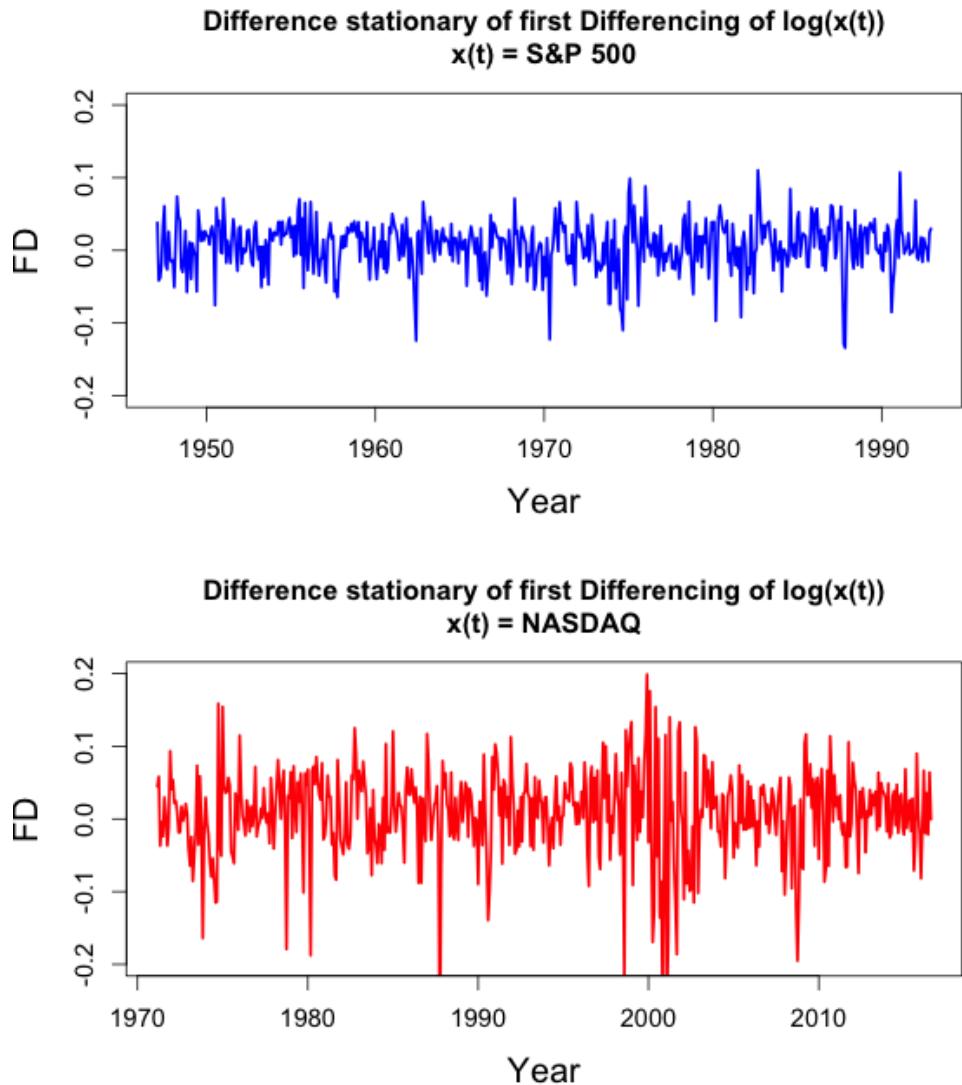


Figure 1.1: Difference stationary of natural log a) SP500 b) NASDAQ. The difference stationary [  $\text{sp500}(t_2)-\text{sp500}(t_1)$  and  $\text{nasdaq}(t_2)-\text{nasdaq}(t_1)$  ] provides insights on the variation of signal. The R program fdplot.R used to generate the graph is given in Appendix A

analysis of the properties of the procedure and allows fast computation, even for a long time series and large amount of trend and seasonal smoothing. Other features of STL are specification of amounts of seasonal and trend smoothing that range, in a nearly continuous way, from a very small amount of smoothing to a very large amount; robust estimates of the trend and seasonal components that are not distorted by divergent behavior in the data.

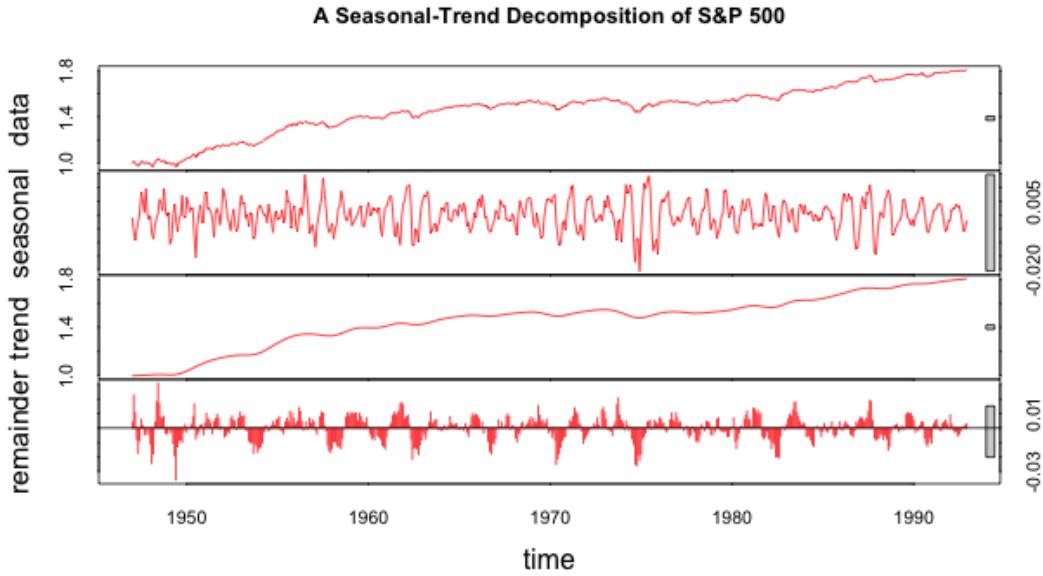


Figure 1.2: STL of Log SP500. a) The natural log of the SP500 b) The cyclical (or called seasonal) pattern c) The business trend of log SP500 d) Noise (remainder) data. The graph was created using the R program named as llt.R and it is attached in the Appendix A.

$$Y_t = f(S_t, T_t, E_t) \quad (1.5)$$

where  $Y_t$  is time series data at time  $t$ ,  $S_t$  is seasonal component at time  $t$ ,  $T_t$  is trend component at time  $t$  and  $E_t$  is remainder (or error or irregular) component of data at time  $t$ . For an additive decomposition by ??

$$Y_t = S_t + T_t + E_t \quad (1.6)$$

The user controls the variations on the trend and seasonal components.

In figure 1.2, trend, seasonal and noise or error remainder data of SP500 are extracted using the seasonal trend decomposition procedure. The seasonal window, in this case the value is done as 5, is used to control the variation on seasonal component.

In the figure 1.3, trend, seasonal and noise or error remainder data of NASDAQ are

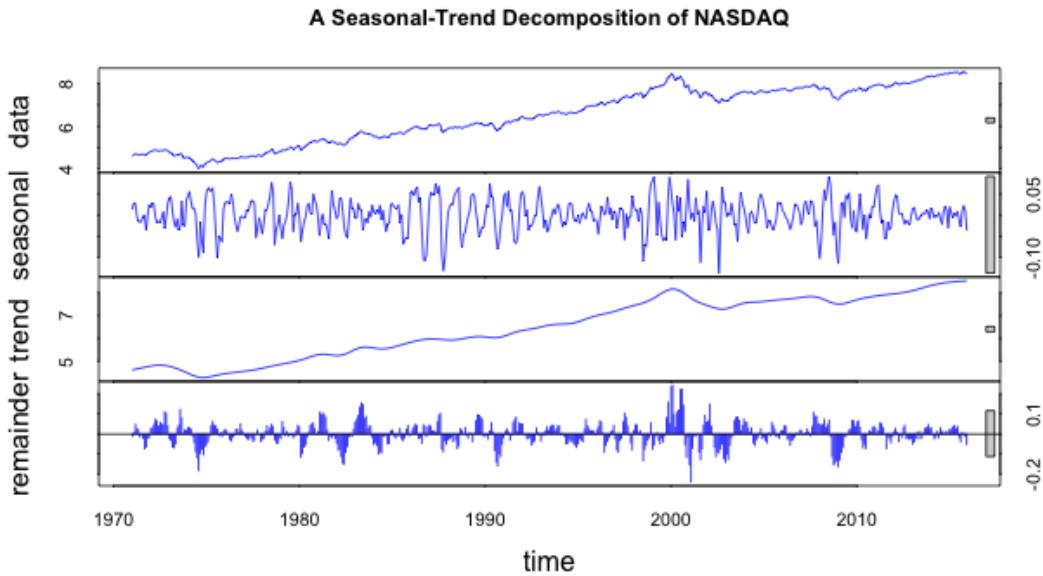


Figure 1.3: STL of NASDAQ. a) The natural log of the NASDAQ b) The cyclical (or called seasonal) pattern c) The business trend of log NASDAQ d) Noise (remainder) data. The graph was created using the R program named as `lilt.R` and it is attached in the Appendix A

extracted using the seasonal trend decomposition procedure. The seasonal window (*value* = 5) used to control the variation on seasonal component.

### 1.0.7 Log-Linear Method

When natural log values used for the dependent variable and independent variable in its original scale, those models are called log linear models.

The following model is of the value in a savings fund that depends on initial investment, growth rate and time duration in which the funds are invested.

$$Y_t = Y_0(1 + r)^t \quad (1.7)$$

where  $Y_t$  represents the value of the fund at the time  $t$ ,  $Y_0$  is the initial investment in the saving fund, and  $r$  is the growth rate.

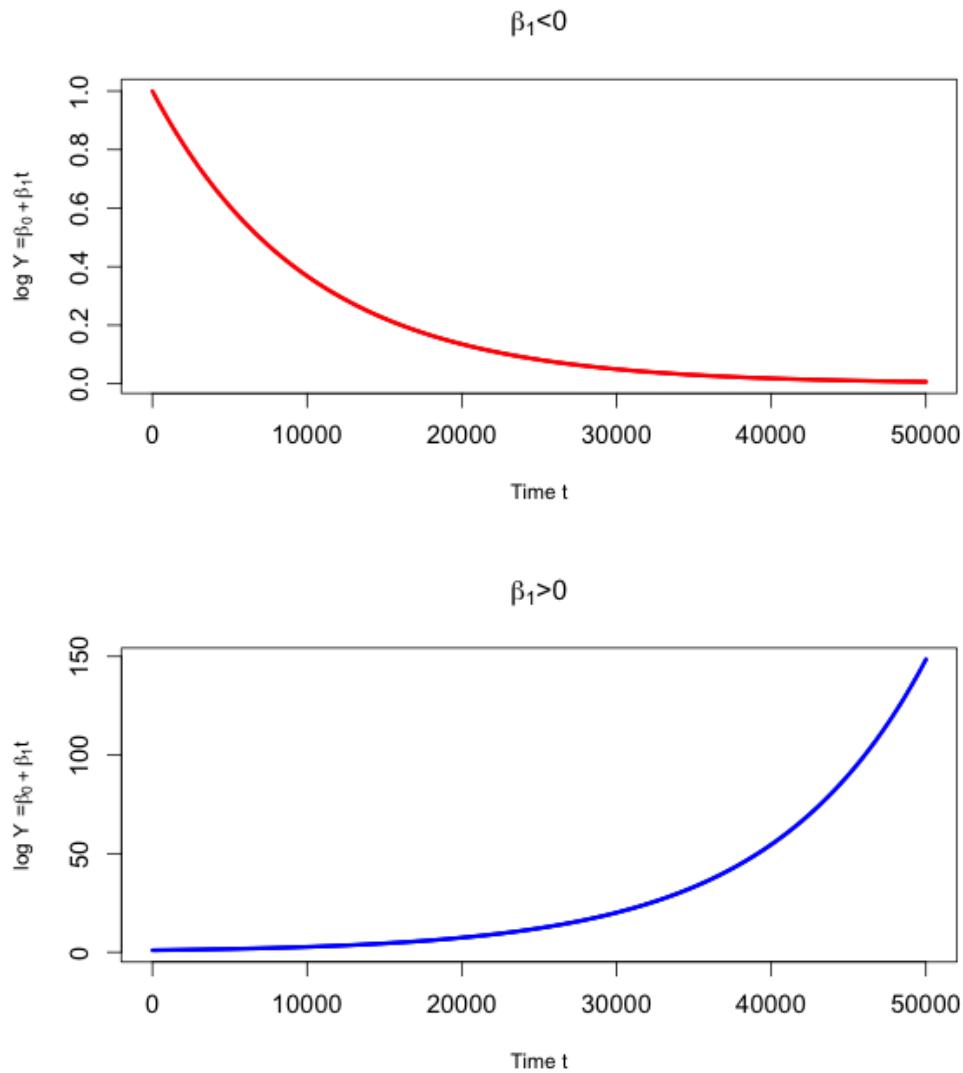


Figure 1.4: Log Linear Model when  $\beta_1 < 0$  and when  $\beta_1 > 0$ . The graph was generated using loglinear.R, and it is attached in the Appendices A

Taking log on both side, we obtain the following.

$$\log Y = \log Y_0 + t \log(1 + r) \quad (1.8)$$

Here  $\log Y_0$  is a constant and is taken to be  $\beta_0$ . Then  $\log(1 + r)$  be  $\beta_1$  and  $\log Y$  is given below:

$$\log Y = \beta_0 + \beta_1 t \quad (1.9)$$

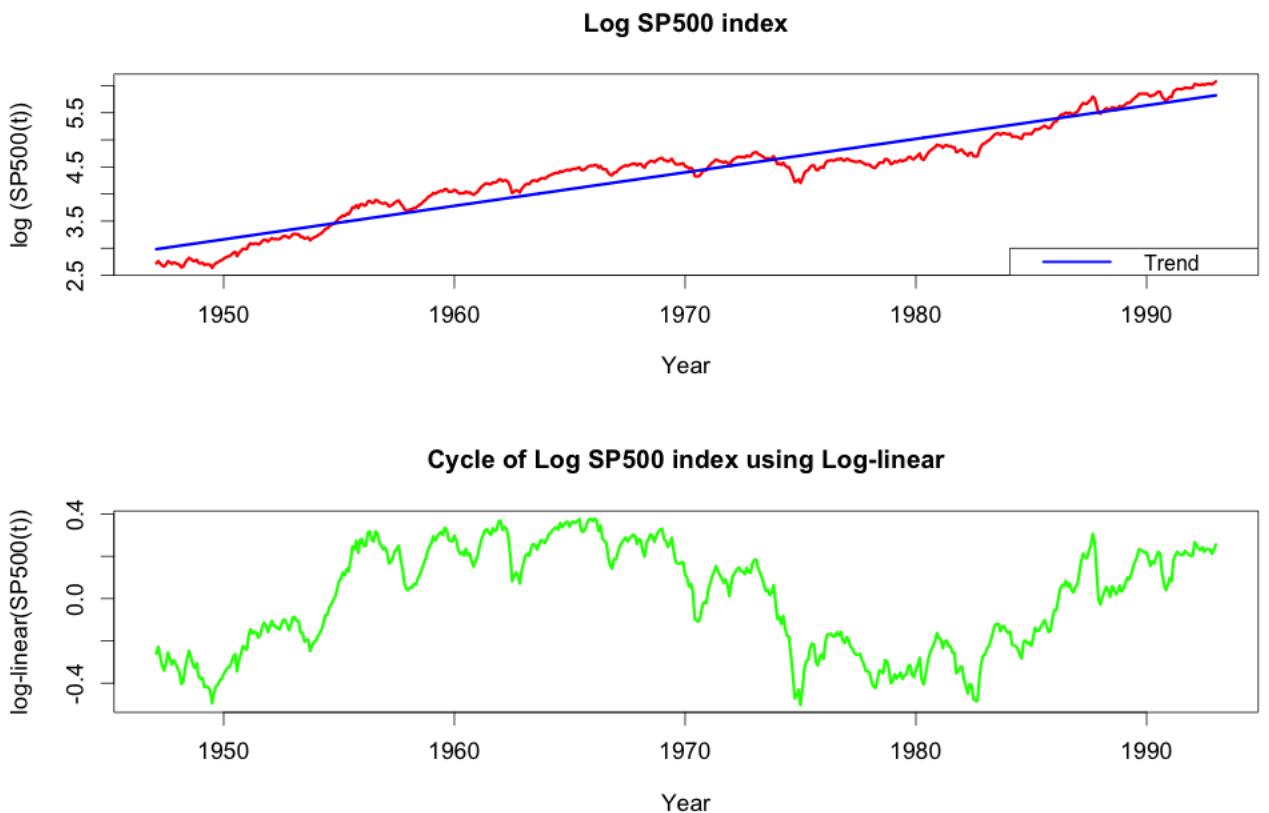


Figure 1.5: Log linear of trend and cycle for log SP500 a) Log SP500 and trend. The trend is calculated based on estimation of slope and y-intercept b) It is cycle of log-linear of SP500 and it is the variance of original  $\log(\text{sp500})$  and estimated trend. AutoCorrelation.R is the R program used to created the graph and it is attached in the Appendix A

After estimation of the log-linear model, the coefficients can be used to determine the impact of the independent variable ( $t$ ) on the dependent variable ( $Y$ ). The coefficients in a log-linear model represent the estimated percent change in the dependent variable for a unit change in independent variable. The coefficient  $\beta_1$  provides the instantaneous rate of growth. The regression coefficients in log-linear model does not represent the slope.

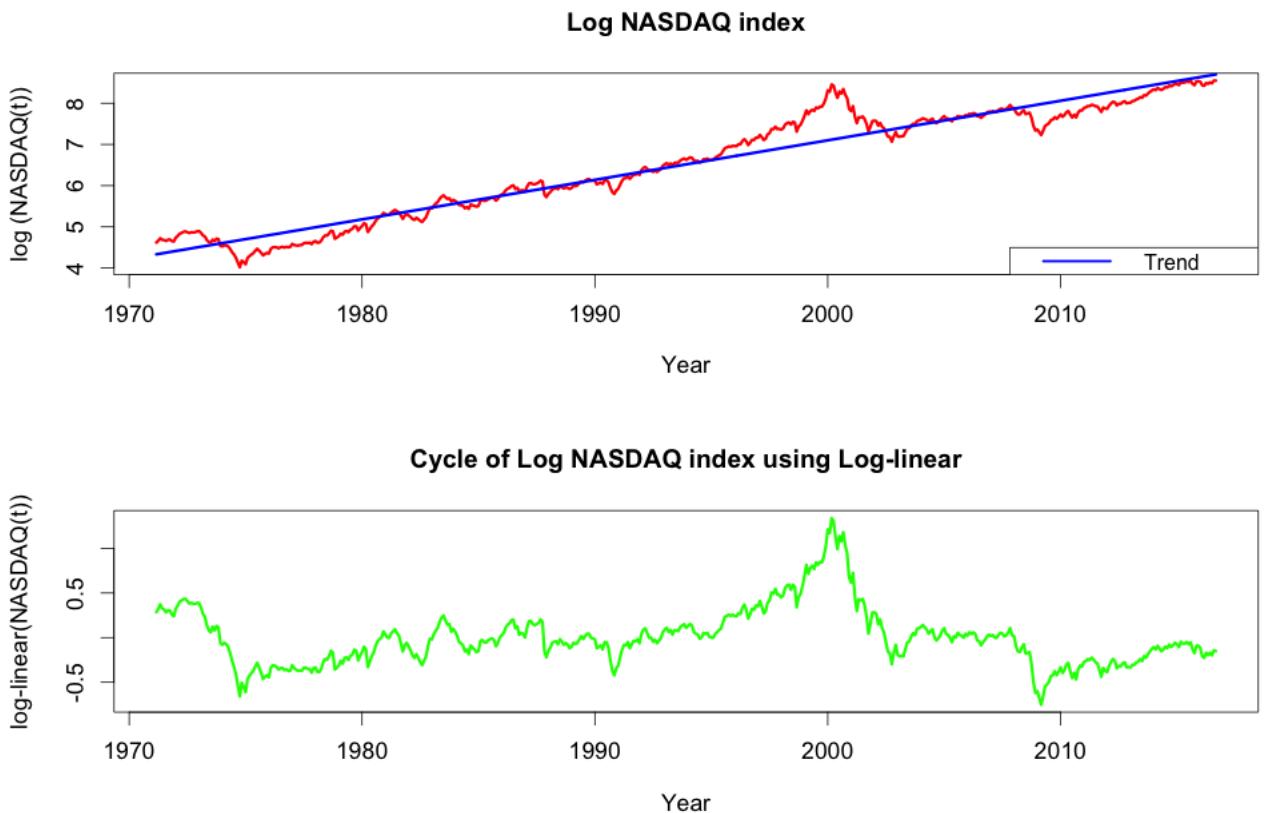


Figure 1.6: Log linear of trend and cycle for log NASDAQ  
 a) Log NASDAQ and trend.  
 The trend is calculated based on estimation of slope and y-intercept  
 b) It is cycle of log-linear of SP500 and it is the difference between the original log(NASDAQ) and estimated trend. AutoCorrelation.R is the R program used to created the graph and it is attached in the Appendix A

When  $\beta_1 > 0$ , the log-linear function illustrates a positive impact from the independent variable and when,  $\beta_1 < 0$ , the log-linear function depicts a negative impact from the independent variable.

### 1.0.8 Auto correlation Functions

An autoregressive model is when a value from a time series is regressed on previous value from that same time series. For example,  $y_t$  on  $y_{t-1}$ :

$$y_t = \beta_0 + \beta_1 y_{t-1} + \epsilon_t \quad (1.10)$$

In this regression model, the response variable in the previous time period has become the predictor and the errors have the same assumptions about errors in a simple linear regression model. The order of an autoregression is the number of immediate preceding values in the series that are used to predict the value at the present time. So, the preceding model is a first-order autoregression, written as  $AR(1)$ .

Let us say, if we want to predict the temperature  $y$  this year ( $y_t$ ) using measurement of global temperature in the previous two years ( $y_{t-1}, y_{t-2}$ ). Then the autoregressive model for doing so would be:

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \epsilon_t \quad (1.11)$$

The above model is a second-order autoregression, written as  $AR(2)$ , since the value at time  $t$  is predicted from the values at times  $t - 1$  and  $t - 2$ . More generally,  $k^{th}$  order autoregression, written as  $AR(k)$ , is a multiple linear regression in which the value of the series at any time  $t$  is a linear function of the values at times  $t - 1, t - 2, \dots, t - k$ .

The coefficient of correlation between two values in a time series is called the **autocorrelation function (ACF)**. Let  $y_t$  be a sample,  $t = 1, 2, \dots, n$  from an ARMA process of possibly unknown order, then the  $j^{th}$  order auto correlation  $\rho(j)$  can be estimated by using the formula

$$\rho(j) = \frac{Cov(y_t, y_{t-j})}{Var(y_t)} \quad (1.12)$$

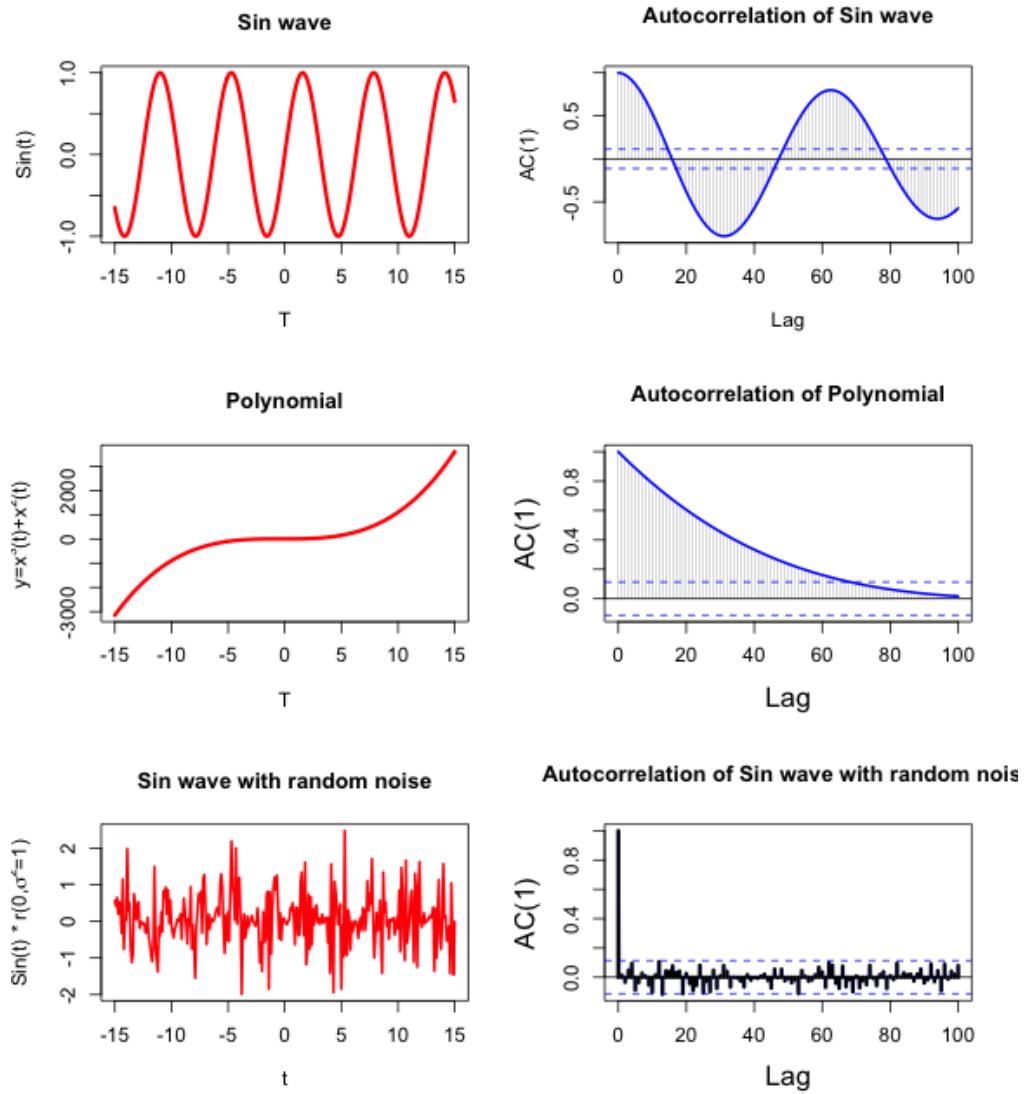


Figure 1.7: Auto correlation for a) Sin wave b) Polynomial equation c) Sin wave with random noise. AutoCorrelation.R is the R program used to created the graph and it is attached in the Appendix A

where

$$Cov(y_t, y_{t-j}) = \frac{1}{n-1} \sum_{t=j+1}^n (y_t - y_\mu)(y_{t-j} - y_\mu) \quad (1.13)$$

$$Var(y_t) = \frac{1}{n-1} \sum_{t=1}^n (y_t - y_\mu)^2 \quad (1.14)$$

$y_\mu$  is the mean of the  $y_t$ .

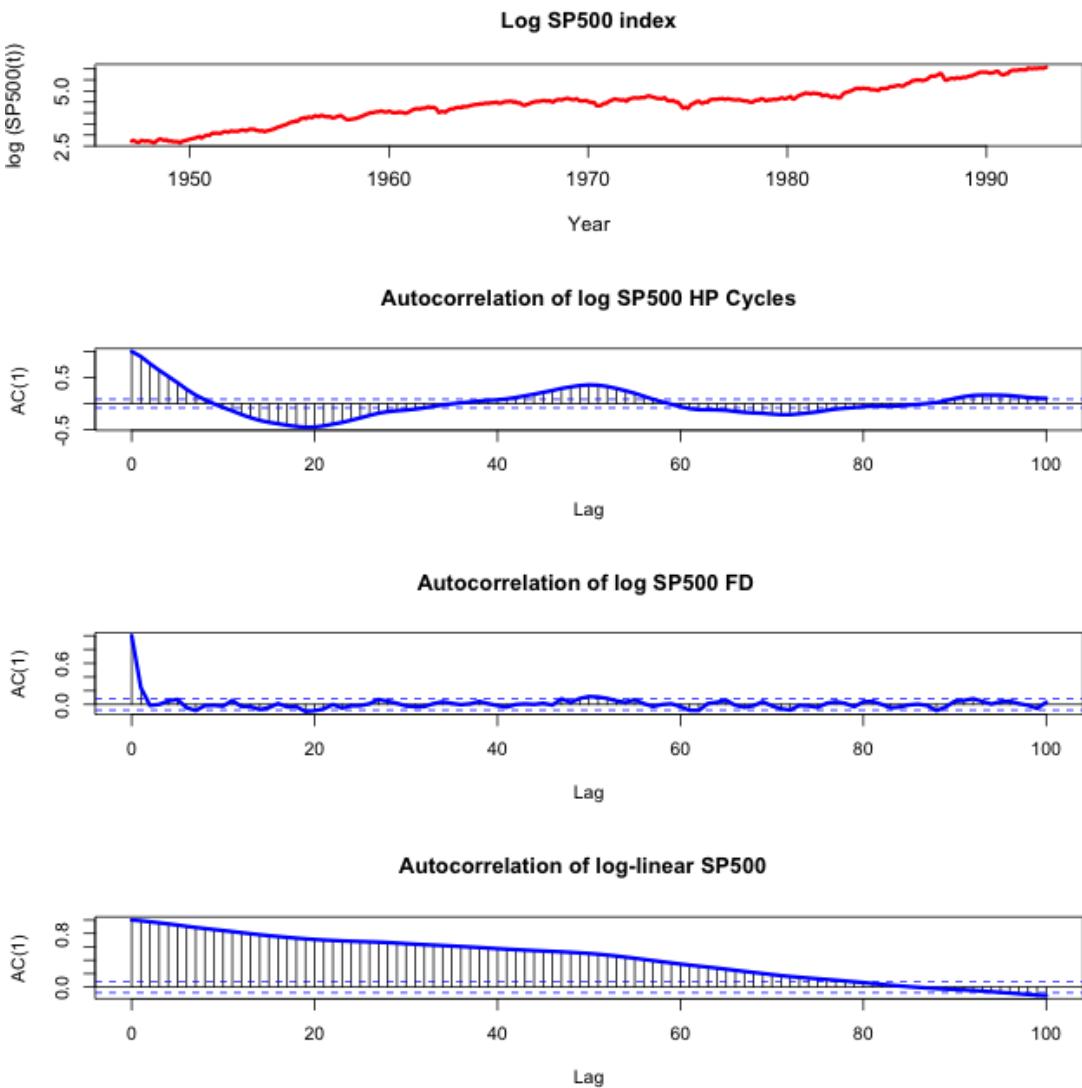


Figure 1.8: Auto correlation for log SP500 a) Log SP500 b)AutoCorrelation of HP cycle c) Autocorrelation of First Differencing d) Autocorrelation of log-linear SP500. AutoCorrelation.R is the R program used to created the graph and it is attached in the Appendix A

### 1.0.9 Hodrick and Prescott (HP) filter

Hodrick and Prescott proposed the HP filter to decompose a macroeconomic time series into a non-stationary trend component and a stationary cyclical residual component. The filter has become popular in applied macro economics in the last 15 years. Given an observed series  $y_i$ , let  $y_i = x_i + c_i$ , with  $y^T = (y_1, y_2, \dots, y_N)$ ,  $x^T =$

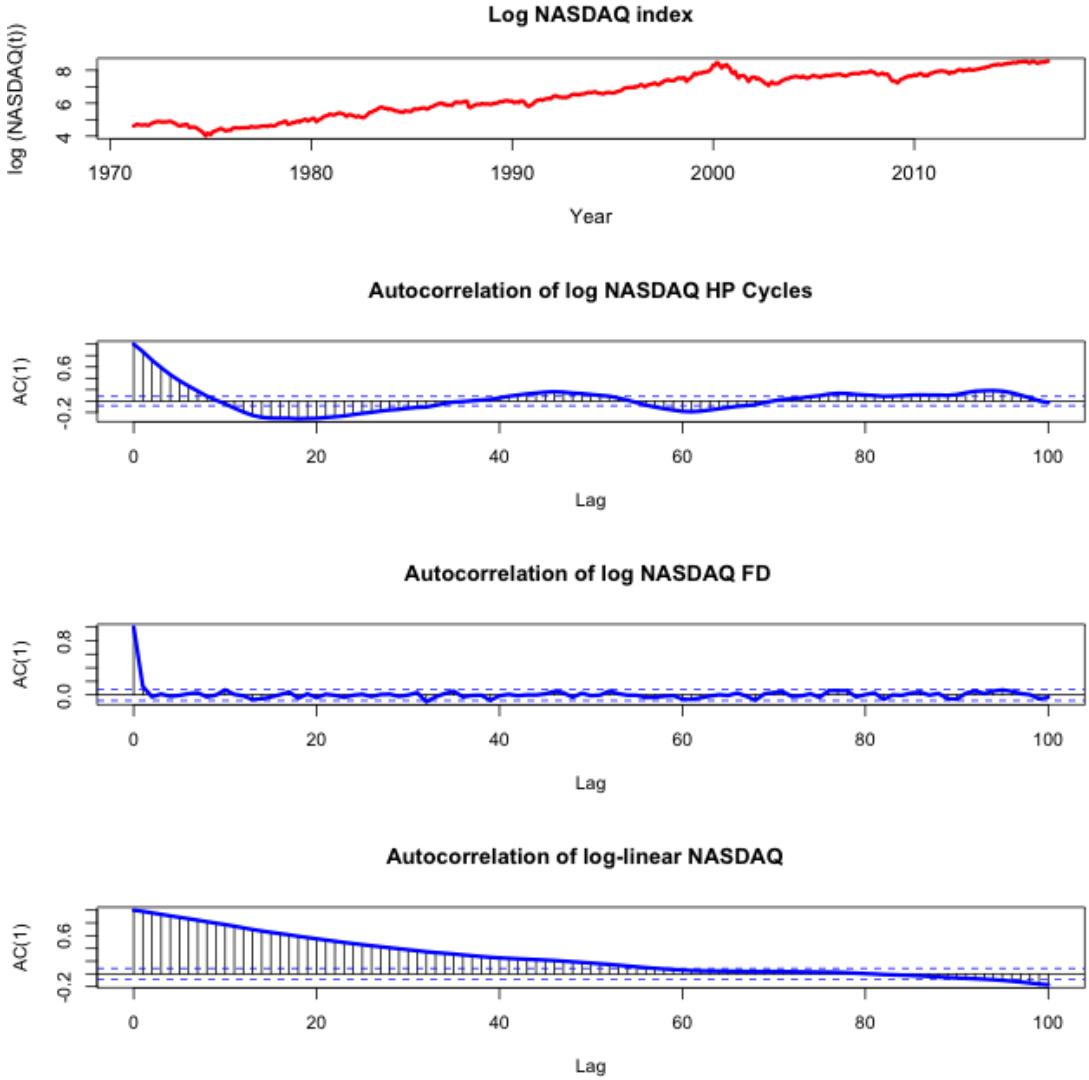


Figure 1.9: Auto correlation for log NASDAQ a) Log NASDAQ b)AutoCorrelation of HP cycle c) Autocorrelation of First Differencing d) Autocorrelation of log-linear NASDAQ. AutoCorrelation.R is the R program used to created the graph and it is attached in the Appendix A

$(x_1, x_2, \dots, x_N)$  and  $c^T = (c_1, c_2, \dots, c_N)$  where  $x_t$  denotes the unobserved trend component at time  $t$  and  $c_t$  the unobserved cyclical residual at time  $t$ . The HP trend  $\hat{x}$  can be obtained as the solution to the following convex minimization problems:

$$\min_{[x_t]_{t=1}^N} \left[ \sum_{t=1}^N (y_t - x_t)^2 + \lambda \sum_{t=2}^{N-1} ((x_{t+1} - x_t) - (x_t - x_{t-1}))^2 \right] \quad (1.15)$$

Here,  $\lambda$  is usually known as the smoothing parameter. As  $\lambda$  becomes larger, the HP estimated trend curve becomes smoother. The term being squared in the second sum of the equation,  $(x_{t+1} - x_t) - (x_t - x_{t-1})$ , or  $\Delta^2 x_i$ , is an approximation to the second derivate of  $x$  at time  $t$ . There are two opposing forces in the HP minimization problem. One force is attempting to minimize the sum of squared cyclical residuals and the other force is attempting to minimize the sum of squared  $\Delta^2 x_i$ . The smoothing parameter,  $\lambda$ , gives relative weight to these two opposing forces.

Detrending	Mean	SD	Variance	$T_0$ (month)	$P_{dc}$ (year)
FD	0.011	0.1123	0.0126	1.94	00.7
HP	0.008	0.2686	0.0722	8.93	02.9
LLD	0.426	0.3265	0.1066	85.6	28.5

Table 1.2: Detrend Statistics on S&P 500. The value for the above table was created by using the Matlab program `DetrendStatistics.m` attached in the appendix.

Detrending	Mean	SD	Variance	$T_0$ (month)	$P_{dc}$ (year)
FD	0.008	0.1072	0.0115	1.81	00.6
HP	0.022	0.2256	0.0509	9.22	03.0
LLD	0.258	0.3159	0.0992	80.5	26.8

Table 1.3: Detrend Statistics on NASDAQ. The value for the above table was created by using the Matlab program `DetrendStatistics.m` attached in the appendix.

The  $\lambda$  parameter determines the smoothness of the trend component. The larger the value of  $\lambda$ , the higher the penalty in the second term. An empirical study indicates that a 5% cyclical component is moderately large, as is a 1/8th of 1% change in the growth rate in a quarter. In the paper [?],  $\lambda$  should be adjusted by multiplying it with the fourth power of the observation frequency ratios. This yields an HP parameter value of 6.25 for annual data given a value of 1600 for quarterly data.

### 1.0.9.1 First-order conditions of the HP minimization problem

The following HP first order conditions are derived by setting the gradient vector of the above minimization equation equal to zero. The first-order conditions are:

$$c_1 = \lambda(x_1 - 2x_2 + x_3)$$

$$c_2 = \lambda(-2x_1 + 5x_2 - 4x_3 + x_4)$$

$$c_t = \lambda(x_{t-2} - 4x_{t-1} + 6x_t - 4x_{t+1} + x_{t+2}), t = 3, 4, 5, \dots, N-2$$

$$c_{N-1} = \lambda(x_{N-3} - 4x_{N-2} + 5x_{N-1} - 2x_N)$$

$$c_N = \lambda(x_{N-2} - 2x_{N-1} + x_N)$$

or more compactly,

$$\mathbf{c} = \boldsymbol{\lambda} \mathbf{F} \mathbf{x}$$

where  $\mathbf{F} =$

$$\left[ \begin{array}{ccccccccc} 1 & -2 & 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ -2 & 5 & -4 & 1 & 0 & \dots & \dots & \dots & 0 \\ 1 & -4 & 6 & -4 & 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & -4 & 6 & -4 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & -4 & 6 & -4 & 1 & 0 & \dots & 0 \\ \vdots & & & & & & & & \vdots \\ \vdots & & & & & & & & \vdots \\ \vdots & & & & & & & & \vdots \\ 0 & \dots & \dots & 0 & 1 & -4 & 6 & -4 & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 1 & -4 & 6 & -4 & 1 \\ 0 & \dots & \dots & \dots & & 0 & 1 & -4 & 5 & -2 \\ 0 & \dots & \dots & \dots & & & 0 & 1 & -2 & 1 \end{array} \right]$$

which implies that

$$\mathbf{y} = (\lambda \mathbf{F} + \mathbf{I})\mathbf{x}$$

Thus, the HP trend is given by:

$$\hat{\mathbf{x}} = (\lambda \mathbf{F} + \mathbf{I})^{-1}\mathbf{y}$$

and

$$\hat{\mathbf{c}} = \mathbf{y} - \hat{\mathbf{x}}$$

In the figure 1.10, the  $\lambda$  value is 80 and the trend and cyclical information is separated from the natural log SP500 index and the trend data is useful for the long term investors like pension fund manager. The long term fund managers require the projection of the performance of an index to strategize the investment to meet the client's retirement objective.

In the figure 1.11, the SP500 trend analysis was performed using different  $\lambda$  values. The trend looks very similar when the  $\lambda$  is above 800 value and there is no significance when the  $\lambda$  values are at 1600, 14400. The above figure 1.11, indicates that when  $\lambda$  is above 800 or above the trend in the HP filter is very close to least square linear regression line.

In the figure 1.12, the  $\lambda$  value is 800 and a similar analysis was performed.

In the figure 1.13, the  $\lambda$  value is 80 and the trend and cyclical information is separated from the natural log NASDAQ index.

In the figure 1.11, the NASDAQ trend analysis was performed using different  $\lambda$  values. The trend looks very similar when the  $\lambda$  is above 800 value and there is no significance when the  $\lambda$  values are at 1600, 14400.

The HP filters is one of the most heavily used econometric methods for measuring business cycles and potential output in empirical research. It is also a smoothing method that belongs to a very general class of nonparametric graduation procedures

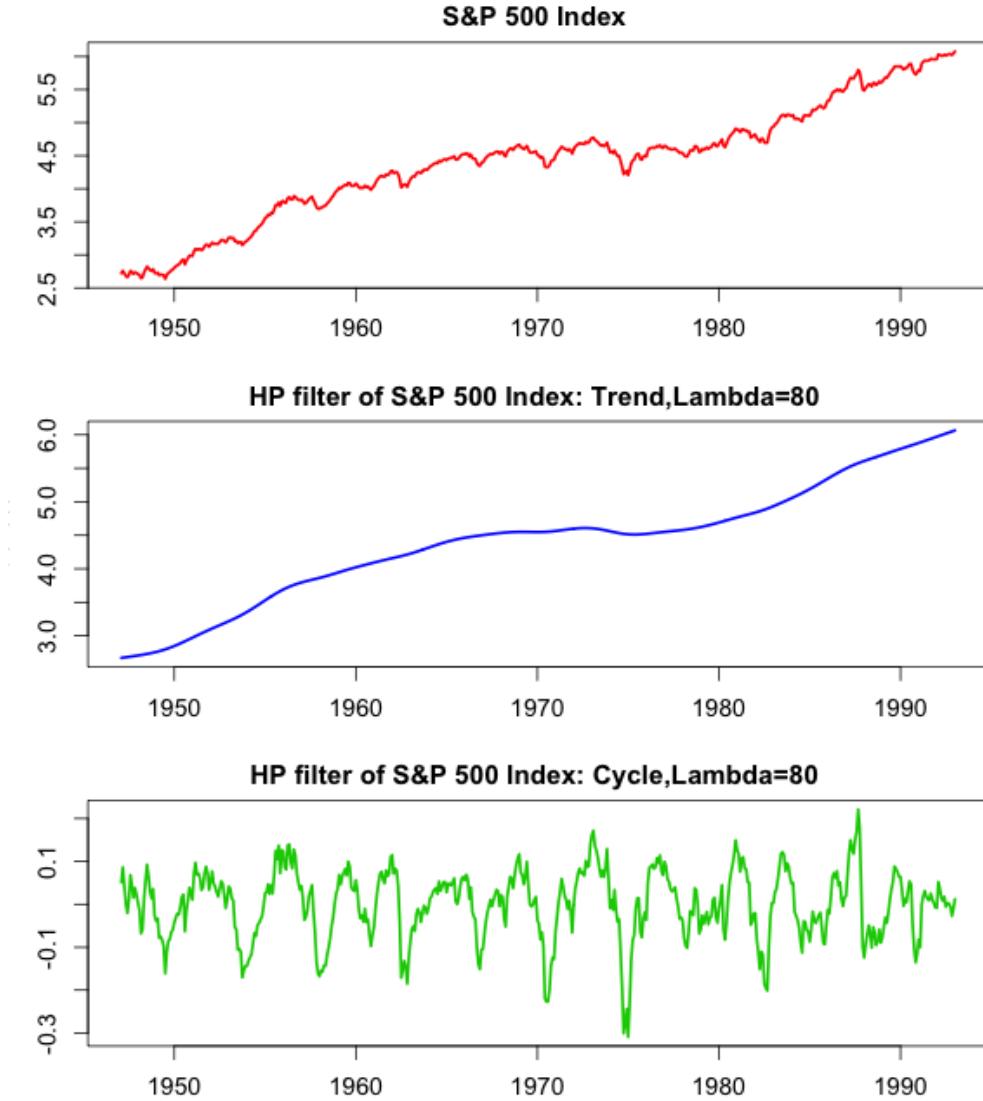


Figure 1.10: The trend and cycle separation from SP500 using HP filter with  $\lambda = 80$ .

- Natural Log of SP500 index
- Trend extracted from Log of SP500 index using HP filter
- Cyclical pattern extracted from SP500 index using HP filter

that depend on a tuning parameter governing the properties of the smoother. The long run potential output of an economy can be substantially influenced by great recessions and depressions, which may sufficiently divert resources to impact long run trend components of output. The HP filter has the advantage that, depending on the smoothing parameter ( $\lambda$ ) choice, it can encompass long run behavior that encompasses a vast range of possibilities -from a deterministic linear trend, to a

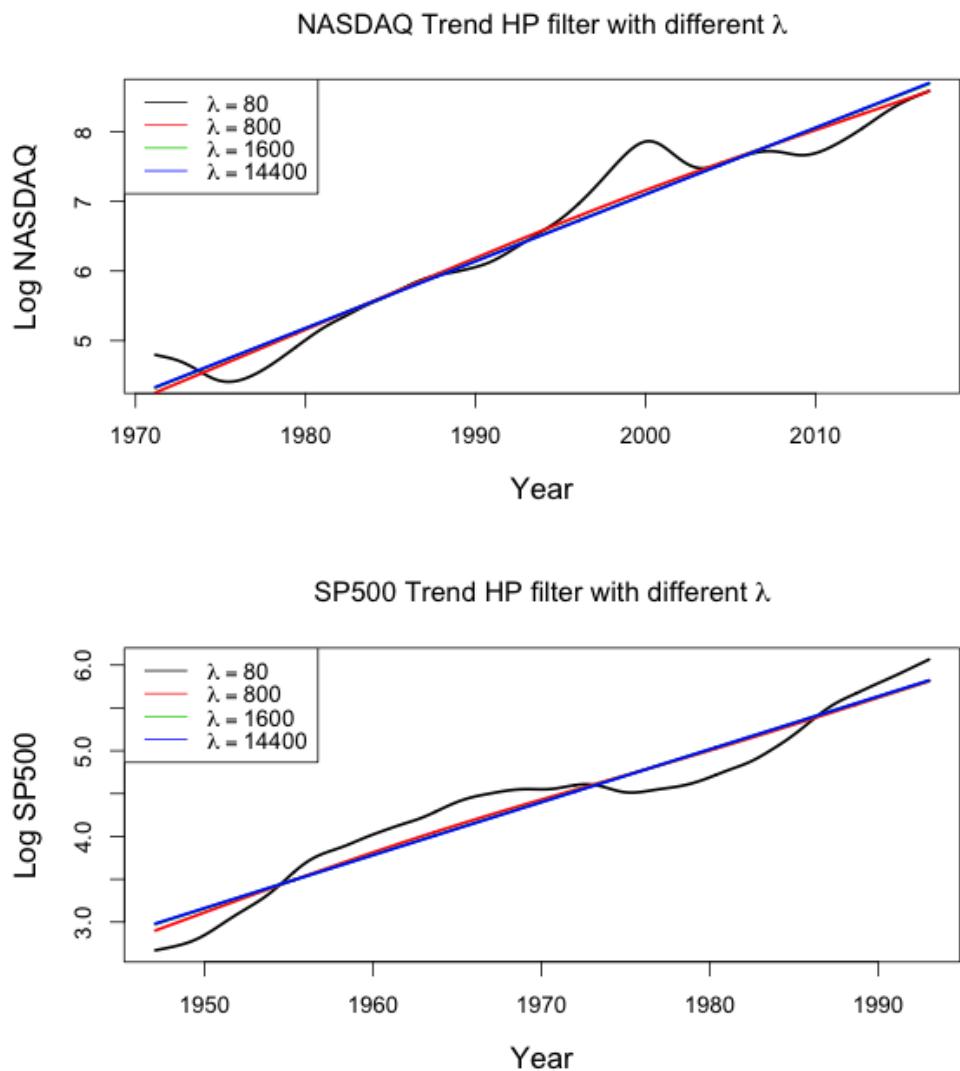


Figure 1.11: HP filter with different value for  $\lambda$  a) NASDAQ b) S&P 500

smooth Gaussian process, through to stochastic trends and combination of stochastic trends and deterministic trends that even include trend breaks.

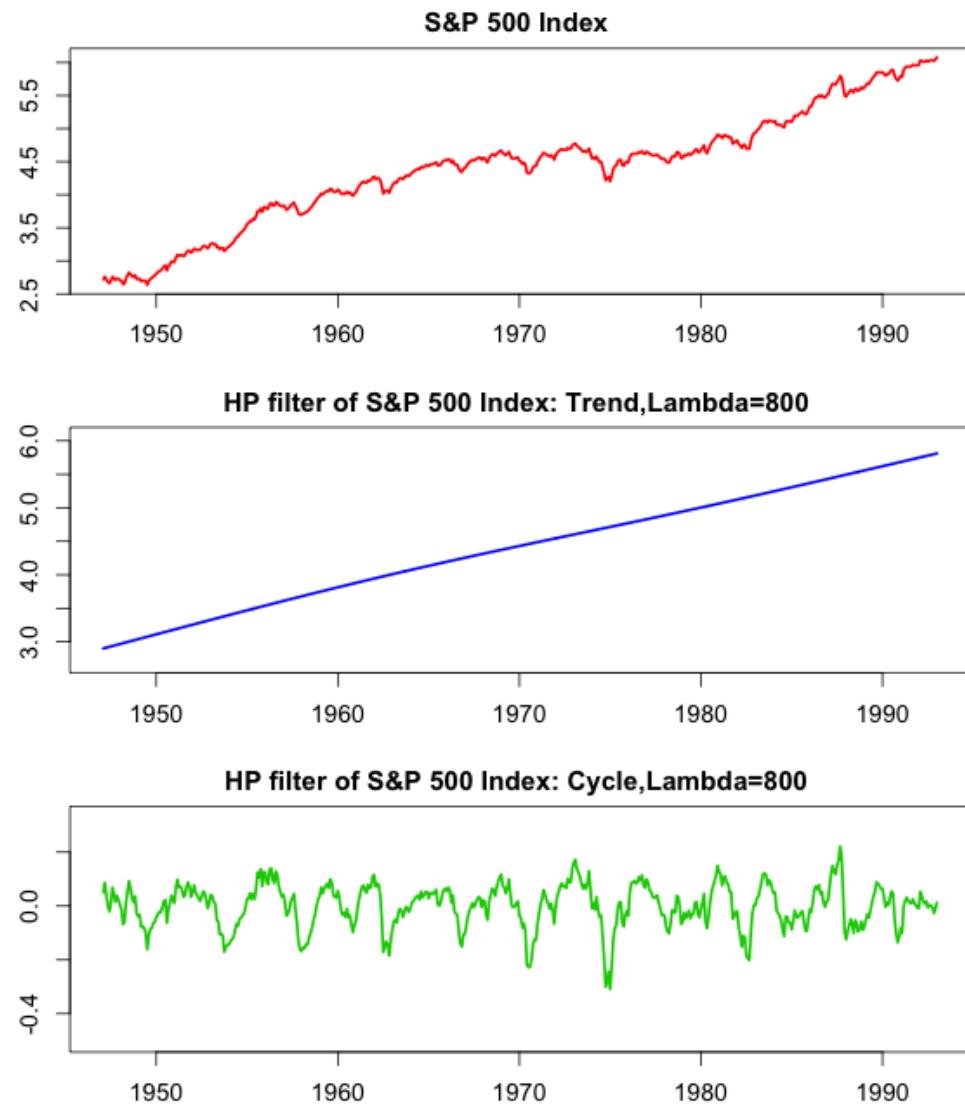


Figure 1.12: The trend and cycle separation from SP500 using HP filter with  $\lambda = 800$ . a) Natural Log of SP500 index b) Trend extracted from Log of SP500 index using HP filter c) Cyclical pattern extracted from SP500 index using HP filter

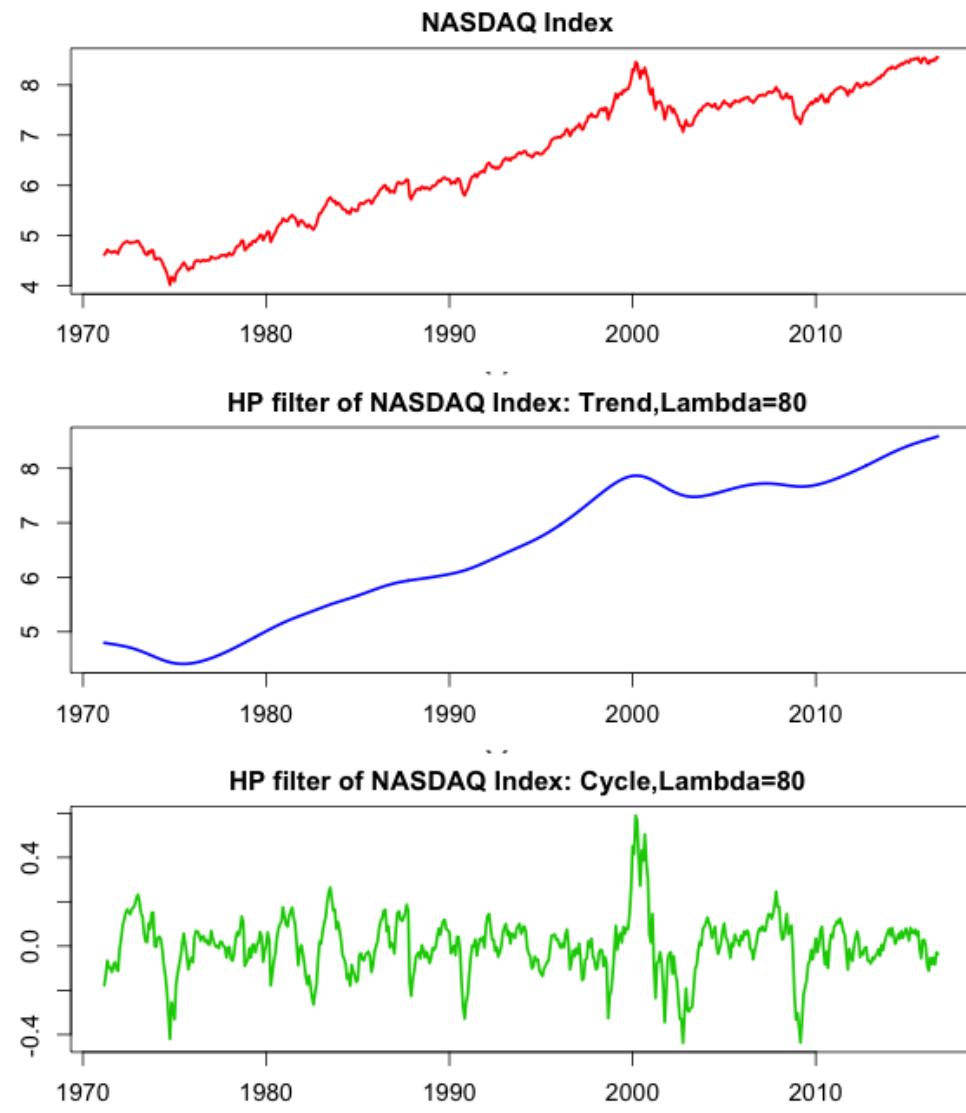


Figure 1.13: The trend and cycle separation from NASDAQ using HP filter with  $\lambda = 80$ . a) Natural Log of NASDAQ index b) Trend extracted from Log of NASDAQ index using HP filter c) Cyclical pattern extracted from NASDAQ index using HP filter

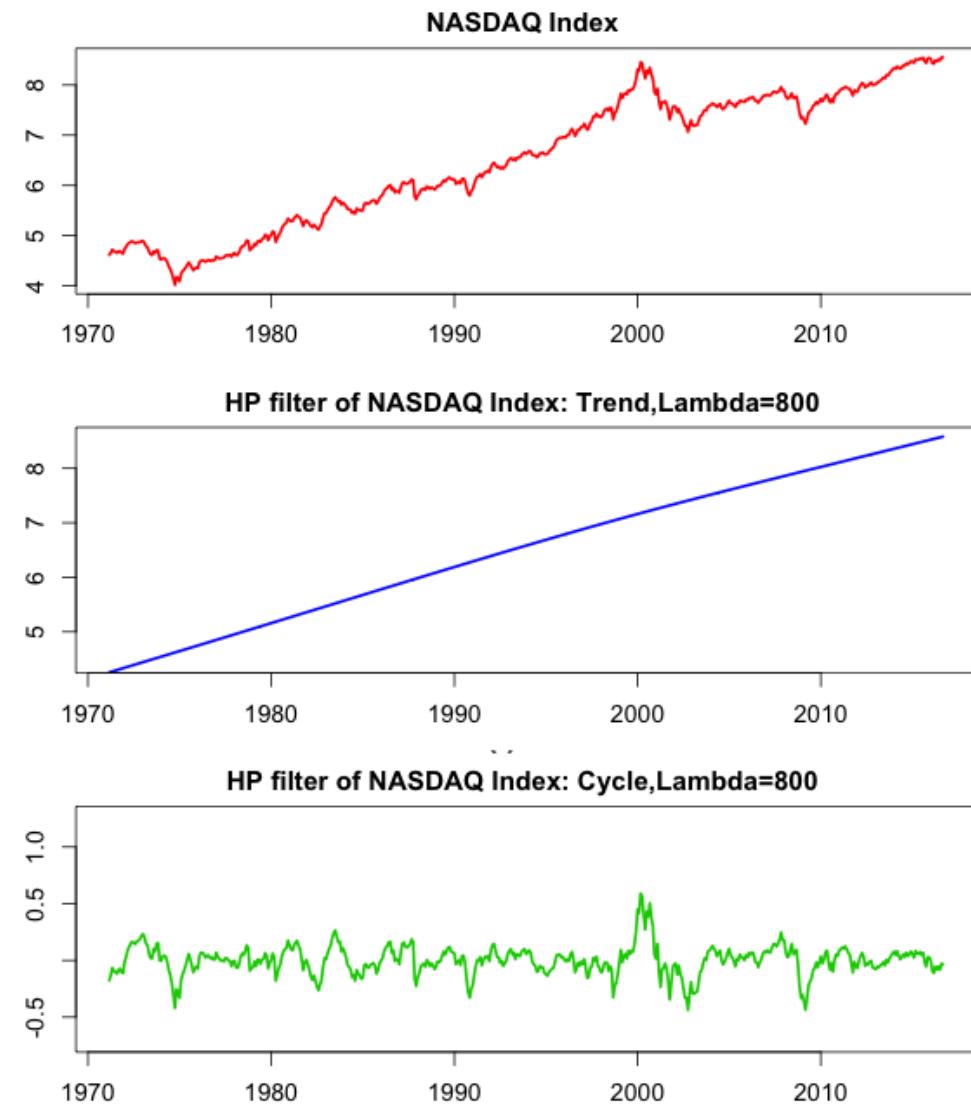


Figure 1.14: The trend and cycle separation from NASDAQ using HP filter with  $\lambda = 800$ . a) Natural Log of NASDAQ index b) Trend extracted from Log of NASDAQ index using HP filter c) Cyclical pattern extracted from NASDAQ index using HP filter

## CHAPTER II

# Gabor Transformation

### 2.1 Signal Processing

An arbitrary signal given by a function  $f(t)$  can be represented by many forms for better understanding of the signal. The representation of the signal  $f(t)$  in a different form depends on the type of application the user is interested in. In few cases, the original signals are perfectly fine as is for a given application.

In signal processing there are two major field of study.

- Signal synthesis - Construction of a signal
- Signal analysis - Study of a signal

Gabor in his original paper - Theory of communication [? ], stated that a time function  $f(t)$  in the time interval  $\tau = \{s : t_1 < s < t_2\}$  contains an infinite amount of data. There are an infinite number of ways to represent  $f(t)$  in the interval  $\tau$  and one of the ways is to represent  $f(t)$  in the interval  $\tau$  by a polynomial of order  $N$ . For example, one can fit  $f(t)$  as closely as possible by the method of least squares and take the coefficients of the polynomial as data. The polynomial coefficients of the curve is the data that represents the curve in the interval. It is equivalent to specifying the polynomial in such a way that its first  $N + 1$  moments  $M_i$  shall be

equal to those of  $f(t)$ .

$$\begin{aligned}
 M_0 &= \int_{t_1}^{t_2} f(t) dt; \\
 M_1 &= \int_{t_1}^{t_2} t f(t) dt; \\
 M_2 &= \int_{t_1}^{t_2} t^2 f(t) dt; \\
 &\dots \\
 M_{N-1} &= \int_{t_1}^{t_2} t^{N-1} f(t) dt \\
 M_N &= \int_{t_1}^{t_2} t^N f(t) dt
 \end{aligned}$$

Instead of the polynomial coefficients, the moments are considered as the specific data.

If the purpose is to transmit the signal, then the moments (equivalent of polynomial coefficients) can be transmitted and the signal can be reconstructed at the other end.

Instead of representing the function  $f(t)$  in the interval  $\tau$  in terms of powers of time functions, it can be represented in terms of orthogonal functions  $\phi_k(t)$  in the interval  $\tau$ . How close the fit will be depends on the set of orthogonal functions selected and type of applications.

### 2.1.1 Sampling Theorem

The Sampling theorem states that for an accurate representation of a signal  $f(t)$  by its time samples  $f(nT)$ , two conditions must be met.

- The signal  $f(t)$  must be band limited, that is, its frequency spectrum must be limited to contain frequencies up to some maximum frequency, say  $S_{max}$ , and no frequencies beyond that.

- The sampling rate  $S_s$  must be chosen to be at least twice the maximum frequency  $S_{max}$ , that is,  $S_s \geq 2S_{max}$  or, in terms of the sampling time interval:  $T \leq \frac{1}{2S_{max}}$ .

The minimum sampling rate allowed by the sampling theorem, that is,  $S_s = 2S_{max}$ , is called the **Nyquist** rate. For arbitrary values of  $S_s$ , the quantity  $\frac{S_s}{2}$  is called the **Nyquist** frequency. It defines the endpoints of the Nyquist frequency interval  $[-\frac{S_s}{2}, \frac{S_s}{2}]$ .

### 2.1.2 Rayleigh Frequency

The lowest resolvable frequency interval is given by the inverse of the length of the analysis window and is denoted by the **Rayleigh** frequency,  $S_r = \frac{1}{L}$ . The frequency resolution of any signal to be analyzed depends on the Rayleigh frequency of the data.

## 2.2 Fourier Transformation

If the orthogonal function set is simple harmonic functions sine and cosine in the interval extending from  $-\infty$  to  $\infty$ , then the given signal is presented in the frequency domain and it is called the Fourier transformation.

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi ts} dt; \quad (2.1)$$

$F(s)$  is the Fourier transform of  $f(t)$ . The inverse Fourier transform function  $F(s)$  provides the function  $f(t)$  and the equation for this is given below.

$$f(t) = \int_{-\infty}^{\infty} F(s)e^{j2\pi ts} ds \quad (2.2)$$

Fourier transformation is a tool to translate a signal from time domain to frequency domain and vice versa. In fact, it is an apt tool to analyze a signal in the

frequency domain given the signal does not evolve over time, but, most of the practical signals like music, seismic and others evolve over time. It is challenging to study and understand the insights of those signals in the frequency domain by just using Fourier transformation. In recent years, there are many ideas proposed to overcome that challenge and gain more insights of the signals in other domains.

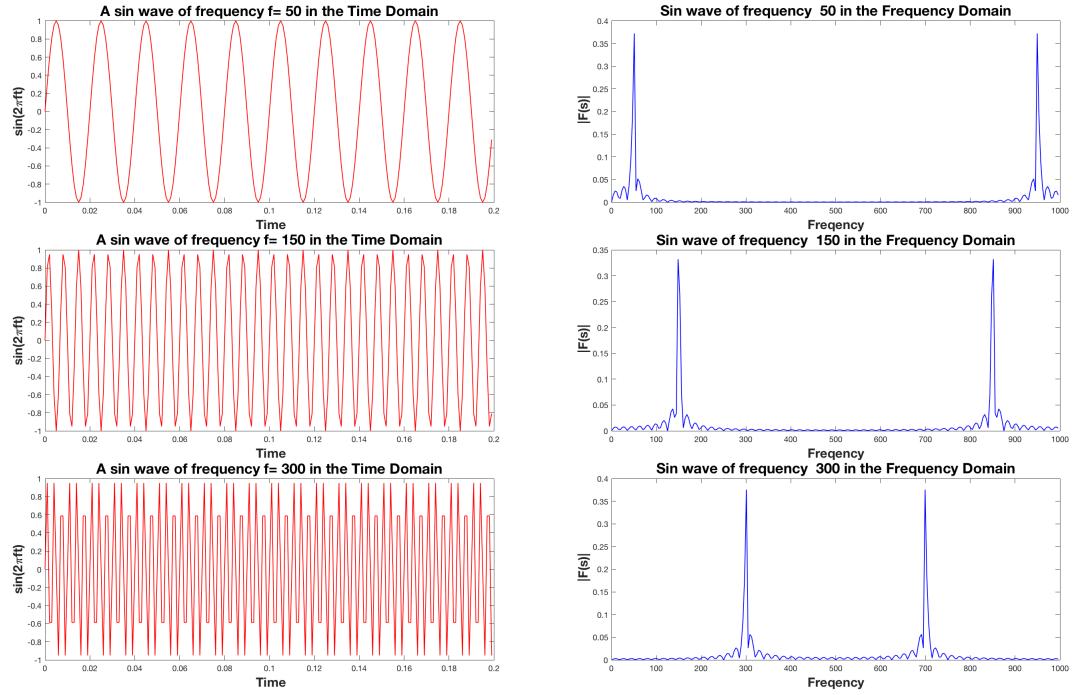


Figure 2.1: Fourier Transforms for discretized time signal sine waves with three different frequencies and respective discrete Fourier transforms are given above. The sampling period used is 0.001 and total length of signal ( $L$ ) is 200. The graphs were created using Matlab code DrawSinFourierGraph.m and it is attached in the Appendix B

In the figure ??, a sine wave with different frequencies  $f = 50, 150, 300$  is transformed into the frequency domain using the discrete Fourier transform. Please note high amplitude in the frequency domain for their respective frequency in each graph.  $\omega = 2\pi f$ .

$$f(t) = \sin(\omega t); \quad (2.3)$$

Three types of sine wave are created with frequencies  $f = 50, 150, 300$ . The Fourier transform of the  $f(t)$  is given by

$$F(s) = \int_{-\infty}^{\infty} \sin(2\pi ft) e^{-j2\pi ts} dt; \quad (2.4)$$

The above function  $F(s)$  works out to be  $\frac{\delta(t-2\pi f) - \delta(t+2\pi f)}{2j}$ , where  $\delta(s)$  is the Dirac Delta function. The Dirac Delta function is defined as

$$\delta(t) = \begin{cases} +\infty & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (2.5)$$

It follows that

$$\int_{-\infty}^{\infty} a\delta(t) dt = a \quad (2.6)$$

where  $a$  is a constant.

For a function  $f(t)$ , being integrable, we have that

$$\int_{-\infty}^{\infty} f(t)\delta(t) dt = f(0) \quad (2.7)$$

It denotes that the integral of any function multiplied by a  $\delta$ -function located about zero is just the value of the function at zero. This concept can be extended to give

the shifting property, again for a function  $f(t)$ , giving,

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a) \quad (2.8)$$

where  $\delta(t-a)$  is just a  $\delta$ -function located at  $t=a$ . In two dimensions, for a function  $f(t, w)$ , we have that,

$$\int \int f(t, w)\delta(t-a, w-b)dtdw = f(a, b) \quad (2.9)$$

where  $\delta(t-a, w-b)$  is a  $\delta$ -function located at position  $a, b$ . This property is central to the idea of convolution, which is used extensively in image formation theory, and in digital image processing. The Fourier transform of a  $\delta$  function can be formed by direct integration using the definition of the Fourier transform, and the shift property. We get,

$$\begin{aligned} F(\delta(t)) &= \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi st}dt \\ &= e^0 \\ &= 1 \end{aligned} \quad (2.10)$$

### 2.2.1 Shifting Theorem

Let  $f(t)$  be a function and its Fourier transform is given by  $F(s)$ . Fourier transform of a function  $f(t - t_0)$  is given  $e^{-j2\pi s t_0} F(s)$ . The proof of the shift theorem is given below.  $F[f(t - t_0)](s) = e^{-j2\pi s t_0} F(s)$

$$F[f(t - t_0)](s) = \int_{-\infty}^{\infty} f(t - t_0)e^{-j2\pi st}dt \quad (2.11)$$

Multiply  $e^{j2\pi st_0}e^{-j2\pi st_0}$  on right hand side of the above equation.

$$\begin{aligned} F[f(t - t_0)](s) &= \int_{-\infty}^{\infty} f(t - t_0)e^{-j2\pi st}e^{j2\pi st_0}e^{-j2\pi st_0}dt \\ &= e^{-j2\pi st_0} \int_{-\infty}^{\infty} f(t - t_0)e^{-j2\pi s(t-t_0)}dt \end{aligned} \quad (2.12)$$

Let  $u = t - t_0$

$$\begin{aligned} F[f(t - t_0)](s) &= \int_{-\infty}^{\infty} f(u)e^{-j2\pi su}du \\ &= e^{-j2\pi st_0}F(s) \end{aligned} \quad (2.13)$$

By applying the shifting theorem in the equation 2.10, we get that,

$$F(\delta(t - a)) = e^{-j2\pi sa} \quad (2.14)$$

so that the Fourier transform of a shifted  $\delta$  function is given by a phase ramp. The modulus squared of above equation is

$$\begin{aligned} |F(\delta(t - a))|^2 &= |e^{-j2\pi sa}|^2 \\ &= 1 \end{aligned} \quad (2.15)$$

The power spectrum of a  $\delta$  function is constant independent of its location in real space. Noting that the Fourier transform is a linear operator, consider two  $\delta$  functions

located at  $\pm a$ . Then the Fourier transform of  $\delta(t - a) + \delta(t + a)$  is given by

$$\begin{aligned} F(\delta(t - a) + \delta(t + a)) &= e^{-j2\pi as} + e^{j2\pi as} \\ &= 2 \cos(2\pi as) \\ \int_{-\infty}^{\infty} \cos(2\pi at) e^{-j2\pi ts} &= \frac{\delta(t - a) + \delta(t + a)}{2} \end{aligned} \tag{2.16}$$

If we take the  $\delta$ -function at  $t = -a$  as negative, the Fourier transform is given below

$$\begin{aligned} F(\delta(t - a) - \delta(t + a)) &= e^{-j2\pi as} - e^{j2\pi as} \\ &= 2j \sin(2\pi as) \\ \int_{-\infty}^{\infty} \sin(2\pi at) e^{-j2\pi ts} &= \frac{\delta(t - a) - \delta(t + a)}{2j} \end{aligned} \tag{2.17}$$

In the figure 2.2, a signal with zero mean random noise, signal with zero mean with no random noise, Gaussian curve, *log sp500*, *log Nasdaq* are also transformed into a frequency domain using the discrete Fourier transform.  $|F(s)|$  is an even function and so only  $s > 0$  is shown in figure 2.2.

### 2.2.2 Discrete Fourier Transform

The Discrete Fourier Transform (DFT) is the equivalent of the continuous Fourier Transform for signals known only at  $N$  instants separated by sample times  $T$  (i.e a finite sequence of data). Let  $f(k)$  be a continuous signal and the  $N$  samples be denoted by  $f(0), f(1), f(2) \dots f(k), \dots f(N - 1)$ . Each sample  $f(k)$  is regarded as an impulse having area  $f(k)$  and the integrand exists only at the sample points. Since there are only a finite number of input data points, the DFT treats the data as if it were periodic i.e  $f(N)$  to  $f(2N - 1)$  is the same as  $f(0)$  to  $f(N - 1)$ . In general, the

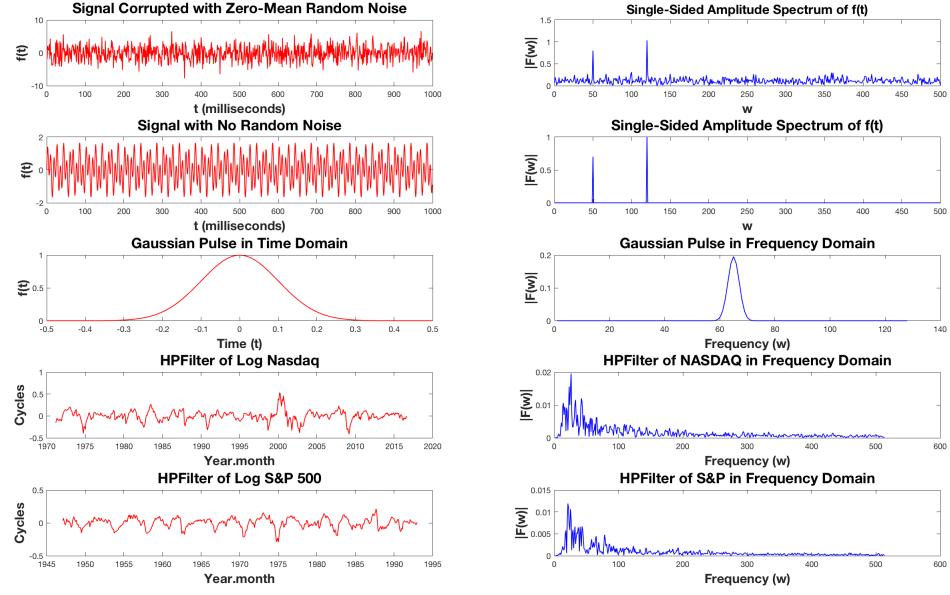


Figure 2.2: Fourier transform for a) Zero mean signal with random noise, b) Zero mean signal  $f(t) = 0.7\sin(2\pi 50t) + \sin(2\pi 120t)$ , c) Gaussian - Frequency are shifted (using `fftshift` function in matlab) in the frequency domain to show that the Fourier transform of Gaussian looks like Gaussian in the frequency domain, d) Log NASDAQ, e) Log S&P 500. The program used to generate the graph is `SPNASDAQFourier.m` and stored in Appendix

discrete Fourier Transform is given by

$$F(s) = \sum_{k=0}^{N-1} f(k)e^{-j\frac{2\pi}{N}sk} \quad (2.18)$$

$$s = 0, 1, \dots, N - 1.$$

$$e^{j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) + j \sin\left(\frac{2\pi}{N}\right) \quad (2.19)$$

is the principal  $N$ -th root of unity. Since  $e^{j\frac{2\pi s}{N}}$  has a period of  $N$ , the DFT coefficients  $F(s)$  are periodic with period  $N$  when  $s$  is taken outside the range  $s = 0, 1, 2, \dots, N - 1$ .  $F(s)$  is the Discrete Fourier Transform of the sequence  $f(t)$ . The matrix-vector format of the Discrete Fourier Transform are given by

$$\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ \vdots \\ F(N-1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & \dots & \dots & 1 \\ 1 & e^{-j(\frac{2\pi}{N})} & e^{-j(\frac{4\pi}{N})} & \dots & e^{-j\frac{2(N-1)\pi}{N}} \\ 1 & e^{-j(\frac{4\pi}{N})} & e^{-j(\frac{8\pi}{N})} & \dots & e^{-j\frac{4(N-1)\pi}{N}} \\ \vdots & & & \vdots & \\ \vdots & & & \vdots & \\ 1 & e^{-j(\frac{2(N-1)\pi}{N})} & e^{-j(\frac{4(N-1)\pi}{N})} & \dots & e^{-j\frac{2(N-1)(N-1)\pi}{N}} \end{bmatrix}}_{\text{DFT Matrix: } F} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ \vdots \\ f(N-1) \end{bmatrix}$$

The above matrix multiplication can also be used to compute the Discrete Fourier

Transform of a given signal  $f(t)$ . The Discrete Fourier Transform matrix F can be precomputed as it is independent of the input signal.

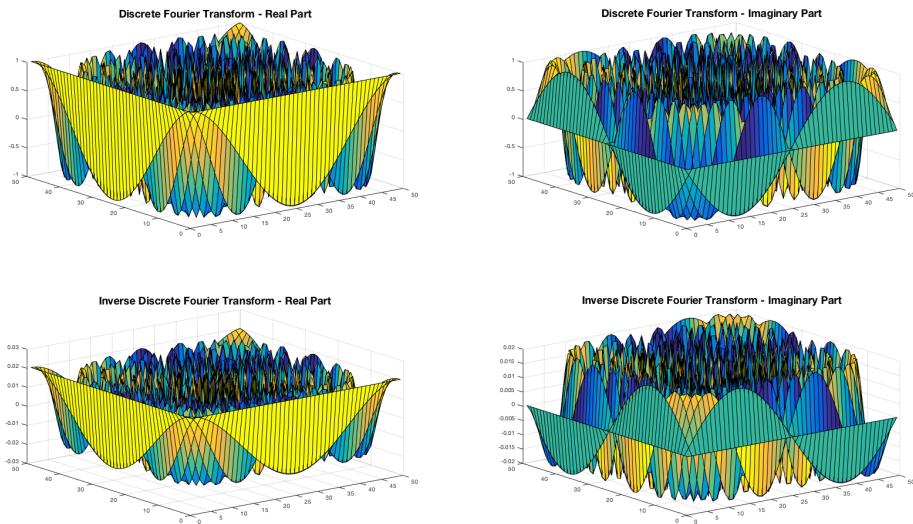


Figure 2.3: Discrete Fourier Transform & Inverse DFT real and imaginary part. The above surface created using the matrix F. The graph is created using Matlab program named mydft.m attached in the Appendix

### 2.2.2.1 Inverse Discrete Fourier Transform

The inverse transform of Discrete Fourier Transform is given by

$$f(t) = \frac{1}{N} \sum_{k=0}^{N-1} F(k) e^{j \frac{2\pi}{N} tk} \quad (2.20)$$

The matrix-vector format of the Inverse Discrete Fourier Transform are given by

$$\underbrace{\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ \vdots \\ f(N-1) \end{bmatrix}}_{\text{Original Signal}} = \frac{1}{N} \underbrace{\begin{bmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & e^{j(\frac{2\pi}{N})} & e^{j(\frac{4\pi}{N})} & \cdots & e^{j(\frac{2(N-1)\pi}{N})} \\ 1 & e^{j(\frac{4\pi}{N})} & e^{j(\frac{8\pi}{N})} & \cdots & e^{j(\frac{4(N-1)\pi}{N})} \\ \vdots & & & \ddots & \vdots \\ \vdots & & & & \vdots \\ 1 & e^{j(\frac{2(N-1)\pi}{N})} & e^{j(\frac{4(N-1)\pi}{N})} & \cdots & e^{j(\frac{2(N-1)(N-1)\pi}{N})} \end{bmatrix}}_{\text{Inverse DFT Matrix:G}} \underbrace{\begin{bmatrix} F(0) \\ F(1) \\ F(2) \\ \vdots \\ \vdots \\ F(N-1) \end{bmatrix}}_{\text{DFT Vector}}$$

The inverse DFT matrix G is inverse of matrix F and scaled by a factor of  $\frac{1}{N}$ .

$G * F = F * G = I$ , where as I is an identify matrix of order N.

### 2.2.3 Short Time Fourier Transformation

One of the ideas is to chop the signal into smaller pieces and perform Fourier transformation for each piece. This technique is called short time Fourier Transform. The smaller pieces in the signal can be chosen by a window function  $w(t - \tau)$

$$F(\tau, s) = \int_{-\infty}^{\infty} f(t) w(t - \tau) e^{-j2\pi ts} dt; \quad (2.21)$$

$w(t)$  represents a window function and there are multiple window functions available to choose.

#### 2.2.3.1 Gaussian Window

The Gaussian Window is defined as

$$w(n) = e^{\frac{-(n-m)^2}{2(\sigma N)^2}} \quad (2.22)$$

for  $n = 0, 1, 2, \dots, N - 1$ , where  $N$  is the length of the window,  $m = \frac{(N-1)}{2.0}$ , and the  $\sigma$  is the standard deviation of the Gaussian window. The Gaussian window is useful for time-frequency analysis because the Fourier transform and the derivative of a Gaussian window both are Gaussian function.

### 2.2.3.2 Hamming Window

The Hamming window is defined as

$$w(n) = \alpha - \beta \cos\left(\frac{2\pi n}{N-1}\right) \quad (2.23)$$

where the constants  $\alpha$  and  $\beta$  are approximations to the values of  $\frac{25}{46}$  and  $\frac{21}{46}$  respectively. The values of  $\alpha$  and  $\beta$  are given as 0.54 and 0.46 respectively.

$$w(n) = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), 0 \leq n \leq N \quad (2.24)$$

The window length is  $L = N + 1$ .

The short time Fourier transform is performed for several signals using Hamming and Gaussian windows and the results are given below.

Simultaneous analysis of signals in both time and frequency domains provides better understanding of the signal and the short time Fourier transform laid the foundation for the joint time-frequency analysis.

## 2.3 Introduction

The Gabor transformation or expansion in signal synthesis uses the Gabor elementary function (GEF) as the base function (equivalent of simple harmonic function in the Fourier transform) and the idea was influenced by Heisenberg's uncertainty principle.

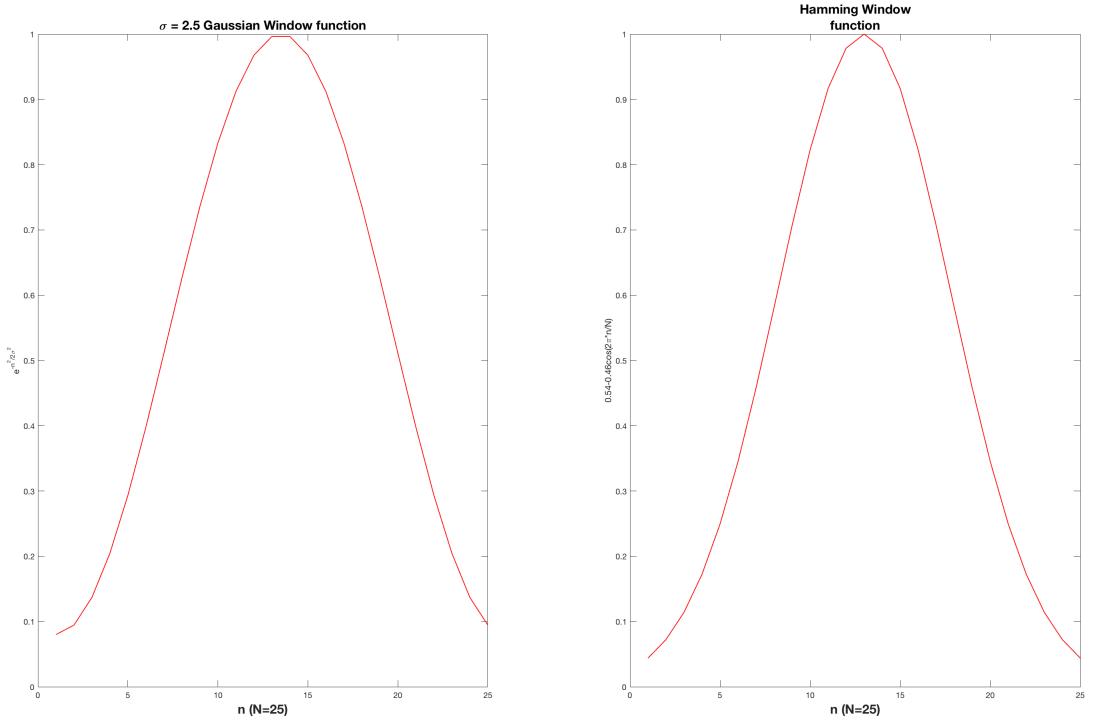


Figure 2.4: Window function (window length =25) for a) Gaussian b) Hamming.  
The graph was created using Matlab program ghamwin.m and attached in Appendix B

### 2.3.1 Heisenberg's uncertainty principle

In quantum mechanics, simultaneously, both the position of a particle and the momentum of the particle can not be measured precisely. Let  $x$  be position and  $p$  be momentum of the particle. The standard deviation of  $x$  and  $p$  are denoted by  $\Delta x$  and  $\Delta p$  respectively. The uncertainty principle states that product of the standard deviations of position and momentum is greater than or equal to  $\frac{\hbar}{2}$

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

$$\Delta x = \sqrt{\langle (x - \langle x \rangle)^2 \rangle}$$

$$\Delta p = \sqrt{\langle (p - \langle p \rangle)^2 \rangle}$$

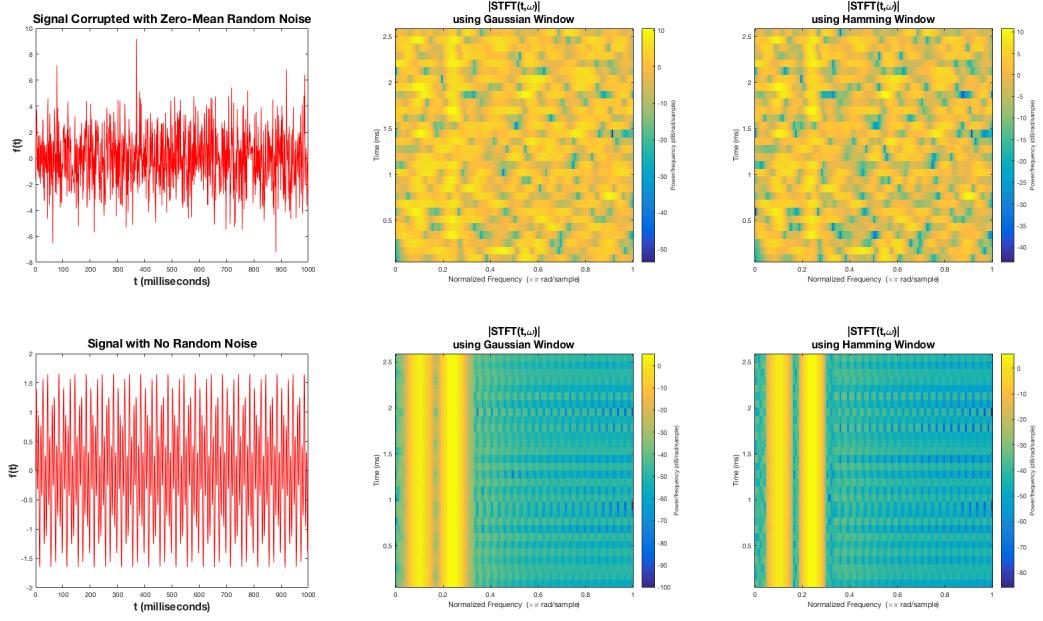


Figure 2.5: Window Length=64 ( $2^6$ ) used for both windows. Gaussian and Hamming Window function used for short time Fourier Transform for a) Zero Mean Signal with Random Noise b) Zero Mean signal. The graph was created using Matlab program drawSpecEg.m and attached in Appendix B

### 2.3.2 Gabor Transformation

Gabor had the insight that the base function with minimum uncertainty in both time and frequency domains best captures temporal information during the frequency analysis.

Gabor in his original study proposed an elementary function in complex form which occupies minimum uncertainty and it is a product of harmonic oscillator of any frequency and a Gaussian function. The area occupied by the following elementary function in the joint time frequency domain is equal to the minimum uncertainty.

$$\psi(t) = \underbrace{e^{-\alpha^2(t)^2}}_v \underbrace{e^{j2\pi f_0 t + \phi}}_w \quad (2.25)$$

An arbitrary function  $f(t)$  can be represented by a series of elementary functions

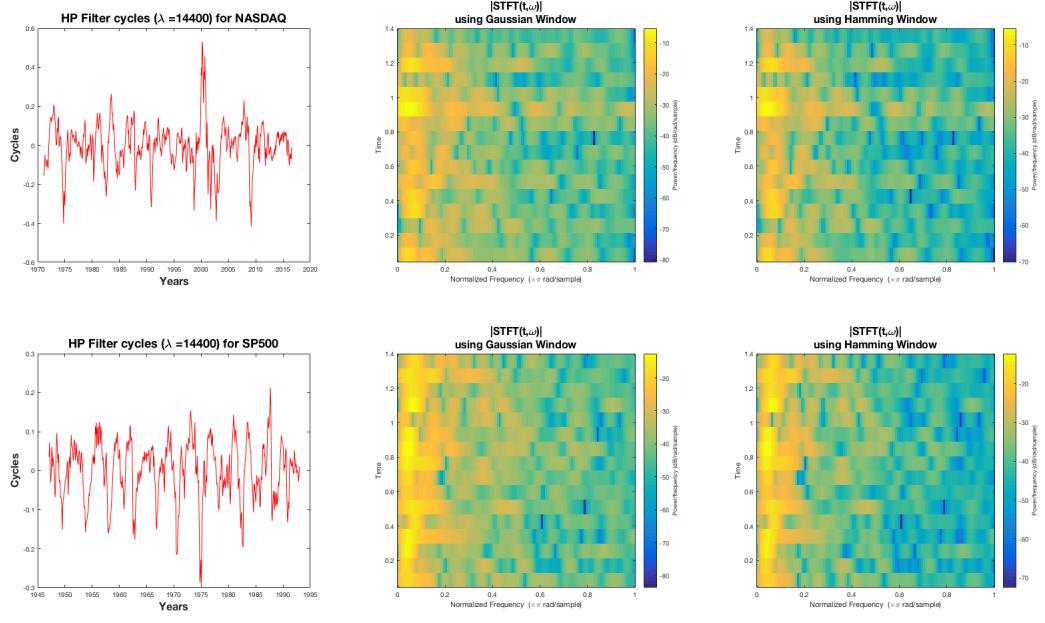


Figure 2.6: Window Length=64 ( $2^6$ ) used for both windows. Gaussian and Hamming Window function used for short time Fourier Transform for a) HP Cycles for NASDAQ b) HP Cycles for SP500. The graph was created using Matlab program drawSpecEco.m and attached in Appendix B

which are constructed by translation in both the time and frequency domains. The function  $f(t)$  is synthesized by the combination of GEFs.

$$f(t) = \sum_{m,n \in Z} C_{m,n} \psi_{m,n}(t) \quad (2.26)$$

where the  $C_{m,n}$  are Gabor coefficients and the  $\psi_{m,n}(t)$  are the bases functions called Gabor elementary functions. A GEF translated by 'na' is denoted by  $\psi_{m,n}$  and given by

$$\psi_n(t) = \psi(t - na)$$

The translation in the frequency domain is also called modulation. A GEF is modu-

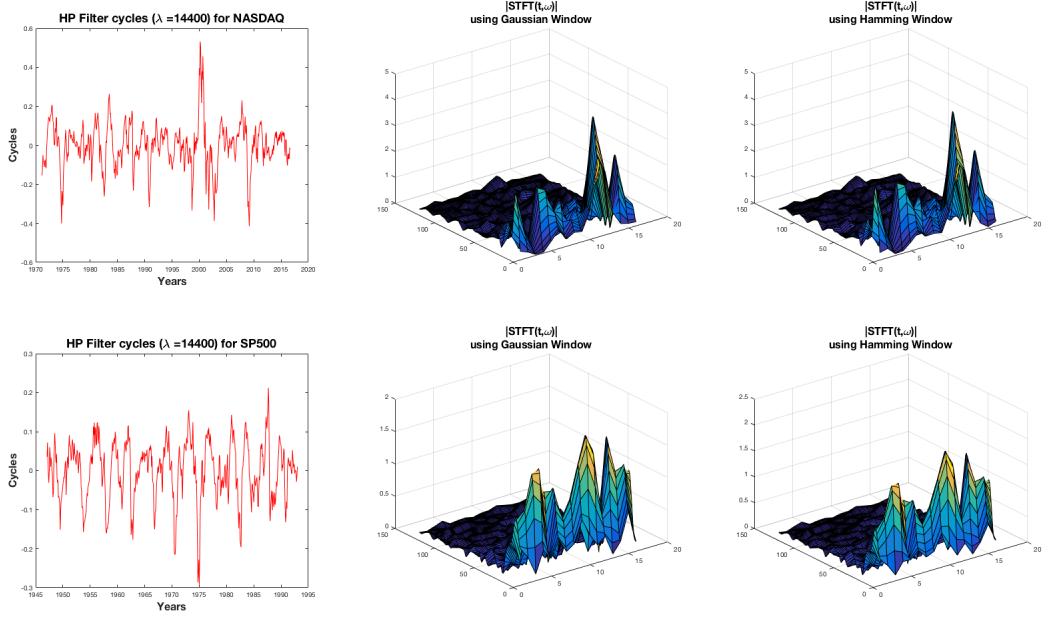


Figure 2.7: Window Length=64 ( $2^6$ ) used for both windows. Gaussian and Hamming Window function used for short time Fourier Transform for a) HP Cycles for NASDAQ b) HP Cycles for SP500. The graph was created using Matlab program drawSTFTEco3D.m and attached in Appendix B

lated by using simple harmonic functions.

$$\psi_{m,n}(t) = \psi(t - na)e^{j2\pi m b t}$$

The original Gabor paper suggested that  $\psi$  is Gaussian. GEFs were studied by using other functions for  $\psi$  like a rectangle.  $a$  and  $b$  are time and frequency shift parameters and  $a,b > 0$ .  $\psi_{m,n}$  is obtained by shifting it by lattice point  $\times mb$  in the time-frequency plane.

The GEFs could be a set of Gaussian functions modulated by simple harmonics. GEF generated in 1D by combining Gaussian and simple harmonic functions. The

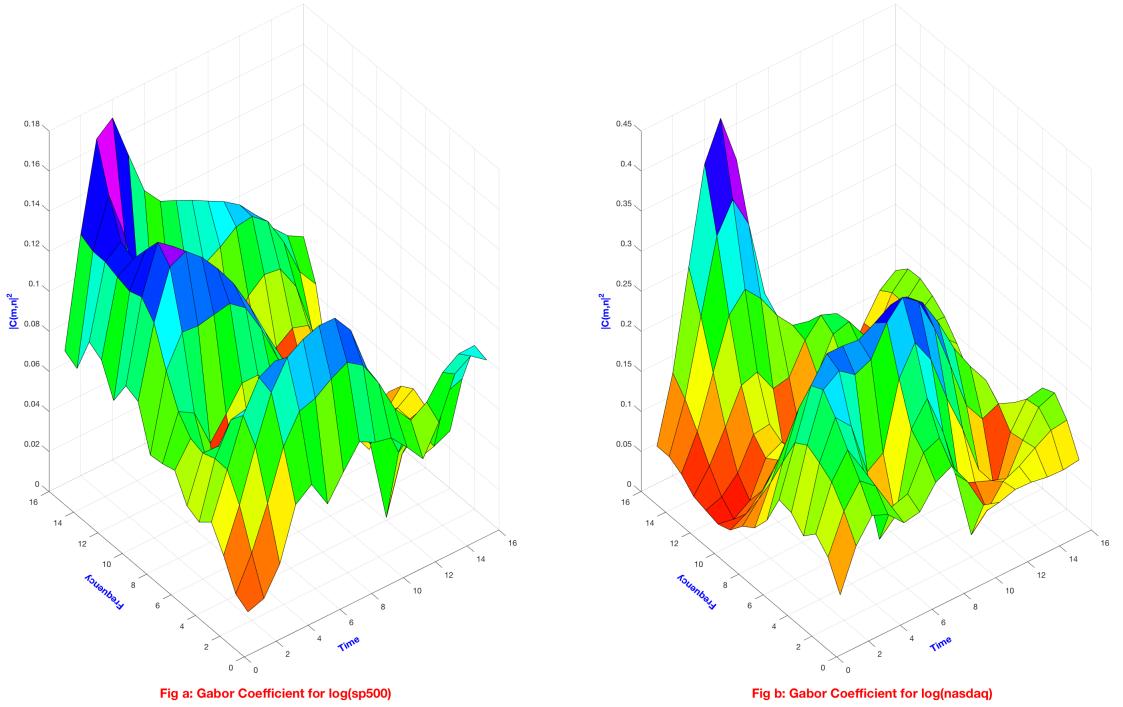


Figure 2.8: Fig a is the Gabor Coefficient for HP Filter cycles of  $\log(\text{SP500})$  and Fig b is the Gabor Coefficient for HP Filter cycles of  $\log(\text{NASDAQ})$ .  $\Delta M = 8$ ;  $\Delta N = 4$ ;  $m = 16$ ;  $n = 16$ ;  $\sigma = \sqrt{\frac{\Delta M L}{\Delta N^2 \pi}}$  The program used to create the graph is mygabor.m and it is attached in the appendix.

Gaussian function remained the same for both GEF and only the sinusoidal functions changed between the two GEFs. Note the change in the frequency between the GEFs. A set of GEFs can be created by varying na and mb.

$$\psi_{m,n}(t) = e^{-\frac{(t-\mu-na)^2}{2\sigma^2}} e^{j2\pi m b t}$$

If the GEF  $\psi$  and its Fourier transform (Frequency representation)  $\Psi$  are localized at the origin then  $g_{m,n}$  is localized at (na,mb) in the joint time-frequency domain. Each GEF occupies a region in time-frequency plane and associated  $C_{m,n}$  represents a quantum of information.

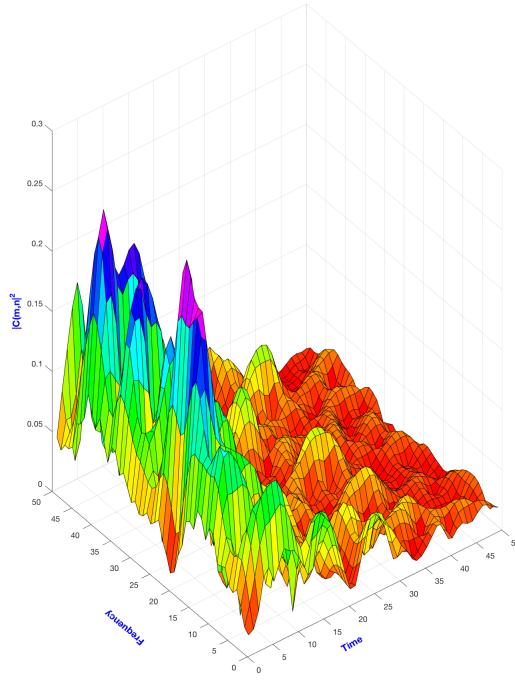


Fig a: Gabor Coefficient for  $\log(\text{sp500})$

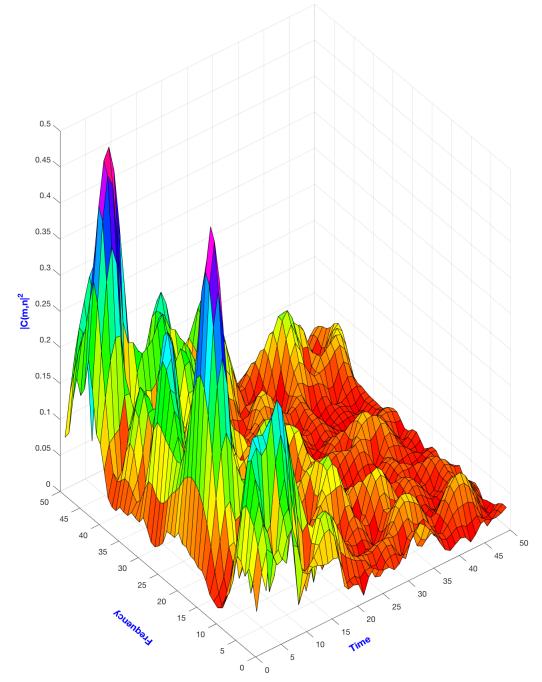


Fig a: Gabor Coefficient for  $\log(\text{nasdaq})$

Figure 2.9: Fig a is the Gabor Coefficient for HP Filter cycles  $\log(\text{SP500})$  and Fig b is the Gabor Coefficient for HP Filter cycles  $\log(\text{NASDAQ})$ .  $\Delta M = 8$ ;  $\Delta N = 4$ ;  $m = 50$ ;  $n = 50$ ;  $\sigma = \sqrt{\frac{\Delta M L}{\Delta N^2 \pi}}$  The program used to create the graph is mygabor.m and it is attached in the appendix.

Let  $\psi(t)$  be a GEF. The GEF in the frequency domain is given by the Fourier transform

$$\Psi(f) = \int_{-\infty}^{\infty} \psi(t) e^{-j2\pi ft} dt$$

These functions have a special property that relates to Heisenberg's uncertainty principle. The product of the standard deviation in time  $\Delta t$  and variance in frequency

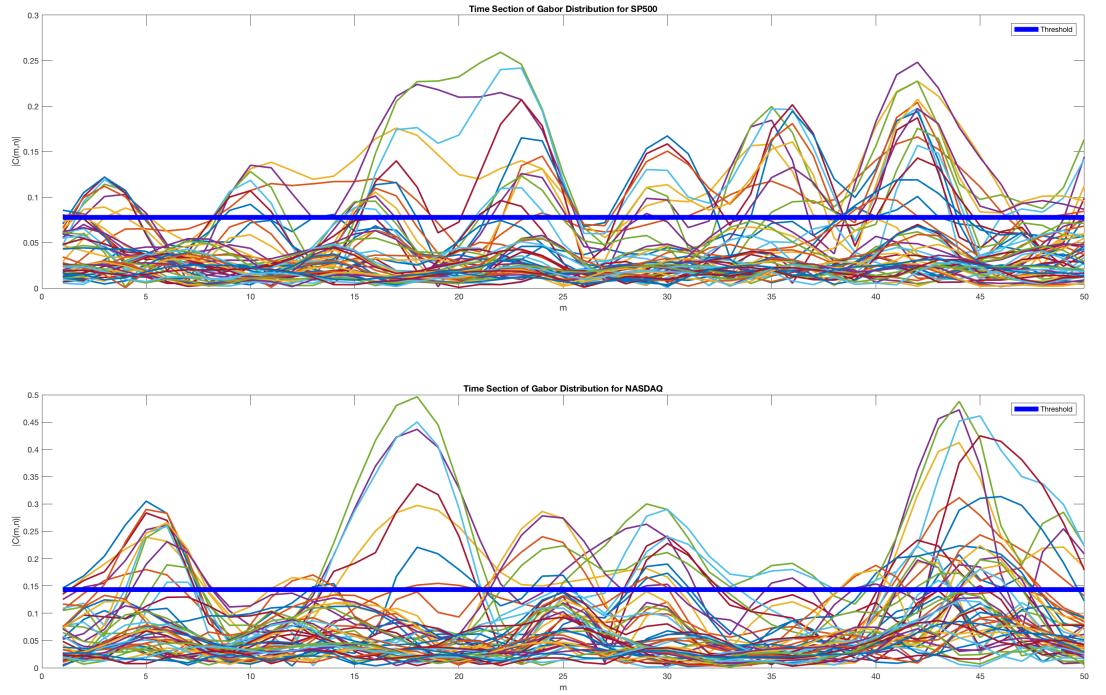


Figure 2.10: Fig a is the time section of Gabor distribution of HP Filter cycles  $\log(\text{SP500})$  and Fig b is the time section of Gabor distribution of for HP Filter cycles  $\log(\text{NASDAQ})$ .  $\Delta M = 8$ ;  $\Delta N = 4$ ;  $m = 50$ ;  $n = 50$ ;  $\sigma = \sqrt{\frac{\Delta M L}{\Delta N 2\pi}}$ . Threshold is calculated as  $\max(|c(m, n)|) - \min(|c(m, n)|))/2$ . The program used to create the graph is mygabor.m and it is attached in the appendix.

$\Delta f$  is always greater than or equal to a certain quantity.

$$\Delta f \Delta t \geq \frac{1}{4\pi} \quad (2.27)$$

The time standard deviation or effective duration and frequency variance or effective frequency width can be calculated by the root mean square (RMS) deviation of the signal from the mean. The effective duration ( $\Delta t$ ) and effective frequency width ( $\Delta f$ )

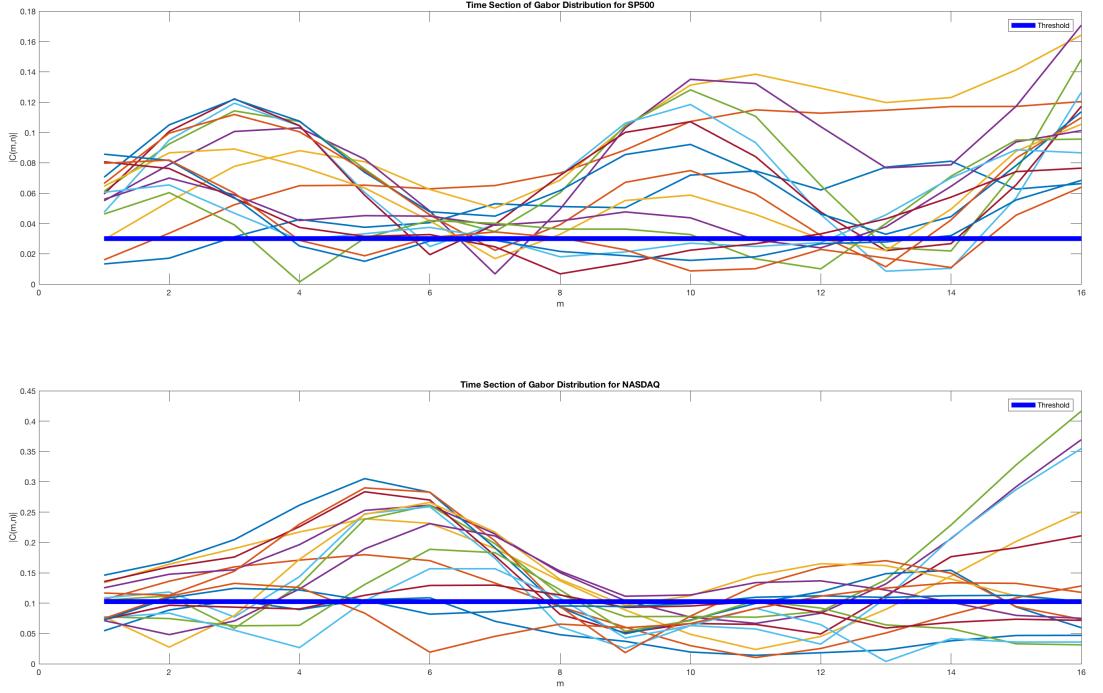


Figure 2.11: Fig a is the time section of Gabor distribution of HP Filter cycles log(SP500) and Fig b is the time section of Gabor distribution of for HP Filter cycles log(NASDAQ).  $\Delta M = 8$ ;  $\Delta N = 4$ ;  $m = 16$ ;  $n = 16$ ;  $\sigma = \sqrt{\frac{\Delta M L}{\Delta N 2\pi}}$ . Threshold is calculated as  $\max(|c(m, n)|) - \min(|c(m, n)|))/2$ . The program used to create the graph is myfiltgabor.m and it is attached in the appendix.

are given by

$$\Delta t = \sqrt{\frac{\int_{-\infty}^{\infty} \psi(t)(\mu_t - t)^2 \psi^*(t) dt}{\int_{-\infty}^{\infty} \psi(t) \psi^*(t) dt}}; \Delta f = \sqrt{\frac{\int_{-\infty}^{\infty} \Psi(f)(\mu_f - f)^2 \Psi^*(f) df}{\int_{-\infty}^{\infty} \Psi(f) \Psi^*(f) df}}$$

where  $\mu_t$  and  $\mu_f$  are mean time and mean frequency and it is given by

$$\mu_t = \frac{\int_{-\infty}^{\infty} \psi(t) t \psi^*(t) dt}{\int_{-\infty}^{\infty} \psi(t) \psi^*(t) dt}; \mu_f = \frac{\int_{-\infty}^{\infty} \Psi(f) f \Psi^*(f) df}{\int_{-\infty}^{\infty} \Psi(f) \Psi^*(f) df}$$

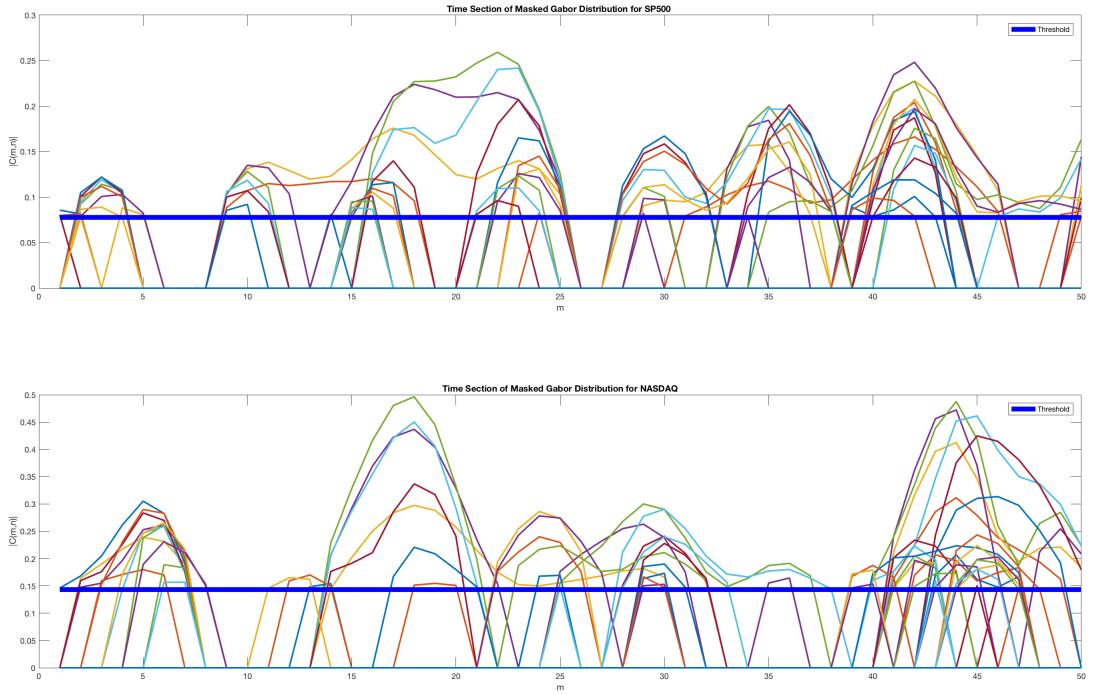


Figure 2.12: Fig a is the time section of Gabor distribution of HP Filter cycles  $\log(\text{SP500})$  and Fig b is the time section of Gabor distribution of for HP Filter cycles  $\log(\text{NASDAQ})$ .  $\Delta M = 8$ ;  $\Delta N = 4$ ;  $m = 50$ ;  $n = 50$ ;  $\sigma = \sqrt{\frac{\Delta M L}{\Delta N^2 \pi}}$ . Mask Operator  $M$  created based on threshold. The program used to create the graph is mygabor.m and it is attached in the appendix.

### 2.3.3 Mask Operator

$$\Delta t \Delta f = \sqrt{\frac{\int_{-\infty}^{\infty} \psi(t)(\mu_t - t)^2 \psi^*(t) dt}{\int_{-\infty}^{\infty} \psi(t)\psi^*(t) dt}} \frac{\int_{-\infty}^{\infty} \Psi(f)(\mu_f - f)^2 \Psi^*(f) df}{\int_{-\infty}^{\infty} \Psi(f)\Psi^*(f) df} \geq \frac{1}{4\pi}$$

There are three possibilities for above equation is given below in table 2.1 .

Analogous to 1D uncertainty principle, there are two 2D uncertainty principles constraining the effective width ( $\Delta x$ ) and the effective length ( $\Delta y$ ) of a signal  $f(x, y)$  and the effective width ( $\Delta u$ ) and the effective length ( $\Delta v$ ) of its 2D Fourier transform

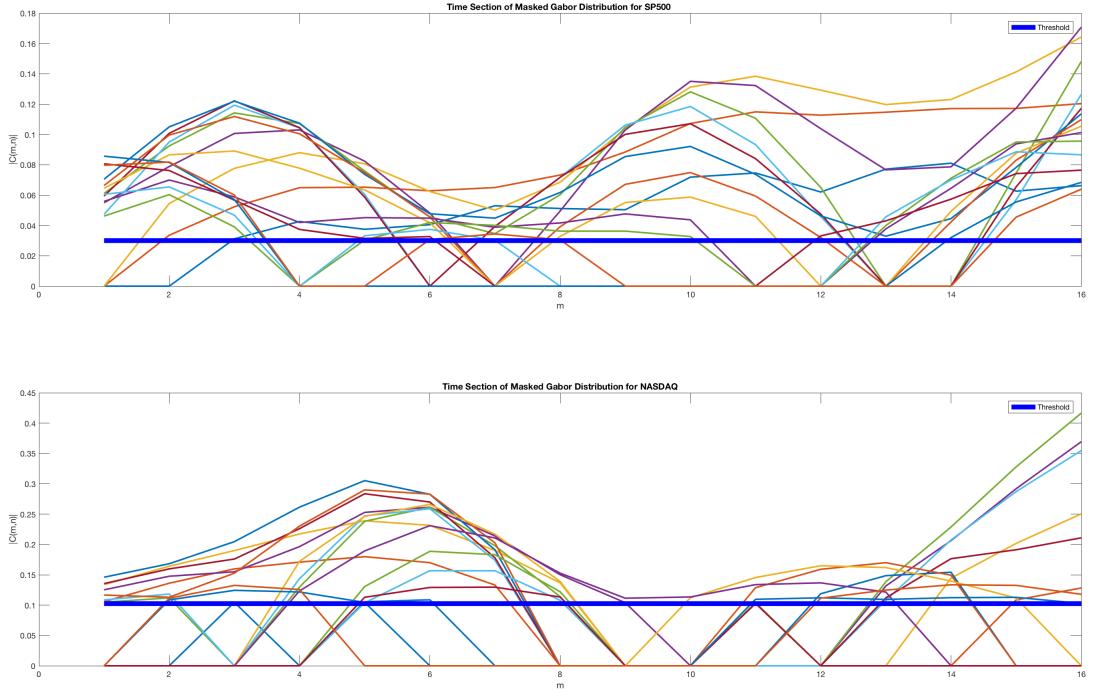


Figure 2.13: Fig a is the time section of Gabor distribution of HP Filter cycles log(SP500) and Fig b is the time section of Gabor distribution of for HP Filter cycles log(NASDAQ).  $\Delta M = 8$ ;  $\Delta N = 4$ ;  $m = 16$ ;  $n = 16$ ;  $\sigma = \sqrt{\frac{\Delta M L}{\Delta N^2 \pi}}$ . Mask Operator  $M$  created based on threshold. The program used to create the graph is mygabor.m and it is attached in the appendix.

$\Delta t \Delta f$	Sampling	Remarks
$= \frac{1}{4\pi}$	Critical	Special functions called GEF
$> \frac{1}{4\pi}$	Over	—
$< \frac{1}{4\pi}$	Under	This case is not possible

Table 2.1: Possible values for  $\Delta t \Delta f$  and special case for Gabor function

$F(u, v)$ . 2D Gabor transformation has wide application in image processing domain and the 2D analysis is out of scope for this project.

$$\Delta x \Delta u = \sqrt{\frac{\int_{-\infty}^{\infty} \psi(x, y)(\mu_x - x)^2 \psi^*(x, y) dx dy}{\int_{-\infty}^{\infty} \psi(x, y) \psi^*(x, y) dx dy} \frac{\int_{-\infty}^{\infty} \Psi(u, v)(\mu_u - u)^2 \Psi^*(u, v) du dv}{\int_{-\infty}^{\infty} \Psi(u, v) \Psi^*(u, v) du dv}} \geq \frac{1}{4\pi}$$

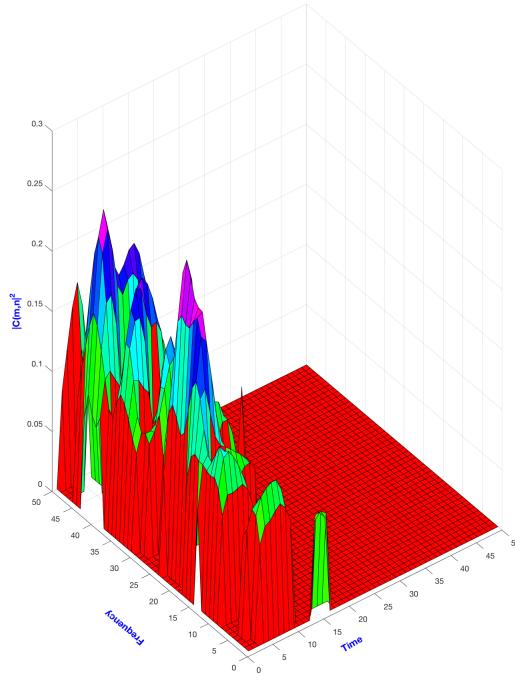


Fig a: Filtered Gabor Coefficient for log(sp500)

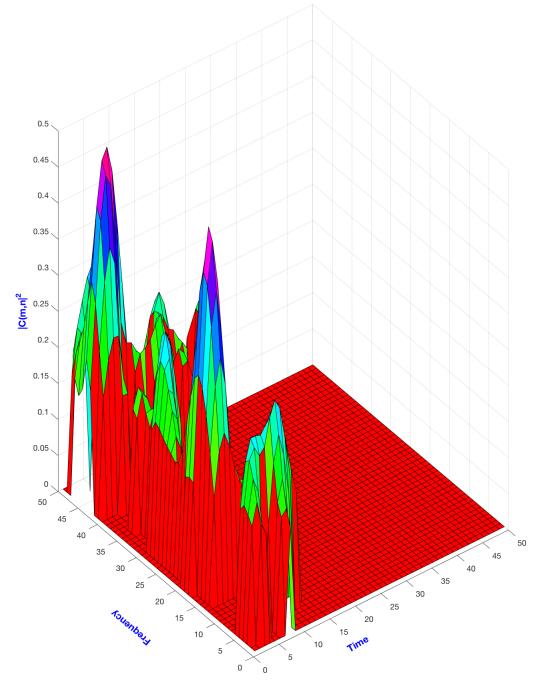


Fig a: Filtered Gabor Coefficient for log(nasdaq)

Figure 2.14: Fig a is the Filtered Gabor Coefficient for HP Filter cycles  $\log(\text{SP500})$  and Fig b is the Filtered Gabor Coefficient for HP Filter cycles  $\log(\text{NASDAQ})$ .  $\Delta M = 8; \Delta N = 4; m = 50; n = 50; \sigma = \sqrt{\frac{\Delta M L}{\Delta N^2 \pi}}$ . The mask operator is created based on the threshold of a peak distribution. The program used to create the graph is myfiltgabor.m and it is attached in the appendix.

$$\Delta y \Delta v = \sqrt{\frac{\int_{-\infty}^{\infty} \psi(x, y)(\mu_y - y)^2 \psi^*(x, y) dx dy}{\int_{-\infty}^{\infty} \psi(x, y) \psi^*(x, y) dx dy} \frac{\int_{-\infty}^{\infty} \Psi(u, v)(\mu_v - v)^2 \Psi^*(u, v) du dv}{\int_{-\infty}^{\infty} \Psi(u, v) \Psi^*(u, v) du dv}} \geq \frac{1}{4\pi}$$

$$\Delta x \Delta u \Delta y \Delta v \geq \frac{1}{16\pi^2}$$

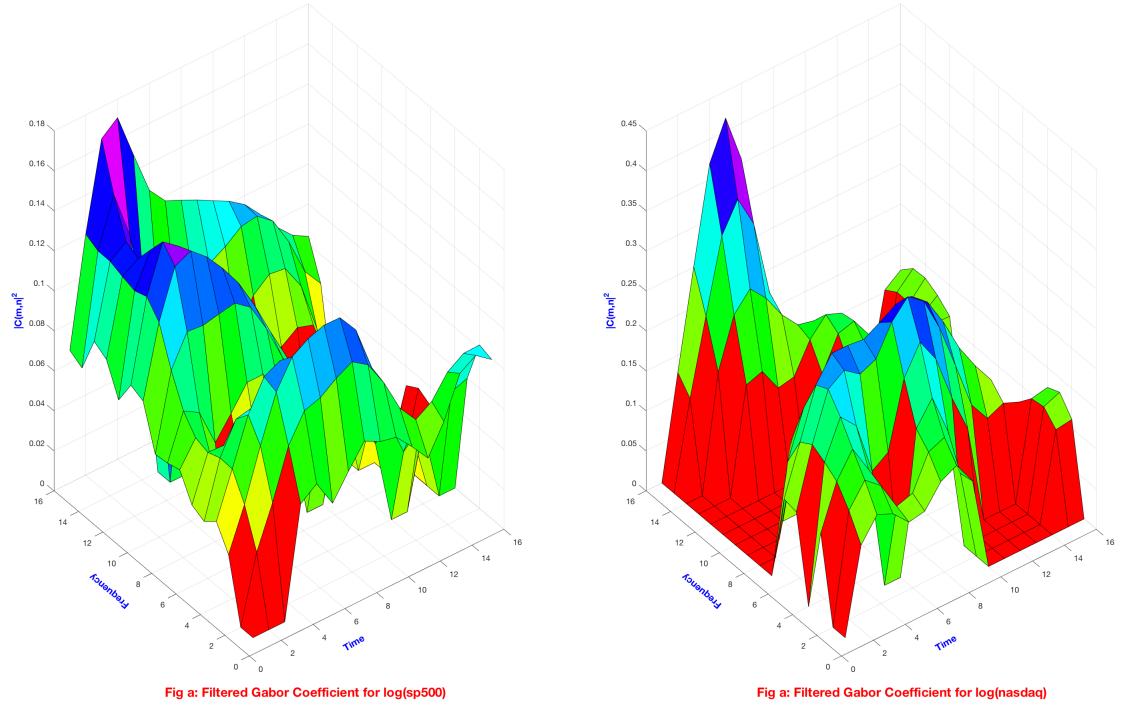


Figure 2.15: Fig a is the Filtered Gabor Coefficient for HP Filter cycles  $\log(\text{SP500})$  and Fig b is the Filtered Gabor Coefficient for HP Filter cycles  $\log(\text{NASDAQ})$ .  $\Delta M = 8$ ;  $\Delta N = 4$ ;  $m = 16$ ;  $n = 16$ ;  $\sigma = \sqrt{\frac{\Delta M L}{\Delta N^2 \pi}}$ . The mask operator is created based on the threshold of a peak distribution. The program used to create the graph is myfiltgabor.m and it is attached in the appendix.

## CHAPTER III

# Wigner Distribution

### 3.1 Wigner Distribution

The Wigner Distribution (WD) introduced by Wigner (Wigner 1932) as a phase representation in Quantum Mechanics gives a simultaneous representation of a signal in space and spatial-frequency variables. The WD belongs to a large class of bilinear distributions known as the Cohen's class, in which each member can be obtained by choosing a different kernel in the generalized bilinear distribution definition.

Let's suppose  $f(t)$  is a continuous, integrable and complex function. The symmetric definition of the Wigner distribution  $W_f(t, \omega)$  is given by

$$W_f(t, \omega) = \int_{-\infty}^{\infty} f(t + \frac{\tau}{2}) f^*(t - \frac{\tau}{2}) e^{-j\omega\tau} d\tau \quad (3.1)$$

where  $t$  and  $\tau$  are spatial variables,  $\omega$  is the spatial frequency variable and  $f^*$  is the complex conjugate of  $f$ . The product function  $r_f(t, \tau)$  is given by

$$r_f(t, \omega) = f(t + \frac{\tau}{2}) f^*(t - \frac{\tau}{2}) \quad (3.2)$$

The auto-Wigner distribution gives a generalized auto convolution at non-zero frequency. From  $W_f(t, \omega)$ , it can be observed that the Wigner Distribution is the Fourier transformation, for a given point  $\tau$ , of the product  $ff^*$ . It may also be obtained from

the Fourier transform  $F$  of  $f$  by

$$W_F(\omega, t) = \int_{-\infty}^{\infty} F(\omega + \frac{\phi}{2}) F^*(\omega - \frac{\phi}{2}) e^{j\phi t} d\phi \quad (3.3)$$

where  $F^*$  is the complex conjugate of  $F$ . Connecting  $W_f(t, \omega)$  and  $W_F(\omega, t)$  the following relation is observed,

$$W_f(t, \omega) = W_F(\omega, t) \quad (3.4)$$

which shows the symmetry between the two conjugate domains. Among various properties of the WD, the interference and inversion properties are most relevant for the current study. The WD computation introduces spurious "auto terms" due to its intrinsic bi-linearity. The WD of the sum of two signals  $f(t) + f'(t)$  is given by

$$W_{f+f'}(t, \omega) = W_f(t, \omega) + W_{f'}(t, \omega) + 2Re[W_{f,f'}(t, \omega)] \quad (3.5)$$

Inversion of the original signal  $f(t)$  from the Wigner Distribution is given by

$$f(t + \frac{\tau}{2}) f^*(t - \frac{\tau}{2}) = \int_{-\infty}^{\infty} W_f(t, \omega) e^{j\omega\tau} d\omega \quad (3.6)$$

Let  $t = \frac{\tau}{2}$  and then setting  $\tau = t$ , we have

$$f(t) f^*(0) = \int_{-\infty}^{\infty} W_f(\frac{t}{2}, \omega) e^{j\omega t} d\omega \quad (3.7)$$

$$f(t) = \frac{1}{f^*(0)} \int_{-\infty}^{\infty} W_f(\frac{t}{2}, \omega) e^{j\omega t} d\omega \quad (3.8)$$

## 3.2 Wigner-Ville Distribution

The Wigner-Ville distribution (WVD) is defined for a signal  $f(t)$  as follows:

$$W_f(t, \omega) = \int_{-\infty}^{\infty} z(t + \frac{\tau}{2}) z^*(t - \frac{\tau}{2}) e^{-j\omega\tau} d\tau \quad (3.9)$$

where  $z(t)$  is the analytical associative of  $f(t)$ . The Wigner distribution is simply defined when the real signal  $f(t)$  is used instead of the analytic one  $z(t)$ . A signal  $f(t)$  is said to be analytical if and only if

$$F(\omega) = 0 \quad (3.10)$$

for all  $\omega < 0$ , where  $F(\omega)$  is Fourier transform of  $f(t)$ . In other words, an analytical signal contains no negative frequencies. It may have a spectral component at zero frequency.

## 3.3 Discrete Wigner Distribution

One of the main disadvantages of the discrete definition is that not all the properties of the continuous WD are preserved by discretization due to aliasing effects. Several alternative definitions have been proposed in the literature in order to overcome this problem (Chan 1982), (Claasen 1983), (Brenner 1983), (Day 1983), (Peyrin 1986). The discrete WD of a sampled function  $f(t)$  is defined by

$$W_f(n, m) = \sum_{k=-\frac{N}{2}}^{\frac{N}{2}-1} f(n+k) f^*(n-k) e^{-j(\frac{2\pi mk}{N+1})} \quad (3.11)$$

where  $n$  and  $m$  are the spatial and frequency variables respectively.

The discrete WD definition given above retains the basic properties of the continuous WD, however, the main differences comes from the inversion property. The

inversion property is the ability to extract the time domain signal from the distribution up to a constant phase factor. The distribution that satisfies this property is given as

$$f(n+k)f^*(n-k) = \sum_{m=-\frac{N}{2}}^{\frac{N}{2}-1} W_f(n, m)e^{j(\frac{2\pi mk}{N+1})} \quad (3.12)$$

In the case of discrete signals, inserting  $k = n$  in the above equation allows one to write

$$f(2n)f^*(0) = \sum_{m=-\frac{N}{2}}^{\frac{N}{2}-1} W_f(n, m)e^{j(\frac{2\pi mn}{N+1})} \quad (3.13)$$

From the above equation only the even samples can be recovered. Inserting  $k-1 = n$  in the discrete WD of sampled function  $f(t)$ ,

$$f(2n+1)f^*(1) = \sum_{m=-\frac{N}{2}}^{\frac{N}{2}-1} W_f(n, m)e^{j(\frac{2\pi m(n+1)}{N+1})} \quad (3.14)$$

leads to the recovering of the odd samples.

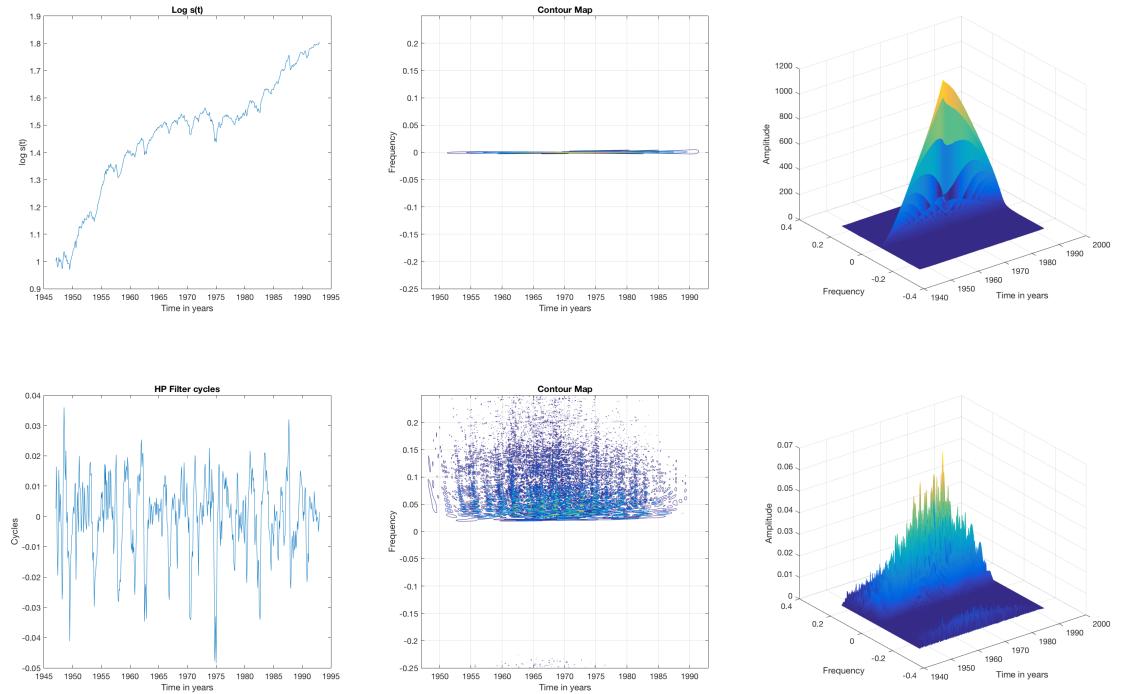


Figure 3.1: Wigner Distribution transform for  $\log(s)$ ; where  $s$  denotes the sp500. Top row is the  $\log(s)$  and its Wigner Distribution transform presentation in the contour and three dimensional mesh. Bottom row is the HP filter cycle of  $\log(s)$  with  $\lambda = 1600$  and its Wigner Distribution transform. The graph was created using Matlab code mywvd.m and it is attached in the Appendix B

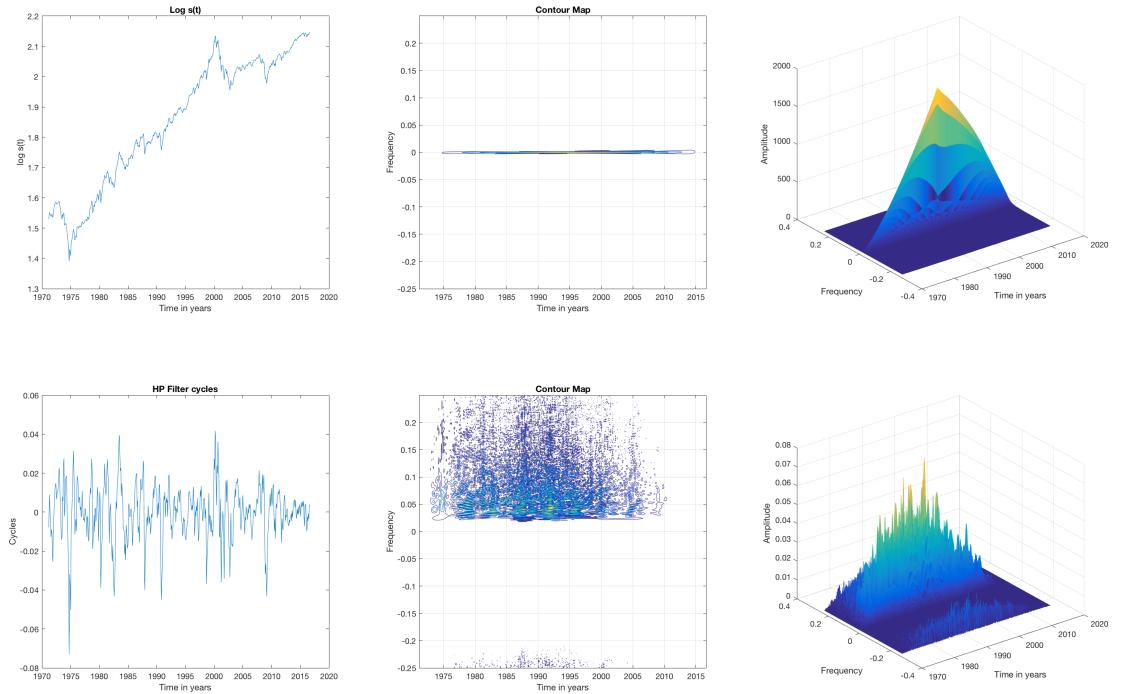


Figure 3.2: Wigner Distribution transform for  $\log(s)$ ; where  $s$  denotes the NASDAQ index. Top row is the  $\log(s)$ , Wigner Distribution transform presentation in the contour and three dimensional mesh. Bottom row is the HP filter cycle of  $\log(s)$  with  $\lambda = 1600$  and its Wigner Distribution transform. The graph was created using Matlab code mywvd.m and it is attached in the Appendix B

## CHAPTER IV

# Time Frequency Distribution Series

### 4.1 Time Frequency Distribution Series

The main drawback of the Wigner Ville distribution is cross term interference and the cross term oscillates and is localized.

Time Frequency Distribution Series (TFDS) was introduced by Chen and Qian [?] as the decomposition of the Wigner Ville distribution via the orthogonal like Gabor expansion. Let me walk through each step to attain the time frequency distribution series.

Let  $g(t)$  be a normalized Gaussian function which is defined as follows.

$$g(t) = \frac{1}{(\pi\sigma^2)^{0.25}} e^{-\frac{t^2}{2\sigma^2}} \quad (4.1)$$

The corresponding Wigner Ville Distribution (WVD) is given below.

$$WVD_g(t, \omega) = 2e^{-(\frac{t^2}{\sigma^2} + \sigma^2\omega^2)} \quad (4.2)$$

The  $WVD_g(t, \omega)$  is centered at origin and it decays exponentially in both the time and frequency domains. The contour plot of the  $WVD_g(t, \omega)$  consists of concentric ellipses and it is given below.

The WVD is time and frequency-shift invariant. Let

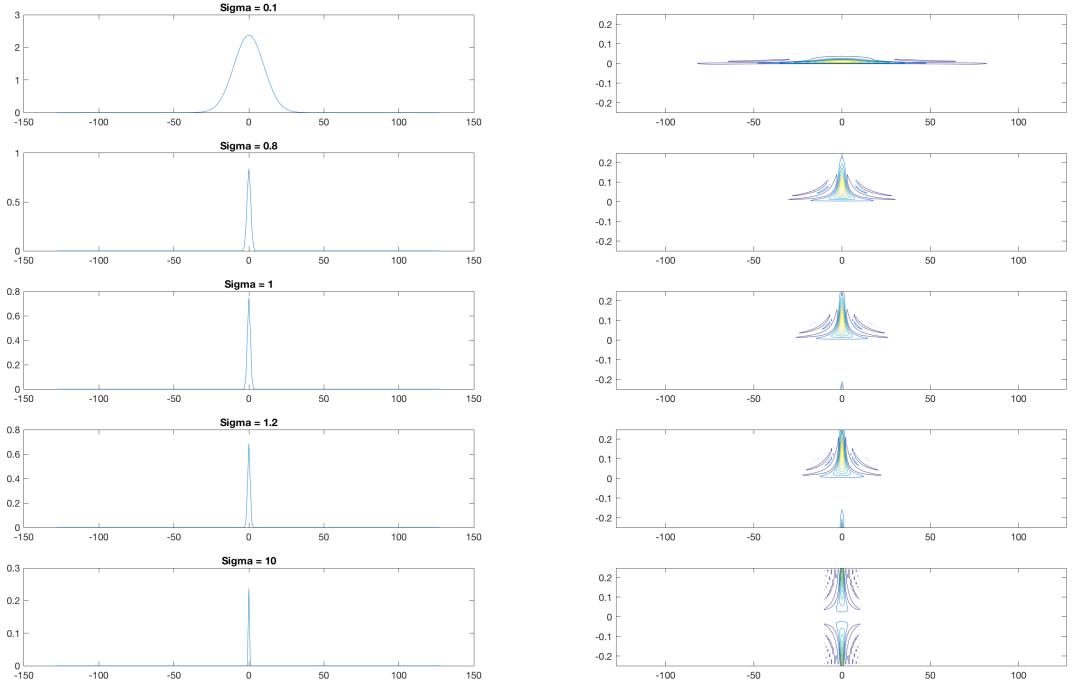


Figure 4.1: Fig in the left hand side represents the Gaussian function  $g(t)$  as defined above for various  $\sigma$  values. Fig in the right hand side represents the Wigner Ville Distribution  $WVD_g$  as defined above. The graph is created using the mywvdgauss.m program attached in the Appendix.  $WVD$  values are created by using the HOSA (Higher Order Spectral Analysis) Matlab toolbox.

$$h(t) = g(t - mT)e^{-jn\Omega t} \quad (4.3)$$

where  $\Omega$  and  $T$  are the frequency and time sampling steps respectively. The WVD of  $h(t)$  is given by,

$$WVD_h(t, \omega) = 2e^{-[\frac{(t-mT)^2}{\sigma^2} + [\sigma(\omega - n\Omega)]^2]} \quad (4.4)$$

$$WVD_h(t, \omega) = WVD_g(t - mT, \omega - n\Omega) \quad (4.5)$$

Let  $s(t) = h(t) + h'(t)$ , then  $WVD_s(t, \omega)$  is given as

$$WVD_s(t, \omega) = WVD_h(t, \omega) + WVD_{h'}(t, \omega) + 2Re[WVD_{h,h'}(t, \omega)] \quad (4.6)$$

where  $WVD_{h,h'}(t, \omega)$  is given by

$$WVD_{h,h'}(t, \omega) = e^{j\omega_d t_\mu} H(t - t_\mu, \omega - \omega_\mu) \quad (4.7)$$

where

$$H(t, \omega) = 2e^{-[\frac{t^2}{\sigma^2} + (\sigma\omega)^2]} e^{-j[t_d\omega - \omega_d t]} \quad (4.8)$$

$$\begin{aligned} t_\mu &= \frac{m + m'}{2} T \\ \omega_\mu &= \frac{n + n'}{2} \Omega \\ t_d &= (m - m')T \\ \omega_d &= (n - n')\Omega \end{aligned} \quad (4.9)$$

The  $WVD_{h,h'}(t, \omega)$  has the same envelope as the  $WVD_g(t, \omega)$  but is oscillating with frequency  $\omega_d$  in the time domain and  $t_d$  in the frequency domain. The location of  $WVD_{h,h'}(t - t_\mu, \omega - \omega_\mu)$  is halfway between  $h$  and  $h'$ . The  $2Re[WVD_{h,h'}(t, \omega)]$  is the cross term. When a signal  $s(t)$  can be decomposed as a linear combination of some elementary functions  $h(t)$  then the cross-terms can be controlled.

Recall from the previous chapter on the Gabor Expansion, for a given signal  $s(t)$ , the orthogonal-like Gabor expansion is defined as follows.

$$s(t) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C_{m,n} g(t - mT) e^{jn\Omega t} \quad (4.10)$$

The Gabor coefficients  $C_{m,n}$  are determined by

$$C_{m,n} = \int s(t)\gamma_{m,n}^*(t)dt = \int s(t)\gamma^*(t - mT)e^{-jn\Omega t}dt = STFT(mT, n\Omega) \quad (4.11)$$

Using the Wigner-Ville distribution of  $s(t)$  from the above equation yields,

$$WVD_s(t, \omega) = \sum_{m,n} \sum_{m',n'} C_{m,n} C_{m',n'} WVD_{h,h'}(t, \omega) \quad (4.12)$$

where

$$WVD_{h,h'}(t, \omega) = e^{j\omega_d t_\mu} 2e^{-[\frac{(t-t_m u)^2}{\sigma^2} + (\sigma(\omega - \omega_\mu))^2]} e^{-j[t_d \omega - \omega_d t]} \quad (4.13)$$

$$WVD_s(t, \omega) = \sum_{m,n} \sum_{m',n'} C_{m,n} C_{m',n'} e^{j\omega_d t_\mu} 2e^{-[\frac{t^2}{\sigma^2} + (\sigma\omega)^2]} e^{-j[t_d \omega - \omega_d t]} \quad (4.14)$$

where  $t_d$  and  $\omega_d$  reflect the degree of oscillation.

When  $m = m'$  and  $n = n'$ ,

$$C_{m,n} C_{m',n'}^* WVD_{h,h'}(t, \omega) = 2|C_{m,n}|^2 e^{-[\frac{(t-mT)^2}{\sigma^2} + \sigma^2(\omega - n\Omega)^2]} \quad (4.15)$$

When  $m \neq m'$  or  $n \neq n'$

$$C_{m,n} C_{m',n'}^* WVD_{h,h'}(t, \omega) = C_{m,n} C_{m',n'}^* e^{j\omega_d t_\mu} H(t - t_\mu, \omega - \omega_\mu) \quad (4.16)$$

Based on the decomposition of the Wigner-Ville distribution defined above, the Time Frequency Distribution Series (TFDS) is defined as follows.

$$TFDS_D(t, \omega) = \sum_{d=0}^D P_d(t, \omega) \quad (4.17)$$

Here  $P_d(t, \omega)$  is the sum of a sequence of  $WVD_{h,h'}(t, \omega)$  which have a similar

contribution to the useful properties and similar influence to the cross terms in which  
 $|m - m'| + |n - n'| = d$

$$P_d(t, \omega) = \sum_{|m-m'|+|n-n'|=d} C_{m,n} C_{m',n'}^* WVD_{h,h'}(t, \omega) \quad (4.18)$$

Substituting the value of  $WVD_{h,h'}(t, \omega)$  in the above equation, we get

$$P_d(t, \omega) = \sum_{|m-m'|+|n-n'|=d} C_{m,n} C_{m',n'}^* e^{j\omega_d t_\mu} 2e^{-[\frac{t^2}{\sigma^2} + (\sigma\omega)^2]} e^{-j[t_d\omega - \omega_d t]} \quad (4.19)$$

Substituting the value of  $t_d$ ,  $\omega_d$ ,  $t_\mu$  and  $\omega_\mu$ , we get

$$P_d(t, \omega) = \sum_{|m-m'|+|n-n'|=d} C_{m,n} C_{m',n'}^* e^{j\frac{n+n'}{2}\Omega(m-m')T} 2e^{-[\frac{t^2}{\sigma^2} + (\sigma\omega)^2]} e^{-j[(m-m')T\omega - (n-n')\Omega t]} \quad (4.20)$$

Both the SP500 and NASDAQ indices are discrete signals used for analysis. The  $P_d$  is further simplified for programming convenience.

The discrete Time Frequency Distribution Series is defined as follows.

$$DTFDS_D[i, k] = TFDSD_d(t, \omega)|_{t=i\Delta t, \omega=\frac{2\pi k}{L\Delta t}} \quad (4.21)$$

for  $-\frac{L}{2} \leq k < \frac{L}{2}$  where  $\frac{1}{\Delta t}$  denotes the sampling frequency.  $L$  denotes the length of the signal. The discrete time frequency distribution series can be summarized as

$$TFDS_d(i, k) = \sum_{d=0}^D P_d[i, k] \quad (4.22)$$

where

$$P_d[i, k] = \sum_{|m-m'|+|n-n'|=d} C_{m,n} C_{m'n'} WVD_{h,h'}[i, k] \quad (4.23)$$

The  $TFDS_D[i, k]$  is the sum of all  $WVD_{h,h'}[i, k]$  in which the distance of the corresponding Gabor elementary functions  $h_{m,n}[i]$  and  $h_{m',n'}[i]$  is less than or equal to  $d$ .  $WVD[i, k]$  is defined as a sampled Wigner-Ville distribution.

$$WVD_s[i, k] = WVD_s(t, \omega)|_{t=i\Delta t, \omega=\frac{2\pi k}{L\Delta t}} \quad (4.24)$$

where  $\Delta t$  denotes the sampling interval. For the Gaussian functions, WVD is obtained by sampling the formula.

$$WVD_{h,h'}[i, k] = 2e^{-\sigma(i-\frac{m+m'}{2}\Delta M)^2 - \frac{1}{\sigma}(k-\frac{n+n'}{2}\Delta N)^2} e^{j\frac{2\pi}{L}[(m-m')\Delta Mk + (n-n')\Delta Ni - \frac{n+n'}{2}\Delta N(m-m')\Delta M]} \quad (4.25)$$

We assume  $\Delta t = 1$ .  $WVD_{h,h'}[i, k]$  in ?? is completely determined by the parameters of the Gabor expansion, such as  $\Delta M, \Delta N, L$  and  $\sigma$  which are independent of the analyzed signal. Therefore, once  $\Delta M, \Delta N, L$  and  $\sigma$  are determined,  $WVD_{h,h'}[i, k]$  can be precomputed and saved in a table.

## CHAPTER V

### Color Chaos Model

#### 5.1 Color Chaos

##### 5.1.1 Chaos models

Chen claimed in [? ], that correlation analysis and spectral analysis are complementary tools in the stationary time-series analysis. Random walk (white noise) has a zero correlation and a flat spectrum while sine wave has an infinite correlation and a sharp spectrum with zero width. Econometric models, such as ARCH and GARCH models with changing means and variance, are parametric models in the non-stationary stochastic approach. A generalized spectral approach is more useful in the studies of deterministic chaos. Based on the detail study of the paper [? ], **Color Chaos** is defined as a time frequency representation as a nonparametric approach for generalized spectral analysis for evolutionary time series. In color chaos, the HP filter is applied for trend-cycle decomposition and time-variant filters in Gabor space for pattern recognition.

Discrete-time white chaos generalized by nonlinear difference equations is tractable analytically and from the needs of empirical analysis, the continuous-time color chaos generated by nonlinear differential equations is more capable of describing business cycles than white chaos, since fluctuations and recurrent patterns can be characterized

by nonlinear oscillations with irregular amplitude and a narrow frequency band in the spectrum. Chen claimed in [? ], the newly decoded deterministic signals from persistent business cycles reveal new sources of market uncertainty, such as changing growth-trend and shifting business cycles.

### 5.1.2 Role of time scale & reference trends in representation of business cycles

**Problem:** A distinctive problem in economic analysis is how to deal with growing trends in an aggregate economic time series. Both level and rate information are important when correlations are not short during business cycles.

*Time Scale:* Chaos theory in nonlinear dynamics emphasizes the role of history, because a nonlinear deterministic system is sensitive to its initial condition. The martingale theory of the stock market ignores the path-dependent information in the stock market. The challenges faced are:

1. Choosing an appropriate time sampling rate is often ignored in econometric analysis. Chaotic cycles in continuous time may look like noise if the sampling time internal is not small compared to its fundamental period of cycle. For example, annual economic data are not capable of revealing the frequency pattern of business cycle.
2. Numerically, a large time unit such as the annual time series can easily obscure a cyclic pattern in the correlation analysis of business cycles.

*Reference Trend:* How to choose a reference trend or a proper transformation to simplify the empirical pattern of business cycles? The core problem in economic analysis is not noise-smoothing but trend-defining in economic observation and decision making. There are two criteria in choosing the proper mathematical representation.

1. Mathematical reliability

## 2. Empirical verifiability.

Unlike experimental economics, macroeconomic time series are not reproducible in history. Traditional tests in econometric analysis have limited power in studies of an evolutionary economy containing deterministic components. A good fit of past data does not guarantee the ability for better future predictions.

There are two extreme approaches in econometric analysis: the trend-stationary (TS) approach of log-linear detrending (LLD) and the difference-stationary (DS) approach of first differencing (FD). A compromise between these two extremes is the Hodrick and Prescott (HP) filter.

In principle, a choice of observation reference is associated with a theory of economic dynamics. Log-linear detrending implies a constant exponential growth as shown in the figure 1.4.

The FD detrending produces a noisy picture that is predicted by the random-walk model with a constant drift (or the so-called unit-root model in econometric literature). Economically speaking, the FD detrending in econometrics implies that the level information in price indicators can be ignored in economic behavior. This assertion may conflict with many economic practices, since traders constantly watch economic trends, and no one will make an investment decision based only on the current rate of price changes. The error-correction model in econometrics tried to remedy the problem by addition some lagged-level information, such as using a one-year-before level as an approximation of the long-run equilibrium. Then comes the problem of what is the long run equilibrium in the empirical sense.

Option traders based on the Black-Scholes model find that it is extremely difficult to predict the mean, variance, and correlations from historical data.

A proper decomposition of trend and cycles may find an appropriate scheme to weigh the short-term and long-run impacts of economic movements in economic dynamics.

The essence of trend-cycle decomposition is finding an appropriate time window, or equivalently, a proper frequency window, for observing time-dependent movements. Log-linear detrending is a low-pass filter or wave detector, while first differencing is a high-pass filter or noise amplifier.

The main drawback of LLD detrending is its over-dependence on historical boundaries, while the DS series is to erratic from amplifying high-frequency noise.

HP filter has two advantages.

1. It is localized approach in detrending, with the problem of boundary dependence.
2. Frequency response is in the range of business cycles.

### 5.1.3 Dimensionality

### 5.1.4 Correlation Dimension

The correlation dimension provides a tool to quantify self-similarity. A larger correlation dimension corresponds to a larger degree of complexity and less self-similarity. The most frequently used procedure to estimate the correlation dimension was introduced by Grassberger and Procaccia (1983). They defined the correlation sum for a collection of points  $X_i$  ( $i = 1, 2, 3, \dots, N$ ) in some phase space to be the fraction of all possible pairs of points which are closer than a given distance  $\epsilon$  in a particular norm:

To compute the correlation integral  $C(m, \epsilon)$  and the correlation dimension  $D_c$ , first delay-coordinate embed the signal as follow as:

$$X(t) = x(t), x(t + \tau), x(t + 2\tau), \dots, x(t + (m - 1)\tau)$$

where  $\tau$  is the delay time, and  $m$  is the embedding dimension.

$$C(m, \epsilon) = \frac{2}{(N - m)(N - m - 1)} \sum_{i=m}^N \sum_{j=i+1}^N \Theta(\epsilon - \|X_i - X_j\|) \quad (5.1)$$

Where  $\Theta$  is Heaviside step function,  $\Theta(x) = 0$  if  $x \leq 0$  and  $\Theta(x) = 1$  for  $x > 0$ . Thus equation 5.1 counts the pairs  $(X_i, X_j)$  whose distance is smaller than  $\epsilon$ . When  $N \rightarrow \infty$ , for small values of  $\epsilon$ ,  $C$  follows a power law;

$$C(\epsilon) \propto \epsilon^{D_c} \quad (5.2)$$

Where  $D_c$  is correlation dimension. Therefore,  $D_c$  is defined as

$$D_c = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \frac{\partial \log_{10} C(\epsilon)}{\partial \log_{10}(\epsilon)} \quad (5.3)$$

Correlation dimension is estimated by computing the slope of the straight line by using least square fit in a plot of  $\log_{10} C(\epsilon)$  vs  $\log_{10}(\epsilon)$ .

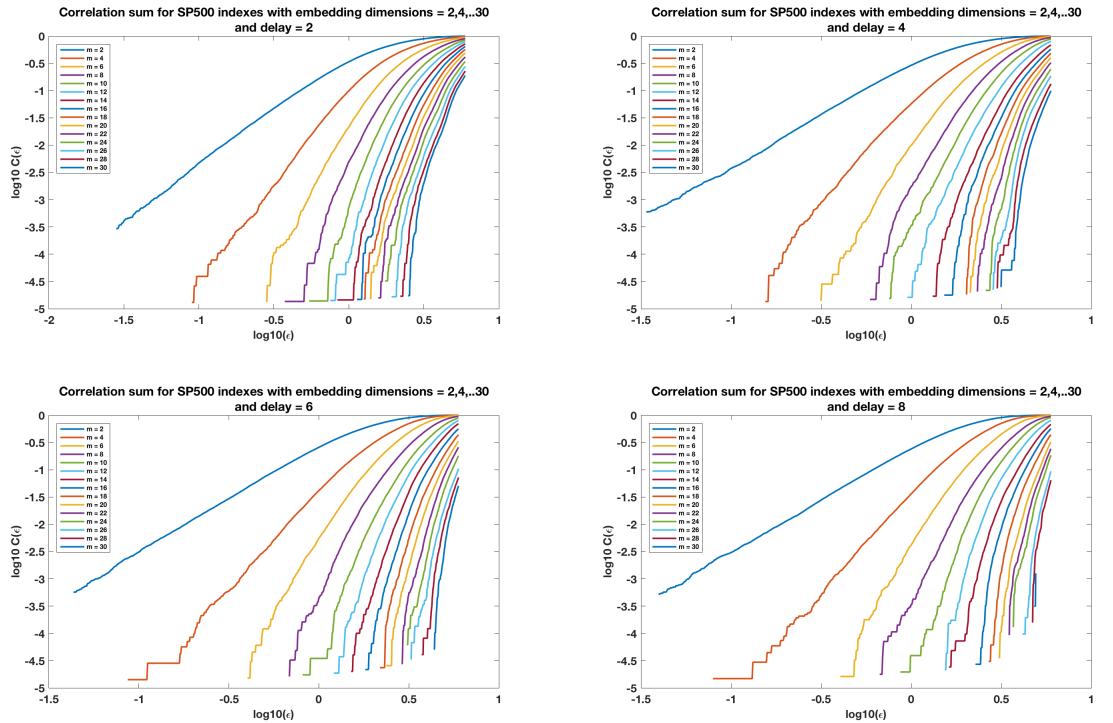


Figure 5.1: Correlation Dimension for NASDAQ. The graph was created by using drawCorrDim.m and it is attached in the Appendix

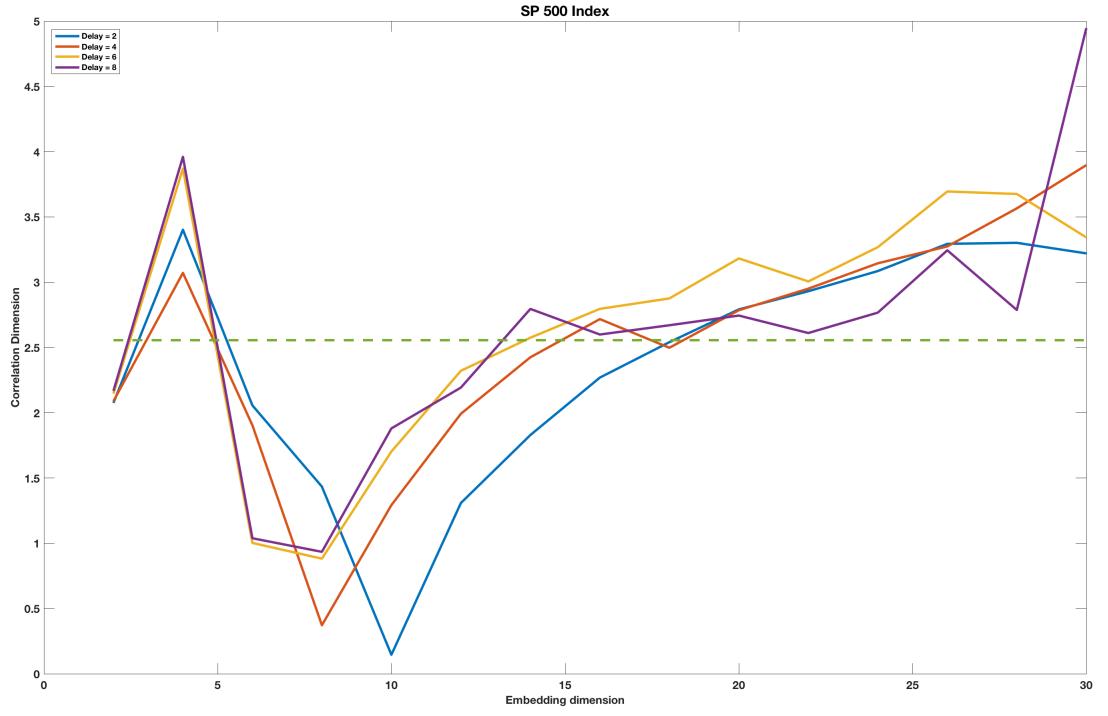


Figure 5.2: Correlation Dimension for NASDAQ. The graph was created by using drawCorrDim.m and it is attached in the Appendix

### 5.1.5 Lyapunov exponent

The Lyapunov exponent is a quantity that describes the rate of separation of two neighbouring trajectories. The divergence of two trajectories at time  $t$  with initial separation  $\delta Z_0$  at  $t = 0$ , is expressed as:

$$|\delta Z(t)| \approx e^{\lambda t} |\delta Z_0| \quad (5.4)$$

where  $\lambda$  is the Lyapunov exponent. Different orientations of initial separation vectors would result in different rate of separation and thus, a whole spectrum of the Lyapunov exponents. The Largest  $\lambda$  is called the Maximum Lyapunov Exponent:

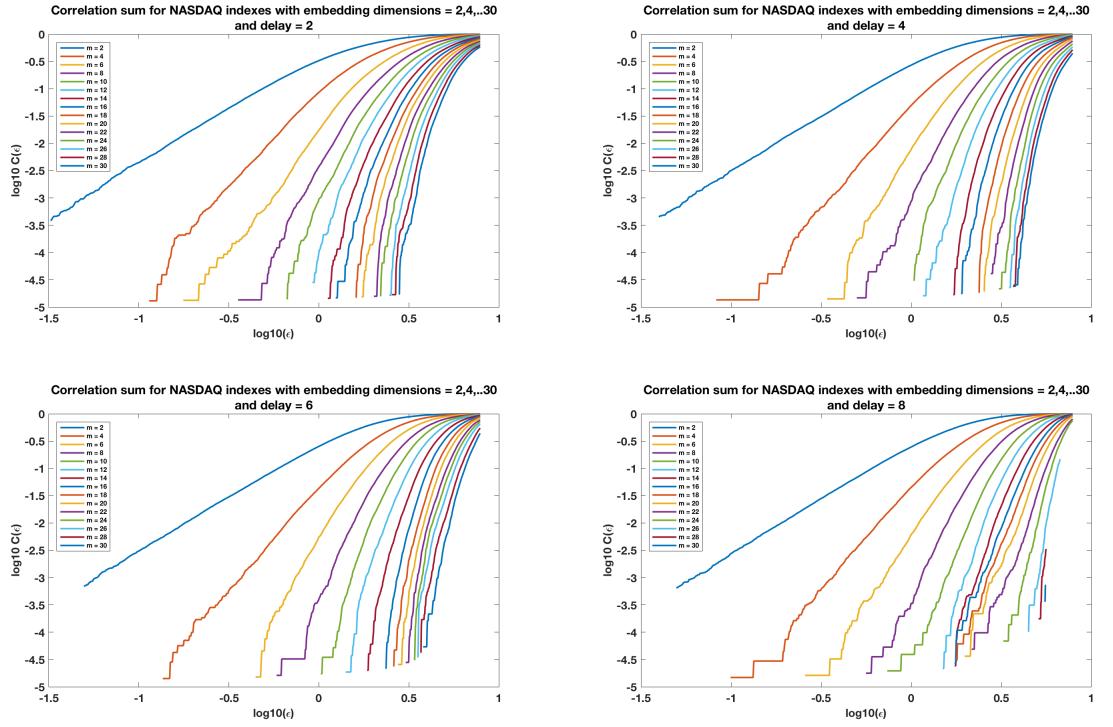


Figure 5.3: Correlation Dimension for NASDAQ. The graph was created by using drawCorrDim.m and it is attached in the Appendix

$$D_c = \lim_{\epsilon \rightarrow 0} \lim_{N \rightarrow \infty} \frac{\partial \log_{10} C(\epsilon)}{\partial \log_{10}(\epsilon)} \quad (5.5)$$

$\lambda_{gt0}$  is usually considered as indicator of chaos,  $\lambda_{lt0}$  is usually considered an indicator of a mean reverting behavior and  $\lambda = 0$  is a characteristic of cyclic behavior.

### 5.1.6 Instantaneous Autocorrelations and instantaneous frequency in Time-frequency representation

The concepts of instantaneous autocorrelation and instantaneous frequency are important in developing generalized spectral analysis. A symmetric window in a localized time interval is introduced in the instantaneous autocorrelation function of the bilinear Wigner distribution (WD); the corresponding time-dependent frequency (or

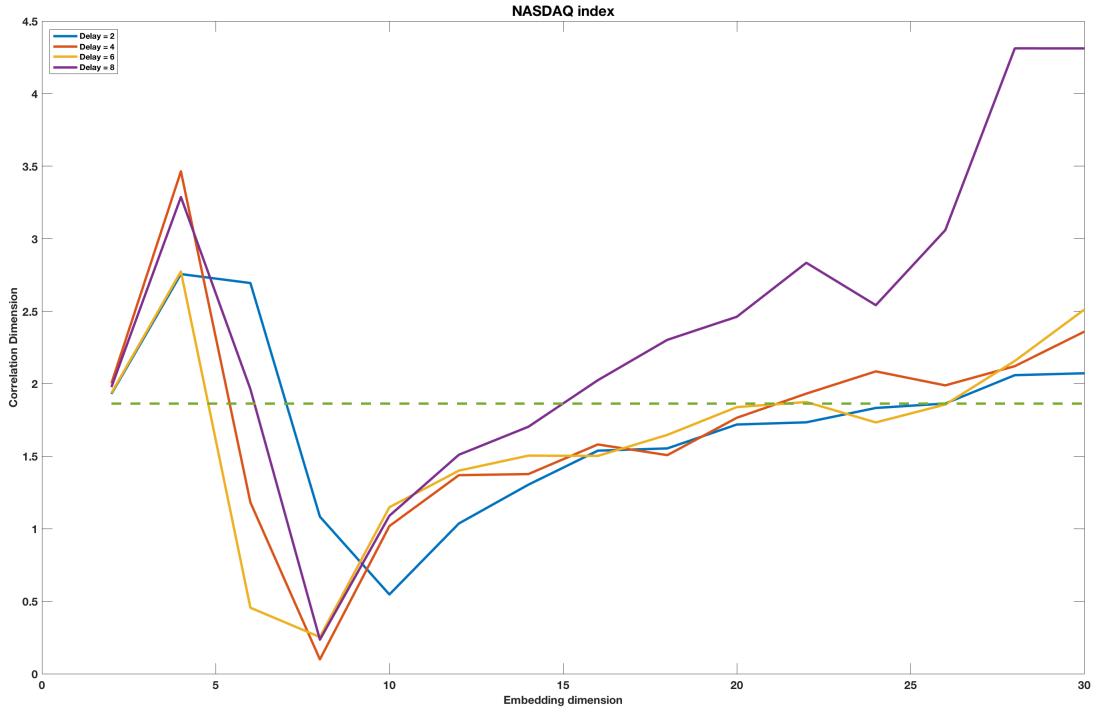


Figure 5.4: Correlation Dimension for NASDAQ. The graph was created by using drawCorrDim.m and it is attached in the Appendix

simply time frequency) can be defined by the Fourier spectrum of its autocorrelations.

In the figure 5.7 the chaos of the SP500 is shown.

In the figure 5.5 the chaos of the SP500 is shown.

### 5.1.7 Time-Variant Filters in the Gabor Space

### 5.1.8 Characteristic Frequency andn Color Chaos

$$X(t) = c_1 + c_2 X(t-1) - c_3 X(t-2) + \mu(t) \quad (5.6)$$

conclusion: We will see that introducing a time-frequency representation and the HP filter does reveal some historical features of business cycles that are not observable through the FD filter.

### NASDAQIn RAW HP Cycles for T=60

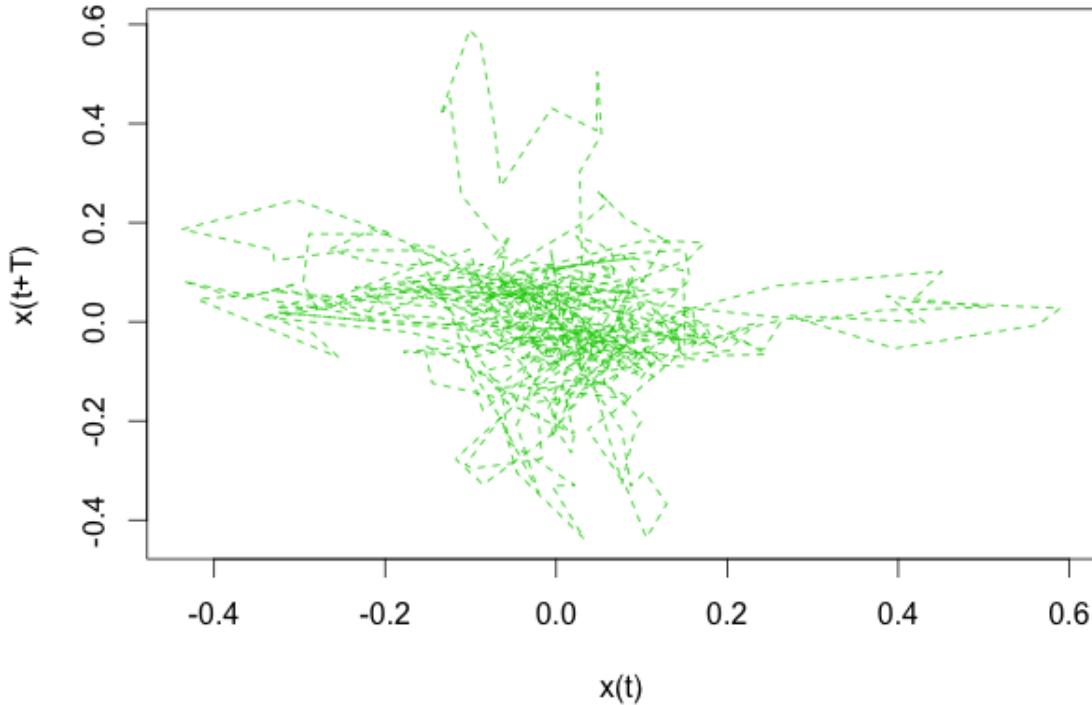


Figure 5.5: Phase Portrait for  $\log(\text{NASDAQ})$  unfiltered series  $T = 60$ . The graph was created by using Attractor.R (R program) and it is attached in the Appendix A

Data	$\eta$	$\nu(\%)$	CCgo	$P_c$	$\phi(\%)$	$P_{dc}$	$\lambda^{-1}$	$\mu$
SP500	1.032	106.6	0.4141	NA	NA	NA	NA	NA
NASDAQ	1.0116	102.33	0.4735	NA	NA	NA	NA	NA

Table 5.1: Detrend Statistics on S&P 500

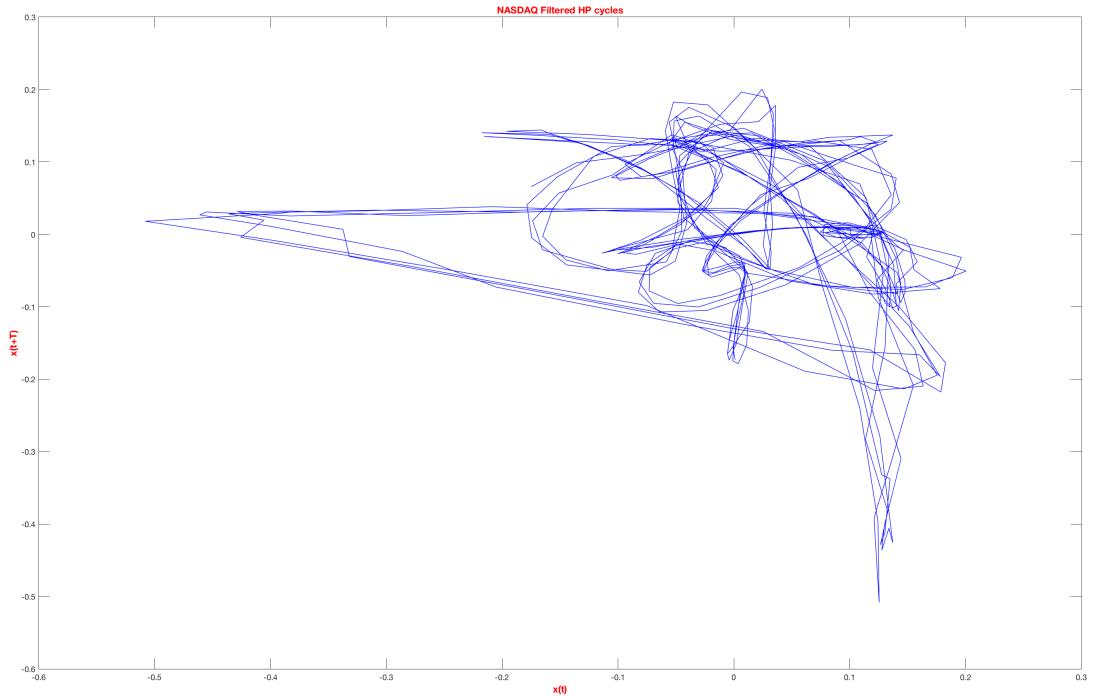


Figure 5.6: Phase Portrait for  $\log(\text{NASDAQ})$  unfiltered series  $T = 60$ . A pattern of strange attractor can be observed in the graph. The threshold value that depends on  $H$  ( $H = 0.5$ ) has less significant impact to the pattern emergence whereas the size of  $m, n$  in  $C(m, n)$ , the Gabor Coefficient has significant impact to the pattern of strange attractor. The  $16 \times 16$  Gabor Coefficient was used. The graph was created by using myAttractor.m (Matlab program) and it is attached in the Appendix B.

**FSPComIn SP 500 RAW HP Cycles for T=60**

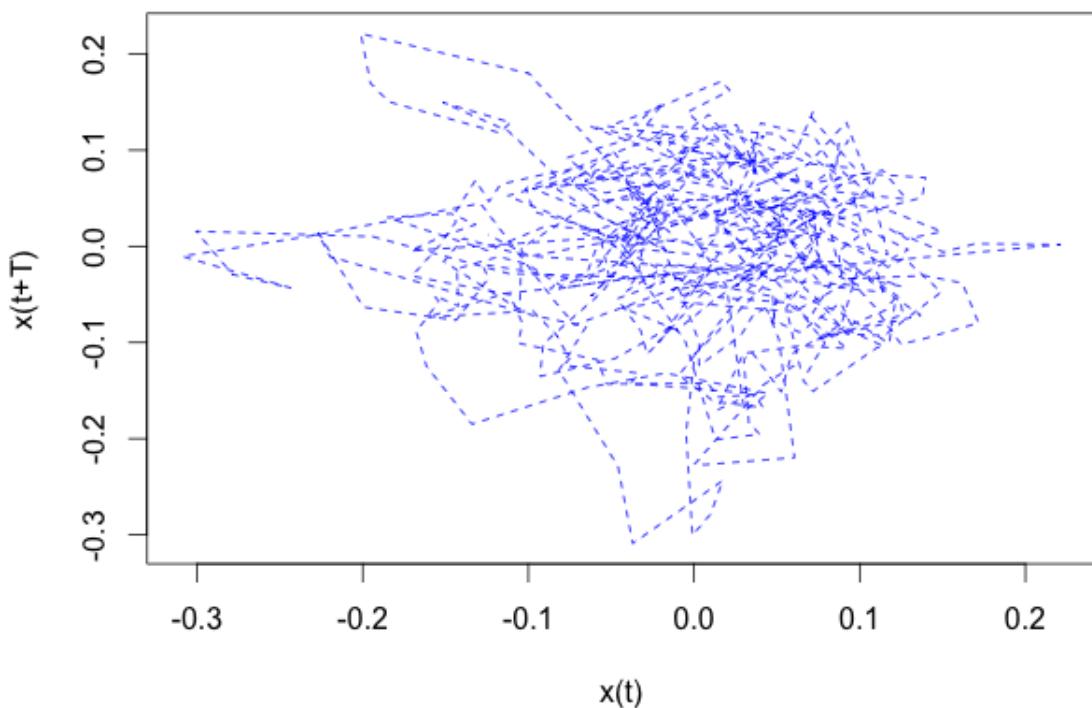


Figure 5.7: Phase Portrait for  $\log(\text{SP500})$  unfiltered series  $T = 60$ . The graph was created by using Attractor.R (R program) and it is attached in the Appendix A.

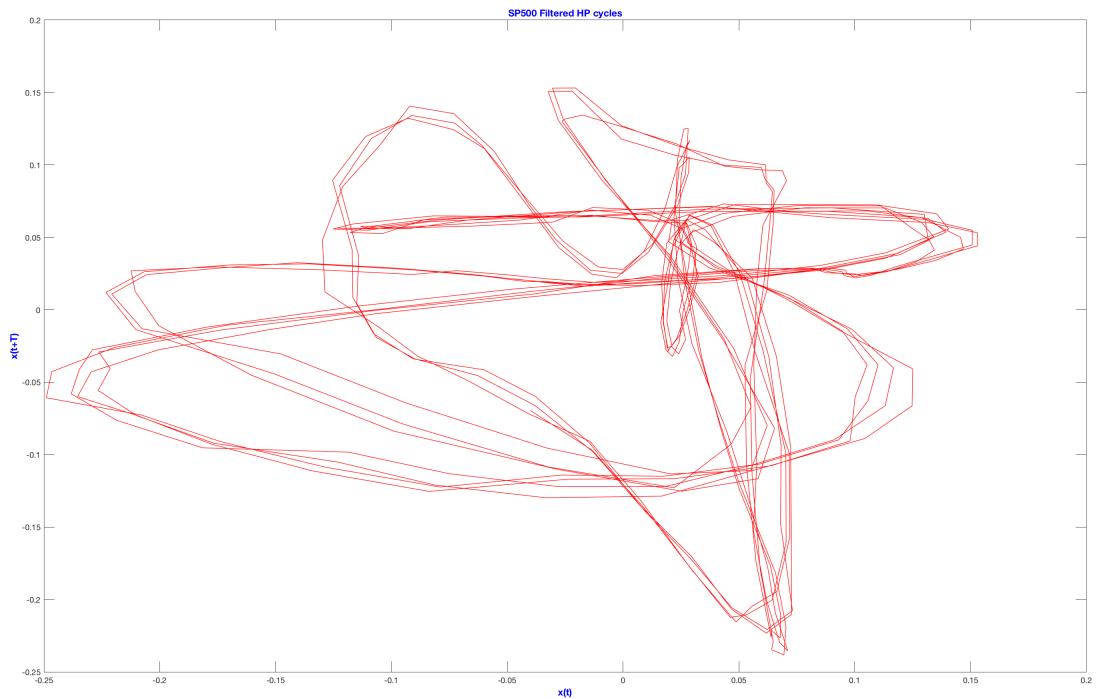


Figure 5.8: Phase Portrait for  $\log(\text{SP500})$  unfiltered series  $T = 60$ . A pattern of strange attractor can be observed in the graph. The threshold value that depends on  $H$  ( $H = 0.5$ ) has less significant impact to the pattern emergence whereas the size of  $m, n$  in  $C(m, n)$ , the Gabor Coefficient has significant impact to the pattern of strange attractor. The  $16 \times 16$  Gabor Coefficient was used. The graph was created by using myAttractor.m (Matlab program) and it is attached in the Appendix B.

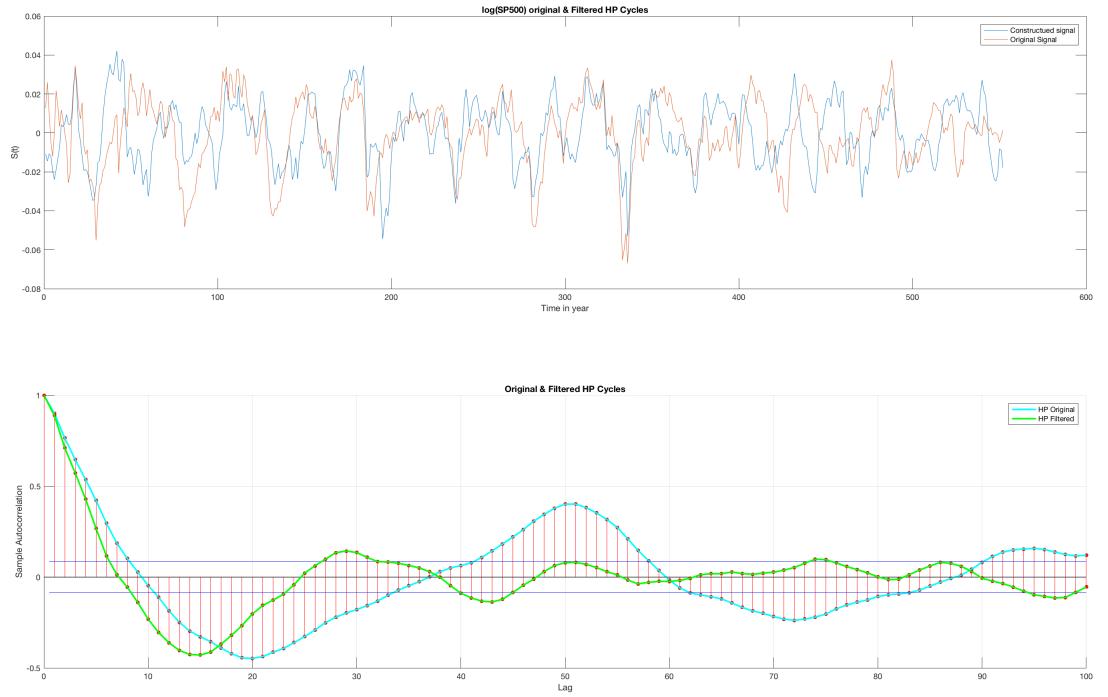


Figure 5.9: Fig a) The original and reconstructed time series of  $\log(\text{SP500})$  HP Cycles. Gabor Coefficients are created using the parameters -  $\Delta M = 8$ ;  $\Delta N = 4$ ;  $m = 16$ ;  $n = 16$ ;  $\sigma = \sqrt{\frac{\Delta M L}{\Delta N^2 \pi}}$ . Fig b) Autocorelations of the original and reconstructed time series of  $\log(\text{SP500})$  HP cycle. The program used to create the graph is myreconstfromgabor.m and it is attached in the appendix B.

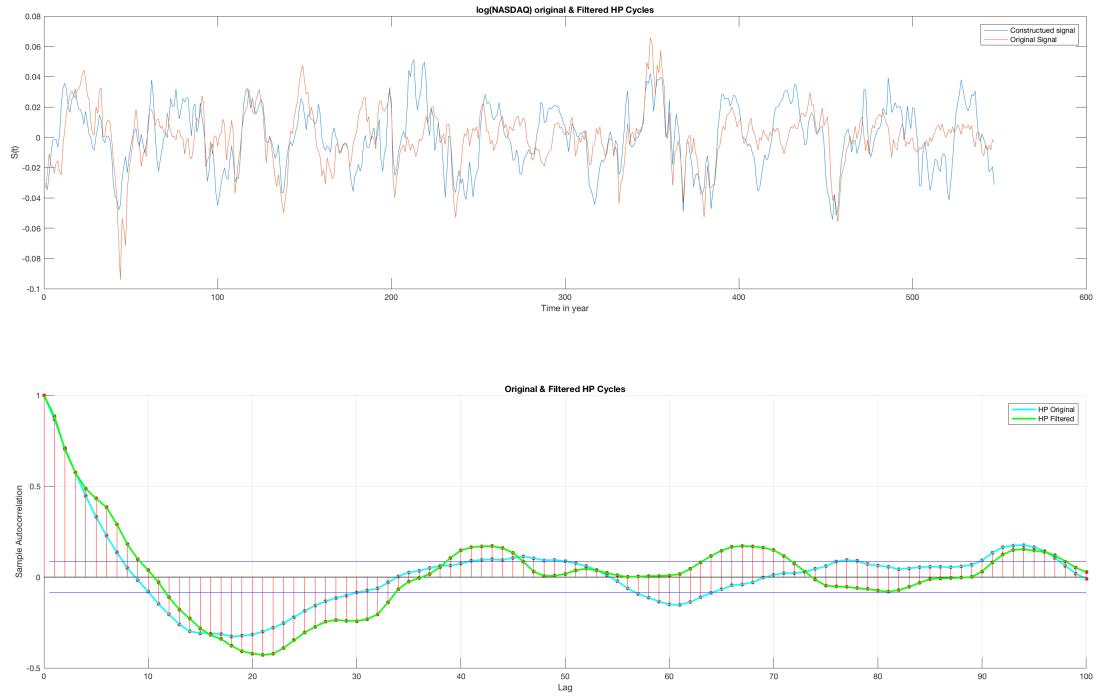


Figure 5.10: Fig a) The original and reconstructed time series of  $\log(\text{NASDAQ})$  HP Cycles. Gabor Coefficients are created using the parameters -  $\Delta M = 8$ ;  $\Delta N = 4$ ;  $m = 16$ ;  $n = 16$ ;  $\sigma = \sqrt{\frac{\Delta M L}{\Delta N^2 \pi}}$ . Fig b) Autocorelations of the original and reconstructed time series of  $\log(\text{NASDAQ})$  HP cycle. The program used to create the graph is myreconstfromgabor.m and it is attached in the appendix B.

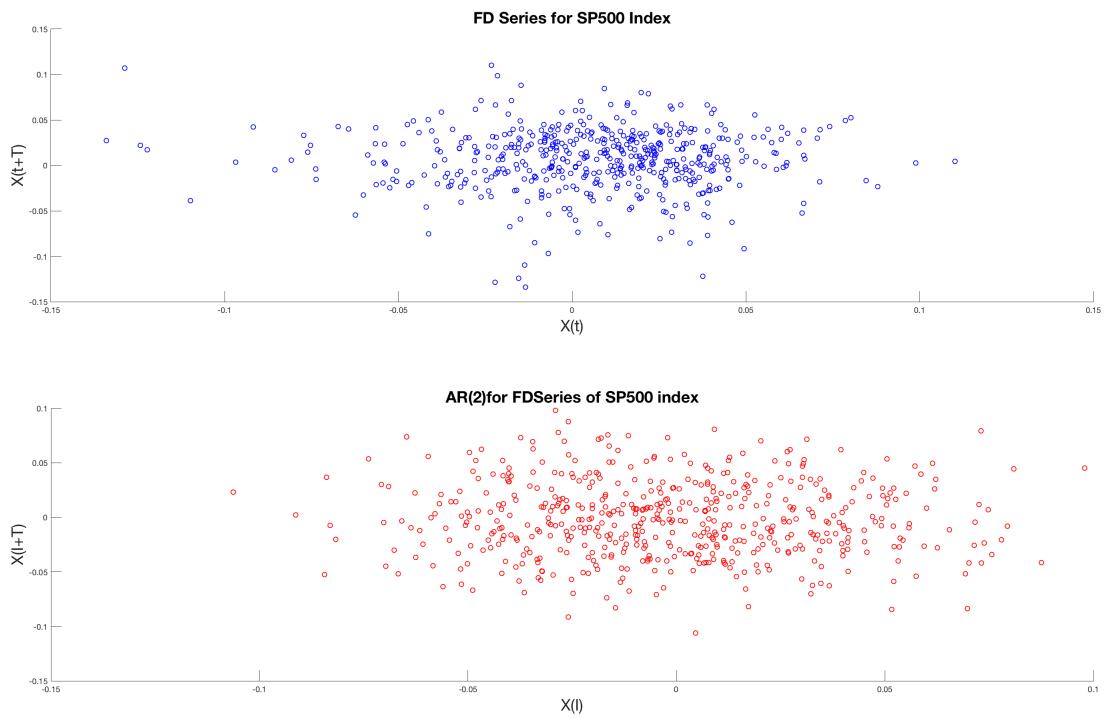


Figure 5.11: Fig a) SP500 FD Series.  $T = 40$ . The pattern demonstrates the existence of dominant of high frequency noise. Fig b) AR(2) for the FD Series of SP500.  $T=5$ . The program used to create the graph is myFDFilter.m and it is attached in the appendix B.

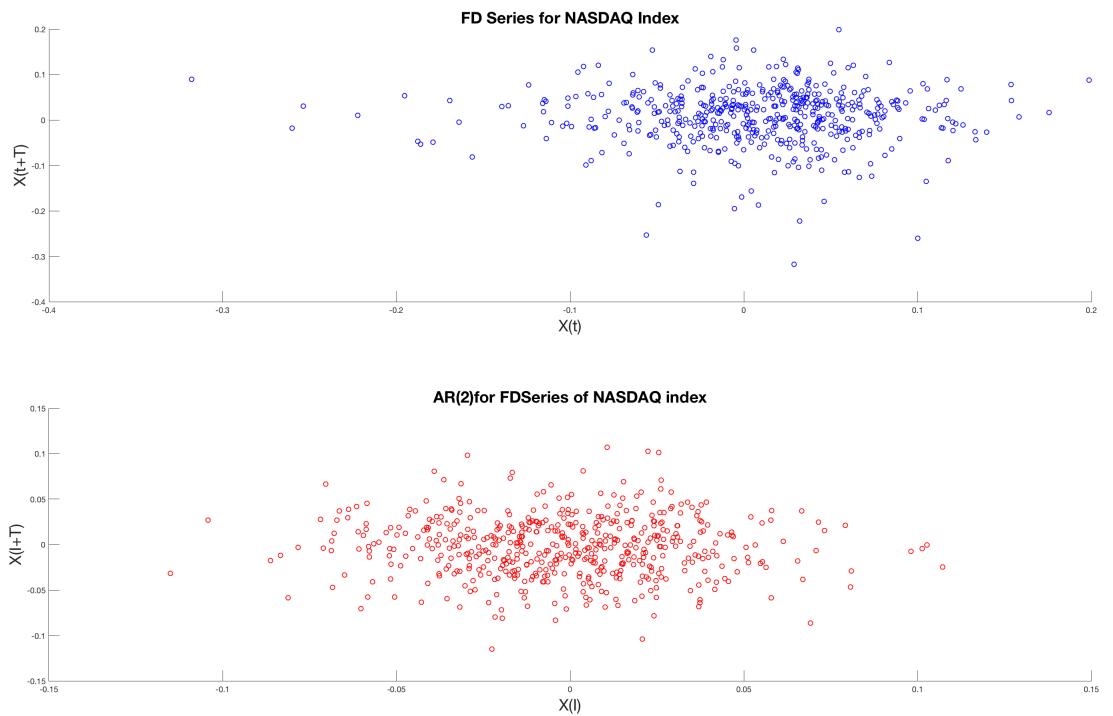


Figure 5.12: Fig a) NASDAQ FD Series.  $T = 40$ . The pattern demonstrates the existence of dominant of high frequency noise. Fig b) AR(2) for the FD Series of NASDAQ.  $T=5$ . The program used to create the graph is myFDFilter.m and it is attached in the appendix B.

## **APPENDICES**

## APPENDIX A

### R code that are used to do analysis

```
1 fdplot <-function()
2 {
3     #Program used to create the Difference stationary for NASDAQ & SP500 indexes
4     # Praba Siva
5     # praba@umich.edu
6     # @prabasiva
7     layout(matrix(c(1,1,2,2), 2, 2, byrow = TRUE))
8     setwd("/Users/sivaspl/Documents/2016/Personal/Praba/MATH599/program")
9     fspcom=read.table('fspcom.dat')
10    dat = log(fspcom[,5])
11    year=fspcom[,2]+1/12*fspcom[,3]
12    t1=dat[1:length(dat)-1]
13    t2=dat[2:length(dat)]
14    plot(year[1:length(year)-1],t2-t1,type='l',
15          main="Difference stationary of first Differencing of log(x(t))\nx(t) = S&P 500",
16          xlab="Year",ylab="FD",col='blue',
17          ylim=c(-.2,.2),cex.axis=1.1,cex.lab=1.5,lwd=2.2)
18    setwd("/Users/sivaspl/Documents/2016/Personal/Praba/MATH599/program")
19    dat <- read.csv(file="nasdaq-ready.csv",head=TRUE,sep=",")
20    year=dat[,1]+1/12*dat[,2]
21    dat=log(dat[,3])
22    t1=dat[1:length(dat)-1]
23    t2=dat[2:length(dat)]
24    plot(year[1:length(year)-1],t2-t1,type='l',
25          main="Difference stationary of first Differencing of log(x(t))\nx(t) = NASDAQ",
26          xlab="Year",ylab="FD",col='red',
27          ylim=c(-.2,.2),cex.axis=1.1,cex.lab=1.5,lwd=2.2)
28
29 }
```

```

1 llt <-function ()
2 {
3   #Program used to Log linear trend and cycles for SP500 & NASDAQ index
4   # Praba Siva
5   # praba@umich.edu
6   # @prabasiva
7 setwd("/Users/sivasp1/Documents/2016/Personal/Praba/MATH599/program")
8 fspcom=read.table('fspcom.dat')
9 year=fspcom[,2]
10 tsfspcom=ts(log(fspcom[,5]),start=year[1],
11             end=c(year[length(year)],12),frequency=12)
12 loglinear=stl(log(tsfspcom),s.window=5)
13 plot(loglinear,main="A Seasonal -Trend Decomposition of S&P 500",
14       col='red')
15 setwd("/Users/sivasp1/Documents/2016/Personal/Praba/MATH599/program")
16 dat <- read.csv(file="nasdaq_ready.csv",head=TRUE,sep=",")
17 year=dat[,1]
18 dat=dat[,3]
19 tsnasdaq=ts(dat,start=year[1],
20              end=c(year[length(year)]-1,12),frequency=12)
21 nloglinear=stl(log(tsnasdaq),s.window=5)
22 plot(nloglinear,main="A Seasonal -Trend Decomposition of NASDAQ",
23       col="blue")
24 strend=loglinear$time.series[,2]
25 ntrend=nloglinear$time.series[,2]
26 plot(year[1:length(strend)],strend[1:length(strend)],
27       col='blue',type='l',ylim=range(strend,ntrend))
28 lines(year[1:length(ntrend)],ntrend[1:length(ntrend)],
29        col='red',type='l')
30 }

```

```

1
2 hpfilt <- function ()
3 {
4   #Program used to create the HP filter for lambda 80 & 800 for S&P 500 index
5   #Load the file and invoke hpfilt()
6   #Filename: hpfilter-slave.R
7   # Praba Siva;praba@umich.edu; @prabasiva
8 library(mFilter)
9 library(latex2exp)
10 opar <- par(no.readonly=TRUE)
11 setwd("/Users/sivasp1/Documents/2016/Personal/Praba/MATH599/program")
12 fspcom=read.table('fspcom.dat')
13 dat = fspcom[,5]
14 syear=fspcom[,2]+1/12*fspcom[,3]
15 ldat=log(dat)
16 dat=ldat
17 dat.hp1 <- hpfilter(dat, freq=80,type="frequency",drift=FALSE)
18 dat.hp2 <- hpfilter(dat, freq=800,type="frequency",drift=FALSE)
19 par(mfrow=c(3,1),mar=c(3,3,2,1),cex=.8)
20 plot(syear,dat,
21       ylim=range(dat),
22       main="S&P 500 Index ")

```

```

22      col=2, ylab="" ,type='1' ,cex.axis=1.1 ,cex.lab=1.3 ,lwd=2.2 )
23 plot(syear,dat.hp1$trend , ylim=range(dat.hp1$trend) ,
24       main="HP filter of S&P 500 Index: Trend ,Lambda=80 " ,
25       col=4, xlab='praba siva' , ylab="log(s(t))" ,type='1' ,cex.axis=1.1 ,cex.lab=1.3 ,lwd=2.2 )
26 plot(syear,dat.hp1$cycle , ylim=range(dat.hp1$cycle) ,
27       main="HP filter of S&P 500 Index: Cycle ,Lambda=80 " ,
28       col=3, ylab="" ,type='1' ,cex.axis=1.1 ,cex.lab=1.3 ,lwd=2.2 )
29 par(mfrow=c(3,1) ,mar=c(3,3,2,1) ,cex=.8)
30 plot(syear,dat , ylim=range(dat) ,
31       main="S&P 500 Index " ,
32       col=2, ylab="" ,type='1' ,cex.axis=1.1 ,cex.lab=1.3 ,lwd=2.2 )
33 plot(syear,dat.hp2$trend , ylim=range(dat.hp2$trend) ,
34       main="HP filter of S&P 500 Index: Trend ,Lambda=800 " ,
35       col=4, xlab='praba siva' , ylab="log(s(t))" ,type='1' ,cex.axis=1.1 ,cex.lab=1.3 ,lwd=2.2 )
36 plot(syear,dat.hp2$cycle , ylim=range(dat.hp2$cycle) ,
37       main="HP filter of S&P 500 Index: Cycle ,Lambda=800 " ,
38       col=3, ylab="" ,type='1' ,cex.axis=1.1 ,cex.lab=1.5 ,lwd=2.2 )
39 par(opar)
40 opar <- par(no.readonly=TRUE)
41 setwd("/Users/sivasap1/Documents/2016/Personal/Praba/MATH599/program")
42 nasda <- read.csv(file="nasdaq-ready.csv",head=TRUE,sep=",")
43 nyear=nasda[,1]+1/12*nasda[,2]
44 nasda=nasda[,3]
45 lnasda=log(nasda)
46 nasda=lnasda
47 nasda.hp1 <- hpfilter(nasda, freq=80,type="frequency",drift=FALSE)
48 nasda.hp2 <- hpfilter(nasda, freq=800,type="frequency",drift=FALSE)
49 par(mfrow=c(3,1) ,mar=c(3,3,2,1) ,cex=.8)
50 plot(nyear,nasda , xlab="Year" ,ylab="log s(t)" ,ylim=range(nasda) ,
51       main="NASDAQ Index " ,
52       col=2, type='1' ,cex.axis=1.1 ,cex.lab=1.5 ,lwd=2.2 )
53 plot(nyear,nasda.hp1$trend , xlab='Year' , ylim=range(nasda.hp1$trend) ,
54       main="HP filter of NASDAQ Index: Trend ,Lambda=80 " ,
55       col=4, type='1' ,cex.axis=1.1 ,cex.lab=1.5 ,lwd=2.2 )
56 plot(nyear,nasda.hp1$cycle , ylim=range(nasda.hp1$cycle) , xlab="Year" ,
57       main="HP filter of NASDAQ Index: Cycle ,Lambda=80 " ,
58       col=3, type='1' ,cex.axis=1.1 ,cex.lab=1.5 ,lwd=2.2 )
59 par(mfrow=c(3,1) ,mar=c(3,3,2,1) ,cex=.8)
60 plot(nyear,nasda , ylim=range(nasda) ,
61       main="NASDAQ Index " ,
62       col=2, ylab="log s(t)" ,type='1' ,cex.axis=1.1 ,cex.lab=1.5 ,lwd=2.2 )
63 plot(nyear,nasda.hp2$trend , ylim=range(nasda.hp2$trend) ,
64       main="HP filter of NASDAQ Index: Trend ,Lambda=800 " ,
65       col=4, xlab='Year' , ylab="log(s(t))" ,type='1' ,cex.axis=1.1 ,cex.lab=1.5 ,lwd=2.2 )
66 plot(nyear,nasda.hp2$cycle , ylim=range(nasda.hp2$cycle) ,
67       main="HP filter of NASDAQ Index: Cycle ,Lambda=800 " ,
68       col=3, ylab="" ,type='1' ,cex.axis=1.1 ,cex.lab=1.5 ,lwd=2.2 )
69 par(opar)
70 nasda.hp3 <- hpfilter(nasda, freq=1600,type="frequency",drift=FALSE)
71 nasda.hp4 <- hpfilter(nasda, freq=14400,type="frequency",drift=FALSE)
72 lambda=c(80,800,1600,14400);
73 c=1:4;
74 layout(matrix(c(1,1,2,2), 2, 2, byrow = TRUE))
75 plot(nyear,nasda.hp1$trend ,ylab='Log NASDAQ' ,
76       main=TeX('NASDAQ Trend HP filter with different $\\lambda$') ,
77       xlab='Year' ,col=c(1) ,type='1' ,cex.axis=1,cex.lab=1.3 ,lwd=2.2 );

```

```

78  lines(nyear, nasda.hp2$trend, main=TeX( 'NASDAQ Trend HP filter with different $\lambda' ) ,
79      col=c(2), type='l', cex.axis=1, cex.lab=1.1, lwd=2.2);
80  lines(nyear, nasda.hp3$trend, main=TeX( 'NASDAQ Trend HP filterwith different $\lambda' ) ,
81      col=c(3), type='l', cex.axis=1, cex.lab=1.1, lwd=2.2);
82  lines(nyear, nasda.hp4$trend, main=TeX( 'NASDAQ Trend HP filterwith different $\lambda' ) ,
83      col=c(4), type='l', cex.axis=1, cex.lab=1.1, lwd=2.2);
84  legend('topleft', legend=TeX(sprintf("$\lambda = %d", lambda)), lwd=1, col=c)
85  dat.hp3 <- hpfilter(dat, freq=1600, type="frequency", drift=FALSE)
86  dat.hp4 <- hpfilter(dat, freq=14400, type="frequency", drift=FALSE)
87  lambda=c(80,800,1600,14400);
88  c=1:4;
89  plot(syear, dat.hp1$trend, ylab='Log SP500', main=TeX( 'SP500 Trend HP filter with different ... 
$\\lambda' ) ,
90      xlab='Year', col=c(1), type='l', cex.axis=1, cex.lab=1.3, lwd=2.2);
91  lines(syear, dat.hp2$trend, main=TeX( 'SP500 Trend HP filter with different $\\lambda' ) ,
92      col=c(2), type='l', cex.axis=1.1, cex.lab=1, lwd=2.2);
93  lines(syear, dat.hp3$trend, main=TeX( 'SP500 Trend HP filterwith different $\\lambda' ) ,
94      col=c(3), type='l', cex.axis=1.1, cex.lab=1, lwd=2.2);
95  lines(syear, dat.hp4$trend, main=TeX( 'SP500 Trend HP filterwith different $\\lambda' ) ,
96      col=c(4), type='l', cex.axis=1.1, cex.lab=1, lwd=2.2);
97  legend('topleft', legend=TeX(sprintf("$\lambda = %d", lambda)), lwd=1, col=c)
98
99 #Statistics of the HP Detrending
100 print('SP500 ')
101 print('HP Filter with Lambda = 80')
102 sprintf("Mean = %5f", mean(dat.hp1$cycle))
103 sprintf("SD = %5f", sd(dat.hp1$cycle))
104 sprintf("Variance = %5f", var(dat.hp1$cycle))
105 print('HP Filter with Lambda = 800')
106 sprintf("Mean = %5f", mean(dat.hp2$cycle))
107 sprintf("SD = %5f", sd(dat.hp2$cycle))
108 sprintf("Variance = %5f", var(dat.hp2$cycle))
109 print('HP Filter with Lambda = 1600')
110 sprintf("Mean = %5f", mean(dat.hp3$cycle))
111 sprintf("SD = %5f", sd(dat.hp3$cycle))
112 sprintf("Variance = %5f", var(dat.hp3$cycle))
113 print('HP Filter with Lambda = 14400')
114 sprintf("Mean = %5f", mean(dat.hp4$cycle))
115 sprintf("SD = %5f", sd(dat.hp4$cycle))
116 sprintf("Variance = %5f", var(dat.hp4$cycle))
117
118 print('NASDAQ ')
119 print('HP Filter with Lambda = 80')
120 xm=sprintf("Mean = %5f", mean(nasda.hp1$cycle))
121 xs=sprintf("SD = %5f", sd(nasda.hp1$cycle))
122 xv=sprintf("Variance = %5f", var(nasda.hp1$cycle))
123
124 print('HP Filter with Lambda = 800')
125 sprintf("Mean = %5f", mean(nasda.hp2$cycle))
126 sprintf("SD = %5f", sd(nasda.hp2$cycle))
127 sprintf("Variance = %5f", var(nasda.hp2$cycle))
128
129 print('HP Filter with Lambda = 1600')
130 sprintf("Mean = %5f", mean(nasda.hp3$cycle))
131 sprintf("SD = %5f", sd(nasda.hp3$cycle))
132 sprintf("Variance = %5f", var(nasda.hp3$cycle))

```

```

133
134 print('HP Filter with Lambda = 14400')
135 sprintf("Mean = %5f",mean(nasda.hp4$cycle))
136 sprintf("SD = %5f",sd(nasda.hp4$cycle))
137 sprintf("Variance = %5f",var(nasda.hp4$cycle))
138
139 }

```

```

1 loglinear<-function()
2 {
3   #Program used to Log linear example
4   #Filename: loglinear.R
5   # Praba Siva
6   # praba@umich.edu
7   # @prabasiva
8 layout(matrix(c(1,1,2,2), 2, 2, byrow = TRUE))
9 t=0:50000;
10 plot(t,exp(t*-.0001),main=TeX('$\backslash\beta_1 < 0$'),
11       xlab='Time t', ylab=TeX('log Y =$\backslash\beta_0+\backslash\beta_1 t$'),
12       type='l',cex.axis=1.1,cex.lab=.9,lwd=3,col='red')
13 plot(t,exp(t*.0001),main=TeX('$\backslash\beta_1 > 0$'),
14       xlab='Time t', ylab=TeX('log Y =$\backslash\beta_0+\backslash\beta_1 t$'),type='l',
15       cex.axis=1.1,cex.lab=.9,lwd=3,col='blue')
16 par(opar)
17 }

```

```

1 ac<-function()
2 {
3   #Program used to create AutoCorrelation Analysis for sample, SP500 & NASDAQ
4   #Filename: AutoCorrelation.R
5   # Praba Siva
6   # praba@umich.edu
7   # @prabasiva
8
9 library(mFilter);
10 library(latex2exp)
11 setwd("~/Users/sivaspl/Downloads/2016/Personal/Praba/MATH599/program")
12 fspcom=read.table('fspcom.dat')
13 dat=(fspcom[,5])
14 mort=log(dat)
15 year=fspcom[,2]+1/12*fspcom[,3]
16 le=length(dat)
17 x=mort[2:le]
18 y=mort[1:le-1]
19 diffxy=x-y
20 #plot(diffxy,type='l')
21 dur=1:length(year)
22 lmr=lm(mort~dur)
23 intercept=coef(lmr)[1]
24 slope=coef(lmr)[2]

```

```

25 dftrend=intercept+slope*dur
26 dfcycle=mort-dftrend
27 dfacf=acf(dfcycle, plot=FALSE, 100);
28 hpf=hpfILTER(mort, freq=14400)
29 layout(matrix(c(1,2,3,4), 4,1, byrow = TRUE))
30 color=c('blue')
31 acl=acf(hpf$cycle, ci.type = "ma", plot=FALSE, 100)
32 plot(year, mort, main='Log SP500 index',
33       xlab='Year', ylab=TeX('log (SP500(t))'),
34       type='l', cex.axis=1.1, cex.lab=1.1, lwd=3, col='red');
35 bcl=acf(diffxy, ci.type="ma", plot=FALSE, 100)
36 plot(acl, main='Autocorrelation of log SP500 HP Cycles'
37       , xlab='Lag', ylab='AC(1)')
38 lines(acl$lag, acl$acf, main='Autocorrelation of log SP500 HP Cycles',
39       xlab='Lag', ylab='AC(1)', type='l',
40       col='blue', cex.axis=1.1, cex.lab=1.1, lwd=3)
41 plot(bcl, main='Autocorrelation of log SP500 FD ',
42       xlab='Lag', ylab='AC(1)')
43 lines(bcl$lag, bcl$acf, main='Autocorrelation of log SP500 FD',
44       xlab='Lag', ylab='AC(1)', type='l', col='blue', lwd=3)
45 plot(dfacf, main='Autocorrelation of log-linear SP500 ',
46       xlab='Lag', ylab='AC(1)')
47 lines(dfacf$lag, dfacf$acf, main='Autocorrelation of log-linear SP500 ',
48       xlab='Lag', ylab='AC(1)', type='l',
49       col='blue', lwd=3)
50 layout(matrix(c(1,2), 2,1, byrow = TRUE))
51 plot(year, mort, main='Log SP500 index',
52       xlab='Year', ylab=TeX('log (SP500(t))'),
53       type='l', cex.axis=1.1, cex.lab=1.1, lwd=3, col='red');
54 lines(year, dftrend, main='Trend of Log SP500 index using Log-linear',
55       xlab='Year', ylab=TeX('log-linear(SP500(t))'),
56       type='l', cex.axis=1.1, cex.lab=1.1, lwd=3, col='blue');
57 legend("bottomright", c("Trend"), lty=c(1), lwd=c(2.5), col=c("blue"))
58 plot(year, dfcycle, main='Cycle of Log SP500 index using Log-linear',
59       xlab='Year', ylab=TeX('log-linear(SP500(t))'),
60       type='l', cex.axis=1.1, cex.lab=1.1, lwd=3, col='green');
61 layout(matrix(c(1,2), 2,1, byrow = TRUE))
62 plot(year, mort, main='Log SP500 index',
63       xlab='Year', ylab=TeX('log (SP500(t))'),
64       type='l', cex.axis=1.1, cex.lab=1.1, lwd=3, col='red');
65 plot(year[1:length(diffxy)], diffxy,
66       main='Cycle of Log SP500 index using Log-linear trend',
67       xlab='Year', ylab=TeX('log-linear(SP500(t))'), type='l',
68       cex.axis=1.1, cex.lab=1.1, lwd=3, col='green');
69 sta.sp500=list(mean(dfacf$acf), sd(dfacf$acf), var(dfacf$acf), corrlength(dfacf),
70                   mean(acl$acf), sd(acl$acf), var(acl$acf), corrlength(acl),
71                   mean(bcl$acf), sd(bcl$acf), var(bcl$acf), corrlength(bcl));
72 layout(matrix(c(1,2,3,4), 4,1, byrow = TRUE))
73 setwd("~/Users/sivaspl/Documents/2016/Personal/Praba/MATH599/program")
74 dat <- read.csv(file="nasdaq-ready.csv", head=TRUE, sep=",")
75 year=dat[,1]+1/12*dat[,2]
76 dat=dat[,3]
77 mort=log(dat)
78 le=length(dat)
79 x=mort[2:le]
80 y=mort[1:le -1]
```

```

81  diffxy=x-y
82  dur=1:length(year)
83  lmr=lm(mort~dur)
84  intercept=coef(lmr)[1]
85  slope=coef(lmr)[2]
86  dftrend=intercept+slope*dur
87  dfcycle=mort-dftrend
88  dfacf=acf(dfcycle, plot=FALSE, 100);
89  hpf=hpfILTER(mort, freq=14400)
90  acl=acf(hpf$cycle, ci.type = "ma", plot=FALSE, 100)
91  bcl=acf(diffxy, ci.type="ma", plot=FALSE, 100)
92  layout(matrix(c(1,2), 2,1, byrow = TRUE))
93  plot(year, mort, main='Log NASDAQ index',
94        xlab='Year', ylab=TeX('log (NASDAQ(t))'),
95        type='l', cex.axis=1.1, cex.lab=1.1, lwd=3, col='red');
96  lines(year, dftrend, main='Trend of Log NASDAQ index using Log-linear',
97        xlab='Year', ylab=TeX('log-linear (NASDAQ(t))'),
98        type='l', cex.axis=1.1, cex.lab=1.1, lwd=3, col='blue');
99  legend("bottomright", c("Trend"), lty=c(1), lwd=c(2.5), col=c("blue"))
100 plot(year, dfcycle, main='Cycle of Log NASDAQ index using Log-linear',
101       xlab='Year', ylab=TeX('log-linear (NASDAQ(t))'),
102       type='l', cex.axis=1.1, cex.lab=1.1, lwd=3, col='green');
103 layout(matrix(c(1,2), 2,1, byrow = TRUE))
104 plot(year, mort, main='Log NASDAQ index',
105       xlab='Year', ylab=TeX('log (NASDAQ(t))'),
106       type='l', cex.axis=1.1, cex.lab=1.1, lwd=3, col='red');
107 plot(year[1:length(diffxy)], diffxy,
108       main='Cycle of Log NASDAQ index using Log-linear trend',
109       xlab='Year', ylab=TeX('log-linear (NASDAQ(t))'), type='l',
110       cex.axis=1.1, cex.lab=1.1, lwd=3, col='green');
111 layout(matrix(c(1,2,3,4), 4,1, byrow = TRUE))
112 plot(year, mort, main='Log NASDAQ index',
113       xlab='Year', ylab=TeX('log (NASDAQ(t))'), type='l',
114       col='red', cex.axis=1.1, cex.lab=1.1, lwd=3);
115 plot(acl, main='Autocorrelation of log NASDAQ HP Cycles',
116       xlab='Lag', ylab='AC(1)')
117 lines(acl$lag, acl$acf, main='Autocorrelation of log NASDAQ HP Cycles',
118       xlab='Lag', ylab='AC(1)', type='l',
119       col='blue', cex.axis=1.1, cex.lab=1.1, lwd=3)
120 plot(bcl, main='Autocorrelation of log NASDAQ FD',
121       xlab='Lag', ylab='AC(1)')
122 lines(bcl$lag, bcl$acf, main='Autocorrelation of log NASDAQ FD',
123       xlab='Lag', ylab='AC(1)', type='l',
124       col='blue', cex.axis=1.1, cex.lab=1.1, lwd=3)
125 plot(dfacf, main='Autocorrelation of log-linear NASDAQ',
126       xlab='Lag', ylab='AC(1)')
127 lines(dfacf$lag, dfacf$acf, main='Autocorrelation of log-linear NASDAQ',
128       xlab='Lag', ylab='AC(1)', type='l', col='blue', lwd=3)
129 layout(matrix(c(1,2,3,4,5,6), 3, 2, byrow = TRUE))
130 #par(mfrow=c(2,1), mar=c(3,3,2,1), cex=.8)
131 x=seq(-15,15,.1);
132 y=sin(x)
133 acl=acf(y, lag.max=100, plot=FALSE);
134 plot(x,y, main='Sin wave', xlab='T', ylab='Sin(t)', type='l',
135       col='red', cex.axis=1.1, cex.lab=1.1, lwd=3)
136 plot(acl, main='Autocorrelation of Sin wave', xlab='Lag', ylab='AC(1)',
```

```

137      cex.axis=1.1 ,cex.lab=1.1 ,lwd=.2 )
138  lines ( ac1$lag ,ac1$acf ,type='l' ,col='blue' ,lwd=2)
139  x=seq (-15 ,15 ,.1 );
140  y=x^2+x^3
141  ac1=acf(y ,lag.max=100 ,plot=FALSE);
142  plot(x,y ,main='Polynomial' ,
143        xlab='T' ,ylab=TeX( 'y=x^3(t)+x^2(t)' ) ,type='l' ,col='red' ,
144        cex.axis=1.1 ,cex.lab=1.1 ,lwd=3)
145  plot(ac1 ,main='Autocorrelation of Polynomial' ,
146        xlab='Lag' ,ylab='AC(1)' ,cex.axis=1.1 ,cex.lab=1.5 ,lwd=.2 )
147  lines ( ac1$lag ,ac1$acf ,type='l' ,col='blue' ,lwd=2)
148  x=seq (-15 ,15 ,.1 );
149  y=sin (x)*rnorm (length (x) ,mean=0 ,sd=1)
150  ac1=acf(y ,lag.max=100 ,plot=FALSE);
151  plot(x,y ,main='Sin wave with random noise' ,
152        xlab='t' ,
153        ylab=TeX( 'Sin(t) * r(\mu=0 ,\sigma^2=1)' ) ,type='l' ,col='red' ,
154        cex.axis=1.1 ,cex.lab=1.1 ,lwd=2)
155  plot(ac1 ,main='Autocorrelation of Sin wave with random noise' ,
156        xlab='Lag' ,ylab='AC(1)' ,cex.axis=1.1 ,cex.lab=1.5 ,lwd=3)
157  lines ( ac1$lag ,ac1$acf ,type='l' ,col='blue' ,lwd=.2 )
158  corrlength (ac1)
159  sta.nasdaq=list (mean (dfacf$acf) ,sd (dfacf$acf) ,var (dfacf$acf) ,corrlength (dfacf) ,
160                     mean (ac1$acf) ,sd (ac1$acf) ,var (ac1$acf) ,corrlength (ac1) ,
161                     mean (bc1$acf) ,sd (bc1$acf) ,var (bc1$acf) ,corrlength (bc1))
162  print("Dtrend statistics for SP500")
163  print(matrix (sta.sp500 ,nrow=4))
164  print("Dtrend statistics for NASDAQ")
165  print(matrix (sta.nasdaq ,nrow=4))
166  }
167
168  corrlength <- function (acfvector)
169  {
170  }
171
172  ind=min (which (acfvector$acf <0));
173
174  return ...
175  ((abs (acfvector$acf [ind]) +abs (acfvector$acf [ind -1]) /10)*(abs (acfvector$acf [ind -1])) +ind -1)
176  }

```

## APPENDIX B

### Matlab code that are used to do analysis

```
1 function DrawSinFourierGraph()
2 Fs = 1000; % Sampling frequency
3 T = 1/Fs; % Sampling period
4 L = 200; % Length of signal
5 t = (0:L-1)*T; % Time vect
6 f = [50,150,300];
7 x1 = sin(2*pi*f(1)*t); % First row wave
8 x2 = sin(2*pi*f(2)*t); % Second row wave
9 x3 = sin(2*pi*f(3)*t); % Third row wave
10
11 X = [x1; x2; x3];
12
13 figure;
14
15 for i = 1:3
16     g=subplot(3,2,i);
17     plot(t(1:L),X(i,1:L),'r','LineWidth',1)
18     ylabel('sin(2\pi f t)', 'FontSize',16, 'FontWeight','bold')
19     xlabel('\fontname{Helvetica} Time', 'FontSize',16, 'FontWeight','bold')
20     title(['A sine wave of frequency f = ', num2str(f(i)), ' in the Time ... '
21             'Domain'], 'FontSize',18, 'FontWeight','bold')
22     p=get(g, 'position');
23     p(1)=.7*p(1);
24     p(4)=1.1*p(4);
25     set(g, 'position',p);
26
27
28 n = 2^nextpow2(L);
29 dim = 2;
30 Y = fft(X,n,dim);
31 P2 = abs(Y/n);
```

```

32 P1 = P2(:,1:n/2+1);
33 P1(:,2:end-1) = 2*P1(:,2:end-1);
34
35
36 for i=1:3
37     g=subplot(3,2,i*2);
38     plot(0:(Fs/n):(Fs-Fs/n),P2(i,1:n),'b','LineWidth',1)
39     title(['Sin wave of frequency ',num2str(f(i)), ' in the Frequency ...',
40             ' Domain'], 'FontSize',18, 'FontWeight','bold')
40     ylabel(' |F(s)| ', 'FontSize',16, 'FontWeight','bold')
41     xlabel('Frequency ', 'FontSize',16, 'FontWeight','bold')
42     p=get(g, 'position');
43     p(1)=.9*p(1);
44     p(4)=1.1*p(4);
45     set(g, 'position',p);
46 end

```

```

1 function SPNASDAQFourier()
2 %Program used to Discrete Fourier transform
3 % Praba Siva
4 % praba@umich.edu
5 % @prabasiva
6 % Filename: SPNASDAQFourier.m
7 Fs = 1000; % Sampling frequency
8 T = 1/Fs; % Sampling period
9 L = 1000; % Length of signal
10 t = (0:L-1)*T; % Time vector
11 S = 0.7*sin(2*pi*50*t) + sin(2*pi*120*t);
12 X = S + 2*randn(size(t));
13 wsize=1000;
14 g=subplot(5,2,1);
15 plot(1000*t(1:wsize),X(1:wsize),'r','LineWidth',1);
16 title('Signal Corrupted with Zero-Mean Random Noise','FontSize',18,'FontWeight','bold')
17 xlabel('t (milliseconds)', 'FontSize',16, 'FontWeight','bold')
18 ylabel('f(t)', 'FontSize',16, 'FontWeight','bold')
19 %N=2^nextpow2(L);
20 Y = fft(X);
21 P2 = abs(Y/L);P1 = P2(1:L/2+1);
22 P1(2:end-1) = 2*P1(2:end-1);
23 f = Fs*(0:(L/2))/L;
24 g=subplot(5,2,2);
25 plot(f,P1,'b','LineWidth',1)
26 title('Single-Sided Amplitude Spectrum of f(t)','FontSize',16,'FontWeight','bold')
27 xlabel('w', 'FontSize',16, 'FontWeight','bold')
28 ylabel(' |F(w)| ', 'FontSize',16, 'FontWeight','bold')
29 g=subplot(5,2,3);
30 plot(1000*t(1:wsize),S(1:wsize),'r','LineWidth',1);
31 title('Signal with No Random Noise','FontSize',18,'FontWeight','bold')
32 xlabel('t (milliseconds)', 'FontSize',16, 'FontWeight','bold')
33 ylabel('f(t)', 'FontSize',16, 'FontWeight','bold')
34 Y = fft(S);
35 P2 = abs(Y/L);
36 P1 = P2(1:L/2+1);

```

```

37 P1(2:end-1) = 2*P1(2:end-1);
38 f = Fs*(0:(L/2))/L;
39 g=subplot(5,2,4);
40 plot(f,P1,'b','LineWidth',1)
41 title('Single - Sided Amplitude Spectrum of f(t)', 'FontSize',18, 'FontWeight', 'bold')
42 xlabel('w', 'FontSize',16, 'FontWeight', 'bold')
43 ylabel('|F(w)|', 'FontSize',16, 'FontWeight', 'bold')
44 Fs = 100; % Sampling frequency
45 t = -0.5:1/Fs:0.5; % Time vector
46 L = length(t); % Signal length
47 X = 1/(4*sqrt(2*pi*0.01))*(exp(-t.^2/(2*0.01)));
48 g=subplot(5,2,5)
49 plot(t,X, 'r', 'LineWidth',1)
50 title('Gaussian Pulse in Time Domain', 'FontSize',18, 'FontWeight', 'bold')
51 xlabel('Time (t)', 'FontSize',16, 'FontWeight', 'bold')
52 ylabel('f(t)', 'FontSize',16, 'FontWeight', 'bold')
53 n = 2^nextpow2(L);
54 Y = fft(X,n);
55 f = Fs*(0:(n/2))/n;
56 P = abs(Y/n);
57 g=subplot(5,2,6)
58 plot(fftshift(P), 'b', 'LineWidth',1)
59 %plot(f, fftshift(P(1:n/2+1)), 'b', 'LineWidth',1)
60 %plot(1:wsize ,P(1:wsize ), 'b', 'LineWidth',1)
61 title('Gaussian Pulse in Frequency Domain', 'FontSize',18, 'FontWeight', 'bold')
62 xlabel('Frequency (w)', 'FontSize',16, 'FontWeight', 'bold')
63 ylabel('|F(w)|', 'FontSize',16, 'FontWeight', 'bold')
64 [naq,year]=getData(2);
65 [dum,naq]=hpfilter(naq,14400);
66 g=subplot(5,2,7)
67 plot(year,naq, 'r', 'LineWidth',1)
68 title('HPFilter of Log Nasdaq', 'FontSize',18, 'FontWeight', 'bold');
69 xlabel('Year.month', 'FontSize',16, 'FontWeight', 'bold');
70 ylabel('Cycles', 'FontSize',16, 'FontWeight', 'bold');
71 Fs = 1000;
72 [L,tp]=size(naq)
73 n = 2^nextpow2(L);
74 Y = fft(naq,n);
75 f = Fs*(0:(n/2))/n;
76 P = abs(Y/n);
77 g=subplot(5,2,8)
78 %plot(f,P(1:n/2+1), 'b', 'LineWidth',1)
79 plot(1:n/2+1,P(1:n/2+1), 'b', 'LineWidth',1)
80 title('HPFilter of NASDAQ in Frequency Domain', 'FontSize',18, 'FontWeight', 'bold')
81 xlabel('Frequency (w)', 'FontSize',16, 'FontWeight', 'bold')
82 ylabel('|F(w)|', 'FontSize',16, 'FontWeight', 'bold')
83 [sp500,year]=getData(1);
84 [dum,sp500]=hpfilter(sp500,14400);
85 g=subplot(5,2,9)
86 plot(year,sp500, 'r', 'LineWidth',1)
87 title('HPFilter of Log S&P 500', 'FontSize',18, 'FontWeight', 'bold');
88 xlabel('Year.month', 'FontSize',16, 'FontWeight', 'bold');
89 ylabel('Cycles', 'FontSize',16, 'FontWeight', 'bold');
90 [L,tp]=size(sp500)
91 n = 2^nextpow2(L);
92 Y = fft(sp500,n);

```

```

93 f = Fs*(0:(n/2))/n;
94 P = abs(Y/n);
95 g=subplot(5,2,10)
96 %plot(f,P(1:n/2+1),'b','LineWidth',1);
97 plot(1:n/2+1,P(1:n/2+1),'b','LineWidth',1);
98 title('HPFilter of S&P in Frequency Domain','FontSize',18,'FontWeight','bold')
99 xlabel('Frequency (w)','FontSize',16,'FontWeight','bold')
100 ylabel('|F(w)|','FontSize',16,'FontWeight','bold')

```

```

1 function ghamwin()
2 s={'Gaussian Window' 'Hamming Window'};
3 wlen=25;
4 figure;
5 for fla=0:1
6 if fla<1
7 % form a periodic hamming window
8 win = hamming(wlen, 'periodic');
9 else
10 win=gausswin(wlen)
11 end
12 g=subplot(1,2,1+fla);
13 plot(abs(win), 'r','LineWidth',1);
14 xlabel('n (N=25)', 'FontSize',16, 'FontWeight','bold')
15 if fla ==0
16 title('\sigma = 2.5 Gaussian Window . .
17 function','interpreter','Tex','FontSize',16,'FontWeight','bold')
17 ylabel('e^{-n^2/2\sigma^2}', 'interpreter','Tex','FontSize',20)
18 else
19 title([s(fla+1), 'function'], 'FontSize',16,'FontWeight','bold')
20 ylabel('0.54-0.46cos(2\pi*n/N)', 'interpreter','Tex','FontSize',20)
21 end
22 end

```

```

1 function drawSpecEg()
2 close all;
3 clear all;
4 Fs = 1000; % Sampling frequency
5 T = 1/Fs; % Sampling period
6 L = 1000; % Length of signal
7 t = (0:L-1)*T; % Time vector
8 S = 0.7*sin(2*pi*50*t) + sin(2*pi*120*t);
9 Y = S + 2*randn(size(t));
10 s={'using Gaussian Window' 'using Hamming Window'};
11 wlen=2^6;
12 figure;
13
14 for fla=0:1
15 g=subplot(2,3,1+(3*fla));
16 if fla==1
17 X=S;

```

```

18     else
19         X=Y;
20     end;
21     plot(L*t,X,'r','LineWidth',1);
22
23     if fla==1
24
25         title('Signal with No Random Noise','FontSize',18,'FontWeight','bold')
26     else
27         title('Signal Corrupted with Zero-Mean Random ...
28             Noise','FontSize',18,'FontWeight','bold')
29     end
30     xlabel('t (milliseconds)','FontSize',16,'FontWeight','bold')
31     ylabel('f(t)','FontSize',16,'FontWeight','bold')
32     g=subplot(2,3,2+(3*fla));
33     spectrogram(X,gausswin(wlen));
34     ylabel(g,'Time (ms)');
35     title(['|STFT(t,\omega)|', s(1)],'FontSize',16,'FontWeight','bold')
36     g=subplot(2,3,3+(3*fla));
37     spectrogram(X,hamming(wlen));
38     title(['|STFT(t,\omega)|', s(2)],'FontSize',16,'FontWeight','bold')
39     ylabel(g,'Time (ms)');
40
41 end

```

```

1 function drawSpecEco()
2 close all;
3 clear all;
4 [sp500,syear]=getData(1);
5 [naq,nyear]=getData(2);
6
7 s={'using Gaussian Window' 'using Hamming Window'};
8 wlen=2^6;
9 figure;
10
11 for fla=0:1
12     g=subplot(2,3,1+(3*fla));
13     if fla==1
14         [trend, X] = hpfilter(sp500,14400);
15         xaxis=syear;
16     else
17         [trend, X] = hpfilter(naq,14400);
18         xaxis=nyear;
19     end;
20     plot(xaxis,X,'r','LineWidth',1);
21
22     if fla==1
23         title('HP Filter cycles (\lambda =14400) for SP500','FontSize',18,'FontWeight','bold')
24     else
25         title('HP Filter cycles (\lambda =14400) for ...
26             NASDAQ','FontSize',18,'FontWeight','bold')
27     end

```

```

27 xlabel('Years','FontSize',16,'FontWeight','bold')
28 ylabel('Cycles','FontSize',16,'FontWeight','bold')
29 g=subplot(2,3,2+(3*f1));
30 spectrogram(X,gausswin(wlen));
31 title(['|STFT(t,\omega)|', s(1)],'FontSize',16,'FontWeight','bold')
32 ylabel(g,'Time');
33 g=subplot(2,3,3+(3*f1));
34 spectrogram(X,hamming(wlen));
35 ylabel(g,'Time');
36 title(['|STFT(t,\omega)|', s(2)],'FontSize',16,'FontWeight','bold')
37 end
38
39 end

```

```

1 function drawSTFTECo3D()
2 close all;
3 clear all;
4 [sp500,syear]=getData(1);
5 [naq,nyear]=getData(2);
6
7 s={'using Gaussian Window' 'using Hamming Window'};
8 wlen=2^6;
9 figure;
10
11 for f1=0:1
12 g=subplot(2,3,1+(3*f1));
13 if f1==1
14 [trend,X]=hpfilter(sp500,14400);
15 xaxis=syear;
16 else
17 [trend,X]=hpfilter(naq,14400);
18 xaxis=nyear;
19 end;
20 plot(xaxis,X,'r','LineWidth',1);
21
22 if f1==1
23 title('HP Filter cycles (\lambda =14400) for SP500','FontSize',18,'FontWeight','bold')
24 else
25 title('HP Filter cycles (\lambda =14400) for ... ...
26 NASDAQ','FontSize',18,'FontWeight','bold')
27 end
28 xlabel('Years','FontSize',16,'FontWeight','bold')
29 ylabel('Cycles','FontSize',16,'FontWeight','bold')
30 g=subplot(2,3,2+(3*f1));
31 out=spectrogram(X,gausswin(wlen));
32 surf(abs(out));
33 title(['|STFT(t,\omega)|', s(1)],'FontSize',16,'FontWeight','bold')
34 g=subplot(2,3,3+(3*f1));
35 out=spectrogram(X,hamming(wlen));
36 surf(abs(out));
37 title(['|STFT(t,\omega)|', s(2)],'FontSize',16,'FontWeight','bold')
38 end
39

```

```
39 end
```

```
1 function [stft , f , t] = stft2(x , wlen , h , nfft , fs ,flag)
2 % function: [stft , f , t] = stft(x , wlen , h , nfft , fs )
3 % x - signal in the time domain
4 % wlen - length of the hamming window
5 % h - hop size
6 % nfft - number of FFT points
7 % flag = 1 for Gaussian window or 0 for Hamming window
8 % fs - sampling frequency , Hz
9 % f - frequency vector , Hz
10 % t - time vector , s
11 % stft - STFT matrix (only unique points , time across columns , freq across rows)
12 % represent x as column-vector if it is not
13 if size(x , 2) > 1
14     x = x';
15 end
16 % length of the signal
17 xlen = length(x);
18 if flag<1
19 % form a periodic hamming window
20 win = hamming(wlen , 'periodic');
21 else
22 win=gausswin(wlen)
23 end
24 % form the stft matrix
25 rown = ceil((1+nfft)/2);           % calculate the total number of rows
26 coln = 1+fix((xlen-wlen)/h);      % calculate the total number of columns
27 stft = zeros(rown , coln);         % form the stft matrix
28
29 % initialize the indexes
30 indx = 0;
31 col = 1;
32
33 % perform STFT
34 while indx + wlen ≤ xlen
35     % windowing
36     xw = x(indx+1:indx+wlen).*win;
37
38     % FFT
39     X = fft(xw , nfft );
40
41     % update the stft matrix
42     stft (: , col) = X(1:rown);
43
44     % update the indexes
45     indx = indx + h;
46     col = col + 1;
47 end
48
49 % calculate the time and frequency vectors
50 t = (wlen/2:h:wlen/2+(coln-1)*h)/fs ;
51 f = (0:rown-1)*fs/nfft ;
```

```

52
53 end

```

```

1 function [index,year]= getData(flag)
2 if flag == 1
3     dat=readtable('~/Users/sivaspl/Documents/2016/Personal/Praba/MATH599/program/fspcom.dat');
4     [maxx,maxy]=size(dat);
5     index=table2array(dat(1:maxx,5));
6     year=(table2array(dat(1:maxx,2)));
7     month=(table2array(dat(1:maxx,3)));
8     index=log(index);
9     year=year+month/12;
10 else
11     dat=csvread('~/Users/sivaspl/Documents/2016/Personal/Praba/MATH599/program/nasdaq_mat.csv');
12     year=dat(:,1);
13     month=dat(:,2);
14     index=dat(:,3);
15     index=log(index);
16     year=year+month/12;
17 end

```

```

1 function mywvd()
2
3 %Program used to Wigner Distribution for NASDAQ & SP500 indexes
4 % Praba Siva
5 % praba@umich.edu
6 % @prabasiva
7 % Filename: mywvd.m
8
9 close all;
10 clear all;
11 [sp500,syear]=getData(1);
12 sp500=log(sp500);
13 [naq,nyear]=getData(2);
14 naq=log(naq);
15
16 for step = 1:2
17
18     if step == 2
19         sp500=naq;
20         syear=nyear;
21     end;
22
23 figure;
24
25 [s2,s1]=hpfilter(sp500,1600);
26 subplot(2,3,1);
27 plot(syear,sp500);
28 xlabel('Time in years');
29 ylabel('log s(t)');

```

```

30      title('Log s(t)');
31
32      [wd,freq]=wig2(sp500);
33      subplot(2,3,2);
34      contour(syear,freq,abs(wd'),8), grid on
35      xlabel('Time in years');
36      ylabel('Frequency');
37      title('Contour Map');
38
39      subplot(2,3,3);
40      mesh(syear,freq,abs(wd'));
41      xlabel('Time in years');
42      ylabel('Frequency');
43      zlabel('Amplitude');
44
45      subplot(2,3,4);
46      plot(syear,s1);
47      xlabel('Time in years');
48      ylabel('Cycles');
49      title('HP Filter cycles');
50
51      [wd,freq]=wig2(s1);
52      subplot(2,3,5);
53      contour(syear,freq,abs(wd'),8), grid on
54      xlabel('Time in years');
55      ylabel('Frequency');
56      title('Contour Map');
57
58      subplot(2,3,6);
59      mesh(syear,freq,abs(wd'));
60      xlabel('Time in years');
61      ylabel('Frequency');
62      zlabel('Amplitude');
63
64 end;

```

```

1 function mywvdgauss()
2
3 %Program used to Wigner Ville Distribution for Gaussian function
4 % Praba Siva
5 % praba@umich.edu
6 % @prabasiva
7 % Filename: mywvdgauss.m
8
9 close all;
10 clear all;
11 sig = [.1,.8,1,1.2,10];
12 j=1;
13 [dummy,winx]=size(sig);
14 winy=2;
15 for i = 1:winx
16     t=-128:127;
17     sigma=sig(i);

```

```

18 coef=nthroot( pi*sigma*sigma , -4 ) ;
19 expo=( t . * t ) / 2 * sigma * sigma ;
20 g=coef*exp( -expo ) ;
21 subplot( winx , winy , j ) ;
22 plot( t , g ) ;
23 title ( [ 'Sigma = ' , num2str( sigma ) ] ) ;
24 set( gca , 'fontsize' , 12 )
25
26 j=j+1;
27 [wd, freq]=wig2( g ) ;
28 subplot( winx , winy , j )
29 j=j+1;
30 contour( t , freq , wd' , 8 )
31 set( gca , 'fontsize' , 12 )
32 end

```

```

1 function mygabor()
2 %Program used to Gabor Coefficients for SP500 & NASDAQ
3 % Praba Siva
4 % praba@umich.edu
5 % @prabasiva
6 % Filename: mygabor.m
7 close all;
8 clear all;
9 [sp500 ,syear]=getData(1);
10 [naq ,nyear]=getData(2);
11     Δm = 8;
12     %M=16;
13     M=50
14     Δn=4;
15     %nn=32;
16     nn=100;
17
18
19 [ s2 , s1 ]= h p f i l t e r ( sp500 , 1600 ) ;
20
21
22 s1=s1';
23 L=length( s1 );
24 t=1:L;
25 N=L/2;
26 nn2=nn/2;
27 sigma=sqrt( (Δm*L)/(Δn * 2 * pi) );
28 c=nthroot( pi*sigma*sigma , -4 ) ;
29 h0 = @(b) c*exp( -((b.*b)/(2*sigma*sigma)) );
30 h = @(ii) h0( mod( ii + N , L ) -N );
31 for m = 1:M
32     for n = 1:nn2
33         c1(m, n)= sum( s1 .*h( mod( t - m*Δm,L ) ) .*exp( -2* pi * i *Δn*n*t/L ) );
34     end
35
36 end
37

```

```

38 [s2,s1]=hpfilter(naq,1600);
39 s1=s1';
40 L=length(s1);
41 t=1:L;
42 N=L/2;
43 nn2=nn/2;
44 sigma=sqrt((Δm*L)/(Δn * 2 * pi));
45 c=nthroot(pi*sigma*sigma,-4);
46 h1=@(b) c*exp(-((b.*b)/(2*sigma*sigma)));
47 h2=@(ii) h1(mod(ii+N,L)-N);
48 for m=1:M
49 for n=1:nn2
50 c2(m,n)=sum(s1.*h2(mod(t-m*Δm,L)).*exp(-2*pi*i*Δn*n*t/L));
51 end
52 end
53 subplot(1,2,1)
54 surf(abs(c1));
55 %Change the text location based on the m & n.
56 %For m=n=16, text(-5,0.4...
57 %For m=n=50 text
58 if M==16
59 text(-5, 0.4, 'Fig a: Gabor Coefficient for ...'
60 log(sp500)', 'FontSize', 16, 'FontWeight', 'bold', 'Color', 'r')
61 else
62 text(-15, 0.4, 'Fig a: Gabor Coefficient for ...'
63 log(sp500)', 'FontSize', 16, 'FontWeight', 'bold', 'Color', 'r')
64 xlabel('Time', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
65 h=get(gca, 'xlabel');
66 set(h, 'rotation', 30)
67 ylabel('Frequency', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
68 h=get(gca, 'ylabel');
69 set(h, 'Position', get(h, 'Position') + [2 4 0])
70 set(h, 'rotation', 140)
71 zlabel('|C(m,n)|^2', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
72 subplot(1,2,2);
73 surf(abs(c2));
74 colormap hsv;
75 if M==16
76 text(-5, 0.4, 'Fig a: Gabor Coefficient for ...'
77 log(nasdaq)', 'FontSize', 16, 'FontWeight', 'bold', 'Color', 'r')
78 else
79 text(-15, 0.4, 'Fig a: Gabor Coefficient for ...'
80 log(nasdaq)', 'FontSize', 16, 'FontWeight', 'bold', 'Color', 'r')
81 xlabel('Time', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
82 h=get(gca, 'xlabel');
83 set(h, 'rotation', 30)
84 ylabel('Frequency', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
85 h=get(gca, 'ylabel');
86 set(h, 'Position', get(h, 'Position') + [2 4 0])
87 set(h, 'rotation', 140)
88 zlabel('|C(m,n)|^2', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
89 end

```

```

1     function myfiltgabor()
2 %Program used to Gabor Coefficients for SP500 & NASDAQ
3 % Praba Siva
4 % praba@umich.edu
5 % @prabasiva
6 % Filename: myfiltgabor.m
7 close all;
8 clear all;
9 [sp500 ,syear]=getData(1);
10 [naq ,nyear]=getData(2);
11     Δm = 8;
12     M=16;
13     %M=50;
14     Δn=4;
15     nn=32;
16     %nn=100;
17     thrcont=3;
18     [s2 ,s1]=hpfilter(sp500 ,1600);
19     s1=s1';
20     L=length(s1);
21     t=1:L;
22     N=L/2;
23     nn2=nn/2;
24     sigma=sqrt((Δm*L)/(Δn * 2 * pi));
25     c=nthroot(pi*sigma*sigma,-4);
26     h0 = @(b) c*exp(-((b.*b)/(2*sigma*sigma)));
27     h = @(ii) h0(mod(ii + N, L)-N);
28     for m = 1:M
29         for n = 1:nn2
30             c1(m, n)= sum(s1.*h(mod(t - m*Δm,L)).*exp(-2*pi*i*Δn*n*t/L));
31         end
32
33     end
34     [s2 ,s1]=hpfilter(naq,1600);
35     s1=s1';
36     L=length(s1);
37     t=1:L;
38     N=L/2;
39     nn2=nn/2;
40     sigma=sqrt((Δm*L)/(Δn * 2 * pi));
41     c=nthroot(pi*sigma*sigma,-4);
42     h1 = @(b) c*exp(-((b.*b)/(2*sigma*sigma)));
43     h2 = @(ii) h1(mod(ii + N, L)-N);
44     for m = 1:M
45         for n = 1:nn2
46             c2(m, n)= sum(s1.*h2(mod(t - m*Δm,L)).*exp(-2*pi*i*Δn*n*t/L));
47         end
48     end
49
50     [m,n]=size(c1);
51     thr=max(abs(c1));
52     spmask = (max(thr)-min(thr))/thrcont;
53     for k = 1:m
54         for j = 1:n
55             if abs(c1(k,j)) < spmask

```

```

56         sp500maskmatrix(k,j)=0;
57     else
58         sp500maskmatrix(k,j)=1;
59     end;
60 end;
62
63 %DRAW THRESHOLD PRESENTATION & MASK OPERATOR
64 figure;
65 subplot(2,1,1);
66 plot(abs(c1), 'LineWidth',2);
67 xlabel('m');
68 ylabel('|C(m,n)|');
69 title('Time Section of Gabor Distribution for SP500');
70 hold on;
71 li(1:m)=spmask;
72 p1=plot(li, 'LineWidth',6, 'Color','b');
73 legend(p1, 'Threshold');
74
75 [m,n]=size(c2);
76 thr(1:m)=max(abs(c2(1:m,:)));
77
78 nasmask = (max(thr)-min(thr))/thrcont;
79 for k = 1:m
80     for j = 1:n
81         if abs(c2(k,j)) < nasmask
82             nasmaskmatrix(k,j)=0;
83         else
84             nasmaskmatrix(k,j)=1;
85         end;
86     end;
87 end;
88 subplot(2,1,2);
89 plot(abs(c2), 'LineWidth',2);
90 xlabel('m');
91 ylabel('|C(m,n)|');
92 title('Time Section of Gabor Distribution for NASDAQ');
93 hold on;
94 li(1:m)=nasmask;
95 p1=plot(li, 'LineWidth',6, 'Color','b');
96 legend(p1, 'Threshold');
97 c1=c1.*sp500maskmatrix;
98 c2=c2.*nasmaskmatrix;
99 figure;
100 subplot(2,1,1);
101 plot(abs(c1), 'LineWidth',2);
102 xlabel('m');
103 ylabel('|C(m,n)|');
104 title('Time Section of Masked Gabor Distribution for SP500');
105 hold on;
106 li(1:m)=spmask;
107 p1=plot(li, 'LineWidth',6, 'Color','b');
108 legend(p1, 'Threshold');
109
110 subplot(2,1,2);
111 plot(abs(c2), 'LineWidth',2);

```

```

112 xlabel('m');
113 ylabel('|C(m,n)|');
114 title('Time Section of Masked Gabor Distribution for NASDAQ');
115 hold on;
116 li(1:m)=nasmask;
117 p1=plot(li,'LineWidth',6,'Color','b');
118 legend(p1,'Threshold');
119
120 figure;
121
122 subplot(1,2,1)
123 surf(abs(c1));
124 colormap hsv;
125 %Change the text location based on the m & n.
126 %For m=n=16, text(-5,0.4...
127 %For m=n=50 text
128 if M==16
129     text(-5, 0.4, 'Fig a: Filtered Gabor Coefficient for ...'
130         log(sp500)', 'FontSize', 16, 'FontWeight', 'bold', 'Color', 'r')
131 else
132     text(-15, 0.4, 'Fig a: Filtered Gabor Coefficient for ...'
133         log(sp500)', 'FontSize', 16, 'FontWeight', 'bold', 'Color', 'r')
134 end;
135 xlabel('Time', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
136 h=get(gca, 'xlabel');
137 set(h, 'rotation', 30)
138 ylabel('Frequency', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
139 h=get(gca, 'ylabel');
140 set(h, 'Position', get(h, 'Position') +[2 4 0])
141 set(h, 'rotation', 140)
142 zlabel('|C(m,n)|^2', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
143
144 subplot(1,2,2);
145 surf(abs(c2));
146 colormap hsv;
147 if M==16
148     text(-5, 0.4, 'Fig a: Filtered Gabor Coefficient for ...'
149         log(nasdaq)', 'FontSize', 16, 'FontWeight', 'bold', 'Color', 'r')
150 else
151     text(-15, 0.4, 'Fig a: Filtered Gabor Coefficient for ...'
152         log(nasdaq)', 'FontSize', 16, 'FontWeight', 'bold', 'Color', 'r')
153 end;
154 xlabel('Time', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
155 h=get(gca, 'xlabel');
156 set(h, 'rotation', 30)
157 ylabel('Frequency', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
158 h=get(gca, 'ylabel');
159 set(h, 'Position', get(h, 'Position') +[2 4 0])
160 set(h, 'rotation', 140)
161 zlabel('|C(m,n)|^2', 'FontSize', 12, 'FontWeight', 'bold', 'Color', 'b')
162
163 end

```

```

1  function myreconsfromgabor()
2  %Program used to Gabor Coefficients for SP500 & NASDAQ
3  % Praba Siva
4  % praba@umich.edu
5  % @prabasiva
6  % Filename: myreconsfromgabor.m
7
8      close all;
9      clear all;
10     [sp500 ,syear]=getData(1);
11     % sp500=log(sp500);
12     [s2 ,s1]=hpfilter(sp500 ,1600);
13
14
15     Δm = 8;
16     M=17;
17     % M=50;
18     Δn=4;
19     nn=84;
20     %nn=100;
21     thrcont=3;
22
23     s1=s1';
24     L=length(s1);
25     t=1:L;
26     N=L/2;
27     nn2=nn/2;
28     sigma=sqrt((Δm*L)/(Δn * 2 * pi));
29     c=nthroot(pi*sigma*sigma,-4);
30     h0 = @(b) c*exp(-((b.*b)/(2*sigma*sigma)));
31     h = @(ii) h0(mod(ii + N, L)-N);
32     for m = 1:M
33         for n = 1:nn2
34             c1(m, n)= sum(s1.*h(mod(t - m*Δm,L)).*exp(-2*pi*i*Δn*n*t/L));
35         end
36
37     end
38
39
40     [m, n]=size(c1);
41
42     H=0.5;
43     %thr=max(abs(c1));
44     %spmask = (max(thr)-min(thr))/thrcont;
45     spmask=mean(c1)+H*std(c1);
46     for k = 1:m
47         for j = 1:n
48             if abs(c1(k, j)) < abs(spmask(k))
49                 sp500maskmatrix(k, j)=0;
50             else
51                 sp500maskmatrix(k, j)=1;
52             end;
53         end;
54     end;
55

```

```

56     for t = 1:L
57         temp=0;
58         for m = 1:M
59             for n = 1:nn2
60                 temp= temp+c1(m,n).*h(mod(t - m*Δm,L)).*exp(2*pi*i*Δn*n*t/L);
61             end
62         end
63         sg(t)=temp;
64
65     end;
66 sg=sg/(2*pi);
67 c1=c1.*sp500maskmatrix;
68
69     for t = 1:L
70         temp=0;
71         for m = 1:M
72             for n = 1:nn2
73                 temp= temp+c1(m,n).*h(mod(t - m*Δm,M)).*exp(2*pi*i*Δn*n*t/L);
74             end
75         end
76         sg2(t)=temp;
77
78     end;
79 sg2=sg2/(2*pi);
80 % plot(real(sg),'LineWidth',3);
81 subplot(2,1,2);
82 hold on;
83 autocorr(s1,100);
84 [c1,c2,c3]=autocorr(s1,100);
85 hold on;
86 p1=plot(c2,c1,'LineWidth',2,'Color','c');
87 hold on;
88 autocorr(real(sg2),100);
89 [d1,d2,d3]=autocorr(real(sg2),100);
90 hold on;
91 p2=plot(d2,d1,'LineWidth',2,'Color','g');
92 legend([p1,p2],'HP Original','HP Filtered');
93 title('Original & Filtered HP Cycles');
94 stdratio=std(real(sg2),1)/std(s1,1)
95 vratio=var(real(sg2),1)/var(s1,1) * 100
96 ccgo=corrcoef(real(sg2),s1)
97
98 subplot(2,1,1);
99 %figure;
100
101 p1=plot(real(sg2),'LineWidth',.5,'DisplayName','Constructed signal');
102 hold on;
103 p2=plot(s1,'DisplayName','Original Signal');
104 legend('show');
105 ylabel('S(t)');
106 xlabel('Time in year');
107 title('log(SP500) original & Filtered HP Cycles');
108
109 figure;
110 clear all;
111
```

```

112      [naq , nyear]=getData ( 2 ) ;
113      naq=log ( naq ) ;
114      [s2 , s1]=hpfilter ( naq , 1600 ) ;
115      Δn = 8 ;
116      % M=16 ;
117      M=50 ;
118      Δn=4 ;
119      %nn=32 ;
120      nn=100 ;
121      thrcont=3 ;
122
123      s1=s1 ' ;
124      L=length ( s1 ) ;
125      t=1:L ;
126      N=L/2 ;
127      nn2=nn/2 ;
128      sigma=sqrt (( Δn*L )/(Δn * 2 * pi )) ;
129      c=nthroot ( pi*sigma*sigma , -4 ) ;
130      h0 = @(b) c*exp ( -(( b.*b )/(2*sigma*sigma))) ;
131      h = @(ii) h0( mod(ii + N , L)-N ) ;
132      for m = 1:M
133          for n = 1:nn2
134              c1(m , n)= sum ( s1 .*h(mod(t - m*Δm,L)) .*exp ( -2*pi*i*Δn*n*t/L) ) ;
135          end
136      end
137
138
139
140      [m , n]=size ( c1 ) ;
141
142      H=0.5 ;
143      %thr=max ( abs ( c1 ) ) ;
144      %spmask = (max ( thr )-min ( thr ))/thrcont ;
145      spmask=mean ( c1 )+H*std ( c1 ) ;
146      for k = 1:m
147          for j = 1:n
148              if abs ( c1(k , j) ) < abs ( spmask(k) )
149                  sp500maskmatrix ( k , j )=0 ;
150              else
151                  sp500maskmatrix ( k , j )=1 ;
152              end ;
153          end ;
154      end ;
155
156      for t = 1:L
157          temp=0 ;
158          for m = 1:M
159              for n = 1:nn2
160                  temp= temp+c1(m , n) .*h(mod(t - m*Δm,L)) .*exp ( 2*pi*i*Δn*n*t/L) ;
161              end
162          end
163          sg ( t )=temp ;
164
165      end ;
166      sg=sg /( 2*pi ) ;
167      c1=c1 .*sp500maskmatrix ;

```

```

168
169     for t = 1:L
170         temp=0;
171         for m = 1:M
172             for n = 1:nn2
173                 temp= temp+c1(m,n).*h(mod(t - m*Δm,M)).*exp(2*pi*i*Δn*n*t/L);
174             end
175         end
176         sg2(t)=temp;
177
178     end;
179     sg2=sg2/(2*pi);
180 % plot(real(sg),'LineWidth',3);
181 subplot(2,1,2);
182 hold on;
183 autocorr(s1,100);
184 [c1,c2,c3]=autocorr(s1,100);
185 hold on;
186 p1=plot(c2,c1,'LineWidth',2,'Color','c');
187 hold on;
188 autocorr(real(sg2),100);
189 [d1,d2,d3]=autocorr(real(sg2),100);
190 hold on;
191 p2=plot(d2,d1,'LineWidth',2,'Color','g');
192 legend([p1,p2],'HP Original','HP Filtered');
193 title('Original & Filtered HP Cycles');
194 stdratio=std(real(sg2),1)/std(s1,1)
195 vratio=var(real(sg2),1)/var(s1,1) * 100
196 ccgo=corrcoef(real(sg2),s1)

197 subplot(2,1,1);
198 %figure;
199
200
201 p1=plot(real(sg2),'LineWidth',.5,'DisplayName','Constructed signal');
202 hold on;
203 p2=plot(s1,'DisplayName','Original Signal');
204 legend('show');
205 ylabel('S(t)');
206 xlabel('Time in year');
207 title('log(NASDAQ) original & Filtered HP Cycles');
208 end

```

```

1 %Program used to Gabor Coefficients for SP500 & NASDAQ
2 % Praba Siva
3 % praba@umich.edu
4 % @prabasiva
5 % Filename: myAttractor.m
6
7 close all;
8 clear all;
9 [sp500,syear]=getData(1);
10 % sp500=log(sp500);
11 [s2,s1]=hpfilter(sp500,1600);

```

```

12
13
14     Δm = 12;
15     M=17;
16     %M=50;
17     Δn=4;
18     nn=84;
19     %nn=100;
20     thrcont=1;
21     s1=s1';
22     L=length(s1);
23     t=1:L;
24     N=L/2;
25     nn2=nn/2;
26     sigma=sqrt((Δn*L)/(Δn * 2 * pi));
27     c=nthroot(pi*sigma*sigma,-4);
28     h0=@(b) c*exp(-((b.*b)/(2*sigma*sigma)));
29     h=@(ii) h0(mod(ii+N,L)-N);
30     for m = 1:M
31         for n = 1:nn2
32             c1(m,n)= sum(s1.*h(mod(t-m*Δm,L)).*exp(-2*pi*i*Δn*n*t/L));
33         end
34     end
35
36     [m,n]=size(c1);
37     H=0.5;
38     %thr=max(abs(c1));
39     %spmask = (max(thr)-min(thr))/thrcont;
40     spmask=mean(c1)+H*std(c1);
41     for k = 1:m
42         for j = 1:n
43             if abs(c1(k,j)) < abs(spmask(k))
44                 if abs(c1(k,j)) < spmask
45
46                     sp500maskmatrix(k,j)=0;
47                 else
48                     sp500maskmatrix(k,j)=1;
49                 end;
50             end;
51         end;
52
53         for t = 1:L
54             temp=0;
55             for m = 1:M
56                 for n = 1:nn2
57                     temp= temp+c1(m,n).*h(mod(t-m*Δm,L)).*exp(2*pi*i*Δn*n*t/L);
58                 end
59             end
60             sg(t)=temp;
61
62         end;
63         sg=sg/(2*pi);
64         c1=c1.*sp500maskmatrix;
65         for t = 1:L
66             temp=0;
67             for m = 1:M

```

```

68         for n = 1:nn2
69             temp= temp+c1(m,n).*h(mod(t - m*Δm,M)).*exp(2*pi*i*Δn*n*t/L);
70         end
71     end
72     sg2(t)=temp;
73 end;
74 Δ=60;
75 % subplot(2,1,1);
76
77 % x=real(s1(1:length(s1)-Δ));
78 % y=real(s1(Δ+1:length(s1)));
79 % plot(x,y);
80 % subplot(2,1,2);
81 figure;
82 x=real(sg2(1:length(sg2)-Δ));
83 y=real(sg2(Δ+1:length(sg2)));
84 plot(x,y,'LineWidth',.7,'Color','r');
85 xlabel('x(t)', 'FontSize',12,'FontWeight','bold','Color','b')
86 ylabel('x(t+T)', 'FontSize',12,'FontWeight','bold','Color','b');
87 title('SP500 Filtered HP cycles','FontSize',12,'FontWeight','bold','Color','b');
88
89
90 sum(sum(sp500maskmatrix))
91 m*n
92
93
94 clear all;
95 [nas,syear]=getData(2);
96 nas=log(nas);
97 [s2,s1]=hpfilter(nas,1600);
98
99
100 Δm = 8;
101 M=16;
102 % M=50;
103 Δn=4;
104 nn=32;
105 %nn=100;
106 thrcont=1;
107 s1=s1';
108 L=length(s1);
109 t=1:L;
110 N=L/2;
111 nn2=nn/2;
112 sigma=sqrt((Δm*L)/(Δn * 2 * pi));
113 c=nthroot(pi*sigma*sigma,-4);
114 h0 = @(b) c*exp(-((b.*b)/(2*sigma*sigma)));
115 h = @(ii) h0(mod(ii + N, L)-N);
116 for m = 1:M
117     for n = 1:nn2
118         c1(m, n)= sum(s1.*h(mod(t - m*Δm,L)).*exp(-2*pi*i*Δn*n*t/L));
119     end
120 end
121 [m,n]=size(c1);
122 H=0.5;

```

```

124      %thr=max( abs(c1));
125      %spmask = (max(thr)-min(thr))/thrcont;
126      spmask=mean(c1)+H*std(c1);
127      for k = 1:m
128          for j = 1:n
129              if abs(c1(k,j)) < abs(spmask(k))
130                  if abs(c1(k,j)) < spmask
131
132                      sp500maskmatrix(k,j)=0;
133                  else
134                      sp500maskmatrix(k,j)=1;
135                  end;
136              end;
137          end;
138
139          for t = 1:L
140              temp=0;
141              for m = 1:M
142                  for n = 1:nn2
143                      temp= temp+c1(m,n).*h(mod(t - m*Δm,L)).*exp(2*pi*i*Δn*n*t/L);
144                  end
145              end
146              sg(t)=temp;
147
148          end;
149          sg=sg/(2*pi);
150          c1=c1.*sp500maskmatrix;
151          for t = 1:L
152              temp=0;
153              for m = 1:M
154                  for n = 1:nn2
155                      temp= temp+c1(m,n).*h(mod(t - m*Δm,M)).*exp(2*pi*i*Δn*n*t/L);
156                  end
157              end
158              sg2(t)=temp;
159          end;
160          Δ=60;
161          figure;
162          x=real(sg2(1:length(sg2)-Δ));
163          y=real(sg2(Δ+1:length(sg2)));
164          plot(x,y,'LineWidth',.7,'Color','b');
165          xlabel('x(t)', 'FontSize',12,'FontWeight','bold','Color','r')
166          ylabel('x(t+T)', 'FontSize',12,'FontWeight','bold','Color','r');
167          title('NASDAQ Filtered HP cycles', 'FontSize',12,'FontWeight','bold','Color','r');
168
169          sum(sum(sp500maskmatrix))
170          m*n

```

```

1 function myFDfilter()
2 %Program used to Gabor Coefficients for SP500 & NASDAQ
3 % Praba Siva
4 % praba@umich.edu
5 % @prabasiva

```

```

6  % Filename: myFDfilter.m
7  close all;
8  clear all;
9  [sp500,syear]=getData(1);
10 process(sp500,1);
11 [naq,nyear]=getData(2);
12 process(naq,2);
13
14
15 function process(sp500,flag)
16
17 % First differencing
18 % Graph looks identifical as shown in the P.Chen's paper
19 % P.Chen's paper says it is log of sp500 data but he used sp500
20 % The graph in his paper shows it all.
21
22 len=size(sp500);
23 t1=sp500(1:len-1);
24 t2=sp500(2:len);
25 fd=t2-t1;
26 lenfd=size(fd);
27 Δ=40;
28 x=fd(1:lenfd-Δ);
29 y=fd(Δ+1:lenfd);
30 figure;
31 subplot(2,1,1);
32 scatter(x,y,'b');
33 xlabel('X(t)', 'Fontsize', 20);
34 ylabel('X(t+T)', 'Fontsize', 20);
35 if flag ==1
36 title('FD Series for SP500 Index','Fontsize',20);
37 else
38 title('FD Series for NASDAQ Index','Fontsize',20);
39 end;
40 y=fd;
41 n=size(y);
42 % C and Phi values are given in the P.Chen's paper.
43
44 c = 0.006*0.002;
45 phi = [0.000265*0.043, -0.81*0.043];
46
47 % White noise created based on the standard deviation value
48 noise = 0.033*randn(n(1),1);
49
50 for t=3:n
51 x(t-2) = c + phi(1)*y(t-1) + phi(2)*y(t-2) + noise(t-2);
52 end
53 lenfd=size(x);
54 Δ=5;
55 x1=x(1:lenfd-Δ);
56 y1=x(Δ+1:lenfd);
57 subplot(2,1,2);
58 scatter(x1,y1,'r');
59 xlabel('X(I)', 'Fontsize', 20);
60 ylabel('X(I+T)', 'Fontsize', 20);
61 if flag ==1

```

```

62         title('AR(2) for FDSeries of SP500 index','FontSize',20);
63     else
64         title('AR(2) for FDSeries of NASDAQ index','FontSize',20);
65     end;
66 end
67 end

```

```

1 function mydft()
2
3 abc=dftmtx(50);
4 subplot(2,2,1);surf(real(abc));
5 title('Discrete Fourier Transform - Real Part','FontSize',18,'FontWeight','bold')
6
7 subplot(2,2,2);surf(imag(abc));
8 title('Discrete Fourier Transform - Imaginary Part','FontSize',18,'FontWeight','bold')
9
10
11 abc=inv(abc);
12 subplot(2,2,3);surf(real(abc));
13 title('Inverse Discrete Fourier Transform - Real Part','FontSize',18,'FontWeight','bold')
14
15 subplot(2,2,4);surf(imag(abc));
16 title('Inverse Discrete Fourier Transform - Imaginary ...
Part','FontSize',18,'FontWeight','bold')

```

```

1 function drawCorrDim()
2 %Program used to Correlation Dimension for both SP500 & NASDAQ
3 % Praba Siva
4 % praba@umich.edu
5 % @prabasiva
6 % Filename: drawCorrDim.m
7
8 close all;
9 clear all;
10 titlestr={' for SP500',' for NASDAQ'};
11 for i = 1:2
12     [index,year]=getData(i);
13     [trend,cycles]=hpfilter(index,14400);
14     N=size(cycles);
15     N=N(1);
16     figure;
17     for delay = 2:2:8
18         subplot(2,2,delay/2);
19         for ed = 2:2:30
20             [r,c] = c2(cycles,ed,delay,400,500);
21             plot(r,c,'LineWidth',2);
22             set(gca,'fontsize',14,'fontweight','bold');
23
24             xlabel('$\log_{10}(\epsilon)$','Interpreter','Latex','FontSize',14,'FontWeight','bold')
25             ylabel('$\log_{10} D$' ...

```

```

26      C(\epsilon)$,'Interpreter','Latex','FontSize',14,'FontWeight','bold')
27      xlabel('log10(\epsilon)', 'FontSize',18,'FontWeight','bold')
28      ylabel('log10 C(\epsilon)', 'FontSize',18,'FontWeight','bold')
29
30      if i==1
31          title({['Correlation sum for SP500 indexes with embedding dimensions = ...',
32                  '2,4,..30'],
33                  ;['and delay = ',num2str(delay)]}, 'fontsize',16);
34      else
35          title({['Correlation sum for NASDAQ indexes with embedding dimensions = ...',
36                  '2,4,..30'],
37                  ;['and delay = ',num2str(delay)]}, 'fontsize',16);
38
39      end
40      textlegmain(ed/2)={['m = ',num2str(ed)]};
41      edmat(delay/2,ed/2)=abs(c/r);
42      hold on;
43
44      legend(textlegmain, 'Location', 'northwest', 'FontSize',8,'FontWeight','bold');
45      legend('show');
46      hold on;
47
48      figure;
49      for delay = 2:2:8
50          plot(2:2:30,edmat(delay/2,:),'LineWidth',3);
51          set(gca, 'fontsize',14,'fontweight','bold');
52          xlabel('Embedding dimension', 'FontSize',18,'FontWeight','bold');
53          ylabel('Correlation Dimension', 'FontSize',18,'FontWeight','bold');
54          if i==1
55              title('SP 500 Index', 'FontSize',24,'FontWeight','bold');
56          else
57              title('NASDAQ index', 'FontSize',24,'FontWeight','bold');
58          end
59          textleg(delay/2)={['Delay = ',num2str(delay)]};
60          mean(edmat(delay/2,:));
61          hold on;
62
63          cdim= mean(mean(edmat));
64          ldata(1:15)=cdim;
65          plot(2:2:30,ldata,'--', 'LineWidth',3);
66          hold on;
67          legend(textleg, 'Location', 'northwest', 'FontSize',10,'FontWeight','bold');
68      end;

```

## APPENDIX C

### Mathematical Proof

#### C.1 Gabor Elementary function

$$\psi(t) = \underbrace{e^{-\alpha^2(t-t_0)^2}}_v \overbrace{e^{j2\pi f_0 t + \phi}}^w \quad (C.1)$$

$v$  represents the probability function and  $w$  represents simple harmonic oscillator.  $\Psi(f)$  is the GEF in the frequency domain. The GEF in the frequency domain is attained by taking the Fourier transform of the GEF.

$$\Psi(f) = \int_{-\infty}^{\infty} \psi(t) e^{-j2\pi f t} dt; \quad \Psi(f) = \int_{-\infty}^{\infty} e^{-\alpha^2(t-t_0)^2} e^{j2\pi f_0 t + \phi} e^{-j2\pi f t} dt$$

$$\Psi(f) = \int_{-\infty}^{\infty} e^{-\alpha^2(t-t_0)^2} e^{j2\pi t(f_0-f) + \phi} dt$$

$$\Psi(f) = e^{\phi} \int_{-\infty}^{\infty} e^{-\alpha^2(t-t_0)^2} e^{j2\pi t(f_0-f)} dt$$

when  $t_0$  is 0, then

$$\Psi(f) = e^\phi \int_{-\infty}^{\infty} e^{-\alpha^2 t^2} e^{j2\pi t(f_0 - f)} dt \quad (\text{C.2})$$

This is of the form.

$$\int_{-\infty}^{\infty} e^{2bx - ax^2} dx = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{a}}$$

where  $b = j\pi(f_0 - f)$  and  $a = \alpha^2$

$$\Psi(f) = \sqrt{\frac{\pi}{\alpha^2}} e^{\frac{(j\pi(f_0 - f))^2}{\alpha^2}} e^\phi$$

$$\Psi(f) = \sqrt{\frac{\pi}{\alpha^2}} e^{-(\frac{\pi}{\alpha})^2(f_0 - f)^2 + \phi}$$

$\alpha$  is connecting the GEF between time and frequency domain.  $\psi(t)$  and  $\Psi(f)$  occupies the minimum uncertainty in time and frequency domain.

### C.1.1 Proof: GEF has minimum uncertainty in the time-frequency domain

I believe, we will better understand physical or mathematical concept by performing a step wise derivation. Let me do a step wise derivation to prove that GEF has a minimum uncertainty for a special case. Let me simplify the GEF by taking GEF at zero frequency,  $t_0 = 0$  and  $\phi = 0$ , the Gabor elementary function and Fourier transform of GEF are given by

$$\psi(t) = e^{-\alpha^2 t^2}$$

$$\Psi(f) = \sqrt{\frac{\pi}{\alpha^2}} e^{-(\frac{\pi}{\alpha})^2 f^2}$$

Effective duration  $\Delta t$  is given by:

$$\Delta t = \sqrt{\frac{\int_{-\infty}^{\infty} e^{-\alpha^2 t^2} t^2 e^{-\alpha^2 t^2} dt}{\int_{-\infty}^{\infty} e^{-\alpha^2 t^2} e^{-\alpha^2 t^2} dt}};$$

Let me take the denominator first

$$\int_{-\infty}^{\infty} e^{-\alpha^2 t^2} e^{-\alpha^2 t^2} dt = \int_{-\infty}^{\infty} e^{-2\alpha^2 t^2} dt$$

The above equation is of the form and it only applies when  $a > 0$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

where  $a = 2\alpha^2$

$$\int_{-\infty}^{\infty} e^{-2\alpha^2 t^2} dt = \sqrt{\frac{\pi}{2\alpha^2}}$$

Let me take the numerator now,

$$\int_{-\infty}^{\infty} e^{-\alpha^2 t^2} t^2 e^{-\alpha^2 t^2} dt = \int_{-\infty}^{\infty} t^2 e^{-2\alpha^2 t^2} dt$$

The above equation is of the form.

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a^3}}$$

where  $a = 2\alpha^2$

$$\int_{-\infty}^{\infty} t^2 e^{-2\alpha^2 t^2} dt = \frac{1}{2} \sqrt{\frac{\pi}{(2\alpha^2)^3}} = \frac{\sqrt{\pi}}{4\sqrt{2}\alpha^3}$$

Let me apply both numerator and denominator value to get the effective duration

$\Delta t$

$$\Delta t = \sqrt{\frac{\frac{\sqrt{\pi}}{4\sqrt{2}\alpha^3}}{\sqrt{\frac{\pi}{2\alpha^2}}}};$$

Straight forward steps to simply the value of  $\Delta t$

$$\Delta t = \sqrt{\frac{\sqrt{\pi}}{4\sqrt{2}\alpha^3}} \sqrt{\frac{2\alpha^2}{\pi}};$$

$$\Delta t = \sqrt{\frac{\sqrt{\pi}}{4\sqrt{2}\alpha^3}} \sqrt{\frac{2\alpha^2}{\pi}};$$

$$\Delta t = \sqrt{\frac{1}{4\alpha^2}};$$

$$\Delta t = \frac{1}{2\alpha} \tag{C.3}$$

Let me do the similar steps to calculate the effective frequency  $\Delta f$ . The frequency representation of the GEF is given by,

$$\Psi(f) = \sqrt{\frac{\pi}{\alpha^2}} e^{-(\frac{\pi}{\alpha})^2 f^2}$$

Effective frequency  $\Delta f$  is given by,

$$\Delta f = \sqrt{\frac{\int_{-\infty}^{\infty} \sqrt{\frac{\pi}{\alpha^2}} e^{-(\frac{\pi}{\alpha})^2 f^2} f^2 \sqrt{\frac{\pi}{\alpha^2}} e^{-(\frac{\pi}{\alpha})^2 f^2} df}{\int_{-\infty}^{\infty} \sqrt{\frac{\pi}{\alpha^2}} e^{-(\frac{\pi}{\alpha})^2 f^2} \sqrt{\frac{\pi}{\alpha^2}} e^{-(\frac{\pi}{\alpha})^2 f^2} df}};$$

Let me take the denominator first.

$$\int_{-\infty}^{\infty} \sqrt{\frac{\pi}{\alpha^2}} e^{-(\frac{\pi}{\alpha})^2 f^2} \sqrt{\frac{\pi}{\alpha^2}} e^{-(\frac{\pi}{\alpha})^2 f^2} df = \frac{\pi}{\alpha^2} \int_{-\infty}^{\infty} e^{-2(\frac{\pi}{\alpha})^2 f^2} df$$

The above equation is of the form and it only applies when  $a > 0$

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$$

where  $a = 2(\frac{\pi}{\alpha})^2$

$$\frac{\pi}{\alpha^2} \int_{-\infty}^{\infty} e^{-2(\frac{\pi}{\alpha})^2 f^2} df = \frac{\pi}{\alpha^2} \sqrt{\frac{\pi}{2(\frac{\pi}{\alpha})^2}}$$

Let  $\beta = \frac{\pi}{\alpha}$

$$\frac{\pi}{\alpha^2} \int_{-\infty}^{\infty} e^{-2(\frac{\pi}{\alpha})^2 f^2} df = \frac{\beta}{\alpha} \sqrt{\frac{\pi}{2\beta^2}} = \frac{1}{\alpha} \sqrt{\frac{\pi}{2}}$$

Let me take the numerator now.

$$\begin{aligned} & \int_{-\infty}^{\infty} \sqrt{\frac{\pi}{\alpha^2}} e^{-(\frac{\pi}{\alpha})^2 f^2} f^2 \sqrt{\frac{\pi}{\alpha^2}} e^{-(\frac{\pi}{\alpha})^2 f^2} df \\ &= \frac{\pi}{\alpha^2} \int_{-\infty}^{\infty} f^2 e^{-2(\frac{\pi}{\alpha})^2 f^2} df \end{aligned}$$

Substitute  $\beta$  in above equation.

$$= \frac{\beta}{\alpha} \int_{-\infty}^{\infty} f^2 e^{-2\beta^2 f^2} df$$

The above equation is of the form.

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{a^3}}$$

where  $a = 2\beta^2$

$$= \frac{\beta}{\alpha} \frac{1}{2} \sqrt{\frac{\pi}{8\beta^6}}$$

$$= \frac{\beta}{\alpha} \frac{\sqrt{\pi}}{4\sqrt{2}\beta^3}$$

Substitute the value of  $\beta$

$$= \frac{1}{\alpha} \frac{\sqrt{\pi}}{4\sqrt{2}\beta^2} = \frac{1}{\alpha} \frac{\sqrt{\pi}\alpha^2}{4\sqrt{2}\pi^2} = \frac{\sqrt{\pi}\alpha}{4\sqrt{2}\pi^2}$$

Apply the value of numerator and denominator of  $\Delta f$

$$\Delta f = \sqrt{\frac{\frac{\sqrt{\pi}\alpha}{4\sqrt{2}\pi^2}}{\frac{1}{\alpha} \sqrt{\frac{\pi}{2}}}} = \sqrt{\frac{\sqrt{\pi}\alpha^2}{4\sqrt{2}\pi^2} \sqrt{\frac{2}{\pi}}}$$

Step wise simplification steps to get the value of  $\Delta f$

$$\Delta f = \sqrt{\frac{\alpha^2}{4\pi^2}}$$

$$\Delta f = \frac{\alpha}{2\pi} \quad (\text{C.4})$$

Apply both the value of  $\Delta f$  and  $\Delta t$  from equation (8) and equation (9)

$$\Delta t \Delta f = \frac{\alpha}{2\pi} \frac{1}{2\alpha}$$

$$\boxed{\Delta t \Delta f = \frac{1}{4\pi}}$$

Hence the proof.

## **BIBLIOGRAPHY**